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ON THE PROBABILITY OF NONEXISTENCE IN BINOMIAL SUBSETS

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Given a hypergraph $\Gamma = (\Omega, \mathcal{X})$ and a sequence $\mathbf{p} = (p_\omega)_{\omega\in\Omega}$ of values in $(0, 1)$, let $\Omega_{\mathbf{p}}$ be the random subset of Ω obtained by keeping every vertex ω independently with probability p_ω. We investigate the general question of deriving fine (asymptotic) estimates for the probability that $\Omega_{\mathbf{p}}$ is an independent set in Γ, which is an omnipresent problem in probabilistic combinatorics. Our main result provides a sequence of upper and lower bounds on this probability, each of which can be evaluated explicitly in terms of the joint cumulants of small sets of edge indicator random variables. Under certain natural conditions, these upper and lower bounds coincide asymptotically, thus giving the precise asymptotics of the probability in question. We demonstrate the applicability of our results with two concrete examples: subgraph containment in random (hyper)graphs and arithmetic progressions in random subsets of the integers.

1. Introduction. Let $\Gamma = (\Omega, \mathcal{X})$ be a hypergraph and, given a sequence $\mathbf{p} = (p_\omega)_{\omega\in\Omega} \in (0, 1)^\Omega$, let $\Omega_{\mathbf{p}}$ be a random subset of Ω formed by including every $\omega \in \Omega$ independently with probability p_ω. What is the probability that $\Omega_{\mathbf{p}}$ is an independent set in Γ? This very general question arises in many different settings.

EXAMPLE 1. Let F be a graph and let n be a positive integer. Define Ω as the edge set $E(K_n) = \binom{[n]}{2}$ of the complete graph with vertex set $[n] = \{1, \dots, n\}$ and let $\mathcal{X}$ be the collection of the edge sets of all copies of F in K_n. Fix some $p \in (0, 1)$ and define $\mathbf{p}$ by setting $p_\omega = p$ for every $\omega \in \Omega$. Then we are asking for the probability that the Erdős–Rényi random graph $G_{n,p}$ is F-free, that is, does not contain F as a (not necessarily induced) subgraph.

EXAMPLE 2. An arithmetic progression of length $r \in \mathbb{N}$ (an r-*AP* for short) is a subset of the integers of the form $\{a + kb : k \in [r]\}$, where $b \neq 0$. Let $\Omega = [n]$ and let $\mathcal{X}$ be the set of all r-APs in $[n]$. Given $p \in (0, 1)$, we define $\mathbf{p}$ by setting $p_\omega = p$ for every $\omega \in \Omega$. Then we are asking for the probability that the p-random subset $[n]_p$ of $[n]$ is r-AP-free.

EXAMPLE 3. Let Ω be a finite set of points in the plane. Include a triple $\{i, j, k\}$ in $\mathcal{X}$ if the points i, j, k lie on a common line. Now we are asking for the probability that the random subset $\Omega_{\mathbf{p}}$ of points is in general position.

It is not hard to find other natural examples that provide further motivation for studying this question. It is convenient to introduce some notation. Given $\Gamma = (\Omega, \mathcal{X})$ and $\mathbf{p} \in (0, 1)^\Omega$, we shall fix an (arbitrary) ordering of the elements of $\mathcal{X}$ as $\gamma_1, \dots, \gamma_N$. We then let X_i denote

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the indicator variable of the event that $\gamma_i \subseteq \Omega_{\mathbf{p}}$ and set $X = X_1 + \cdots + X_N$. Thus, X counts the number of edges of Γ that are fully contained in $\Omega_{\mathbf{p}}$ and our goal is to compute the probability that $X = 0$. Of course, these notations all depend on the given pair $(\Gamma, \mathbf{p})$, but we shall always suppress this dependence as it will be clear from the context.

Most of the time, we will be interested in the case where $\Gamma = \Gamma(n)$ and $\mathbf{p} = \mathbf{p}(n)$ (and hence also $X = X(n)$) depend on some parameter n that tends to infinity and ask:

What are the asymptotics of the probability $\mathbb{P}[X = 0]$ *as* $n \to \infty$*?*

The above question can also be viewed as a computational problem: we want to derive closed formulas that are asymptotic to $\mathbb{P}[X = 0]$, at least for various ranges of the density parameter $\mathbf{p}$.

For technical convenience, we shall exclude the border case where $p_\omega \in \{0, 1\}$ for some ω. That case can always be addressed by changing Γ or by a continuity argument.

1.1. *The Harris and Janson inequalities.* The main reason why computing $\mathbb{P}[X = 0]$ is challenging is that the random variables $X_1, \dots, X_N$ are usually not independent. However, this is not to say that there is no structure at all: each random variable X_i is a nondecreasing function on the product space $\{0, 1\}^\Omega$. An important inequality that applies in this case is the *Harris inequality*:

THEOREM 4 (Harris inequality [10]). *Let Ω be a finite set and let X and Y be random variables defined on a product probability space over $\{0, 1\}^\Omega$. If X and Y are both nondecreasing* (*or nonincreasing*), *then*

$$\mathbb{E}[XY] \geq \mathbb{E}[X]\mathbb{E}[Y].$$

If X is nondecreasing and Y is nonincreasing, *then*

$$\mathbb{E}[XY] \leq \mathbb{E}[X]\mathbb{E}[Y].$$

In our setting, for every $V \subseteq [N]$, the random variable $\prod_{i \in V}(1 - X_i)$ is nonincreasing, so we easily deduce from Harris's inequality that

$$\mathbb{P}[X = 0] = \mathbb{E}\Big[\prod_{i \in [N]} (1 - X_i)\Big] \geq \prod_{i \in [N]} \big(1 - \mathbb{E}[X_i]\big). \tag{1}$$

Note that (1) would be true with equality if $X_1, \dots, X_N$ were independent. An upper bound on $\mathbb{P}[X = 0]$ is given by *Janson's inequality*, which states that the reverse of (1) holds up to a multiplicative error term that is an explicit function of the pairwise dependencies between the indicator random variables $X_1, \dots, X_N$. Formally, we write $i \sim j$ if $i \neq j$ and $\gamma_i \cap \gamma_j \neq \varnothing$, and define the sum of joint moments

$$\Delta_2 = \sum_{i \sim j} \mathbb{E}[X_i X_j]. \tag{2}$$

THEOREM 5 (Janson's inequality [2, 15]). *For all Γ and $\mathbf{p}$ as above*,

$$\mathbb{P}[X = 0] \leq \exp\big(-\mathbb{E}[X] + \Delta_2\big).$$

To compare this with (1), we will now assume that the individual probabilities of $X_i = 1$ are not too large, say $\mathbb{E}[X_i] \leq 1 - \varepsilon$ for some positive constant ε. In this case, we may use the inequality $1 - x \geq \exp(-x - x^2/\varepsilon)$ for $x \in [0, 1 - \varepsilon]$ to obtain from (1)

$$\mathbb{P}[X = 0] \geq \prod_{i \in [N]} \big(1 - \mathbb{E}[X_i]\big) \geq \exp\big(-\mathbb{E}[X] - \delta_1/\varepsilon\big), \tag{3}$$

where

$$\delta_1 = \sum_{i \in [N]} \mathbb{E}[X_i]^2. \tag{4}$$

Combining this lower bound with the upper bound given by Janson's inequality, we get the approximation

$$\mathbb{P}[X = 0] = \exp\bigl(-\mathbb{E}[X] + O(\delta_1 + \Delta_2)\bigr). \tag{5}$$

If $\delta_1 + \Delta_2 = o(1)$, then (5) gives the correct asymptotics of $\mathbb{P}[X = 0]$. The condition $\Delta_2 = o(1)$ in particular requires that the pairwise correlations between the indicator variables X_i vanish asymptotically in a well-defined sense. This rather strict requirement is not satisfied in many natural settings, including the ones presented in Examples 1–3 for certain choices of p. It is therefore an important question to obtain better approximations of $\mathbb{P}[X = 0]$ in cases when the pairwise dependencies among the X_i are not negligible. This is the starting point of our investigations.

1.2. *Triangles in random graphs*. Even though our results will be phrased in the general framework introduced above and are thus widely applicable, we believe that it is useful to keep in mind the following well-studied instance of the problem that will serve as a guiding example.

EXAMPLE 6. Assume $p = p(n) \in (0, 1)$ and let $X = X(n)$ denote the number of triangles in $G_{n,p}$, as in Example 1 with $F = K_3$. Since each triangle has three edges, we have $\mathbb{E}[X_i] = p^3$ for all i. Thus, $\mathbb{E}[X] = \binom{n}{3} p^3$ and $\delta_1 = O(n^3 p^6)$. Moreover, we have $\Delta_2 = O(n^4 p^5)$, because if two distinct triangles intersect, then their union is the graph with 4 vertices and 5 edges. Thus, (5) implies that as long as $p = o(n^{-4/5})$, we have

$$\mathbb{P}[X = 0] = \exp\bigl(-n^3 p^3/6 + o(1)\bigr).$$

Extending this result, Wormald [25] and later Stark and Wormald [23] obtained asymptotic expressions for $\mathbb{P}[X = 0]$ even when $p = \Omega(n^{-4/5})$ and thus (5) no longer gives an asymptotic bound. In particular, it was shown by Stark and Wormald in [23] that if $p = o(n^{-7/11})$, then

$$\mathbb{P}[X = 0] = \exp\left(-\frac{n^3 p^3}{6} + \frac{n^4 p^5}{4} - \frac{7n^5 p^7}{12} + \frac{n^2 p^3}{2} - \frac{3n^4 p^6}{8} + \frac{27 n^6 p^9}{16} + o(1)\right).$$

One goal of this paper is to give a simple interpretation of the individual terms in this formula. Indeed, we will formulate a general result from which the above formula may be obtained by a few short calculations. More precisely, we will prove a generalisation of (5) that takes into account the k-wise dependencies between the variables X_i for all $k \geq 2$.

1.3. *Joint cumulants*, *clusters*, *dependency graphs*. Let $A = \{Z_1, \ldots, Z_m\}$ be a finite set of real-valued random variables. The *joint moment* of the variables in A is

$$\Delta(A) = \mathbb{E}[Z_1 \cdots Z_m]. \tag{6}$$

The *joint cumulant* of the variables is

$$\kappa(A) = \sum_{\pi \in \Pi(A)} \bigl(|\pi| - 1\bigr)!(-1)^{|\pi|-1} \prod_{P \in \pi} \Delta(P), \tag{7}$$

where $\Pi(A)$ denotes the set of all partitions of A into nonempty sets. In particular,

$$\begin{aligned}\kappa(\{X\}) &= \mathbb{E}[X],\\ \kappa(\{X,Y\}) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y],\\ \kappa(\{X,Y,Z\}) &= \mathbb{E}[XYZ] - \mathbb{E}[X]\mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[XZ] - \mathbb{E}[Z]\mathbb{E}[XY]\\ &\quad + 2\mathbb{E}[X]\mathbb{E}[Y]\mathbb{E}[Z].\end{aligned}$$

The joint cumulant $\kappa(A)$ can be regarded as a measure of the mutual dependence of the variables in A. For example, $\kappa(\{X,Y\})$ is simply the covariance of X and Y, and so $\kappa(\{X,Y\}) = 0$ if X and Y are independent. More generally, the following holds.

PROPOSITION 7. *Let A be a finite set of real-valued random variables. If A can be partitioned into two subsets A_1 and A_2 such that all variables in A_1 are independent of all variables in A_2, then $\kappa(A) = 0$.*

In fact, Proposition 7 remains valid when one replaces the independence assumption with the weaker assumption that $\Delta(B_1 \cup B_2) = \Delta(B_1)\Delta(B_2)$ for all $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$. An elegant proof of Proposition 7 can be found in [1]. The proposition motivates the definition of the following notion.

DEFINITION 8 (decomposable, cluster). A set A of random variables is *decomposable* if there exists a partition $A = A_1 \cup A_2$ such that the variables in A_1 are independent of the variables in A_2. A nondecomposable set is also called a *cluster*.

In our setting, the notion of a cluster has a natural combinatorial interpretation. Given $\Gamma = (\Omega, \mathcal{X})$ and $\mathbf{p} \in (0,1)^\Omega$, we define the *dependency graph* G_Γ as the graph on the vertex set $[N]$ whose edges are all pairs $\{i,j\}$ such that $i \sim j$, that is, $\gamma_i \cap \gamma_j \neq \varnothing$. It is then clear that a set $V \subseteq [N]$ induces a connected subgraph in G_Γ if and only if the set of random variables $\{X_i : i \in V\}$ is a cluster (this is one reason why it is convenient to assume $p_\omega \notin \{0,1\}$ for all $\omega \in \Omega$). In particular, the joint cumulant $\kappa(\{X_i : i \in V\})$ vanishes unless $G_\Gamma[V]$ is connected.

Motivated by this, we shall write $\mathcal{C}_k$ for the collection of all k-element subsets $V \subseteq [N]$ such that $G_\Gamma[V]$ is connected, and define

$$\kappa_k = \sum_{V \in \mathcal{C}_k} \kappa(\{X_i : i \in V\}) \quad \text{and} \quad \Delta_k = \sum_{V \in \mathcal{C}_k} \Delta(\{X_i : i \in V\}). \tag{8}$$

Note that this definition of Δ_k is consistent with the definition of Δ_2 given by (2). Moreover, it follows from (7) and Harris's inequality that $|\kappa_k| \le K_k \Delta_k$ for some K_k depending only on k.

1.4. *Our main result.* Let $\Gamma = (\Omega, \mathcal{X})$ and $\mathbf{p} \in (0,1)^\Omega$ be as above. Given a subset $V \subseteq [N]$, we write

$$\partial(V) = N_{G_\Gamma}(V) \setminus V$$

for the external neighbourhood of V in the dependency graph and let

$$\lambda(V) = \sum_{i \in \partial(V)} \mathbb{E}\Big[X_i \,\Big|\, \prod_{j \in V} X_j = 1\Big]$$

be the expected number of external neighbours i of V in the dependency graph such that $\gamma_i \subseteq \Omega_{\mathbf{p}}$, conditioned on $\gamma_j \subseteq \Omega_{\mathbf{p}}$ for all $j \in V$. For all $k \in \mathbb{N}$, we define

$$\Lambda_k(\Gamma, \mathbf{p}) = \max\{\lambda(V) : V \subseteq [N] \text{ and } 1 \le |V| \le k\}.$$

It can be intuitively helpful to think of $\Lambda_k(\Gamma, \mathbf{p})$ as a measure of (non)expansion of the dependency graph G_Γ.

THEOREM 9. *For every $n \in \mathbb{N}$, let $\Gamma(n) = (\Omega(n), \mathcal{X}(n))$ be a hypergraph and let $\mathbf{p}(n) \in (0,1)^{\Omega(n)}$. Assume that for every constant $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \max_{\omega\in\Omega(n)} p_\omega(n) = 0 \quad \textit{and} \quad \limsup_{n\to\infty} \Lambda_k\big(\Gamma(n), \mathbf{p}(n)\big) < \infty.$$

Let $X(n)$ denote the number of edges of $\Gamma(n)$ that are fully contained in $\Omega(n)_{\mathbf{p}(n)}$. Then, for every constant $k \in \mathbb{N}$,

$$\mathbb{P}\big[X(n) = 0\big] = \exp\big(-\kappa_1 + \kappa_2 - \cdots + (-1)^k \kappa_k + O(\delta_1 + \Delta_{k+1})\big)$$

as $n \to \infty$, where $\delta_1, \kappa_1, \dots, \kappa_k$, and Δ_{k+1} are defined as above.

The condition $\max\{p_\omega(n) : \omega \in \Omega(n)\} = o(1)$ implies $\kappa_k = \Delta_k + o(\Delta_k)$ for every fixed k, as can be seen from the definition (7) of κ_k. In such cases, the first-order behaviour of κ_k is thus given by Δ_k. However, this does *not* mean that we can then replace $\kappa_1, \dots, \kappa_k$ by $\Delta_1, \dots, \Delta_k$ in the formula for $\mathbb{P}[X(n) = 0]$ given by Theorem 9, because the lower-order terms can be nonnegligible; see the proof of Corollary 15 below, for instance.

The fact that $\kappa_1 = \mathbb{E}[X]$ shows that the case $k = 1$ of Theorem 9 gives (a slight weakening of) Janson's inequality (5). Unlike (5), Theorem 9 requires the additional assumptions $\max_{\omega\in\Omega(n)} p_\omega(n) = o(1)$ and $\Lambda_k(\Gamma(n), \mathbf{p}(n)) = O(1)$ for all constant k. Both conditions are perhaps not strictly necessary. As we will see further below, the latter condition implies that $\Delta_{k+1} = O(\Delta_k)$ for all constant k, which gives at least an indication of the type of assumption that is involved.

It is natural to ask under which conditions Theorem 9 can give asymptotically sharp bounds. While computing the first error term δ_1 is generally straightforward, it is not so obvious how one should estimate Δ_{k+1}. Here we will focus on the rather common situation where each edge of $\Gamma(n)$ has bounded size and there is some $p(n) \in (0,1)$ such that $\mathbf{p}_\omega(n) = p(n)$ for all $\omega \in \Omega(n)$. We then write simply $\Omega(n)_{p(n)}$ instead of $\Omega(n)_{\mathbf{p}(n)}$. This is the situation that we encounter in all of our applications.

Generally, for a hypergraph $\Gamma = (\Omega, \mathcal{X})$ and a subset $\Omega' \subseteq \Omega$, define the jth codegree of Ω' by

$$d_j(\Omega') = \big|\{\gamma \in \mathcal{X} : \Omega' \subseteq \gamma \text{ and } |\gamma| = |\Omega'| + j\}\big|,$$

and let

$$D(\Gamma, p) = \max_{j\ge 1} \max_{\varnothing\ne\Omega'\subseteq\Omega} d_j(\Omega') p^j;$$

one can think of this as a weighted maximum codegree of Γ. The following is a specialised version of Theorem 9 that gives an easily verifiable condition ensuring $\Delta_{k+1} = o(1)$ for some constant k.

THEOREM 10. *Let r be a fixed positive integer. For every $n \in \mathbb{N}$, let $\Gamma(n) = (\Omega(n), \mathcal{X}(n))$ be a hypergraph whose edges all have size at most r and let $p(n)$ be a real number in $(0,1)$. Assume*

$$\lim_{n\to\infty} p(n) = 0 \quad \textit{and} \quad \limsup_{n\to\infty} D\big(\Gamma(n), p(n)\big) < \infty.$$

Let $X(n)$ denote the number of edges of $\Gamma(n)$ that are fully contained in $\Omega(n)_{p(n)}$. Then, for every constant $k \in \mathbb{N}$,

$$\mathbb{P}\big[X(n) = 0\big] = \exp\big(-\kappa_1 + \kappa_2 - \cdots + (-1)^k \kappa_k + O(\delta_1 + \Delta_{k+1})\big)$$

as $n \to \infty$, where $\delta_1, \kappa_1, \dots, \kappa_k$, and Δ_{k+1} are defined as above.

Moreover, if $D(\Gamma(n), p(n)) \le |\Omega(n)|^{-\varepsilon}$ for some positive ε, then there is a positive integer $k = k(\varepsilon, r)$ such that $\Delta_{k+1} = o(1)$.

Let us briefly illustrate the applicability of this result by considering again the example of triangle-free random graphs.

EXAMPLE 6 (continuing from page 495). The hypergraph Γ of triangles in K_n is 3-uniform, so we can choose $r = 3$ in Theorem 10. One easily verifies that $D(\Gamma, p) \leq p + np^2$. We recall from our earlier discussion that $\delta_1 \leq n^3 p^6$. Therefore, Theorem 10 implies that for every fixed positive integer k and all $p = o(n^{-1/2})$, we have

$$\mathbb{P}[X = 0] = \exp\bigl(-\kappa_1 + \kappa_2 - \cdots + (-1)^k \kappa_k + O(\Delta_{k+1}) + o(1)\bigr).$$

Moreover, if $p \leq n^{-1/2-\varepsilon}$ for some positive constant ε, then there exists a constant k such that

$$\mathbb{P}[X = 0] = \exp\bigl(-\kappa_1 + \kappa_2 - \cdots + (-1)^k \kappa_k + o(1)\bigr),$$

that is, the asymptotics of $\mathbb{P}[X = 0]$ are given by a finite formula that we could in principle compute by analysing the finitely many possible "shapes" of clusters formed by at most k triangles in K_n.

We shall derive both of the above theorems from a more general result, Theorem 11 below, which has the advantage that it can be applied in certain nonsparse settings. Its disadvantage lies in the fact that the error terms are somewhat less transparent. For a set A of random variables, we define

$$\delta(A) = \Delta(A) \cdot \max\bigl\{\mathbb{E}[X] : X \in A\bigr\}.$$

Given $k \in \mathbb{N}$, we set

$$\delta_k = \sum_{V \in \mathcal{C}_k} \delta\bigl(\{X_i : i \in V\}\bigr), \tag{9}$$

analogously to (8), and

$$\rho_k = \max_{\substack{V \subseteq [N] \\ 1 \leq |V| \leq k}} \mathbb{P}\bigl[X_i = 1 \text{ for some } i \in V \cup \partial(V)\bigr]. \tag{10}$$

Observe that this definition of δ_k generalises (4).

THEOREM 11. *For every $k \in \mathbb{N}$ and $\varepsilon > 0$, there is a $K = K(k, \varepsilon)$ such that the following holds. Let $\Gamma = (\Omega, \mathcal{X})$ be a hypergraph and let $\mathbf{p} \in (0, 1)^{\Omega}$. If $\rho_{k+1} \leq 1 - \varepsilon$, then*

$$\bigl|\log \mathbb{P}[X = 0] + \kappa_1 - \kappa_2 + \kappa_3 - \cdots + (-1)^{k+1} \kappa_k\bigr| \leq K \cdot (\delta_{1,K} + \Delta_{k+1,K}),$$

where

$$\delta_{1,K} = \sum_{i=1}^{K} \delta_i \quad \textit{and} \quad \Delta_{k+1,K} = \sum_{i=k+1}^{K} \Delta_i.$$

We will derive Theorems 9 and 10 from Theorem 11 in Section 2. The proof of Theorem 11, which is the main part of this paper, will be presented in Section 3.

1.5. *Application*: *Random graphs and hypergraphs*. A fundamental question studied by the random graphs community, raised already in the seminal paper of Erdős and Rényi [8], is to determine the probability that $G_{n,p}$ contains no copies of a given "forbidden" graph F (as in Example 1). The classical result of Bollobás [5], proved independently by Karoński and Ruciński [16], determines this probability asymptotically for every strictly balanced[1] F, but only for p such that the expected number of copies of F in $G_{n,p}$ is constant. (In the case when F is a tree or a cycle, this was done earlier by Erdős and Rényi [8] and in the case when F is a complete graph, by Schürger [22].) It was later proved by Frieze [9] that the same estimate remains valid as long as the expected number of copies of F in $G_{n,p}$ is $o(n^{\varepsilon})$ for some positive constant ε that depends only on F. Wormald [25] and later Stark and Wormald [23] obtained asymptotic formulas for significantly larger ranges of p in the special case where F is a triangle. Prior to those papers and the present work, the strongest result of this form (i.e., determining the probability of being F-free asymptotically) for a general graph F followed from Harris's and Janson's inequalities; see (5). Finally, we remark that for several special graphs F, the probability that $G_{n,p}$ is F-free can be computed very precisely either when $p = 1/2$ or, in some cases, even for all sufficiently large $p = o(1)$ using the known precise structural characterisations of F-free graphs; see [4, 11, 17–19].

Using Theorem 10, we can answer this question for a large class of graphs and a wide range of densities. We will take a rather general point of view and consider the analogous problem in random r-uniform hypergraphs, where instead of just avoiding a single graph F, our goal is to avoid every graph in some finite family $\mathcal{F}$. Let $G_{n,p}^{(r)}$ denote the random r-uniform hypergraph (r-graph for short) on n vertices containing every possible edge (r-element subset of the vertices) with probability p, independently of other edges. In particular, $G_{n,p}^{(2)}$ is simply the binomial random graph $G_{n,p}$. Given a family $\mathcal{F} = \{F_1, \dots, F_t\}$ of r-graphs, we consider the problem of determining the probability that $G_{n,p}^{(r)}$ is $\mathcal{F}$-*free*, that is, it simultaneously avoids all copies of all r-graphs in $\mathcal{F}$.

Since removing isomorphic duplicates from $\mathcal{F}$ does not affect the probability that we are interested in, we can assume that the r-graphs in $\mathcal{F}$ are pairwise nonisomorphic. Similarly, we can assume that no hypergraph in $\mathcal{F}$ has isolated vertices.

We encode this problem in a hypergraph $\Gamma = (\Omega, \mathcal{X})$ by proceeding similarly as we did in Example 1. That is, we let $\Omega = \binom{[n]}{r}$ be the edge set of $K_n^{(r)}$, the complete r-graph with vertex set $[n]$, and we let $\mathcal{X}$ be the collection of edge sets of subhypergraphs of $K_n^{(r)}$ that are isomorphic to one of the r-graphs in $\mathcal{F}$. The probability that $G_{n,p}$ is $\mathcal{F}$-free is then precisely the probability that the p-random subset Ω_p contains no edges of Γ.

Note that the maximal size of an edge in Γ is bounded from above by the largest number of edges of an r-graph in $\mathcal{F}$, which does not depend on n. By applying Theorem 10 to this hypergraph, we can therefore get the asymptotics for the probability that $G_{n,p}^{(r)}$ is $\mathcal{F}$-free in a certain range of p. To quantify this range, given an r-graph F, define

$$m_*(F) = \min\left\{\frac{e_F - e_H}{v_F - v_H} : H \subseteq F \text{ with } v_H < v_F \text{ and } e_H > 0\right\},$$

where we use the convention $\min\varnothing = \infty$ and where v_G and e_G denote, respectively, the numbers of vertices and edges in a (hyper)graph G. For a family $\mathcal{F}$ of r-graphs, we then set

$$m_*(\mathcal{F}) = \min\{m_*(F) : F \in \mathcal{F}\} \quad \text{and} \quad d(\mathcal{F}) = \min\{e_F/v_F : F \in \mathcal{F}\}.$$

It is easy to see that $\delta_1 \le |\mathcal{F}| \cdot \max\{n^{v_F} p^{2e_F} : F \in \mathcal{F}\}$ and thus $\delta_1 = o(1)$ if $np^{2d(\mathcal{F})} = o(1)$. Moreover, for any nonempty set Ω' of edges in $K_n^{(r)}$ whose union forms an r-graph H with

[1] A graph F is strictly balanced if $e_F/v_F > e_H/v_H$ for every proper nonempty subgraph H of F.

$e_H > 0$ edges, we have

$$\max_{j\geq 1} d_j(\Omega')p^j = O\big(\max\{n^{v_F-v_H}p^{e_F-e_H} : H \subseteq F \in \mathcal{F} \text{ and } v_H < v_F\}\big).$$

It follows that $D(\Gamma, p) = (np^{m_*(\mathcal{F})})^{\Theta(1)}$. Theorem 10 then immediately implies the following result.

COROLLARY 12. *Let $\mathcal{F}$ be a finite family of r-uniform hypergraphs and assume that $p = p(n) \in (0,1)$ satisfies*

$$np^{m_*(\mathcal{F})} = o(1) \quad \textit{and} \quad np^{2d(\mathcal{F})} = o(1). \tag{11}$$

Then, for every constant $k \in \mathbb{N}$, we have

$$\mathbb{P}\big[G_{n,p}^{(r)} \textit{ is } \mathcal{F}\textit{-free}\big] = \exp\big(-\kappa_1 + \kappa_2 - \cdots + (-1)^k\kappa_k + O(\Delta_{k+1}) + o(1)\big)$$

as $n \to \infty$. Moreover, if $np^{m_*(\mathcal{F})} \leq n^{-\varepsilon}$ for some positive ε, then there is a positive integer $k = k(\varepsilon, \mathcal{F})$ such that $\Delta_{k+1} = o(1)$.

The conditions in (11) can be further simplified under certain natural assumptions on the family $\mathcal{F}$. Recall that the *r-density* of an r-graph F with at least two edges is

$$m_r(F) = \max\left\{\frac{e_H - 1}{v_H - r} : H \subseteq F \text{ with } e_H > 1\right\}$$

and that F is *r-balanced* if this maximum is achieved with $H = F$, that is, if $m_r(F) = (e_F - 1)/(v_F - r)$. Observe that for every F with at least two edges, we have

$$m_r(F) \geq \frac{e_F - 1}{v_F - r} \geq m_*(F).$$

We claim that if F is r-balanced, then in fact $m_r(F) = m_*(F)$. Indeed, writing $\alpha_K = (e_K - 1)/(v_K - r)$, we see that for every $H \subseteq F$ with $v_H < v_F$ and $e_H > 1$,

$$\frac{e_F - e_H}{v_F - v_H} = \frac{\alpha_F(v_F - r) - \alpha_H(v_H - r)}{(v_F - r) - (v_H - r)} \geq m_r(F),$$

since $m_r(F) = \alpha_F \geq \alpha_H$ (as F is r-balanced) and this inequality continues to hold if $e_H = 1$. Thus, $m_*(F) \geq m_r(F)$ and so $m_*(F) = m_r(F)$.

Another simplification is possible in the important special case $r = 2$. In this case, the second condition in (11) follows from the first condition, since $2e_F/v_F \geq (e_F - 1)/(v_F - 2)$ for every graph F and consequently $m_*(\mathcal{F}) \leq 2d(\mathcal{F})$ for every family of graphs $\mathcal{F}$.

COROLLARY 13. *Let $\mathcal{F}$ be a finite family of 2-balanced graphs with at least two edges each and let $p = p(n) \in (0,1)$ be such that $p = o(n^{-1/m_2(F)})$ for every $F \in \mathcal{F}$. Then, for every fixed $k \in \mathbb{N}$, we have*

$$\mathbb{P}[G_{n,p} \textit{ is } \mathcal{F}\textit{-free}] = \exp\big(-\kappa_1 + \kappa_2 - \cdots + (-1)^k\kappa_k + O(\Delta_{k+1}) + o(1)\big),$$

as $n \to \infty$. Moreover, if $p \leq n^{-1/m_2(F)-\varepsilon}$ for some positive ε and all $F \in \mathcal{F}$, then there is a positive integer $k = k(\varepsilon, \mathcal{F})$ such that $\Delta_{k+1} = o(1)$.

Of course, neither Corollary 12 nor Corollary 13 would be particularly useful if one could not compute the values κ_k for at least several small integers k. In Section 4, we outline a general approach for doing so and perform the calculations for two special cases.

COROLLARY 14. *If $p = o(n^{-4/5})$, then the probability that $G_{n,p}$ is simultaneously K_3-free and C_4-free is asymptotically*

$$\exp\left(-\frac{n^3p^3}{6} - \frac{n^4p^4}{8} + \frac{n^6p^7}{4} + \frac{2n^5p^6}{3}\right).$$

COROLLARY 15. *If $p = o(n^{-7/11})$, then the probability that $G_{n,p}$ is triangle-free is asymptotically*

$$\exp\left(-\frac{n^3p^3}{6} + \frac{n^4p^5}{4} - \frac{7n^5p^7}{12} + \frac{n^2p^3}{2} - \frac{3n^4p^6}{8} + \frac{27n^6p^9}{16}\right).$$

As mentioned above, Corollary 15 was obtained independently by Stark and Wormald [23]. It extends a result of Wormald [25] that applies to a smaller range of p. However, the derivation of Corollary 15 from Theorem 10 is very short compared to the proofs in [23] and [25].

1.6. *Application*: *Arithmetic progressions*. As a second application, we will estimate the probability that $[n]_p$, the p-random subset of $[n]$, is r-AP-free, that is, does not contain any arithmetic progression of length r. As in Example 2, we encode this problem in the hypergraph $\Gamma = (\Omega, \mathcal{X})$ on $\Omega = [n]$ whose edge set is the collection $\mathcal{X}$ of r-APs in $[n]$.

Since any two distinct integers are contained at most $\binom{r}{2} = O(1)$ common r-APs, it is easy to see that $\delta_1 = O(n^2p^{2r})$ and $D(\Gamma, p) = O(p + np^{r-1})$. Therefore, Theorem 10 has the following corollary.

COROLLARY 16. *Let $r \geq 3$ be a fixed integer and assume $p = p(n) \in (0, 1)$ satisfies $p = o(n^{-1/(r-1)})$. Then, for every fixed $k \in \mathbb{N}$, we have*

$$\mathbb{P}\big[[n]_p \text{ is } r\text{-AP-free}\big] = \exp\big(-\kappa_1 + \kappa_2 - \cdots + (-1)^k\kappa_k + O(\Delta_{k+1}) + o(1)\big)$$

as $n \to \infty$. Moreover, if $p = o(n^{-1/(r-1)-\varepsilon})$ for some positive constant ε, then there exists a positive integer $k = k(\varepsilon, r)$ such that $\Delta_{k+1} = o(1)$.

In Section 4, we will perform the necessary calculations to determine the precise asymptotics of $\mathbb{P}[[n]_p$ is r-AP-free] for $p = o(n^{-4/7})$.

COROLLARY 17. *If $p = o(n^{-4/7})$, then the probability that $[n]_p$ is 3-AP-free is asymptotically*

$$\exp\left(-\frac{n^2p^3}{4} + \frac{7n^3p^5}{24}\right).$$

1.7. *Related work and open problems*. Janson's inequality was first proved (by Svante Janson himself) during the 1987 conference on random graphs in Poznań, in response to Bollobás's announcement of his estimate [6] for the chromatic number of random graphs, which requires a strong upper bound on the probability that a random graph contains no large cliques. A related estimate was found, during the same conference, by Łuczak. Janson's original proof was based on the analysis of the moment-generating function of X, whereas Łuczak's proof used martingales. Both of these arguments can be found in [14]. Our proof of Theorem 11 is inspired by a subsequent proof of Janson's inequality that was found soon afterwards by Boppana and Spencer [7]; it uses only the Harris inequality. Somewhat later, Janson [12] showed that his proof actually gives bounds for the whole lower tail, and not just for the probability $\mathbb{P}[X=0]$. Around the same time, Suen [24] proved a correlation inequality that is very similar to Janson's. Suen's inequality gives a slightly weaker estimate (which was later sharpened by Janson [13]), but is applicable in a much more general context. Another generalisation of Janson's inequality was obtained recently by Riordan and Warnke [20].

In [25], Wormald proved that if $p=o(n^{-2/3})$, then

$$\mathbb{P}[G_{n,p} \text{ is } K_3\text{-free}] = \exp\left(-\frac{n^3p^3}{6}+\frac{n^4p^5}{4}-\frac{7n^5p^7}{12}+o(1)\right), \tag{12}$$

whereas for $G_{n,m}$ with $m=d\binom{n}{2}$ and $d=o(n^{-2/3})$, we have

$$\mathbb{P}[G_{n,m} \text{ is } K_3\text{-free}] = \exp\left(-\frac{n^3d^3}{6}+o(1)\right).$$

These results were strengthened recently by Stark and Wormald [23], who obtained the approximation in Corollary 15 (which implies (12)) and also

$$\mathbb{P}[G_{n,m} \text{ is } K_3\text{-free}] = \exp\left(-\frac{n^3d^3}{6}+\frac{n^2d^3}{2}-\frac{n^4d^6}{8}+o(1)\right),$$

where $m=d\binom{n}{2}$, which holds when $d=o(n^{-7/11})$. In fact, they were able to obtain a more general result, which states that in the range where Corollary 13 is applicable, the probability that $G_{n,p}$ or $G_{n,m}$ is F-free is approximated by the exponential of the first few terms of a power series in n and p (resp. d) whose terms depend only on F. However, the way in which these terms are computed is rather implicit. In contrast, in the setting of binomial random subsets such as $G_{n,p}$, our Theorem 9 explains what these terms are.

While our results (and our methods) apply only to binomial subsets (e.g., $G_{n,p}$ and not $G_{n,m}$), the results for $G_{n,p}$ could conceivably be transferred to $G_{n,m}$ using the identity

$$\mathbb{P}[G_{n,m} \text{ is } F\text{-free}] = \frac{\mathbb{P}[G_{n,p} \text{ is } F\text{-free}]\cdot \mathbb{P}[e(G_{n,p})=m \mid G_{n,p} \text{ is } F\text{-free}]}{\mathbb{P}[e(G_{n,p})=m]}.$$

It was shown by Stark and Wormald [23] that the conditional probability in the right-hand side can be computed explicitly for a carefully chosen p of the same order of magnitude as d. However, this is not at all an easy task.

It would be interesting to establish a similar relationship in the more abstract and general setting of random induced subhypergraphs. If this was possible, Theorem 9 could be used to count independent sets of a given (sufficiently small) cardinality in general hypergraphs. In some sense, this would complement the counting results that can be obtained with the so-called hypergraph container method developed by Balogh, Morris, and Samotij [3] and by Saxton and Thomason [21]. Whereas the container method applies to somewhat large independent sets, which exhibit a "global" structure, our Theorem 9 would yield estimates on the number of smaller independent sets that only exhibit "local" structure. In particular, the container method can be used to estimate the probability that $G_{n,p}$ is F-free whenever $p=\omega(n^{-1/m_2(F)})$ for every nonbipartite graph F. For p in this range, $G_{n,p}$ conditioned on being F-free is approximately $(\chi(F)-1)$-partite with very high probability. On the other hand, our method (and the method of [23]) applies whenever $p=o(n^{-1/m_2(F)})$, provided that F is 2-balanced. For p in this range, the edges of $G_{n,p}$ conditioned on being F-free are still distributed very uniformly with probability close to one.

2. Proofs of Theorems 9 and 10. In this section, we will show that Theorem 11 implies Theorems 9 and 10. To prove Theorem 9, we need the following lemma, which also clarifies the definition of Λ_k.

LEMMA 18. *For every hypergraph* $\Gamma = (\Omega, \mathcal{X})$, *every* $\mathbf{p} \in (0,1)^\Omega$, *and every positive integer* k, *we have*

$$\Delta_{k+1}/\Delta_k \le \Lambda_k(\Gamma, \mathbf{p}) \quad \textit{and} \quad \delta_{k+1}/\delta_k \le \Lambda_k(\Gamma, \mathbf{p}).$$

PROOF. For every $V \in \mathcal{C}_{k+1}$ there exist at least two distinct $i \in V$ such that $V \setminus \{i\} \in \mathcal{C}_k$. Indeed, every connected graph with at least two vertices has at least two noncut vertices. Therefore, for each $V \in \mathcal{C}_{k+1}$, we can make a canonical choice of a set $V^- \subset V$ such that $V^- \in \mathcal{C}_k$ and

$$\max\{\mathbb{E}[X_i] : i \in V\} = \max\{\mathbb{E}[X_i] : i \in V^-\}. \tag{13}$$

Denoting by i_V the unique element in $V \setminus V^-$, we have $i_V \in \partial(V^-)$ because $G_\Gamma[V]$ is connected. Moreover,

$$\Delta(\{X_i : i \in V\}) = \Delta(\{X_i : i \in V^-\}) \cdot \mathbb{E}\Big[X_{i_V} \,\Big|\, \prod_{i \in V^-} X_i = 1\Big]$$

and, analogously,

$$\delta(\{X_i : i \in V\}) = \delta(\{X_i : i \in V^-\}) \cdot \mathbb{E}\Big[X_{i_V} \,\Big|\, \prod_{i \in V^-} X_i = 1\Big].$$

It follows that

$$\begin{aligned}\Delta_{k+1} &\le \sum_{V^- \in \mathcal{C}_k} \Delta(\{X_i : i \in V^-\}) \sum_{j \in \partial(V^-)} \mathbb{E}\Big[X_j \,\Big|\, \prod_{i \in V^-} X_i = 1\Big] \\ &= \sum_{V^- \in \mathcal{C}_k} \Delta(\{X_i : i \in V^-\}) \cdot \lambda(V^-) \le \Delta_k \cdot \Lambda_k(\Gamma, \mathbf{p})\end{aligned}$$

and, similarly, $\delta_{k+1} \le \delta_k \cdot \Lambda_k(\Gamma, \mathbf{p})$. □

PROOF OF THEOREM 9 FROM THEOREM 11. Assume that $\Gamma(n) = (\Omega(n), \mathcal{X}(n))$ and $\mathbf{p}(n) = (p_\omega(n))_{\omega \in \Omega(n)}$ are as in the statement of the theorem.

Fix any $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$ and let $K = K(k, \varepsilon)$ be as given by Theorem 11. We verify that $\Gamma(n)$ and $\mathbf{p}(n)$ satisfy the assumption of Theorem 11 for all sufficiently large n. For this, consider some nonempty $V \subseteq [N]$ of size at most $k+1$. Since $p = o(1)$, we have $\sum_{i \in V} \mathbb{E}[X_i] \le (1-\varepsilon)/2$ for all sufficiently large n. Additionally, if $i \in \partial(V)$, then γ_i intersects $\bigcup_{j \in V} \gamma_j$. Therefore,

$$\sum_{i \in \partial(V)} \mathbb{E}[X_i] \le \lambda(V) \cdot \max\Big\{p_\omega(n) : \omega \in \bigcup_{j \in V} \gamma_j\Big\} \le (1-\varepsilon)/2.$$

By the union bound, this implies

$$\rho_{k+1} = \max_{\substack{V \subseteq [N] \\ 1 \le |V| \le k+1}} \mathbb{P}\big[X_i = 1 \text{ for some } i \in V \cup \partial(V)\big] \le 1 - \varepsilon.$$

Therefore, Theorem 11 yields

$$\big|\log \mathbb{P}[X=0] + \kappa_1 - \kappa_2 + \cdots + (-1)^{k+1}\kappa_k\big| \le K \cdot (\delta_{1,K} + \Delta_{k+1,K}).$$

Using Lemma 18 and our assumption that $\Lambda_i(\Gamma(n), \mathbf{p}(n)) = O(1)$ for all constant i (in particular, for all $1 \le i \le K$), we get

$$K \cdot \delta_{1,K} = K \cdot \sum_{i=1}^K \delta_i = O(\delta_1) \quad \text{and} \quad K \cdot \Delta_{k+1,K} = K \cdot \sum_{i=k+1}^K \Delta_i = O(\Delta_{k+1}),$$

which completes the proof. □

LEMMA 19. *For all positive integers k and r, there exist $k' = k'(k,r) \geq 1$ and $K = K(k,r)$ such that, for every $p \in (0,1)$ and every hypergraph $\Gamma = (\Omega, \mathcal{X})$ with all edges of size at most r,*

$$\Delta_{k'}/\Delta_k \leq K \cdot \max\{D(\Gamma, p), D(\Gamma, p)^{k'}\}.$$

PROOF. Define $D^{(j)} = \max_{\varnothing \neq \Omega' \subseteq \Omega} d_j(\Omega')$ for every $j \geq 1$ and note that then $D(\Gamma, p) = \max_{j \geq 1} D^{(j)} p^j$. It is convenient to also define $D^{(0)} = 1$.

We choose $k' = 2^{rk}$. Note that if $V \in \mathcal{C}_{k'}$, then there is an ordering of the elements of V as $i_1, \dots, i_{k'}$ such that the set $\{i_1, \dots, i_\ell\}$ belongs to $\mathcal{C}_\ell$ for all $\ell \in [k']$. For every ℓ, let $j_\ell = |\gamma_{i_\ell} \setminus (\gamma_{i_1} \cup \dots \cup \gamma_{i_{\ell-1}})|$. Since $|\gamma_i| \leq r$ for all i, there are at most $2^{rk} - 1$ edges of Γ that are completely contained in $\gamma_{i_1} \cup \dots \cup \gamma_{i_k}$. Therefore, by our choice of k', at least one of $j_{k+1}, \dots, j_{k'}$ must be nonzero. Since there at most $2^{r\ell}$ choices for the intersection of γ_{i_ℓ} and $\gamma_{i_1} \cup \dots \cup \gamma_{i_{\ell-1}}$, it then follows that

$$\Delta_{k'}/\Delta_k \leq \sum_{\substack{0 \leq j_{k+1}, \dots, j_{k'} \leq r \\ j_{k+1} + \dots + j_{k'} \geq 1}} \prod_{\ell=k+1}^{k'} 2^{r\ell} D^{(j_\ell)} p^{j_\ell} \leq K \cdot \max\{D(\Gamma, p), D(\Gamma, p)^{k'}\}$$

for an appropriate choice of K. $\square$

PROOF OF THEOREM 10 FROM THEOREM 9. Suppose that $\Gamma(n) = (\Omega(n), \mathcal{X}(n))$ and $p(n) \in (0,1)$ are as in the statement of the theorem. Define the sequence $\mathbf{p}(n) = (p_\omega(n))_{\omega \in \Omega(n)}$ by $p_\omega(n) = p(n)$ for all $\omega \in \Omega(n)$. For every $V \subseteq [N]$, we have $|\bigcup_{i \in V} \gamma_i| \leq r|V|$, and so

$$\begin{aligned}
\lambda(V) &= \sum_{i \in \partial(V)} \mathbb{E}\Big[X_i \,\Big|\, \prod_{j \in V} X_j = 1\Big] \\
&\leq 2^{r|V|} + \sum_{\varnothing \neq \Omega' \subseteq \bigcup_{i \in V} \gamma_i} (r-1) \max_{j \geq 1} d_j(\Omega') p(n)^j \\
&\leq 2^{r|V|}\big(1 + (r-1) D(\Gamma(n), p(n))\big).
\end{aligned}$$

Using our assumption on $D(\Gamma(n), p(n))$, this implies $\Lambda_k(\Gamma(n), \mathbf{p}(n)) = O(1)$ for every fixed $k \in \mathbb{N}$. Since we also assume $p(n) \to 0$, Theorem 9 implies the first statement of Theorem 10.

To see the second statement, assume $D(\Gamma(n), p(n)) \leq |\Omega(n)|^{-\varepsilon}$ for a positive ε. By Lemma 19, iterated r/ε times, we find that there are $k = k(\varepsilon, r)$ and $K = K(\varepsilon, r)$ such that $\Delta_k \leq K \cdot |\Omega(n)|^{-r} \cdot \Delta_1$. Since $\Delta_1 \leq |\Omega(n)|^r p(n)$, we obtain $\Delta_k \leq K p(n) = o(1)$. $\square$

3. Proof of Theorem 11. Let Γ and $\mathbf{p}$ be as in the statement of the theorem. We start the proof by establishing some notational conventions. Given a subset $V \subseteq [N]$, we use the abbreviations

$$X_V = \prod_{i \in V} X_i \quad \text{and} \quad \overline{X}_V = \prod_{i \in V} (1 - X_i).$$

Note that these are the indicator variables for the events "$\gamma_i \subseteq \Omega_{\mathbf{p}}$ for all $i \in V$" and "$\gamma_i \nsubseteq \Omega_{\mathbf{p}}$ for all $i \in V$", respectively. Besides being positively correlated by Harris's inequality, the variables X_V satisfy the stronger FKG lattice condition

$$\mathbb{E}[X_U]\mathbb{E}[X_V] \leq \mathbb{E}[X_{U \cup V}]\mathbb{E}[X_{U \cap V}] \quad \text{for all } U, V \subseteq \mathcal{X}. \tag{14}$$

To see that this is true, rewrite (14) using $\mathbb{E}[X_W] = \prod_{\omega \in \bigcup W} p_\omega$, take logarithms of both sides, and note that

$$\begin{aligned}
&\sum_{\omega \in \bigcup_{i \in U \cup V} \gamma_i} \log p_\omega \\
&\quad = \sum_{\omega \in \bigcup_{i \in U} \gamma_i} \log p_\omega + \sum_{\omega \in \bigcup_{i \in V} \gamma_i} \log p_\omega - \sum_{\omega \in (\bigcup_{i \in U} \gamma_i) \cap (\bigcup_{i \in V} \gamma_i)} \log p_\omega \\
&\quad \geq \sum_{\omega \in \bigcup_{i \in U} \gamma_i} \log p_\omega + \sum_{\omega \in \bigcup_{i \in V} \gamma_i} \log p_\omega - \sum_{\omega \in \bigcup_{i \in U \cap V} \gamma_i} \log p_\omega,
\end{aligned}$$

since $\log p_\omega < 0$ for all $\omega \in \Omega$ and $\bigcup_{i \in U \cap V} \gamma_i \subseteq \bigcup_{i \in U} \gamma_i \cap \bigcup_{i \in V} \gamma_i$.

We will also use the notation

$$\mu_\pi = \prod_{P \in \pi} \mathbb{E}[X_P]$$

whenever π is a set of subsets of $[N]$ (usually a partition of some subset of $[N]$). Thus, for a nonempty subset $V \subseteq [N]$, the value

$$\kappa(V) = \sum_{\pi \in \Pi(V)} (-1)^{|\pi|-1} \big(|\pi| - 1\big)! \mu_\pi \tag{15}$$

is the joint cumulant of $\{X_i : i \in V\}$. For the sake of brevity, we will from now on write $\kappa(V)$ instead of the more cumbersome $\kappa(\{X_i : i \in V\})$.

Recall that we denote by $\partial(V)$ the external neighbourhood of V in the dependency graph, that is,

$$\partial(V) = N_{G_\Gamma}(V) \setminus V$$

for every nonempty subset $V \subseteq [N]$. We define

$$\rho_V = \mathbb{P}\big[X_i = 1 \text{ for some } i \in V \cup \partial(V)\big], \tag{16}$$

so that $\rho_{k+1} = \max\{\rho_V : V \subseteq [N] \text{ and } 1 \leq |V| \leq k+1\}$. Moreover, we set

$$I(V) = [N] \setminus \big(V \cup \partial(V)\big).$$

Neglecting the distinction between an index i and the variable X_i, we may say that $\partial(V)$ contains the variables outside of of V that are dependent on V and $I(V)$ contains those that are independent of V. As above, we write $\mathcal{C}_i$ for the collection of all i-element sets $V \subseteq [N]$ such that $G_\Gamma[V]$ is connected. We will also write $\mathcal{C}_i(\ell)$ for the subset of $\mathcal{C}_i$ comprising all $A \in \mathcal{C}_i$ with $\max A = \ell$.

Assume that $k \in \mathbb{N}$ and $\varepsilon > 0$ are such that $\rho_{k+1} \leq 1 - \varepsilon$. Note that this implies, in particular, that $\mathbb{E}[X_i] \leq 1 - \varepsilon$ for all $i \in [N]$. Then we need to show that, for some $K = K(k, \varepsilon)$,

$$\left| \log \mathbb{P}[X = 0] + \sum_{i \in [k]} (-1)^{i+1} \kappa_i \right| \leq K \cdot (\delta_{1,K} + \Delta_{k+1,K}),$$

where

$$\delta_{1,K} = \sum_{i=1}^{K} \delta_i \quad \text{and} \quad \Delta_{k+1,K} = \sum_{i=k+1}^{K} \Delta_i.$$

To do so, we first write out the probability that $X = 0$ using the chain rule:

$$\mathbb{P}[X = 0] = \prod_{\ell \in [N]} \mathbb{P}[X_\ell = 0 \mid \overline{X}_{[\ell-1]} = 1] = \prod_{\ell \in [N]} \big(1 - \mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1]\big).$$

Note that by the Harris inequality, $\mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] \le \mathbb{E}[X_\ell] \le 1 - \varepsilon$. Taking logarithms of both sides of the above equality and using the fact that $|\log(1-x) + x| \le x^2/\varepsilon$ for $x \in [0, 1-\varepsilon]$, we get

$$\left|\log \mathbb{P}[X = 0] + \sum_{\ell \in [N]} \mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1]\right| \le \sum_{\ell \in [N]} \mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1]^2/\varepsilon.$$

Hence, using again $\mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] \le \mathbb{E}[X_\ell]$,

$$\left|\log \mathbb{P}[X = 0] + \sum_{\ell \in [N]} \mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1]\right| \le \sum_{\ell \in [N]} \mathbb{E}[X_\ell]^2/\varepsilon = \delta_1/\varepsilon. \tag{17}$$

Thus, our main goal becomes estimating the sum

$$\sum_{\ell \in [N]} \mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1]. \tag{18}$$

We shall do this by approximating (18) by an expression involving the quantities

$$q(V, S) = \frac{(-1)^{|V|-1}\mathbb{E}[X_V]}{\mathbb{E}[\overline{X}_{S \setminus I(V)} \mid \overline{X}_{S \cap I(V)} = 1]}. \tag{19}$$

This ratio is well defined for all $V, S \subseteq [N]$ because

$$\mathbb{E}[\overline{X}_{S \setminus I(V)} \mid \overline{X}_{S \cap I(V)} = 1] \ge \mathbb{E}[\overline{X}_{S \setminus I(V)}] > 0,$$

which is a consequence of the Harris inequality and the assumption that $p_\omega < 1$ for all $\omega \in \Omega$. The relationship between (18) and (19) is made precise in the following lemma:

LEMMA 20. *Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be such that $\rho_{k+1} \le 1 - \varepsilon$. Then*

$$\left|\sum_{\ell \in [N]} \mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] - \sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} q(V, [\ell - 1])\right| \le \Delta_{k+1}/\varepsilon.$$

We postpone the proof of Lemma 20 to Section 3.1 and instead show how it implies the assertion of the theorem. Before we can do this, we need several additional definitions.

DEFINITION 21 (Attachment). Given subsets $U, V \subseteq [N]$, we say that U *attaches* to V, in symbols $U \hookrightarrow V$, if every connected component of $G_\Gamma[U \cup V]$ contains a vertex of V (see Figure 1).

We state the following simple facts for future reference:

(i) We have $\varnothing \hookrightarrow V$ for every $V \subseteq [N]$.
(ii) If $i \in \partial(V)$, then $\{i\} \hookrightarrow V$.
(iii) If $U \hookrightarrow V$ and $W \hookrightarrow V$, then also $U \cup W \hookrightarrow V$.
(iv) If $V \in \mathcal{C}_{|V|}$ and $U \hookrightarrow V$, then $U \cup V \in \mathcal{C}_{|U \cup V|}$.

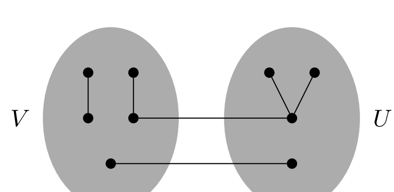

FIG. 1. *The set U attaches to V, that is, $U \hookrightarrow V$, but not vice-versa.*

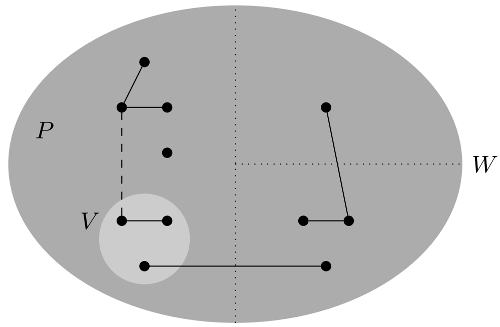

FIG. 2. *A partition in $\Pi_V^{\mathsf{C}}(W)$. Note that V is the union of components of the subgraph induced by the part P containing it. If the dashed edge were in G_Γ, then the partition would no longer be in $\Pi_V^{\mathsf{C}}(W)$.*

DEFINITION 22. Suppose that $\varnothing \neq V \subseteq W \subseteq [N]$. We define

$$\Pi_V^{\mathsf{C}}(W) \subseteq \Pi(W)$$

to be the set of all partitions π of W that contain a part $P \in \pi$ such that $V \subseteq P$ and V is the union of connected components of $G_\Gamma[P]$ (see Figure 2).

Next, for $\varnothing \neq V \subseteq W \subseteq [N]$, we define

$$\kappa_V(W) = \sum_{\pi \in \Pi_V^{\mathsf{C}}(W)} (-1)^{|\pi|-1}(|\pi|-1)!\mu_\pi. \tag{20}$$

Note that this is very similar to the definition (15) of $\kappa(W)$, except that we sum over $\Pi_V^{\mathsf{C}}(W)$ instead of $\Pi(W)$. For every $k \in \mathbb{N}$ and all $V, S \subseteq [N]$ with $V \neq \varnothing$, we set

$$\kappa_V^{(k)}(S) = \sum_{\substack{V \subseteq W \subseteq V \cup S \\ W \hookrightarrow V \\ |W| \leq k}} (-1)^{|W|-1}\kappa_V(W). \tag{21}$$

Certainly, this is a very complicated definition, whose meaning is far from clear at this point. However, it serves as a convenient "bridge" between $q(V, [\ell-1])$ and the values κ_i, as shown by the following two lemmas:

LEMMA 23. *Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be such that $\rho_{k+1} \leq 1-\varepsilon$. Then there is some $K = K(k, \varepsilon)$ such that*

$$\left|\sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} \big(q(V, [\ell-1]) - \kappa_V^{(k)}([\ell-1])\big)\right| \leq K \cdot (\delta_{1,K} + \Delta_{k+1,K}).$$

LEMMA 24. *For every $k \in \mathbb{N}$, we have*

$$\sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} \kappa_V^{(k)}([\ell-1]) = \sum_{i \in [k]} (-1)^{i+1}\kappa_i.$$

Theorem 11 is an easy consequence of Lemmas 20, 23, and 24. Indeed, assume $k \in \mathbb{N}$ and $\varepsilon > 0$ are such that $\rho_{k+1} \leq 1-\varepsilon$. It follows from (17), the above three lemmas, and the triangle inequality that

$$\left|\log \mathbb{P}[X=0] + \sum_{i \in [k]} (-1)^{i+1}\kappa_i\right| \leq \delta_1/\varepsilon + \Delta_{k+1}/\varepsilon + K' \cdot (\delta_{1,K'} + \Delta_{k+1,K'})$$

for some $K' = K'(k, \varepsilon)$. The assertion of the theorem now follows simply by observing that the right-hand side above is at most $K \cdot (\delta_{1,K} + \Delta_{k+1,K})$ for $K = K' + 1/\varepsilon$.

3.1. *Proof of Lemma* 20. We derive Lemma 20 from the following auxiliary lemma, which will also be used in the proof of Lemma 23.

LEMMA 25. *Assume that $V, S \subseteq [N]$ are disjoint. Then for every nonnegative integer k,*

$$(-1)^k \cdot \mathbb{E}[X_V \mid \overline{X}_S = 1] \le (-1)^{k+|V|-1} \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ |U| \le k}} q(V \cup U, S). \tag{22}$$

PROOF. We claim that it suffices to prove that for every integer $k \ge 0$,

$$(-1)^k \cdot \mathbb{E}[X_V \overline{X}_S] \le \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 0 \le |U| \le k}} (-1)^{k+|U|} \mathbb{E}[X_{V \cup U}] \mathbb{E}[\overline{X}_{S \cap I(V \cup U)}]. \tag{23}$$

Indeed, (23) implies (22) because

$$\mathbb{E}[\overline{X}_{S \cap I(V \cup U)}] = \mathbb{P}[\overline{X}_S = 1] \cdot \mathbb{E}[\overline{X}_{S \setminus I(V \cup U)} \mid \overline{X}_{S \cap I(V \cup U)} = 1]^{-1}$$

and because definition (19) gives

$$q(V \cup U, S) = \frac{(-1)^{|V|+|U|-1} \mathbb{E}[X_{V \cup U}]}{\mathbb{E}[\overline{X}_{S \setminus I(V \cup U)} \mid \overline{X}_{S \cap I(V \cup U)} = 1]}.$$

We prove (23) by induction on k. When $k = 0$, the inequality simplifies to

$$\mathbb{E}[X_V \overline{X}_S] \le \mathbb{E}[X_V] \mathbb{E}[\overline{X}_{S \cap I(V)}],$$

which holds because $\overline{X}_S \le \overline{X}_{S \cap I(V)}$ and because X_V and $X_{S \cap I(V)}$ are independent. Assume now that $k \ge 1$ and that (23) holds for all k' with $0 \le k' < k$. It follows from the Bonferroni inequalities that

$$(-1)^k \cdot \overline{X}_{S \cap \partial(V)} \le (-1)^k \cdot \sum_{\substack{U' \subseteq S \cap \partial(V) \\ |U'| \le k}} (-1)^{|U'|} X_{U'}. \tag{24}$$

Since S and V are disjoint and $\partial(V) \cup I(V) = [N] \setminus V$, then multiplying (24) through by $X_V \overline{X}_{S \cap I(V)}$ and taking expectations yields

$$(-1)^k \cdot \mathbb{E}[X_V \overline{X}_S] \le \sum_{\substack{U' \subseteq S \cap \partial(V) \\ |U'| \le k}} (-1)^{k+|U'|} \mathbb{E}[X_{V \cup U'} \overline{X}_{S \cap I(V)}]. \tag{25}$$

Observe that for every $U' \subseteq S \cap \partial(V)$, the sets $V \cup U'$ and $S \cap I(V)$ are disjoint. In particular, if U' is nonempty, then we may appeal to the induction hypothesis (with $k \leftarrow k - |U'|$) to bound each term in the right-hand side of (25) as follows. As $S \cap I(V) \cap I(V \cup U' \cup U'') = S \cap I(V \cup U' \cup U'')$,

$$\begin{aligned} &(-1)^{k+|U'|} \cdot \mathbb{E}[X_{V \cup U'} \overline{X}_{S \cap I(V)}] \\ &\quad \le \sum_{\substack{U'' \subseteq S \cap I(V) \\ U'' \hookrightarrow V \cup U' \\ 0 \le |U''| \le k - |U'|}} (-1)^{k+|U'|+|U''|} \mathbb{E}[X_{V \cup U' \cup U''}] \mathbb{E}[\overline{X}_{S \cap I(V \cup U' \cup U'')}]. \end{aligned} \tag{26}$$

Finally, observe that every nonempty $U \subseteq S$ such that $U \hookrightarrow V$ can be partitioned into a nonempty $U' \subseteq S \cap \partial(V)$ and an $U'' \subseteq S \cap I(V)$ such that $U'' \hookrightarrow (V \cup U')$ in a unique way. Indeed, one sets $U' = U \cap \partial(V)$ and $U'' = U \setminus U'$; this is the only such partition. Since $\varnothing \hookrightarrow V$ by definition, then bounding each term in (25) that corresponds to a nonempty U' using (26) and rearranging the sum gives (23). $\square$

PROOF OF LEMMA 20. Fix $\ell \in [N]$ and assume $k \in \mathbb{N}$ and $\varepsilon > 0$ are such that $\rho_{k+1} \le 1-\varepsilon$. Invoking Lemma 25 with $V = \{\ell\}$ and $S = [\ell-1]$ twice, first with $k \leftarrow k-1$ and then with $k \leftarrow k$, to get both an upper and a lower bound on $\mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]}]$, we obtain

$$\begin{aligned} &\left|\mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] - \sum_{\substack{U \subseteq [\ell-1], U \hookrightarrow \{\ell\} \\ |U| \le k-1}} q\big(U \cup \{\ell\}, [\ell-1]\big)\right| \\ &\quad \le \left|\sum_{\substack{U \subseteq [\ell-1], U \hookrightarrow \{\ell\} \\ |U| = k}} q\big(U \cup \{\ell\}, [\ell-1]\big)\right|. \end{aligned} \tag{27}$$

Since the sets $U \cup \{\ell\}$ with $U \subseteq [\ell-1]$, $U \hookrightarrow \{\ell\}$, and $|U| = i-1$ are precisely the elements of $\mathcal{C}_i(\ell)$, we can rewrite the above inequality as

$$\left|\mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] - \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} q\big(V, [\ell-1]\big)\right| \le \sum_{V \in \mathcal{C}_{k+1}(\ell)} \big|q\big(V, [\ell-1]\big)\big|. \tag{28}$$

It follows from definition (19) and Harris's inequality that

$$\begin{aligned} \big|q(V,S)\big| &= \frac{\mathbb{E}[X_V]}{\mathbb{E}[\overline{X}_{S \setminus I(V)} \mid \overline{X}_{S \cap I(V)} = 1]} \\ &= \frac{\mathbb{E}[X_V]}{1 - \mathbb{P}[X_i = 1 \text{ for some } i \in S \setminus I(V) \mid \overline{X}_{S \cap I(V)} = 1]} \le \frac{\mathbb{E}[X_V]}{1-\rho_V}. \end{aligned}$$

Since $\rho_V \le \rho_{k+1} \le 1-\varepsilon$ for all V with $|V| = k+1$, summing (28) over all $\ell \in [N]$ yields

$$\left|\sum_{\ell \in [N]} \mathbb{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] - \sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} q\big(V, [\ell-1]\big)\right| \le \Delta_{k+1}/\varepsilon,$$

which is precisely the assertion of the lemma. $\square$

3.2. *Proof of Lemma 23—Preliminaries.* The goal of this subsection is to derive a recursive formula for $\kappa_V(W)$, stated in Lemma 30 below, which will be used in the proof of Lemma 23.

DEFINITION 26. Suppose that $\varnothing \ne V \subseteq W \subseteq [N]$. We define $\Pi_V(W)$ and $\Pi_V^{\hookrightarrow}(W)$ as follows:

1. $\Pi_V(W)$ is the set of all partitions of W that contain V as a part;
2. $\Pi_V^{\hookrightarrow}(W)$ is the set of all partitions $\pi \in \Pi_V(W)$ such that $P \hookrightarrow V$ for every part $P \in \pi$.

Since by now we have defined several different classes of partitions of a set W, it is a good moment to pause and convince ourselves that

$$\Pi_V^{\hookrightarrow}(W) \subseteq \Pi_V(W) \subseteq \Pi_V^{\mathsf{C}}(W) \subseteq \Pi(W).$$

As a first step towards the promised recursive formula, we give an alternative expression for $\kappa_V(W)$.

DEFINITION 27 (Degree of a part in a partition). For a partition π of a subset of $[N]$ and any part $P \in \pi$, let $d_\pi(P)$ denote the number of parts $P' \in \pi \setminus \{P\}$ such that G_Γ contains an edge between P' and P. We call $d_\pi(P)$ the *degree* of P in π.

LEMMA 28. *If* $\varnothing \neq V \subseteq W \subseteq [N]$, *then*

$$\kappa_V(W) = \sum_{\pi \in \Pi_V(W)} (-1)^{|\pi|-1} \chi_V(\pi) \mu_\pi,$$

where

$$\chi_V(\pi) = \begin{cases} 1 & \text{if } |\pi| = 1, \\ d_\pi(V)\big(|\pi| - 2\big)! & \text{if } |\pi| \geq 2. \end{cases}$$

PROOF. Given a $\pi \in \Pi_V^{\mathsf{C}}(W)$, let P denote the part of π containing V. Define a map $f \colon \Pi_V^{\mathsf{C}}(W) \to \Pi_V(W)$ as follows. If $P = V$, then let $f(\pi) = \pi$. Otherwise, let $f(\pi)$ be the partition obtained from π by splitting P into V and $P \setminus V$. Clearly,

$$\begin{aligned} \kappa_V(W) &= \sum_{\pi \in \Pi_V^{\mathsf{C}}(W)} (-1)^{|\pi|-1} \big(|\pi| - 1\big)! \mu_\pi \\ &= \sum_{\pi \in \Pi_V(W)} \sum_{\pi' \in f^{-1}(\pi)} (-1)^{|\pi'|-1} \big(|\pi'| - 1\big)! \mu_{\pi'}. \end{aligned}$$

Observe that every $\pi \in \Pi_V(W)$ has exactly $|\pi| - d_\pi(V)$ preimages via f. One of them is π itself and there are $|\pi| - 1 - d_\pi(V)$ additional partitions obtained from π by merging V with some other part $Q \in \pi$ such that G_Γ contains no edges between V and Q. In particular, there is one preimage of size $|\pi|$ and there are $|\pi| - 1 - d_\pi(V)$ preimages of size $|\pi| - 1$. Furthermore, note that $\mu_{\pi'} = \mu_\pi$ for every $\pi' \in f^{-1}(\pi)$. Indeed, for every $Q \in \pi$ with no edges of G_Γ between Q and V, we have

$$\mathbb{E}[X_V] \cdot \mathbb{E}[X_Q] = \mathbb{E}[X_V X_Q] = \mathbb{E}[X_{V \cup Q}].$$

It follows that

$$\begin{aligned} \kappa_V(W) &= \sum_{\pi \in \Pi_V(W)} (-1)^{|\pi|-1} \Big(\big(|\pi| - 1\big)! - \big(|\pi| - 1 - d_\pi(V)\big) \cdot \big(|\pi| - 2\big)! \Big) \mu_\pi \\ &= \sum_{\pi \in \Pi_V(W)} (-1)^{|\pi|-1} \chi_V(\pi) \mu_\pi, \end{aligned}$$

as claimed. $\square$

Our next lemma is the main result of this subsection and the essential combinatorial ingredient of the proof of Lemma 23. Stating it requires the following definition (illustrated in Figure 3).

DEFINITION 29 ($\mathrm{Cut}_V(P)$). Suppose that $V \subseteq [N]$ is nonempty and $P \subseteq [N]$ is disjoint from V and satisfies $P \hookrightarrow V$. Then we write $\mathrm{Cut}_V(P)$ for the collection of all sets $C \subseteq [N]$ satisfying $\partial(V) \cap P \subseteq C \subseteq P$ and $C \hookrightarrow V$.

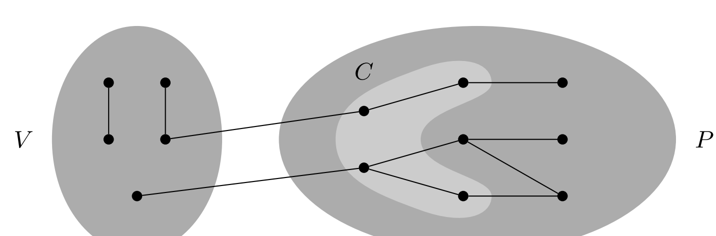

FIG. 3. *A set C in* $\mathrm{Cut}_V(P)$. *Every element of* $\mathrm{Cut}_V(P)$, *except for P itelf, is a cutset in $G_\Gamma(V \cup P)$ that disconnects V from some vertices in P.*

LEMMA 30. *Suppose that* $\varnothing \neq V \subseteq W \subseteq [N]$ *and* $W \hookrightarrow V$. *Then*

$$\kappa_V(W) = \mathbb{E}[X_V] \sum_{\pi \in \Pi_V^{\hookrightarrow}(W)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{\substack{P \in \pi \\ P \neq V}} \sum_{C \in \mathrm{Cut}_V(P)} \kappa_C(P). \tag{29}$$

PROOF. Denote the right-hand side of (29) by $r_V(W)$. We need to show $\kappa_V(W) = r_V(W)$. Let us first rewrite the inner sum in (29). To this end, fix some nonempty $P \subseteq W \setminus V$ such that $P \hookrightarrow V$. By the definition of $\kappa_C(P)$ (see (20)),

$$\sum_{C \in \mathrm{Cut}_V(P)} \kappa_C(P) = \sum_{C \in \mathrm{Cut}_V(P)} \sum_{\pi \in \Pi_C^{\mathsf{C}}(P)} (-1)^{|\pi|-1}(|\pi|-1)!\mu_\pi. \tag{30}$$

We may write this double sum more compactly as follows. For brevity, let $\partial_P(V) := \partial(V) \cap P$. Denote by $\tilde{\Pi}_V(P)$ the set of all partitions $\pi \in \Pi(P)$ such that some $Q \in \pi$ contains all neighbours of V in P, that is, such that $\partial_P(V) \subseteq Q$ for some $Q \in \pi$. We claim that

$$\sum_{C \in \mathrm{Cut}_V(P)} \kappa_C(P) = \sum_{\pi \in \tilde{\Pi}_V(P)} (-1)^{|\pi|-1}(|\pi|-1)!\mu_\pi. \tag{31}$$

Indeed, this follows from (30) because, letting

$$\mathcal{Q}(V, P) = \{(C, \pi) : C \in \mathrm{Cut}_V(P) \text{ and } \pi \in \Pi_C^{\mathsf{C}}(P)\},$$

the projection $p_2 \colon \mathcal{Q}(V, P) \ni (C, \pi) \mapsto \pi \in \Pi(P)$ is a bijection between $\mathcal{Q}(V, P)$ and $\tilde{\Pi}_V(P)$. This is because for every $(C, \pi) \in \mathcal{Q}(V, P)$, C is the union of those connected components of $G_\Gamma(Q)$ that intersect $\partial_P(V)$. Furthermore, observe that the right-hand side of (31) is simply the joint cumulant of the set

$$P_V = \{X_i : i \in P \setminus \partial_P(V)\} \cup \{X_{\partial_P(V)}\},$$

which is obtained from P by replacing $\{X_i : i \in \partial_P(V)\}$ with the single variable $X_{\partial_P(V)}$. Therefore, it follows from (31) that

$$r_V(W) = \mathbb{E}[X_V] \sum_{\pi \in \Pi_V^{\hookrightarrow}(W)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{\substack{P \in \pi \\ P \neq V}} \kappa(P_V). \tag{32}$$

Let $\Pi'_V(W)$ be the set of all partitions in $\Pi_V(W)$ whose every part, except possibly V itself, contains a neighbour of V. We claim that the product in the right-hand side of (32) is zero for every $\pi \in \Pi'_V(W) \setminus \Pi_V^{\hookrightarrow}(W)$ and hence we may replace $\Pi_V^{\hookrightarrow}(W)$ with $\Pi'_V(W)$ in the range of summation in (32). Indeed, if $\pi \in \Pi'_V(W) \setminus \Pi_V^{\hookrightarrow}(W)$, then there is a $P \in \pi \setminus \{V\}$ such that $\partial_P(V) \neq \varnothing$ but $P \not\hookrightarrow V$. In particular, some connected component of $G_\Gamma[P]$ is disjoint from $\partial_P(V)$ and hence $\kappa(P_V) = 0$. Expanding $\kappa(P_V)$ again, we obtain

$$\begin{aligned} r_V(W) = \mathbb{E}[X_V] &\sum_{\pi \in \Pi'_V(W)} (-1)^{|\pi|-1}(|\pi|-1)! \\ &\times \prod_{\substack{P \in \pi \\ P \neq V}} \sum_{\pi' \in \tilde{\Pi}_V(P)} (-1)^{|\pi'|-1}(|\pi'|-1)!\mu_{\pi'}. \end{aligned} \tag{33}$$

Let us write $\mathcal{P}$ to denote the set of all pairs $(\pi, \pi^*) \in \Pi'_V(W) \times \Pi_V(W)$ obtained as follows. Choose an arbitrary partition $\pi \in \Pi'_V(W)$ and refine every $P \in \pi \setminus \{V\}$ by replacing it by some $\pi_P \in \tilde{\Pi}_V(P)$, so that $\partial_P(V)$ is contained in a single part of π_P; finally, let π^* be the resulting partition of W.

Suppose that $(\pi, \pi^*) \in \mathcal{P}$. Enumerate the parts of π as $V, P_1, \dots, P_t$ and suppose that π^* was obtained from π by refining each P_j into $i_j + 1$ parts, so that $|\pi^*| = t + 1 + i_1 + \cdots + i_t$. Then, letting

$$f(\pi, \pi^*) = f_t(i_1, \dots, i_t) := (-1)^t t! \prod_{j \in [t]} (-1)^{i_j} i_j! = (-1)^{|\pi^*|-1} t! \prod_{j \in [t]} i_j!,$$

we may rewrite (33) as

$$r_V(W) = \sum_{(\pi,\pi^*) \in \mathcal{P}} f(\pi, \pi^*) \mu_{\pi^*}. \tag{34}$$

Fix some $\pi^* \in \Pi_V(W)$ and note that π^* contains $d_{\pi^*}(V)$ parts other than V that intersect $\partial(V)$. Let us write $s = |\pi^*|$, $t = d_{\pi^*}(V)$, and $\pi^* = \{V, P_1^*, \dots, P_{s-1}^*\}$, so that $P_1^*, \dots, P_t^*$ are the parts intersecting $\partial(V)$. Fix an arbitrary permutation σ of $[s-1]$ such that $\sigma(1) \in [t]$. Such a σ can be used to define a π such that $(\pi, \pi^*) \in \mathcal{P}$ in the following way. Consider the sequence $P_\sigma^* = (P_{\sigma(1)}^*, \dots, P_{\sigma(s-1)}^*)$. For every $i \in [t]$, let P_i be the union of P_i^* and all the P_j^*, with $j \in [s-1] \setminus [t]$, for which P_i^* is the right-most element among $P_1^*, \dots, P_t^*$ that is to the left of P_j^* in P_σ^*. (Since $\sigma(1) \in [t]$, then each P_j^* with $j \in [s-1] \setminus [t]$ has one of $P_1^*, \dots, P_t^*$ left of it.) A moments thought reveals that each partition π with $(\pi, \pi^*) \in \mathcal{P}$ is obtained this way from exactly $|f(\pi, \pi^*)|$ permutations σ. It follows that

$$\begin{aligned}
&r_V(W) \\
&\quad = \sum_{\pi^* \in \Pi_V(W)} (-1)^{|\pi^*|-1} \mu_{\pi^*} \sum_{\substack{\pi \in \Pi'_V(W) \\ (\pi,\pi^*) \in \mathcal{P}}} |f(\pi, \pi^*)| \\
&\quad = \sum_{\pi^* \in \Pi_V(W)} (-1)^{|\pi^*|-1} \mu_{\pi^*} \cdot \bigl|\bigl\{\sigma \in \operatorname{Sym}(|\pi^*| - 1) : \sigma(1) \in \{1, \dots, d_{\pi^*}(V)\}\bigr\}\bigr| \\
&\quad = \sum_{\pi^* \in \Pi_V(W)} (-1)^{|\pi^*|-1} \mu_{\pi^*} \cdot \chi_V(\pi^*),
\end{aligned}$$

where $\chi_V(\pi^*)$ is as defined in Lemma 28. By Lemma 28, we conclude that $r_V(W) = \kappa_V(W)$, as required. $\square$

3.3. *Proof of Lemma* 23. For $V, S \subseteq [N]$ and $k \in \mathbb{N}$ such that $|V| \le k$, we define

$$\tilde{\kappa}_V^{(k)}(S) = (-1)^{|V|-1} \mathbb{E}[X_V] \sum_{0 \le i \le k-|V|} \Biggl(\sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} \kappa_U^{(k-|V|)}\bigl(S \cap I(V)\bigr) \Biggr)^i \tag{35}$$

and

$$q^{(k)}(V, S) = (-1)^{|V|-1} \mathbb{E}[X_V] \sum_{0 \le i \le k-|V|} \Biggl(\sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} q\bigl(U, S \cap I(V)\bigr) \Biggr)^i. \tag{36}$$

Our proof of Lemma 23 consists of three steps. First, in Lemma 31, we show that $q(V, S) \approx q^{(k)}(V, S)$. Second, in Lemma 32, we show that $\kappa_V^{(k)}(S) \approx \tilde{\kappa}_V^{(k)}(S)$. Finally, the fact that $q^{(k)}(V, S)$ and $\tilde{\kappa}_V^{(k)}(S)$ satisfy similar recurrences (given the above approximate equalities) allows us to prove that also $q(V, S) \approx \kappa_V^{(k)}(S)$. Lemma 23 then follows easily. The precise definition of "$\approx$" above will be expressed by the following quantities. For integers k and K satisfying $1 \le k \le K$, define

$$\Delta_k(V) = \sum_{\substack{U \hookrightarrow V \\ |U \cup V| = k}} \mathbb{E}[X_{U \cup V}] \quad \text{and} \quad \Delta_{k,K}(V) = \sum_{i=k}^{K} \Delta_i(V), \tag{37}$$

and

$$\delta_{k,K}(V) = \sum_{\substack{U \hookrightarrow V \\ k \le |U \cup V| \le K}} \mathbb{E}[X_{U \cup V}] \max\{\mathbb{E}[X_i] : i \in U \cup V\}. \tag{38}$$

LEMMA 31. *Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be such that $\rho_k \le 1 - \varepsilon$. Then there exists $K = K(k, \varepsilon)$ such that for all $V, S \subseteq [N]$ with $1 \le |V| \le k$,*

$$|q(V, S) - q^{(k)}(V, S)| \le K \cdot (\delta_{1,K}(V) + \Delta_{k+1,K}(V)).$$

PROOF. Fix V and S as in the statement of the lemma and set

$$\rho = \mathbb{P}[X_i = 1 \text{ for some } i \in S \setminus I(V) \mid \overline{X}_{S \cap I(V)} = 1].$$

Then, by definition,

$$q(V, S) = \frac{(-1)^{|V|-1}\mathbb{E}[X_V]}{\mathbb{E}[\overline{X}_{S \setminus I(V)} \mid \overline{X}_{S \cap I(V)} = 1]} = \frac{(-1)^{|V|-1}\mathbb{E}[X_V]}{1 - \rho}. \tag{39}$$

Since, by Harris's inequality and $|V| \le k$, we have $0 \le \rho \le \rho_V \le \rho_k \le 1 - \varepsilon$, then (39) and the identity $(1 - \rho)^{-1} = 1 + \rho + \cdots + \rho^{k-|V|} + \rho^{k-|V|+1}(1 - \rho)^{-1}$ yield

$$|q(V, S) - (-1)^{|V|-1}\mathbb{E}[X_V] \cdot (1 + \rho + \cdots + \rho^{k-|V|})| \le \varepsilon^{-1}\mathbb{E}[X_V]\rho_V^{k-|V|+1}. \tag{40}$$

We now observe that

$$\begin{aligned} \mathbb{E}[X_V]\rho_V^{k-|V|+1} &\le \mathbb{E}[X_V]\Big(\sum_{i \in V \cup \partial(V)} \mathbb{E}[X_i]\Big)^{k-|V|+1} \\ &= \mathbb{E}[X_V] \sum_{i_1, \dots, i_{k-|V|+1}} \prod_{j=1}^{k-|V|+1} \mathbb{E}[X_{i_j}] \end{aligned}$$

and note that if $i_1, \dots, i_{k-|V|+1}$ are distinct elements of $\partial(V)$, then

$$\mathbb{E}[X_V] \prod_{j=1}^{k-|V|+1} \mathbb{E}[X_{i_j}] \le \mathbb{E}[X_{V \cup \{i_1, \dots, i_{k-|V|+1}\}}]$$

by Harris's inequality; if, on the other hand, either $i_j \in V$ for some j or some two i_j are equal, then Harris's inequality and the fact that $|\mathbb{E}[X_i]| \le 1$ for each i imply the stronger bound

$$\begin{aligned} &\mathbb{E}[X_V] \prod_{j=1}^{k-|V|+1} \mathbb{E}[X_{i_j}] \\ &\quad \le \mathbb{E}[X_{V \cup \{i_1, \dots, i_{k-|V|+1}\}}] \cdot \max\{\mathbb{E}[X_i] : i \in V \cup \{i_1, \dots, i_{k-|V|+1}\}\}. \end{aligned}$$

In particular, the right-hand side of (40) is bounded from above by

$$\varepsilon^{-1} \cdot (k - |V| + 1)! \cdot \Delta_{k+1}(V) + \varepsilon^{-1} \cdot k^{k-|V|+1} \cdot \delta_{1,k}(V),$$

which yields

$$\begin{aligned} &|q(V, S) - (-1)^{|V|-1}\mathbb{E}[X_V] \cdot (1 + \rho + \cdots + \rho^{k-|V|})| \\ &\quad \le K_1 \cdot (\Delta_{k+1}(V) + \delta_{1,k}(V)) \end{aligned} \tag{41}$$

for some constant K_1 that depends only on k and ε.

We claim that there is a constant $K_2 = K_2(k, \varepsilon)$ such that, for all $0 \le i \le k - |V|$,

$$\mathbb{E}[X_V] \cdot \left| \rho^i - \Bigg(\sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k - |V|}} q\big(U, S \cap I(V)\big) \Bigg)^i \right| \le K_2 \cdot \big(\delta_{1,K_2}(V) + \Delta_{k+1,K_2}(V)\big). \tag{42}$$

Observe that (41) and (42) imply that

$$\big|q(V, S) - q^{(k)}(V, S)\big| \le K \cdot \big(\delta_{1,K}(V) + \Delta_{k+1,K}(V)\big)$$

for some $K = K(k, \varepsilon)$, giving the assertion of the lemma. It thus remains to prove (42).

We first consider the case $i = 1$. By the Bonferroni inequalities, for every positive j,

$$(-1)^{j-1} \cdot \rho \le (-1)^{j-1} \cdot \sum_{\substack{U' \subseteq S \setminus I(V) \\ 1 \le |U'| \le j}} (-1)^{|U'|-1} \mathbb{E}[X_{U'} \mid \overline{X}_{S \cap I(V)} = 1].$$

Applying Lemma 25 with $k \leftarrow j - |U'|$, $V \leftarrow U'$, and $S \leftarrow S \cap I(V)$, we get that for each $U' \subseteq S \setminus I(V)$ with $1 \le |U'| \le j$,

$$(-1)^{j-|U'|} \mathbb{E}[X_{U'} \mid \overline{X}_{S \cap I(V)} = 1] \le \sum_{\substack{U'' \subseteq S \cap I(V), U'' \hookrightarrow U' \\ |U''| \le j - |U'|}} (-1)^{j-1} q\big(U' \cup U'', S \cap I(V)\big).$$

Next, observe that any nonempty $U \subseteq S$ with $U \hookrightarrow V$ of size at most j can be written uniquely as the disjoint union of U' and U'', where $U' \subseteq V \cup \partial(V)$ and $U'' \subseteq I(V)$ and $U'' \hookrightarrow U'$. The previous two inequalities then imply that

$$(-1)^{j-1} \cdot \rho \le (-1)^{j-1} \cdot \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le j}} q\big(U, S \cap I(V)\big). \tag{43}$$

Invoking (43) twice, first with $j \leftarrow k - |V|$ and then with $j \leftarrow k - |V| + 1$, to get both an upper and a lower bound on ρ, we obtain

$$\begin{aligned} \left| \rho - \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k - |V|}} q\big(U, S \cap I(V)\big) \right| &\le \left| \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ |U| = k - |V| + 1}} q\big(U, S \cap I(V)\big) \right| \\ &\le \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ |U| = k - |V| + 1}} \varepsilon^{-1} \mathbb{E}[X_U], \end{aligned} \tag{44}$$

where the last inequality uses the definition of $q(U, S \cap I(V))$ and the assumption that $\rho_k \le 1 - \varepsilon$; see the discussion below (39).

Finally, we show how to deduce (42) from (44). Let

$$y = \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k - |V|}} q\big(U, S \cap I(V)\big),$$

so that the left-hand side of (42) is $\mathbb{E}[X_V] \cdot |\rho^i - y^i|$, and observe that, as in (44),

$$|y| \le z := \sum_{\substack{U \hookrightarrow V \\ 1 \le |U| \le k - |V|}} \varepsilon^{-1} \mathbb{E}[X_U].$$

Fix an $i \in \{1, \dots, k - |V|\}$. Since $|\rho| \le 1$, then

$$|\rho^i - y^i| \le |\rho - y| \cdot \sum_{j=0}^{i-1} |\rho^j y^{i-1-j}| \le (1 + z)^{i-1} \cdot |\rho - y|,$$

which together with (44) implies that

$$\mathbb{E}[X_V] \cdot \left|\rho^i - y^i\right| \le (1+z)^{i-1}\mathbb{E}[X_V] \sum_{\substack{U \hookrightarrow V \\ |U|=k-|V|+1}} \varepsilon^{-1}\mathbb{E}[X_U].$$

Note that for pairwise disjoint $U_1, \dots, U_j \subseteq [N]$, Harris's inequality gives

$$\prod_{\ell=1}^{j} \mathbb{E}[X_{U_\ell}] \le \mathbb{E}[X_{U_1 \cup \dots \cup U_j}]$$

and if $U_1, \dots, U_j \subseteq [N]$ are not pairwise disjoint, then the stronger FKG lattice condition (14) implies that

$$\prod_{\ell=1}^{j} \mathbb{E}[X_{U_\ell}] \le \mathbb{E}[X_{U_1 \cup \dots \cup U_j}] \cdot \max\{\mathbb{E}[X_i] : i \in U_1 \cup \dots \cup U_j\}.$$

In particular, using a similar reasoning as used for deriving the bound (41) from (40), we obtain

$$(1+z)^{i-1}\mathbb{E}[X_V] \sum_{\substack{U \hookrightarrow V \\ |U|=k-|V|+1}} \varepsilon^{-1}\mathbb{E}[X_U] \le K_4 \cdot \left(\delta_{1,ik}(V) + \Delta_{k+1,ik+1}(V)\right)$$

for sufficiently large $K_4 = K_4(k, \varepsilon)$. This shows (42) and hence the lemma. □

LEMMA 32. *For every $k \in \mathbb{N}$ there exists some $K = K(k)$ such that, for all $V, S \subseteq [N]$ with $1 \le |V| \le k$, we have*

$$\left|\kappa_V^{(k)}(S) - \tilde{\kappa}_V^{(k)}(S)\right| \le K \cdot \left(\delta_{1,K}(V) + \Delta_{k+1,K}(V)\right).$$

PROOF. Fix k, S, and V as in the statement of the lemma and let

$$x = \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} \kappa_U^{(k-|V|)}\left(S \cap I(V)\right),$$

so that

$$\tilde{\kappa}_V^{(k)}(S) = (-1)^{|V|-1}\mathbb{E}[X_V]\left(1 + x + x^2 + \dots + x^{k-|V|}\right). \tag{45}$$

Using the definition (21), we may rewrite

$$x = \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} \sum_{\substack{U \subseteq W \subseteq U \cup (S \cap I(V)) \\ W \hookrightarrow U, |W| \le k-|V|}} (-1)^{|W|-1}\kappa_U(W). \tag{46}$$

Recalling from Definition 29 that

$$\mathrm{Cut}_V(W) = \left\{U \subseteq W : U \hookrightarrow V \text{ and } \partial(V) \cap W \subseteq U\right\},$$

we may switch the order of summation in (46) to obtain

$$x = \sum_{\substack{W \subseteq S, W \hookrightarrow V \\ 1 \le |W| \le k-|V|}} \sum_{U \in \mathrm{Cut}_V(W)} (-1)^{|W|-1}\kappa_U(W).$$

For the sake of brevity, write

$$f(W) = \sum_{U \in \mathrm{Cut}_V(W)} (-1)^{|W|-1}\kappa_U(W).$$

We may now rewrite (45) as

$$\tilde{\kappa}_V^{(k)}(S) = (-1)^{|V|-1}\mathbb{E}[X_V]\sum_{i=0}^{k-|V|}\sum_{\substack{W_1,\dots,W_i\subseteq S\\ W_1,\dots,W_i\hookrightarrow V\\ 1\le|W_1|,\dots,|W_i|\le k-|V|}} f(W_1)\cdots f(W_i). \tag{47}$$

Consider first the total contribution $\tilde{\kappa}_1$ to the right-hand side of (47) coming from terms corresponding to $W_1,\dots,W_i \subseteq S\setminus V$ that are pairwise disjoint and whose union has size at most $k-|V|$. Each such term may be regarded as a partition of the set $W = V\cup W_1\cup\cdots\cup W_i$, which satisfies $V\subseteq W\subseteq S$ and $|W|\le k$; this partition $\{V, W_1,\dots,W_i\}$ belongs to $\Pi_V^{\hookrightarrow}(W)$. Conversely, given a W with these properties, every partition $\pi\in\Pi_V^{\hookrightarrow}(W)$ corresponds to exactly $(|\pi|-1)!$ such terms; this is the number of ways to order the elements of $\pi\setminus\{V\}$ as $W_1,\dots,W_i$. Therefore,

$$\tilde{\kappa}_1 = (-1)^{|V|-1}\mathbb{E}[X_V]\sum_{\substack{V\subseteq W\subseteq V\cup S\\ W\hookrightarrow V, |W|\le k}}\sum_{\pi\in\Pi_V^{\hookrightarrow}(W)}\bigl(|\pi|-1\bigr)!\prod_{\substack{P\in\pi\\ P\neq V}} f(P).$$

In particular, Lemma 30 gives

$$\tilde{\kappa}_1 = (-1)^{|V|-1}\sum_{\substack{V\subseteq W\subseteq V\cup S\\ W\hookrightarrow V, |W|\le k}}(-1)^{|W|-|V|}\kappa_V(W) = \kappa_V^{(k)}(S).$$

Every term in the right-hand side of (47) corresponding to $W_1,\dots,W_i$ that is not included in $\tilde{\kappa}_1$ either satisfies $|V\cup W_1\cup\cdots\cup W_i| > k$ or the sets $V, W_1,\dots,W_i$ are not pairwise disjoint. Let $\tilde{\kappa}_2 = \tilde{\kappa}_V^{(k)}(S) - \tilde{\kappa}_1$ denote the total contribution of these terms. Since for every W, Harris's inequality implies

$$\bigl|f(W)\bigr| \le \sum_{U\subseteq W}\bigl|\kappa_U(W)\bigr| \le \sum_{\pi\in\Pi(W)}|\pi|!\mu_\pi \le |W|^{|W|}\mathbb{E}[X_W],$$

there is a constant K_1 that depends only on k such that

$$|\tilde{\kappa}_2| \le K_1\mathbb{E}[X_V]\sum_{W_1,\dots,W_i}\prod_{j=1}^{i}\mathbb{E}[X_{W_j}],$$

where the sum ranges over all $i\le k-|V|$ and $W_1,\dots,W_i\subseteq S$, each of size at most $k-|V|$ and attaching to V, such that either $|V\cup W_1\cup\cdots\cup W_i| > k$ or the sets $V, W_1,\dots,W_i$ are not pairwise disjoint. An argument analogous to the one given at the end of the proof of Lemma 31, employing Harris's inequality and the stronger FKG lattice condition (14), gives

$$|\tilde{\kappa}_2| \le K\cdot\bigl(\delta_{1,K}(V) + \Delta_{k+1,K}(V)\bigr)$$

for some K that depends only on k. $\square$

LEMMA 33. *Let $k\in\mathbb{N}$ be such that $\rho_k \le 1-\varepsilon$. Then there exists $K = K(k,\varepsilon)$ such that for all $V, S\subseteq[N]$ with $1\le|V|\le k$, we have*

$$\bigl|q(V,S) - \kappa_V^{(k)}(S)\bigr| \le K\cdot\bigl(\delta_{1,K}(V) + \Delta_{k+1,K}(V)\bigr).$$

PROOF. We prove the lemma by complete induction on k. To this end, let $k\ge 0$ and suppose that the statement holds for all $k'\in\mathbb{N}$ with $k' < k$. By the triangle inequality

$$\begin{aligned}\bigl|q(V,S) - \kappa_V^{(k)}(S)\bigr| \le{}& \bigl|q(V,S) - q^{(k)}(V,S)\bigr|\\ &+ \bigl|q^{(k)}(V,S) - \tilde{\kappa}_V^{(k)}(S)\bigr|\\ &+ \bigl|\tilde{\kappa}_V^{(k)}(S) - \kappa_V^{(k)}(S)\bigr|.\end{aligned}$$

Lemmas 31 and 32 imply that

$$\left|q(V,S) - q^{(k)}(V,S)\right| + \left|\tilde{\kappa}_V^{(k)}(S) - \kappa_V^{(k)}(S)\right| \le K_1 \cdot \left(\delta_{1,K_1}(V) + \Delta_{k+1,K_1}(V)\right)$$

for some sufficiently large $K_1 = K_1(k,\varepsilon)$ and thus it suffices to show that there is some $K_2 = K_2(k,\varepsilon)$ such that

$$\left|q^{(k)}(V,S) - \tilde{\kappa}_V^{(k)}(S)\right| \le K_2 \cdot \left(\delta_{1,K_2}(V) + \Delta_{k+1,K_2}(V)\right). \tag{48}$$

To this end, observe first that since $k - |V| < k$, then the induction hypothesis states that there is a constant $K' = K'(k,\varepsilon)$ such that

$$\left|q\left(U, S \cap I(V)\right) - \kappa_U^{(k-|V|)}\left(S \cap I(V)\right)\right| \le K' \cdot \left(\delta_{1,K'}(U) + \Delta_{k-|V|+1,K'}(U)\right) \tag{49}$$

for all U such that $1 \le |U| \le k - |V|$. Let

$$x = \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} \kappa_U^{(k-|V|)}\left(S \cap I(V)\right)$$

and, as in the proof of Lemma 31,

$$y = \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} q\left(U, S \cap I(V)\right).$$

Observe that

$$|y| \le z := \sum_{\substack{U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} \varepsilon^{-1} \mathbb{E}[X_U],$$

as in the proof of Lemma 31, and that (49) implies that

$$|x - y| \le w := K' \cdot \sum_{\substack{U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} \left(\delta_{1,K'}(U) + \Delta_{k-|V|+1,K'}(U)\right). \tag{50}$$

For any $i \ge 1$, we have

$$\left|x^i - y^i\right| \le |x - y| \cdot \sum_{j=0}^{i-1} \left|x^j y^{i-1-j}\right| \le |x - y| \cdot \left(|x| + |y|\right)^{i-1} \le w(2z + w)^{i-1}.$$

It follows that

$$\left|q^{(k)}(V,S) - \tilde{\kappa}_V^{(k)}(S)\right| \le \sum_{1 \le i \le k-|V|} \mathbb{E}[X_V] \cdot w(2z + w)^{i-1}. \tag{51}$$

Similarly as in the proofs of Lemmas 31 and 32, one sees that the FKG lattice condition (14) implies that the right hand side of (51) is bounded from above by $K_2 \cdot (\delta_{1,K_2}(V) + \Delta_{k+1,K_2}(V))$, provided $K_2 = K_2(k,\varepsilon)$ is sufficiently large, as claimed. □

PROOF OF LEMMA 23. It follows from Lemma 33 that there is $K_1 = K_1(k,\varepsilon)$ such that

$$\begin{aligned} &\left|\sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} \left(q\left(V, [\ell - 1]\right) - \kappa_V^{(k)}(S)\right)\right| \\ &\qquad \le \sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} K_1 \cdot \left(\delta_{1,K_1}(V) + \Delta_{k+1,K_1}(V)\right). \end{aligned} \tag{52}$$

But if we choose K sufficiently large then the right-hand side is at most $K \cdot (\delta_{1,K} + \Delta_{k+1,K})$, as required. □

3.4. *Proof of Lemma* 24. Fix an integer k and an $\ell \in [N]$. Recalling (21), we rewrite the ℓth term of the sum from the statement of the lemma as follows:

$$\sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\kappa_V^{(k)}([\ell-1]) = \sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\sum_{\substack{V\subseteq W\subseteq V\cup[\ell-1]\\ W\hookrightarrow V\\ |W|\le k}}(-1)^{|W|-1}\kappa_V(W).$$

It follows from Definition 21 that if V is connected then $W \hookrightarrow V$ if and only if W is connected. Therefore, changing the order of the last two sums in the right-hand side of the above identity yields

$$\sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\kappa_V^{(k)}([\ell-1]) = \sum_{i\in[k]}\sum_{W\in\mathcal{C}_i(\ell)}\sum_{V\in\mathcal{C}_W}(-1)^{|W|-1}\kappa_V(W), \tag{53}$$

where $\mathcal{C}_W$ denotes the collection of all connected sets $V \subseteq W$ satisfying $\max V = \max W$.

We claim that for each $W \in \mathcal{C}_i(\ell)$,

$$\kappa(W) = \sum_{V\in\mathcal{C}_W}\kappa_V(W). \tag{54}$$

Observe first that establishing this claim completes the proof of the lemma. Indeed, substituting (54) into (53) and summing over all ℓ gives

$$\begin{aligned}\sum_{\ell\in[N]}\sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\kappa_V^{(k)}([\ell-1]) &= \sum_{i\in[k]}\sum_{\ell\in[N]}\sum_{W\in\mathcal{C}_i(\ell)}(-1)^{|W|-1}\kappa(W)\\ &= \sum_{i\in[k]}(-1)^{i-1}\sum_{W\in\mathcal{C}_i}\kappa(W) = \sum_{i\in[k]}(-1)^{i-1}\kappa_i.\end{aligned}$$

Therefore, we only need to prove the claim. To this end, fix a $W \in \mathcal{C}_i(\ell)$. Recalling (15) and (20), it clearly suffices to show that $\{\Pi_V^{\mathsf{C}}(W) : V \in \mathcal{C}_W\}$ is a partition of $\Pi(W)$. Obviously, $\Pi_V^{\mathsf{C}}(W) \subseteq \Pi(W)$ for each $V \in \mathcal{C}_W$. Conversely, given an arbitrary $\pi \in \Pi(W)$, let $P \in \pi$ be the part containing $\max W$ and let V be the connected component of $\max W$ in $G_\Gamma[P]$. Clearly, $V \in \mathcal{C}_W$ and $\pi \in \Pi_V^{\mathsf{C}}(W)$. Moreover, the connected component of $\max W$ in $G_\Gamma[P]$ is the only set V with this property, and so the sets $\Pi_V^{\mathsf{C}}(W)$ and $\Pi_U^{\mathsf{C}}(W)$ are disjoint for distinct $U, V \in \mathcal{C}_W$.

4. Computations. The goal of this section is to carry out the necessary computations for proving Corollaries 14, 15, and 17.

4.1. *Corollaries* 14 *and* 15. Assume that $\mathcal{F} = \{F_1, \dots, F_t\}$ is a collection of pairwise nonisomorphic r-graphs without isolated vertices and let the associated hypergraph $\Gamma = (\Omega, \mathcal{X})$ be defined as in Section 1.5. To prove Corollaries 14 and 15, we need to compute the quantities κ_k for small values of k. This can be done using the following general approach: We first enumerate all "isomorphism types" of clusters in $\mathcal{C}_k$. Then we compute the joint cumulant for each isomorphism type. Finally we multiply each value with the size of the respective isomorphism class. This is made more precise as follows.

DEFINITION 34. An $\mathcal{F}$*-complex* is a nonempty set of subgraphs of K_n, each of which is isomorphic to a graph in $\mathcal{F}$. An $\mathcal{F}$-complex B is *irreducible* if it cannot be written as the union of two $\mathcal{F}$-complexes B_1 and B_2 where every graph in B_1 is edge-disjoint from every graph in B_2. The set of all irreducible $\mathcal{F}$-complexes of cardinality k is denoted by $\mathcal{C}_k(\mathcal{F})$. The *underlying graph* G_B of an $\mathcal{F}$-complex B is the subgraph of K_n formed by taking the union of (the edge sets of) the graphs in B.

Note that there is a natural bijection ϕ between the sets $V \subseteq [N]$ of size k and the $\mathcal{F}$-complexes of size k: ϕ maps $V = \{i_1, \dots, i_k\}$ to the $\mathcal{F}$-complex $B = \{G_1, \dots, G_k\}$, where G_j is the subgraph of K_n spanned by the edges in γ_{i_j} (recall that γ_{i_j} is a set of edges in K_n and that we assume that none of the graphs in $\mathcal{F}$ have isolated vertices). Note also that $\phi|_{\mathcal{C}_k}$ is a bijection between $\mathcal{C}_k$ and $\mathcal{C}_k(\mathcal{F})$. We can therefore write $\kappa(B)$ for the joint cumulant of $\{X_i : i \in \phi^{-1}(B)\}$ without ambiguity, obtaining

$$\kappa_k = \sum_{B \in \mathcal{C}_k(\mathcal{F})} \kappa(B).$$

Using (7) we easily express $\kappa(B)$ in terms of G_B:

$$\kappa(B) = \sum_{\pi \in \Pi(B)} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B' \in \pi} p^{e_{G_{B'}}}. \tag{55}$$

DEFINITION 35. Let B_1 and B_2 be $\mathcal{F}$-complexes. A map $f : V(G_{B_1}) \to V(G_{B_2})$ is a *homomorphism* from B_1 to B_2 if for every graph $H \in B_1$, the graph $f(H)$ (with vertex set $f(V(H))$ and edge set $\{\{f(u), f(v)\} : \{u, v\} \in E(H)\}$) belongs to B_2. If f is bijective and both f and f^{-1} are homomorphisms, then f is an *isomorphism*. We denote by $\mathrm{Aut}(B)$ the group of automorphisms of B, that is of isomorphisms from B to B.

It is easy to see that κ assigns equal values to isomorphic $\mathcal{F}$-complexes. The following simple lemma can then be used to compute the values κ_k. In the sequel, we will denote by $n^{\underline{i}}$ the falling factorial $n(n-1)\cdots(n-i+1)$.

LEMMA 36. *Let $\mathcal{C}_k(\mathcal{F})/{\cong}$ be the set of isomorphism types of $\mathcal{F}$-complexes in $\mathcal{C}_k(\mathcal{F})$. Then*

$$\sum_{B \in \mathcal{C}_k(\mathcal{F})} \kappa(B) = \sum_{[B] \in \mathcal{C}_k(\mathcal{F})/\cong} \kappa(B) \cdot \frac{n^{\underline{v_{G_B}}}}{|\mathrm{Aut}(B)|}.$$

PROOF. For each isomorphism type $[B]$, there are $n^{\underline{v_{G_B}}}$ ways to place the vertices of G_B into K_n; this way, every element of $\mathcal{C}_k(\mathcal{F})$ isomorphic to B is counted once for every automorphism of B. □

PROOF OF COROLLARY 14. Suppose that $\mathcal{F} = \{K_3, C_4\}$ and that $p = o(n^{-4/5})$. Since both K_3 and C_4 are 2-balanced and

$$\min\{m_2(K_3), m_2(C_4)\} = \min\{2, 3/2\} \geq 5/4,$$

we can apply Corollary 13, which states that the probability that $G_{n,p}$ is simultaneously K_3-free and C_4-free is

$$\exp(-\kappa_1 + \kappa_2 - \kappa_3 + O(\Delta_4) + o(1)).$$

Figure 4 shows all seven nonisomorphic irreducible $\mathcal{F}$-complexes of size at most two. Using Lemma 36, the contribution to κ_k from a given $\mathcal{F}$-complex B of size k is

$$\kappa(B) \cdot \frac{n^{\underline{v_{G_B}}}}{|\mathrm{Aut}(B)|}.$$

For the complexes shown in Figure 4, we can easily calculate $|\mathrm{Aut}(B)|$ manually; going through the figure from the top left to the bottom right, we obtain the values

$$6, 8, 4, 4, 4, 2, 2.$$

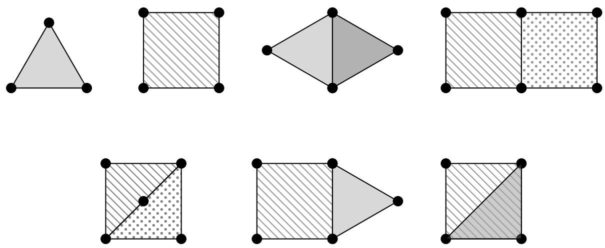

FIG. 4. *The irreducible $\{K_3, C_4\}$-complexes of size at most two. Copies of K_3 are represented by solid triangles and copies of C_4 by hatched or dotted quadrilaterals.*

Therefore,

$$\kappa_1 = \frac{n^{\underline{3}} p^3}{6} + \frac{n^{\underline{4}} p^4}{8}$$

and, since $p = o(n^{-4/5})$,

$$\begin{aligned}
\kappa_2 &= \frac{n^{\underline{4}}(p^5 - p^6)}{4} + \frac{n^{\underline{6}}(p^7 - p^8)}{4} + \frac{n^{\underline{5}}(p^6 - p^8)}{4} \\
&\quad + \frac{n^{\underline{5}}(p^6 - p^7)}{2} + \frac{n^{\underline{4}}(p^5 - p^7)}{2} \\
&= \frac{n^{\underline{6}} p^7}{4} + \frac{3n^{\underline{5}} p^6}{4} + o(1).
\end{aligned}$$

When calculating κ_3, we first observe that the underlying graphs of the third $\mathcal{F}$-complex and the fifth $\mathcal{F}$-complex in Figure 4 each contain a C_4 that is not already part of the complex and that the graph of the bottom right $\mathcal{F}$-complex contains a triangle that is not a part of the complex. Let κ_3' denote the contribution of the two $\mathcal{F}$-complexes of size three that are obtained from one of these three complexes of size two by adding the "extra" C_4 or K_3. Then

$$\kappa_3' = \frac{n^{\underline{4}}(p^5 - 2p^8 - p^9 + 2p^{10})}{4} + \frac{n^{\underline{5}}(p^6 - 3p^{10} + 2p^{12})}{12} = \frac{n^{\underline{5}} p^6}{12} + o(1).$$

On the other hand, the contribution of every other $\mathcal{F}$-complex of to κ_3 is at most in the order of $(p + np^2 + n^2 p^3) \cdot \kappa_2$, because, except in the two cases mentioned above, the graph of a complex of size three is obtained from the graph of a complex of size two by adding either a new edge, or a new vertex and two new edges, or two new vertices and three new edges. Using the assumption $p = o(n^{-4/5})$, we get

$$\left(p + np^2 + n^2 p^3\right) \cdot \kappa_2 = O\left(n^6 p^8 + n^5 p^7 + n^7 p^9 + n^8 p^{10}\right) = o(1),$$

and therefore

$$\kappa_3 = \frac{n^{\underline{5}} p^6}{12} + o(1).$$

Since the $\mathcal{F}$-complexes accounted for by κ_3' are "complete" (in the sense that their graphs do not contain either a K_3 or a C_4 that is not already a part of the complex), a similar reasoning shows that

$$\Delta_4 \le O\left(\left(p + np^2 + n^2 p^3\right) \cdot \kappa_3'\right) + O\left(\left(1 + p + np^2 + n^2 p^3\right) \cdot \left(\kappa_3 - \kappa_3'\right)\right) = o(1).$$

Since our assumption on p implies that $\max\{\kappa_1, \kappa_2, \kappa_3\} = o(n)$, we can replace $n^{\underline{i}}$ by n^i in the expressions for κ_1, κ_2, κ_3, incurring only an additive error of $o(1)$. Thus, the probability that $G_{n,p}$ with $p = o(n^{-4/5})$ is simultaneously triangle-free and C_4-free is asymptotically

$$\exp\left(-\frac{n^3p^3}{6} - \frac{n^4p^4}{8} + \frac{n^6p^7}{4} + \frac{2n^5p^6}{3}\right),$$

as claimed. □

PROOF OF COROLLARY 15. Suppose that $\mathcal{F} = \{K_3\}$ and $p = o(n^{-7/11})$. Since K_3 is 2-balanced and $m_2(K_3) = 2 \geq 11/7$, we can apply Corollary 13, which tells us that the probability that $G_{n,p}$ is triangle-free is

$$\exp\bigl(-\kappa_1 + \kappa_2 - \kappa_3 + \kappa_4 + O(\Delta_5) + o(1)\bigr).$$

In Figure 5, we see representations of all isomorphism types of irreducible $\mathcal{F}$-complexes of size up to four. Generating a similar list of complexes of size five would most likely require the help of a computer.

By Lemma 36, the contribution to κ_k from the isomorphism type of an $\mathcal{F}$-complex B of size k is

$$\kappa(B) \cdot \frac{n^{v_{G_B}}}{|\operatorname{Aut}(B)|}.$$

For the complexes shown in Figure 5, it is not too difficult to calculate $|\operatorname{Aut}(B)|$ by hand. In fact, since the automorphism group of K_3 comprises all 3! permutations of $V(K_3)$, automorphisms of $\{K_3\}$-complexes are simply automorphisms of the 3-uniform hypergraphs involved.[2] For example, the leftmost $\mathcal{F}$-complex in the second row has exactly two automorphisms: the trivial one, and the unique automorphism exchanging the vertices belonging to

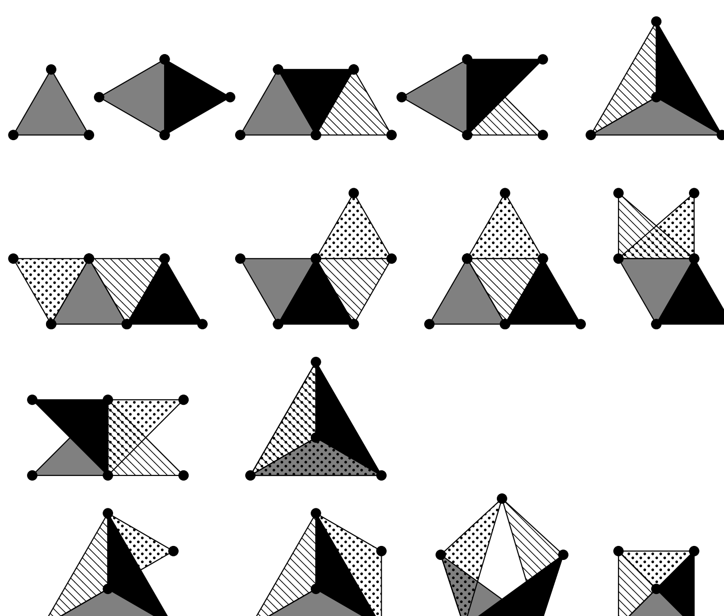

FIG. 5. *The irreducible $\{K_3\}$-complexes of size at most four. The four complexes in the bottom row are negligible when $p = o(n^{-7/11})$.*

[2]But for general $\mathcal{F}$, it is wrong to think of an $\mathcal{F}$-complex isomorphism as a hypergraph isomorphism.

exactly one triangle. Under our assumptions on p, we have $\kappa_k = \Delta_k + o(1)$ for $k \in \{3, 4\}$. This is the case because $|\kappa_k - \Delta_k| = O(p\Delta_k)$ and

$$p\Delta_3 \le O\big(n^5 p^8 + n^4 p^7\big) = o(1) \quad \text{and} \quad p\Delta_4 \le p \cdot O\big(1 + p + p^2 n\big) \cdot \Delta_3 = o(1),$$

as can be seen by looking at Figure 5.

Now we just work through the figure row by row (from the top left to the bottom right) and in this order, we compute (using the first row)

$$\begin{aligned}\kappa_1 &= \frac{n^{\underline{3}} p^3}{6},\\ \kappa_2 &= \frac{n^{\underline{4}}(p^5 - p^6)}{4},\\ \kappa_3 &= \Delta_3 + o(1) = \frac{n^{\underline{5}} p^7}{2} + \frac{n^{\underline{5}} p^7}{12} + \frac{n^{\underline{4}} p^6}{6} + o(1),\end{aligned}$$

and (using the other rows)

$$\begin{aligned}\kappa_4 &= \Delta_4 + o(1)\\ &= \frac{n^{\underline{6}} p^9}{2} + \frac{n^{\underline{6}} p^9}{2} + \frac{n^{\underline{6}} p^9}{6} + \frac{n^{\underline{6}} p^9}{2} + \frac{n^{\underline{6}} p^9}{48} + \frac{n^{\underline{4}} p^6}{24} + O\big(n^5 p^8\big) + o(1).\end{aligned}$$

The term $O(n^5 p^8)$ represents the contribution of the four complexes in the bottom row of Figure 5, which is $o(1)$, as $p = o(n^{-7/11})$. Finally, we have

$$\begin{aligned}\Delta_5 &= O\big(p\Delta_4 + np^2\Delta_4 + n^5 p^8 + n^5 p^9\big)\\ &= O\big(n^4 p^7 + n^5 p^8 + n^6 p^{10} + n^7 p^{11}\big) = o(1),\end{aligned}$$

since the graph of an $\mathcal{F}$-complex of size five must be obtained by adding either a new edge or a new vertex and two new edges to one of the graphs in Figure 5, or else it must be isomorphic to one of the first three graphs in the bottom row of Figure 5 (as the graphs of the remaining complexes of size four contain only triangles that are already in the complex).

Finally, $\kappa_1 = n^{\underline{3}} p^3/6 = (n^3 - 3n^2)p^3/6 + o(1)$ and, since $\max\{\kappa_2, \kappa_3, \kappa_4\} = o(n)$, we may replace the falling factorials $n^{\underline{i}}$ in the remaining expressions by n^i. Adding up the terms in $-\kappa_1 + \kappa_2 - \kappa_3 + \kappa_4$, we obtain that the probability that $G_{n,p}$ with $p = o(n^{-7/11})$ is triangle-free is asymptotically

$$\exp\Big(-\frac{n^3 p^3}{6} + \frac{n^4 p^5}{4} - \frac{7n^5 p^7}{12} + \frac{n^2 p^3}{2} - \frac{3n^4 p^6}{8} + \frac{27 n^6 p^9}{16}\Big),$$

as claimed. □

4.2. *Corollary* 17. It only remains to prove Corollary 17.

PROOF OF COROLLARY 17. Let Γ be the hypergraph of r-APs in $[n]$, as defined in Section 1.6, and assume that $p = o(n^{-4/7})$. Then by Corollary 16 with $r = 3$ and $k = 2$,

$$\mathbb{P}[X = 0] = \exp\big(-\kappa_1 + \kappa_2 + O(\Delta_3) + o(1)\big).$$

It remains to calculate κ_1, κ_2, and Δ_3. For $i \in [n]$, the number of 3-APs containing i is

$$f(i) = \frac{n}{2} + \min\{i, n - i\} + O(1),$$

where $\min\{i, n-i\}$ counts the 3-APs that have i as their midpoint, and $n/2$ counts the others. Thus, the total number of 3-APs in $[n]$ is

$$\frac{1}{3}\sum_{i=1}^{n} f(i) = \frac{n^2}{4} + O(n),$$

and therefore (using $np^3 = o(1)$)

$$\kappa_1 = \frac{n^2p^3}{4} + o(1).$$

If $\{i, j\}$ is an edge in the dependency graph, then $|\gamma_i \cap \gamma_j|$ is either 1 or 2. The number of pairs γ_i, γ_j intersecting in two elements is at most $\binom{n}{2}\binom{3}{2}^2$, so the contribution of these pairs to κ_2 is $O(n^2p^4)$, which is $o(1)$ by our assumption on p. The number of pairs $\{\gamma_i, \gamma_j\}$ with $i \neq j$ and $|\gamma_i \cap \gamma_j| \geq 1$ is precisely $\sum_{i=1}^{n}\binom{f(i)}{2}$ and hence the number M of pairs with $|\gamma_i \cap \gamma_j| = 1$ satisfies

$$M = \sum_{i=1}^{n}\binom{f(i)}{2} + O(n^2) = \frac{1}{2}\sum_{i=1}^{n} f(i)^2 + O(n^2).$$

Since

$$\begin{aligned}\sum_{i=1}^{n} f(i)^2 &= \sum_{i=1}^{n}(n/2 + \min\{i, n-i\})^2 + O(n^2) = 2\sum_{i=1}^{\lfloor n/2\rfloor}(n/2+i)^2 + O(n^2)\\ &= 2\left(\frac{n^3}{3} - \frac{(n/2)^3}{3}\right) + O(n^2) = \frac{7n^3}{12} + O(n^2)\end{aligned}$$

and $n^2p^4 = o(1)$, we have

$$\kappa_2 = M(p^5 - p^6) + O(n^2(p^4 - p^6)) = \frac{7n^3p^5}{24} + o(1).$$

Lastly, we claim that $\Delta_3 = O(n^4p^7) = o(1)$. Since any two distinct numbers are contained in at most three 3-APs, we have $|\mathcal{C}_3| = O(n^4)$. Moreover, let $\mathcal{C}_3^*$ be the family of all $\{i, j, k\} \in \mathcal{C}_3$ such that $|\gamma_i \cup \gamma_j \cup \gamma_k| < 7$. A simple case analysis shows that

$$\sum_{V \in \mathcal{C}_3^*} \Delta(\{X_i : i \in V\}) = O(n^2p^5 + n^3p^6) = o(1).$$

On the other hand, $\Delta(\{X_i : i \in V\}) = p^7$ for every $V \in \mathcal{C}_3 \setminus \mathcal{C}_3^*$. Thus,

$$\Delta_3 \leq |\mathcal{C}_3|p^7 + \sum_{V \in \mathcal{C}_3^*} \Delta(\{X_i : i \in V\}) = O(n^4p^7 + n^2p^4 + n^3p^6) = o(1)$$

and we conclude that the probability that $[n]_p$ is 3-AP-free is asymptotically

$$\exp\left(-\frac{n^2p^3}{4} + \frac{7n^3p^5}{24}\right),$$

as claimed. $\square$

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