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CAPACITY OF THE RANGE IN DIMENSION 5

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We prove a central limit theorem for the capacity of the range of a symmetric random walk on $\mathbb{Z}^5$, under only a moment condition on the step distribution. The result is analogous to the central limit theorem for the size of the range in dimension three, obtained by Jain and Pruitt in 1971. In particular, an atypical logarithmic correction appears in the scaling of the variance. The proof is based on new asymptotic estimates, which hold in any dimension $d \geq 5$, for the probability that the ranges of two independent random walks intersect. The latter are then used for computing covariances of some intersection events at the leading order.

1. Introduction. Consider a random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$, that is, a process of the form $S_n = S_0 + X_1 + \cdots + X_n$ where the $(X_i)_{i\geq 1}$ are independent and identically distributed. A general question is to understand the geometric properties of its range, that is, the random set $\mathcal{R}_n := \{S_0, \dots, S_n\}$ and, more specifically, to analyze its large scale limiting behavior as the time n is growing. In their pioneering work, Dvoretzky and Erdős [10] proved a strong law of large numbers for the number of distinct sites in $\mathcal{R}_n$ in any dimension $d \geq 1$. Later, a central limit theorem was obtained first by Jain and Orey [13] in dimensions $d \geq 5$, then by Jain and Pruitt [14] in dimension 3 and higher and finally by Le Gall [17] in dimension 2 under fairly general hypotheses on the common law of the $(X_i)_{i\geq 1}$. Furthermore, a lot of activity has been focused on analyzing the large and moderate deviations, which we will not discuss here.

More recently, some papers were concerned with other functionals of the range, including its entropy [5] and its boundary [1, 5, 6, 8, 18]. Here, we will be interested in another natural way to measure the size of the range which also captures some properties of its shape. Namely, we will consider its Newtonian capacity, defined for a finite subset $A \subset \mathbb{Z}^d$, as

$$\mathrm{Cap}(A) := \sum_{x\in A} \mathbb{P}_x[H_A^+ = \infty], \tag{1.1}$$

where $\mathbb{P}_x$ is the law of the walk starting from x and H_A^+ denotes the first return time to A (see (2.1) below). Actually, the first study of the capacity of the range goes back to the earlier work by Jain and Orey [13], who proved a law of large numbers in any dimension $d \geq 3$ and, more precisely, that almost surely, as $n \to \infty$,

$$\frac{1}{n}\mathrm{Cap}(\mathcal{R}_n) \to \gamma_d \tag{1.2}$$

for some constant γ_d, which is nonzero if and only if $d \geq 5$—the latter observation being actually directly related to the fact that it is only in dimension 5 and higher that two independent ranges have a positive probability not to intersect each other. However, until very recently, to our knowledge there were no other work on the capacity of the range, even though the results

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of Lawler on the intersection of random walks incidentally gave a sharp asymptotic behavior of the mean in dimension four; see [15].

In a series of recent papers [3, 4, 7], the central limit theorem has been established for the simple random walk in any dimension $d \geq 3$, except for the case of dimension 5 which remained unsolved so far. The main goal of this paper is to fill this gap, but in the mean time we obtain general results on the probability that the ranges of two independent walks intersect which might be of independent interest. Furthermore, we obtain estimates for the covariances between such events which is, arguably, one of the main novelties of our work, but we shall come back on this point a bit later.

Our hypotheses on the random walk are quite general: we only require that the distribution of the $(X_i)_{i\geq 1}$ is a symmetric and irreducible probability measure on $\mathbb{Z}^d$ which has a finite dth moment. Under these hypotheses our first result is the following.

THEOREM A. *Assume $d = 5$. There exists a constant $\sigma > 0$ such that as $n \to \infty$,*

$$\operatorname{Var}\big(\operatorname{Cap}(\mathcal{R}_n)\big) \sim \sigma^2 n \log n.$$

We then deduce a central limit theorem.

THEOREM B. *Assume $d = 5$. Then,*

$$\frac{\operatorname{Cap}(\mathcal{R}_n) - \gamma_5 n}{\sigma\sqrt{n \log n}} \quad \overset{(\mathcal{L})}{\underset{n\to\infty}{\Longrightarrow}} \quad \mathcal{N}(0,1).$$

As already mentioned, along the proof we also obtain a precise asymptotic estimate for the probability that the ranges of two independent walks starting from far away intersect. Previously, to our knowledge only the order of magnitude up to multiplicative constants had been established; see [15]. Since our proof works the same in any dimension $d \geq 5$, we state our result in this general setting. Recall that to each random walk one can associate a norm (see below for a formal definition), which we denote here by $\mathcal{J}(\cdot)$ (in particular in the case of the simple random walk it coincides with the Euclidean norm).

THEOREM C. *Assume $d \geq 5$. Let S and $\widetilde{S}$ be two independent random walks starting from the origin (with the same distribution). There exists a constant $c > 0$ such that as $\|x\| \to \infty$,*

$$\mathbb{P}\big[\mathcal{R}_\infty \cap (x + \widetilde{\mathcal{R}}_\infty) \neq \varnothing\big] \sim \frac{c}{\mathcal{J}(x)^{d-4}}.$$

In fact, we obtain a stronger and more general result. Indeed, first we get some control on the second order term and show that it is $\mathcal{O}(\|x\|^{4-d-\nu})$, for some constant $\nu > 0$. Moreover, we also consider some functionals of the position of one of the two walks at its hitting time of the other range. More precisely, we obtain asymptotic estimates for quantities of the form $\mathbb{E}[F(S_\tau)\mathbf{1}\{\tau < \infty\}]$, with τ the hitting time of the range $x + \widetilde{\mathcal{R}}_\infty$, for functions F satisfying some regularity property; see (4.1). In particular, it applies to functions of the form $F(x) = 1/\mathcal{J}(x)^\alpha$ for any $\alpha \in [0, 1]$, for which we obtain that, for some constants $\nu > 0$ and $c > 0$,

$$\mathbb{E}\bigg[\frac{\mathbf{1}\{\tau < \infty\}}{1 + \mathcal{J}(S_\tau)^\alpha}\bigg] = \frac{c}{\mathcal{J}(x)^{d-4+\alpha}} + \mathcal{O}\big(\|x\|^{4-\alpha-d-\nu}\big).$$

Moreover, the same kind of estimates is obtained when one considers rather τ as the hitting time of $x + \widetilde{\mathcal{R}}[0, \ell]$ with ℓ a finite integer. These results are then used to derive asymptotic estimates for covariances of hitting events in the following four situations: let S, S^1, S^2 and S^3 be four independent random walks on $\mathbb{Z}^5$, all starting from the origin and consider either:

(i) $A=\{\mathcal{R}_\infty^1\cap\mathcal{R}[k,\infty)\neq\varnothing\}$, and $B=\{\mathcal{R}_\infty^2\cap(S_k+\mathcal{R}_\infty^3)\neq\varnothing\}$,
(ii) $A=\{\mathcal{R}_\infty^1\cap\mathcal{R}[k,\infty)\neq\varnothing\}$, and $B=\{(S_k+\mathcal{R}_\infty^2)\cap\mathcal{R}[k+1,\infty)\neq\varnothing\}$,
(iii) $A=\{\mathcal{R}_\infty^1\cap\mathcal{R}[k,\infty)\neq\varnothing\}$, and $B=\{(S_k+\mathcal{R}_\infty^2)\cap\mathcal{R}[0,k-1]\neq\varnothing\}$,
(iv) $A=\{\mathcal{R}_\infty^1\cap\mathcal{R}[1,k]\neq\varnothing\}$, and $B=\{(S_k+\mathcal{R}_\infty^2)\cap\mathcal{R}[0,k-1]\neq\varnothing\}$.

In all of these cases, we show that, for some constant $c>0$, as $k\to\infty$,

$$\mathrm{Cov}(A,B)\sim\frac{c}{k}.$$

Case (i) is the easiest and follows directly from Theorem C, since, actually, one can see that in this case both $\mathbb{P}[A\cap B]$ and $\mathbb{P}[A]\cdot\mathbb{P}[B]$ are asymptotically equivalent to a constant times the inverse of k. However, the other cases are more intricate, partly due to some cancellations that occur between the two terms, which, if estimated separately, are both of order $1/\sqrt{k}$ in cases (ii) and (iii) or even of order 1 in case (iv). In these cases we rely on the extensions of Theorem C that we just mentioned above. More precisely, in case (ii) we rely on the general result applied with the functions $F(x)=1/\|x\|$ and its convolution with the distribution of S_k, while in cases (iii) and (iv) we use the extension to hitting times of finite windows of the range. We stress also that showing the positivity of the constants c here is a delicate part of the proof, especially in case (iv), where it relies on the following inequality:

$$\int_{0\le s\le t\le 1}\left(\mathbb{E}\left[\frac{1}{\|\beta_s-\beta_1\|^3\cdot\|\beta_t\|^3}\right]-\mathbb{E}\left[\frac{1}{\|\beta_s-\beta_1\|^3}\right]\mathbb{E}\left[\frac{1}{\|\beta_t\|^3}\right]\right)ds\,dt>0,$$

with $(\beta_u)_{u\ge0}$ a standard Brownian motion in $\mathbb{R}^5$.

The paper is organized as follows. Section 2 is devoted to preliminaries, in particular we fix the main notation, recall known results on the transition kernel and the Green's function and derive some basic estimates. In Section 3 we give the plan of the proof of Theorem A which is cut into a number of intermediate results, Propositions 3.3–3.7. Propositions 3.3–3.6 are proved in the companion paper [19]. The last one, which is also the most delicate one, requires Theorem C and its extensions. Its proof is therefore postponed to Section 5, while we first prove our general results on the intersection of two independent ranges in Section 4 which is written in the general setting of random walks on $\mathbb{Z}^d$, for any $d\ge5$, and can be read independently of the rest of the paper. Finally, Section 6 is devoted to the proof of Theorem B, which is done by following a relatively well-established general scheme, based on the Lindeberg–Feller theorem for triangular arrays.

2. Preliminaries.

2.1. *Notation.* We recall that we assume the law of the $(X_i)_{i\ge1}$ to be a symmetric and irreducible probability measure[1] on $\mathbb{Z}^d$, $d\ge5$ with a finite dth moment.[2] The walk is called aperiodic, if the probability to be at the origin at time n is nonzero for all n large enough, and it is called bipartite, if this probability is nonzero only when n is even. Note that only these two cases may appear for a symmetric random walk.

Recall also that for $x\in\mathbb{Z}^d$, we denote by $\mathbb{P}_x$ the law of the walk starting from $S_0=x$. When $x=0$, we simply write it as $\mathbb{P}$. We denote its total range as $\mathcal{R}_\infty:=\{S_k\}_{k\ge0}$ and, for $0\le k\le n\le+\infty$, set $\mathcal{R}[k,n]:=\{S_k,\dots,S_n\}$.

For an integer $k\ge2$, the law of k independent random walks (with the same step distribution) starting from some $x_1,\dots,x_k\in\mathbb{Z}^5$ is denoted by $\mathbb{P}_{x_1,\dots,x_k}$ or simply by $\mathbb{P}$, when they all start from the origin.

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[1] Symmetric means that for all $x\in\mathbb{Z}^d$, $\mathbb{P}[X_1=x]=\mathbb{P}[X_1=-x]$, and irreducible means that for all x, $\mathbb{P}[S_n=x]>0$ for some $n\ge1$.

[2] This means that $\mathbb{E}[\|X_1\|^d]<\infty$, with $\|\cdot\|$ the Euclidean norm.

We define

$$H_A := \inf\{n \geq 0 : S_n \in A\} \quad \text{and} \quad H_A^+ := \inf\{n \geq 1 : S_n \in A\}, \tag{2.1}$$

respectively, for the hitting time and first return time to a subset $A \subset \mathbb{Z}^d$ that we abbreviate, respectively, as H_x and H_x^+ when A is a singleton $\{x\}$.

We let $\|x\|$ be the Euclidean norm of $x \in \mathbb{Z}^d$. If X_1 has covariance matrix $\Gamma = \Lambda\Lambda^t$, we define its associated norm as

$$\mathcal{J}^*(x) := |x \cdot \Gamma^{-1}x|^{1/2} = \|\Lambda^{-1}x\|$$

and set $\mathcal{J}(x) = d^{-1/2}\mathcal{J}^*(x)$ (see [16] p.4 for more details).

For a and b, some nonnegative reals, we let $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. We use the letters c and C to denote constants (which could depend on the covariance matrix of the walk), whose values might change from line to line. We also use standard notation for the comparison of functions: we write $f = \mathcal{O}(g)$ or, sometimes, $f \lesssim g$, if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for all x. Likewise, $f = o(g)$ means that $f/g \to 0$, and $f \sim g$ means that f and g are equivalent, that is, if $|f - g| = o(f)$. Finally, we write $f \asymp g$, when both $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$.

2.2. *Transition kernel and Green's function.* We denote by $p_n(x)$ the probability that a random walk starting from the origin ends up at position $x \in \mathbb{Z}^d$ after n steps, that is, $p_n(x) := \mathbb{P}[S_n = x]$, and note that, for any $x, y \in \mathbb{Z}^d$, one has $\mathbb{P}_x[S_n = y] = p_n(y - x)$. Recall the definitions of Γ and $\mathcal{J}^*$ from the previous subsection, and define

$$\overline{p}_n(x) := \frac{1}{(2\pi n)^{d/2}\sqrt{\det \Gamma}} \cdot e^{-\frac{\mathcal{J}^*(x)^2}{2n}}. \tag{2.2}$$

The first tool we shall need is a local central limit theorem, roughly saying that $p_n(x)$ is well approximated by $\overline{p}_n(x)$ under appropriate hypotheses. Such result has a long history; see, in particular, the standard books by Feller [12] and Spitzer [20]. We refer here to the more recent book of Lawler and Limic [16] and, more precisely, to their Theorem 2.3.5 in the case of an aperiodic random walk and to (the proof of) their Theorem 2.1.3 in the case of bipartite walks which provide the result we need under minimal hypotheses (in particular, it only requires a finite fourth moment for $\|X_1\|$).

THEOREM 2.1 (Local Central Limit Theorem). *There exists a constant $C > 0$ such that, for all $n \geq 1$ and all $x \in \mathbb{Z}^d$,*

$$|p_n(x) - \overline{p}_n(x)| \leq \frac{C}{n^{(d+2)/2}}$$

in the case of an aperiodic walk, and, for bipartite walks,

$$|p_n(x) + p_{n+1}(x) - 2\overline{p}_n(x)| \leq \frac{C}{n^{(d+2)/2}}.$$

In addition, under our hypotheses (in particular assuming $\mathbb{E}[\|X_1\|^d] < \infty$) there exists a constant $C > 0$ such that, for any $n \geq 1$ and any $x \in \mathbb{Z}^d$ (see Proposition 2.4.6 in [16]),

$$p_n(x) \leq C \cdot \begin{cases} n^{-d/2} & \text{if } \|x\| \leq \sqrt{n}, \\ \|x\|^{-d} & \text{if } \|x\| > \sqrt{n}. \end{cases} \tag{2.3}$$

It is also known (see the proof of Proposition 2.4.6 in [16]) that

$$\mathbb{E}[\|S_n\|^d] = \mathcal{O}(n^{d/2}). \tag{2.4}$$

Together with the reflection principle (see Proposition 1.6.2 in [16]), and Markov's inequality, this gives that for any $n \geq 1$ and $r \geq 1$,

$$\mathbb{P}\Big[\max_{0\leq k\leq n} \|S_k\| \geq r\Big] \leq C \cdot \left(\frac{\sqrt{n}}{r}\right)^d. \tag{2.5}$$

Now, we define for $\ell \geq 0$, $G_\ell(x) := \sum_{n\geq \ell} p_n(x)$. The *Green's function* is the function $G := G_0$. A union bound gives

$$\mathbb{P}\big[x \in \mathcal{R}[\ell, \infty)\big] \leq G_\ell(x). \tag{2.6}$$

By (2.3) there exists a constant $C > 0$ such that, for any $x \in \mathbb{Z}^d$, and $\ell \geq 0$,

$$G_\ell(x) \leq \frac{C}{\|x\|^{d-2} + \ell^{\frac{d-2}{2}} + 1}. \tag{2.7}$$

It follows from this bound (together with the corresponding lower bound $G(x) \geq c\|x\|^{2-d}$ which can be deduced from Theorem 2.1) and the fact that G is harmonic on $\mathbb{Z}^d \setminus \{0\}$, that the hitting probability of a ball is bounded as follows (see the proof of [16], Proposition 6.4.2):

$$\mathbb{P}_x[\eta_r < \infty] = \mathcal{O}\left(\frac{r^{d-2}}{1 + \|x\|^{d-2}}\right), \quad \text{with } \eta_r := \inf\{n \geq 0 : \|S_n\| \leq r\}. \tag{2.8}$$

We shall need as well some control on the overshoot. We state the result we need as a lemma and provide a short proof for the sake of completeness.

2.3. *Basic tools.* We prove here some elementary facts, which will be needed throughout the paper and which are immediate consequences of the results from the previous subsection.

LEMMA 2.2. *There exists $C > 0$ such that, for all $x \in \mathbb{Z}^d$ and $\ell \geq 0$,*

$$\sum_{z\in\mathbb{Z}^d} G_\ell(z)G(z - x) \leq \frac{C}{\|x\|^{d-4} + \ell^{\frac{d-4}{2}} + 1}.$$

PROOF. Assume first that $\ell = 0$. Then, by (2.7)

$$\sum_{z\in\mathbb{Z}^d} G(z)G(z - x) \lesssim \frac{1}{1 + \|x\|^{d-2}}\left(\sum_{\|z\|\leq 2\|x\|} \frac{1}{1 + \|z\|^{d-2}} + \sum_{\|z-x\|\leq\frac{\|x\|}{2}} \frac{1}{1 + \|z - x\|^{d-2}}\right)$$

$$+ \sum_{\|z\|\geq 2\|x\|} \frac{1}{1 + \|z\|^{2(d-2)}} \lesssim \frac{1}{1 + \|x\|^{d-4}}.$$

Assume next that $\ell \geq 1$. We distinguish two cases: if $\|x\| \leq \sqrt{\ell}$, then by using (2.7) again we deduce

$$\sum_{z\in\mathbb{Z}^d} G_\ell(z)G(z - x) \lesssim \frac{1}{\ell^{d/2}} \cdot \sum_{\|z\|\leq 2\sqrt{\ell}} \frac{1}{1 + \|z - x\|^{d-2}} + \sum_{\|z\|\geq 2\sqrt{\ell}} \frac{1}{\|z\|^{2(d-2)}} \lesssim \frac{1}{\ell^{\frac{d-4}{2}}}.$$

When $\|x\| > \sqrt{\ell}$, the result follows from case $\ell = 0$, since $G_\ell(z) \leq G(z)$. □

LEMMA 2.3. *One has*

$$\sup_{x\in\mathbb{Z}^d} \mathbb{E}\big[G(S_n - x)\big] = \mathcal{O}\left(\frac{1}{n^{\frac{d-2}{2}}}\right), \tag{2.9}$$

and, for any $\alpha \in [0, d)$,

$$\sup_{n\geq 0} \mathbb{E}\left[\frac{1}{1 + \|S_n - x\|^\alpha}\right] = \mathcal{O}\left(\frac{1}{1 + \|x\|^\alpha}\right). \tag{2.10}$$

PROOF. For (2.9) we proceed similarly as in the proof of Lemma 2.2. If $\|x\| \le \sqrt{n}$, one has, using (2.3) and (2.7),

$$\begin{aligned}\mathbb{E}\big[G(S_n - x)\big] &= \sum_{z\in\mathbb{Z}^d} p_n(z)G(z-x)\\ &\lesssim \frac{1}{n^{d/2}} \sum_{\|z\|\le 2\sqrt{n}} \frac{1}{1+\|z-x\|^{d-2}} + \sum_{\|z\|>2\sqrt{n}} \frac{1}{\|z\|^{2d-2}} \lesssim n^{\frac{2-d}{2}},\end{aligned}$$

while if $\|x\| > \sqrt{n}$, we get as well

$$\mathbb{E}\big[G(S_n - x)\big] \lesssim \frac{1}{n^{d/2}} \sum_{\|z\|\le \sqrt{n}/2} \frac{1}{\|x\|^{d-2}} + \sum_{\|z\|>\sqrt{n}/2} \frac{1}{\|z\|^{d}(1+\|z-x\|)^{d-2}} \lesssim n^{\frac{2-d}{2}}.$$

Considering now (2.10), we write

$$\begin{aligned}&\mathbb{E}\bigg[\frac{1}{1+\|S_n - x\|^\alpha}\bigg]\\ &\quad\le \frac{C}{1+\|x\|^\alpha} + \sum_{\|z-x\|\le\|x\|/2} \frac{p_n(z)}{1+\|z-x\|^\alpha}\\ &\quad\overset{(2.3)}{\lesssim} \frac{1}{1+\|x\|^\alpha} + \frac{1}{1+\|x\|^d} \sum_{\|z-x\|\le\|x\|/2} \frac{1}{1+\|z-x\|^\alpha} \lesssim \frac{1}{1+\|x\|^\alpha}.\end{aligned}$$

□

The next result deals with the probability that two independent ranges intersect. Despite its proof is a rather straightforward consequence of the previous results, it already provides upper bounds of the right order (only off by a multiplicative constant).

LEMMA 2.4. *Let S and $\widetilde{S}$ be two independent walks starting, respectively, from the origin and some $x \in \mathbb{Z}^d$. Let also ℓ and m be two given nonnegative integers (possibly infinite for m). Define*

$$\tau := \inf\{n \ge 0 : \widetilde{S}_n \in \mathcal{R}[\ell, \ell+m]\}.$$

Then, for any function $F : \mathbb{Z}^d \to \mathbb{R}_+$,

$$\mathbb{E}_{0,x}\big[\mathbb{1}\{\tau<\infty\}F(\widetilde{S}_\tau)\big] \le \sum_{i=\ell}^{\ell+m} \mathbb{E}\big[G(S_i - x)F(S_i)\big]. \tag{2.11}$$

In particular, uniformly in ℓ and m,

$$\mathbb{P}_{0,x}[\tau<\infty] = \mathcal{O}\bigg(\frac{1}{1+\|x\|^{d-4}}\bigg). \tag{2.12}$$

Moreover, uniformly in $x \in \mathbb{Z}^d$,

$$\mathbb{P}_{0,x}[\tau<\infty] = \begin{cases}\mathcal{O}(m\cdot \ell^{\frac{2-d}{2}}) & \text{if } m<\infty,\\ \mathcal{O}(\ell^{\frac{4-d}{2}}) & \text{if } m=\infty.\end{cases} \tag{2.13}$$

PROOF. The first statement follows from (2.6). Indeed, using this and the independence between S and $\widetilde{S}$, we deduce that

$$\mathbb{E}_{0,x}\big[\mathbb{1}\{\tau<\infty\}F(\widetilde{S}_\tau)\big] \le \sum_{i=\ell}^{\ell+m} \mathbb{E}_{0,x}\big[\mathbb{1}\{S_i \in \widetilde{\mathcal{R}}_\infty\}F(S_i)\big] \overset{(2.6)}{\le} \sum_{i=\ell}^{\ell+m} \mathbb{E}\big[G(S_i - x)F(S_i)\big].$$

For (2.12) note first that it suffices to consider the case when $\ell = 0$ and $m = \infty$, as, otherwise, the probability is just smaller. Taking now $F \equiv 1$ in (2.11) and using Lemma 2.2 gives the result. Similarly, (2.13) directly follows from (2.11) and (2.9). □

3. Scheme of proof of Theorem A.

3.1. *A last passage decomposition for the capacity of the range.* We provide here a last passage decomposition for the capacity of the range, in the same fashion as the well-known decomposition for the size of the range, which goes back to the seminal paper by Dvoretzky and Erdős [10] and which was also used by Jain and Pruitt [14] for their proof of the central limit theorem. We note that Jain and Orey [13] used as well a similar decomposition in their analysis of the capacity of the range (in fact, they used instead a first passage decomposition).

So let $(S_n)_{n\geq 0}$ be some random walk starting from the origin, and set

$$\varphi_k^n := \mathbb{P}_{S_k}\big[H^+_{\mathcal{R}_n} = \infty \mid \mathcal{R}_n\big] \quad \text{and} \quad Z_k^n := \mathbb{1}\{S_\ell \neq S_k \text{ for all } \ell = k+1, \dots, n\},$$

for all $0 \leq k \leq n$. By definition of the capacity (1.1), one can write by recording the sites of $\mathcal{R}_n$ according to their last visit,

$$\mathrm{Cap}(\mathcal{R}_n) = \sum_{k=0}^{n} Z_k^n \cdot \varphi_k^n.$$

A first simplification is to remove the dependance in n in each of the terms in the sum. To do this, we need some additional notation: we consider $(S_n)_{n\in\mathbb{Z}}$ a two-sided random walk starting from the origin (i.e., $(S_n)_{n\geq 0}$ and $(S_{-n})_{n\geq 0}$ are two independent walks starting from the origin) and denote its total range by $\overline{\mathcal{R}}_\infty := \{S_n\}_{n\in\mathbb{Z}}$. Then, for $k \geq 0$, let

$$\varphi(k) := \mathbb{P}_{S_k}\big[H^+_{\overline{\mathcal{R}}_\infty} = \infty \mid (S_n)_{n\in\mathbb{Z}}\big] \quad \text{and} \quad Z(k) := \mathbb{1}\{S_\ell \neq S_k, \text{ for all } \ell \geq k+1\}.$$

We note that $\varphi(k)$ can be zero with nonzero probability but that $\mathbb{E}[\varphi(k)] \neq 0$ (see the proof of Theorem 6.5.10 in [16]). We then define

$$\mathcal{C}_n := \sum_{k=0}^{n} Z(k)\varphi(k) \quad \text{and} \quad W_n := \mathrm{Cap}(\mathcal{R}_n) - \mathcal{C}_n.$$

The following lemma is proved in the companion paper [19].

LEMMA 3.1. *One has*

$$\mathbb{E}\big[W_n^2\big] = \mathcal{O}(n).$$

Given this result, Theorem A reduces to an estimate of the variance of $\mathcal{C}_n$. To this end, we first observe that

$$\mathrm{Var}(\mathcal{C}_n) = 2 \sum_{0\leq \ell < k \leq n} \mathrm{Cov}\big(Z(\ell)\varphi(\ell), Z(k)\varphi(k)\big) + \mathcal{O}(n).$$

Furthermore, by translation invariance, for any $\ell < k$

$$\mathrm{Cov}\big(Z(\ell)\varphi(\ell), Z(k)\varphi(k)\big) = \mathrm{Cov}\big(Z(0)\varphi(0), Z(k-\ell)\varphi(k-\ell)\big),$$

so that, in fact,

$$\mathrm{Var}(\mathcal{C}_n) = 2\sum_{\ell=1}^{n}\sum_{k=1}^{\ell} \mathrm{Cov}\big(Z(0)\varphi(0), Z(k)\varphi(k)\big) + \mathcal{O}(n).$$

Thus, Theorem A is a direct consequence of the following theorem.

THEOREM 3.2. *There exists a constant $\sigma > 0$ such that*

$$\mathrm{Cov}\big(Z(0)\varphi(0), Z(k)\varphi(k)\big) \sim \frac{\sigma^2}{2k}.$$

This result is the core of the paper and uses, in particular, Theorem C (in fact, for some more general statement see Theorem 4.1). More details about its proof are given in the next subsection.

3.2. *Scheme of proof of Theorem* 3.2. We provide here some decomposition of $\varphi(0)$ and $\varphi(k)$ into a sum of terms involving intersection and nonintersection probabilities of different parts of the path $(S_n)_{n\in\mathbb{Z}}$. For this, we consider some sequence of integers $(\varepsilon_k)_{k\geq 1}$ satisfying $k > 2\varepsilon_k$, for all $k \geq 3$, and whose value will be fixed later. A first step in our analysis is to reduce the influence of the random variables $Z(0)$ and $Z(k)$ which play a very minor role in the whole proof. Thus, we define

$$Z_0 := \mathbb{1}\{S_\ell \neq 0, \forall \ell = 1,\dots,\varepsilon_k\} \quad \text{and} \quad Z_k := \mathbb{1}\{S_\ell \neq S_k, \forall \ell = k+1,\dots,k+\varepsilon_k\}.$$

Note that these notation are slightly misleading (as, in fact, Z_0 and Z_k depend on ε_k, but this shall hopefully not cause any confusion). One has

$$\mathbb{E}[|Z(0) - Z_0|] = \mathbb{P}[0 \in \mathcal{R}[\varepsilon_k + 1, \infty)] \overset{(2.6)}{\leq} G_{\varepsilon_k}(0) \overset{(2.7)}{=} \mathcal{O}(\varepsilon_k^{-3/2}),$$

and the same estimate holds for $\mathbb{E}[|Z(k) - Z_k|]$ by the Markov property. Therefore,

$$\operatorname{Cov}(Z(0)\varphi(0), Z(k)\varphi(k)) = \operatorname{Cov}(Z_0\varphi(0), Z_k\varphi(k)) + \mathcal{O}(\varepsilon_k^{-3/2}).$$

Then, recall that we consider a two-sided walk $(S_n)_{n\in\mathbb{Z}}$, and that $\varphi(0) = \mathbb{P}[H^+_{\mathcal{R}(-\infty,\infty)} = \infty \mid S]$. Thus, one can decompose $\varphi(0)$ as follows:

$$\varphi(0) = \varphi_0 - \varphi_1 - \varphi_2 - \varphi_3 + \varphi_{1,2} + \varphi_{1,3} + \varphi_{2,3} - \varphi_{1,2,3},$$

with

$$\begin{aligned}
\varphi_0 &:= \mathbb{P}[H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],\\
\varphi_1 &:= \mathbb{P}[H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],\\
\varphi_2 &:= \mathbb{P}[H^+_{\mathcal{R}[\varepsilon_k+1,k]} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],\\
\varphi_3 &:= \mathbb{P}[H^+_{\mathcal{R}[k+1,\infty)} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],\\
\varphi_{1,2} &:= \mathbb{P}[H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]} < \infty, H^+_{\mathcal{R}[\varepsilon_k+1,k]} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],\\
\varphi_{1,3} &:= \mathbb{P}[H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]} < \infty, H^+_{\mathcal{R}[k+1,\infty)} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],\\
\varphi_{2,3} &:= \mathbb{P}[H^+_{\mathcal{R}[\varepsilon_k+1,k]} < \infty, H^+_{\mathcal{R}[k+1,\infty)} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],\\
\varphi_{1,2,3} &:= \mathbb{P}[H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]} < \infty, H^+_{\mathcal{R}[\varepsilon_k+1,k]} < \infty, H^+_{\mathcal{R}[k+1,\infty)} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S].
\end{aligned}$$

We decompose, similarly,

$$\varphi(k) = \psi_0 - \psi_1 - \psi_2 - \psi_3 + \psi_{1,2} + \psi_{1,3} + \psi_{2,3} - \psi_{1,2,3},$$

where index 0 refers to the event of avoiding $\mathcal{R}[k-\varepsilon_k, k+\varepsilon_k]$, index 1 to the event of hitting $\mathcal{R}(-\infty,-1]$, index 2 to the event of hitting $\mathcal{R}[0, k-\varepsilon_k-1]$ and index 3 to the event of hitting $\mathcal{R}[k+\varepsilon_k+1,\infty)$ (for a walk starting from S_k this time). Note that φ_0 and ψ_0 are independent. Then, write

$$\begin{aligned}
&\operatorname{Cov}(Z_0\varphi(0), Z_k\varphi(k))\\
&\quad = -\sum_{i=1}^{3}(\operatorname{Cov}(Z_0\varphi_i, Z_k\psi_0) + \operatorname{Cov}(Z_0\varphi_0, Z_k\psi_i)) + \sum_{i,j=1}^{3}\operatorname{Cov}(Z_0\varphi_i, Z_k\psi_j)\\
&\qquad + \sum_{1\leq i<j\leq 3}(\operatorname{Cov}(Z_0\varphi_{i,j}, Z_k\psi_0) + \operatorname{Cov}(Z_0\varphi_0, Z_k\psi_{i,j})) + R_{0,k},
\end{aligned} \tag{3.1}$$

where $R_{0,k}$ is an error term. One first task is to show that it is negligible.

PROPOSITION 3.3. *One has* $|R_{0,k}| = \mathcal{O}(\varepsilon_k^{-3/2})$.

The second step is the following.

PROPOSITION 3.4. *One has*:

(i) $|\mathrm{Cov}(Z_0\varphi_{1,2}, Z_k\psi_0)| + |\mathrm{Cov}(Z_0\varphi_0, Z_k\psi_{2,3})| = \mathcal{O}(\frac{\sqrt{\varepsilon_k}}{k^{3/2}})$,

(ii) $|\mathrm{Cov}(Z_0\varphi_{1,3}, Z_k\psi_0)| + |\mathrm{Cov}(Z_0\varphi_0, Z_k\psi_{1,3})| = \mathcal{O}(\frac{\sqrt{\varepsilon_k}}{k^{3/2}} \cdot \log(\frac{k}{\varepsilon_k}) + \frac{1}{\varepsilon_k^{3/4}\sqrt{k}})$,

(iii) $|\mathrm{Cov}(Z_0\varphi_{2,3}, Z_k\psi_0)| + |\mathrm{Cov}(Z_0\varphi_0, Z_k\psi_{1,2})| = \mathcal{O}(\frac{\sqrt{\varepsilon_k}}{k^{3/2}} \cdot \log(\frac{k}{\varepsilon_k}) + \frac{1}{\varepsilon_k^{3/4}\sqrt{k}})$.

In the same fashion as Part (i) of the previous proposition, we show

PROPOSITION 3.5. *For any* $1 \le i < j \le 3$,

$$\big|\mathrm{Cov}(Z_0\varphi_i, Z_k\psi_j)\big| = \mathcal{O}\bigg(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\bigg), \qquad \big|\mathrm{Cov}(Z_0\varphi_j, Z_k\psi_i)\big| = \mathcal{O}\bigg(\frac{1}{\varepsilon_k}\bigg).$$

The next step deals with the first sum in the right-hand side of (3.1).

PROPOSITION 3.6. *There exists a constant* $\alpha \in (0, 1)$ *such that*

$$\mathrm{Cov}(Z_0\varphi_1, Z_k\psi_0) = \mathrm{Cov}(Z_0\varphi_0, Z_k\psi_3) = 0,$$

$$\big|\mathrm{Cov}(Z_0\varphi_2, Z_k\psi_0)\big| + \big|\mathrm{Cov}(Z_0\varphi_0, Z_k\psi_2)\big| = \mathcal{O}\bigg(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\bigg),$$

$$\big|\mathrm{Cov}(Z_0\varphi_3, Z_k\psi_0)\big| + \big|\mathrm{Cov}(Z_0\varphi_0, Z_k\psi_1)\big| = \mathcal{O}\bigg(\frac{\varepsilon_k^{\alpha}}{k^{1+\alpha}}\bigg).$$

At this point one can already deduce the bound $\mathrm{Var}(\mathrm{Cap}(\mathcal{R}_n)) = \mathcal{O}(n \log n)$, just applying the previous propositions with say $\varepsilon_k := \lfloor k/4 \rfloor$. The proofs of Propositions 3.3–3.6 are done in the companion paper [19].

Our goal now is to obtain the finer asymptotic result stated in Theorem 3.2, for which it remains to estimate the leading terms in (3.1), This is, of course, the most delicate (and interesting) part. The result reads as follows:

PROPOSITION 3.7. *There exists* $\delta > 0$ *such that if* $\varepsilon_k \ge k^{1-\delta}$ *and* $\varepsilon_k = o(k)$, *then*, *for some positive constants* $(\sigma_{i,j})_{1 \le i \le j \le 3}$,

$$\mathrm{Cov}(Z_0\varphi_j, Z_k\psi_i) \sim \mathrm{Cov}(Z_0\varphi_{4-i}, Z_k\psi_{4-j}) \sim \frac{\sigma_{i,j}}{k}.$$

Note that Theorem 3.2 is a direct consequence of (3.1) and Propositions 3.3–3.7.

4. Intersection of two random walks and proof of Theorem C. In this section we prove a general result, which will be needed for proving Proposition 3.7 and which also gives Theorem C as a corollary. First, we introduce some general condition for a function $F : \mathbb{Z}^d \to \mathbb{R}$, namely,

$$\text{there exists a constant } C_F > 0 \text{ such that}$$
$$\big|F(y) - F(x)\big| \le C_F \frac{\|y - x\|}{1 + \|y\|} \cdot \big|F(x)\big|, \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{4.1}$$

Note that any function satisfying (4.1) is automatically bounded. Observe also that this condition is satisfied by functions which are equivalent to $c/\mathcal{J}(x)^\alpha$, for some constants $\alpha \in [0,1]$, and $c > 0$. On the other hand, it is not satisfied by functions which are $o(1/\|x\|)$, as $\|x\| \to \infty$. However, this is fine, since the only two cases that will be of interest for us here are when either F is constant or when $F(x)$ is of order $1/\|x\|$. Now, for a general function $F : \mathbb{Z}^d \to \mathbb{R}$, we define for $r > 0$,

$$\overline{F}(r) := \sup_{r \le \|x\| \le r+1} |F(x)|.$$

Then, set

$$I_F(r) := \frac{\log(2+r)}{r^{d-2}} \int_0^r s \cdot \overline{F}(s)\, ds + \int_r^\infty \frac{\overline{F}(s)\log(2+s)}{s^{d-3}}\, ds,$$

and, with $\chi_d(r) := 1 + (\log(2+r))\mathbb{1}_{\{d=5\}}$,

$$J_F(r) := \frac{\chi_d(r)}{r^{d-2}} \int_0^r \overline{F}(s)\, ds + \int_r^\infty \frac{\overline{F}(s)\chi_d(s)}{s^{d-2}}\, ds.$$

THEOREM 4.1. *Let $(S_n)_{n\ge0}$ and $(\widetilde{S}_n)_{n\ge0}$ be two independent random walks on $\mathbb{Z}^d$, $d \ge 5$, starting, respectively, from the origin and some $x \in \mathbb{Z}^d$. Let $\ell \in \mathbb{N} \cup \{\infty\}$, and define*

$$\tau := \inf\{n \ge 0 : \widetilde{S}_n \in \mathcal{R}[0,\ell]\}.$$

There exists $\nu \in (0,1)$ such that, for any $F : \mathbb{Z}^d \to \mathbb{R}$ satisfying (4.1),

$$\begin{aligned} \mathbb{E}_{0,x}\big[F(\widetilde{S}_\tau)\mathbb{1}\{\tau < \infty\}\big] &= \frac{\gamma_d}{\kappa} \cdot \mathbb{E}\Bigg[\sum_{i=0}^{\ell} G(S_i - x)F(S_i)\Bigg] \\ &\quad + \mathcal{O}\bigg(\frac{I_F(\|x\|)}{(\ell \wedge \|x\|)^\nu} + (\ell \wedge \|x\|)^\nu J_F(\|x\|)\bigg), \end{aligned} \tag{4.2}$$

where γ_d is as in (1.2) and κ is some positive constant given by

$$\kappa := \mathbb{E}\bigg[\Big(\sum_{n\in\mathbb{Z}} G(S_n)\Big) \cdot \mathbb{P}\big[H^+_{\overline{\mathcal{R}}_\infty} = +\infty \mid \overline{\mathcal{R}}_\infty\big] \cdot \mathbb{1}\{S_n \ne 0, \forall n \ge 1\}\bigg],$$

with $(S_n)_{n\in\mathbb{Z}}$ a two-sided walk starting from the origin and $\overline{\mathcal{R}}_\infty := \{S_n\}_{n\in\mathbb{Z}}$.

REMARK 4.2. Note that, when $F(x) \sim c/\mathcal{J}(x)^\alpha$, for some constants $\alpha \in [0,1]$ and $c > 0$, then $I_F(r)$ and $J_F(r)$ are, respectively, of order $1/r^{d-4+\alpha}$, and $1/r^{d-3+\alpha}$ (up to logarithmic factors), while one could show that

$$\mathbb{E}\Bigg[\sum_{i=0}^{\ell} G(S_i - x)F(S_i)\Bigg] \sim \frac{c'}{\mathcal{J}(x)^{d-4+\alpha}} \quad \text{as } \|x\| \to \infty \text{ and } \ell/\|x\|^2 \to \infty$$

for some other constant $c' > 0$ (see below for a proof at least when $\ell = \infty$ and $\alpha = 0$). Therefore, in these cases Theorem 4.1 provides a true equivalent for the term on the left-hand side of (4.2).

REMARK 4.3. This theorem strengthens Theorem C in two aspects: on one hand, it allows to consider functionals of the position of one of the two walks at its hitting time of the other path, and, on the other hand, it also allows to consider only a finite time horizon for one of the two walks (not mentioning the fact that it gives, additionally, some bound on the error term). Both these aspects will be needed later (the first one in the proof of Lemma 5.2 and the second one in the proofs of Lemmas 5.3 and 5.4).

Given this result, one obtains Theorem C as a corollary. To see this, we first recall an asymptotic result on the Green's function: in any dimension $d \geq 5$, under our hypotheses on μ, there exists a constant $c_d > 0$ such that as $\|x\| \to \infty$,

$$G(x) = \frac{c_d}{\mathcal{J}(x)^{d-2}} + \mathcal{O}(\|x\|^{1-d}). \tag{4.3}$$

This result is proved in [21] under only the hypothesis that X_1 has a finite $(d-1)$th moment (we refer also to Theorem 4.3.5 in [16] for a proof under the stronger hypothesis that X_1 has a finite $(d+1)$th moment). One also needs the following elementary fact.

LEMMA 4.4. *There exists a positive constant c such that, as $\|x\| \to \infty$,*

$$\sum_{y \in \mathbb{Z}^d \setminus \{0,x\}} \frac{1}{\mathcal{J}(y)^{d-2} \cdot \mathcal{J}(y-x)^{d-2}} = \frac{c}{\mathcal{J}(x)^{d-4}} + \mathcal{O}\left(\frac{1}{\|x\|^{d-3}}\right).$$

PROOF. The proof follows by first an approximation by an integral and then a change of variables. More precisely, letting $u := x/\mathcal{J}(x)$, one has

$$\begin{aligned}
&\sum_{y \in \mathbb{Z}^d \setminus \{0,x\}} \frac{1}{\mathcal{J}(y)^{d-2}\mathcal{J}(y-x)^{d-2}} \\
&\quad = \int_{\mathbb{R}^d} \frac{1}{\mathcal{J}(y)^{d-2}\mathcal{J}(y-x)^{d-2}}\, dy + \mathcal{O}(\|x\|^{3-d}) \\
&\quad = \frac{1}{\mathcal{J}(x)^{d-4}} \int_{\mathbb{R}^5} \frac{1}{\mathcal{J}(y)^{d-2}\mathcal{J}(y-u)^{d-2}}\, dy + \mathcal{O}(\|x\|^{3-d}),
\end{aligned}$$

and it suffices to observe that, by rotational invariance, the last integral is independent of x. □

PROOF OF THEOREM C. The result follows from Theorem 4.1 by taking $F \equiv 1$ and $\ell = \infty$, and then by using (4.3) together with Lemma 4.4. □

It amounts now to prove Theorem 4.1. For this we need some technical estimates that we gather in Lemma 4.5 below. Since we believe this is not the most interesting part, we defer its proof to the end of this section.

LEMMA 4.5. *Assume that F satisfies* (4.1). *Then*:

1. *There exists a constant $C > 0$ such that, for any $x \in \mathbb{Z}^d$,*

$$\sum_{i=0}^{\infty} \mathbb{E}\left[\left(\sum_{j=0}^{\infty} G(S_j - S_i)\frac{\|S_j - S_i\|}{1 + \|S_j\|}\right) \cdot |F(S_i)| G(S_i - x)\right] \leq C J_F(\|x\|). \tag{4.4}$$

2. *There exists $C > 0$ such that, for any $R > 0$ and any $x \in \mathbb{Z}^d$,*

$$\sum_{i=0}^{\infty} \mathbb{E}\left[\left(\sum_{|j-i| \geq R} G(S_j - S_i)\right)|F(S_i)| G(S_i - x)\right] \leq \frac{C}{R^{\frac{d-4}{2}}} \cdot I_F(\|x\|), \tag{4.5}$$

$$\sum_{i=0}^{\infty} \mathbb{E}\left[\left(\sum_{|j-i| \geq R} G(S_j - S_i)|F(S_j)|\right) G(S_i - x)\right] \leq \frac{C}{R^{\frac{d-4}{2}}} \cdot I_F(\|x\|). \tag{4.6}$$

One also need some standard results from (discrete) potential theory. If Λ is a nonempty finite subset of $\mathbb{Z}^d$, containing the origin, we define

$$\mathrm{rad}(\Lambda) := 1 + \sup_{x \in \Lambda} \|x\|$$

and also consider, for $x \in \Lambda$,

$$e_\Lambda(x) := \mathbb{P}_x[H_\Lambda^+ = \infty] \quad \text{and} \quad \overline{e}_\Lambda(x) := \frac{e_\Lambda(x)}{\mathrm{Cap}(\Lambda)}.$$

The measure $\overline{e}_\Lambda$ is sometimes called the harmonic measure of Λ from infinity, due to the next result.

LEMMA 4.6. *There exists a constant $C > 0$ such that, for any finite subset $\Lambda \subseteq \mathbb{Z}^d$ containing the origin and any $y \in \mathbb{Z}^d$ with $\|y\| > 2\,\mathrm{rad}(\Lambda)$,*

$$\mathbb{P}_y[H_\Lambda < \infty] \le C \cdot \frac{\mathrm{Cap}(\Lambda)}{1 + \|y\|^{d-2}}. \tag{4.7}$$

Furthermore, for any $x \in \Lambda$ and any $y \in \mathbb{Z}^d$,

$$\bigl|\mathbb{P}_y[S_{H_\Lambda} = x \mid H_\Lambda < \infty] - \overline{e}_\Lambda(x)\bigr| \le C \cdot \frac{\mathrm{rad}(\Lambda)}{1 + \|y\|}. \tag{4.8}$$

This lemma is proved in [16] for finite range random walks. The proof extends to our setting, but some little care is needed, so we shall give some details at the end of this section. Assuming this, one can now give the proof of our main result.

PROOF OF THEOREM 4.1. The proof consists in computing the quantity

$$A := \mathbb{E}_{0,x}\left[\sum_{i=0}^{\ell}\sum_{j=0}^{\infty} \mathbb{1}\{S_i = \widetilde{S}_j\} F(S_i)\right], \tag{4.9}$$

in two different ways.[3] On one hand, by integrating with respect to the law of $\widetilde{S}$ first, we obtain

$$A = \mathbb{E}\left[\sum_{i=0}^{\ell} G(S_i - x) F(S_i)\right]. \tag{4.10}$$

On the other hand, the double sum in (4.9) is nonzero only when τ is finite. Therefore, using also the Markov property at time τ, we get

$$\begin{aligned} A &= \mathbb{E}_{0,x}\left[\left(\sum_{i=0}^{\ell}\sum_{j=0}^{\infty} \mathbb{1}\{S_i = \widetilde{S}_j\} F(S_i)\right) \mathbb{1}\{\tau < \infty\}\right] \\ &= \sum_{i=0}^{\ell} \mathbb{E}_{0,x}\left[\left(\sum_{j=0}^{\ell} G(S_j - S_i) F(S_j)\right) Z_i^\ell \cdot \mathbb{1}\{\tau < \infty, \widetilde{S}_\tau = S_i\}\right], \end{aligned}$$

where we recall that $Z_i^\ell = \mathbb{1}\{S_j \neq S_i, \forall j = i+1, \dots, \ell\}$. The computation of this last expression is divided in a few steps:

---

[3] This idea goes back to the seminal paper of Erdős and Taylor [11], even though it was not used properly there and was corrected only a few years later by Lawler; see [15].

Step 1. Set

$$B := \sum_{i=0}^{\ell} \mathbb{E}_{0,x}\left[\left(\sum_{j=0}^{\ell} G(S_j - S_i)\right) F(S_i) Z_i^{\ell} \cdot \mathbb{1}\{\tau < \infty, \widetilde{S}_\tau = S_i\}\right],$$

and note that

$$\begin{aligned} |A - B| &\overset{(4.1)}{\le} C_F \sum_{i=0}^{\ell} \mathbb{E}_{0,x}\left[\left(\sum_{j=0}^{\ell} G(S_j - S_i)\frac{\|S_j - S_i\|}{(1 + \|S_j\|)}\right)\bigl|F(S_i)\bigr| \mathbb{1}\{S_i \in \widetilde{\mathcal{R}}_\infty\}\right] \\ &\overset{(2.6)}{\le} C_F \sum_{i=0}^{\ell} \mathbb{E}\left[\left(\sum_{j=0}^{\ell} G(S_j - S_i)\frac{\|S_j - S_i\|}{(1 + \|S_j\|)}\right)\bigl|F(S_i)\bigr| G(S_i - x)\right] \overset{(4.4)}{=} \mathcal{O}\bigl(J_F(\|x\|)\bigr). \end{aligned}$$

Step 2. Consider now some positive integer R, and define

$$D_R := \sum_{i=0}^{\ell} \mathbb{E}_{0,x}\bigl[\mathcal{G}_{i,R,\ell} F(S_i) Z_i^{\ell} \cdot \mathbb{1}\{\tau < \infty, \widetilde{S}_\tau = S_i\}\bigr],$$

with $\mathcal{G}_{i,R,\ell} := \sum_{j=(i-R)\vee 0}^{(i+R)\wedge \ell} G(S_j - S_i)$. One has

$$|B - D_R| \overset{(2.6)}{\le} \sum_{i=0}^{\ell} \mathbb{E}\left[\left(\sum_{|j-i|>R} G(S_j - S_i)\right)\bigl|F(S_i)\bigr| G(S_i - x)\right] \overset{(4.5)}{\lesssim} \frac{I_F(\|x\|)}{R^{\frac{d-4}{2}}}.$$

Step 3. Let R be an integer larger than 2 and such that $\ell \wedge \|x\|^2 \ge R^6$. Let $M := \lfloor \ell / R^5 \rfloor - 1$, and define, for $0 \le m \le M$,

$$I_m := \{mR^5 + R^3, \dots, (m+1)R^5 - R^3\} \quad \text{and} \quad J_m := \{mR^5, \dots, (m+1)R^5 - 1\}.$$

Define further

$$E_R := \sum_{m=0}^{M} \sum_{i \in I_m} \mathbb{E}_{0,x}\bigl[\mathcal{G}_{i,R} F(S_i) Z_i^{\ell} \cdot \mathbb{1}\{\tau < \infty, \widetilde{S}_\tau = S_i\}\bigr],$$

with $\mathcal{G}_{i,R} := \sum_{j=i-R}^{i+R} G(S_j - S_i)$. One has, bounding $\mathcal{G}_{i,R}$ by $(2R+1)G(0)$,

$$\begin{aligned} |D_R - E_R| &\le (2R+1)G(0) \\ &\quad \times \left\{\sum_{m=0}^{M} \sum_{i \in J_m \setminus I_m} \mathbb{E}\bigl[\bigl|F(S_i)\bigr| G(S_i - x)\bigr] + \sum_{i=(M+1)R^5}^{\ell} \mathbb{E}\bigl[\bigl|F(S_i)\bigr| G(S_i - x)\bigr]\right\}, \end{aligned}$$

with the convention that the last sum is zero when ℓ is infinite. Using $\ell \ge R^6$, we get

$$\begin{aligned} &\sum_{i=(M+1)R^5}^{\ell} \mathbb{E}\bigl[\bigl|F(S_i)\bigr| G(S_i - x)\bigr] \\ &\quad \le \sum_{z \in \mathbb{Z}^d} \bigl|F(z)\bigr| G(z - x) \sum_{i=(M+1)R^5}^{(M+2)R^5} p_i(z) \\ &\quad \overset{(2.3),(2.7)}{\lesssim} \frac{R^5}{\ell} \sum_{z \in \mathbb{Z}^d} \frac{|F(z)|}{(1 + \|z - x\|^{d-2})(1 + \|z\|^{d-2})} \lesssim \frac{R^5}{\ell} \cdot I_F\bigl(\|x\|\bigr). \end{aligned}$$

Likewise, since $\|x\|^2 \geq R^6$,

$$\begin{aligned}
&\sum_{m=0}^{M} \sum_{i \in J_m \setminus I_m} \mathbb{E}\big[\big|F(S_i)\big| G(S_i - x)\big] \\
&\quad \leq \sum_{z \in \mathbb{Z}^d} \frac{|F(z)|}{1 + \|z - x\|^{d-2}} \sum_{m=0}^{M} \sum_{i \in J_m \setminus I_m} p_i(z) \\
&\quad \overset{(2.7)}{\lesssim} \frac{1}{1 + \|x\|^{d-2}} \sum_{\|z\|^2 \leq R^5} \frac{1}{1 + \|z\|^{d-2}} \qquad (4.11)\\
&\qquad + \sum_{\|z\|^2 \geq R^5} \frac{|F(z)|}{1 + \|z - x\|^{d-2}} \sum_{m=0}^{M} \sum_{i \in J_m \setminus I_m} \left(\frac{\mathbb{1}\{i \leq \|z\|^2\}}{1 + \|z\|^d} + \frac{\mathbb{1}\{i \geq \|z\|^2\}}{i^{d/2}} \right) \\
&\quad \lesssim \frac{R^5}{1 + \|x\|^{d-2}} + \frac{1}{R^2} \cdot I_F\big(\|x\|\big),
\end{aligned}$$

using for the last inequality that the proportion of indices i, which are not in one of the I_m's, is of order $1/R^2$.

Step 4. For $0 \leq m \leq M + 1$, set

$$\mathcal{R}^{(m)} := \mathcal{R}\big[mR^5, (m + 1)R^5 - 1\big] \quad \text{and} \quad \tau_m := \inf\{n \geq 0 : \widetilde{S}_n \in \mathcal{R}^{(m)}\}.$$

Then, let

$$F_R := \sum_{m=0}^{M} \sum_{i \in I_m} \mathbb{E}_{0,x}\big[\mathcal{G}_{i,R} F(S_i) Z_i^{\ell} \cdot \mathbb{1}\{\tau_m < \infty, \widetilde{S}_{\tau_m} = S_i\}\big].$$

Since by definition $\tau \leq \tau_m$, for any m, one has for any $i \in I_m$,

$$\begin{aligned}
&\big|\mathbb{P}_{0,x}[\tau < \infty, \widetilde{S}_\tau = S_i \mid S] - \mathbb{P}_{0,x}[\tau_m < \infty, \widetilde{S}_{\tau_m} = S_i \mid S]\big| \\
&\quad \leq \mathbb{P}_{0,x}[\tau < \tau_m < \infty, \widetilde{S}_{\tau_m} = S_i \mid S] \\
&\quad \leq \sum_{j \notin J_m} \mathbb{P}_{0,x}[\tau < \tau_m < \infty, \widetilde{S}_\tau = S_j, \widetilde{S}_{\tau_m} = S_i \mid S] \\
&\quad \overset{(2.6)}{\leq} \sum_{j \notin J_m} G(S_j - x) G(S_i - S_j).
\end{aligned}$$

Therefore, bounding again $\mathcal{G}_{i,R}$ by $(2R + 1)G(0)$, we get

$$\begin{aligned}
|E_R - F_R| &\lesssim R \sum_{m=0}^{M} \sum_{i \in I_m} \mathbb{E}\bigg[\bigg(\sum_{j \notin J_m} G(S_i - S_j) G(S_j - x)\bigg) \cdot \big|F(S_i)\big|\bigg] \\
&\lesssim R \sum_{i=0}^{\infty} \mathbb{E}\bigg[\bigg(\sum_{j : |j - i| \geq R^3} G(S_i - S_j) G(S_j - x)\bigg) \cdot \big|F(S_i)\big|\bigg] \\
&\overset{(4.6)}{\lesssim} \frac{1}{R^{3\frac{d-4}{2} - 1}} \cdot I_F\big(\|x\|\big) \lesssim \frac{1}{\sqrt{R}} \cdot I_F\big(\|x\|\big).
\end{aligned}$$

Step 5. For $m \geq 0$ and $i \in I_m$, define

$$e_i^m := \mathbb{P}_{S_i}\big[H^+_{\mathcal{R}^{(m)}} = \infty \mid S\big] \quad \text{and} \quad \overline{e}_i^m := \frac{e_i^m}{\mathrm{Cap}(\mathcal{R}^{(m)})}.$$

Then, let

$$H_R := \sum_{m=0}^{M} \sum_{i \in I_m} \mathbb{E}_{0,x}\big[\mathcal{G}_{i,R} F(S_i) Z_i^{\ell} \overline{e}_i^m \cdot \mathbb{1}\{\tau_m < \infty\}\big].$$

Applying (4.8) to the sets $\Lambda_m := \mathcal{R}^{(m)} - S_{i_m}$, we get, for any $m \geq 0$ and any $i \in I_m$,

$$\big|\mathbb{P}_{0,x}[\widetilde{S}_{\tau_m} = S_i \mid \tau_m < \infty, S] - \overline{e}_i^m\big| \leq C \frac{\mathrm{rad}(\Lambda_m)}{1 + \|x - S_{i_m}\|}. \tag{4.12}$$

By (4.7) it also holds

$$\begin{aligned} \mathbb{P}_{0,x}[\tau_m < \infty \mid S] &\leq \frac{CR^5}{1 + \|x - S_{i_m}\|^{d-2}} + \mathbb{1}\{\|x - S_{i_m}\| \leq 2\,\mathrm{rad}(\Lambda_m)\} \\ &\lesssim \frac{R^5 + \mathrm{rad}(\Lambda_m)^{d-2}}{1 + \|x - S_{i_m}\|^{d-2}}, \end{aligned} \tag{4.13}$$

using that $\mathrm{Cap}(\Lambda_m) \leq |\Lambda_m| \leq R^5$. Note also that by (2.4) and Doob's L^p-inequality (see Theorem 4.3.3 in [9]) one has, for any $1 < p \leq d$,

$$\mathbb{E}\big[\mathrm{rad}(\Lambda_m)^p\big] = \mathcal{O}(R^{\frac{5p}{2}}). \tag{4.14}$$

Therefore,

$$\begin{aligned} |F_R - H_R| &\overset{(4.12)}{\lesssim} R \sum_{m=0}^{M} \sum_{i \in I_m} \mathbb{E}_{0,x}\bigg[\frac{|F(S_i)| \cdot \mathrm{rad}(\Lambda_m)}{1 + \|x - S_{i_m}\|} \mathbb{1}\{\tau_m < \infty\}\bigg] \\ &\overset{(4.1)}{\lesssim} R^6 \sum_{m=0}^{M} \mathbb{E}_{0,x}\bigg[\frac{|F(S_{i_m})| \cdot \mathrm{rad}(\Lambda_m)^2}{1 + \|x - S_{i_m}\|} \mathbb{1}\{\tau_m < \infty\}\bigg] \\ &\overset{(4.13),(4.14)}{\lesssim} R^{6+\frac{5d}{2}} \sum_{m=0}^{M} \mathbb{E}\bigg[\frac{|F(S_{i_m})|}{1 + \|x - S_{i_m}\|^{d-1}}\bigg] \lesssim R^{6+\frac{5d}{2}} \sum_{z \in \mathbb{Z}^d} \frac{|F(z)| G(z)}{1 + \|x - z\|^{d-1}} \\ &\overset{(2.7)}{\lesssim} \frac{R^{6+\frac{5d}{2}}}{1 + \|x\|} \cdot I_F(\|x\|). \end{aligned}$$

Step 6. Let

$$K_R := \sum_{m=0}^{M} \sum_{i \in I_m} \mathbb{E}\big[\mathcal{G}_{i,R} Z_i^{\ell} \overline{e}_i^m\big] \cdot \mathbb{E}\big[F(S_{i_m}) \mathbb{1}\{\tau_m < \infty\}\big].$$

One has, using the Markov property and a similar argument as in the previous step,

$$\begin{aligned} |K_R - H_R| &\overset{(4.1)}{\lesssim} R \sum_{m=0}^{M} \sum_{i \in I_m} \mathbb{E}_{0,x}\bigg[\frac{|F(S_{i_m})| \cdot (1 + \|S_i - S_{i_m}\|^2)}{1 + \|S_{i_m}\|} \cdot \mathbb{1}\{\tau_m < \infty\}\bigg] \\ &\overset{(4.13),(2.5)}{\lesssim} R^{6+\frac{5d}{2}} \sum_{m=0}^{M} \mathbb{E}\bigg[\frac{|F(S_{i_m})|}{(1 + \|S_{i_m}\|)(1 + \|x - S_{i_m}\|^{d-2})}\bigg] \lesssim R^{6+\frac{5d}{2}} \cdot J_F(\|x\|). \end{aligned}$$

Step 7. Finally, we define

$$\widetilde{A} := \frac{\kappa}{\gamma_d} \cdot \mathbb{E}_{0,x}\big[F(\widetilde{S}_\tau) \mathbb{1}\{\tau < \infty\}\big].$$

We recall that one has (see Lemmas 2.1 and 2.2 in [2])

$$\mathbb{E}\big[\big(\mathrm{Cap}(\mathcal{R}_n)-\gamma_d n\big)^2\big]=\mathcal{O}\big(n(\log n)^2\big). \tag{4.15}$$

It also holds for any nonempty subset $\Lambda \subseteq \mathbb{Z}^d$,

$$\mathrm{Cap}(\Lambda) \geq c|\Lambda|^{1-\frac{2}{d}} \geq c|\Lambda|^3, \tag{4.16}$$

using $d \geq 5$ for the second inequality (while the first inequality follows from [16], Proposition 6.5.5, applied to the constant function equal to $c/|\Lambda|^{2/d}$, with $c > 0$ small enough). As a consequence, for any $m \geq 0$ and any $i \in I_m$,

$$\begin{aligned}
&\left|\mathbb{E}[\mathcal{G}_{i,R} Z_i^\ell \overline{e}_i^m] - \frac{\mathbb{E}[\mathcal{G}_{i,R} Z_i^\ell e_i^m]}{\gamma_d R^5}\right| \\
&\quad\lesssim \frac{1}{R^4}\mathbb{E}\left[\frac{|\mathrm{Cap}(\mathcal{R}^{(m)})-\gamma_d R^5|}{\mathrm{Cap}(\mathcal{R}^{(m)})}\right] \\
&\quad\overset{(4.15)}{\lesssim} \frac{\log R}{R^{3/2}}\mathbb{E}\left[\frac{1}{\mathrm{Cap}(\mathcal{R}^{(m)})^2}\right]^{1/2} \overset{(4.16)}{\lesssim} \frac{\log R}{R^{3/2}}\left(\frac{\mathbb{P}[\mathrm{Cap}(\mathcal{R}^{(m)}) \leq \gamma_d R^5/2]}{R^6}+\frac{1}{R^{10}}\right)^{1/2} \\
&\quad\overset{(4.15)}{\lesssim} \frac{\log R}{R^{3/2}}\left(\frac{(\log R)^2}{R^{11}}+\frac{1}{R^{10}}\right)^{1/2} \lesssim \frac{1}{R^6}.
\end{aligned}$$

Next, recall that $Z(i)=\mathbb{1}\{S_j \neq S_i, \forall j > i\}$, and note that

$$\big|\mathbb{E}[\mathcal{G}_{i,R} Z_i^\ell e_i^m] - \mathbb{E}[\mathcal{G}_{i,R} Z(i) e_i^m]\big| \overset{(2.6),(2.7)}{\lesssim} \frac{1}{R^{7/2}}.$$

Moreover, letting $e_i := \mathbb{P}_{S_i}[H^+_{\overline{\mathcal{R}}_\infty} = \infty \mid \overline{\mathcal{R}}_\infty]$ (where we recall $\overline{\mathcal{R}}_\infty$ is the range of a two-sided random walk), one has

$$\big|\mathbb{E}[\mathcal{G}_{i,R} Z_i e_i^m] - \mathbb{E}[\mathcal{G}_{i,R} Z_i e_i]\big| \overset{(2.13)}{\lesssim} \frac{1}{\sqrt{R}},$$

$$\big|\mathbb{E}[\mathcal{G}_{i,R} Z_i e_i] - \kappa\big| \leq 2\mathbb{E}\Big[\sum_{j>R} G(S_j)\Big] \overset{(2.9)}{\lesssim} \frac{1}{\sqrt{R}}.$$

Altogether, this gives, for any $m \geq 0$ and any $i \in I_m$,

$$\left|\mathbb{E}[\mathcal{G}_{i,R} Z_i^\ell \overline{e}_i^m] - \frac{\kappa}{\gamma_d R^5}\right| \lesssim \frac{1}{R^{5+\frac{1}{2}}},$$

and, thus, for any $m \geq 0$,

$$\left|\Big(\sum_{i \in I_m} \mathbb{E}[\mathcal{G}_{i,R} Z_i^\ell \overline{e}_i^m]\Big) - \frac{\kappa}{\gamma_d}\right| \lesssim \frac{1}{\sqrt{R}}.$$

Now, a similar argument as in Step 6 shows that

$$\sum_{m=0}^{M} \big|\mathbb{E}_{0,x}\big[F(S_{i_m})\mathbb{1}\{\tau_m < \infty\}\big] - \mathbb{E}_{0,x}\big[F(\widetilde{S}_{\tau_m})\mathbb{1}\{\tau_m < \infty\}\big]\big| \lesssim R^{\frac{5d}{2}} J_F(\|x\|).$$

Furthermore, using that

$$\begin{aligned}
F(\widetilde{S}_\tau)\mathbb{1}\{\tau < \infty\} &= \sum_{m=0}^{M+1} F(\widetilde{S}_{\tau_m})\mathbb{1}\{\tau = \tau_m < \infty\} \\
&= \sum_{m=0}^{M+1} F(\widetilde{S}_{\tau_m})\big(\mathbb{1}\{\tau_m < \infty\} - \mathbb{1}\{\tau < \tau_m < \infty\}\big),
\end{aligned}$$

(with the convention that the term corresponding to index $M+1$ is zero when $\ell = \infty$) we get

$$\left|\sum_{m=0}^{M} \mathbb{E}_{0,x}\big[F(\widetilde{S}_{\tau_m})\mathbf{1}\{\tau_m < \infty\}\big] - \mathbb{E}_{0,x}\big[F(\widetilde{S}_{\tau})\mathbf{1}\{\tau < \infty\}\big]\right|$$
$$\lesssim \mathbb{P}_{0,x}[\tau_{M+1} < \infty] + \sum_{m=0}^{M} \mathbb{E}_{0,x}\big[\big|F(\widetilde{S}_{\tau_m})\big|\mathbf{1}\{\tau < \tau_m < \infty\}\big].$$

Using (4.13), (4.14) and (2.10), we get

$$\mathbb{P}_{0,x}[\tau_{M+1} < \infty] \lesssim \frac{R^{\frac{5(d-2)}{2}}}{1 + \|x\|^{d-2}}.$$

On the other hand, for any $m \geq 0$,

$$\begin{aligned}
&\mathbb{E}\big[\big|F(\widetilde{S}_{\tau_m})\big|\mathbf{1}\{\tau < \tau_m < \infty\}\big] \\
&\quad \leq \sum_{j\in J_m} \sum_{i\notin J_m} \mathbb{E}\big[\big|F(S_j)\big|G(S_i - S_j)G(S_i - x)\big] \\
&\quad \leq \sum_{j\in I_m} \sum_{|j-i|>R^3} \mathbb{E}\big[\big|F(S_j)\big|G(S_i - S_j)G(S_i - x)\big] \\
&\qquad + \sum_{j\in J_m\setminus I_m} \sum_{i\notin J_m} \mathbb{E}\big[\big|F(S_j)\big|G(S_i - S_j)G(S_i - x)\big].
\end{aligned}$$

The first sum is handled as in Step 4. Namely,

$$\begin{aligned}
&\sum_{m=0}^{M} \sum_{j\in I_m} \sum_{|j-i|>R^3} \mathbb{E}\big[\big|F(S_j)\big|G(S_i - S_j)G(S_i - x)\big] \\
&\quad \leq \sum_{j\geq 0} \sum_{|j-i|>R^3} \mathbb{E}\big[\big|F(S_j)\big|G(S_i - S_j)G(S_i - x)\big] \overset{(4.6)}{\lesssim} \frac{I_F(\|x\|)}{R^{3/2}}.
\end{aligned}$$

Similarly, defining $\widetilde{J}_m := \{mR^5, \dots, mR^5 + R\} \cup \{(m+1)R^5 - R, \dots, (m+1)R^5 - 1\}$, one has

$$\begin{aligned}
&\sum_{m=0}^{M} \sum_{j\in J_m\setminus I_m} \sum_{i\notin J_m} \mathbb{E}\big[\big|F(S_j)\big|G(S_i - S_j)G(S_i - x)\big] \\
&\quad \leq \sum_{m=0}^{M} \sum_{j\in J_m\setminus I_m} \sum_{|i-j|>R} \mathbb{E}\big[\big|F(S_j)\big|G(S_i - S_j)G(S_i - x)\big] \\
&\qquad + \sum_{m=0}^{M} \sum_{j\in J_m\setminus I_m} \sum_{i\notin J_m, |i-j|\leq R} \mathbb{E}\big[\big|F(S_j)\big|G(S_i - S_j)G(S_i - x)\big] \\
&\quad \overset{(4.6),(4.1)}{\lesssim} \frac{I_F(\|x\|)}{\sqrt{R}} + \sum_{m=0}^{M} \sum_{j\in J_m\setminus I_m} \sum_{i\notin J_m, |i-j|\leq R} \mathbb{E}\big[\big|F(S_i)\big|G(S_i - x)\big] \\
&\quad \lesssim \frac{I_F(\|x\|)}{\sqrt{R}} + R \sum_{m=0}^{M} \sum_{i\in \widetilde{J}_m} \mathbb{E}\big[\big|F(S_i)\big|G(S_i - x)\big] \lesssim \frac{I_F(\|x\|)}{\sqrt{R}} + \frac{R^5}{1 + \|x\|^{d-2}},
\end{aligned}$$

using for the last inequality the same argument as in (4.11). Note also that

$$\mathbb{E}\big[|F(\widetilde{S}_\tau)|\mathbb{1}\{\tau<\infty\}\big]\overset{(2.11)}{\le}\sum_{i\ge 0}\mathbb{E}\big[|F(S_i)|G(S_i-x)\big]\lesssim I_F(\|x\|).$$

Therefore, putting all pieces together yields

$$|K_R-\widetilde{A}|\lesssim\frac{I_F(\|x\|)}{\sqrt{R}}+R^{\frac{5d}{2}}\cdot J_F(\|x\|)+\frac{R^{\frac{5(d-2)}{2}}}{1+\|x\|^{d-2}}.$$

Step 8. Altogether the previous steps show that, for any R large enough, any $\ell\ge 1$ and any $x\in\mathbb{Z}^d$, satisfying $\ell\wedge\|x\|^2\ge R^6$,

$$|A-\widetilde{A}|\lesssim\left(\frac{1}{\sqrt{R}}+\frac{R^{6+\frac{5d}{2}}}{1+\|x\|}\right)\cdot I_F(\|x\|)+\frac{R^{\frac{5(d-2)}{2}}}{1+\|x\|^{d-2}}+R^{6+\frac{5d}{2}}\cdot J_F(\|x\|).$$

The proof of the theorem follows by taking for R a sufficiently small power of $\|x\|\wedge\ell$, and observing that for any function F satisfying (4.1), one has $\liminf_{\|z\|\to\infty}|F(z)|/\|z\|>0$ and, thus, also $I_F(\|x\|)\ge\frac{c}{1+\|x\|^{d-3}}$. $\square$

It amounts now to give the proofs of Lemmas 4.5 and 4.6.

PROOF OF LEMMA 4.5. We start with the proof of (4.4). Recall the definition of χ_d given just above Theorem 4.1. One has for any $i\ge 0$,

$$\begin{aligned}\mathbb{E}&\left[\sum_{j=i+1}^{\infty}G(S_j-S_i)\frac{\|S_j-S_i\|}{1+\|S_j\|}\,\middle|\,S_i\right]\\&\overset{(2.7)}{\lesssim}\mathbb{E}\left[\sum_{j=i+1}^{\infty}\frac{1}{(1+\|S_j-S_i\|^{d-3})(1+\|S_j\|)}\,\middle|\,S_i\right]\\&\lesssim\sum_{z\in\mathbb{Z}^d}G(z)\frac{1}{(1+\|z\|^{d-3})(1+\|S_i+z\|)}\overset{(2.7)}{\lesssim}\frac{\chi_d(\|S_i\|)}{1+\|S_i\|},\end{aligned}$$

and, moreover,

$$\begin{aligned}&\sum_{i=0}^{\infty}\mathbb{E}\left[\frac{|F(S_i)|\chi_d(\|S_i\|)}{1+\|S_i\|}G(S_i-x)\right]\\&\quad=\sum_{z\in\mathbb{Z}^d}G(z)\frac{|F(z)|\chi_d(\|z\|)}{1+\|z\|}G(z-x)\\&\quad\overset{(2.7)}{\lesssim}\frac{\chi_d(\|x\|)}{1+\|x\|^{d-2}}\sum_{\|z\|\le\frac{\|x\|}{2}}\frac{|F(z)|}{1+\|z\|^{d-1}}+\sum_{\|z\|\ge\frac{\|x\|}{2}}\frac{|F(z)|\chi_d(\|z\|)}{1+\|z\|^{2d-3}}\\&\quad\quad+\frac{\chi_d(\|x\|)}{1+\|x\|^{d-1}}\sum_{\|z-x\|\le\frac{\|x\|}{2}}\frac{|F(z)|}{1+\|z-x\|^{d-2}}\\&\quad\overset{(4.1)}{\lesssim}J_F(\|x\|/2)+\frac{|F(x)|\chi_d(\|x\|)}{1+\|x\|^{d-3}}\lesssim J_F(\|x\|),\end{aligned}\tag{4.17}$$

where the last inequality follows from the fact that by (4.1),

$$\int_{\|x\|/2}^{\|x\|}\frac{\overline{F}(s)\chi_d(s)}{s^{d-2}}\,ds\asymp\frac{|F(x)|\chi_d(\|x\|)}{1+\|x\|^{d-3}}\asymp\frac{\chi_d(\|x\|)}{1+\|x\|^{d-2}}\int_{\|x\|/2}^{\|x\|}\overline{F}(s)\,ds.$$

Thus,

$$\sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{E}\Big[G(S_j-S_i)\frac{\|S_j-S_i\|}{1+\|S_j\|}\big|F(S_i)\big|G(S_i-x)\Big]=\mathcal{O}\big(J_F(\|x\|)\big).$$

On the other hand, for any $j\geq 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i=j+1}^{\infty}G(S_j-S_i)\|S_j-S_i\|\cdot\big|F(S_i)\big|G(S_i-x)\,\Big|\,S_j\Big]\\
&\overset{(2.7)}{\lesssim}\sum_{i=j+1}^{\infty}\mathbb{E}\Big[\frac{|F(S_i)|G(S_i-x)}{1+\|S_j-S_i\|^{d-3}}|S_j\Big]\overset{(2.7)}{\lesssim}\sum_{z\in\mathbb{Z}^d}\frac{|F(S_j+z)|G(S_j+z-x)}{1+\|z\|^{2d-5}}\\
&\overset{(4.1),(2.7)}{\lesssim}\sum_{z\in\mathbb{Z}^d}\frac{|F(S_j)|}{(1+\|z\|^{2d-5})(1+\|S_j+z-x\|^{d-2})}\\
&\qquad+\frac{1}{1+\|S_j\|^{2d-5}}\sum_{\|u\|\leq\|S_j\|}\frac{|F(u)|}{1+\|u-x\|^{d-2}} \qquad (4.18)\\
&\overset{(4.1)}{\lesssim}\frac{|F(S_j)|\chi_d(\|S_j-x\|)}{1+\|S_j-x\|^{d-2}}+\frac{\mathbb{1}\{\|S_j\|\leq\|x\|/2\}\cdot|F(S_j)|}{(1+\|x\|^{d-2})(1+\|S_j\|^{d-5})}\\
&\qquad+\frac{\mathbb{1}\{\|S_j\|\geq\|x\|/2\}}{1+\|S_j\|^{2d-5}}\big(\big|F(x)\big|(1+\|x\|^2)+\big|F(S_j)\big|(1+\|S_j\|^2)\big)\\
&\lesssim\frac{|F(S_j)|\chi_d(\|S_j-x\|)}{1+\|S_j-x\|^{d-2}}+\frac{\mathbb{1}\{\|S_j\|\leq\|x\|\}|F(S_j)|}{1+\|x\|^{d-2}}+\frac{\mathbb{1}\{\|S_j\|\geq\|x\|\}|F(S_j)|}{1+\|S_j\|^{d-2}},
\end{aligned}$$

where for the last two inequalities we used that by (4.1) if $\|u\|\leq\|v\|$, then $|F(u)|\lesssim|F(v)|(1+\|v\|)/(1+\|u\|)$ and also that $d\geq 5$ for the last one. Moreover, for any $r\geq 0$

$$\begin{aligned}
&\sum_{j=0}^{\infty}\mathbb{E}\Big[\frac{\mathbb{1}\{\|S_j\|\leq r\}\cdot|F(S_j)|}{1+\|S_j\|}\Big]=\sum_{\|z\|\leq r}\frac{G(z)|F(z)|}{1+\|z\|}\overset{(2.7)}{=}\mathcal{O}\Big(\int_0^r\overline{F}(s)\,ds\Big),\\
&\sum_{j=0}^{\infty}\mathbb{E}\Big[\frac{\mathbb{1}\{\|S_j\|\geq r\}\cdot|F(S_j)|}{1+\|S_j\|^{d-1}}\Big]=\sum_{\|z\|\geq r}\frac{G(z)|F(z)|}{1+\|z\|^{d-1}}\overset{(2.7)}{=}\mathcal{O}\Big(\int_r^{\infty}\frac{\overline{F}(s)}{s^{d-2}}\,ds\Big).
\end{aligned}$$

Using also similar computations as in (4.17) to handle the first term in (4.18), we conclude that

$$\sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty}\mathbb{E}\Big[G(S_j-S_i)\frac{\|S_j-S_i\|}{1+\|S_j\|}\big|F(S_i)\big|G(S_i-x)\Big]=\mathcal{O}\big(J_F(\|x\|)\big),$$

which finishes the proof of (4.4).

We then move to the proof of (4.5). First, note that, for any $i\geq 0$,

$$\mathbb{E}\Big[\sum_{j\geq i+R}G(S_j-S_i)|S_i\Big]=\mathbb{E}\Big[\sum_{j\geq R}G(S_j)\Big]\overset{(2.9)}{=}\mathcal{O}(R^{\frac{4-d}{2}}),$$

and, furthermore,

$$\sum_{i=0}^{\infty}\mathbb{E}\big[\big|F(S_i)\big|G(S_i-x)\big]=\sum_{z\in\mathbb{Z}^d}\big|F(z)\big|G(z-x)G(z)\overset{(4.1),(2.7)}{=}\mathcal{O}\big(I_F(\|x\|)\big), \tag{4.19}$$

which together give the desired upper bound for the sum on the set $\{0 \le i \le j - R\}$. On the other hand, for any $j \ge 0$, we get as for (4.18),

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i\ge j+R} G(S_j - S_i)\big|F(S_i)\big|G(S_i - x) \mid S_j\Big] \\
&\quad = \sum_{z\in\mathbb{Z}^d} G(z)\big|F(S_j + z)\big|G(S_j + z - x)G_R(z) \\
&\quad \overset{(2.7)}{\lesssim} \frac{1}{R^{\frac{d-4}{2}}} \cdot \sum_{z\in\mathbb{Z}^d} \frac{|F(S_j + z)|}{(1 + \|z\|^d)(1 + \|S_j + z - x\|^{d-2})} \\
&\quad \overset{(4.1)}{\lesssim} \frac{1}{R^{\frac{d-4}{2}}} \Bigg\{\sum_{z\in\mathbb{Z}^d} \frac{|F(S_j)|}{(1 + \|z\|^d)(1 + \|S_j + z - x\|^{d-2})} \\
&\qquad + \frac{1}{1 + \|S_j\|^d} \sum_{\|u\|\le\|S_j\|} \frac{|F(u)|}{1 + \|u - x\|^{d-2}}\Bigg\} \\
&\quad \lesssim \frac{1}{R^{\frac{d-4}{2}}} \Bigg\{\frac{|F(S_j)| \log(2 + \|S_j - x\|)}{1 + \|S_j - x\|^{d-2}} + \frac{|F(S_j)|}{1 + \|x\|^{d-2} + \|S_j\|^{d-2}}\Bigg\}.
\end{aligned}$$

Then, similar computation as above, see, for example, (4.19), give

$$\begin{aligned}
&\sum_{j\ge 0} \mathbb{E}\Big[\frac{|F(S_j)| \log(2 + \|S_j - x\|)}{1 + \|S_j - x\|^{d-2}}\Big] = \mathcal{O}\big(I_F(\|x\|)\big), \\
&\sum_{j\ge 0} \mathbb{E}\Big[\frac{|F(S_j)|}{1 + \|x\|^{d-2} + \|S_j\|^{d-2}}\Big] = \mathcal{O}\big(I_F(\|x\|)\big),
\end{aligned} \tag{4.20}$$

which altogether proves (4.5).

The proof of (4.6) is entirely similar: on one hand, for any $i \ge 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{j=i+R}^{\infty} G(S_j - S_i)\big|F(S_j)\big| \mid S_i\Big] \\
&\quad \overset{(4.1)}{\lesssim} \mathbb{E}\Big[\sum_{j=i+R}^{\infty} G(S_j - S_i)\frac{\|S_j - S_i\|}{1 + \|S_j\|} \;\Big|\; S_i\Big]\big|F(S_i)\big| \\
&\quad \lesssim \sum_{z\in\mathbb{Z}^d} G_R(z)\frac{|F(S_i)|}{(1 + \|z\|^{d-3})(1 + \|S_i + z\|)} \\
&\quad \lesssim \sum_{z\in\mathbb{Z}^d} \frac{|F(S_i)|}{(R^{\frac{d-2}{2}} + \|z\|^{d-2})(1 + \|z\|^{d-3})(1 + \|S_i + z\|)} \lesssim \frac{|F(S_i)|}{R^{\frac{d-4}{2}}},
\end{aligned}$$

and, together with (4.20), this yields

$$\sum_{i=0}^{\infty} \sum_{j=i+R}^{\infty} \mathbb{E}\big[G(S_j - S_i)\big|F(S_j)\big|G(S_i - x)\big] \lesssim \frac{I_F(\|x\|)}{R^{\frac{d-4}{2}}}.$$

On the other hand, for any $j \ge 0$, using (2.7),

$$\mathbb{E}\Big[\sum_{i\ge j+R} G(S_j - S_i)G(S_i - x) \mid S_j\Big]$$

$$\lesssim \sum_{z\in\mathbb{Z}^d} \frac{G(S_j+z-x)}{R^{\frac{d-4}{2}}(1+\|z\|^d)} \lesssim \frac{\log(2+\|S_j-x\|)}{R^{\frac{d-4}{2}}(1+\|S_j-x\|^{d-2})},$$

and we conclude the proof of (4.6) using (4.20) again. □

PROOF OF LEMMA 4.6. The first statement follows directly from (4.3) and the last-exit decomposition (see Proposition 4.6.4 (c) in [16])

$$\mathbb{P}_y[H_\Lambda<\infty]=\sum_{x\in\Lambda} G(y-x)e_\Lambda(x).$$

Indeed, if $\|y\|>2\operatorname{rad}(\Lambda)$, using (2.7) we get $G(y-x)\le C\|y\|^{2-d}$, for some constant $C>0$ independent of $x\in\Lambda$, which gives well (4.7), since by definition $\sum_{x\in\Lambda} e_\Lambda(x)=\operatorname{Cap}(\Lambda)$.

The second statement is more involved. Note that one can always assume $\mathcal{J}(y)>C\operatorname{rad}(\Lambda)$, for some constant $C>0$, for, otherwise, the result is trivial. We use similar notation as in [16]. In particular, $G_A(x,y)$ denotes the Green's function restricted to a subset $A\subseteq\mathbb{Z}^d$, that is, the expected number of visits to y before exiting A for a random walk starting from x, and $H_A(x,y)=\mathbb{P}_x[H_{A^c}=y]$. We also let $\mathcal{C}_n$ denote the (discrete) ball of radius n for the norm $\mathcal{J}(\cdot)$. Then, exactly as in [16] (see Lemma 6.3.3 and Proposition 6.3.5 thereof), one can see using (4.3) that, for all $n\ge 1$,

$$|G_{\mathcal{C}_n}(x,w)-G_{\mathcal{C}_n}(0,w)|\le C\frac{\|x\|}{1+\|w\|}G_{\mathcal{C}_n}(0,w), \tag{4.21}$$

for all $x\in\mathcal{C}_{n/4}$ and all w satisfying $2\mathcal{J}(x)\le\mathcal{J}(w)\le n/2$. One can then derive an analogous estimate for the (discrete) derivative of $H_{\mathcal{C}_n}$. Define $A_n=\mathcal{C}_n\setminus\mathcal{C}_{n/2}$ and $\rho=H^+_{A_n^c}$. By the last-exit decomposition (see [16], Lemma 6.3.6), one has, for $x\in\mathcal{C}_{n/8}$ and $z\notin\mathcal{C}_n$,

$$\begin{aligned}
&|H_{\mathcal{C}_n}(x,z)-H_{\mathcal{C}_n}(0,z)|\\
&\quad\le\sum_{w\in\mathcal{C}_{n/2}}|G_{\mathcal{C}_n}(x,w)-G_{\mathcal{C}_n}(0,w)|\cdot\mathbb{P}_w[S_\rho=z]\\
&\quad\overset{(4.21),(2.7)}{\lesssim}\frac{\|x\|}{n}\cdot H_{\mathcal{C}_n}(0,z)+\sum_{2\mathcal{J}(x)\le\mathcal{J}(w)\le\frac{n}{4}}\frac{\|x\|}{\|w\|^{d-1}}\mathbb{P}_w[S_\rho=z]\\
&\qquad+\sum_{\mathcal{J}(w)\le 2\mathcal{J}(x)}\left(\frac{1}{1+\|w-x\|^{d-2}}+\frac{1}{1+\|w\|^{d-2}}\right)\mathbb{P}_w[S_\rho=z].
\end{aligned}$$

Now, observe that for any $y\notin\mathcal{C}_n$, any $w\in\mathcal{C}_{n/4}$ and any $A\subseteq\mathbb{Z}^d$,

$$\sum_{z\notin\mathcal{C}_n}G_A(y,z)\mathbb{P}_w[S_\rho=z]\lesssim\sum_{z\notin\mathcal{C}_n}\mathbb{P}_w[S_\rho=z]\lesssim\mathbb{P}_w\Big[\mathcal{J}(S_1)>\frac{n}{2}\Big]\lesssim\mathbb{P}\Big[\mathcal{J}(X_1)>\frac{n}{4}\Big]\lesssim n^{-d},$$

using that by hypothesis $\mathcal{J}(X_1)$ has a finite dth moment. It follows from the last two displays that

$$\begin{aligned}
&\sum_{z\notin\mathcal{C}_n}G_A(y,z)H_{\mathcal{C}_n}(x,z)\\
&\quad=\Big(\sum_{z\notin\mathcal{C}_n}G_A(y,z)H_{\mathcal{C}_n}(0,z)\Big)\Big(1+\mathcal{O}\Big(\frac{\|x\|}{n}\Big)\Big)+\mathcal{O}\Big(\frac{\|x\|}{n^{d-1}}\Big).
\end{aligned}\tag{4.22}$$

Now, let Λ be some finite subset of $\mathbb{Z}^d$ containing the origin, and let $m:=\sup\{\mathcal{J}(u):\|u\|\le 2\operatorname{rad}(\Lambda)\}$. Note that $m=\mathcal{O}(\operatorname{rad}(\Lambda))$, and, thus, one can assume $\mathcal{J}(y)>16m$. Set

$n := \mathcal{J}(y) - 1$. Using again the last-exit decomposition and symmetry of the step distribution, we get for any $x \in \Lambda$,

$$\mathbb{P}_y[S_{H_\Lambda} = x, H_\Lambda < \infty] = \sum_{z \notin \mathcal{C}_n} G_{\Lambda^c}(y,z)\mathbb{P}_x\big[S_{\tau_n} = z, \tau_n < H_\Lambda^+\big], \tag{4.23}$$

with $\tau_n := H_{\mathcal{C}_n^c}$. We then write, using the Markov property,

$$\begin{aligned} &\mathbb{P}_x\big[S_{\tau_n} = z, \tau_n < H_\Lambda^+\big] \\ &\quad = \sum_{x' \in \mathcal{C}_{n/8} \setminus \mathcal{C}_m} \mathbb{P}_x\big[\tau_m < H_\Lambda^+, S_{\tau_m} = x'\big] \cdot \mathbb{P}_{x'}\big[S_{\tau_n} = z, \tau_n < H_\Lambda^+\big] \\ &\qquad + \mathbb{P}_x\bigg[\mathcal{J}(S_{\tau_m}) > \frac{n}{8}, S_{\tau_n} = z\bigg], \end{aligned} \tag{4.24}$$

with $\tau_m := H_{\mathcal{C}_m^c}$. Concerning the last term, we note that

$$\begin{aligned} &\sum_{z \notin \mathcal{C}_n} G_{\Lambda^c}(y,z)\mathbb{P}_x\bigg[\mathcal{J}(S_{\tau_m}) > \frac{n}{8}, S_{\tau_n} = z\bigg] \\ &\quad \overset{(2.6)}{\leq} \sum_{z \notin \mathcal{C}_n} G(z-y)\bigg\{\mathbb{P}_x[S_{\tau_m} = z] + \sum_{u \in \mathcal{C}_n \setminus \mathcal{C}_{n/8}} \mathbb{P}_x[S_{\tau_m} = u]G(z-u)\bigg\} \\ &\quad \overset{\text{Lemma 2.2}}{\lesssim} \sum_{z \notin \mathcal{C}_n} G(z-y)\mathbb{P}_x[S_{\tau_m} = z] + \sum_{u \in \mathcal{C}_n \setminus \mathcal{C}_{n/8}} \frac{\mathbb{P}_x[S_{\tau_m} = u]}{\|y-u\|^{d-4}} \\ &\quad \lesssim \mathbb{P}_x[\mathcal{J}(S_{\tau_m}) > n/8] \lesssim \sum_{u \in \mathcal{C}_m} G_{\mathcal{C}_m}(x,u)\mathbb{P}\bigg[J(X_1) > \frac{n}{8} - m\bigg] \\ &\quad \overset{(2.7)}{=} \mathcal{O}\bigg(\frac{m^2}{n^d}\bigg) = \mathcal{O}\bigg(\frac{m}{n^{d-1}}\bigg), \end{aligned} \tag{4.25}$$

applying once more the last-exit decomposition at the penultimate line and the hypothesis that $\mathcal{J}(X_1)$ has a finite dth moment at the end. Next, we handle the sum in the right-hand side of (4.24). First, note that (4.22) gives, for any $x' \in \mathcal{C}_{n/8}$,

$$\begin{aligned} &\sum_{z \notin \mathcal{C}_n} G_{\Lambda^c}(y,z)\mathbb{P}_{x'}[S_{\tau_n} = z] \\ &\quad = \sum_{z \notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(x',z) \\ &\quad = \bigg(\sum_{z \notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\bigg)\bigg(1 + \mathcal{O}\bigg(\frac{\|x'\|}{n}\bigg)\bigg) + \mathcal{O}\bigg(\frac{\|x'\|}{n^{d-1}}\bigg). \end{aligned} \tag{4.26}$$

Observe then two facts. On one hand, by the last exit-decomposition and symmetry of the step distribution,

$$\begin{aligned} \sum_{z \notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z) &\leq \sum_{z \notin \mathcal{C}_n} G_{\mathbb{Z}^d \setminus \{0\}}(y,z)H_{\mathcal{C}_n}(0,z) \\ &= \mathbb{P}[H_y < \infty] \overset{(2.6),(2.7)}{\lesssim} n^{2-d}, \end{aligned} \tag{4.27}$$

and, on the other hand, by Proposition 4.6.2 in [16],

$$
\begin{aligned}
&\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\\
&\quad = \sum_{z\notin \mathcal{C}_n} G_{\mathbb{Z}^d\setminus\{0\}}(y,z)H_{\mathcal{C}_n}(0,z) + \sum_{z\notin \mathcal{C}_n}\bigl(G_{\Lambda^c}(y,z)-G_{\mathbb{Z}^d\setminus\{0\}}(y,z)\bigr)H_{\mathcal{C}_n}(0,z)\\
&\quad \ge \mathbb{P}[H_y<\infty] - \mathcal{O}\biggl(\mathbb{P}_y[H_\Lambda<\infty]\sum_{z\notin \mathcal{C}_n} G(z)H_{\mathcal{C}_n}(0,z)\biggr) \qquad (4.28)\\
&\quad \overset{(2.8)}{\ge} \mathbb{P}[H_y<\infty] - \mathcal{O}\biggl(n^{2-d}\sum_{z\notin \mathcal{C}_n} G(z)^2\biggr) \overset{(4.3)}{\ge} \frac{c}{n^{d-2}}.
\end{aligned}
$$

This last fact, combined with (4.26), gives, therefore, for any $x'\in \mathcal{C}_{n/8}$,

$$
\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)\mathbb{P}_{x'}[S_{\tau_n}=z] = \biggl(\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\biggr)\biggl(1+\mathcal{O}\biggl(\frac{\|x'\|}{n}\biggr)\biggr). \tag{4.29}
$$

By the Markov property we get as well

$$
\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)\mathbb{P}_{x'}[S_{\tau_n}=z \mid H_\Lambda<\tau_n] = \biggl(\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\biggr)\biggl(1+\mathcal{O}\biggl(\frac{m}{n}\biggr)\biggr),
$$

since, by definition, $\Lambda\subseteq \mathcal{C}_m\subset \mathcal{C}_{n/8}$ and, thus,

$$
\begin{aligned}
&\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)\mathbb{P}_{x'}[S_{\tau_n}=z, H_\Lambda<\tau_n]\\
&\quad = \mathbb{P}_{x'}[H_\Lambda<\tau_n]\biggl(\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\biggr)\biggl(1+\mathcal{O}\biggl(\frac{m}{n}\biggr)\biggr).
\end{aligned}
$$

Subtracting this from (4.29), we get for $x'\in \mathcal{C}_{n/8}\setminus \mathcal{C}_m$,

$$
\begin{aligned}
&\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)\mathbb{P}_{x'}[S_{\tau_n}=z, \tau_n<H_\Lambda]\\
&\quad = \mathbb{P}_{x'}[\tau_n<H_\Lambda]\biggl(\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\biggr)\biggl(1+\mathcal{O}\biggl(\frac{\|x'\|}{n}\biggr)\biggr),
\end{aligned}
$$

since, by (2.8), one has $\mathbb{P}_{x'}[\tau_n<H_\Lambda]>c$, for some constant $c>0$, for any $x'\notin \mathcal{C}_m$ (note that the stopping time theorem gives in fact $\mathbb{P}_{x'}[H_\Lambda<\infty]\le G(x')/\inf_{\|u\|\le \mathrm{rad}(\Lambda)} G(u)$, and, thus, by using (4.3) one can ensure $\mathbb{P}_{x'}[H_\Lambda<\infty]\le 1-c$, by taking $\|x'\|$ large enough, which is always possible). Combining this with (4.23), (4.24) and (4.25), and using as well (4.27) and (4.28), we get

$$
\begin{aligned}
&\mathbb{P}_y[S_{H_\Lambda}=x, H_\Lambda<\infty]\\
&\quad = \mathbb{P}_x[\tau_n<H_\Lambda]\biggl(\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\biggr)\\
&\qquad + \mathcal{O}\biggl(\frac{1}{n^{d-1}}\sum_{x'\in \mathcal{C}_{n/8}\setminus \mathcal{C}_m} \mathbb{P}_x[S_{\tau_m}=x']\cdot\|x'\|\biggr) + \mathcal{O}\biggl(\frac{m}{n^{d-1}}\biggr)\\
&\quad \overset{(2.8)}{=} e_\Lambda(x)\biggl(\sum_{z\notin \mathcal{C}_n} G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\biggr)\biggl(1+\mathcal{O}\biggl(\frac{m}{n}\biggr)\biggr)
\end{aligned}
$$

$$+\mathcal{O}\left(\frac{1}{n^{d-1}}\sum_{r=2m}^{n/8}\frac{m^2}{r^{d-1}}\right)+\mathcal{O}\left(\frac{m}{n^{d-1}}\right)$$
$$=e_\Lambda(x)\left(\sum_{z\notin\mathcal{C}_n}G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\right)\left(1+\mathcal{O}\left(\frac{m}{n}\right)\right),$$

using the same argument as in (4.25) for bounding $\mathbb{P}_{x'}[\mathcal{J}(S_{\tau_m})\geq r]$, when $r\geq 2m$. Summing over $x\in\Lambda$ gives

$$\mathbb{P}_y[H_\Lambda<\infty]=\operatorname{Cap}(\Lambda)\left(\sum_{z\notin\mathcal{C}_n}G_{\Lambda^c}(y,z)H_{\mathcal{C}_n}(0,z)\right)\left(1+\mathcal{O}\left(\frac{m}{n}\right)\right),$$

and the proof of the lemma follows from the last two displays. □

5. Proof of Proposition 3.7. The proof is divided into four steps corresponding to the next four lemmas:

LEMMA 5.1. *Assume that $\varepsilon_k\to\infty$, and $\varepsilon_k/k\to 0$. There exists a constant $\sigma_{1,3}>0$ such that*

$$\operatorname{Cov}(Z_0\varphi_3,Z_k\psi_1)\sim\frac{\sigma_{1,3}}{k}.$$

LEMMA 5.2. *There exist positive constants δ and $\sigma_{1,1}$ such that, when $\varepsilon_k\geq k^{1-\delta}$ and $\varepsilon_k/k\to 0$,*

$$\operatorname{Cov}(Z_0\varphi_1,Z_k\psi_1)\sim\operatorname{Cov}(Z_0\varphi_3,Z_k\psi_3)\sim\frac{\sigma_{1,1}}{k}.$$

LEMMA 5.3. *There exist positive constants δ and $\sigma_{1,2}$ such that, when $\varepsilon_k\geq k^{1-\delta}$ and $\varepsilon_k/k\to 0$,*

$$\operatorname{Cov}(Z_0\varphi_2,Z_k\psi_1)\sim\operatorname{Cov}(Z_0\varphi_3,Z_k\psi_2)\sim\frac{\sigma_{1,2}}{k}.$$

LEMMA 5.4. *There exist positive constants δ and $\sigma_{2,2}$ such that, when $\varepsilon_k\geq k^{1-\delta}$ and $\varepsilon_k/k\to 0$,*

$$\operatorname{Cov}(Z_0\varphi_2,Z_k\psi_2)\sim\frac{\sigma_{2,2}}{k}.$$

5.1. *Proof of Lemma 5.1.* We assume now to simplify notation that the distribution μ is aperiodic, but it should be clear from the proof that the case of a bipartite walk could be handled similarly.

The first step is to show that

$$\operatorname{Cov}(Z_0\varphi_3,Z_k\psi_1)=\rho^2\left\{\sum_{x\in\mathbb{Z}^5}p_k(x)\varphi_x^2-\left(\sum_{x\in\mathbb{Z}^5}p_k(x)\varphi_x\right)^2\right\}+o\left(\frac{1}{k}\right),\tag{5.1}$$

where ρ and φ_x are defined, respectively, as

$$\rho:=\mathbb{E}\left[\mathbb{P}\left[H^+_{\mathcal{R}_\infty}=\infty\mid(S_n)_{n\in\mathbb{Z}}\right]\cdot\mathbb{1}\{S_\ell\neq 0,\forall\ell\geq 1\}\right]\tag{5.2}$$

and

$$\varphi_x:=\mathbb{P}_{0,x}[\mathcal{R}_\infty\cap\widetilde{\mathcal{R}}_\infty\neq\varnothing].$$

To see this, one needs to dissociate Z_0 and Z_k as well as the events of avoiding $\mathcal{R}[-\varepsilon_k, \varepsilon_k]$ and $\mathcal{R}[k-\varepsilon_k, k+\varepsilon_k]$ by two independent walks starting, respectively, from the origin and from S_k, which are local events (in the sense that they only concern small parts of the different paths), from the events of hitting $\mathcal{R}[k+1,\infty)$ and $\mathcal{R}(-\infty,-1]$ by these two walks which involve different parts of the trajectories.

To be more precise, consider $(S_n^1)_{n\geq 0}$ and $(S_n^2)_{n\geq 0}$, two independent random walks starting from the origin and independent of $(S_n)_{n\in\mathbb{Z}}$. Then, define

$$\tau_1 := \inf\{n \geq \varepsilon_k : S_n^1 \in \mathcal{R}[k+\varepsilon_k, \infty)\}, \qquad \tau_2 := \inf\{n \geq \varepsilon_k : S_k + S_n^2 \in \mathcal{R}(-\infty, -\varepsilon_k]\}.$$

We first consider the term $\mathbb{E}[Z_0\varphi_3]$. Let

$$\tau_{0,1} := \inf\{n \geq \varepsilon_k : S_n^1 \in \mathcal{R}[-\varepsilon_k, \varepsilon_k]\}$$

and

$$\Delta_{0,3} := \mathbb{E}[Z_0 \cdot \mathbf{1}\{\mathcal{R}^1[1,\varepsilon_k] \cap \mathcal{R}[-\varepsilon_k,\varepsilon_k] = \varnothing\} \cdot \mathbf{1}\{\tau_1 < \infty\}].$$

One has

$$\begin{aligned}|\mathbb{E}[Z_0\varphi_3] - \Delta_{0,3}| &\leq \mathbb{P}[\tau_{0,1} < \infty, \tau_1 < \infty] + \mathbb{P}[\mathcal{R}^1[0,\varepsilon_k] \cap \mathcal{R}[k,\infty) \neq \varnothing] \\ &\quad + \mathbb{P}[\mathcal{R}^1_\infty \cap \mathcal{R}[k, k+\varepsilon_k] \neq \varnothing] \\ &\overset{(2.13)}{\leq} \mathbb{P}[\tau_1 \leq \tau_{0,1} < \infty] + \mathbb{P}[\tau_{0,1} \leq \tau_1 < \infty] + \mathcal{O}\left(\frac{\varepsilon_k}{k^{3/2}}\right).\end{aligned}$$

Next, conditioning on $\mathcal{R}[-\varepsilon_k, \varepsilon_k]$ and using the Markov property at time $\tau_{0,1}$, we get, with $X = S_{\varepsilon_k} - S^1_{\tau_{0,1}}$,

$$\begin{aligned}\mathbb{P}[\tau_{0,1} \leq \tau_1 < \infty] &\leq \mathbb{E}[\mathbb{P}_{0,X}[\mathcal{R}[k,\infty) \cap \widetilde{\mathcal{R}}_\infty \neq \varnothing] \cdot \mathbf{1}\{\tau_{0,1} < \infty\}] \\ &\overset{(2.13)}{=} \mathcal{O}\left(\frac{\mathbb{P}[\tau_{0,1} < \infty]}{\sqrt{k}}\right) \overset{(2.13)}{=} \mathcal{O}\left(\frac{1}{\sqrt{k\varepsilon_k}}\right).\end{aligned}$$

Likewise, using the Markov property at time τ_1, we get

$$\begin{aligned}&\mathbb{P}[\tau_1 \leq \tau_{0,1} < \infty] \\ &\quad \overset{(2.11)}{\leq} \mathbb{E}\left[\left(\sum_{j=-\varepsilon_k}^{\varepsilon_k} G(S_j - S^1_{\tau_1})\right)\mathbf{1}\{\tau_1 < \infty\}\right] \\ &\quad \overset{(2.11)}{\leq} \sum_{i=k+\varepsilon_k}^{\infty} \sum_{j=-\varepsilon_k}^{\varepsilon_k} \mathbb{E}[G(S_j - S_i)G(S_i - S^1_{\varepsilon_k})] \\ &\quad \leq (2\varepsilon_k + 1) \sup_{x\in\mathbb{Z}^5} \sum_{i=k}^{\infty} \mathbb{E}[G(S_i)G(S_i - x)] \\ &\quad \leq (2\varepsilon_k + 1) \sup_{x\in\mathbb{Z}^5} \sum_{z\in\mathbb{Z}^5} G(z)G(z-x)G_k(z) \overset{(2.7),\text{Lemma } 2.2}{=} \mathcal{O}\left(\frac{\varepsilon_k}{k^{3/2}}\right).\end{aligned}$$

Now, define for any $y_1, y_2 \in \mathbb{Z}^5$,

$$H(y_1,y_2) := \mathbb{E}[Z_0 \mathbf{1}\{\mathcal{R}^1[1,\varepsilon_k] \cap \mathcal{R}[-\varepsilon_k,\varepsilon_k] = \varnothing, S_{\varepsilon_k} = y_1, S^1_{\varepsilon_k} = y_2\}]. \tag{5.3}$$

One has by the Markov property

$$\Delta_{0,3} = \sum_{x\in\mathbb{Z}^5} \sum_{y_1,y_2\in\mathbb{Z}^5} H(y_1,y_2) p_k(x + y_2 - y_1)\varphi_x.$$

Observe that typically $\|y_1\|$ and $\|y_2\|$ are much smaller than $\|x\|$, and, thus, $p_k(x+y_2-y_1)$ should be also typically close to $p_k(x)$. To make this precise, consider $(\chi_k)_{k\ge 1}$ some sequence of positive integers such that $\varepsilon_k\chi_k^3\le k$, for all $k\ge 1$ and $\chi_k\to\infty$, as $k\to\infty$. One has, using Cauchy–Schwarz at the third line,

$$\sum_{\|x\|^2\le k/\chi_k}\ \sum_{y_1,y_2\in\mathbb{Z}^5} H(y_1,y_2)p_k(x+y_2-y_1)\varphi_x$$

$$\le \sum_{\|x\|^2\le k/\chi_k}\ \sum_{y_2\in\mathbb{Z}^5} p_{\varepsilon_k}(y_2)p_{k+\varepsilon_k}(x)\varphi_{x-y_2} \overset{(2.12)}{\lesssim} \mathbb{E}\left[\frac{\mathbf{1}\{\|S_{k+\varepsilon_k}\|^2\le k/\chi_k\}}{1+\|S_{k+\varepsilon_k}-S^1_{\varepsilon_k}\|}\right]$$

$$\lesssim \mathbb{E}\left[\frac{1}{1+\|S_{k+2\varepsilon_k}\|^2}\right]^{1/2}\cdot\mathbb{P}\left[\|S_{k+\varepsilon_k}\|^2\le k/\chi_k\right]^{1/2} \overset{(2.3)}{\lesssim} \frac{1}{\sqrt{k}\cdot\chi_k^{5/4}}.$$

Likewise, using just (2.5) at the end instead of (2.3), we get

$$\sum_{\|x\|^2\ge k\chi_k}\ \sum_{y_1,y_2\in\mathbb{Z}^5} H(y_1,y_2)p_k(x+y_2-y_1)\varphi_x \lesssim \frac{1}{\sqrt{k}\cdot\chi_k^{5/4}},$$

and one can handle the sums on the sets $\{\|y_1\|^2\ge\varepsilon_k\chi_k\}$ and $\{\|y_2\|^2\ge\varepsilon_k\chi_k\}$ similarly. Therefore, it holds

$$\Delta_{0,3} = \sum_{k/\chi_k\le\|x\|^2\le k\chi_k}\ \sum_{\|y_1\|^2\le\varepsilon_k\chi_k}\ \sum_{\|y_2\|^2\le\varepsilon_k\chi_k} H(y_1,y_2)p_k(x+y_2-y_1)\varphi_x + \mathcal{O}\left(\frac{1}{\sqrt{k}\cdot\chi_k^{5/4}}\right).$$

Moreover, Theorem 2.1 shows that, for any x, y_1, y_2 as in the three sums above, one has

$$\left|p_k(x+y_2-y_1)-p_k(x)\right| = \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}\cdot\chi_k}{\sqrt{k}}\cdot p_k(x)+\frac{1}{k^{7/2}}\right).$$

Note also that by (2.12) one has

$$\sum_{x,y_1,y_2\in\mathbb{Z}^5} H(y_1,y_2)p_k(x)\varphi_x \le \sum_{x\in\mathbb{Z}^5} p_k(x)\varphi_x = \mathcal{O}\left(\frac{1}{\sqrt{k}}\right). \tag{5.4}$$

Using as well that $\sqrt{\varepsilon_k}\chi_k\le\sqrt{k/\chi_k}$, and $\sum_{\|x\|^2\le k\chi_k}\varphi_x=\mathcal{O}(k^2\chi_k^2)$, we get

$$\Delta_{0,3} = \rho_k\sum_{x\in\mathbb{Z}^5} p_k(x)\varphi_x + \mathcal{O}\left(\frac{1}{\sqrt{k\cdot\chi_k}}+\frac{\chi_k^2}{k^{3/2}}\right),$$

with

$$\rho_k := \sum_{y_1,y_2\in\mathbb{Z}^5} H(y_1,y_2) = \mathbb{E}\left[Z_0\cdot\mathbf{1}\{\mathcal{R}^1[1,\varepsilon_k]\cap\mathcal{R}[-\varepsilon_k,\varepsilon_k]=\varnothing\}\right].$$

Furthermore, note that one can always take χ_k such that $\chi_k=o(\sqrt{k})$, and that by (2.6), (2.7) and (2.13), one has $|\rho_k-\rho|\lesssim\varepsilon_k^{-1/2}$. This gives

$$\mathbb{E}[Z_0\varphi_3] = \rho\sum_{x\in\mathbb{Z}^5} p_k(x)\varphi_x + o\left(\frac{1}{\sqrt{k}}\right). \tag{5.5}$$

By symmetry the same estimate holds for $\mathbb{E}[Z_k\psi_1]$, and, thus, using again (5.4), it entails

$$\mathbb{E}[Z_0\varphi_3]\cdot\mathbb{E}[Z_k\psi_1] = \rho^2\left(\sum_{x\in\mathbb{Z}^5} p_k(x)\varphi_x\right)^2 + o\left(\frac{1}{k}\right).$$

The estimate of $\mathbb{E}[Z_0\varphi_3 Z_k\psi_1]$ is done along the same line but is a bit more involved. Indeed, let

$$\begin{aligned}\Delta_{1,3} := \mathbb{E}\big[&Z_0 Z_k \mathbf{1}\{\mathcal{R}^1[1,\varepsilon_k]\cap\mathcal{R}[-\varepsilon_k,\varepsilon_k]=\varnothing\}\\ &\times\mathbf{1}\big\{\big(S_k+\mathcal{R}^2[1,\varepsilon_k]\big)\cap\mathcal{R}[k-\varepsilon_k,k+\varepsilon_k]=\varnothing,\tau_1<\infty,\tau_2<\infty\big\}\big].\end{aligned}$$

The difference between $\mathbb{E}[Z_0\varphi_3 Z_k\psi_1]$ and $\Delta_{1,3}$ can be controlled roughly as above, but one needs additionally to handle the probability of τ_2 being finite. Namely, one has using symmetry,

$$\begin{aligned}&\big|\mathbb{E}[Z_0\varphi_3 Z_k\psi_1]-\Delta_{1,3}\big|\\ &\quad\le 2\big(\mathbb{P}[\tau_{0,1}<\infty,\tau_1<\infty,\overline{\tau}_2<\infty]\\ &\qquad+\mathbb{P}[\mathcal{R}^1[0,\varepsilon_k]\cap\mathcal{R}[k,\infty)\neq\varnothing,\overline{\tau}_2<\infty]\\ &\qquad+\mathbb{P}[\mathcal{R}^1_\infty\cap\mathcal{R}[k,k+\varepsilon_k]\neq\varnothing,\overline{\tau}_2<\infty]\big),\end{aligned}\tag{5.6}$$

with

$$\overline{\tau}_2:=\inf\{n\ge 0: S_k+S^2_n\in\mathcal{R}(-\infty,0]\}.$$

The last term in (5.6) is handled as follows:

$$\begin{aligned}&\mathbb{P}\big[\mathcal{R}^1_\infty\cap\mathcal{R}[k,k+\varepsilon_k]\neq\varnothing,\overline{\tau}_2<\infty\big]\\ &\quad=\sum_{x\in\mathbb{Z}^5}\mathbb{P}\big[\mathcal{R}^1_\infty\cap\mathcal{R}[k,k+\varepsilon_k]\neq\varnothing,\overline{\tau}_2<\infty,S_k=x\big]\\ &\quad\overset{(2.11)}{\le}\sum_{x\in\mathbb{Z}^5}p_k(x)\varphi_x\sum_{i=0}^{\varepsilon_k}\mathbb{E}\big[G(S_i+x)\big]\\ &\quad\overset{(2.7),(2.12),(2.10)}{\lesssim}\varepsilon_k\sum_{x\in\mathbb{Z}^5}\frac{p_k(x)}{1+\|x\|^4}\overset{(2.3)}{\lesssim}\frac{\varepsilon_k}{k^2}.\end{aligned}$$

The same arguments give as well

$$\mathbb{P}[\mathcal{R}^1[0,\varepsilon_k]\cap\mathcal{R}[k,\infty)\neq\varnothing,\overline{\tau}_2<\infty]\lesssim\frac{\varepsilon_k}{k^2},$$

$$\mathbb{P}[\tau_{0,1}<\infty,\tau_1<\infty,\overline{\tau}_2<\infty]=\mathbb{P}[\tau_{0,1}<\infty,\tau_1<\infty,\tau_2<\infty]+\mathcal{O}\bigg(\frac{\varepsilon_k}{k^2}\bigg).$$

Then, we can write

$$\begin{aligned}&\mathbb{P}[\tau_{0,1}\le\tau_1<\infty,\tau_2<\infty]\\ &\quad=\mathbb{E}\big[\mathbb{P}_{0,S_{k+\varepsilon_k}-S_{\tau_{0,1}}}[\mathcal{R}_\infty\cap\widetilde{\mathcal{R}}_\infty\neq\varnothing]\mathbf{1}\{\tau_{0,1}<\infty,\tau_2<\infty\}\big]\\ &\quad\overset{(2.11),(2.12)}{\lesssim}\sum_{i=-\varepsilon_k}^{\varepsilon_k}\mathbb{E}\bigg[\frac{1}{1+\|S_{k+\varepsilon_k}-S_i\|}\cdot\frac{G(S_i-S^1_{\varepsilon_k})}{1+\|S_k-S_{-\varepsilon_k}\|}\bigg]\\ &\quad\overset{(2.9)}{\lesssim}\frac{1}{\varepsilon_k^{3/2}}\sum_{i=-\varepsilon_k}^{\varepsilon_k}\mathbb{E}\bigg[\frac{1}{1+\|S_k-S_i\|}\cdot\frac{1}{1+\|S_k-S_{-\varepsilon_k}\|}\bigg]\\ &\quad\lesssim\frac{1}{\sqrt{\varepsilon_k}}\max_{k-\varepsilon_k\le j\le k+\varepsilon_k}\sup_{u\in\mathbb{Z}^d}\mathbb{E}\bigg[\frac{1}{1+\|S_j\|}\cdot\frac{1}{1+\|S_j+u\|}\bigg]\lesssim\frac{1}{k\sqrt{\varepsilon_k}},\end{aligned}$$

where the last equality follows from straightforward computations, using (2.3). On the other hand,

$$
\begin{aligned}
&\mathbb{P}[\tau_1 \le \tau_{0,1} < \infty, \tau_2 < \infty] \\
&\overset{(2.11),(2.12)}{\lesssim} \sum_{i=k+\varepsilon_k}^{\infty} \sum_{j=-\varepsilon_k}^{\varepsilon_k} \mathbb{E}\Big[\frac{G(S_j - S_i)G(S_i - S^1_{\varepsilon_k})}{1 + \|S_k - S_{-\varepsilon_k}\|}\Big] \\
&\overset{(2.7),(2.10)}{\lesssim} \sum_{j=-\varepsilon_k}^{\varepsilon_k} \sum_{i=k+\varepsilon_k}^{\infty} \mathbb{E}\Big[\frac{G(S_j - S_i)}{(1 + \|S_i\|^3)(1 + \|S_k - S_{-\varepsilon_k}\|)}\Big] \\
&\lesssim \sum_{j=-\varepsilon_k}^{\varepsilon_k} \sum_{z\in\mathbb{Z}^d} G_{\varepsilon_k}(z)\mathbb{E}\Big[\frac{G(z + S_k - S_j)}{(1 + \|z + S_k\|^3)(1 + \|S_k - S_{-\varepsilon_k}\|)}\Big].
\end{aligned}
$$

Note now that, for $x, y \in \mathbb{Z}^5$, by (2.7) and Lemma 2.2,

$$
\sum_{z\in\mathbb{Z}^d} \frac{G_{\varepsilon_k}(z)}{(1 + \|z - x\|^3)(1 + \|z - y\|^3)} \lesssim \frac{1}{1 + \|x\|^3}\Big(\frac{1}{\sqrt{\varepsilon_k}} + \frac{1}{1 + \|y - x\|}\Big).
$$

It follows that

$$
\begin{aligned}
&\mathbb{P}[\tau_1 \le \tau_{0,1} < \infty, \tau_2 < \infty] \\
&\lesssim \sum_{j=-\varepsilon_k}^{\varepsilon_k} \mathbb{E}\Big[\frac{1}{(1 + \|S_k\|^3)(1 + \|S_k - S_{-\varepsilon_k}\|)}\Big(\frac{1}{\sqrt{\varepsilon_k}} + \frac{1}{1 + \|S_j\|}\Big)\Big] \\
&\overset{(2.10)}{\lesssim} \mathbb{E}\Big[\frac{\sqrt{\varepsilon_k}}{1 + \|S_k\|^4}\Big] + \sum_{j=-\varepsilon_k}^{0} \mathbb{E}\Big[\frac{1}{(1 + \|S_k\|^3)(1 + \|S_k - S_j\|)(1 + \|S_j\|)}\Big] \\
&\quad + \sum_{j=1}^{\varepsilon_k} \mathbb{E}\Big[\frac{1}{(1 + \|S_k\|^4)(1 + \|S_j\|)}\Big] \\
&\lesssim \frac{1}{k^2}\Big(\sqrt{\varepsilon_k} + \sum_{j=-\varepsilon_k}^{\varepsilon_k} \mathbb{E}\Big[\frac{1}{1 + \|S_j\|}\Big]\Big) \lesssim \frac{\sqrt{\varepsilon_k}}{k^2},
\end{aligned}
$$

using for the third inequality that by (2.3), it holds uniformly in $x \in \mathbb{Z}^5$ and $j \le \varepsilon_k$,

$$
\mathbb{E}\Big[\frac{1}{1 + \|S_k - S_j + x\|^4}\Big] \lesssim k^{-2}, \qquad \mathbb{E}\Big[\frac{1}{(1 + \|S_k\|^3)(1 + \|S_k + x\|)}\Big] \lesssim k^{-2}.
$$

Now, we are left with computing $\Delta_{1,3}$. This step is essentially the same as above, so we omit to give all the details. We first define for $y_1, y_2, y_3 \in \mathbb{Z}^5$,

$$
H(y_1, y_2, y_3) := \mathbb{E}[Z_0 \mathbb{1}\{\mathcal{R}^1[1, \varepsilon_k] \cap \mathcal{R}[-\varepsilon_k, \varepsilon_k] = \varnothing, S_{\varepsilon_k} = y_1, S^1_{\varepsilon_k} = y_2, S_{-\varepsilon_k} = y_3\}]
$$

and note that

$$
\Delta_{1,3} = \sum_{\substack{y_1,y_2,y_3\in\mathbb{Z}^5 \\ z_1,z_2,z_3\in\mathbb{Z}^5 \\ x\in\mathbb{Z}^5}} H(y_1, y_2, y_3)H(z_1, z_2, z_3)p_{k-2\varepsilon_k}(x - y_1 + z_3)\varphi_{x+z_1-y_2}\varphi_{x+z_2-y_3}.
$$

Observe here that by Theorem C, $\varphi_{x+z_1-y_2}$ is equivalent to φ_x, when $\|z_1\|$ and $\|y_2\|$ are small when compared to $\|x\|$ and, similarly, for $\varphi_{x+z_2-y_3}$. Thus, using similar arguments as above and, in particular, that by (2.3) and (2.12)

$$\sum_{x\in\mathbb{Z}^5} p_k(x)\varphi_x^2 = \mathcal{O}\Big(\frac{1}{k}\Big), \tag{5.7}$$

we obtain

$$\Delta_{1,3} = \rho^2 \sum_{x\in\mathbb{Z}^5} p_k(x)\varphi_x^2 + o\Big(\frac{1}{k}\Big).$$

Putting all pieces together gives (5.1). Using in addition (2.3), (2.12) and Theorem 2.1, we deduce that

$$\operatorname{Cov}(Z_0\varphi_3, Z_k\psi_1) = \rho^2\Big\{\sum_{x\in\mathbb{Z}^5} \overline{p}_k(x)\varphi_x^2 - \Big(\sum_{x\in\mathbb{Z}^5} \overline{p}_k(x)\varphi_x\Big)^2\Big\} + o\Big(\frac{1}{k}\Big).$$

Then, Theorem C, together with (5.4) and (5.7), show that

$$\operatorname{Cov}(Z_0\varphi_3, Z_k\psi_1) = \sigma\Big\{\sum_{x\in\mathbb{Z}^5} \frac{\overline{p}_k(x)}{1+\mathcal{J}(x)^2} - \Big(\sum_{x\in\mathbb{Z}^5} \frac{\overline{p}_k(x)}{1+\mathcal{J}(x)}\Big)^2\Big\} + o\Big(\frac{1}{k}\Big),$$

for some constant $\sigma > 0$. Finally, an approximation of the series with an integral and a change of variables gives, with $c_0 := (2\pi)^{-5/2}(\det\Gamma)^{-1/2}$,

$$\operatorname{Cov}(Z_0\varphi_3, Z_k\psi_1) = \frac{\sigma c_0}{k}\Big\{\int_{\mathbb{R}^5} \frac{e^{-5\mathcal{J}(x)^2/2}}{\mathcal{J}(x)^2}\,dx - c_0\Big(\int_{\mathbb{R}^5} \frac{e^{-5\mathcal{J}(x)^2/2}}{\mathcal{J}(x)}\,dx\Big)^2\Big\} + o\Big(\frac{1}{k}\Big).$$

The last step of the proof is to observe that the difference between the two terms in the curly bracket is well a positive real. This follows simply by Cauchy–Schwarz, once we observe that $c_0\int_{\mathbb{R}^5} e^{-5\mathcal{J}(x)^2/2}\,dx = 1$, which itself can be deduced for instance from the fact that $1 = \sum_{x\in\mathbb{Z}^5} p_k(x) \sim c_0\int_{\mathbb{R}^5} e^{-5\mathcal{J}(x)^2/2}\,dx$, by the above arguments. This concludes the proof of Lemma 5.1.

5.2. *Proof of Lemma* 5.2. Let us concentrate on the term $\operatorname{Cov}(Z_0\varphi_3, Z_k\psi_3)$, the estimate of $\operatorname{Cov}(Z_0\varphi_1, Z_k\psi_1)$ being entirely similar. We also assume to simplify notation that the walk is aperiodic.

We consider as in the proof of the previous lemma $(S_n^1)_{n\ge 0}$ and $(S_n^2)_{n\ge 0}$ two independent random walks starting from the origin, independent of $(S_n)_{n\in\mathbb{Z}}$, and define this time

$$\tau_1 := \inf\{n \ge k+\varepsilon_k : S_n \in \mathcal{R}^1[\varepsilon_k,\infty)\},$$
$$\tau_2 := \inf\{n \ge k+\varepsilon_k : S_n \in S_k + \mathcal{R}^2[\sqrt{\varepsilon_k},\infty)\}.$$

Define as well

$$\overline{\tau}_1 := \inf\{n \ge k+\varepsilon_k : S_n \in \mathcal{R}^1_\infty\}, \qquad \overline{\tau}_2 := \inf\{n \ge k+\varepsilon_k : S_n \in S_k + \mathcal{R}^2_\infty\}.$$

Step 1. Our first task is to show that

$$\operatorname{Cov}(Z_0\varphi_3, Z_k\psi_3) = \rho^2\cdot\operatorname{Cov}\big(\mathbb{1}\{\overline{\tau}_1<\infty\}, \mathbb{1}\{\overline{\tau}_2<\infty\}\big) + o\Big(\frac{1}{k}\Big), \tag{5.8}$$

with ρ as defined in (5.2). This step is essentially the same as in the proof of Lemma 5.1 but with some additional technical difficulties, so let us give some details. First, the proof of Lemma 5.1 shows that (using the same notation)

$$\mathbb{E}[Z_0\varphi_3] = \Delta_{0,3} + \mathcal{O}\Big(\frac{1}{\sqrt{k\varepsilon_k}} + \frac{\varepsilon_k}{k^{3/2}}\Big),$$

and that for any sequence $(\chi_k)_{k\geq 1}$ going to infinity with $\varepsilon_k \chi_k^{2+\frac{1}{4}} \leq k$,

$$\Delta_{0,3} = \sum_{k/\chi_k \leq \|x\|^2 \leq k\chi_k} \sum_{\substack{\|y_1\|^2 \leq \varepsilon_k \chi_k \\ \|y_2\|^2 \leq \varepsilon_k \chi_k}} H(y_1, y_2) p_k(x + y_2 - y_1)\varphi_x + \mathcal{O}\Big(\frac{1}{\sqrt{k}\cdot \chi_k^{5/4}}\Big).$$

Observe moreover, that by symmetry $H(y_1, y_2) = H(-y_1, -y_2)$ and that by Theorem 2.1, for any x, y_1, and y_2 as above, for some constant $c > 0$,

$$\big|p_k(x + y_2 - y_1) + p_k(x + y_1 - y_2) - p_k(x)\big| = \mathcal{O}\Big(\frac{\varepsilon_k \chi_k}{k}\overline{p}_k(cx) + \frac{1}{k^{7/2}}\Big).$$

It follows that one can improve the bound (5.5) into

$$\begin{aligned}\mathbb{E}[Z_0\varphi_3] &= \rho \sum_{x\in\mathbb{Z}^5} p_k(x)\varphi_x + \mathcal{O}\Big(\frac{\varepsilon_k\chi_k}{k^{3/2}} + \frac{\chi_k^2}{k^{3/2}} + \frac{1}{\sqrt{k}\cdot\chi_k^{5/4}} + \frac{1}{\sqrt{k\varepsilon_k}} + \frac{\varepsilon_k}{k^{3/2}}\Big) \\ &= \rho\mathbb{P}[\overline{\tau}_1 < \infty] + \mathcal{O}\Big(\frac{\varepsilon_k\chi_k}{k^{3/2}} + \frac{\chi_k^2}{k^{3/2}} + \frac{1}{\sqrt{k}\cdot\chi_k^{5/4}} + \frac{1}{\sqrt{k\varepsilon_k}} + \frac{\varepsilon_k}{k^{3/2}}\Big).\end{aligned} \tag{5.9}$$

Since by (2.13) one has

$$\mathbb{E}[Z_k\psi_3] \leq \mathbb{E}[\psi_3] = \mathcal{O}\Big(\frac{1}{\sqrt{\varepsilon_k}}\Big),$$

this yields by taking $\chi_k^{2+1/4} := k/\varepsilon_k$, and $\varepsilon_k \geq k^{2/3}$ (but still $\varepsilon_k = o(k)$),

$$\mathbb{E}[Z_0\varphi_3]\cdot\mathbb{E}[Z_k\psi_3] = \rho\mathbb{P}[\overline{\tau}_1 < \infty]\cdot\mathbb{E}[Z_k\psi_3] + o\Big(\frac{1}{k}\Big). \tag{5.10}$$

We next seek an analogous estimate for $\mathbb{E}[Z_k\psi_3]$. Define $Z_k' := 1\{S_{k+i} \neq S_k, \forall i = 1,\dots,\varepsilon_k^{3/4}\}$ and

$$\Delta_0 := \mathbb{E}\big[Z_k'\cdot \mathbf{1}\{\mathcal{R}[k-\varepsilon_k, k+\varepsilon_k^{3/4}] \cap (S_k + \mathcal{R}^2[1,\sqrt{\varepsilon_k}]) = \varnothing, \tau_2 < \infty\}\big].$$

Note that (with $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ two independent walks)

$$\begin{aligned}\big|\mathbb{E}[Z_k\psi_3] - \Delta_0\big| \leq{} & \mathbb{P}[0 \in \mathcal{R}[\varepsilon_k^{3/4}, \varepsilon_k]] + \mathbb{P}[\widetilde{\mathcal{R}}[0,\sqrt{\varepsilon_k}] \cap \mathcal{R}[\varepsilon_k,\infty) \neq \varnothing] \\ & + \mathbb{P}[\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}[\varepsilon_k^{3/4},\varepsilon_k] \neq \varnothing, \widetilde{\mathcal{R}}_\infty \cap \mathcal{R}[\varepsilon_k,\infty) \neq \varnothing] \\ & + \mathbb{P}[\widetilde{\mathcal{R}}[\sqrt{\varepsilon_k},\infty) \cap \mathcal{R}[-\varepsilon_k,\varepsilon_k] \neq \varnothing, \widetilde{\mathcal{R}}[\sqrt{\varepsilon_k},\infty) \cap \mathcal{R}[\varepsilon_k,\infty) \neq \varnothing].\end{aligned}$$

Moreover,

$$\mathbb{P}[0 \in \mathcal{R}[\varepsilon_k^{3/4},\varepsilon_k]] \overset{(2.6),(2.9)}{\lesssim} \varepsilon_k^{-9/8}, \qquad \mathbb{P}[\widetilde{\mathcal{R}}[0,\sqrt{\varepsilon_k}] \cap \mathcal{R}[\varepsilon_k,\infty) \neq \varnothing] \overset{(2.13)}{\lesssim} \varepsilon_k^{-1}. \tag{5.11}$$

Using also the same computation as in the proof of Lemma 4.1 from [19], we get

$$\begin{aligned}\mathbb{P}[\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}[\varepsilon_k^{3/4},\varepsilon_k] \neq \varnothing, \widetilde{\mathcal{R}}_\infty \cap \mathcal{R}[\varepsilon_k,\infty) \neq \varnothing] &\lesssim \varepsilon_k^{-\frac{3}{8}-\frac{1}{2}}, \\ \mathbb{P}[\widetilde{\mathcal{R}}[\sqrt{\varepsilon_k},\infty) \cap \mathcal{R}[-\varepsilon_k,\varepsilon_k] \neq \varnothing, \widetilde{\mathcal{R}}[\sqrt{\varepsilon_k},\infty) \cap \mathcal{R}[\varepsilon_k,\infty) \neq \varnothing] &\lesssim \varepsilon_k^{-\frac{1}{4}-\frac{1}{2}}.\end{aligned} \tag{5.12}$$

As a consequence

$$\mathbb{E}[Z_k\psi_3] = \Delta_0 + \mathcal{O}(\varepsilon_k^{-3/4}). \tag{5.13}$$

Introduce now

$$\begin{aligned}\widetilde{H}(y_1,y_2) := \mathbb{E}\big[Z'_k \cdot \mathbf{1}\{\mathcal{R}[k-\varepsilon_k, k+\varepsilon_k^{3/4}] \cap (S_k + \mathcal{R}^2[1,\sqrt{\varepsilon_k}]) = \varnothing\}\\ \times \mathbf{1}\{S_{k+\varepsilon_k^{3/4}} - S_k = y_1, S^2_{\sqrt{\varepsilon_k}} = y_2\}\big],\end{aligned}$$

and note that

$$\Delta_0 = \sum_{x\in\mathbb{Z}^d} \sum_{y_1,y_2\in\mathbb{Z}^d} \widetilde{H}(y_1,y_2)\, p_{\varepsilon_k - \varepsilon_k^{3/4}}(x+y_2-y_1)\varphi_x.$$

Let $\chi_k := \varepsilon_k^{1/8}$. As above, we can see that

$$\begin{aligned}\Delta_0 &= \sum_{\substack{\varepsilon_k/\chi_k \le \|x\|^2 \le \varepsilon_k\chi_k\\ \|y_1\|^2 \le \varepsilon_k^{3/4}\chi_k\\ \|y_2\|^2 \le \sqrt{\varepsilon_k}\chi_k}} \widetilde{H}(y_1,y_2)\, p_{\varepsilon_k - \varepsilon_k^{3/4}}(x+y_2-y_1)\varphi_x + \mathcal{O}\Big(\frac{1}{\sqrt{\varepsilon_k}\chi_k^{5/4}}\Big)\\ &= \Big(\sum_{y_1,y_2\in\mathbb{Z}^d} \widetilde{H}(y_1,y_2)\Big)\Big(\sum_{x\in\mathbb{Z}^d} p_{\varepsilon_k}(x)\varphi_x\Big) + \mathcal{O}\Big(\frac{\chi_k}{\varepsilon_k^{3/4}} + \frac{\chi_k^2}{\varepsilon_k^{3/2}} + \frac{1}{\sqrt{\varepsilon_k}\chi_k^{5/4}}\Big)\\ &= \rho \cdot \mathbb{P}[\overline{\tau}_2 < \infty] + \mathcal{O}(\varepsilon_k^{-5/8}).\end{aligned}$$

Then, by taking $\varepsilon_k \ge k^{5/6}$ and recalling (5.10) and (5.13), we obtain

$$\mathbb{E}[Z_0\varphi_3]\cdot\mathbb{E}[Z_k\psi_3] = \rho^2 \cdot \mathbb{P}[\overline{\tau}_1 < \infty]\cdot\mathbb{P}[\overline{\tau}_2 < \infty] + o\Big(\frac{1}{k}\Big). \tag{5.14}$$

Finally, let

$$\begin{aligned}\Delta_{3,3} := \mathbb{E}\big[Z_0 Z'_k \mathbf{1}\{\mathcal{R}^1[1,\varepsilon_k] \cap \mathcal{R}[-\varepsilon_k,\varepsilon_k] = \varnothing\}\\ \times \mathbf{1}\{(S_k + \mathcal{R}^2[1,\sqrt{\varepsilon_k}]) \cap \mathcal{R}[k-\varepsilon_k^{\frac{3}{4}}, k+\varepsilon_k^{\frac{3}{4}}] = \varnothing, \tau_1 < \infty, \tau_2 < \infty\}\big].\end{aligned}$$

It amounts to estimate the difference between $\Delta_{3,3}$ and $\mathbb{E}[Z_0 Z_k \varphi_3 \psi_3]$. Define

$$\widetilde{\tau}_1 := \inf\{n \ge k+\varepsilon_k : S_n \in \mathcal{R}^1[0,\varepsilon_k]\},\ \widetilde{\tau}_2 := \inf\{n \ge k+\varepsilon_k : S_n \in S_k + \mathcal{R}^2[0,\sqrt{\varepsilon_k}]\}.$$

Observe first that

$$\begin{aligned}\mathbb{P}[\widetilde{\tau}_1 \le \overline{\tau}_2 < \infty] &\overset{(2.12)}{\lesssim} \mathbb{E}\Big[\frac{\mathbf{1}\{\widetilde{\tau}_1 < \infty\}}{1+\|S_{\widetilde{\tau}_1} - S_k\|}\Big] \overset{(2.11)}{\lesssim} \sum_{i=0}^{\varepsilon_k} \mathbb{E}\Big[\frac{G(S_i^1 - S_{k+\varepsilon_k})}{1+\|S_i^1 - S_k\|}\Big]\\ &\lesssim \sum_{i=0}^{\varepsilon_k}\sum_{z\in\mathbb{Z}^5} p_i(z)\mathbb{E}\Big[\frac{G(z-S_{k+\varepsilon_k})}{1+\|z-S_k\|}\Big] \overset{(2.3)}{\lesssim} \sum_{z\in\mathbb{Z}^5} \frac{\sqrt{\varepsilon_k}}{1+\|z\|^4}\mathbb{E}\Big[\frac{G(z-S_{k+\varepsilon_k})}{1+\|z-S_k\|}\Big]\\ &\overset{(2.7)}{\lesssim} \mathbb{E}\Big[\frac{\sqrt{\varepsilon_k}}{(1+\|S_{k+\varepsilon_k}\|^2)(1+\|S_k\|)}\Big] \overset{(2.10)}{\lesssim} \mathbb{E}\Big[\frac{\sqrt{\varepsilon_k}}{1+\|S_k\|^3}\Big] \overset{(2.9)}{\lesssim} \frac{\sqrt{\varepsilon_k}}{k^{3/2}},\end{aligned} \tag{5.15}$$

and, likewise,

$$\begin{aligned}\mathbb{P}[\overline{\tau}_1 \le \widetilde{\tau}_2 < \infty] &\overset{(2.11)}{\le} \sum_{j\ge 0}\sum_{i=0}^{\varepsilon_k} \mathbb{E}\big[G(S_k + S_i^2 - S_j^1)G(S_j^1 - S_{k+\varepsilon_k})\big]\\ &= \sum_{i=0}^{\varepsilon_k}\sum_{z\in\mathbb{Z}^5} \mathbb{E}\big[G(z)G(S_k + S_i^2 - z)G(z - S_{k+\varepsilon_k})\big]\end{aligned}$$

$$\lesssim \sum_{i=0}^{\varepsilon_k} \mathbb{E}\Big[\frac{1}{1+\|S_k+S_i^2\|^3}\Big(\frac{1}{1+\|S_{k+\varepsilon_k}\|}+\frac{1}{1+\|S_{k+\varepsilon_k}-S_k-S_i^2\|}\Big)\Big]$$

$$\overset{(2.9),(2.10)}{\lesssim} \mathbb{E}\Big[\frac{\varepsilon_k}{1+\|S_k\|^4}\Big]+\mathbb{E}\Big[\frac{\sqrt{\varepsilon_k}}{1+\|S_k\|^3}\Big] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.$$

Additionally, it follows directly from (2.13) that

$$\mathbb{P}[\overline{\tau}_2 \le \widetilde{\tau}_1 < \infty] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}} \quad \text{and} \quad \mathbb{P}[\widetilde{\tau}_2 \le \overline{\tau}_1 < \infty] \lesssim \frac{1}{\varepsilon_k\sqrt{k}},$$

which altogether yields

$$\big|\mathbb{P}[\overline{\tau}_1 < \infty, \overline{\tau}_2 < \infty] - \mathbb{P}[\tau_1 < \infty, \tau_2 < \infty]\big| \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}} + \frac{1}{\varepsilon_k\sqrt{k}}.$$

Similar computations give also

$$\mathbb{P}[\overline{\tau}_1 < \infty, \overline{\tau}_2 < \infty] \lesssim \frac{1}{\sqrt{k\varepsilon_k}}. \tag{5.16}$$

Next, using (5.11) and the Markov property, we get

$$\mathbb{E}\big[|Z_k - Z_k'|\mathbb{1}\{\tau_1 < \infty\}\big] \lesssim \frac{1}{\varepsilon_k^{9/8}\sqrt{k}}.$$

Thus, for $\varepsilon_k \ge k^{5/6}$,

$$\begin{aligned}
&\big|\mathbb{E}[Z_0 Z_k \varphi_3 \psi_3] - \Delta_{3,3}\big| \\
&\quad \le \mathbb{P}[\tau_{0,1} < \infty, \tau_1 < \infty, \tau_2 < \infty] + \mathbb{P}[\tau_{0,2} < \infty, \tau_1 < \infty, \tau_2 < \infty] \\
&\qquad + \mathbb{P}[\widetilde{\tau}_{0,2} < \infty, \tau_1 < \infty, \tau_2 < \infty] + o\Big(\frac{1}{k}\Big),
\end{aligned}$$

where $\tau_{0,1}$ is, as defined in the proof of Lemma 5.1,

$$\tau_{0,2} := \inf\{n \ge \sqrt{\varepsilon_k} : S_k + S_n^2 \in \mathcal{R}[k-\varepsilon_k, k+\varepsilon_k]\}$$

and

$$\widetilde{\tau}_{0,2} := \inf\{n \le \sqrt{\varepsilon_k} : S_k + S_n^2 \in \mathcal{R}[k-\varepsilon_k, k-\varepsilon_k^{3/4}] \cup \mathcal{R}[k+\varepsilon_k^{3/4}, k+\varepsilon_k]\}.$$

Applying (2.13) twice already shows that

$$\mathbb{P}[\widetilde{\tau}_{0,2} < \infty, \tau_1 < \infty] \lesssim \frac{1}{\sqrt{k}} \cdot \mathbb{P}[\widetilde{\tau}_{0,2} < \infty] \lesssim \frac{1}{\sqrt{k}\varepsilon_k^{5/8}} = o\Big(\frac{1}{k}\Big).$$

Then, notice that (5.15) entails

$$\mathbb{P}[\mathcal{R}[k+\varepsilon_k, \infty) \cap \mathcal{R}^1[0, \tau_{0,1}] \neq \varnothing, S^1_{\tau_{0,1}} \in \mathcal{R}[-\varepsilon_k, 0]] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}[\mathcal{R}[k+\varepsilon_k, \infty) \cap \mathcal{R}^1[0, \tau_{0,1}] \neq \varnothing, S^1_{\tau_{0,1}} \in \mathcal{R}[0, \varepsilon_k]] \\
&\quad \overset{(2.11)}{\le} \sum_{i=0}^{\varepsilon_k} \sum_{j=k+\varepsilon_k}^{\infty} \mathbb{E}[G(S_i - S_{k+j})G(S_{k+j} - S_k)]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\varepsilon_k} \sum_{z\in\mathbb{Z}^5} \mathbb{E}\big[G(S_i - S_k + z)G(z)G_{\varepsilon_k}(z)\big] \\
&\overset{(2.9)}{\lesssim} \frac{\varepsilon_k}{k^{3/2}} \sum_{z\in\mathbb{Z}^5} G(z)G_{\varepsilon_k}(z) \overset{\text{Lemma 2.2}}{\lesssim} \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.
\end{aligned}$$

By (2.11) and (2.9) one has with $\widetilde{\mathcal{R}}_\infty$ an independent copy of $\mathcal{R}_\infty$,

$$\begin{aligned}
&\mathbb{P}[\tau_{0,1} < \infty, \tau_2 < \infty, \mathcal{R}[k+\varepsilon_k, \infty) \cap \mathcal{R}^1[\tau_{0,1}, \infty) \neq \varnothing] \\
&\quad \lesssim \frac{1}{\sqrt{\varepsilon_k}} \max_{-\varepsilon_k \le i \le \varepsilon_k} \mathbb{P}\big[\tau_2 < \infty, \mathcal{R}\big[k+\varepsilon_k, \infty) \cap (S_i + \widetilde{\mathcal{R}}_\infty) \neq \varnothing\big] \lesssim \frac{1}{\varepsilon_k \sqrt{k}},
\end{aligned}$$

where the last equality follows from (5.16). Thus,

$$\mathbb{P}[\tau_{0,1} < \infty, \tau_1 < \infty, \tau_2 < \infty] = o\left(\frac{1}{k}\right).$$

In a similar fashion one has

$$\mathbb{P}[\tau_{0,2} < \infty, \tau_2 \le \tau_1 < \infty] \overset{(2.13)}{\lesssim} \frac{1}{\sqrt{k}} \mathbb{P}[\tau_{0,2} < \infty, \tau_2 < \infty] \overset{(5.12)}{\lesssim} \frac{1}{\varepsilon_k^{3/4}\sqrt{k}}$$

as well as

$$\begin{aligned}
&\mathbb{P}\big[\tau_{0,2} < \infty, \tau_1 \le \tau_2 < \infty, S_{\tau_2} \in \big(S_k + \mathcal{R}^2[0, \tau_{0,2}]\big)\big] \\
&\overset{(2.11)}{\le} \sum_{i=k-\varepsilon_k}^{k+\varepsilon_k} \sum_{j\ge 0} \sum_{\ell \ge 0} \mathbb{E}\big[G(S_i - \widetilde{S}_j - S_\ell^1)G(\widetilde{S}_j + S_\ell^1 - S_k)G(S_\ell^1 - S_{k+\varepsilon_k})\big] \\
&\le \sum_{i=k-\varepsilon_k}^{k+\varepsilon_k} \sum_{\ell\ge 0} \sum_{z\in\mathbb{Z}^5} \mathbb{E}\big[G(z)G(S_i - S_\ell^1 - z)G(z + S_\ell^1 - S_k)G(S_\ell^1 - S_{k+\varepsilon_k})\big] \\
&\overset{\text{Lemma 2.2}}{\lesssim} \sum_{i=k-\varepsilon_k}^{k+\varepsilon_k} \sum_{\ell\ge 0} \mathbb{E}\bigg[\frac{G(S_\ell^1 - S_{k+\varepsilon_k})}{1 + \|S_\ell^1 - S_k\|^3}\bigg(\frac{1}{1+\|S_\ell^1 - S_i\|} + \frac{1}{1+\|S_i - S_k\|}\bigg)\bigg] \\
&\overset{(2.9),(2.10)}{\lesssim} \sum_{i=0}^{\varepsilon_k} \sum_{\ell\ge 0} \bigg\{\mathbb{E}\bigg[\frac{\varepsilon_k^{-3/2}}{1 + \|S_\ell^1 - S_k\|^3}\bigg(\frac{1}{1+\|S_\ell^1 - S_{k-i}\|} + \frac{1}{1+\|S_{k-i} - S_k\|}\bigg)\bigg] \\
&\qquad + \mathbb{E}\bigg[\frac{1}{(1+\|S_\ell^1 - S_{k+i}\|^3)(1+\|S_\ell^1 - S_k\|^3)}\bigg(\frac{1}{1+\|S_\ell^1 - S_{k+i}\|} + \frac{1}{1+\|S_{k+i} - S_k\|}\bigg)\bigg]\bigg\} \\
&\overset{(2.3),(2.10)}{\lesssim} \sum_{i=0}^{\varepsilon_k} \sum_{\ell\ge 0} \bigg\{\mathbb{E}\bigg[\frac{\varepsilon_k^{-3/2}}{1 + \|S_\ell^1 - S_{k-i}\|^3}\bigg(\frac{1}{1+\|S_\ell^1 - S_{k-i}\|} + \frac{1}{1+\sqrt{i}}\bigg)\bigg] \\
&\qquad + \mathbb{E}\bigg[\frac{(1+i)^{-1/2}}{1+\|S_\ell^1 - S_k\|^6}\bigg]\bigg\} \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\big[\tau_{0,2} < \infty, \tau_1 \le \tau_2 < \infty, S_{\tau_2} \in \big(S_k + \mathcal{R}^2[\tau_{0,2}, \infty)\big)\big] \\
&\overset{(2.6)}{\le} \sum_{i=-\varepsilon_k}^{\varepsilon_k} \mathbb{E}\big[G\big(S_{k+i} - S_k - S^2_{\sqrt{\varepsilon_k}}\big)\mathbf{1}\{\tau_1 < \infty, \mathcal{R}[\tau_1, \infty) \cap (S_{k+i} + \widetilde{\mathcal{R}}_\infty) \neq \varnothing\}\big]
\end{aligned}$$

$$\begin{aligned}
&\overset{(2.12)}{\lesssim} \sum_{i=-\varepsilon_k}^{\varepsilon_k} \mathbb{E}\Big[G\big(S_{k+i}-S_k-S^2_{\sqrt{\varepsilon_k}}\big)\frac{1\{\tau_1<\infty\}}{1+\|S_{\tau_1}-S_{k+i}\|}\Big]\\
&\overset{(2.11)}{\lesssim} \sum_{i=-\varepsilon_k}^{\varepsilon_k}\sum_{j\ge k+\varepsilon_k} \mathbb{E}\Big[\frac{G(S_{k+i}-S_k-S^2_{\sqrt{\varepsilon_k}})G(S_j)}{1+\|S_j-S_{k+i}\|}\Big]\\
&\lesssim \sum_{i=0}^{\varepsilon_k}\sum_{z\in\mathbb{Z}^5}\Big\{\mathbb{E}\Big[\frac{G(S_{k-i}-S_k-S^2_{\sqrt{\varepsilon_k}})G(S_k+z)G(z)}{1+\|z+S_k-S_{k-i}\|}\Big]\\
&\qquad+\mathbb{E}\Big[\frac{G(S_{k+i}-S_k-S^2_{\sqrt{\varepsilon_k}})G(S_{k+i}+z)G(z)}{1+\|z\|}\Big]\Big\}\\
&\lesssim \sum_{i=0}^{\varepsilon_k}\Big\{\mathbb{E}\Big[\frac{G(S_{k-i}-S_k-S^2_{\sqrt{\varepsilon_k}})}{1+\|S_k\|^2}\Big]+\mathbb{E}\Big[\frac{G(S_{k+i}-S_k-S^2_{\sqrt{\varepsilon_k}})}{1+\|S_{k+i}\|^2}\Big]\Big\}\\
&\overset{(2.9),(2.10)}{\lesssim} \frac{1}{\varepsilon_k^{3/4}}\sum_{i=0}^{\sqrt{\varepsilon_k}}\mathbb{E}\Big[\frac{1}{1+\|S_k\|^2}+\frac{1}{1+\|S_{k+i}\|^2}\Big]\\
&\qquad+\sum_{i=\sqrt{\varepsilon_k}}^{\varepsilon_k}\mathbb{E}\Big[\frac{G(S_{k-i}-S_k)}{1+\|S_k\|^2}+\frac{G(S_{k+i}-S_k)}{1+\|S_{k+i}\|^2}\Big]\\
&\overset{(2.3),(2.9)}{\lesssim} \frac{1}{\varepsilon_k^{1/4}k}+\sum_{i=\sqrt{\varepsilon_k}}^{\varepsilon_k}\frac{1}{i^{3/2}}\cdot\mathbb{E}\Big[\frac{1}{1+\|S_{k-i}\|^2}+\frac{1}{1+\|S_k\|^2}\Big]\lesssim\frac{1}{\varepsilon_k^{1/4}k}.
\end{aligned}$$

Thus, at this point we have shown that

$$\big|\mathbb{E}[Z_0Z_k\varphi_3\psi_3]-\Delta_{3,3}\big|=o\Big(\frac{1}{k}\Big). \tag{5.17}$$

Now, define

$$\begin{aligned}
\widetilde{H}(z_1,z_2,z_3):=\mathbb{P}\big[0\notin\mathcal{R}[1,\varepsilon_k^{3/4}],\widetilde{\mathcal{R}}[1,\sqrt{\varepsilon_k}]\cap\mathcal{R}[-\varepsilon_k^{3/4},\varepsilon_k^{3/4}]=\varnothing,\\
S_{\varepsilon_k^{3/4}}=z_1,S_{-\varepsilon_k^{3/4}}=z_3,\widetilde{S}_{\sqrt{\varepsilon_k}}=z_3\big],
\end{aligned}$$

and recall also the definition of $H(y_1,y_2)$ given in (5.3). One has

$$\Delta_{3,3}=\sum H(y_1,y_2)\widetilde{H}(z_1,z_2,z_3)p_{k-\varepsilon_k-\varepsilon_k^{3/4}}(x-y_1+y_2+z_3-z_2)p_{\varepsilon_k-\varepsilon_k^{3/4}}(u-z_1+z_2)\varphi_{x,u},$$

where the sum runs over all $x,u,y_1,y_2,z_1,z_2,z_3\in\mathbb{Z}^5$, and

$$\varphi_{x,u}:=\mathbb{P}[\overline{\tau}_1<\infty,\overline{\tau}_2<\infty\mid S_k=x,S_{k+\varepsilon_k}=x+u].$$

Note that the same argument as for (5.16) gives also

$$\varphi_{x,u}\lesssim\frac{1}{1+\|u\|}\Big(\frac{1}{1+\|x+u\|}+\frac{1}{1+\|x\|}\Big). \tag{5.18}$$

Using this, it is possible to see that in the expression of $\Delta_{3,3}$ given just above, one can restrict the sum to typical values of the parameters. Indeed, consider, for instance, the sum on atypically large values of x. More precisely, take χ_k, such that $\varepsilon_k\chi_k^{2+1/4}=k$, and note that by (5.18)

$$
\begin{aligned}
&\sum_{\substack{\|x\|^2\geq k\chi_k\\ u,y_1,y_2,z_1,z_2,z_3}} H(y_1,y_2)\widetilde{H}(z_1,z_2,z_3)p_{k-\varepsilon_k-\varepsilon_k^{3/4}}(x-y_1+y_2+z_3-z_2)\\
&\qquad \times p_{\varepsilon_k-\varepsilon_k^{3/4}}(u-z_1+z_2)\varphi_{x,u}\\
&\qquad \leq \mathbb{P}\big[\|S_k-S^1_{\varepsilon_k}\|\geq\sqrt{k\chi_k},\tau_1<\infty,\tau_2<\infty\big]\leq \mathbb{P}\big[\|S_k-S^1_{\varepsilon_k}\|\geq\sqrt{k\chi_k},\tau_1<\infty,\overline{\tau}_2<\infty\big]\\
&\qquad \lesssim \mathbb{E}\bigg[\frac{\mathbf{1}\{\|S_k-S^1_{\varepsilon_k}\|\geq\sqrt{k\chi_k}\}}{1+\|S_{k+\varepsilon_k}-S_k\|}\bigg(\frac{1}{1+\|S_k-S^1_{\varepsilon_k}\|}+\frac{1}{1+\|S_{k+\varepsilon_k}-S^1_{\varepsilon_k}\|}\bigg)\bigg]\\
&\qquad \lesssim \frac{1}{\chi_k^{5/4}\sqrt{k\varepsilon_k}},
\end{aligned}
$$

where the last equality follows by applying Cauchy–Schwarz inequality and (2.5). The other cases are entirely similar. Thus, $\Delta_{3,3}$ is well approximated by the sums on typical values of the parameters (similarly as for Δ_0, for instance), and then we can deduce with Theorem 2.1 and (5.18) that

$$
\Delta_{3,3}=\rho^2\cdot\mathbb{P}[\overline{\tau}_1<\infty,\overline{\tau}_2<\infty]+o\bigg(\frac{1}{k}\bigg).
$$

Together with (5.17) and (5.14), this proves (5.8).

Step 2. For a (possibly random) time T, set

$$
\overline{\tau}_1\circ T:=\inf\{n\geq T\vee\varepsilon_k:S_n\in\mathcal{R}^1_\infty\},\qquad \overline{\tau}_2\circ T:=\inf\{n\geq T\vee\varepsilon_k:S_n\in\big(S_k+\mathcal{R}^2_\infty\big)\}.
$$

Observe that

$$
\mathbb{P}[\overline{\tau}_1\leq\overline{\tau}_2<\infty]=\mathbb{P}[\overline{\tau}_1\leq\overline{\tau}_2\circ\overline{\tau}_1<\infty]-\mathbb{P}[\overline{\tau}_2\leq\overline{\tau}_1\circ\overline{\tau}_2\leq\overline{\tau}_2\circ\overline{\tau}_1\circ\overline{\tau}_2<\infty], \tag{5.19}
$$

and, symmetrically,

$$
\mathbb{P}[\overline{\tau}_2\leq\overline{\tau}_1<\infty]=\mathbb{P}[\overline{\tau}_2\leq\overline{\tau}_1\circ\overline{\tau}_2<\infty]-\mathbb{P}[\overline{\tau}_1\leq\overline{\tau}_2\circ\overline{\tau}_1\leq\overline{\tau}_1\circ\overline{\tau}_2\circ\overline{\tau}_1<\infty]. \tag{5.20}
$$

Our aim here is to show that the two error terms appearing in (5.19) and (5.20) are negligible. Applying repeatedly (2.11) gives

$$
\begin{aligned}
E_1&:=\mathbb{P}[\overline{\tau}_1\leq\overline{\tau}_2\circ\overline{\tau}_1\leq\overline{\tau}_1\circ\overline{\tau}_2\circ\overline{\tau}_1<\infty]\\
&\lesssim\sum_{j\geq0}\sum_{\ell\geq0}\sum_{m\geq0}\mathbb{E}\big[G(S^1_j-S_k-S^2_\ell)G(S_k+S^2_\ell-S^1_m)G(S^1_m-S_{k+\varepsilon_k})\big]\\
&\overset{(2.10)}{\lesssim}\sum_{j\geq0}\sum_{\ell\geq0}\sum_{m\geq0}\mathbb{E}\big[G(S^1_j-S_k-S^2_\ell)G(S_k+S^2_\ell-S^1_m)G(S^1_m-S_k)\big]\\
&\lesssim\sum_{j\geq0}\sum_{m\geq0}G(z)\mathbb{E}\big[G(S^1_j-S_k-z)G(S_k+z-S^1_m)G(S^1_m-S_k)\big].
\end{aligned}
$$

Note also that, by using Lemma 2.2 and (2.7), we get

$$
\sum_{z\in\mathbb{Z}^5}G(z-x)G(z-y)G(z)\lesssim\frac{1}{1+\|x\|^3}\bigg(\frac{1}{1+\|y\|}+\frac{1}{1+\|y-x\|}\bigg).
$$

Thus, distinguishing also the two cases $j \le m$ and $m \le j$, we obtain

$$\begin{aligned}
E_1 &\lesssim \sum_{j\ge 0}\sum_{m\ge 0} \mathbb{E}\Big[\frac{G(S_m^1 - S_k)}{1+\|S_j^1 - S_k\|^3}\Big(\frac{1}{1+\|S_m^1 - S_k\|} + \frac{1}{1+\|S_m^1 - S_j^1\|}\Big)\Big] \\
&\lesssim \sum_{j\ge 0}\sum_{z\in\mathbb{Z}^5} G(z)\Big\{\mathbb{E}\Big[\frac{G(z+S_j^1 - S_k)}{1+\|S_j^1 - S_k\|^3}\Big(\frac{1}{1+\|z+S_j^1 - S_k\|} + \frac{1}{1+\|z\|}\Big)\Big] \\
&\qquad + \mathbb{E}\Big[\frac{G(S_j^1 - S_k)}{1+\|z+S_j^1 - S_k\|^3}\Big(\frac{1}{1+\|S_j^1 - S_k\|} + \frac{1}{1+\|z\|}\Big)\Big]\Big\} \\
&\lesssim \sum_{j\ge 0}\mathbb{E}\Big[\frac{1}{1+\|S_j^1 - S_k\|^5}\Big] \lesssim \mathbb{E}\Big[\frac{\log(1+\|S_k\|)}{1+\|S_k\|^3}\Big] \lesssim \frac{\log k}{k^{3/2}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{P}[\overline{\tau}_2 \le \overline{\tau}_1 \circ \overline{\tau}_2 \le \overline{\tau}_2 \circ \overline{\tau}_1 \circ \overline{\tau}_2 < \infty] \\
&\quad\lesssim \sum_{j\ge 0}\sum_{\ell\ge 0}\sum_{m\ge 0} \mathbb{E}\big[G(S_j^2 + S_k - S_\ell^1)G(S_\ell^1 - S_k - S_m^2)G(S_m^2 + S_k - S_{k+\varepsilon_k})\big] \\
&\quad\overset{(2.9),(2.10)}{\lesssim} \frac{1}{\sqrt{\varepsilon_k}} \sum_{j\ge 0}\sum_{\ell\ge 0}\sum_{m\ge 0} \mathbb{E}\Big[\frac{G(S_j^2 + S_k - S_\ell^1)G(S_\ell^1 - S_k - S_m^2)}{1+\|S_m^2\|^2}\Big] \\
&\quad\lesssim \frac{1}{\sqrt{\varepsilon_k}} \sum_{j\ge 0}\sum_{m\ge 0} \mathbb{E}\Big[\frac{1}{(1+\|S_m^2\|^2)(1+\|S_j^2 + S_k\|^3)}\Big(\frac{1}{1+\|S_m^2 + S_k\|} + \frac{1}{1+\|S_m^2 - S_j^2\|}\Big)\Big] \\
&\quad\lesssim \frac{1}{\sqrt{\varepsilon_k}} \sum_{j\ge 0} \mathbb{E}\Big[\frac{1}{(1+\|S_j^2\|)(1+\|S_j^2 + S_k\|^3)} + \frac{1}{(1+\|S_j^2\|^2)(1+\|S_j^2 + S_k\|^2)}\Big] \\
&\quad\lesssim \frac{1}{\sqrt{\varepsilon_k}} \cdot \mathbb{E}\Big[\frac{\log(1+\|S_k\|)}{1+\|S_k\|^2}\Big] \lesssim \frac{\log k}{k\sqrt{\varepsilon_k}}.
\end{aligned}$$

Step 3. We now come to the estimate of the two main terms in (5.19) and (5.20). In fact, it will be convenient to replace $\overline{\tau}_1$ in the first one by

$$\widehat{\tau}_1 := \inf\{n \ge k : S_n \in \mathcal{R}_\infty^1\}.$$

The error made by doing this is bounded as follows: by shifting the origin to S_k and using symmetry of the step distribution, we can write

$$\begin{aligned}
&\big|\mathbb{P}[\overline{\tau}_1 \le \overline{\tau}_2 \circ \overline{\tau}_1 < \infty] - \mathbb{P}[\widehat{\tau}_1 \le \overline{\tau}_2 \circ \widehat{\tau}_1 < \infty]\big| \\
&\quad \le \mathbb{P}[\mathcal{R}_\infty^1 \cap \mathcal{R}[k, k+\varepsilon_k] \ne \varnothing, \overline{\tau}_2 < \infty] \\
&\quad \overset{(2.6)}{\le} \mathbb{E}\Big[\Big(\sum_{i=0}^{\varepsilon_k} G(S_i - \widetilde{S}_k)\Big)\Big(\sum_{j=\varepsilon_k}^{\infty} G(S_j)\Big)\Big] \\
&\quad = \mathbb{E}\Big[\Big(\sum_{i=0}^{\varepsilon_k} G(S_i - \widetilde{S}_k)\Big)\Big(\sum_{z\in\mathbb{Z}^5} G(z)G(z+S_{\varepsilon_k})\Big)\Big] \\
&\quad \overset{\text{Lemma 2.2}}{\lesssim} \sum_{i=0}^{\varepsilon_k} \mathbb{E}\Big[\frac{G(S_i - \widetilde{S}_k)}{1+\|S_{\varepsilon_k}\|}\Big] \overset{(2.9)}{\lesssim} \frac{\varepsilon_k}{k^{3/2}} \cdot \mathbb{E}\Big[\frac{1}{1+\|S_{\varepsilon_k}\|}\Big] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.
\end{aligned}$$

Moreover, using Theorem C, the Markov property and symmetry of the step distribution, we get, for some constant $c > 0$,

$$\begin{aligned}
&\mathbb{P}[\widehat{\tau}_1 \le \overline{\tau}_2 \circ \widehat{\tau}_1 < \infty] \\
&\quad = c\mathbb{E}\left[\frac{\mathbb{1}\{\widehat{\tau}_1 < \infty\}}{1 + \mathcal{J}(S_{\widehat{\tau}_1} - S_k)}\right] + o\left(\frac{1}{k}\right) \\
&\quad = c\mathbb{E}\left[\frac{\mathbb{1}\{\widehat{\tau}_1 < \infty\}}{1 + \mathcal{J}(S_{\widehat{\tau}_1})}\right] + o\left(\frac{1}{k}\right) \\
&\quad = c\sum_{x \in \mathbb{Z}^5} p_k(x)\mathbb{E}_{0,x}\big[F(S_\tau)\mathbb{1}\{\tau < \infty\}\big] + o\left(\frac{1}{k}\right),
\end{aligned}$$

with τ the hitting time of two independent walks starting, respectively, from the origin and from x, and $F(z) := 1/(1 + \mathcal{J}(z))$. Note that the bound $o(1/k)$ on the error term in the last display comes from the fact that

$$\mathbb{E}\left[\frac{\mathbb{1}\{\widehat{\tau}_1 < \infty\}}{1 + \mathcal{J}(S_{\widehat{\tau}_1})}\right] \overset{(2.11)}{\lesssim} \sum_{j \ge 0} \mathbb{E}\left[\frac{G(\widetilde{S}_j - S_k)}{1 + \|\widetilde{S}_j\|}\right] \lesssim \sum_{z \in \mathbb{Z}^5} \mathbb{E}\left[\frac{G(z)G(z - S_k)}{1 + \|z\|}\right] \lesssim \frac{1}{k}.$$

Then, by applying Theorem 4.1 we get

$$\mathbb{P}[\widehat{\tau}_1 \le \overline{\tau}_2 \circ \widehat{\tau}_1 < \infty] = c_0 \sum_{x \in \mathbb{Z}^5} p_k(x) \sum_{z \in \mathbb{Z}^5} \frac{G(z)G(z - x)}{1 + \mathcal{J}(z)} + o\left(\frac{1}{k}\right), \tag{5.21}$$

for some constant $c_0 > 0$. Likewise, by Theorem 4.1 one has, for some constant $\nu \in (0, 1)$,

$$\mathbb{P}[\overline{\tau}_2 \le \overline{\tau}_1 \circ \overline{\tau}_2 < \infty] = c\mathbb{E}\left[\frac{\mathbb{1}\{\overline{\tau}_2 < \infty\}}{1 + \mathcal{J}(S_{\overline{\tau}_2})}\right] + \mathcal{O}\left(\mathbb{E}\left[\frac{\mathbb{1}\{\overline{\tau}_2 < \infty\}}{1 + \mathcal{J}(S_{\overline{\tau}_2})^{1+\nu}}\right]\right).$$

Furthermore,

$$\begin{aligned}
&\mathbb{E}\left[\frac{\mathbb{1}\{\overline{\tau}_2 < \infty\}}{1 + \mathcal{J}(S_{\overline{\tau}_2})^{1+\nu}}\right] \\
&\quad \lesssim \sum_{j \ge 0} \mathbb{E}\left[\frac{G(S_j^2 + S_k - S_{k+\varepsilon_k})}{1 + \|S_j^2 + S_k\|^{1+\nu}}\right] \\
&\quad \overset{(2.9),(2.10)}{\lesssim} \frac{1}{\sqrt{\varepsilon_k}} \sum_{j \ge 0} \mathbb{E}\left[\frac{1}{(1 + \|S_j^2\|^2)(1 + \|S_j^2 + S_k\|^{1+\nu})}\right] \\
&\quad \lesssim \frac{1}{\sqrt{\varepsilon_k}} \mathbb{E}\left[\frac{\log(1 + \|S_k\|)}{1 + \|S_k\|^{1+\nu}}\right] \lesssim \frac{\log k}{k^{(1+\nu)/2}\sqrt{\varepsilon_k}}.
\end{aligned}$$

Therefore, taking $\varepsilon_k \ge k^{1-\nu/2}$, we get

$$\begin{aligned}
\mathbb{P}[\overline{\tau}_2 \le \overline{\tau}_1 \circ \overline{\tau}_2 < \infty] &= c\mathbb{E}\left[\frac{\mathbb{1}\{\overline{\tau}_2 < \infty\}}{1 + \mathcal{J}(S_{\overline{\tau}_2})}\right] + o\left(\frac{1}{k}\right) \\
&= c\sum_{u \in \mathbb{Z}^5} p_{\varepsilon_k}(u)\mathbb{E}_{0,u}\left[\frac{\mathbb{1}\{\tau < \infty\}}{1 + \mathcal{J}(S_\tau - S_k)}\right] + o\left(\frac{1}{k}\right) \qquad (5.22) \\
&= c\sum_{u \in \mathbb{Z}^5} p_{\varepsilon_k}(u)\mathbb{E}_{0,u}\big[\widetilde{F}(S_\tau)\mathbb{1}\{\tau < \infty\}\big] + o\left(\frac{1}{k}\right),
\end{aligned}$$

with τ the hitting time of two independent walks starting, respectively, from the origin, from u and

$$\widetilde{F}(z) := \mathbb{E}\Big[\frac{1}{1+\mathcal{J}(z-S_k)}\Big].$$

We claim that this function $\widetilde{F}$ satisfies (4.1), for some constant $C_{\widetilde{F}}$ which is independent of k. Indeed, first notice that

$$\widetilde{F}(z) \asymp \frac{1}{1+\|z\|+\sqrt{k}} \quad \text{and} \quad \mathbb{E}\Big[\frac{1}{1+\mathcal{J}(z-S_k)^2}\Big] \asymp \frac{1}{1+\|z\|^2+k},$$

which can be seen by using Theorem 2.1. Moreover, by triangle inequality and Cauchy–Schwarz,

$$\begin{aligned}|\widetilde{F}(y)-\widetilde{F}(z)| &\lesssim \mathbb{E}\Big[\frac{\|y-z\|}{(1+\|y-S_k\|)(1+\|z-S_k\|)}\Big]\\ &\lesssim \|y-z\|\mathbb{E}\Big[\frac{1}{1+\|y-S_k\|^2}\Big]^{\frac{1}{2}}\mathbb{E}\Big[\frac{1}{1+\|z-S_k\|^2}\Big]^{\frac{1}{2}}\\ &\lesssim \frac{\|y-z\|}{(1+\|y\|+\sqrt{k})(1+\|z\|+\sqrt{k})} \lesssim \frac{\|y-z\|}{1+\|y\|}\cdot \widetilde{F}(z),\end{aligned}$$

which is the desired condition (4.1). Therefore, coming back to (5.22) and applying Theorem 4.1 once more gives

$$\begin{aligned}\mathbb{P}[\overline{\tau}_2 \le \overline{\tau}_1 \circ \overline{\tau}_2 < \infty] &= c_0 \sum_{u\in\mathbb{Z}^5} p_{\varepsilon_k}(u) \sum_{z\in\mathbb{Z}^5} G(z)G(z-u)\widetilde{F}(z) + o\Big(\frac{1}{k}\Big)\\ &= c_0 \sum_{u\in\mathbb{Z}^5}\sum_{x\in\mathbb{Z}^5} p_{\varepsilon_k}(u)p_k(x) \sum_{z\in\mathbb{Z}^5} \frac{G(z)G(z-u)}{1+\mathcal{J}(z-x)} + o\Big(\frac{1}{k}\Big).\end{aligned}\tag{5.23}$$

Similarly, one has

$$\begin{aligned}\mathbb{P}[\overline{\tau}_1<\infty]\cdot\mathbb{P}[\overline{\tau}_2<\infty] &= \mathbb{P}[\widehat{\tau}_1<\infty]\cdot\mathbb{P}[\overline{\tau}_2<\infty] + \mathcal{O}\Big(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\Big)\\ &= c_0 \sum_{u\in\mathbb{Z}^5}\sum_{x\in\mathbb{Z}^5} p_{\varepsilon_k}(u)p_k(x) \sum_{z\in\mathbb{Z}^5} \frac{G(z)G(z-u)}{1+\mathcal{J}(x)} + o\Big(\frac{1}{k}\Big).\end{aligned}\tag{5.24}$$

Note, in particular, that the constant c_0 that appears here is the same as in (5.21) and (5.23).

Step 4. We claim now that when one takes the difference between the two expressions in (5.23) and (5.24), one can remove the parameter u from the factor $G(z-u)$ (and then absorb the sum over u). Indeed, note that, for any z with $\mathcal{J}(z) \le \mathcal{J}(x)/2$, one has

$$\Big|\frac{1}{1+\mathcal{J}(z+x)} + \frac{1}{1+\mathcal{J}(z-x)} - \frac{2}{1+\mathcal{J}(x)}\Big| \lesssim \frac{\|z\|^2}{1+\|x\|^3}.$$

It follows that, for any $\chi_k \ge 2$,

$$\begin{aligned}&\sum_{\substack{u,x\in\mathbb{Z}^5\\ \mathcal{J}(z)\le\frac{\mathcal{J}(x)}{\chi_k}}} p_{\varepsilon_k}(u)p_k(x)G(z)G(z-u)\Big|\frac{1}{1+\mathcal{J}(z-x)} + \frac{1}{1+\mathcal{J}(z+x)} - \frac{2}{1+\mathcal{J}(x)}\Big|\\ &\quad\lesssim \sum_{x\in\mathbb{Z}^5} \frac{p_k(x)}{1+\|x\|^3} \sum_{\mathcal{J}(z)\le\mathcal{J}(x)/\chi_k} \frac{\mathbb{E}[G(z-S_{\varepsilon_k})]}{1+\|z\|} \overset{(2.10)}{\lesssim} \frac{1}{k\chi_k}.\end{aligned}$$

In the same way, for any z with $\mathcal{J}(z) \geq 2\mathcal{J}(u)$, one has

$$\big|G(z-u) - G(z)\big| \lesssim \frac{\|u\|}{1+\|z\|^4},$$

$$\left|\frac{1}{1+\mathcal{J}(z-x)} - \frac{1}{1+\mathcal{J}(x)}\right| \lesssim \frac{\|z\|}{(1+\|x\|)(1+\|z-x\|)}.$$

Therefore, for any $\chi_k \geq 2$,

$$\sum_{\substack{u,x\in\mathbb{Z}^5 \\ \mathcal{J}(z)\geq(\mathcal{J}(u)\chi_k)\vee\frac{\mathcal{J}(x)}{\chi_k}}} p_{\varepsilon_k}(u)p_k(x)G(z)\big|G(z-u)-G(z)\big|\left|\frac{1}{1+\mathcal{J}(z-x)} - \frac{1}{1+\mathcal{J}(x)}\right|$$

$$\lesssim \sqrt{\varepsilon_k}\sum_{x\in\mathbb{Z}^5}\frac{p_k(x)}{1+\|x\|}\sum_{\mathcal{J}(z)\geq\mathcal{J}(x)/\chi_k}\frac{1}{\|z\|^6(1+\|z-x\|)} \overset{(2.10)}{\lesssim} \frac{\chi_k^2\sqrt{\varepsilon_k}}{k^{3/2}}.$$

On the other hand, by taking $\chi_k = (k/\varepsilon_k)^{1/6}$ we get, using (2.3) and (2.5),

$$\sum_{\substack{x,z\in\mathbb{Z}^5 \\ \mathcal{J}(u)\geq\sqrt{\varepsilon_k}\chi_k}} p_{\varepsilon_k}(u)p_k(x)G(z)G(z-u)\left(\frac{1}{1+\mathcal{J}(z-x)} + \frac{1}{1+\mathcal{J}(x)}\right) \lesssim \frac{1}{\chi_k^5\sqrt{k\varepsilon_k}} = o\left(\frac{1}{k}\right),$$

$$\sum_{\substack{u,z\in\mathbb{Z}^5 \\ \mathcal{J}(x)\leq\sqrt{k}/\chi_k}} p_{\varepsilon_k}(u)p_k(x)G(z)G(z-u)\left(\frac{1}{1+\mathcal{J}(z-x)} + \frac{1}{1+\mathcal{J}(x)}\right) = o\left(\frac{1}{k}\right).$$

As a consequence, since $\mathcal{J}(u) \leq \sqrt{\varepsilon_k}\chi_k$ and $\mathcal{J}(x) \geq \sqrt{k}/\chi_k$ implies $\mathcal{J}(u) \leq \mathcal{J}(x)/\chi_k$, with our choice of χ_k we get as wanted (using also symmetry of the step distribution) that

$$\begin{aligned} &\mathbb{P}[\overline{\tau}_2 \leq \overline{\tau}_1 \circ \overline{\tau}_2 < \infty] - \mathbb{P}[\overline{\tau}_1 < \infty]\cdot\mathbb{P}[\overline{\tau}_2 < \infty] \\ &= c_0\sum_{x,z\in\mathbb{Z}^5} p_k(x)G(z)^2\left(\frac{1}{1+\mathcal{J}(z-x)} - \frac{1}{1+\mathcal{J}(x)}\right) + o\left(\frac{1}{k}\right) \qquad (5.25)\\ &= \frac{c_0}{2}\sum_{x,z\in\mathbb{Z}^5} p_k(x)G(z)^2\left(\frac{1}{1+\mathcal{J}(z-x)} + \frac{1}{1+\mathcal{J}(z+x)} - \frac{2}{1+\mathcal{J}(x)}\right) + o\left(\frac{1}{k}\right). \end{aligned}$$

Step 5. The previous steps show that

$$\operatorname{Cov}\big(\{\overline{\tau}_1 < \infty\}, \{\overline{\tau}_2 < \infty\}\big) = c_0\sum_{x,z\in\mathbb{Z}^5} p_k(x)\left(\frac{G(z)G(z-x)}{1+\mathcal{J}(z)} + \frac{G(z)^2}{1+\mathcal{J}(z-x)} - \frac{G(z)^2}{1+\mathcal{J}(x)}\right).$$

Now, by approximating the series with an integral (recall (4.3)) and doing a change of variables, we get with $u := x/\mathcal{J}(x)$ and $v := \Lambda^{-1}u$, and, for some constant $c > 0$ (that might change from line to line),

$$\begin{aligned} &\sum_{z\in\mathbb{Z}^5}\left(\frac{G(z)G(z-x)}{1+\mathcal{J}(z)} + \frac{G(z)^2}{1+\mathcal{J}(z-x)} - \frac{G(z)^2}{1+\mathcal{J}(x)}\right) \\ &\sim c\int_{\mathbb{R}^5}\left\{\frac{1}{\mathcal{J}(z)^4\cdot\mathcal{J}(z-x)^3} + \frac{1}{\mathcal{J}(z)^6}\left(\frac{1}{\mathcal{J}(z-x)} - \frac{1}{\mathcal{J}(x)}\right)\right\}dz \qquad (5.26)\\ &= \frac{c}{\mathcal{J}(x)^2}\int_{\mathbb{R}^5}\left\{\frac{1}{\mathcal{J}(z)^4\cdot\mathcal{J}(z-u)^3} + \frac{1}{\mathcal{J}(z)^6}\left(\frac{1}{\mathcal{J}(z-u)} - 1\right)\right\}dz \\ &= \frac{c}{\mathcal{J}(x)^2}\int_{\mathbb{R}^5}\left\{\frac{1}{\|z\|^4\cdot\|z-v\|^3} + \frac{1}{\|z\|^6}\left(\frac{1}{\|z-v\|} - 1\right)\right\}dz. \end{aligned}$$

Note that the last integral is convergent and independent of v (and thus of x as well) by rotational invariance. Therefore, since $\sum_{x\in\mathbb{Z}^5} p_k(x)/\mathcal{J}(x)^2 \sim \sigma/k$ for some constant $\sigma > 0$ (for instance, by applying Theorem 2.1), it only remains to show that the integral above is positive. To see this, we use that the map $z \mapsto \|z\|^{-3}$ is harmonic outside the origin and, thus, satisfies the mean value property on $\mathbb{R}^5 \setminus \{0\}$. In particular, using also the rotational invariance, this shows (with $\mathcal{B}_1$ the unit Euclidean ball and $\partial\mathcal{B}_1$ the unit sphere)

$$\begin{aligned}\int_{\mathcal{B}_1^c} \frac{1}{\|z\|^4\cdot\|z-v\|^3}\,dz &= \frac{1}{|\partial\mathcal{B}_1|}\int_{\partial\mathcal{B}_1} dv \int_{\mathcal{B}_1^c} \frac{1}{\|z\|^4\cdot\|z-v\|^3}\,dz \\ &= \int_{\mathcal{B}_1^c} \frac{1}{\|z\|^7}\,dz = c_1\int_1^\infty \frac{1}{r^3}\,dr = \frac{c_1}{2},\end{aligned} \tag{5.27}$$

for some constant $c_1 > 0$. Likewise,

$$\int_{\mathcal{B}_1} \frac{1}{\|z\|^4\cdot\|z-v\|^3}\,dz = \frac{c_1}{|\partial\mathcal{B}_1|}\int_0^1 dr \int_{\partial\mathcal{B}_1} \frac{du}{\|ru-v\|^3} = c_1, \tag{5.28}$$

with the same constant c_1 as in the previous display. On the other hand,

$$\int_{\mathcal{B}_1^c} \frac{1}{\|z\|^6}\,dz = c_1\int_1^\infty \frac{1}{r^2}\,dr = c_1. \tag{5.29}$$

Furthermore, using again the rotational invariance,

$$\begin{aligned}&\int_{\mathcal{B}_1} \frac{1}{\|z\|^6}\left(\frac{1}{\|z-v\|}-1\right)dz \\ &\quad= \int_{\mathcal{B}_1} \frac{1}{\|z\|^6}\left(\frac{1}{2\|z-v\|}+\frac{1}{2\|z+v\|}-1\right)dz \\ &\quad= \frac{c_1}{|\partial\mathcal{B}_1|}\int_0^1 \frac{dr}{r^2}\int_{\partial\mathcal{B}_1}\left(\frac{1}{2\|v-ru\|}+\frac{1}{2\|v+ru\|}-1\right)du.\end{aligned} \tag{5.30}$$

Now, we claim that, for any $u, v \in \partial\mathcal{B}_1$ and any $r \in (0,1)$,

$$\frac{1}{2}\left(\frac{1}{\|v-ru\|}+\frac{1}{\|v+ru\|}\right) \geq \frac{1}{\sqrt{1+r^2}}. \tag{5.31}$$

Before we prove this claim, let us see how we can conclude the proof. It suffices to notice that if $f(s) = (1+s^2)^{-1/2}$, then $f'(s) \geq -s$ for all $s \in (0,1)$, and, thus,

$$\frac{1}{\sqrt{1+r^2}} - 1 = f(r) - f(0) \geq -\int_0^r s\,ds \geq -r^2/2. \tag{5.32}$$

Inserting this and (5.31) in (5.30) gives

$$\int_{\mathcal{B}_1} \frac{1}{\|z\|^6}\left(\frac{1}{\|z-v\|}-1\right)dz \geq -\frac{c_1}{2}.$$

Together with (5.27), (5.28) and (5.29), this shows that the integral in (5.26) is well positive. Thus, all that remains to do is proving the claim (5.31). Since the origin, v, $v+ru$, and $v-ru$ all lie in a common two-dimensional plane, one can always work in the complex plane and assume, for simplicity, that $v = 1$ and $u = e^{i\theta}$, for some $\theta \in [0, \pi/2]$. In this case, the claim is equivalent to showing that

$$\frac{1}{2}\left(\frac{1}{\sqrt{1+r^2+2r\cos\theta}}+\frac{1}{\sqrt{1+r^2-2r\cos\theta}}\right) \geq \frac{1}{\sqrt{1+r^2}},$$

which is easily obtained using that the left-hand side is a decreasing function of θ. This concludes the proof of Lemma 5.2.

REMARK 5.5. Note that the estimate of the covariance mentioned in the Introduction in case (ii), can now be done as well. Indeed, denoting by

$$\widehat{\tau}_2 := \inf\{n \geq k+1 : S_n \in S_k + \mathcal{R}_\infty^2\},$$

it only remains to show that

$$\big|\mathbb{P}[\widehat{\tau}_2 \leq k+\varepsilon_k, \overline{\tau}_1 < \infty] - \mathbb{P}[\widehat{\tau}_2 \leq k+\varepsilon_k] \cdot \mathbb{P}[\overline{\tau}_1 < \infty]\big| = o\Big(\frac{1}{k}\Big).$$

Using similar estimates as above, we get, with $\chi_k = (k/\varepsilon_k)^{4/5}$,

$$\begin{aligned}
&\big|\mathbb{P}[\widehat{\tau}_2 \leq k+\varepsilon_k, \overline{\tau}_1 < \infty] - \mathbb{P}[\widehat{\tau}_2 \leq k+\varepsilon_k] \cdot \mathbb{P}[\overline{\tau}_1 < \infty]\big| \\
&\overset{(2.5)}{=} \big|\mathbb{P}\big[\widehat{\tau}_2 \leq k+\varepsilon_k, \|S_{\widehat{\tau}_2} - S_k\| \leq \sqrt{\varepsilon_k\chi_k}, \overline{\tau}_1 < \infty\big] - \mathbb{P}[\widehat{\tau}_2 \leq k+\varepsilon_k]\mathbb{P}[\overline{\tau}_1 < \infty]\big| \\
&\quad + \mathcal{O}\Big(\frac{1}{\sqrt{k}\chi_k^{\frac{5}{2}}}\Big) \\
&= \sum_{\substack{x\in\mathbb{Z}^5 \\ \|y\|\leq\sqrt{\varepsilon_k\chi_k}}} \bigg|\frac{p_k(x+y)+p_k(x-y)}{2} - p_k(x)\bigg| \mathbb{P}[\widehat{\tau}_2 \leq k+\varepsilon_k, S_{\widehat{\tau}_2} - S_k = y]\varphi_x \\
&\quad + \mathcal{O}\Big(\frac{1}{\sqrt{k}\chi_k^{\frac{5}{2}}}\Big) \\
&\lesssim \frac{1}{k^{\frac{3}{2}}}\mathbb{E}\big[\|S_{\widehat{\tau}_2} - S_k\|^2 \mathbf{1}\{\|S_{\widehat{\tau}_2} - S_k\| \leq \sqrt{\varepsilon_k\chi_k}\}\big] + \frac{1}{\sqrt{k}\chi_k^{\frac{5}{2}}} \lesssim \frac{1}{\sqrt{k}\chi_k^{\frac{5}{2}}} + \frac{\sqrt{\varepsilon_k\chi_k}}{k^{\frac{3}{2}}},
\end{aligned}$$

using that by (2.12) and the Markov property, one has $\mathbb{P}[\|S_{\widehat{\tau}_2} - S_k\| \geq t] \lesssim \frac{1}{t}$.

5.3. *Proof of Lemma* 5.3. We consider only the case of $\mathrm{Cov}(Z_0\varphi_2, Z_k\psi_1)$, the other one being entirely similar. Define

$$\tau_1 := \inf\{n \geq 0 : S_n^1 \in \mathcal{R}[\varepsilon_k, k]\}, \qquad \tau_2 := \inf\{n \geq 0 : S_k + S_n^2 \in \mathcal{R}(-\infty, 0]\},$$

with S^1 and S^2 two independent walks, independent of S. The first step is to see that

$$\mathrm{Cov}(Z_0\varphi_3, Z_k\psi_2) = \rho^2 \cdot \mathrm{Cov}\big(\mathbf{1}\{\tau_1 < \infty\}, \mathbf{1}\{\tau_2 < \infty\}\big) + o\Big(\frac{1}{k}\Big),$$

with ρ as in (5.2). Since the proof of this fact has exactly the same flavor as in the two previous lemmas, we omit the details and directly move to the next step.

Let $\eta \in (0, 1/2)$ be some fixed constant (which will be sent to zero later). Notice first that

$$\begin{aligned}
&\mathbb{P}\big[S_{\tau_1}^1 \in \mathcal{R}\big[(1-\eta)k, k\big], \tau_2 < \infty\big] \\
&\overset{(2.6),(2.12)}{\lesssim} \sum_{i=\lfloor(1-\eta)k\rfloor}^{k} \mathbb{E}\Big[\frac{G(S_i)}{1+\|S_k\|}\Big] \\
&\overset{(2.9)}{\lesssim} \sum_{i=\lfloor(1-\eta)k\rfloor}^{k} \frac{\mathbb{E}[G(S_i)]}{1+\sqrt{k-i}} \overset{(2.9)}{\lesssim} \frac{\sqrt{\eta}}{k}.
\end{aligned} \tag{5.33}$$

Next, fix another constant $\delta \in (0, 1/4)$ (which will be soon chosen small enough). Then, let $N := \lfloor(1-\eta)k/\varepsilon_k^{1-\delta}\rfloor$, and, for $i = 1, \dots, N$, define

$$\tau_1^i := \inf\{n \geq 0 : S_n^1 \in \mathcal{R}[k_i, k_{i+1}]\}, \quad \text{with } k_i := \varepsilon_k + i\lfloor\varepsilon_k^{1-\delta}\rfloor.$$

We claim that with sufficiently high probability, at most one of these hitting times is finite. Indeed, for $i \leq N$, set $I_i := \{k_i, \dots, k_{i+1}\}$, and notice that

$$\begin{aligned}
&\sum_{1 \leq i < j \leq N} \mathbb{P}\bigl[\tau_1^i < \infty, \tau_1^j < \infty, \tau_2 < \infty\bigr] \\
&\quad \leq \sum_{1 \leq i < j \leq N} \bigl(\mathbb{P}\bigl[\tau_1^i \leq \tau_1^j < \infty, \tau_2 < \infty\bigr] + \mathbb{P}\bigl[\tau_1^j \leq \tau_1^i < \infty, \tau_2 < \infty\bigr]\bigr) \\
&\quad \overset{(2.11),(2.12)}{\lesssim} \sum_{\substack{i=1,\dots,N, j \neq i \\ \ell \in I_i, m \in I_j}} \mathbb{E}\biggl[\frac{G(S_\ell - S_m)G(S_m)}{1 + \|S_k\|}\biggr] \lesssim \frac{1}{\sqrt{k}} \sum_{\substack{i=1,\dots,N, j \neq i \\ \ell \in I_i, m \in I_j}} \mathbb{E}\bigl[G(S_\ell - S_m)G(S_m)\bigr] \\
&\quad \overset{(2.9),(2.10)}{\lesssim} \frac{1}{\sqrt{k}} \sum_{\substack{i=1,\dots,N, j \neq i \\ \ell \in I_i, m \in I_j}} \frac{1}{(1 + |m - \ell|^{3/2})(m \wedge \ell)^{3/2}} \lesssim \frac{N \varepsilon_k^{(1-\delta)/2}}{\varepsilon_k^{3/2}\sqrt{k}} = o\biggl(\frac{1}{k}\biggr),
\end{aligned}$$

where the last equality follows by assuming $\varepsilon_k \geq k^{1-c}$, with $c > 0$ small enough. Therefore, as claimed

$$\mathbb{P}[\tau_1 < \infty, \tau_2 < \infty] = \sum_{i=1}^{N} \mathbb{P}\bigl[\tau_1^i < \infty, \tau_2 < \infty\bigr] + o\biggl(\frac{1}{k}\biggr),$$

and one can show as well that

$$\mathbb{P}[\tau_1 < \infty] \cdot \mathbb{P}[\tau_2 < \infty] = \sum_{i=1}^{N-2} \mathbb{P}\bigl[\tau_1^i < \infty\bigr] \cdot \mathbb{P}[\tau_2 < \infty] + o\biggl(\frac{1}{k}\biggr).$$

Next, observe that, for any $i \leq N$, using Hölder's inequality at the third line,

$$\begin{aligned}
&\mathbb{P}\bigl[\tau_1^i < \infty, \tau_2 < \infty, \|S_{k_{i+1}} - S_{k_i}\|^2 \geq \varepsilon_k^{1-\delta/2}\bigr] \\
&\quad \overset{(2.6),(2.12)}{\lesssim} \sum_{j=k_i}^{k_{i+1}} \mathbb{E}\biggl[\frac{G(S_j)\mathbf{1}\{\|S_{k_{i+1}} - S_{k_i}\|^2 \geq \varepsilon_k^{1-\delta/2}\}}{1 + \|S_k\|}\biggr] \\
&\quad \overset{(2.9)}{\lesssim} \frac{1}{\sqrt{k}} \sum_{j=k_i}^{k_{i+1}} \mathbb{E}\bigl[G(S_j)\mathbf{1}\bigl\{\|S_{k_{i+1}} - S_{k_i}\|^2 \geq \varepsilon_k^{1-\delta/2}\bigr\}\bigr] \\
&\quad \lesssim \frac{1}{\sqrt{k}} \Biggl(\sum_{j=k_i}^{k_{i+1}} \mathbb{E}\biggl[\frac{1}{1 + \|S_j\|^4}\biggr]^{3/4}\Biggr) \cdot \mathbb{P}\bigl[\|S_{k_{i+1}} - S_{k_i}\|^2 \geq \varepsilon_k^{1-\delta/2}\bigr]^{1/4} \\
&\quad \overset{(2.5)}{\lesssim} \frac{\varepsilon_k^{1-\delta}}{k_i^{3/2}\sqrt{k}} \cdot \frac{1}{\varepsilon_k^{5\delta/16}} = o\biggl(\frac{1}{Nk}\biggr),
\end{aligned}$$

by choosing again $\varepsilon_k \geq k^{1-c}$, with c small enough. Similarly, one has, using Cauchy–Schwarz,

$$\begin{aligned}
&\mathbb{P}\bigl[\tau_1^i < \infty, \tau_2 < \infty, \|S_k - S_{k_{i+1}}\|^2 \geq k\varepsilon_k^{\delta/2}\bigr] \\
&\quad \lesssim \sum_{j=k_i}^{k_{i+1}} \mathbb{E}\biggl[\frac{G(S_j)\mathbf{1}\{\|S_k - S_{k_{i+1}}\|^2 \geq k\varepsilon_k^{\delta/2}\}}{1 + \|S_k\|}\biggr] \\
&\quad \lesssim \frac{1}{\varepsilon_k^{5\delta/8}} \sum_{j=k_i}^{k_{i+1}} \mathbb{E}\biggl[G(S_j)\mathbb{E}\biggl[\frac{1}{1 + \|S_k\|^2} \biggm| S_j\biggr]^{1/2}\biggr] \lesssim \frac{\varepsilon_k^{1-\delta}}{k_i^{3/2}\sqrt{k}} \cdot \frac{1}{\varepsilon_k^{5\delta/8}} = o\biggl(\frac{1}{Nk}\biggr).
\end{aligned}$$

As a consequence, using also Theorem 2.1, one has for $i \le N$ and with $\ell := k_{i+1} - k_i$,

$$\begin{aligned}
&\mathbb{P}[\tau_1^i < \infty, \tau_2 < \infty] \\
&\quad = \sum_{x \in \mathbb{Z}^5} \sum_{\substack{\|z\|^2 \le k\varepsilon_k^{\delta/2} \\ \|y\|^2 \le \varepsilon_k^{1-\delta/2}}} p_{k_i}(x) \mathbb{P}_{0,x}[\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}[0,\ell] \ne \varnothing, \widetilde{S}_\ell = y] p_{k-k_{i+1}}(z-y) \varphi_{x+z} + o\Big(\frac{1}{Nk}\Big) \\
&\quad = \sum_{x \in \mathbb{Z}^5} \sum_{\substack{\|z\|^2 \le k\varepsilon_k^{\delta/2} \\ \|y\|^2 \le \varepsilon_k^{1-\delta/2}}} p_{k_i}(x) \mathbb{P}_{0,x}[\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}[0,\ell] \ne \varnothing, \widetilde{S}_\ell = y] p_{k-k_i}(z) \varphi_{x+z} + o\Big(\frac{1}{Nk}\Big) \\
&\quad = \sum_{x,z \in \mathbb{Z}^5} p_{k_i}(x) \mathbb{P}_{0,x}[\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}[0,\ell] \ne \varnothing] p_{k-k_i}(z) \varphi_{x+z} + o\Big(\frac{1}{Nk}\Big).
\end{aligned}$$

Moreover, Theorem 4.1 yields for any nonzero $x \in \mathbb{Z}^5$ and some $\nu > 0$,

$$\mathbb{P}_{0,x}[\mathcal{R}_\infty \cap \widetilde{\mathcal{R}}[0,\ell] \ne \varnothing] = \frac{\gamma_5}{\kappa} \cdot \mathbb{E}\Big[\sum_{j=0}^{\ell} G(x + \widetilde{S}_j)\Big] + \mathcal{O}\Big(\frac{\log(1+\|x\|)}{\|x\|(\|x\| \wedge \ell)^\nu}\Big). \tag{5.34}$$

Note also that, for any $\varepsilon \in [0,1]$,

$$\sum_{x,z \in \mathbb{Z}^5} \frac{p_{k_i}(x)}{1+\|x\|^{1+\varepsilon}} p_{k-k_i}(z) \varphi_{x+z} = \mathbb{E}\Big[\frac{1}{(1+\|S_{k_i}\|^{1+\varepsilon})(1+\|S_k\|)}\Big] \lesssim \frac{1}{\sqrt{k_i}^{1+\varepsilon}\sqrt{k}},$$

and, thus,

$$\sum_{i=1}^{N} \sum_{x,z \in \mathbb{Z}^5} \frac{p_{k_i}(x)}{1+\|x\|^{1+\varepsilon}} p_{k-k_i}(z) \varphi_{x+z} = \mathcal{O}\Big(\frac{1}{\ell k^\varepsilon}\Big).$$

In particular, the error term in (5.34) can be neglected, as we take for instance $\delta = \nu/2$ and $\varepsilon_k \ge k^{1-c}$, with c small enough. It amounts now to estimate the other term in (5.34). By (2.3), for any $x \in \mathbb{Z}^5$ and $j \ge 0$,

$$\mathbb{E}[G(x+S_j)] = G_j(x) = G(x) - \mathcal{O}\Big(\frac{j}{1+\|x\|^d}\Big).$$

As will become clear, the error term can be neglected here. Furthermore, similar computations as above show that, for any $j \in \{k_i, \dots, k_{i+1}\}$,

$$\sum_{x,z \in \mathbb{Z}^5} p_{k_i}(x) G(x) p_{k-k_i}(z) \varphi_{x+z} = \sum_{x,z \in \mathbb{Z}^5} p_j(x) G(x) p_{k-j}(z) \varphi_{x+z} + o\Big(\frac{1}{Nk}\Big).$$

Altogether and applying once more Theorem 4.1, this gives, for some $c_0 > 0$,

$$\begin{aligned}
\sum_{i=1}^{N} \mathbb{P}[\tau_1^i < \infty, \tau_2 < \infty] &= \sum_{j=\varepsilon_k}^{(1-\eta)k} \mathbb{E}[G(S_j) \varphi_{S_k}] + o\Big(\frac{1}{k}\Big) \\
&= c_0 \sum_{j=\varepsilon_k}^{\lfloor (1-\eta)k \rfloor} \mathbb{E}\Big[\frac{G(S_j)}{1+\mathcal{J}(S_k)}\Big] + o\Big(\frac{1}{k}\Big).
\end{aligned} \tag{5.35}$$

We treat the first terms of the sum separately. Concerning the other ones, notice that, by (4.3) and Donsker's invariance principle, one has

$$\begin{aligned}\sum_{j=\lfloor \eta k\rfloor}^{\lfloor (1-\eta)k\rfloor} \mathbb{E}\Big[\frac{G(S_j)}{1+\mathcal{J}(S_k)}\Big] &= \frac{1}{k}\int_\eta^{1-\eta} \mathbb{E}\Big[\frac{G(\Lambda\beta_s)}{\mathcal{J}(\Lambda\beta_1)}\Big]\,ds + o\Big(\frac{1}{k}\Big)\\ &= \frac{c_5}{k}\int_\eta^{1-\eta} \mathbb{E}\Big[\frac{1}{\|\beta_s\|^3\cdot\|\beta_1\|}\Big]\,ds + o\Big(\frac{1}{k}\Big),\end{aligned}$$

with $(\beta_s)_{s\geq 0}$ a standard Brownian motion and $c_5 > 0$ the constant that appears in (4.3). In the same way, one has

$$\begin{aligned}\sum_{i=1}^{N} \mathbb{P}[\tau_1^i < \infty]\cdot\mathbb{P}[\tau_2<\infty] = c_0 &\sum_{j=\varepsilon_k}^{\lfloor \eta k\rfloor} \mathbb{E}[G(S_j)]\mathbb{E}\Big[\frac{1}{1+\mathcal{J}(S_k)}\Big]\\ &+ \frac{c_0c_5}{k}\int_\eta^{1-\eta} \mathbb{E}\Big[\frac{1}{\|\beta_s\|^3}\Big]\mathbb{E}\Big[\frac{1}{\|\beta_1\|}\Big]\,ds + o\Big(\frac{1}{k}\Big)\end{aligned}$$

with the same constant c_0, as in (5.35). We next handle the sum of the first terms in (5.35) and show that its difference with the sum from the previous display is negligible. Indeed, observe already that, with $\chi_k := k/(\eta\varepsilon_k)$,

$$\sum_{j=\varepsilon_k}^{\lfloor \eta k\rfloor} \mathbb{E}\Big[\frac{G(S_j)\mathbb{1}\{\|S_j\|\geq \eta^{1/4}\sqrt{k}\}}{1+\mathcal{J}(S_k)}\Big] + \mathbb{E}\Big[\frac{G(S_j)\mathbb{1}\{\|S_k\|\geq \sqrt{k\chi_k}\}}{1+\mathcal{J}(S_k)}\Big] \lesssim \frac{\eta^{1/4}}{k}.$$

Thus, one has, using Theorem 2.1,

$$\begin{aligned}&\sum_{j=\varepsilon_k}^{\lfloor \eta k\rfloor}\Big|\mathbb{E}\Big[\frac{G(S_j)}{1+\mathcal{J}(S_k)}\Big] - \mathbb{E}[G(S_j)]\cdot\mathbb{E}\Big[\frac{1}{1+\mathcal{J}(S_k)}\Big]\Big|\\ &\lesssim \sum_{j=\varepsilon_k}^{\lfloor \eta k\rfloor}\sum_{\substack{\|x\|\leq \eta^{1/4}\sqrt{k}\\ \|z\|\leq\sqrt{k\chi_k}}} \frac{p_j(x)G(x)}{1+\|z\|}\big|\overline{p}_{k-j}(z-x)+\overline{p}_{k-j}(z+x)-2\overline{p}_k(z)\big| + \frac{\eta^{1/4}}{k} \qquad (5.36)\\ &\lesssim \frac{\eta^{1/4}}{k}.\end{aligned}$$

Define now for $s\in(0,1]$,

$$H_s := \mathbb{E}\Big[\frac{1}{\|\beta_s\|^3\|\beta_1\|}\Big] - \mathbb{E}\Big[\frac{1}{\|\beta_s\|^3}\Big]\cdot\mathbb{E}\Big[\frac{1}{\|\beta_1\|}\Big].$$

Let $f_s(\cdot)$ be the density of β_s, and notice that, as $s\to 0$,

$$\begin{aligned}H_s &= \int_{\mathbb{R}^5}\int_{\mathbb{R}^5}\frac{f_s(x)f_{1-s}(y)}{\|x\|^3\|x+y\|}\,dx\,dy - \int_{\mathbb{R}^5}\int_{\mathbb{R}^5}\frac{f_s(x)f_1(y)}{\|x\|^3\|y\|}\,dx\,dy\\ &= \frac{1}{s^{3/2}}\int_{\mathbb{R}^5}\int_{\mathbb{R}^5}\frac{f_1(x)f_1(y)}{\|x\|^3}\Big(\frac{1}{\|y\sqrt{1-s}+x\sqrt{s}\|} - \frac{1}{\|y\|}\Big)\,dx\,dy\\ &= \frac{1}{s^{3/2}}\int_{\mathbb{R}^5}\int_{\mathbb{R}^5}\frac{f_1(x)f_1(y)}{\|x\|^3\|y\|}\Big\{\Big(\frac{1}{2}+\frac{\|x\|^2}{2\|y\|^2}+\frac{\langle x,y\rangle^2}{\|y\|^4}\Big)s + \mathcal{O}(s^{3/2})\Big\}\,dx\,dy\\ &= \frac{c}{\sqrt{s}} + \mathcal{O}(1),\end{aligned}$$

with $c > 0$. Thus, the map $s \mapsto H_s$ is integrable at 0, and, since it is also continuous on $(0, 1]$, its integral on this interval is well defined. Since η can be taken arbitrarily small in (5.33) and (5.36), in order to finish the proof it just remains to show that the integral of H_s on $(0, 1]$ is positive.

To this end, note first that $\widetilde{\beta}_{1-s} := \beta_1 - \beta_s$ is independent of β_s. We use then (5.31), which implies with $q = \mathbb{E}[1/\|\beta_1\|^3]$,

$$\mathbb{E}\left[\frac{1}{\|\beta_s\|^3\|\beta_1\|}\right] = \mathbb{E}\left[\frac{1}{\|\beta_s\|^3\|\beta_s + \widetilde{\beta}_{1-s}\|}\right] \geq \mathbb{E}\left[\frac{1}{\|\beta_s\|^3\sqrt{\|\beta_s\|^2 + \|\widetilde{\beta}_{1-s}\|^2}}\right]$$

$$= \frac{q^2}{5s^{3/2}} \int_0^\infty \int_0^\infty \frac{re^{-\frac{r^2}{2}}u^4e^{-\frac{u^2}{2}}}{\sqrt{sr^2 + (1-s)u^2}}\,dr\,du.$$

We split the double integral into two parts—one on the set $\{sr^2 \leq (1-s)u^2\}$ and the other one on the complementary set $\{sr^2 \geq (1-s)u^2\}$. Call, respectively, I_s^1 and I_s^2 the integrals on these two sets. For I_s^1, (5.32) gives

$$I_s^1 \geq \frac{1}{\sqrt{1-s}} \int_0^\infty u^3 e^{-\frac{u^2}{2}} \int_0^{\sqrt{\frac{1-s}{s}}u} re^{-\frac{r^2}{2}}\,drdu$$

$$- \frac{s}{2(1-s)^{3/2}} \int_0^\infty ue^{-\frac{u^2}{2}} \int_0^{\sqrt{\frac{1-s}{s}}u} r^3e^{-\frac{r^2}{2}}\,dr\,du$$

$$= \frac{2(1-s^2)}{\sqrt{1-s}} + \frac{s^2}{\sqrt{1-s}} - \frac{s}{\sqrt{1-s}} = \frac{2-s-s^2}{\sqrt{1-s}}.$$

For I_s^2 we simply use the rough bound

$$I_s^2 \geq \frac{1}{\sqrt{2s}} \int_0^\infty \int_0^\infty e^{-\frac{r^2}{2}}u^4e^{-\frac{u^2}{2}}\mathbf{1}\{sr^2 \geq (1-s)u^2\}\,dr\,du,$$

which entails

$$\int_0^1 \frac{I_s^2}{s^{3/2}}\,ds \geq \frac{1}{\sqrt{2}} \int_0^\infty \int_0^\infty e^{-\frac{r^2}{2}}u^4e^{-\frac{u^2}{2}}\left(\int_{\frac{u^2}{u^2+r^2}}^1 \frac{1}{s^2}\,ds\right)dr\,du$$

$$= \frac{1}{\sqrt{2}}\left(\int_0^\infty r^2e^{-\frac{r^2}{2}}\,dr\right)^2 = \frac{1}{\sqrt{2}}\left(\int_0^\infty e^{-\frac{r^2}{2}}\,dr\right)^2 = \frac{\pi}{2\sqrt{2}} > 1,$$

where for the last inequality we use $\sqrt{2} < 3/2$. Note now that

$$\mathbb{E}\left[\frac{1}{\|\beta_s\|^3}\right] \cdot \mathbb{E}\left[\frac{1}{\|\beta_1\|}\right] = \frac{2q^2}{5s^{3/2}},$$

and

$$\int_0^1 \frac{I_s^1 - 2}{s^{3/2}}\,ds \geq \int_0^1 s^{-3/2}\left\{(2-s-s^2)\left(1 + \frac{s}{2} + \frac{3s^2}{8}\right) - 2\right\}ds$$

$$= -\int_0^1 \left(\frac{3}{4}\sqrt{s} + \frac{7}{8}s^{3/2} + \frac{3}{8}s^{5/2}\right)ds$$

$$= -\left(\frac{1}{2} + \frac{7}{20} + \frac{3}{28}\right) = -\frac{134}{140} > -1.$$

Altogether, this shows that the integral of H_s on $(0, 1]$ is well positive as wanted. This concludes the proof of the lemma.

5.4. *Proof of Lemma* 5.4. We define here

$$\tau_1 := \inf\{n \geq 0 : S_n^1 \in \mathcal{R}[\varepsilon_k, k-\varepsilon_k]\}, \tau_2 := \inf\{n \geq 0 : S_k + S_n^2 \in \mathcal{R}[\varepsilon_k, k-\varepsilon_k]\},$$

with S^1 and S^2 two independent walks, independent of S. As in the previous lemma, we omit the details of the fact that

$$\mathrm{Cov}(Z_0\varphi_2, Z_k\psi_2) = \rho^2 \cdot \mathrm{Cov}(\mathbf{1}\{\tau_1 < \infty\}, \mathbf{1}\{\tau_2 < \infty\}) + o\left(\frac{1}{k}\right).$$

Then, we define $N := \lfloor (k-3\varepsilon_k)/\varepsilon_k \rfloor$ and let $(\tau_1^i)_{i=1,\dots,N}$ be as in the proof of Lemma 5.3. Define also $(\tau_2^i)_{i=1,\dots,N}$ analogously. Similarly, as before one can see that

$$\mathbb{P}[\tau_1 < \infty, \tau_2 < \infty] = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbb{P}[\tau_1^i < \infty, \tau_2^j < \infty] + o\left(\frac{1}{k}\right). \tag{5.37}$$

Note also that, for any i and j, with $|i-j| \leq 1$, by (2.6) and (2.9)

$$\mathbb{P}[\tau_1^i < \infty, \tau_2^j < \infty] = \mathcal{O}\left(\frac{\varepsilon_k^{2(1-\delta)}}{k_i^{3/2}(k-k_i)^{3/2}}\right),$$

so that in (5.37) one can consider only the sum on the indices i and j satisfying $|i-j| \geq 2$. Furthermore, when $i < j$, the events $\{\tau_1^i < \infty\}$ and $\{\tau_2^j < \infty\}$ are independent. Thus, altogether this gives

$$\begin{aligned}&\mathrm{Cov}(\mathbf{1}\{\tau_1 < \infty\}, \mathbf{1}\{\tau_2 < \infty\})\\ &\quad= \sum_{i=1}^{N-2}\sum_{j=i+2}^{N} (\mathbb{P}[\tau_1^j < \infty, \tau_2^i < \infty] - \mathbb{P}[\tau_1^j < \infty]\mathbb{P}[\tau_2^i < \infty]) + o\left(\frac{1}{k}\right).\end{aligned}$$

Then, by following carefully the same steps as in the proof of the previous lemma we arrive at

$$\mathrm{Cov}(\mathbf{1}\{\tau_1 < \infty\}, \mathbf{1}\{\tau_2 < \infty\}) = \frac{c}{k}\int_0^1 \widetilde{H}_t\,dt + o\left(\frac{1}{k}\right),$$

with $c > 0$ some positive constant and,

$$\widetilde{H}_t := \int_0^t \left(\mathbb{E}\left[\frac{1}{\|\beta_s - \beta_1\|^3 \cdot \|\beta_t\|^3}\right] - \mathbb{E}\left[\frac{1}{\|\beta_s - \beta_1\|^3}\right] \cdot \mathbb{E}\left[\frac{1}{\|\beta_t\|^3}\right]\right) ds,$$

at least provided we show first that $\widetilde{H}_t$ is well defined and that its integral over $[0, 1]$ is convergent. However, observe that, for any $t \in (0, 1)$, one has with $q = \mathbb{E}[\|\beta_1\|^{-3}]$,

$$\int_0^t \mathbb{E}\left[\frac{1}{\|\beta_s - \beta_1\|^3}\right] \cdot \mathbb{E}\left[\frac{1}{\|\beta_t\|^3}\right] = \frac{q^2}{t^{3/2}}\int_0^t \frac{1}{(1-s)^{3/2}}\,ds = \frac{2q^2(1-\sqrt{1-t})}{t^{3/2}\sqrt{1-t}},$$

and, therefore, this part is integrable on $[0, 1]$. This implies, in fact, that the other part in the definition of $\widetilde{H}_t$ is also well defined and integrable, since we already know that $\mathrm{Cov}(\mathbf{1}\{\tau_1 < \infty\}, \mathbf{1}\{\tau_2 < \infty\}) = \mathcal{O}(1/k)$. Thus, it only remains to show that the integral of $\widetilde{H}_t$ on $[0, 1]$ is positive. To this end, we write $\beta_t = \beta_s + \gamma_{t-s}$ and $\beta_1 = \beta_s + \gamma_{t-s} + \delta_{1-t}$, with $(\gamma_u)_{u\geq 0}$ and $(\delta_u)_{u\geq 0}$, two independent Brownian motions, independent of β. Furthermore, knowing that the map $z \mapsto 1/\|z\|^3$ is harmonic outside the origin, we can compute

$$I_1 := \mathbb{E}\left[\frac{\mathbf{1}\{\|\beta_s\| \geq \|\gamma_{t-s}\| \geq \|\delta_{1-t}\|\}}{\|\beta_s - \beta_1\|^3 \cdot \|\beta_t\|^3}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\biggl[\frac{\mathbf{1}\{\|\beta_s\| \ge \|\gamma_{t-s}\| \ge \|\delta_{1-t}\|\}}{\|\gamma_{t-s} + \delta_{1-t}\|^3 \cdot \|\beta_s\|^3}\biggr] \\
&= \frac{5q}{s^{3/2}}\mathbb{E}\biggl[\frac{\mathbf{1}\{\|\gamma_{t-s}\| \ge \|\delta_{1-t}\|\}}{\|\gamma_{t-s} + \delta_{1-t}\|^3}\int_{\frac{\|\gamma_{t-s}\|}{\sqrt{s}}}^{\infty} r e^{-\frac{5}{2}r^2}\,dr\biggr] \\
&= \frac{q}{s^{3/2}}\mathbb{E}\biggl[\frac{\mathbf{1}\{\|\gamma_{t-s}\| \ge \|\delta_{1-t}\|\}}{\|\gamma_{t-s} + \delta_{1-t}\|^3}e^{-\frac{5}{2s}\|\gamma_{t-s}\|^2}\biggr] \\
&= \frac{q}{s^{3/2}}\mathbb{E}\biggl[\frac{\mathbf{1}\{\|\gamma_{t-s}\| \ge \|\delta_{1-t}\|\}}{\|\gamma_{t-s}\|^3}e^{-\frac{5}{2s}\|\gamma_{t-s}\|^2}\biggr] \\
&= \frac{5q^2}{s^{3/2}(t-s)^{3/2}}\mathbb{E}\biggl[\int_{\frac{\|\delta_{1-t}\|}{\sqrt{t-s}}}^{\infty} r e^{-\frac{5}{2}r^2(1+\frac{t-s}{s})}\,dr\biggr] \\
&= \frac{q^2}{\sqrt{s}(t-s)^{3/2}t}\mathbb{E}\bigl[e^{-\frac{\|\delta_{1-t}\|^2 t}{s(t-s)}}\bigr] = \frac{5q^3}{\sqrt{s}(t-s)^{3/2}t}\int_0^{\infty} r^4 e^{-\frac{5}{2}r^2(1+\frac{t(1-t)}{s(t-s)})}\,dr \\
&= \frac{q^2 s^2 (t-s)}{t\Delta^{5/2}},
\end{aligned}
$$

with

$$
\Delta := t(1-t) + s(t-s) = (1-t)(t-s) + s(1-s).
$$

Likewise,

$$
\begin{aligned}
I_2 &:= \mathbb{E}\biggl[\frac{\mathbf{1}\{\|\beta_s\| \ge \|\gamma_{t-s}\|, \|\delta_{1-t}\| \ge \|\gamma_{t-s}\|\}}{\|\beta_s - \beta_1\|^3 \cdot \|\beta_t\|^3}\biggr] \\
&= \frac{q}{s^{3/2}}\mathbb{E}\biggl[\frac{\mathbf{1}\{\|\gamma_{t-s}\| \le \|\delta_{1-t}\|\}}{\|\gamma_{t-s} + \delta_{1-t}\|^3}e^{-\frac{5}{2s}\|\gamma_{t-s}\|^2}\biggr] \\
&= \frac{q}{s^{3/2}}\mathbb{E}\biggl[\frac{\mathbf{1}\{\|\gamma_{t-s}\| \le \|\delta_{1-t}\|\}}{\|\delta_{1-t}\|^3}e^{-\frac{5}{2s}\|\gamma_{t-s}\|^2}\biggr] \\
&= \frac{5q^2}{s^{3/2}(1-t)^{3/2}}\mathbb{E}\biggl[e^{-\frac{5}{2s}\|\gamma_{t-s}\|^2}\int_{\frac{\|\gamma_{t-s}\|}{\sqrt{1-t}}}^{\infty} r e^{-\frac{5}{2}r^2}\,dr\biggr] \\
&= \frac{q^2}{s^{3/2}(1-t)^{3/2}}\mathbb{E}\bigl[e^{-\frac{5}{2}\|\gamma_{t-s}\|^2(\frac{1}{s}+\frac{1}{1-t})}\bigr] = \frac{q^2 s(1-t)}{\Delta^{5/2}}.
\end{aligned}
$$

Define as well

$$
\begin{aligned}
&I_3 := \mathbb{E}\biggl[\frac{\mathbf{1}\{\|\beta_s\| \le \|\gamma_{t-s}\| \le \|\delta_{1-t}\|\}}{\|\beta_s - \beta_1\|^3 \cdot \|\beta_t\|^3}\biggr], \\
&I_4 := \mathbb{E}\biggl[\frac{\mathbf{1}\{\|\delta_{1-t}\| \le \|\beta_s\| \le \|\gamma_{t-s}\|\}}{\|\beta_s - \beta_1\|^3 \cdot \|\beta_t\|^3}\biggr], \qquad I_5 := \mathbb{E}\biggl[\frac{\mathbf{1}\{\|\beta_s\| \le \|\delta_{1-t}\| \le \|\gamma_{t-s}\|\}}{\|\beta_s - \beta_1\|^3 \cdot \|\beta_t\|^3}\biggr].
\end{aligned}
$$

Note that by symmetry one has

$$
\int_{0\le s\le t\le 1} I_1\,ds\,dt = \int_{0\le s\le t\le 1} I_3\,ds\,dt \quad \text{and} \quad \int_{0\le s\le t\le 1} I_4\,ds\,dt = \int_{0\le s\le t\le 1} I_5\,ds\,dt.
$$

Observe also that

$$
I_1 + I_2 = \frac{q^2 s}{t\Delta^{3/2}}.
$$

Moreover, using symmetry again, we can see that

$$\int_0^t \frac{s-t/2}{\Delta^{3/2}}\,ds = 0,$$

and, thus,

$$\int_0^t (I_1+I_2)\,ds = \frac{q^2}{2}\int_0^t \frac{1}{\Delta^{3/2}}\,ds.$$

Likewise,

$$\begin{aligned}\int_{0\le s\le t\le 1} I_1\,ds\,dt &= \int_{0\le s\le t\le 1} \frac{q^2 s(t-s)^2}{t\Delta^{5/2}}\,ds\,dt = \frac{1}{2}\int_{0\le s\le t\le 1}\frac{q^2 s(t-s)}{\Delta^{5/2}}\,ds\,dt \\ &= \int_{0\le s\le t\le 1}\frac{q^2(1-t)(t-s)}{2\Delta^{5/2}}\,dsdt = \int_{0\le s\le t\le 1}\frac{q^2 t(1-t)}{4\Delta^{5/2}}\,ds\,dt \\ &= \int_{0\le s\le t\le 1}\frac{q^2}{6\Delta^{3/2}}\,ds\,dt.\end{aligned}$$

It follows that

$$\int_{0\le s\le t\le 1}(I_1+I_2+I_3)\,ds\,dt = \frac{2q^2}{3}\int_{0\le s\le t\le 1}\Delta^{-3/2}\,ds\,dt.$$

We consider now the term I_4, which is a bit more complicated to compute, thus we only give a lower bound on a suitable interval. To be more precise, we first define for $r\ge 0$ and $\lambda\ge 0$,

$$F(r) := \int_0^r s^4 e^{-5s^2/2}\,ds \quad \text{and} \quad F_2(\lambda,r) := \int_0^r F(\lambda s)s^4 e^{-5s^2/2}\,ds,$$

and then we write

$$\begin{aligned} I_4 &= \mathbb{E}\left[\frac{\mathbb{1}\{\|\delta_{1-t}\|\le\|\beta_s\|\le\|\gamma_{t-s}\|\}}{\|\gamma_{t-s}\|^6}\right] \\ &= 5q\cdot\mathbb{E}\left[\frac{\mathbb{1}\{\|\beta_s\|\le\|\gamma_{t-s}\|\}}{\|\gamma_{t-s}\|^6}F\left(\frac{\|\beta_s\|}{\sqrt{1-t}}\right)\right] \\ &= \mathbb{E}\left[\frac{(5q)^2}{\|\gamma_{t-s}\|^6}F_2\left(\frac{\sqrt{s}}{\sqrt{1-t}},\frac{\|\gamma_{t-s}\|}{\sqrt{s}}\right)\right] = \frac{(5q)^3}{(t-s)^3}\int_0^\infty \frac{e^{-\frac{5r^2}{2}}}{r^2}F_2\left(\frac{\sqrt{s}}{\sqrt{1-t}}, r\frac{\sqrt{t-s}}{\sqrt{s}}\right)dr \\ &= \frac{(5q)^3}{(t-s)^3}\left\{\frac{\sqrt{t-s}}{\sqrt{s}}\int_0^\infty F\left(r\frac{\sqrt{t-s}}{\sqrt{1-t}}\right)r^3 e^{-\frac{5r^2}{2}}\,dr\right. \\ &\quad \left. -5\int_0^\infty F_2\left(\frac{\sqrt{s}}{\sqrt{1-t}}, r\frac{\sqrt{t-s}}{\sqrt{s}}\right)e^{-\frac{5r^2}{2}}\,dr\right\} \\ &\ge \frac{(5q)^3}{(t-s)^3}\left\{\frac{(t-s)^{\frac{3}{2}}}{s^{3/2}}\int_0^\infty F\left(r\frac{\sqrt{t-s}}{\sqrt{1-t}}\right)r^3 e^{-\frac{5r^2t}{2s}}\,dr\right. \\ &\quad \left. +\frac{(2s-t)\sqrt{t-s}}{s^{3/2}}\int_0^\infty F\left(r\frac{\sqrt{t-s}}{\sqrt{1-t}}\right)r^3 e^{-\frac{5r^2}{2}}\,dr\right\},\end{aligned}$$

using that

$$F_2(\lambda,r)\le\frac{1}{5}r^3F(\lambda r)\big(1-e^{-5r^2/2}\big).$$

Therefore, if $t/2 \le s \le t$,

$$\begin{aligned} I_4 &\ge \frac{(5q)^3}{[s(t-s)]^{3/2}} \int_0^\infty r^3 F\biggl(r\frac{\sqrt{t-s}}{\sqrt{1-t}}\biggr) e^{-\frac{5r^2 t}{2s}}\,dr \\ &= \frac{(5q)^3\sqrt{s}}{t^2(t-s)^{3/2}} \int_0^\infty r^3 F\biggl(r\frac{\sqrt{s(t-s)}}{\sqrt{t(1-t)}}\biggr) e^{-5r^2/2}\,dr \\ &\ge \frac{2\cdot 5^2 q^3\sqrt{s}}{t^2(t-s)^{3/2}} \int_0^\infty F\biggl(r\frac{\sqrt{s(t-s)}}{\sqrt{t(1-t)}}\biggr) r e^{-\frac{5r^2}{2}}\,dr \\ &= \frac{2\cdot 5 q^3 s^3(t-s)}{t^2[t(1-t)]^{5/2}} \int_0^\infty r^4 e^{-\frac{5r^2\Delta}{2t(1-t)}}\,dr \\ &= \frac{2q^2 s^3(t-s)}{t^2\Delta^{5/2}} \ge \frac{q^2 s(t-s)}{2\Delta^{5/2}}, \end{aligned}$$

and as a consequence,

$$\begin{aligned} \int_{0\le s\le t\le 1} I_4\,ds\,dt &\ge \int_{t/2\le s\le t\le 1} I_4\,ds\,dt \ge \frac{q^2}{2}\int_{t/2\le s\le t\le 1} \frac{s(t-s)}{\Delta^{5/2}}\,ds\,dt \\ &= \frac{q^2}{4}\int_{0\le s\le t\le 1} \frac{s(t-s)}{\Delta^{5/2}}\,ds\,dt = \frac{q^2}{12}\int_{0\le s\le t\le 1} \Delta^{-3/2}\,ds\,dt. \end{aligned}$$

Putting all these estimates together yields

$$\int_{0\le s\le t\le 1} \mathbb{E}\biggl[\frac{1}{\|\beta_s-\beta_1\|^3\cdot\|\beta_t\|^3}\biggr]\,ds\,dt = \sum_{k=1}^{5}\int_{0\le s\le t\le 1} I_k\,ds\,dt \ge \frac{5}{6}\int_{0\le s\le t\le 1}\Delta^{-3/2}\,ds\,dt.$$

Thus, it just remains to show that

$$\int_{0\le s\le t\le 1}\Delta^{-3/2}\,ds\,dt \ge \frac{6}{5}\int_{0\le s\le t\le 1}\widetilde{\Delta}^{-3/2}\,ds\,dt, \tag{5.38}$$

where $\widetilde{\Delta} := t(1-s)$. Note that $\Delta = \widetilde{\Delta} + (t-s)^2$. Recall also that, for any $\alpha \in \mathbb{R}$, and any $x \in (-1,1)$,

$$(1+x)^\alpha = 1 + \sum_{i\ge 1}\frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}x^i. \tag{5.39}$$

Thus,

$$\frac{1}{\Delta^{3/2}} = \frac{1}{\widetilde{\Delta}^{3/2}}\biggl(1+\sum_{k\ge 1}\frac{(3/2)(5/2)\dots(k+1/2)}{k!}\cdot\frac{(t-s)^{2k}}{\widetilde{\Delta}^k}\biggr).$$

One needs now to compute the coefficients C_k defined by

$$C_k := \frac{(3/2)(5/2)\dots(k+1/2)}{k!}\int_{0\le s\le t\le 1}\frac{(t-s)^{2k}}{\widetilde{\Delta}^{k+3/2}}\,ds\,dt.$$

We claim that one has for any $k \ge 0$,

$$C_k = \frac{2^{2k+2}}{2k+1}(-1)^k\Sigma_k, \tag{5.40}$$

with $\Sigma_0 = 1$, and for $k \ge 1$,

$$\Sigma_k = 1 + \sum_{i=1}^{2k}(-1)^i\frac{(k+1/2)(k-1/2)\dots(k-i+3/2)}{i!}.$$

We will prove this formula in a moment, but let us conclude the proof of the lemma first, assuming it is true. Straightforward computations show by (5.40) that

$$C_0 = 4, \qquad C_1 = \frac{2}{3} \quad \text{and} \quad C_2 = \frac{3}{10},$$

and $C_0 + C_1 + C_2 \geq 6C_0/5$, gives (5.38) as wanted.

So let us prove (5.40) now. Note that one can assume $k \geq 1$, as the result for $k = 0$ is immediate. By (5.39) one has

$$(1-s)^{-k-3/2} = 1 + \sum_{i \geq 1} \frac{(k+3/2)(k+5/2)\dots(k+i+1/2)}{i!} s^i.$$

Thus, by integrating parts we get

$$\int_0^t \frac{(t-s)^{2k}}{(1-s)^{k+3/2}} \, ds = (2k)! \sum_{i \geq 0} \frac{(k+3/2)\dots(k+i+1/2)}{(2k+i+1)!} \cdot t^{2k+i+1},$$

and then

$$\int_0^1 \int_0^t \frac{(t-s)^{2k}}{t^{k+3/2}(1-s)^{k+3/2}} \, ds \, dt = (2k)! \sum_{i \geq 0} \frac{(k+3/2)\dots(k+i-1/2)}{(2k+i+1)!}.$$

As a consequence,

$$\begin{aligned} C_k &= \frac{(2k)!}{k!} \sum_{i \geq 0} \frac{(3/2)(5/2)\dots(k+i-1/2)}{(2k+i+1)!} \\ &= \frac{(2k)!}{(k+1/2)(k-1/2)\dots(3/2)(1/2)^2 \cdot k!} \sum_{i \geq 0} \frac{|(k+1/2)(k-1/2)\dots(-k-i+1/2)|}{(2k+i+1)!} \\ &= \frac{2^{2k+2}}{2k+1} \sum_{i \geq 0} \frac{|(k+1/2)(k-1/2)\dots(-k-i+1/2)|}{(2k+i+1)!}, \end{aligned}$$

and it just remains to observe that the last sum is well equal to Σ_k. The latter is obtained by taking the limit as t goes to 1 in the formula (5.39) for $(1-t)^{k+1/2}$. This concludes the proof of Lemma 5.4.

REMARK 5.6. It would be interesting to show that the covariance between $1/\|\beta_s - \beta_1\|^3$ and $1/\|\beta_t\|^3$ itself is positive for all $0 \leq s \leq t \leq 1$ and not just its integral, as we have just shown.

6. Proof of Theorem B.

The proof of Theorem B is based on the Lindeberg–Feller theorem for triangular arrays that we recall for convenience (see Theorem 3.4.5 in [9]).

THEOREM 6.1 (Lindeberg–Feller). *For each n let $(X_{n,i} : 1 \leq i \leq n)$ be a collection of independent random variables with zero mean. Suppose that the following two conditions are satisfied*:

(i) $\sum_{i=1}^n \mathbb{E}[X_{n,i}^2] \to \sigma^2 > 0$ *as* $n \to \infty$, *and*
(ii) $\sum_{i=1}^n \mathbb{E}[(X_{n,i})^2 \mathbf{1}\{|X_{n,i}| > \varepsilon\}] \to 0$, *as* $n \to \infty$, *for all* $\varepsilon > 0$.

Then, $S_n = X_{n,1} + \dots + X_{n,n} \Longrightarrow \mathcal{N}(0, \sigma^2)$, *as* $n \to \infty$.

In order to apply this result, one needs three ingredients. The first one is an asymptotic estimate for the variance of the capacity of the range which is given by our Theorem A. The second ingredient is a decomposition of the capacity of two sets as a sum of the capacities of the two sets minus some error term, in the spirit of the inclusion-exclusion formula for the cardinality of a set, which allows to decompose the capacity of the range up to time n into a sum of independent pieces having the law of the capacity of the range up to a smaller time index and, finally, the last ingredient is a sufficiently good bound on the centered fourth moment.

This strategy has already been employed successfully for the capacity of the range in dimension six and more in [3] (and for the size of the range as well; see [13, 14]). In this case the asymptotic of the variance followed simply from a subadditivity argument, but the last two ingredients are entirely similar in dimension 5 and in higher dimensions. In particular, one has the following decomposition (see Proposition 1.6 in [4]): for any two subsets $A, B \subset \mathbb{Z}^d$, $d \geq 3$,

$$\mathrm{Cap}(A \cup B) = \mathrm{Cap}(A) + \mathrm{Cap}(B) - \chi(A, B), \tag{6.1}$$

where $\chi(A, B)$ is some error term. Its precise expression is not so important here. All one needs to know is that

$$\left|\chi(A, B)\right| \leq 3 \sum_{x \in A} \sum_{y \in B} G(x, y),$$

so that by [3], Lemma 3.2, if $\mathcal{R}_n$ and $\widetilde{\mathcal{R}}_n$ are the ranges of two independent walks in $\mathbb{Z}^5$, then

$$\mathbb{E}\big[\chi(\mathcal{R}_n, \widetilde{\mathcal{R}}_n)^4\big] = \mathcal{O}(n^2). \tag{6.2}$$

We note that the result is shown for the simple random walk only in [3], but the proof applies as well to our setting (in particular Lemma 3.1 thereof also follows from (2.9)). Now, as noticed already by Le Gall in his paper [17] (see his remark (iii) p.503), a good bound on the centered fourth moment follows from (6.1) and (6.2) and the triangle inequality in L^4. More precisely, in dimension 5 one obtains (see, for instance, the proof of Lemma 4.2 in [3] for some more details)

$$\mathbb{E}\big[\big(\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)]\big)^4\big] = \mathcal{O}\big(n^2 (\log n)^4\big). \tag{6.3}$$

Actually, we would even obtain the slightly better bound $\mathcal{O}(n^2(\log n)^2)$, using our new bound on the variance $\mathrm{Var}(\mathrm{Cap}(\mathcal{R}_n)) = \mathcal{O}(n \log n)$, but this is not needed here. Using next a dyadic decomposition of n, one can write with $T := \lfloor n/(\log n)^4 \rfloor$,

$$\mathrm{Cap}(\mathcal{R}_n) = \sum_{i=0}^{\lfloor n/T \rfloor} \mathrm{Cap}\big(\mathcal{R}_T^{(i)}\big) - R_n, \tag{6.4}$$

where the $(\mathcal{R}_T^{(i)})_{i=0,\dots,n/T}$ are independent pieces of the range of length either T or $T+1$ and

$$R_n = \sum_{\ell=1}^{L} \sum_{i=0}^{2^{\ell-1}} \chi\big(\mathcal{R}_{n/2^\ell}^{(2i)}, \mathcal{R}_{n/2^\ell}^{(2i+1)}\big)$$

is a triangular array of error terms (with $L = \log_2 (\log n)^4$). Then, it follows from (6.2) that

$$\begin{aligned}\mathrm{Var}(R_n) &\leq L \sum_{\ell=1}^{L} \mathrm{Var}\left(\sum_{i=1}^{2^{\ell-1}} \chi\big(\mathcal{R}_{n/2^\ell}^{(2i)}, \mathcal{R}_{n/2^\ell}^{(2i+1)}\big)\right) \leq L \sum_{\ell=1}^{L} \sum_{i=1}^{2^{\ell-1}} \mathrm{Var}\big(\chi\big(\mathcal{R}_{n/2^\ell}^{(2i)}, \mathcal{R}_{n/2^\ell}^{(2i+1)}\big)\big) \\ &= \mathcal{O}\big(L^2 n\big) = \mathcal{O}\big(n (\log \log n)^2\big).\end{aligned}$$

In particular, $(R_n - \mathbb{E}[R_n])/\sqrt{n \log n}$ converges in probability to 0. Thus, one is just led to show the convergence in law of the remaining sum in (6.4). For this one can apply Theorem 6.1 with

$$X_{n,i} := \frac{\operatorname{Cap}(\mathcal{R}_T^{(i)}) - \mathbb{E}[\operatorname{Cap}(\mathcal{R}_T^{(i)})]}{\sqrt{n \log n}}.$$

Indeed, Condition (i) of the theorem follows from Theorem A, and Condition (ii) follows from (6.3) and Markov's inequality (more details can be found in [3]). This concludes the proof of Theorem B.

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SUPPLEMENTARY MATERIAL

Supplement to "Capacity of the range in dimension 5" (DOI: 10.1214/20-AOP1442SUPP; .pdf). Capacity of the range in dimension 5: rough variance bounds [19]. This companion paper provides the proofs of Propositions 3.3–3.6, as well as the proof of Lemma 3.1.

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