

SYMMETRIC EXCLUSION AS A RANDOM ENVIRONMENT: INVARIANCE PRINCIPLE

BY MILTON JARA¹ AND OTÁVIO MENEZES²

¹IMPA, mjara@impa.br

²CAMGSD, Instituto Superior Técnico, otavio.menezes@tecnico.ulisboa.pt

We establish an invariance principle for a one-dimensional random walk in a dynamic random environment given by a speed-change exclusion process. The jump probabilities of the walk depend on the configuration of the exclusion in a finite box around the walker. The environment starts from equilibrium. After a suitable space-time rescaling, the random walk converges to a sum of two independent processes: a Brownian motion and a Gaussian process with stationary increments.

1. Introduction. The present paper establishes an invariance principle for a family of random walks in dynamic random environments (RWDRE). The model we analyse was introduced in [6], where the authors proved laws of large numbers both for the random walk and for the environment as seen by the walker. Later, in [7], they proved the corresponding large deviations principle. Our article completes the picture by proving an invariance principle. We define the model in Section 2, but a good picture to keep in mind for now is the following: color the integers with the colors L and R. Use the coloring to define a continuous time random walk on \mathbb{Z} , that has a drift to the left on sites of the color L and a drift to the right on sites of the color R. Now let the coloring change in a Markovian way by swapping the colors of sites x and $x + 1$ at some positive rate $r_{x,x+1}$, possibly dependent on the colors of $x - 1$ and $x + 2$. This gives rise to a continuous time RWDRE. We show that when space, time and swapping rates are scaled in a certain way the trajectory of the walk converges to the sum of a Brownian motion and a Gaussian process of stationary increments, independent of the Brownian motion. For certain choices of the parameters, the limiting Gaussian process is a fractional Brownian motion of Hurst parameter $\frac{3}{4}$. Moreover, the same scaling limit holds if the drift of the walk at x is allowed to depend not only on the color of x but also on the colors of $x - k, \dots, x + k$ for some fixed k . The variance of the limiting Brownian motion and the covariances of the limiting Gaussian process can be computed explicitly.

Our article fits into two niches in the literature: random walks on dynamic random environments (RWDRE) and scaling limits of interacting particle systems. The symmetric exclusion in [5] was introduced as an example of dynamic random environment with slowly decaying time correlations. This followed a series of works dealing with random walks on “fast mixing” environments, which are models where, in some sense, the environment refreshes itself after a finite (but maybe random) number of jumps of the walk. In this setting, one expects the walk to behave as if the environment were deterministic. That is, a law of large numbers holds, fluctuations around the limit are Gaussian and large deviation probabilities decay exponentially fast. See [1] for an overview. Fast mixing environments are the opposite, in a sense, to static environments, where the (random) transition kernel for the walk at each site is the same at all times. In the static scenario, the walk can get trapped for a long time in small regions, leading to scaling limits different from those of the fast-mixing case. For

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instance, it can present subdiffusive behavior and polynomial decay of the large deviation probabilities, see [36]. In the fast mixing scenario, the traps dissolve before the walk can get stuck for too long. What happens then in the intermediate situation, where the environment is dynamic but not too fast? This question motivated the study of symmetric exclusion as a random environment, as well as of other conservative interacting particle systems, see [8, 10, 12, 21, 23]. The goal of these works is to prove laws of large numbers, central limit theorems and large deviation principles, and most results hold only in a subset of the space of parameters. Simulations reported in [8] indicate that trapping may happen when the dynamic random environment is the one-dimensional exclusion processes, indicating that the random walk should have anomalous scaling on some region of parameters. We also mention the recent works [4] and [3], analysing a new family of random environments interpolating between static and fast mixing.

The model introduced in [6] plays with the idea of slow mixing in a different way. Let n be a scaling parameter, that will be sent to ∞ . When the environment is given by the symmetric exclusion process, it is reasonable to introduce a diffusive space-time scaling $x \mapsto \frac{x}{n}$, $t \mapsto tn^2$. Under this scaling, the evolution of the exclusion process satisfies a law of large numbers (the so-called *hydrodynamic limit*) and a central limit theorem. In [6], the jump rate of the random walk is slowed down by a factor $\frac{\lambda}{n}$, where $\lambda > 0$. Then, at least heuristically, between two jumps of the random walker the environment achieves local equilibrium in a region of size \sqrt{n} around the walker, which is exactly the size at which fluctuations appear. Therefore, the walker should see a randomly evolving equilibrium of the environment process. This heuristics can be made rigorous by means of the formalism of hydrodynamic limits of interacting particle systems, which yields laws of large numbers [6] and large deviation principles [7].

Here we show a central limit theorem for the random walk under the scaling introduced in [6], assuming the dynamic random environment is stationary in time. The scaling limit is then a mixture of two independent Gaussian processes: a Brownian motion and the process with stationary increments introduced in [19] as the scaling limit of the occupation time of the origin in the weakly asymmetric exclusion process. The role of the weak asymmetry in [19] is played here by the asymptotic speed of the random walk. When the asymptotic speed is zero, this Gaussian process is a fractional Brownian motion of Hurst exponent $H = 3/4$. An earlier example of non-Brownian scaling limits of additive functionals is the work [33].

From the hydrodynamic limits side, we compute the scaling limit of an additive functional of an interacting particle system without explicit knowledge of the invariant measures. On our way to obtain this result, we prove an estimate on the relative entropy between the environment process at time t and a product measure, using a modification of Yau's Relative Entropy method, introduced in [35]. This method is nowadays a standard tool for proving hydrodynamic limits. However, the current state of the art only yields a bound of order $o(tn)$. This bound is enough to derive a law of large numbers and also a large deviations principle, but it is far from what is required in order to prove a central limit theorem. Our main technical innovation is the derivation of a bound of order $\mathcal{O}(t)$, obtained with a different implementation of the Relative Entropy method, which is of independent interest. With this bound on the entropy we are able to prove a replacement lemma without spatial averaging, as in [25].

The problem we address can also be seen as a variation on the problem of the tagged particle. The seminal article on this problem is [28], where a method for establishing scaling limits of tagged particles was introduced. It looks at the *environment as seen from the particle*, $\xi_t(x) := \eta_t(x + x_t)$ (η_t is the particle system and x_t is the tagged particle) and writes x_t as a martingale plus an additive functional. The martingale part can be handled by the Martingale Functional Central Limit Theorem (MFCLT), see Theorem 2.7. The problem reduces, therefore, to the study of the scaling limit of the additive functional. The work [28] gives

sufficient conditions to approximate this additive functional by a martingale, thus establishing Brownian motion as the scaling limit of the tagged particle. We point the reader to [29] for a comprehensive exposition of the martingale approximation method and to [2] for an application in RWRE. In our model, the additive functional does not converge to Brownian motion, but to a singular functional of the density fluctuation field associated to the environment process. This functional turns out to be identical to the scaling limit of the occupation time of the origin of a stationary, weakly asymmetric exclusion process. The problem of the asymptotic behavior of the occupation time was already considered in the 60s [31] in the case of independent particles and generalized to the case of interaction by branching [22] and of the exclusion process, [33] and [19]. We follow the approach of [19], adapted to deal with the lack of knowledge of the invariant measure of the environment as seen by the walker.

2. Notation and results.

2.1. *A warm-up example.* Let $(\eta_t)_{t \geq 0}$ be the simple symmetric exclusion process (SSEP) on \mathbb{Z} , namely the Markov process that takes values in $\{0, 1\}^{\mathbb{Z}}$ and is generated by the operator

$$L^{ex} f(\eta) = \sum_{x \in \mathbb{Z}} [f(\eta^{x,x+1}) - f(\eta)],$$

where $f : \{0, 1\}^{\mathbb{Z}}$ is a local function (i.e., $f(\eta)$ depends on finitely many $\eta(x)$) and $\eta^{x,x+1}$ is obtained from η by interchanging the values of $\eta(x)$ and $\eta(x + 1)$. Let $\rho \in (0, 1)$ and let ν_ρ denote the Bernoulli product measure in $\{0, 1\}^{\mathbb{Z}}$, that is, the $\eta(x)$ are i.i.d. and their mean is ρ . We assume that η_0 has law ν_ρ . In that case, the law of η_t is ν_ρ for any $t \geq 0$.

The process $(\eta_t)_{t \geq 0}$ will be a dynamic random environment for the random walk that is defined as follows. Let $n \in \mathbb{N}$ be a scaling parameter and let $\eta_t^n = \eta_{tn^2}$. Let $p, q \geq 0$ be such that $p + q = 1$ and let $\lambda > 0$. Given a realization of the rescaled exclusion process $(\eta_t^n)_{t \geq 0}$, let $(x_t^n)_{t \geq 0}$ be the time-inhomogeneous chain with the following dynamics: the chain waits an exponential time of rate λn , at the end of which it jumps to one of its two neighbors. To make its choice, it looks at the value $\eta_t^n(x)$ of the SSEP at its current location x . If $\eta_t^n(x) = 1$, the chain jumps to the right with probability p and to the left with probability q . If $\eta_t^n(x) = 0$, the probabilities are reversed: the chain jumps to the right with probability q and to the left with probability p .

In [6], it was proved

$$\lim_{n \rightarrow \infty} \frac{x_t^n}{n} = v(\rho)t$$

in probability, where $v(\rho) = \lambda(p - q)(2\rho - 1)$, that is, the random walk $(x_t^n)_{t \geq 0}$ satisfies a law of large numbers. The corresponding large deviations principle was proved in [5]. Our goal is to prove the corresponding central limit theorem:

$$\lim_{n \rightarrow \infty} \frac{x_t^n - v(\rho)tn}{\sqrt{n}} = \sqrt{\lambda}B_t + 2\lambda(p - q)Z_t =: X_t$$

in distribution, where $(B_t)_{t \geq 0}$ is a standard Brownian motion and $(Z_t)_{t \geq 0}$ is a Gaussian process with stationary increments, independent of $(B_t)_{t \geq 0}$.

The variance of the process $(Z_t)_{t \geq 0}$ can be explicitly computed and it is equal to

$$\rho(1 - \rho)\sqrt{\frac{2}{\pi}} \int_0^t \frac{(t - s)e^{-\frac{v(\rho)^2}{2}s}}{\sqrt{s}} ds.$$

Recently, in [20], the authors proved a central limit theorem for the *unscaled* random walk in dynamic random environment, which in our setting corresponds to a random walk jumping

at rate λn^2 instead of λn . Notice that this also corresponds to taking $n = 1$. Let us write x_t instead of x_t^1 . In our setting, [20] shows that there exists a nondecreasing function $\tilde{v}(\rho)$ such that

$$\lim_{n \rightarrow \infty} \frac{x_{nt}}{n} = \tilde{v}(\rho)t$$

for any $t \neq 0$ and any $\rho \in (0, 1)$ with the exception of at most two densities. They also prove that whenever $\tilde{v}(\rho) \neq 0$, the corresponding central limit theorem holds with a Brownian limit. It is believed that when $\tilde{v}(\rho) = 0$, the limit is not Brownian. The following heuristics provides support for this claim, and also gives a conjectured limit for the fluctuations of the random walk in that case.

As we pointed out above, removing the slow scale of the random walk is the same as taking a jump rate λ that grows with n . Therefore, it makes sense to study the behaviour of the process $(X_t)_{t \geq 0}$ as $\lambda \rightarrow \infty$. The process $(X_t)_{t \geq 0}$ is Gaussian, so it is enough to look at its variance. Although we expect $v(\rho)$ to be different from $\tilde{v}(\rho)$, it is reasonable to expect that they are nonzero in the same density region. Notice that

$$\int_0^t \frac{(t-s)e^{-\frac{v(\rho)^2}{2}s}}{\sqrt{s}} ds = \frac{1}{v(\rho)} \int_0^{v(\rho)^2 t} \frac{(t-u/v(\rho)^2)e^{-\frac{1}{2}u}}{\sqrt{u}} du.$$

Since the integral

$$\int_0^\infty \frac{e^{-\frac{1}{2}u}}{\sqrt{u}} du$$

is finite, we see that there exists a positive, finite constant $\sigma(\rho)$ such that

$$\lim_{v(\rho) \rightarrow \infty} v(\rho)E[Z_t^2] = \sigma(\rho)t.$$

Since $v(\rho)$ is proportional to λ , $2\sqrt{\lambda}(p-q)Z_t$ converges to a Brownian motion as $\lambda \rightarrow \infty$, and therefore the limit of $X_t/\sqrt{\lambda}$ is Brownian, which is coherent with the central limit theorem proved in [20].

If $v(\rho) = 0$, which is the case for $\rho = 1/2$, then the term $2\lambda(p-q)Z_t$ is dominant over $\sqrt{\lambda}B_t$ and we have that

$$\lim_{\lambda \rightarrow \infty} \frac{X_t}{\lambda} = 2(p-q)Z_t.$$

Moreover,

$$E[Z_t^2] = \frac{1}{3} \sqrt{\frac{2}{\pi}} t^{3/2},$$

so the process $(Z_t)_{t \geq 0}$ is a fractional Brownian motion of Hurst exponent $3/4$. We formulate the following conjecture.

CONJECTURE 2.1. At $\rho = 1/2$,

$$\lim_{n \rightarrow \infty} \frac{x_{tn}}{n^{3/4}} = Z_t,$$

where $(Z_t)_{t \geq 0}$ is a fractional Brownian motion of Hurst index $3/4$.

2.2. *General setting.* Let us first describe the dynamic environment. Let $\Omega = \{0, 1\}^{\mathbb{Z}}$. For $x \in \mathbb{Z}$ let $\tau_x : \Omega \rightarrow \Omega$ denote the canonical shift: $\tau_x \eta(y) = \eta(x + y)$ for any $\eta \in \Omega$ and any $y \in \mathbb{Z}$. We say that a function $f : \Omega \rightarrow \mathbb{R}$ has *support contained* in a set $A \subseteq \mathbb{Z}$ if $f(\eta) = f(\xi)$ whenever $\eta(x) = \xi(x)$ for every $x \in A$. We say that f is a *local function* if its support is contained in some finite set.

ASSUMPTION 2.2. Let $c : \Omega \rightarrow [0, \infty)$ satisfy:

- (i) *Finite range:* $c(\cdot)$ is a local function;
- (ii) *Ellipticity:* There exists $\varepsilon_0 > 0$ such that $c(\eta) \geq \varepsilon_0$ for any $\eta \in \Omega$;
- (iii) *Reversibility:* $c(\eta) = c(\xi)$ whenever $\eta(x) = \xi(x)$ for all $x \neq 0, 1$, that is, the support of $c(\cdot)$ is contained in $\mathbb{Z} \setminus \{0, 1\}$.

Let $c_x : \Omega \rightarrow \mathbb{R}$ be defined as $c_x(\eta) = c(\tau_x \eta)$ for any $\eta \in \Omega$. For a local function $f : \Omega \rightarrow \mathbb{R}$, define $L_b f : \Omega \rightarrow \mathbb{R}$ as

$$L_b f(\eta) = \sum_{x \in \mathbb{Z}} c_x(\eta) [f(\eta^{x,x+1}) - f(\eta)],$$

where $\eta^{x,x+1}$ is defined as

$$\eta^{x,x+1}(z) = \begin{cases} \eta(x + 1), & z = x, \\ \eta(x), & z = x + 1, \\ \eta(z), & z \neq x, x + 1. \end{cases}$$

Since f is local, only a finite number of terms in the sum defining $L_b f$ are nonzero.

The lattice gas with interaction rate $c(\cdot)$ is the Markov process $(\eta_t)_{t \geq 0}$ defined in Ω and generated by the operator L_b . Notice that the SSEP corresponds to the choice $c \equiv 1$.

For $\rho \in [0, 1]$, let ν_ρ be the Bernoulli product measure in Ω with density ρ : for any $x_1, \dots, x_\ell \in \mathbb{Z}$,

$$\nu_\rho \{ \eta(x_1) = \dots = \eta(x_\ell) = 1 \} = \rho^\ell.$$

Thanks to the reversibility condition iii), these measures are invariant under the evolution of $(\eta_t)_{t \geq 0}$. From now on, we fix $\rho \in (0, 1)$ and we assume that η_0 (and therefore η_t for any $t \geq 0$) has law ν_ρ .

Now let us describe how our random walk moves. Let $\mathcal{R} \subseteq \mathbb{Z} \setminus \{0\}$ be a finite set. For each $z \in \mathcal{R}$, let $r_z : \Omega \rightarrow [0, \infty)$ be a local function. Let $n \in \mathbb{N}$ be a scaling parameter and let $(\eta_t^n)_{t \geq 0}$ be the lattice gas defined above, speeded up by n^2 , that is, $\eta_t^n = \eta_{tn^2}$. We denote by \mathbb{P}_n the law of $(\eta_t^n)_{t \geq 0}$ and by \mathbb{E}_n the expectation with respect to \mathbb{P}_n . For $x \in \mathbb{Z}$ and $z \in \mathcal{R}$, define $r_z(\cdot, x) : \Omega \rightarrow [0, \infty)$ as $r_z(\eta, x) = r_z(\tau_x \eta)$.

We define the process $(x_t^n)_{t \geq 0}$ as the random walk that jumps from x to $x + z$ with instantaneous rate $n r_z(\eta_t^n, x)$. If we do not keep track of the environment $(\eta_t^n)_{t \geq 0}$, the process $(x_t^n)_{t \geq 0}$ is not a Markov process. However, the couple $((\eta_t^n, x_t^n))_{t \geq 0}$ turns out to be a Markov process, generated by the operator

$$\begin{aligned} \mathcal{L}_n f(\eta, x) &= n^2 \sum_{y \in \mathbb{Z}} c_y(\eta) [f(\eta^{y,y+1}, x) - f(\eta, x)] \\ &\quad + n \sum_{z \in \mathcal{R}} r_z(\eta, x) [f(\eta, x + z) - f(\eta, x)]. \end{aligned}$$

We are now able to state the law of large numbers and the central limit theorem for $(x_t^n)_{t \geq 0}$. Define

$$(2.1) \quad v(\rho) = \int \sum_{z \in \mathcal{R}} z r_z d\nu_\rho.$$

Fix a finite time horizon $T > 0$ and denote by $\mathcal{D}([0, T], \mathbb{R})$ the space of càdlàg, real-valued trajectories. The following result was proved in [6] in the case $c \equiv 1$.

PROPOSITION 2.3. *Let $v(\rho)$ be as in (2.1). Then, for any $T > 0$,*

$$\lim_{n \rightarrow \infty} \frac{x_t^n}{n} = v(\rho)t$$

in law with respect to the J_1 -Skorohod topology of $\mathcal{D}([0, T], \mathbb{R})$.

Here we prove the corresponding central limit theorem.

THEOREM 2.4. *Under Assumption 2.10, for any $T > 0$,*

$$\lim_{n \rightarrow \infty} \frac{x_t^n - v(\rho)tn}{\sqrt{n}} = \sigma B_t + v'(\rho)Z_t,$$

in law with respect to the J_1 -Skorohod topology of $\mathcal{D}([0, T], \mathbb{R})$. In the above display,

$$(2.2) \quad \sigma^2 = \int \sum_{z \in \mathcal{R}} z^2 r_z d\nu_\rho,$$

$(B_t)_{t \geq 0}$ is a standard Brownian motion and $(Z_t)_{t \geq 0}$ is a Gaussian process of stationary increments, independent of $(B_t)_{t \geq 0}$, with variance as given in (5.2).

REMARK 2.5. The parameters $D(\rho)$ and $\chi(\rho)$ that appear in (5.2) are functions of $(\eta_t^n)_{t \geq 0}$ alone.

REMARK 2.6. The so-called *gradient condition*, stated in Assumption 2.10 is only needed to prove that the processes $(B_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ are independent.

2.3. *The environment process.* A classical idea in the context of random walks in random environments is to consider the *environment as seen by the random walk*. Here we follow the approach of [28]. The process $(\xi_t^n)_{t \geq 0}$ with values in Ω , defined as

$$\xi_t^n(x) = \eta_t^n(x + x_t^n) \quad \text{for any } x \in \mathbb{Z} \text{ and any } t \geq 0$$

is a Markov process generated by the operator $L_n = n^2 L_b + n L^{rw}$, where

$$L^{rw} f(\xi) = \sum_{z \in \mathcal{R}} r_z(\xi)(f(\tau_z \xi) - f(\xi)).$$

The process $(x_t^n)_{t \geq 0}$ can be recovered from $(\xi_t^n)_{t \geq 0}$ as follows: for each $z \in \mathcal{R}$, let $N_t^{z,n}$ be the number of shifts in direction z the process $(\xi_t^n)_{t \geq 0}$ has performed up to time t . On one hand,

$$x_t^n = \sum_{z \in \mathcal{R}} z N_t^{z,n},$$

and on the other hand, $(N_t^{z,n})_{t \geq 0}$ is a (time-inhomogeneous) Poisson process of rate $(n r_z(\xi_t^n))_{t \geq 0}$. Therefore,

$$M_t^{z,n} := \frac{1}{\sqrt{n}} N_t^{z,n} - \sqrt{n} \int_0^t r_z(\xi_s^n) ds$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\xi_s^n : s \leq t\}$. Its predictable quadratic variation is given by

$$\langle M_t^{z,n} \rangle = \int_0^t r_z(\xi_s^n) ds.$$

Moreover, since the jumps of these Poisson processes are disjoint, these martingales are mutually orthogonal.

Adding the martingales $(M_t^{n,z})_{t \geq 0}$, we can write the position of the random walk as a sum of a martingale and an integral term, namely

$$(2.3) \quad \frac{x_t^n - v(\rho)nt}{\sqrt{n}} = M_t^n + A_t^n,$$

where $M_t^n := \sum_{z \in \mathcal{R}} z M_t^{n,z}$ and

$$(2.4) \quad A_t^n = \sqrt{n} \int_0^t (\omega(\xi_s^n) - v(\rho)) ds \quad \text{with } \omega(\xi) := \sum_{z \in \mathcal{R}} z r_z(\xi).$$

Besides,

$$\langle M_t^n \rangle = \int_0^t \sum_{z \in \mathcal{R}} z^2 r_z(\xi_s^n) ds.$$

The process $(A_t^n)_{t \geq 0}$ is an instance of what is known in the literature as an *additive functional* of the chain $(\xi_t^n)_{t \geq 0}$. Theorem 2.4 is an immediate consequence of the following result.

THEOREM 2.7. *Consider the decomposition (2.3) and recall the definition of σ in (2.2) and $(Z_t)_{t \geq 0}$ in Theorem 2.4. Fix $T > 0$.*

- (i) *As $n \rightarrow \infty$, $\{(M_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ converges in law to $(\sigma B_t)_{t \in [0, T]}$ with respect to the J_1 -Skorohod topology of $\mathcal{D}([0, T], \mathbb{R})$;*
- (ii) *as $n \rightarrow \infty$, $\{(A_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ converges in law to $(v'(\rho)Z_t)_{t \in [0, T]}$ with respect to the J_1 -Skorohod topology of $\mathcal{D}([0, T], \mathbb{R})$;*
- (iii) *under Assumption 2.10, the processes $(B_t)_{t \in [0, T]}$ and $(Z_t)_{t \in [0, T]}$ are independent.*

The rest of the paper is devoted to the proof of Theorem 2.7. Part (i) follows from the Martingale FCLT (Proposition 2.8). It is necessary to check that $\langle M_t^n \rangle \rightarrow \sigma^2 t$ in probability as $n \rightarrow \infty$, and this can be proved by combining Theorem 3.5 in Section 3 with Theorem 1.3 of [6]. The proof of (ii) spans Sections 3, 4 and 5; the strategy is to show that A^n has the same scaling limit as a certain additive functional of the lattice gas, studied in [19]. The proof of (iii) is in Section 6; the strategy is to show that the limiting processes $(B_t)_{t \in [0, T]}$ and $(Z_t)_{t \in [0, T]}$ are uncorrelated and that their joint law is Gaussian.

2.4. Auxiliary results.

2.4.1. Invariance principle for martingales. To prove convergence of the sequence $\{(M_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ and to prove that the limiting processes $(B_t)_{t \in [0, T]}$ and $(Z_t)_{t \in [0, T]}$ are independent, we will use the following result.

PROPOSITION 2.8 (Martingale FCLT). *Let $\{(\mathcal{M}_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ be a sequence of square-integrable martingales. Assume that:*

- (i) *the sequence of predictable quadratic variation processes $\{(\mathcal{M}_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ converges in law to an increasing, deterministic function $H : [0, T] \rightarrow \mathbb{R}$;*
- (ii) *the size of the largest jump of $(\mathcal{M}_t^n)_{t \in [0, T]}$ converges in probability to 0.*

Then the sequence $\{(\mathcal{M}_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ converges in law to a continuous martingale of quadratic variation H . In addition, let $\{(\mathcal{N}_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ be another sequence of square-integrable martingales satisfying (i), (ii), possibly for a different function H . If $\{(\mathcal{M}_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ is orthogonal to $\{(\mathcal{N}_t^n)_{t \in [0, T]}\}_{n \in \mathbb{N}}$ for each n , then the limiting martingales are independent.

A proof of this result for the case $H(t) = \sigma t$ can be found in [16], Theorem 1.4, or in [34], Theorem 2.1. The proof for general $H(t)$ can be found in Chapter VIII.3.a of [24].

2.4.2. *Density fluctuation field.* This section presents some results on $(\eta_t^n)_{t \geq 0}$ that are needed in the proof of Theorem 2.7.

Let $\mathcal{S}(\mathbb{R})$ be the Schwarz space of test functions in \mathbb{R} . Let $\phi : \Omega \rightarrow \mathbb{R}$ be a local function. Denote $\bar{\phi}(\rho) := \int \phi \, d\nu_\rho$. For $f \in \mathcal{S}(\mathbb{R})$, $n \in \mathbb{N}$ and $t \geq 0$, let

$$(2.5) \quad X_t^n(f; \phi) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} (\phi(\tau_x \eta) - \bar{\phi}(\rho)) f\left(\frac{x}{n}\right).$$

This defines a process $(X_t^n(\cdot; \phi))_{t \geq 0}$ with values in the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions. We will use the shorthand $X_t^n(f) := X_t^n(f; \eta_0)$. The process $(X_t^n)_{t \geq 0}$ defined in this way is known as the *density fluctuation field* associated to the particle system $(\eta_t^n)_{t \geq 0}$.

Since the topology of the space of tempered distributions is not very strong, it is sometimes more convenient to consider the *Sobolev spaces* $H_\ell(\mathbb{R})$ instead of $\mathcal{S}'(\mathbb{R})$. Those are defined as the closure of $\mathcal{S}(\mathbb{R})$ with respect to the norms

$$\|f\|_{H_\ell(\mathbb{R})} = \left(\int f(x) (-\Delta + x^2)^\ell f(x) \, dx \right)^{1/2}.$$

One can check that $(X_t^n)_{t \geq 0}$ is a well-defined process in $H_{-2}(\mathbb{R})$ (see, for example, Chapter 11 of [27]). The following result was proved in [32]. Under different assumptions on the rates $c(\eta)$, it was proved in [14, 15] and [30].

PROPOSITION 2.9. *Let $(\eta_t^n)_{t \geq 0}$ the lattice gas with initial law ν_ρ . There exists a constant $D(\rho)$ such that for any $T > 0$,*

$$\lim_{n \rightarrow \infty} X_t^n = X_t$$

in law with respect to the J_1 -Skorohod topology of $\mathcal{D}([0, T], H_{-2}(\mathbb{R}))$, where $(X_t)_{t \geq 0}$ is the stationary solution of

$$(2.6) \quad \partial_t X = D(\rho) \Delta X + \sqrt{2D(\rho)\rho(1-\rho)} \nabla \dot{W}_t.$$

In this equation, \dot{W}_t denotes a standard, space-time white noise.

For the proof in Section 6 that the martingale and the additive functional in (2.3) are independent in the limit, we need an additional assumption on the exchange rates $c_x(\eta)$.

ASSUMPTION 2.10 (Gradient condition). *There exists a finite family of local functions $g_1, \dots, g_k : \Omega \rightarrow \mathbb{R}$ and finitely supported functions $q_1, \dots, q_k : \mathbb{Z} \rightarrow \mathbb{R}$ such that*

$$\sum_{x \in \mathbb{Z}} q_j(x) = 0 = \sum_{x \in \mathbb{Z}} x q_j(x) \quad \text{for all } j \in \{1, \dots, k\}$$

and

$$L_b \eta_0 = \sum_{j=1}^k \sum_{x \in \mathbb{Z}} q_j(x) g_j(\tau_x \eta).$$

As examples of rates that satisfy Assumptions 2.2 and 2.10, one can consider $c(\eta) = 1$ (simple symmetric exclusion) or $c(\eta) = 1 + \eta_{-1} + \eta_2$. Our assumption follows [30]. It is slightly different from the gradient condition as stated in [27], page 61, and in [18]. We only need it for the following lemma.

LEMMA 2.11. *Let $T > 0$ and let $H : [0, T] \rightarrow \mathcal{S}(\mathbb{R})$ be a smooth function. Denote $H_s(u) := H(s, u)$. Then, for any $t \in [0, T]$,*

$$\int_0^t (\partial_s + n^2 L_b) X_s^n(H_s) ds - \int_0^t X_s^n((\partial_s + D(\rho)\Delta)H_s) ds \longrightarrow 0$$

in probability, as $n \rightarrow \infty$.

We defer the proof to the end of Section 3, because it uses Corollary 3.6 as an input.

3. Replacement lemma and entropy bound. In this section, we will establish two estimates that are fundamental to the proof of Theorem 2.7. First, we obtain a sharp bound on the entropy production for the environment process. Then, we prove the so-called *replacement lemma*, that allows to write A_t as a function of the density of particles plus an error that vanishes in the limit.

3.1. *Entropy bound.* Let us recall that the processes $(\eta_t^n)_{t \geq 0}$ and $(\xi_t^n)_{t \geq 0}$ start from the Bernoulli product law ν_ρ . We recall that ν_ρ is invariant under the evolution of $(\eta_t^n)_{t \geq 0}$ and stress that it is not invariant under $(\xi_t^n)_{t \geq 0}$, unless very delicate cancellations occur. In fact, invariance of ν_ρ under $(\xi_t^n)_{t \geq 0}$ is equivalent to $\psi = 0$, where ψ is defined in (3.5). Let μ_t^n be the law of ξ_t^n and define

$$H_n(t) := H(\mu_t^n | \nu_\rho),$$

where

$$H(\mu | \nu) := \int f \log f d\nu, \quad f = \frac{d\mu}{d\nu}$$

is the relative entropy (or Kullback–Leibler divergence) of μ with respect to ν . The main result of this section is the following bound.

THEOREM 3.1 (Entropy bound). *There exists C depending only on $\rho, \{r_z; z \in \mathcal{R}\}$ and ε_0 such that $H'_n(t) \leq C$ for any $t \geq 0$. In particular, $H_n(t) \leq Ct$ for any $t \geq 0$.*

REMARK 3.2. In [6], it is proved that $H'_n(t) \leq Cn$. As observed in [9], a bound of this type is enough (aside from the usual model-dependent technical points) to adapt Varadhan’s approach to obtain hydrodynamic limits and the associated large deviations principle. In [5, 6], this strategy was successfully applied for the process $(\xi_t^n)_{t \geq 0}$. Actually, the bound $H'_n(t) \leq Cn$ is not hard to prove (see Lemma 2.2 in [6], Lemma 3.2 in [9] or Lemma 6.1 in [17]). A bound of the form

$$\lim_{n \rightarrow \infty} \frac{H_n(t)}{n} = 0$$

is more difficult to obtain, and it is the main point of the so-called Yau’s *relative entropy* method in hydrodynamic limits, see [35] and Chapter 6 of [27]. An adaptation of Yau’s method to the model considered in this article only gives a bound of the form

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq t} \frac{H'_n(t)}{n} = 0,$$

which is very far from Theorem 3.1.

PROOF. Let f_t be the Radon–Nykodim derivative of μ_t^n with respect to ν_ρ (we are not indexing in n in order to not overcharge the notation). By Theorem A.9.2 in [27], we have that

$$(3.1) \quad H'_n(t) \leq 2\langle \sqrt{f_t}, L_n \sqrt{f_t} \rangle.$$

In the last equation and throughout the rest of the article, $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\nu_\rho)$.

Recall that $L_n = n^2 L_b + n L^{rw}$. Since ν_ρ is invariant under L_b , we have that $\langle \sqrt{f_t}, L_b \sqrt{f_t} \rangle \leq 0$. Even more, ν_ρ is reversible for L_b . From reversibility, one can show that

$$(3.2) \quad \langle \sqrt{f_t}, L_b \sqrt{f_t} \rangle = -\frac{1}{2} \sum_{x \in \mathbb{Z}} \int c_x(\xi) (\sqrt{f_t(\xi^{x,x+1})} - \sqrt{f_t(\xi)})^2 d\nu_\rho.$$

Let us introduce the Dirichlet form $\mathcal{D}(\cdot)$, defined as

$$(3.3) \quad \mathcal{D}(h) = \frac{1}{2} \sum_{x \in \mathbb{Z}} \int (h(\xi^{x,x+1}) - h(\xi))^2 d\nu_\rho$$

for any $h : \Omega \rightarrow \mathbb{R}$. Thanks to the ellipticity condition $c_x \geq \varepsilon_0$, we have tat

$$\langle \sqrt{f_t}, L_b \sqrt{f_t} \rangle \leq -\varepsilon_0 \mathcal{D}(\sqrt{f_t}).$$

Using (3.2), we see that

$$H'_n(t) \leq -\varepsilon_0 n^2 \mathcal{D}(\sqrt{f_t}) + 2n \langle \sqrt{f_t}, L^{rw} \sqrt{f_t} \rangle.$$

Therefore, if we are able to control $\langle \sqrt{f_t}, L^{rw} \sqrt{f_t} \rangle$ in terms of the Dirichlet form of $\sqrt{f_t}$, the theorem will be proved. The following lemma provides the required bound, which is going to be used several times in the rest of the article.

LEMMA 3.3. For any $f \geq 0$ such that $\int f d\nu_\rho = 1$, the following inequality holds:

$$(3.4) \quad \langle \sqrt{f}, L_n \sqrt{f} \rangle \leq -\varepsilon_0 n^2 \mathcal{D}(\sqrt{f}) + \langle \psi, nf \rangle,$$

where

$$(3.5) \quad \psi(\xi) := \frac{1}{2} \sum_{z \in \mathcal{R}} (r_z(\tau_{-z}\xi) - r_z(\xi)).$$

In addition, for all $\beta > 0$,

$$(3.6) \quad \langle \psi, nf \rangle \leq \beta \mathcal{D}(\sqrt{f}) + \frac{Cn^2}{\beta},$$

where $C > 0$ does not depend on n .

Before we prove Lemma 3.3, let us show how it implies a bound on $H'_n(t)$ that is uniform in n . Putting together (3.1), (3.4) and (3.6), we get

$$H'_n(t) \leq -\varepsilon_0 n^2 \mathcal{D}(\sqrt{f_t}) + \beta \mathcal{D}(\sqrt{f_t}) + \frac{Cn^2}{\beta},$$

for any positive β . The choice $\beta = \varepsilon_0 n^2$ yields $H'_n(t) \leq \frac{C}{\varepsilon_0}$, which is the inequality we were aiming at. \square

PROOF OF LEMMA 3.3. Combining (3.2) with the ellipticity assumption, we get

$$2\langle \sqrt{f}, L_n \sqrt{f} \rangle \leq -\varepsilon_0 n^2 \mathcal{D}(\sqrt{f}) + 2n \langle \sqrt{f}, L^{rw} \sqrt{f} \rangle.$$

Using the identity $\sqrt{a}(\sqrt{b} - \sqrt{a}) = -\frac{1}{2}(\sqrt{b} - \sqrt{a})^2 + \frac{1}{2}(b - a)$, we get

$$\begin{aligned} \langle \sqrt{f}, L^{rw} \sqrt{f} \rangle &= \sum_{z \in \mathcal{R}} \int r_z(\xi) \sqrt{f(\xi)} (\sqrt{f(\tau_z \xi)} - \sqrt{f(\xi)}) dv_\rho \\ &= -\frac{1}{2} \sum_{z \in \mathcal{R}} \int r_z(\xi) (\sqrt{f(\tau_z \xi)} - \sqrt{f(\xi)})^2 dv_\rho \\ &\quad + \frac{1}{2} \sum_{z \in \mathcal{R}} \int r_z(\xi) (f(\tau_z \xi) - f(\xi)) dv_\rho. \end{aligned}$$

Neglecting the first term and performing the change of variables $\xi \mapsto \tau_z \xi$, we conclude that

$$\langle \sqrt{f}, L^{rw} \sqrt{f} \rangle \leq \frac{1}{2} \sum_{z \in \mathcal{R}} \langle f, r_z \circ \tau_{-z} - r_z \rangle,$$

and this finishes the proof of (3.4).

It remains to prove (3.6). The strategy is to split the integrand $\psi(\xi)$ into several terms of the form $h(\xi^{x,x+1}) - h(\xi)$, for appropriate local functions h . To each of these terms, we then apply Lemma A.1. Let us start with the function $r \circ \tau_1 - r$, where r is local. To simplify the notation, assume that r has support in $\{0, \dots, k\}$ and denote by $\nabla^{x,x+1} h$ the function $\xi \mapsto h(\xi^{x,x+1}) - h(\xi)$. Then $r(\tau_1 \xi) = r(\nabla^{k,k+1} \dots \nabla^{0,1} \xi)$. Therefore,

$$r(\tau_1 \xi) - r(\xi) = r(\nabla^{0,1} \xi) - r(\xi) + \sum_{y=1}^k r(\nabla^{y,y+1} \dots \nabla^{0,1} \xi) - r(\nabla^{y-1,y} \dots \nabla^{0,1} \xi).$$

Applying Lemma A.1, we get, for any $\beta > 0$,

$$\langle nf, r \circ \tau_1 - r \rangle \leq \beta \mathcal{D}(\sqrt{f}) + \frac{n^2}{\beta} \|r\|_\infty^2 k.$$

Using a telescoping argument, we can obtain a similar bound for $\langle nf, r \circ \tau_z - r \rangle$. Adding over $z \in \mathcal{R}$ we finish the proof. Notice that the constant C in the statement depends on the dynamics of the random walk alone: it is a function of the size of \mathcal{R} , the sizes of the supports of the rates r_z and the numbers $\|r_z\|_\infty$. \square

REMARK 3.4. If the random walk moves at a rate smaller than n , our proof breaks down in the inequality

$$\langle \sqrt{f}, -L^{rw} \sqrt{f} \rangle \leq n \langle \sqrt{f}, L_b \sqrt{f} \rangle + O(1)$$

in Lemma 3.3. The constant n is the best that we could manage (the precise point in the proof where we use it is (3.5)). Without this inequality, we are not able to show $H(\xi_t^n | \nu_\rho) = O(1)$. We could still get $H(\xi_t^n | \nu_\rho) = o(n)$, though, and use this to get a law of large numbers when the walk moves at rate much smaller than $n^{\frac{2}{3}}$.

3.2. *Replacement lemma.* Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a smooth function with compact support in $(0, 1)$ and such that $\int_{\mathbb{R}} \varphi(u) du = 1$. Let $\varphi_\varepsilon(u) := \varepsilon^{-1} \varphi(u/\varepsilon)$.

In this section, we will prove that the additive functional A_t^n is asymptotically equivalent to a function of the density of particles around the origin. More precisely, we will prove that

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_n \left(\left| \sqrt{n} \int_0^t (\omega(\xi_s^n) - v(\rho) - v'(\rho)(\xi_s^n \star \varphi_\varepsilon)(0)) ds \right| > \delta \right) = 0,$$

where we use the notation

$$(3.7) \quad (\xi \star \varphi_\varepsilon)(x) := \frac{1}{n} \sum_{y \in \mathbb{Z}} \varphi_\varepsilon\left(\frac{y}{n}\right) (\xi(x+y) - \rho).$$

In this theorem, the particular form of the function ω does not play a fundamental role. In fact, this result is a particular instance of what is known in the literature as the *replacement lemma*, which roughly states that any local function of ξ_t^n is asymptotically equivalent to a function of the density of particles. We take averages using a smooth function instead of the usual arithmetic mean for technical reasons having to do with the topology of Skorohod space. This issue shows up in Section 5, where we characterize the limiting trajectories of the random walk.

THEOREM 3.5 (Replacement lemma). *Let $\phi : \Omega \rightarrow \mathbb{R}$ be a local function. For $\lambda \in [0, 1]$, define $\bar{\phi}(\lambda) = \int \phi \, d\nu_\lambda$. Then, for any $\delta > 0$ and any $t \geq 0$,*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_n \left(\left| \sqrt{n} \int_0^t (\phi(\xi_s^n) - \bar{\phi}(\rho) - \bar{\phi}'(\rho)(\xi_s^n \star \varphi_\varepsilon)(0)) \, ds \right| > \delta \right) = 0.$$

The same statement holds when ξ_s^n is replaced by η_s^n .

COROLLARY 3.6. *Fix $t \in [0, T]$. Let $H : [0, T] \rightarrow \mathcal{S}(\mathbb{R})$ be a smooth function and $\phi : \Omega \rightarrow \mathbb{R}$ be a local function. Then*

$$\lim_{n \rightarrow \infty} \int_0^t (X_s^n(H_s; \phi) - \bar{\phi}'(\rho) X_s^n(H_s)) \, ds \longrightarrow 0$$

in probability.

PROOF. First, we observe that for any random variable X ,

$$P(|X| > \delta) \leq P(X > \delta) + P(-X > \delta).$$

Considering ϕ and $-\phi$, it is enough to prove that

$$(3.8) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}_n \left(\sqrt{n} \int_0^t (\phi(\xi_s^n) - \bar{\phi}(\rho) - \bar{\phi}'(\rho)(\xi_s^n \star \varphi_\varepsilon)(0)) \, ds > \delta \right) = 0.$$

Let $V : \Omega \rightarrow \mathbb{R}$ be a bounded function. Combining Theorem A1.7.2 and equation A3.1.1 of [27], we have the following:

$$\log \mathbb{E}_n [e^{\int_0^t V(\xi_s^n) \, ds}] \leq t \sup_f \{ \langle V, f \rangle + \langle \sqrt{f}, L_n \sqrt{f} \rangle \},$$

where the supremum is over all densities f with respect to ν_ρ . Combining this proposition with (3.4), we get the following estimate:

$$(3.9) \quad \log \mathbb{E}_n [e^{\int_0^t V(\xi_s^n) \, ds}] \leq t \sup_f \{ \langle V + n\psi, f \rangle - \varepsilon_0 n^2 \mathcal{D}(\sqrt{f}) \}.$$

We are going to need this inequality later on in order to prove tightness. Here, we proceed by substituting (3.6), getting

$$\log \mathbb{E}_n [e^{\int_0^t V(\xi_s^n) \, ds}] \leq t \sup_f \left\{ \langle V, f \rangle + (\beta - \varepsilon_0) n^2 \mathcal{D}(\sqrt{f}) + \frac{C}{\beta} \right\},$$

for any $\beta > 0$. The constant C does not depend on n or β .

Now we can go back to (3.8): for any bounded function V and any positive γ and β ,

$$(3.10) \quad \begin{aligned} \log \mathbb{P}_n \left(\int_0^t V(\xi_s^n) ds > \delta \right) &\leq -\gamma\delta + \log \mathbb{E}_n [e^{\gamma \int_0^t V(\xi_s^n) ds}] \\ &\leq -\gamma\delta + t \sup_f \left\{ \langle \gamma V, f \rangle + (\beta - \varepsilon_0)n^2 \mathcal{D}(\sqrt{f}) + \frac{C}{\beta} \right\}, \end{aligned}$$

where the supremum is taken over all $f \geq 0$ such that $\int f d\nu_\rho = 1$ and where $C > 0$ does not depend on n . The expression above becomes easier to understand if one keeps in mind that the term $-\varepsilon_0 \mathcal{D}(\sqrt{f})$ comes from the reversible dynamics and the term with β comes from the random walk dynamics.

Now the proof of (3.8) consists in writing the integrand as a sum of terms of the form $V(\xi)$ for which good bounds of $\langle V, f \rangle$ in terms of $\mathcal{D}(\sqrt{f})$ are available. As we have seen in the proof of the entropy bound, such terms are of the form $h(\xi^{x,x+1}) - h(\xi)$ for bounded local functions h .

Let $\mathcal{R} \subset \mathbb{Z}$ be the support of ϕ . The first thing to notice is that every mean-zero local function ϕ can be written as a linear combination of the simpler variables $\{\xi(A) : A \subset \mathcal{R}\}$, where

$$\xi(A) := \prod_{x \in A} (\xi(x) - \rho).$$

It is enough, then, to prove inequality (3.8) when $\phi(\xi)$ is of the form $\phi(\xi) = \xi(A)$ for some finite set A .

We start with the simplest case, in which the local function is $\phi(\xi) = \xi_0 - \rho$. The statement reads

$$(3.11) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \mathbb{P} \left(\sqrt{n} \left| \int_0^t (\xi_s^n(0) - \rho - (\xi_s^n \star \varphi_\varepsilon)(0)) ds \right| > \delta \right) = 0.$$

Since time will not play any role in the computations that follow, we will omit it from the notation for a while. Denote $\xi_x := \xi(x)$ and $\bar{\xi}_0 := \xi_0 - \rho$. Recall that $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\nu_\rho)$ and that $\mathcal{D}(\sqrt{f})$ denotes the Dirichlet form of symmetric exclusion, as defined in (3.3).

In view of (3.10), we need to estimate the integral $\langle \bar{\xi}_0 - (\xi \star \varphi_\varepsilon)(0), f \rangle$ in terms of $\mathcal{D}(\sqrt{f})$. We are going to prove that, for any ν_ρ -density f , the following inequality holds:

$$(3.12) \quad \gamma n^{\frac{1}{2}} \langle \bar{\xi}_0 - (\xi \star \varphi_\varepsilon)(0), f \rangle \leq \alpha n^2 \mathcal{D}(\sqrt{f}) + \varepsilon \gamma^2 \frac{C'}{\alpha} + o_n(1),$$

where $\alpha > 0$ is arbitrary and C' does not depend on n .

We would like to write the difference inside the inner product in (3.12) as a linear combination of the functions $\xi \mapsto \xi_x - \xi_{x+1}$, in order to apply Lemma A.1. For that, we need the coefficients of the $\xi_x - \rho$ to sum up to 1. Define

$$m_n(\varepsilon) := \frac{1}{n} \sum_{x \in \mathbb{Z}} \varphi_\varepsilon \left(\frac{x}{n} \right)$$

and write the telescoping sum

$$(3.13) \quad \begin{aligned} &\gamma n^{\frac{1}{2}} \left\langle \bar{\xi}_0 - \frac{1}{n} \sum_{x \in \mathbb{Z}} \varphi_\varepsilon \left(\frac{x}{n} \right) \bar{\xi}_x, f \right\rangle \\ &= \gamma n^{\frac{1}{2}} (1 - m_n(\varepsilon)) \langle \bar{\xi}_0, f \rangle + \sum_{x \in \mathbb{Z}} \sum_{y=x+1}^{\infty} \varphi_\varepsilon \left(\frac{y}{n} \right) \gamma n^{-\frac{1}{2}} \langle \bar{\xi}_x - \bar{\xi}_{x+1}, f \rangle. \end{aligned}$$

Since m_n is a Riemman sum for $\int_0^\varepsilon \varphi_\varepsilon(u) du = 1$, the first term is of order $n^{-\frac{1}{2}}$. As for the second term, notice that, since φ_ε has support contained in $(0, \varepsilon)$, only finitely many terms of the sum over x are not null, namely those with $0 < x < \varepsilon n$.

Let $\alpha > 0$. Applying Lemma A.1, we can bound the second term of (3.13) by

$$(3.14) \quad \alpha n^2 \mathcal{D}(\sqrt{f}) + \frac{2\gamma^2}{\alpha n} \sum_{x=0}^{\varepsilon n} \left(\frac{1}{n} \sum_{y=x+1}^{\varepsilon n} \varphi_\varepsilon\left(\frac{y}{n}\right) \right)^2.$$

Using $\|\varphi_\varepsilon\|_\infty \leq \varepsilon^{-1} \|\varphi\|_\infty$, we get the inequality

$$(3.15) \quad \sum_{x=0}^{\varepsilon n} \left(\frac{1}{n} \sum_{y=x+1}^{\varepsilon n} \varphi_\varepsilon\left(\frac{y}{n}\right) \right)^2 \leq \|\varphi\|_\infty^2 \varepsilon n.$$

Therefore, the number (3.14) is smaller than

$$\alpha n^2 \mathcal{D}(\sqrt{f}) + \frac{2\gamma^2}{\alpha} \|\varphi\|_\infty^2 \varepsilon,$$

and this finishes the proof of (3.12). Plugging this inequality into (3.10), with the choices $\alpha = \beta = \frac{\varepsilon_0}{2}$ and $V(\xi) = \gamma n^{\frac{1}{2}} [\bar{\xi}_0 - (\xi \star \varphi_\varepsilon)(0)]$, we get the replacement lemma when the local function is $\phi(\xi) = \xi_0 - \rho$. In an analogous manner, one can prove the lemma for $\phi(\xi) = \xi_x - \rho$ for any $x \in \mathcal{R}$.

Next, we show that the higher order monomials vanish. More precisely, we show that if $A \subset \mathbb{Z}$ is a finite set and $|A| \geq 2$ then

$$(3.16) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\left|\sqrt{n} \int_0^t \xi_s^n(A) ds\right| > \delta\right) = 0.$$

Write the set A in the form $A = \{x_0\} \cup A' \cup \{y_0\}$, where we assume that $x_0 < y_0$ and $A' \subset \{x_0 + 1, \dots, y_0 - 1\}$. Denote by $(\xi \star \tilde{\varphi}_\varepsilon)(x_0)$ the weighted average of the centered configuration ξ in a box to the left of x_0 :

$$(\xi \star \tilde{\varphi}_\varepsilon)(x_0) = \frac{1}{n} \sum_{y \in \mathbb{Z}} (\xi(x_0 - y) - \rho) \varphi_\varepsilon\left(\frac{y}{n}\right) = \frac{1}{n} \sum_{y \in \mathbb{Z}} \varphi_\varepsilon\left(\frac{y}{n}\right) \bar{\xi}_{x_0 - y}.$$

To prove assertion (3.16), we prove that each of the probabilities below converges to zero as first $n \rightarrow \infty$ then $\varepsilon \rightarrow 0$.

$$\begin{aligned} & \mathbb{P}\left(\left|\sqrt{n} \int_0^t \xi_s^n(x_0) \xi_s^n(A') (\bar{\xi}_s^n(y_0) - (\xi_s^n \star \varphi_\varepsilon)(y_0)) ds\right| > \delta\right), \\ & \mathbb{P}\left(\left|\sqrt{n} \int_0^t (\bar{\xi}_s^n(x_0) - (\xi_s^n \star \tilde{\varphi}_\varepsilon)(x_0)) \xi_s^n(A') (\xi_s^n \star \varphi_\varepsilon)(y_0) ds\right| > \delta\right), \\ & \mathbb{P}\left(\left|\sqrt{n} \int_0^t (\xi_s^n \star \tilde{\varphi}_\varepsilon)(x_0) \xi_s^n(A') (\xi_s^n \star \varphi_\varepsilon)(y_0) ds\right| > \delta\right). \end{aligned}$$

To bound the first probability, we mimic the proof of (3.11). That is, first we use (3.10) to reduce the proof to a variational problem, then we write the difference $\bar{\xi}_s^n(y_0) - (\xi_s^n \star \varphi_\varepsilon)(y_0)$ as a telescoping sum as in (3.13) and apply Lemma A.1 to each term of the sum, with the roles of g and h in that lemma being played by $\bar{\xi}_x$ and $\bar{\xi}(x_0)\xi(A')$ respectively. The proof can be replicated to bound the second inequality, this time with the roles of g and h being played by $\bar{\xi}_x$ and $\xi_s^n(A')(\xi_s^n \star \varphi_\varepsilon)(y_0)$, respectively. In both cases, one can use the bounds $\|h\|_\infty \leq 1$ and (3.15).

It remains to deal with the last probability. For that, recall that μ_s^n denotes the law of ξ_s^n , the environment as seen from the random walk at time s . We claim that there exists a large $D = D(t)$ such that, for all $s \leq t$,

$$\lim_{n \rightarrow \infty} \mathbb{E}_n [|\sqrt{n}(\xi_s^n \star \tilde{\varphi}_\varepsilon)(x_0)\xi_s^n(A')(\xi_s^n \star \varphi_\varepsilon)(y_0)|] = 0.$$

To prove that, we apply the bounds $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ and $|\xi(A')| \leq 1$ and get

$$\begin{aligned} \mathbb{E}_{\mu_s^n} [|\sqrt{n}(\xi \star \tilde{\varphi}_\varepsilon)(x_0)\xi(A')(\xi \star \varphi_\varepsilon)(y_0)|] &\leq \frac{1}{2} \mathbb{E}_{\mu_s^n} [\sqrt{n}(\xi \star \tilde{\varphi}_\varepsilon)^2(x_0)] \\ &\quad + \frac{1}{2} \mathbb{E}_{\mu_s^n} [\sqrt{n}(\xi \star \varphi_\varepsilon)^2(y_0)]. \end{aligned}$$

Now we use relative entropy to replace the μ_s^n expectation by a ν_ρ expectation, using the following argument: for any $\alpha > 0$, it holds

$$(3.17) \quad \mathbb{E}_{\mu_s^n} [\sqrt{n}(\xi \star \varphi_\varepsilon)^2(y_0)] \leq \frac{H(\mu_s^n | \nu_\rho)}{\alpha} + \frac{1}{\alpha} \log \mathbb{E}_{\nu_\rho} [e^{\alpha \sqrt{n}(\xi \star \varphi_\varepsilon)^2(y_0)}].$$

Under the product measure ν_ρ , the random variable $(\xi \star \varphi_\varepsilon)(y_0)$ is a linear combination of i.i.d., bounded random variables, see (3.7). The variance of this sum is at most $\frac{\|\varphi\|_\infty^2}{\varepsilon n} =: \sigma^2$. By Lemma B.2, the logarithm is bounded by $8\alpha\sqrt{n}\sigma^2$ whenever $4\alpha\sqrt{n}\sigma^2 < 1$, that is, whenever $\alpha \leq \frac{\varepsilon\sqrt{n}}{4\|\varphi\|_\infty^2}$. Going back to (3.17), we can choose such an α of order \sqrt{n} . Combining the resulting inequality with the fact that the entropy is of order 1, we conclude that

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mu_s^n} [\sqrt{n}(\xi \star \varphi_\varepsilon)^2(y_0)] = 0.$$

An analogous argument shows that

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mu_s^n} [\sqrt{n}(\xi \star \tilde{\varphi}_\varepsilon)^2(x_0)] = 0,$$

and this finishes the proof of (3.16). \square

PROOF OF LEMMA 2.11. It is only here that we need Assumption 2.10. It allows us to make a summation by parts on $L_b X_s^n(H_s)$ and write

$$\begin{aligned} n^2 L_b X_s^n(H_s) &= \frac{1}{\sqrt{n}} \sum_{j=1}^k \left(\frac{1}{2} \sum_{y \in \mathbb{Z}} y^2 q_j(y) \right) \sum_{x \in \mathbb{Z}} g_j(\tau_x \eta_s^n) \Delta H_s \left(\frac{x}{N} \right) g_j(\tau^x \eta_s^n) \\ &\quad + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

The statement follows from Corollary 3.6, with

$$D(\rho) := \sum_{j=1}^k \sum_{y \in \mathbb{Z}} y^2 q_j(y) \bar{g}'_j(\rho). \quad \square$$

4. Tightness. In this section, we prove that the sequence of additive functionals

$$\{A_t^n : t \in [0, T]\}_{n \in \mathbb{N}},$$

defined in (2.4), is tight in $\mathcal{C}([0, T], \mathbb{R})$. Since $A_0^n = 0$ for all $n \in \mathbb{N}$, we only need to prove equicontinuity.

The proof is an application of the Kolmogorov–Centov criterion, see Problem 2.4.11 in [26]:

PROPOSITION 4.1. Assume that the sequence of stochastic processes $\{X_t^n : t \in [0, T]\}_{n \in \mathbf{N}}$ satisfies

$$\mathbf{E}[|X_t^n - X_s^n|^\lambda] \leq C|t - s|^{1+\lambda'}$$

for some positive constants λ, λ' and C and for all $s, t \in [0, T]$. Then it also satisfies

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \mathbf{P}\left(\sup_{\substack{|t-s| \leq \delta \\ s, t \in [0, T]}} |X_t^n - X_s^n| > \varepsilon\right) = 0 \quad \text{for all } \varepsilon > 0.$$

Tightness is a corollary of the following statement.

THEOREM 4.2. For any $\lambda \in (1, 2)$, there exists a constant $C = C(\lambda)$ such that

$$\mathbf{E}\left[\left|\sqrt{n} \int_t^{t+\tau} \omega(\xi_s^n) - v(\rho) ds\right|^\lambda\right] \leq C\tau^{3\lambda/4}$$

holds for every $t, \tau \in [0, T]$ and for every $n \in \mathbf{N}$.

The rest of this section deals with the proof of Theorem 4.2. The plan is the following: first, we notice that $\omega(\xi) - v(\rho)$ is the sum of finitely many mean-zero (with respect to ν_ρ) local functions: $\omega(\xi) - v(\rho) = \sum_{z \in \mathcal{R}} z(r_z(\xi) - \int r_z d\nu_\rho)$. Every mean-zero local function can be written as a polynomial in the variables $\{\bar{\xi}_x := \xi_x - \rho\}_{x \in \mathcal{R}}$. The number of terms in this polynomial does not depend on n . Therefore, it is enough to prove that, for all $\{x_1, \dots, x_k\} \subset \mathcal{R}$,

$$(4.1) \quad \mathbf{E}_n\left[\left|\sqrt{n} \int_t^{t+\tau} \bar{\xi}_s^n(x_1) \cdots \bar{\xi}_s^n(x_k) ds\right|^\lambda\right] \leq C \cdot \tau^{3\lambda/4} \quad \text{for all } t, \tau \leq T.$$

As we will see, the proof of (4.1) amounts to a careful reproving of the replacement lemma. From a technical point of view, the proofs of the replacement lemma and the entropy estimate are very similar, hinging upon the estimation of certain time integrals of the process. The estimate is always done in two steps: one first replaces local functions by their spatial averages and then controls the moments of those averages with the help of concentration inequalities.

It will be more convenient to work with tail bounds instead of moments, because our tool for estimating time integrals, the Feynman–Kac formula, gives bounds on the exponential moments. The following lemma converts tail bounds into moment estimates.

LEMMA 4.3. Let X be a nonnegative random variable. Assume $\mathbf{P}(X > \delta) \leq A/\delta^2$ for any $\delta > 0$. Then, for any $\lambda \in (1, 2)$, there exists a constant $C(\lambda)$ such that $\mathbf{E}[X^\lambda] \leq C(\lambda)A^{\lambda/2}$.

PROOF. Fix $\varepsilon > 0$. Then

$$\begin{aligned} \mathbf{E}[X^\lambda] &= \int_0^\infty \lambda \delta^{\lambda-1} \mathbf{P}(X > \delta) d\delta \\ &\leq \varepsilon^\lambda + \int_\varepsilon^\infty \lambda A \delta^{\lambda-3} d\delta \\ &= \varepsilon^\lambda + A \frac{\lambda}{2-\lambda} \varepsilon^{\lambda-2}. \end{aligned}$$

Choosing $\varepsilon = A^{1/2}$ we get $\mathbf{E}[X^\lambda] \leq (1 + \lambda/(2 - \lambda))A^{\lambda/2}$. \square

STEP 1 (Concentration). Given $\ell \in \mathbb{N}$ and $x \in \mathbb{Z}$, denote

$$\bar{\xi}_s^{n,\ell}(x) := \frac{\bar{\xi}_s^n(x-1) + \dots + \bar{\xi}_s^n(x-\ell)}{\ell}.$$

Then, for $\ell = \lfloor n\sqrt{\tau} \rfloor$ and $\lambda \in (1, 2)$,

$$(4.2) \quad \mathbb{E}_n \left[\left| \sqrt{n} \int_t^{t+\tau} (\bar{\xi}_s^{n,\ell}(x))^2 ds \right|^\lambda \right] \leq C(\lambda) \tau^{3\lambda/4}.$$

STEP 2 (Replacement). Let $x_1, \dots, x_k \in \mathbb{Z}$ with $x_1 < x_2 < \dots < x_k$. Then, for $\ell = \lfloor n\sqrt{\tau} \rfloor$,

$$\mathbb{E}_n \left[\left| \sqrt{n} \int_t^{t+\tau} \bar{\xi}_s^n(x_1) \dots \bar{\xi}_s^n(x_k) - (\bar{\xi}_s^{n,\ell}(x_1))^2 ds \right|^\lambda \right] \leq C(\lambda) \cdot \tau^{3\lambda/4}.$$

PROOF OF STEP 1. During the proof, C will denote a positive number that may change from line to line. It depends on λ and T but not on any other parameter. Since $|\bar{\xi}_s^{n,\ell}(x)| \leq 1$, we can prove (4.2) with $|\bar{\xi}_s^{n,\ell}(x)|$ in place of $(\bar{\xi}_s^{n,\ell}(x))^2$.

By the entropy inequality (B.3),

$$(4.3) \quad \mathbb{P}_n(n^{\frac{1}{2}} |\bar{\xi}_s^{n,\ell}(x)| > \delta) \leq \frac{H_n(s) + \log 2}{\log\{1 + \mathbb{P}_n(n^{\frac{1}{2}} |\bar{\xi}_0^{n,\ell}(x)| > \delta)^{-1}\}}.$$

Recall that ξ_0 has law ν_ρ by assumption, therefore the variables $\{\bar{\xi}_0(x)\}_{x \in \mathbb{Z}}$ are independent. From Lemma B.1 and (B.1), it follows

$$(4.4) \quad \mathbb{P}_n(n^{\frac{1}{2}} |\bar{\xi}_0^{n,\ell}(x)| > \delta) \leq 2e^{-2\ell\delta^2/n}.$$

We have already proved in Section 3.1 that $H_n(s) \leq Cs$ for some universal constant C . Combining this fact with (4.3) and (4.4), we can prove

$$(4.5) \quad \mathbb{P}_n(n^{\frac{1}{2}} |\bar{\xi}_s^{n,\ell}(x)| > \delta) \leq \frac{H_n(s) + \log 2}{\log\{1 + \frac{1}{2}e^{\frac{2\ell}{n}\delta^2}\}} \leq \frac{2(H_n(s) + \log 2)}{\frac{2\ell}{n}\delta^2} \leq \frac{Cn}{\ell\delta^2}.$$

Applying Lemma 4.3 and recalling our choice $\ell = n\sqrt{\tau}$,

$$\mathbb{E}_n[|n^{\frac{1}{2}} \bar{\xi}_s^{n,\ell}(x)|^\lambda] \leq C/\tau^{\lambda/4} \quad \text{for all } s \leq T.$$

We finish the proof with an application of Jensen’s inequality:

$$\begin{aligned} \mathbb{E}_n \left[\left| \sqrt{n} \int_t^{t+\tau} \bar{\xi}_s^{n,\ell}(x) ds \right|^\lambda \right] &\leq \tau^\lambda \cdot \frac{1}{\tau} \int_t^{t+\tau} \mathbb{E}_n[|n^{\frac{1}{2}} \bar{\xi}_s^{n,\ell}(x)|^\lambda] ds \\ &\leq C\tau^{3\lambda/4}. \end{aligned} \quad \square$$

PROOF OF STEP 2. Write

$$\begin{aligned} \bar{\xi}_s^n(x_1) \dots \bar{\xi}_s^n(x_k) - (\bar{\xi}_s^{n,\ell}(x_1))^2 &= (\bar{\xi}_s^n(x_1) - \bar{\xi}_s^{n,\ell}(x_1)) \bar{\xi}_s^n(x_2) \dots \bar{\xi}_s^n(x_k) \\ &\quad + \bar{\xi}_s^{n,\ell}(x_1) (\bar{\xi}_s^n(x_2) \dots \bar{\xi}_s^n(x_k) - \bar{\xi}_s^{n,\ell}(x_1)). \end{aligned}$$

An application of the inequalities $|\xi_s^n(x)| \leq 1$ and $|a + b|^\lambda \leq 2^{\lambda-1}(|a|^\lambda + |b|^\lambda)$ gives

$$\begin{aligned} &\mathbb{E} \left[\left| \sqrt{n} \int_t^{t+\tau} \bar{\xi}_s^n(x_1) \dots \bar{\xi}_s^n(x_k) - (\bar{\xi}_s^{n,\ell}(x_1))^2 ds \right|^\lambda \right] \\ &\leq 2^{\lambda-1} \mathbb{E} \left[\left| \sqrt{n} \int_t^{t+\tau} (\bar{\xi}_s^n(x_1) - \bar{\xi}_s^{n,\ell}(x_1)) \bar{\xi}_s^n(x_2) \dots \bar{\xi}_s^n(x_k) ds \right|^\lambda \right] \\ &\quad + 2^\lambda \mathbb{E} \left[\left(\sqrt{n} \int_t^{t+\tau} |\bar{\xi}_s^{n,\ell}(x_1)| ds \right)^\lambda \right]. \end{aligned}$$

We have already proved that the second expectation is bounded by $C(\lambda)\tau^{3\lambda/4}$ for some constant $C(\lambda)$. It remains to prove the same for the first expectation. From Lemma 4.3, we see that it suffices to prove, for all $\delta > 0$ and $\tau \leq T$,

$$(4.6) \quad \mathbb{P}_n \left(\sqrt{n} \left| \int_t^{t+\tau} (\bar{\xi}_s^n(x_1) - \bar{\xi}_s^{n,\ell}(x_1)) \bar{\xi}_s^n(x_2) \cdots \bar{\xi}_s^n(x_k) ds \right| > \delta \right) \leq \frac{C\tau^{3/2}}{\delta^2}$$

for some C that does not depend on n . During the rest of the section, we use the notation

$$W(\xi) = \bar{\xi}_{x_2} \cdots \bar{\xi}_{x_k}$$

and we keep the convention the value of C can change from line to line but does not depend on n . \square

LEMMA 4.4. *Recall the notation (3.5). There exists $\theta_0 > 0$ such that*

$$(4.7) \quad \log \mathbb{P}_n \left(\int_0^\tau \pm \sqrt{n} \cdot (\bar{\xi}_s^n(x_1) - \bar{\xi}_s^{n,\ell}(x_1)) W(\xi_s^n) - \theta_0 n \psi(\xi_s^n) ds > \delta \right) \leq -\frac{C\delta^2}{\tau^{3/2}}.$$

In fact, we can take $\theta_0 = 2\tau^{3/2}/\delta\varepsilon_0$. The same θ_0 satisfies

$$(4.8) \quad \mathbb{P}_n \left(\left| \int_t^{t+\tau} \theta_0 n \psi(\xi_s^n) ds \right| > \delta \right) \leq \frac{C\tau^{3/2}}{\delta^2}.$$

Before proving the lemma, let us use it to deduce (4.6). Following the three-line computation in (4.5), we can deduce from (4.7) that

$$\mathbb{P}_n \left(\int_t^{t+\tau} \pm \sqrt{n} \cdot (\bar{\xi}_s^n(x_1) - \bar{\xi}_s^{n,\ell}(x_1)) W(\xi_s^n) - \theta_0 n \psi(\xi_s^n) ds > \delta \right) \leq -C'\tau^{3/2}/\delta^2,$$

for a constant C' that does not depend on n or τ . Putting this together with (4.8), we get (4.6).

PROOF OF LEMMA 4.4. Let $\theta > 0$. To bound the probability in (4.7), we first apply the inequality $\mathbb{P}(X > \delta) \leq e^{-\theta\delta} \mathbb{E}[e^{\theta X}]$ and then the inequality (3.9). Thus, (4.7) is bounded by

$$(4.9) \quad -\theta\delta + \tau \sup_f \{ \pm (\bar{\xi}_{x_1} - \bar{\xi}_{x_1}^\ell) W(\xi), \theta\sqrt{n}f \} - \varepsilon_0 n^2 \mathcal{D}(\sqrt{f}) + (1 - \theta\theta_0) \langle \psi, nf \rangle \},$$

where the supremum is taken over the set of probability densities with respect to ν_ρ , and $\bar{\xi}_{x_1}^\ell = \frac{1}{\ell}(\bar{\xi}_{x_1-1}^\ell + \cdots + \bar{\xi}_{x_1-\ell}^\ell)$. To bound the first term, we split the difference $\bar{\xi}_{x_1} - \bar{\xi}_{x_1}^\ell$ as

$$\bar{\xi}_{x_1} - \bar{\xi}_{x_1}^\ell = \sum_{j=0}^{\ell-1} \frac{\ell-j}{\ell} (\bar{\xi}_{x_1-j} - \bar{\xi}_{x_1-j-1})$$

and apply Lemma A.1 to each piece, with $g = \bar{\xi}_{x_1-j-1}$. Using the bound $|W(\xi)| \leq 1$, it is possible to prove

$$|(\bar{\xi}_{x_1} - \bar{\xi}_{x_1}^\ell) W(\xi), \theta\sqrt{n}f| \leq \varepsilon_0 n^2 \mathcal{D}(\sqrt{f}) + C \frac{\theta^2 \ell}{\varepsilon_0 n}.$$

Going back to (4.9), choose $\theta_0 = \theta^{-1}$. Recall that $\ell = n\sqrt{\tau}$. Then (4.9) is bounded by $-\theta\delta + \frac{C\theta^2\tau^{3/2}}{\varepsilon_0}$. We can choose $\theta = \delta\varepsilon_0/2C\tau^{3/2}$. This proves (4.7).

With this choice of θ_0 , inequality (4.8) will follow if we can prove

$$\mathbb{E}_n \left[\left| \int_t^{t+\tau} n \psi(\xi_s^n) ds \right| \right] \leq C.$$

The entropy inequality (B.2) gives the bound

$$\mathbb{E}_n \left[\left| \int_t^{t+\tau} n \psi(\xi_s^n) ds \right| \right] \leq H_n(t) + \log \mathbb{E} \left[\exp \left| \int_0^\tau n \psi(\xi_s^n) ds \right| \right].$$

To bound this quantity, apply four inequalities in succession; first, the entropy bound (Theorem 3.1); second, $e^{|a|} \leq e^a + e^{-a}$; third, inequality (3.9); finally, inequality (3.6). \square

5. Limit points of the additive functional. In the previous section, we proved that the sequence of additive functionals

$$\left\{ A_t^n := \int_0^t \sqrt{n}(\omega(\xi_s^n) - v(\rho)) ds : t \in [0, T] \right\}_{n \in \mathbb{N}}$$

is tight. In this section, we identify its limit points, in Proposition 5.2. For that, we will rely strongly on the results of [19].

By the replacement lemma 3.5, we can approximate A_t^n by the additive functional $v'(\rho)\sqrt{n} \int_0^t (\xi_s^n \star \varphi_\varepsilon)(0) ds$. Following [19], we relate this functional to the density fluctuation field of the underlying particle system. One can write

$$\sqrt{n} \int_0^t (\xi_s^n \star \varphi_\varepsilon)(0) ds = \int_0^t X_s^n(\tau_{-x_s^n/n} \varphi_\varepsilon) ds,$$

where X^n is the density fluctuation field of the lattice gas, (2.5), and $\tau_b \varphi(u) := \varphi(b + u)$. By Theorem 2.3, the rescaled random walk $(\frac{x_t^n}{n})_{0 \leq t \leq T}$ converges to the deterministic trajectory $(v(\rho)t)_{0 \leq t \leq T}$. Because of that, we expect the integral $\int_0^t \sqrt{n}(\omega(\xi_s^n) - v(\rho)) ds$ to have the same scaling limit as the integral $v'(\rho) \int_0^t X_s^n(\tau_{-v(\rho)s} \varphi_\varepsilon) ds$. To find the scaling limit of this last process, we use the following result.

THEOREM 5.1. *Let $a \in \mathbb{R}$. Denote by $(X_t^a)_{t \geq 0}$ the stationary solution of the Ornstein–Uhlenbeck equation with drift a :*

$$(5.1) \quad dX_t^a := D(\rho)\Delta X_t^a dt + a\nabla X_t^a dt + \sqrt{2D(\rho)\chi(\rho)} d\nabla \dot{W}_t.$$

For $\varepsilon \in (0, 1)$, let $i_\varepsilon(u) = \varepsilon^{-1}\mathbf{1}_{(0,\varepsilon]}$ and let $\{Z_t^\varepsilon : t \in [0, T]\}$ be the process defined by

$$Z_t^\varepsilon := \int_0^t X_s^a(i_\varepsilon) ds.$$

Then the sequence of processes $\{Z^\varepsilon\}_{\varepsilon > 0}$ converges in the uniform topology of $\mathcal{C}([0, T]; \mathbb{R})$ to a Gaussian process $\{Z_t : t \in [0, T]\}$ of stationary increments and variance

$$(5.2) \quad \mathbb{E}[Z_t^2] = D(\rho)\chi(\rho)\sqrt{\frac{2}{\pi}} \int_0^t \frac{(t-s)e^{-\frac{a^2}{2}s}}{\sqrt{s}} ds.$$

The same statement holds if i_ε is replaced by a smooth function φ_ε with support contained in $(0, \varepsilon)$.

This corresponds to Theorem 6.3 of [19]. Now we have all the definitions needed to characterize the limit points of the additive functional A_t^n .

PROPOSITION 5.2. *Let $\{A_t : t \in [0, T]\}$ be a limit point of the sequence A^n , defined in (2.4). Let Z^ε, Z be the processes defined in Theorem 5.1, with $a = v(\rho)$. Then A and $v'(\rho)Z$ have the same finite-dimensional distributions.*

LEMMA 5.3. *Let X be the stationary solution of the Ornstein–Uhlenbeck equation (2.6) and $v(\rho)$ be as in (2.1). Denote $\tau_x f(u) := f(x + u)$. Then the process $\{X_t^a : t \in [0, T]\}$ defined by*

$$X_t^a(f) := X_t(\tau_{v(\rho)t} f)$$

is a solution of the Ornstein–Uhlenbeck equation with drift (5.1), with drift $a = v(\rho)$.

PROOF. Our goal is to prove that, for any sufficiently smooth $H : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, the process $\{M_t(H) : t \in [0, T]\}$ defined by

$$M_t(H) := X_t^a(H_t) - X_0^a(H_0) - \int_0^t X_s^a((\partial_s + D(\rho)\Delta + v(\rho)\nabla)H_s) ds$$

is a martingale with quadratic variation

$$\left\{ \int_0^t 2D(\rho)\chi(\rho)\|\nabla H_s\|_{L^2(\mathbb{R})}^2 ds : t \in [0, T] \right\}.$$

Substituting the definition of X^a in the formula for the martingale, we find

$$M_t(H) = X_t(\tau_{v(\rho)t} H_t) - X_0(H_0) - \int_0^t X_s((\partial_s + D(\rho)\Delta)\tau_{v(\rho)s} H_s) ds.$$

Since X solves the Ornstein–Uhlenbeck equation without drift (2.6), the expression above is a martingale with quadratic variation

$$\begin{aligned} \langle M_t(H) \rangle &= \int_0^t 2D(\rho)\chi(\rho)\|\nabla(\tau_{v(\rho)s} H_s)\|_{L^2(\mathbb{R})}^2 ds \\ &= \int_0^t 2D(\rho)\chi(\rho)\|\nabla(H_s)\|_{L^2(\mathbb{R})}^2 ds, \end{aligned}$$

as we wanted to show. \square

Define the auxiliary process

$$A_t^{n,\varepsilon} := \sqrt{n} \int_0^t (\xi_s^n \star \varphi_\varepsilon)(0) ds.$$

LEMMA 5.4. *Let $\varepsilon > 0$. The sequence $A^{n,\varepsilon}$ converges weakly in \mathcal{C} to the process Z^ε of Theorem 5.1.*

PROOF. Consider the mapping $\Phi : \mathcal{D}([0, T], H_{-2}) \times \mathcal{D}([0, T], \mathbb{R}) \rightarrow \mathcal{C}([0, T], \mathbb{R})$ defined by $\Phi(X, x)(t) = \int_0^t X_s(\tau_{x_s} \varphi_\varepsilon) ds$. Write $A^{n,\varepsilon} = v'(\rho)\Phi(X^n, \frac{x^n}{n})$.

We know from Proposition 2.9 that X^n converges weakly to Y in $\mathcal{D}([0, T], H_{-2})$ and that $X \in \mathcal{C}([0, T], H_{-2})$ almost surely and from Theorem 2.3 that $\frac{x^n}{n}$ converges weakly in $\mathcal{D}([0, T], \mathbb{R})$ to the deterministic continuous trajectory $t \mapsto v(\rho)t$.

Besides, it is possible to prove that the mapping Φ is continuous at all points of $\mathcal{C}([0, T], H_{-2}) \times \mathcal{C}([0, T], \mathbb{R})$, taking advantage of the smoothness and compact support of φ_ε . If a sequence converges in \mathcal{D} to a point of \mathcal{C} , than it also converges in the uniform topology. Translation is a continuous operation in H_{-2} , at least on the set of smooth compactly supported functions.

Therefore, the sequence $\Phi(X^n, \frac{x^n}{n})$ converges weakly in \mathcal{C} to the process Z^ε \square

PROOF OF PROPOSITION 5.2. Let $S \subset [0, T]$ be a finite set, say $S = \{s_1, \dots, s_k\}$. For a function $x \in \mathcal{C}$, denote by x_S the vector $(x_{s_1}, \dots, x_{s_k}) \in \mathbb{R}^k$ and by $|x_S|$ the norm $|x_S| :=$

$\sum_{s \in S} |x(s)|$. Our goal is to show that A_S and Z_S have the same law. It follows from the replacement lemma that

$$(5.3) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(|A_S^n - A_S^{n,\varepsilon}| > \delta) = 0.$$

Let $\varepsilon > 0$. Since the sequences A^n and $A^{n,\varepsilon}$ are tight in $\mathcal{C}([0, T], \mathbb{R})$, the vector $(A^n, A^{n,\varepsilon})$ is tight in $\mathcal{C}([0, T], \mathbb{R}) \times \mathcal{C}([0, T], \mathbb{R})$. It is not true in general that tightness of a sequence of random variables is implied by tightness of the marginals. This is a special feature of the space $\mathcal{C}([0, T], \mathbb{R})$ and follows from the fact that, in $\mathcal{C}([0, T], \mathbb{R})$, tightness is equivalent to equicontinuity, see, for instance, [11], p. 81. Let $\{n'\}$ be a subsequence of n such that $A^{n'}$ converges weakly to \tilde{A} . Since $(A^{n'}, A^{n',\varepsilon})$ is tight, there is a subsequence $\{n''\}$ of $\{n'\}$ along which $(A^{n''}, A^{n'',\varepsilon})$ converges weakly. The limit point is a coupling between A and the process Z^ε of Theorem 5.1. Under this coupling,

$$\mathbb{P}(|A_S - Z_S^\varepsilon| \geq \delta) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(|A_S^n - A_S^{n,\varepsilon}| \geq \delta).$$

Moreover, we know from Theorem 5.1 that the sequence Z^ε converges weakly to Z . Therefore, the sequence of vectors $(\tilde{A}, Z^\varepsilon)$ is tight in $\mathcal{C}([0, T], \mathbb{R}) \times \mathcal{C}([0, T], \mathbb{R})$, and each limit point is a coupling between \tilde{A} and Z . Under such a coupling,

$$\mathbb{P}(|A_S - Z_S| \geq \delta) \leq \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(|A_S - Z_S^\varepsilon| \geq \delta).$$

Together with (5.3), this implies that \tilde{A}_S and Z_S have the same marginals. \square

6. Asymptotic independence. The starting point in the study of our random walk was the decomposition (2.3). We proved that the martingale part converges to a Brownian motion and that the additive functional converges to a Gaussian process with stationary increments. Now we prove that these limiting processes are independent, meaning that the sequence $(M_t^n, A_t^n)_{t \in [0, T]}$ of random elements of $(\mathcal{C}([0, T], \mathbb{R}))^2$ converges in law to a product measure.

First, we tackle the problem of proving that M_t is independent of A_t for a fixed $t \in [0, T]$. In view of the replacement lemma 3.5 and the Law of Large Numbers 2.3, we can try to approximate A_t by

$$(6.1) \quad v'(\rho) \sqrt{n} \int_0^t (\eta_s^n \star \varphi_\varepsilon)(v(\rho)s) ds.$$

The functional (6.1) depends only on the environment. We will construct a martingale $(N_{s,t}^n)_{s \leq t}$ such that $N_{t,t}^n$ approximates the integral in (6.1). This martingale will be a function of the lattice gas process only, so it will never jump at the same time as the walker. Since M^n jumps only when the walker jumps, the martingales $(M_s^n)_{s \leq t}$ and $(N_{s,t}^n)_{s \leq t}$ will be orthogonal. We will prove that the quadratic variation $(\langle N_{s,t}^n \rangle)_{s \leq t}$ converges to an increasing function of s , and apply the Martingale FCLT to conclude that $\{(M_s^n, N_{s,t}^n) : s \leq t\}_{n \in \mathbb{N}}$ converges to a pair of independent continuous martingales M and N . In particular, M_t is independent of $N_{t,t} = A_t$.

LEMMA 6.1. *Let (M, A) be a limit point of the sequence (M^n, A^n) and $t \in [0, T]$. Then M_t is independent of A_t .*

PROOF. Let $\varepsilon > 0$ and $t \in [0, T]$. Let $H^{\varepsilon,t} : [0, t] \times \mathbb{R} \rightarrow \mathbb{R}$ be the solution of

$$(6.2) \quad \begin{cases} \partial_s H^{\varepsilon,t}(s, u) + D(\rho) \partial_{uu} H^{\varepsilon,t}(s, u) \varphi_\varepsilon(v(\rho)s + u) & \text{for all } s \in [0, t], u \in \mathbb{R}, \\ H^{\varepsilon,t}(t, u) = 0 & \text{for all } u \in \mathbb{R}. \end{cases}$$

Define, for $s \in [0, t]$,

$$N_{s,t}^n := -X_s^n(H_s^{\varepsilon,t}) + \int_0^s (\partial_r + n^2 L_b) X_r^n(H_r^{\varepsilon,t}) dr,$$

so that $(N_{s,t}^n)_{s \in [0,t]}$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\xi_s^n : s \leq t\}$ and $\langle N_{s,t}^n, M_s^n \rangle = 0$ for all $s \in [0, t]$, because $(M_s^n)_{s \in [0,t]}$ and $(N_{s,t}^n)_{s \in [0,t]}$ never jump simultaneously. We prove in Lemma C.1 that $\langle N_{s,t}^n \rangle$ converges in probability, as $n \rightarrow \infty$, to a non-decreasing continuous function of s . It follows from the Martingale FCLT, Proposition 2.8, that $(M_s^n, N_{s,t}^n)_{s \in [0,t]}$ converges in law to a \mathbb{R}^2 -valued continuous Gaussian process with independent marginals.

To conclude the lemma, we need to show that $v'(\rho)N_{t,t}^n - A_t^n \rightarrow 0$ in probability, as $n \rightarrow \infty$. We can prove this by combining Lemma 2.11, Proposition 2.3 and Theorem 3.5. \square

THEOREM 6.2. *Let (M, A) be a limit point of the sequence (M^n, A^n) . Let $0 \leq t_1 < \dots < t_k \leq T$. Then $(M_{t_1}, \dots, M_{t_k})$ and $(A_{t_1}, \dots, A_{t_k})$ are independent.*

PROOF. To simplify the notation, we will prove the theorem only for $k = 2$. It is sufficient to prove that (M_{t_1}, M_{t_2}) is independent of $b_1 A_{t_1} + b_2 A_{t_2}$ for any choice of real numbers b_1 and b_2 , as one can check by computing the characteristic functions. Fix $\varepsilon > 0$ and define functions H_s^{ε,t_1} and H_s^{ε,t_2} by (6.2). Extend the definition of H_s^{ε,t_1} to $s \in [0, t_2]$ by declaring $H_s^{\varepsilon,t_1} = H_{t_1}^{\varepsilon,t_1}$ if $s \in [t_1, t_2]$. Define, for $s \in [0, t_2]$,

$$N_s^n := -X_s^n(b_1 H_s^{\varepsilon,t_1} + b_2 H_s^{\varepsilon,t_2}) + \int_0^s (\partial_r + n^2 L_b) X_r^n(b_1 H_r^{\varepsilon,t_1} + b_2 H_r^{\varepsilon,t_2}) dr.$$

Then $(N_s^n)_{s \in [0,t_2]}$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\xi_s^n : s \leq t\}$ and $N_{t_2}^n = \int_0^{t_2} (\partial_r + n^2 L_b) X_r^n(b_1 H_r^{\varepsilon,t_1} + b_2 H_r^{\varepsilon,t_2}) dr$. We prove in Lemma C.1 that $\langle N_s^n \rangle$ converges in probability, as $n \rightarrow \infty$, to an increasing function of s . By Proposition 2.8, $(M_s^n, N_s^n)_{s \in [0,t_2]}$ converges in law to a \mathbb{R}^2 -valued continuous Gaussian process with independent marginals. To conclude the proof of the theorem, we need to show that $v'(\rho)N_{t_2}^n - [b_1 A_{t_1}^n - b_2 A_{t_2}^n] \rightarrow 0$ in probability, as $n \rightarrow \infty$. We can prove this by combining Lemma 2.11, Proposition 2.3 and Theorem 3.5. \square

APPENDIX A: A VARIATIONAL INEQUALITY

In this section, we prove variational inequalities relating the Dirichlet form $\mathcal{D}(\sqrt{f})$ with various integrals of interest. We start with some definitions. Recall the definition of the Dirichlet form:

$$\mathcal{D}(\sqrt{f}) = \sum_{x \in \mathbb{Z}} \int (\sqrt{f^{x,x+1}} - \sqrt{f})^2 dv_\rho.$$

We have the following result.

LEMMA A.1. *Let f be a density with respect to ν_ρ , that is, $f \geq 0$ and $\int f dv_\rho = 1$. Fix $x \in \mathbb{Z}$ and $\beta > 0$. Let g be a local function and let h be a bounded function such that $h(\eta^{x,x+1}) = h(\eta)$ for all $\eta \in \Omega$. Then*

$$\langle fh, g^{x,x+1} - g \rangle \leq \beta \mathcal{D}^{x,x+1}(\sqrt{f}) + \frac{1}{\beta} (g^2 + (g^{x,x+1})^2, fh^2).$$

PROOF. Since ν_ρ is invariant with respect to the change of variables $\xi \mapsto \xi^{x,x+1}$, we have

$$\langle f, g^{x,x+1} - g \rangle = \frac{1}{2} \langle f - f^{x,x+1}, g^{x,x+1} - g \rangle.$$

Write $A = \frac{1}{2}h(g^{x,x+1} - g)$, $B = f$ and $C = f^{x,x+1}$. We have that

$$B - C = (\sqrt{B} - \sqrt{C})(\sqrt{B} + \sqrt{C}),$$

and using the weighted Cauchy–Schwarz inequality we get

$$A(B - C) \leq \beta(\sqrt{B} - \sqrt{C})^2 + \frac{A^2(\sqrt{B} + \sqrt{C})^2}{4\beta}.$$

Notice that $(\sqrt{B} + \sqrt{C})^2 \leq 2(B + C)$, whence

$$(A.1) \quad A(B - C) \leq \beta(\sqrt{B} - \sqrt{C})^2 + \frac{A^2(B + C)}{2\beta}.$$

Recall the definitions of A , B and C . We have that $A^2 \leq h^2(g^2 + (g^{x,x+1})^2)$. Integrating (A.1) with respect to ν_ρ we obtain the lemma. \square

APPENDIX B: INEQUALITIES INVOLVING SUBGAUSSIAN RANDOM VARIABLES AND RELATIVE ENTROPY

Both lemmas below are proved in [13], Section 2.3.

LEMMA B.1 (Hoeffding’s Inequality). *Let X be a mean-zero random variable taking values in the interval $[a, b]$. Then*

$$\mathbb{E}[e^{\theta X}] \leq e^{\frac{\theta^2(b-a)^2}{8}}.$$

LEMMA B.2 (Subgaussianity). *Let X be a random variable. If*

$$\mathbb{E}[e^{\theta X}] \leq e^{\frac{\theta^2 \sigma^2}{2}} \quad \text{for all } \theta > 0$$

then

$$(B.1) \quad \mathbb{P}_n(X > \delta) \leq e^{-\frac{\delta^2}{2\sigma^2}}$$

and

$$\log \mathbb{E}[e^{cX^2}] \leq 8c\sigma^2 \quad \text{for all } 0 < c < (4\sigma^2)^{-1}.$$

Recall the definition of *relative entropy* between two probability measures μ and ν on the same space:

$$H(\mu|\nu) = \begin{cases} \int f \log f \, d\nu & \text{if } d\mu = f \, d\nu, \\ +\infty & \text{if } \mu \text{ does not have a density w.r.t. } \nu. \end{cases}$$

Then the following inequalities hold (proofs in [27], A.1.8):

$$(B.2) \quad \int g \, d\mu \leq \frac{H(\mu|\nu)}{\gamma} + \frac{1}{\gamma} \log \int e^{\gamma g} \, d\nu \quad \text{for all positive } \gamma$$

and

$$(B.3) \quad \mu(A) \leq \frac{H(\mu|\nu) + \log 2}{\log(1 + \frac{1}{\nu(A)})} \quad \text{for all sets } A \text{ such that } \nu(A) > 0.$$

APPENDIX C: CONVERGENCE OF THE QUADRATIC VARIATIONS

LEMMA C.1. Let $G : [0, T] \rightarrow \mathcal{S}(\mathbb{R})$ be a smooth function. Define the martingale

$$M_t^n(G) := X_t^n - \int_0^t (\partial_s + n^2 L_b) X_s^n(G_s) ds.$$

Then $\langle M_t^n \rangle$ converges in probability, as $n \rightarrow \infty$, to a continuous nondecreasing function of t .

PROOF. One can compute the quadratic variation and find that

$$\begin{aligned} \langle M_t^n(G) \rangle &= \int_0^t \sum_{x \in \mathbb{Z}} n \left[G_r \left(\frac{x+1}{n} \right) - G_r \left(\frac{x}{n} \right) \right]^2 r_x(\eta_r^n) [\eta_r^n(x+1) - \eta_r^n(x)]^2 dr \\ &\quad + \int_0^t \sum_{x \in \mathbb{Z}} n \left[G_r \left(\frac{x}{n} \right) - G_r \left(\frac{x-1}{n} \right) \right]^2 r_{x-1}(\eta_r^n) [\eta_r^n(x) - \eta_r^n(x-1)]^2 dr. \end{aligned}$$

We will prove that $\langle M_t^n(G) \rangle$ converges in probability to

$$2\bar{r}_0(\rho)\rho(1-\rho) \int_0^t \int_{\mathbb{R}} |\partial_u G_s(u)|^2 du ds.$$

While there is an elementary proof that $\lim_{n \rightarrow \infty} \text{Var} \langle M_t^n(G) \rangle = 0$, Corollary 3.6 gives a shorter proof.

Let $\phi(\eta) = r_0(\eta_r^n)[\eta_r^n(1) - \eta_r^n(0)]^2 + r_1(\eta_r^n)[\eta_r^n(0) - \eta_r^n(-1)]^2$. Notice that $\bar{\phi}(\rho) = 2\bar{r}_0(\rho)\rho(1-\rho)$. Then, by a Taylor expansion of G_r ,

$$\langle M_t^n(G) \rangle - \int_0^t \frac{1}{n} \sum_{x \in \mathbb{Z}} \left(\partial_u G_r \left(\frac{x}{n} \right) \right)^2 \bar{\phi}(\rho) dr - \frac{1}{\sqrt{n}} \int_0^t X^n((\partial_u G_r)^2; \phi) \rightarrow 0$$

in probability, as $n \rightarrow \infty$. The last term converges in probability to zero, as $n \rightarrow \infty$, by Corollary 3.6 and Proposition 2.9. \square

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