The Annals of Probability
2020, Vol. 48, No. 1, 264–295
https://doi.org/10.1214/19-AOP1360

CONVERGENCE OF TRANSPORT NOISE TO ORNSTEIN–UHLENBECK FOR 2D EULER EQUATIONS UNDER THE ENSTROPHY MEASURE

BY FRANCO FLANDOLI[1] AND DEJUN LUO[2]

[1] *Scuola Normale Superiore, franco.flandoli@sns.it*
[2] *Key Laboratory of RCSDS, AMSS, Chinese Academy of Sciences, luodj@amss.ac.cn*

We consider the vorticity form of the 2D Euler equations which is perturbed by a suitable transport type noise and has white noise initial condition. It is shown that stationary solutions of this equation converge to the unique stationary solution of the 2D Navier–Stokes equation driven by the space-time white noise.

1. Introduction. Navier–Stokes equations in dimension 2 with periodic boundary conditions and additive space-time white noise

$$\begin{gathered} \mathrm{d}u + (u \cdot \nabla u + \nabla p)\,\mathrm{d}t = \nu \Delta u\,\mathrm{d}t + \alpha\,\mathrm{d}W, \\ \operatorname{div} u = 0 \end{gathered} \tag{1.1}$$

have been the object of several investigations, [4, 5, 9, 11, 21, 24, 25] among others and, with its first-stage renormalization, even contributed to the development of some of the ideas around regularity structures. One of the main features is the Gaussian invariant measure formally given by

$$\mu(\mathrm{d}\omega) = Z^{-1} \exp\bigl(-\beta \|\omega\|_{L^2}^2\bigr)\,\mathrm{d}\omega \tag{1.2}$$

($\beta > 0$ related to the constants of equations (1.1) and the domain) where we have denoted by ω the vorticity associated to the velocity field u and where $\|\omega\|_{L^2}^2$ denotes the enstrophy (hence μ is often called enstrophy measure). This equation is well posed in suitable function spaces, even in the strong probabilistic sense. For the purpose of the next description, it is convenient to reformulate the equation in vorticity form

$$\mathrm{d}\omega + u \cdot \nabla \omega\,\mathrm{d}t = \nu \Delta \omega\,\mathrm{d}t + \alpha \nabla^{\perp} \cdot \mathrm{d}W, \tag{1.3}$$

where, as said above, $\omega = \nabla^{\perp} \cdot u$ and, for a vector field v, $\nabla^{\perp} \cdot v$ denotes $\partial_2 v_1 - \partial_1 v_2$. Here, W is a solenoidal vector valued cylindrical Brownian motion.

A related model is 2D Euler equations, that in vorticity form is

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

with $\omega = \nabla^{\perp} \cdot u$, $\operatorname{div} u = 0$. In the sense described in [2, 13], the enstrophy measure μ is invariant also for this equation (for every $\beta > 0$, in this case). The same fact holds for a stochastic version of 2D Euler equations, but with transport type noise, as described in [14, 15]:

$$\mathrm{d}\omega + u \cdot \nabla \omega\,\mathrm{d}t = \sum_k \sigma_k \cdot \nabla \omega \circ \mathrm{d}W^k,$$

Received June 2018; revised February 2019.
MSC2010 subject classifications. Primary 35Q35; secondary 60H40.
Key words and phrases. Navier–Stokes equations, Euler equations, space-time white noise, vorticity formulation, weak convergence.

where $\sigma_k(x)$ are divergence-free vector fields and W^k independent Brownian motions. We focus our discussion on the 2D torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and choose, to fix notation,

$$\sigma_k(x) = \frac{1}{\sqrt{2}} \frac{k^\perp}{|k|^\gamma} e_k(x), \quad k \in \mathbb{Z}_0^2,$$

where $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$, $k^\perp = (k_2, -k_1)$ and $e_k(x)$ is the orthonormal basis of sine and cosine functions; see (2.1) below. In [14, 15], the problem has been studied for $\gamma > 2$.

The purpose of this paper is to present a rather unexpected link between stationary solutions of these two models. Based on [14, 15], it is interesting to ask what happens when $\gamma = 2$, the limiting case where certain terms diverge. For instance, the Itô–Stratonovich correction of the multiplicative noise above diverges proportionally to $\sum_{|k|\le N} \frac{1}{|k|^2}$ as $N \to \infty$. We therefore investigate whether this divergence may be compensated by an infinitesimal coefficient in front of the noise:

$$\mathrm{d}\omega + u \cdot \nabla\omega \,\mathrm{d}t = 2\sqrt{\nu}\varepsilon_N \sum_{|k|\le N} \frac{k^\perp}{|k|^2} e_k \cdot \nabla\omega \circ \mathrm{d}W^k, \tag{1.4}$$

where $\varepsilon_N = (\sum_{|k|\le N} \frac{1}{|k|^2})^{-1/2} \sim \frac{1}{\sqrt{\log N}}$. Our main result can be stated as follows (see the next section for relevant notation and Theorem 2.13 for a more precise statement).

THEOREM 1.1. *The stationary solutions of the model* (1.4) *with marginal* (1.2) *converge to the unique stationary solution of* (1.3) *with* $\alpha = \sqrt{2\nu}$.

Notice that the system (1.4) is hyperbolic in nature, while (1.3) is of parabolic type. Let us explain a vague physical intuition about this result, which however is not sufficient to state a firm conjecture, without a due detailed investigation. Transport multiplicative noise $\sum_k \sigma_k \cdot \nabla\omega \circ \mathrm{d}W^k$ provokes a random Lagrangian displacement of "fluid particles." Assume that the space-covariance of the Gaussian field $\sum_k \sigma_k(x) W_t^k$ is concentrated around zero, as it is in the scaling limit investigated in this work. Look at fluid particles as an interacting system of particles; the effect of the Gaussian field on different particles is almost independent, when the distance between particles is not too small (see [8], Introduction, for related discussions). Thus, approximatively, it is like driving each particle with an independent noise, and we know from mean field theories that independent Brownian perturbation of particles reflects into a Laplacian in the limit PDE. This intuitively explains the presence of the Laplacian in the limit equation, but the presence of a white noise is less clear. The latter fact seems to be related to the white noise structure of the stationary solutions considered here. It would be interesting to investigate similar questions for other stationary solutions of SPDEs like [17, 18]. It seems that for other SPDE models different results are possible; see [16]. Another interesting example of special limit with diffusion and noise can be found in [7, 12].

Let us also emphasize another nontrivial aspect that could be misunderstood. Technically speaking, a Laplacian (or a more complicated second- order differential operator) arises when rewriting a Stratonovich multiplicative transport noise in Itô's form (see Section 2 below). This does not mean that the original equation, with transport noise, is parabolic. The original equation is hyperbolic, and the solution (when smooth enough) is the stochastic Lagrangian transport of the initial condition. Thus it is a nontrivial fact that a truly parabolic equation is obtained in the scaling limit investigated in the present work.

This paper is organized as follows. In Section 2, we prove the main result (Theorem 2.13) which states that the white noise solutions of a sequence of stochastic Euler equations converge weakly to the stationary solution of the Navier–Stokes equation driven by space-time white noise. We solve in Section 3 the corresponding Kolmogorov equation by using the

Galerkin approximation. Finally, in the first part of the Appendix we recall a decomposition formula which plays an important role in the proof, and in Section A.2 we prove the coincidence of two different definitions of the nonlinear part in the Euler equation.

2. Convergence of the equations (1.4).

First, we introduce some notation. We denote by $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the two-dimensional torus, which will be understood as $[-1/2, 1/2]^2$ endowed with periodic boundary condition. Set

$$e_k(x) = \sqrt{2}\begin{cases}\cos(2\pi k\cdot x), & k \in \mathbb{Z}^2_+,\\ \sin(2\pi k\cdot x), & k\in\mathbb{Z}^2_-,\end{cases} \quad x \in \mathbb{T}^2, \tag{2.1}$$

where $\mathbb{Z}^2_+ = \{k \in \mathbb{Z}^2_0 : (k_1 > 0) \text{ or } (k_1 = 0, k_2 > 0)\}$ and $\mathbb{Z}^2_- = -\mathbb{Z}^2_+$. Then $\{e_k : k \in \mathbb{Z}^2_0\}$ constitute a CONS of $L^2_0(\mathbb{T}^2)$, the space of square integrable functions with zero mean. Define

$$\sigma_k(x) = \frac{1}{\sqrt{2}}\frac{k^\perp}{|k|^2}e_k(x), \quad k \in \mathbb{Z}^2_0, \tag{2.2}$$

with $k^\perp = (k_2, -k_1)$. Let $\nu > 0$ be fixed and, for $N \geq 1$, define $\Lambda_N = \{k \in \mathbb{Z}^2_0 : |k| \leq N\}$. We rewrite the equation (1.4) as

$$\mathrm{d}\omega^N_t + u^N_t \cdot \nabla\omega^N_t = 2\sqrt{2\nu}\varepsilon_N \sum_{k\in\Lambda_N} \sigma_k \cdot \nabla\omega^N_t \circ \mathrm{d}W^k_t. \tag{2.3}$$

Here, $\omega^N_t = \nabla^\perp \cdot u^N_t$ and conversely, u^N_t is represented by ω^N_t via the Biot–Savart law:

$$u^N_t(x) = (\omega^N_t * K)(x) = \langle\omega^N_t, K(x-\cdot)\rangle,$$

with K being the Biot–Savart kernel on $\mathbb{T}^2$:

$$K(x) = 2\pi \mathrm{i}\sum_{k\in\mathbb{Z}^2_0}\frac{k^\perp}{|k|^2}\mathrm{e}^{2\pi \mathrm{i}k\cdot x} = -2\pi\sum_{k\in\mathbb{Z}^2_0}\frac{k^\perp}{|k|^2}\sin(2\pi k\cdot x).$$

We assume that the initial data ω^N_0 of (2.3) is a white noise on $\mathbb{T}^2$; namely, ω^N_0 is a random variable defined on some probability space $(\Theta, \mathcal{F}, \mathbb{P})$, taking values in the space of distributions $C^\infty(\mathbb{T}^2)'$ on $\mathbb{T}^2$, such that, for any $\phi \in C^\infty(\mathbb{T}^2)$, $\langle\omega^N_0, \phi\rangle$ is a centered Gaussian random variable with variance $\|\phi\|^2_{L^2(\mathbb{T}^2)}$. From the definition, we easily deduce that

$$\mathbb{E}\langle\omega^N_0,\phi\rangle\langle\omega^N_0,\psi\rangle = \langle\phi,\psi\rangle_{L^2(\mathbb{T}^2)} \quad \text{for any } \phi,\psi\in C^\infty(\mathbb{T}^2).$$

We denote the law of ω^N_0 by μ, which is also called the enstrophy measure with the heuristic expression (1.2). It is not difficult to show that μ is supported by $H^{-1-}(\mathbb{T}^2) = \bigcap_{s>0} H^{-1-s}(\mathbb{T}^2)$, where, for any $r \in \mathbb{R}$, $H^r(\mathbb{T}^2)$ is the usual Sobolev space on $\mathbb{T}^2$.

For any fixed $N \geq 1$, following the proof of [14], Theorem 1.3, we can show that the equation (2.3) has a white noise solution $\omega^N \in C([0,T], H^{-1-}(\mathbb{T}^2))$ (possibly defined on a new probability space with new Brownian motions); namely, for any $t \in [0,T]$, ω^N_t is distributed as the white noise measure μ, and for any $\phi \in C^\infty(\mathbb{T}^2)$,

$$\begin{aligned}\langle\omega^N_t,\phi\rangle = {}& \langle\omega^N_0,\phi\rangle + \int_0^t \langle\omega^N_r\otimes\omega^N_r, H_\phi\rangle\,\mathrm{d}r\\ &- 2\sqrt{2\nu}\varepsilon_N\sum_{k\in\Lambda_N}\int_0^t\langle\omega^N_r, \sigma_k\cdot\nabla\phi\rangle\,\mathrm{d}W^k_r\\ &+ 4\nu\varepsilon_N^2\sum_{k\in\Lambda_N}\int_0^t\langle\omega^N_r, \sigma_k\cdot\nabla(\sigma_k\cdot\nabla\phi)\rangle\,\mathrm{d}r.\end{aligned} \tag{2.4}$$

Moreover, it is easy to show that ω^N is a stationary process, which is a consequence of the same result for the stochastic point vortex dynamics proved in [14], Proposition 2.3. Our purpose is to show that the equations (2.3) converge in some sense to

$$\mathrm{d}\omega_t + u_t \cdot \nabla\omega_t \,\mathrm{d}t = \nu\Delta\omega_t \,\mathrm{d}t + \sqrt{2\nu}\nabla^\perp \cdot \mathrm{d}W_t, \quad \omega_0 \overset{d}{\sim} \mu. \tag{2.5}$$

REMARK 2.1. Some explanations for the nonlinear term in (2.4) are necessary. For $\phi \in C^\infty(\mathbb{T}^2)$,

$$H_\phi(x, y) := \frac{1}{2} K(x - y) \cdot \big(\nabla\phi(x) - \nabla\phi(y)\big), \quad x, y \in \mathbb{T}^2,$$

with K being the Biot–Savart kernel and the convention that $H_\phi(x, x) = 0$. It is well known that, for all $x \in \mathbb{T}^2 \setminus \{0\}$, $K(-x) = -K(x)$ and $|K(x)| \le C/|x|$ for some constant $C > 0$; thus H_ϕ is symmetric and

$$\|H_\phi\|_\infty \le C\|\nabla^2\phi\|_\infty. \tag{2.6}$$

Since ω_r^N is a white noise on $\mathbb{T}^2$ for any $r \in [0, T]$, the quantity $\langle\omega_r^N \otimes \omega_r^N, H_\phi\rangle$ is well defined as a limit in $L^2(\Theta, \mathbb{P})$ of an approximating sequence; see [13], Theorem 8, for details. According to the arguments in Section A.2, this definition is consistent with that defined by the Galerkin approximation; the latter will be used in Section 3.

First, we follow the arguments in [14], Section 3, to show that the family of distributions $\{Q^N\}_{N\ge1}$ of ω^N on $\mathcal{X} := C([0, T], H^{-1-}(\mathbb{T}^2))$ is tight. To this end, we need to apply the compactness criterion proved in [22], page 90, Corollary 9. We state it here in our context.

Take $\delta \in (0, 1)$ and $\kappa > 5$ (this choice is due to estimates below) and consider the spaces

$$X = H^{-1-\delta/2}(\mathbb{T}^2), \quad B = H^{-1-\delta}(\mathbb{T}^2), \quad Y = H^{-\kappa}(\mathbb{T}^2).$$

Then $X \subset B \subset Y$ with compact embeddings and we also have, for a suitable constant $C > 0$ and for

$$\theta = \frac{\delta/2}{\kappa - 1 - \delta/2}, \tag{2.7}$$

the interpolation inequality

$$\|\omega\|_B \le C\|\omega\|_X^{1-\theta}\|\omega\|_Y^\theta, \quad \omega \in X.$$

These are the preliminary assumptions of [22], page 90, Corollary 9. We consider here a particular case:

$$\mathcal{S} = L^{p_0}(0, T; X) \cap W^{1/3,4}(0, T; Y),$$

where for $0 < \alpha < 1$ and $p \ge 1$,

$$W^{\alpha,p}(0, T; Y) = \left\{ f : f \in L^p(0, T; Y) \,\&\, \int_0^T \int_0^T \frac{\|f(t) - f(s)\|_Y^p}{|t - s|^{\alpha p+1}} \,\mathrm{d}t \,\mathrm{d}s < \infty \right\}.$$

The next result is taken from [14], Lemma 3.1.

LEMMA 2.2. *Let $\delta \in (0, 1)$ and $\kappa > 5$ be given. If*

$$p_0 > \frac{12(\kappa - 1 - 3\delta/2)}{\delta},$$

then $\mathcal{S}$ is compactly embedded into $C([0, T], H^{-1-\delta}(\mathbb{T}^2))$.

PROOF. Recall that θ is defined in (2.7). In our case, we have $s_0 = 0$, $r_0 = p_0$ and $s_1 = 1/3$, $r_1 = 4$. Hence $s_\theta = (1-\theta)s_0 + \theta s_1 = \theta/3$ and

$$\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} = \frac{1-\theta}{p_0} + \frac{\theta}{4}.$$

It is clear that for p_0 given above, it holds $s_\theta > 1/r_\theta$, thus the desired result follows from the second assertion of [22], Corollary 9. □

Next, since $H^{-1-}(\mathbb{T}^2)$ is endowed with the Fréchet topology, one can prove the following.

LEMMA 2.3. *The family* $\{Q^N\}_{N\ge 1}$ *is tight in* $\mathcal{X}$ *if and only if it is tight in* $C([0,T], H^{-1-\delta}(\mathbb{T}^2))$ *for any* $\delta > 0$.

The proof is similar to Step 1 of the proof of [14], Proposition 2.2, and we omit it here. In view of the above two lemmas, it is sufficient to prove that $\{Q^N\}_{N\ge 1}$ is bounded in probability in $W^{1/3,4}(0,T;H^{-\kappa}(\mathbb{T}^2))$ and in each $L^p(0,T;H^{-1-\delta}(\mathbb{T}^2))$ for any $p > 0$ and $\delta > 0$.

Before moving further, we recall some properties of the white noise which will be frequently used below.

LEMMA 2.4. *Let* $\xi : (\Theta, \mathcal{F}, \mathbb{P}) \to C^\infty(\mathbb{T}^2)'$ *be a white noise on* $\mathbb{T}^2$. *Then for any* $p > 1$ *and* $\delta > 0$, *there exist* $C_p > 0$, $C_{p,\delta} > 0$ *such that*:

(1) $\mathbb{E}(|\langle \xi, \phi\rangle|^p) \le C_p \|\phi\|_\infty^p$ *for all* $\phi \in C^\infty(\mathbb{T}^2)$;
(2) $\mathbb{E}(\|\xi\|_{H^{-1-\delta}}^p) \le C_{p,\delta}$;
(3) $\mathbb{E}(|\langle \xi\otimes\xi, H_\phi\rangle|^p) \le C_p \|\nabla^2\phi\|_\infty^p$ *for all* $\phi \in C^\infty(\mathbb{T}^2)$.

PROOF. The first assertion follows from the fact that $\langle \xi, \phi\rangle$ is a centered Gaussian random variable with variance $\|\phi\|_{L^2(\mathbb{T}^2)}^2$. Applying this result to $\phi = e_k$, we can deduce the second estimate from the definition of the Sobolev norm $\|\cdot\|_{H^{-1-\delta}}$.

We turn to prove the last one. Let $H_\phi^n, n \ge 1$ be the smooth approximations of H_ϕ constructed in [13], Remark 9, satisfying

$$\|H_\phi^n\|_\infty \le \|H_\phi\|_\infty \le C\|\nabla^2\phi\|_\infty,$$

where the last inequality is due to (2.6). By [13], Corollary 6(i), we have

$$\mathbb{E}\big(|\langle \xi\otimes\xi, H_\phi^n\rangle|^p\big) \le C_p \|H_\phi^n\|_\infty^p \le C_p' \|\nabla^2\phi\|_\infty^p.$$

This implies the family $\{\langle \xi\otimes\xi, H_\phi^n\rangle\}_{n\ge 1}$ is bounded in any $L^p(\Theta,\mathbb{P})$, $p > 1$, which, combined with the fact that $\langle \xi\otimes\xi, H_\phi^n\rangle$ converges to $\langle \xi\otimes\xi, H_\phi\rangle$ in $L^2(\Theta,\mathbb{P})$ (see [13], Theorem 8), yields the desired result. □

We first note that, for any $p > 1$ and $\delta > 0$, by (2) of Lemma 2.4,

$$\mathbb{E}\bigg[\int_0^T \|\omega_t^N\|_{H^{-1-\delta}}^p \,\mathrm{d}t\bigg] = \int_0^T \mathbb{E}\big[\|\omega_t^N\|_{H^{-1-\delta}}^p\big]\,\mathrm{d}t \le C_{p,\delta} T, \quad \text{for all } N \ge 1. \tag{2.8}$$

Next, similar to [14], Lemma 3.3, we can prove the following.

LEMMA 2.5. *There exists* $C > 0$ *such that for any* $\phi \in C^\infty(\mathbb{T}^2)$, *we have*

$$\mathbb{E}\big[\langle \omega_t^N - \omega_s^N, \phi\rangle^4\big] \le C(t-s)^2\big(\|\nabla\phi\|_\infty^4 + \|\nabla^2\phi\|_\infty^4\big).$$

PROOF. The proof is similar to that of [14], Lemma 3.3. By (2.4), we have

$$
\begin{aligned}
\langle \omega_t^N - \omega_s^N, \phi \rangle = &\int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle \, \mathrm{d}r \\
&- 2\sqrt{2\nu}\varepsilon_N \sum_{k\in\Lambda_N} \int_s^t \langle \omega_r^N, \sigma_k \cdot \nabla\phi \rangle \, \mathrm{d}W_r^k \\
&+ 4\nu\varepsilon_N^2 \sum_{k\in\Lambda_N} \int_s^t \langle \omega_r^N, \sigma_k \cdot \nabla(\sigma_k \cdot \nabla\phi) \rangle \, \mathrm{d}r.
\end{aligned} \tag{2.9}
$$

First, Hölder's inequality leads to

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle \, \mathrm{d}r\bigg)^4\bigg] \\
&\quad \le (t-s)^3 \mathbb{E}\bigg[\int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle^4 \, \mathrm{d}r\bigg] \\
&\quad \le (t-s)^3 \int_s^t C\|\nabla^2\phi\|_\infty^4 \, \mathrm{d}r = C(t-s)^4 \|\nabla^2\phi\|_\infty^4,
\end{aligned} \tag{2.10}
$$

where in the second step we used the fact that ω_r^N is a white noise and Lemma 2.4(3).

Next, by Burkholder's inequality,

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg(\varepsilon_N \sum_{k\in\Lambda_N} \int_s^t \langle \omega_r^N, \sigma_k \cdot \nabla\phi \rangle \, \mathrm{d}W_r^k\bigg)^4\bigg] \\
&\quad \le C\varepsilon_N^4 \mathbb{E}\bigg[\bigg(\int_s^t \sum_{k\in\Lambda_N} \langle \omega_r^N, \sigma_k \cdot \nabla\phi \rangle^2 \, \mathrm{d}r\bigg)^2\bigg] \\
&\quad \le C\varepsilon_N^4 (t-s) \int_s^t \mathbb{E}\bigg[\bigg(\sum_{k\in\Lambda_N} \langle \omega_r^N, \sigma_k \cdot \nabla\phi \rangle^2\bigg)^2\bigg] \, \mathrm{d}r.
\end{aligned}
$$

We have by Cauchy's inequality and Lemma 2.4(1) that

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg(\sum_{k\in\Lambda_N} \langle \omega_r^N, \sigma_k \cdot \nabla\phi \rangle^2\bigg)^2\bigg] \\
&\quad = \sum_{k,l\in\Lambda_N} \mathbb{E}\big[\langle \omega_r^N, \sigma_k \cdot \nabla\phi \rangle^2 \langle \omega_r^N, \sigma_l \cdot \nabla\phi \rangle^2\big] \\
&\quad \le \sum_{k,l\in\Lambda_N} \big[\mathbb{E}\langle \omega_r^N, \sigma_k \cdot \nabla\phi \rangle^4\big]^{1/2} \big[\mathbb{E}\langle \omega_r^N, \sigma_l \cdot \nabla\phi \rangle^4\big]^{1/2} \\
&\quad \le C\bigg(\sum_{k\in\Lambda_N} \|\sigma_k \cdot \nabla\phi\|_\infty^2\bigg)^2 \\
&\quad \le \tilde{C}\|\nabla\phi\|_\infty^4 \bigg(\sum_{k\in\Lambda_N} \|\sigma_k\|_\infty^2\bigg)^2.
\end{aligned}
$$

Note that, by (2.2),

$$
\sum_{k\in\Lambda_N} \|\sigma_k\|_\infty^2 = \sum_{k\in\Lambda_N} \frac{1}{|k|^2} = \varepsilon_N^{-2},
$$

hence,

$$\mathbb{E}\bigg[\bigg(\sum_{k\in\Lambda_N}\langle\omega_r^N,\sigma_k\cdot\nabla\phi\rangle^2\bigg)^2\bigg]\le C\|\nabla\phi\|_\infty^4\varepsilon_N^{-4}.$$

This implies

$$\mathbb{E}\bigg[\bigg(\varepsilon_N\sum_{k\in\Lambda_N}\int_s^t\langle\omega_r^N,\sigma_k\cdot\nabla\phi\rangle\,\mathrm{d}W_r^k\bigg)^4\bigg]\le C(t-s)^2\|\nabla\phi\|_\infty^4. \tag{2.11}$$

Finally, by Hölder's inequality,

$$\begin{aligned}&\mathbb{E}\bigg[\bigg(\varepsilon_N^2\sum_{k\in\Lambda_N}\int_s^t\langle\omega_r^N,\sigma_k\cdot\nabla(\sigma_k\cdot\nabla\phi)\rangle\,\mathrm{d}r\bigg)^4\bigg]\\&\quad\le\varepsilon_N^8(t-s)^3\int_s^t\mathbb{E}\bigg[\bigg(\sum_{k\in\Lambda_N}\langle\omega_r^N,\sigma_k\cdot\nabla(\sigma_k\cdot\nabla\phi)\rangle\bigg)^4\bigg]\mathrm{d}r.\end{aligned}$$

Since $\sigma_k\cdot\nabla\sigma_k\equiv 0$, we have $\sigma_k\cdot\nabla(\sigma_k\cdot\nabla\phi)=\mathrm{Tr}[(\sigma_k\otimes\sigma_k)\nabla^2\phi]$. Therefore, by Lemma 2.6 below,

$$\sum_{k\in\Lambda_N}\sigma_k\cdot\nabla(\sigma_k\cdot\nabla\phi)=\frac{1}{4}\varepsilon_N^{-2}\Delta\phi.$$

As a result,

$$\begin{aligned}&\mathbb{E}\bigg[\bigg(\varepsilon_N^2\sum_{k\in\Lambda_N}\int_s^t\langle\omega_r^N,\sigma_k\cdot\nabla(\sigma_k\cdot\nabla\phi)\rangle\,\mathrm{d}r\bigg)^4\bigg]\\&\quad\le C(t-s)^3\int_s^t\mathbb{E}\big[\langle\omega_r^N,\Delta\phi\rangle^4\big]\,\mathrm{d}r\le C(t-s)^4\|\Delta\phi\|_\infty^4.\end{aligned}$$

Combining this estimate together with (2.9)–(2.11), we obtain the desired estimate. □

LEMMA 2.6. *It holds that*

$$\sum_{k\in\Lambda_N}\sigma_k\otimes\sigma_k=\frac{1}{4}\varepsilon_N^{-2}I_2,$$

where I_2 is the two-dimensional identity matrix.

PROOF. We have

$$\begin{aligned}Q_N(x)&:=\sum_{k\in\Lambda_N}\sigma_k(x)\otimes\sigma_k(x)\\&=\sum_{k\in\Lambda_N\cap\mathbb{Z}_+^2}\frac{k^\perp\otimes k^\perp}{|k|^4}\big[\cos^2(2\pi k\cdot x)+\sin^2(2\pi k\cdot x)\big]\\&=\sum_{k\in\Lambda_N\cap\mathbb{Z}_+^2}\frac{1}{|k|^4}\begin{pmatrix}k_2^2&-k_1k_2\\-k_1k_2&k_1^2\end{pmatrix}\\&=\frac{1}{2}\sum_{k\in\Lambda_N}\frac{1}{|k|^4}\begin{pmatrix}k_2^2&-k_1k_2\\-k_1k_2&k_1^2\end{pmatrix}.\end{aligned}$$

So Q_N is independent on x. First, we have

$$Q_N^{1,2} = -\frac{1}{2} \sum_{k \in \Lambda_N} \frac{k_1 k_2}{|k|^4} = 0$$

since we can sum the four terms involving (k_1, k_2), $(-k_1, k_2)$, $(k_1, -k_2)$ and $(-k_1, -k_2)$ at one time. Next,

$$Q_N^{1,1} = \frac{1}{2} \sum_{k \in \Lambda_N} \frac{k_2^2}{|k|^4} = \frac{1}{2} \sum_{k \in \Lambda_N} \frac{k_1^2}{|k|^4} = Q_N^{2,2}$$

since the points (k_1, k_2) and (k_2, k_1) appear in pair. Therefore,

$$Q_N^{1,1} = Q_N^{2,2} = \frac{1}{4} \sum_{k \in \Lambda_N} \frac{k_1^2 + k_2^2}{|k|^4} = \frac{1}{4} \sum_{k \in \Lambda_N} \frac{1}{|k|^2} = \frac{1}{4} \varepsilon_N^{-2}.$$

The proof is complete. □

Applying Lemma 2.5 with $\phi(x) = e_k(x)$ leads to

$$\mathbb{E}\big[|\langle \omega_t^N - \omega_s^N, e_k \rangle|^4\big] \le C(t-s)^2 |k|^8, \quad k \in \mathbb{Z}_0^2.$$

As a result, by Cauchy's inequality,

$$\begin{aligned}
&\mathbb{E}\big(\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4\big) \\
&= \mathbb{E}\bigg[\bigg(\sum_k (1+|k|^2)^{-\kappa} |\langle \omega_t^N - \omega_s^N, e_k \rangle|^2\bigg)^2\bigg] \\
&\le \bigg(\sum_k (1+|k|^2)^{-\kappa}\bigg) \sum_k (1+|k|^2)^{-\kappa} \mathbb{E}\big[|\langle \omega_t^N - \omega_s^N, e_k \rangle|^4\big] \\
&\le \tilde{C}(t-s)^2 \sum_k (1+|k|^2)^{-\kappa} |k|^8 \le \hat{C}(t-s)^2,
\end{aligned}$$

since $2\kappa - 8 > 2$ due to the choice of κ. Consequently,

$$\mathbb{E}\bigg[\int_0^T \int_0^T \frac{\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4}{|t-s|^{7/3}} \,\mathrm{d}t\,\mathrm{d}s\bigg] \le \hat{C} \int_0^T \int_0^T \frac{|t-s|^2}{|t-s|^{7/3}} \,\mathrm{d}t\,\mathrm{d}s < \infty.$$

This implies the family $\{Q^N\}_{N \ge 1}$ of probability measures is bounded in probability in $W^{1/3,4}(0, T; H^{-\kappa}(\mathbb{T}^2))$.

Combining this result with (2.8) and the discussions below Lemma 2.3, we conclude that $\{Q^N\}_{N \ge 1}$ is tight in $\mathcal{X} = C([0, T], H^{-1-}(\mathbb{T}^2))$.

Since we are dealing with the SDEs (2.3), we need to consider Q^N together with the distribution of Brownian motions. Although we use only finitely many Brownian motions in (2.3), here we consider for simplicity the whole family $\{(W_t^k)_{0 \le t \le T} : k \in \mathbb{Z}_0^2\}$. To this end, we assume $\mathbb{R}^{\mathbb{Z}_0^2}$ is endowed with the metric

$$d_{\mathbb{Z}_0^2}(a, b) = \sum_{k \in \mathbb{Z}_0^2} \frac{|a_k - b_k| \wedge 1}{2^{|k|}}, \quad a, b \in \mathbb{R}^{\mathbb{Z}_0^2}.$$

Then $(\mathbb{R}^{\mathbb{Z}_0^2}, d_{\mathbb{Z}_0^2})$ is separable and complete (see [6], page 9, Example 1.2). The distance in $\mathcal{Y} := C([0, T], \mathbb{R}^{\mathbb{Z}_0^2})$ is given by

$$d_{\mathcal{Y}}(w, \hat{w}) = \sup_{t \in [0,T]} d_{\mathbb{Z}_0^2}\big(w(t), \hat{w}(t)\big), \quad w, \hat{w} \in \mathcal{Y},$$

which makes $\mathcal{Y}$ a Polish space. Denote by $\mathcal{W}$ the law on $\mathcal{Y}$ of the sequence of independent Brownian motions $\{(W_t^k)_{0\le t\le T} : k \in \mathbb{Z}_0^2\}$.

To simplify the notation, we write $W_\cdot = (W_t)_{0\le t\le T}$ for the whole sequence of processes $\{(W_t^k)_{0\le t\le T} : k \in \mathbb{Z}_0^2\}$ in $\mathcal{Y}$. Denote by P^N the joint law of $(\omega_\cdot^N, W_\cdot)$ on $\mathcal{X}\times\mathcal{Y}$, $N \ge 1$. Since the marginal laws $\{Q^N\}_{N\in\mathbb{N}}$ and $\{\mathcal{W}\}$ are respectively tight on $\mathcal{X}$ and $\mathcal{Y}$, we conclude that $\{P^N\}_{N\in\mathbb{N}}$ is tight on $\mathcal{X}\times\mathcal{Y}$. The Prohorov theorem (see [6], page 59, Theorem 5.1) implies that there exists a subsequence $\{N_i\}_{i\in\mathbb{N}}$ of integers, such that P^{N_i} converge weakly to some probability measure on $\mathcal{X}\times\mathcal{Y}$. By Skorohod's representation theorem ([6], page 70, Theorem 6.7), there exist a probability space $(\tilde\Theta, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and stochastic processes $(\tilde\omega_\cdot^{N_i}, \tilde W_\cdot^{N_i})$ on this space with the corresponding laws P^{N_i}, and converging $\tilde{\mathbb{P}}$-a.s. in $\mathcal{X}\times\mathcal{Y}$ to a limit $(\tilde\omega_\cdot, \tilde W_\cdot)$. We are going to prove that $\tilde\omega_\cdot$ solves equation (2.5) with a suitable cylindrical Brownian motion.

First, we have the following simple result.

LEMMA 2.7. *The process $\tilde\omega_\cdot$ is stationary with paths in $\mathcal{X}$, and for every $t\in[0,T]$, the law μ_t of $\tilde\omega_t$ on $H^{-1-}(\mathbb{T}^2)$ is the white noise measure μ.*

PROOF. Recall that, for every $i\ge 1$, $\tilde\omega_\cdot^{N_i}$ has the same law as the stationary process $\omega_\cdot^{N_i}$ which solves (2.3) with $N = N_i$, and has white noise measure μ as their marginal distributions. For every $m\ge 1$ and $F\in C_b((H^{-1-}(\mathbb{T}^2))^m)$, $0\le t_1<\cdots<t_m\le T$ and $h>0$ such that $t_m+h\le T$, since $\tilde\omega_\cdot^{N_i}$ converges to $\tilde\omega_\cdot$ a.s. in $C([0,T], H^{-1-}(\mathbb{T}^2))$, one has

$$\begin{aligned}\tilde{\mathbb{E}}\big[F(\tilde\omega_{t_1},\dots,\tilde\omega_{t_m})\big] &= \lim_{i\to\infty}\tilde{\mathbb{E}}\big[F\big(\tilde\omega_{t_1}^{N_i},\dots,\tilde\omega_{t_m}^{N_i}\big)\big]\\ &= \lim_{i\to\infty}\tilde{\mathbb{E}}\big[F\big(\tilde\omega_{t_1+h}^{N_i},\dots,\tilde\omega_{t_m+h}^{N_i}\big)\big]\\ &= \tilde{\mathbb{E}}\big[F(\tilde\omega_{t_1+h},\dots,\tilde\omega_{t_m+h})\big],\end{aligned}$$

where $\tilde{\mathbb{E}}$ is the expectation on $(\tilde\Theta, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Hence $\tilde\omega_\cdot$ is stationary. Similarly, for any $F\in C_b(H^{-1-}(\mathbb{T}^2))$,

$$\int F(\omega)\,\mathrm{d}\mu_t(\omega) = \tilde{\mathbb{E}}\big[F(\tilde\omega_t)\big] = \lim_{i\to\infty}\tilde{\mathbb{E}}\big[F\big(\tilde\omega_t^{N_i}\big)\big] = \int F(\omega)\,\mathrm{d}\mu(\omega). \qquad \square$$

Next, we show that $(\tilde\omega_\cdot^{N_i}, \tilde W_\cdot^{N_i})$ satisfies an equation similar to that for $(\omega_\cdot^{N_i}, W_\cdot)$. By (2.4) and Lemma 2.6,

$$\begin{aligned}\langle\omega_t^{N_i},\phi\rangle &= \langle\omega_0^{N_i},\phi\rangle + \int_0^t\langle\omega_r^{N_i}\otimes\omega_r^{N_i}, H_\phi\rangle\,\mathrm{d}r + \nu\int_0^t\langle\omega_r^{N_i},\Delta\phi\rangle\,\mathrm{d}r\\ &\quad - 2\sqrt{2\nu}\varepsilon_{N_i}\sum_{k\in\Lambda_{N_i}}\int_0^t\langle\omega_r^{N_i},\sigma_k\cdot\nabla\phi\rangle\,\mathrm{d}W_r^k.\end{aligned}\tag{2.12}$$

For any $\phi\in C^\infty(\mathbb{T}^2)$, let $\{H_\phi^n\}_{n\ge1}\subset H^{2+}(\mathbb{T}^2\times\mathbb{T}^2)$ be an approximation of H_ϕ satisfying (cf. [13], Remark 9)

$$\lim_{n\to\infty}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}(H_\phi^n - H_\phi)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0, \qquad \int_{\mathbb{T}^2}H_\phi^n(x,x)\,\mathrm{d}x = 0, \quad n\ge 1.$$

Note that $(\tilde{\omega}_\cdot^{N_i}, \tilde{W}_\cdot^{N_i})$ has the same law as $(\omega_\cdot^{N_i}, W_\cdot)$, and the latter satisfies the equation (2.12), therefore, it is easy to show that

$$\begin{aligned}\tilde{\mathbb{E}}\bigg\{\sup_{t\in[0,T]}\bigg|\langle\tilde{\omega}_t^{N_i},\phi\rangle-\langle\tilde{\omega}_0^{N_i},\phi\rangle-\int_0^t\langle\tilde{\omega}_r^{N_i}\otimes\tilde{\omega}_r^{N_i},H_\phi\rangle\,\mathrm{d}r\\ -\nu\int_0^t\langle\tilde{\omega}_r^{N_i},\Delta\phi\rangle\,\mathrm{d}r+2\sqrt{2\nu}\varepsilon_{N_i}\sum_{k\in\Lambda_{N_i}}\int_0^t\langle\tilde{\omega}_r^{N_i},\sigma_k\cdot\nabla\phi\rangle\,\mathrm{d}\tilde{W}_r^{N_i,k}\bigg|\bigg\}\\ \le\tilde{\mathbb{E}}\bigg\{\sup_{t\in[0,T]}\bigg|\int_0^t\langle\tilde{\omega}_r^{N_i}\otimes\tilde{\omega}_r^{N_i},H_\phi-H_\phi^n\rangle\,\mathrm{d}r\bigg|\bigg\}\\ +\mathbb{E}\bigg\{\sup_{t\in[0,T]}\bigg|\int_0^t\langle\omega_r^{N_i}\otimes\omega_r^{N_i},H_\phi-H_\phi^n\rangle\,\mathrm{d}r\bigg|\bigg\},\end{aligned}$$

which, since both $\tilde{\omega}_r^{N_i}$ and $\omega_r^{N_i}$ are distributed as the white noise measure μ, is dominated by

$$2T\mathbb{E}_\mu\big|\langle\omega\otimes\omega,H_\phi-H_\phi^n\rangle\big|\le2\sqrt{2}T\bigg(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}(H_\phi^n-H_\phi)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y\bigg)^{1/2},$$

where the inequality can be found in the proof of [13], Theorem 8. Letting $n\to\infty$ yields, $\tilde{\mathbb{P}}$-a.s., for all $t\in[0,T]$,

$$\begin{aligned}\langle\tilde{\omega}_t^{N_i},\phi\rangle=\langle\tilde{\omega}_0^{N_i},\phi\rangle+\int_0^t\langle\tilde{\omega}_r^{N_i}\otimes\tilde{\omega}_r^{N_i},H_\phi\rangle\,\mathrm{d}r+\nu\int_0^t\langle\tilde{\omega}_r^{N_i},\Delta\phi\rangle\,\mathrm{d}r\\ -2\sqrt{2\nu}\varepsilon_{N_i}\sum_{k\in\Lambda_{N_i}}\int_0^t\langle\tilde{\omega}_r^{N_i},\sigma_k\cdot\nabla\phi\rangle\,\mathrm{d}\tilde{W}_r^{N_i,k}.\end{aligned}\tag{2.13}$$

REMARK 2.8. Using the almost sure convergence of $\tilde{\omega}^{N_i}$ to $\tilde{\omega}$ in $C([0,T], H^{-1-}(\mathbb{T}^2))$, we can show that the quantities in the first line of (2.13) converge respectively in $L^2(\tilde{\Theta},\tilde{\mathbb{P}})$ to

$$\langle\tilde{\omega}_t,\phi\rangle,\qquad\langle\tilde{\omega}_0,\phi\rangle,\qquad\int_0^t\langle\tilde{\omega}_r\otimes\tilde{\omega}_r,H_\phi\rangle\,\mathrm{d}r,\qquad\int_0^t\langle\tilde{\omega}_r,\Delta\phi\rangle\,\mathrm{d}r;$$

see [14], Proposition 3.6, for details. However, the term involving stochastic integrals does not converge strongly to the last term of (2.5). Therefore, we can only seek for a weaker form of convergence.

Before proceeding further, we introduce some notation. By $\Lambda\Subset\mathbb{Z}_0^2$, we mean that Λ is a finite subset. Let $\Pi_\Lambda: H^{-1-}(\mathbb{T}^2)\to\mathrm{span}\{e_k:k\in\Lambda\}$ be the projection operator: $\Pi_\Lambda\omega=\sum_{l\in\Lambda}\langle\omega,e_l\rangle e_l$. We shall use the family of cylindrical functions below:

$$\mathcal{F}C_b^2=\big\{F(\omega)=f\big(\langle\omega,e_l\rangle;l\in\Lambda\big)\text{ for some }\Lambda\Subset\mathbb{Z}_0^2\text{ and }f\in C_b^2(\mathbb{R}^\Lambda)\big\},$$

where $\mathbb{R}^\Lambda$ is the $(\#\Lambda)$-dimensional Euclidean space. To simplify the notation, sometimes we write the cylindrical functions as $F=f\circ\Pi_\Lambda$, and for $l,m\in\Lambda$, $f_l(\omega)=(\partial_l f)(\Pi_\Lambda\omega)$ and $f_{l,m}(\omega)=(\partial_l\partial_m f)(\Pi_\Lambda\omega)$. Denote by $\mathcal{L}_\infty$ the generator of the equation (2.5): for any cylindrical function $F=f\circ\Pi_\Lambda$ with $\Lambda\Subset\mathbb{Z}_0^2$,

$$\mathcal{L}_\infty F=4\nu\pi^2\sum_{l\in\Lambda}|l|^2\big[f_{l,l}(\omega)-f_l(\omega)\langle\omega,e_l\rangle\big]-\big\langle u(\omega)\cdot\nabla\omega,DF\big\rangle,\tag{2.14}$$

where the drift part

$$\big\langle u(\omega)\cdot\nabla\omega,DF\big\rangle=-\sum_{l\in\Lambda}f_l(\omega)\langle\omega\otimes\omega,H_{e_l}\rangle.$$

Finally, we introduce the notation

$$C_{k,l}=\frac{k^\perp\cdot l}{|k|^2},\quad k,l\in\mathbb{Z}_0^2. \tag{2.15}$$

Now we prove that the limit process $\tilde\omega$ is a martingale solution of the operator $\mathcal{L}_\infty$.

PROPOSITION 2.9. *For any $F\in\mathcal{F}C_b^2$,*

$$\tilde M_t^F:=F(\tilde\omega_t)-F(\tilde\omega_0)-\int_0^t\mathcal{L}_\infty F(\tilde\omega_s)\,\mathrm{d}s \tag{2.16}$$

is an $\tilde{\mathcal{F}}_t=\sigma(\tilde\omega_s:s\le t)$-martingale.

PROOF. Recall the CONS defined in (2.1). Taking $\phi=e_l$ in (2.13) for some $l\in\mathbb{Z}_0^2$, we have

$$\begin{aligned}\mathrm{d}\langle\tilde\omega_t^{N_i},e_l\rangle&=\langle\tilde\omega_t^{N_i}\otimes\tilde\omega_t^{N_i},H_{e_l}\rangle\,\mathrm{d}t-4\nu\pi^2|l|^2\langle\tilde\omega_t^{N_i},e_l\rangle\,\mathrm{d}t\\&\quad-2\sqrt{2\nu}\varepsilon_{N_i}\sum_{k\in\Lambda_{N_i}}\langle\tilde\omega_t^{N_i},\sigma_k\cdot\nabla e_l\rangle\,\mathrm{d}\tilde W_t^{N_i,k}.\end{aligned} \tag{2.17}$$

For $l,m\in\mathbb{Z}_0^2$, we write $\mathrm{d}\langle\tilde\omega_t^{N_i},e_l\rangle\cdot\mathrm{d}\langle\tilde\omega_t^{N_i},e_m\rangle$ for the differential of the cross-variation of the two processes $\langle\tilde\omega_t^{N_i},e_l\rangle$ and $\langle\tilde\omega_t^{N_i},e_m\rangle$. Then

$$\mathrm{d}\langle\tilde\omega_t^{N_i},e_l\rangle\cdot\mathrm{d}\langle\tilde\omega_t^{N_i},e_m\rangle=8\nu\varepsilon_{N_i}^2\sum_{k\in\Lambda_{N_i}}\langle\tilde\omega_t^{N_i},\sigma_k\cdot\nabla e_l\rangle\langle\tilde\omega_t^{N_i},\sigma_k\cdot\nabla e_m\rangle\,\mathrm{d}t.$$

Direct computation leads to $\sigma_k\cdot\nabla e_l=\sqrt{2}\pi C_{k,l}e_ke_{-l}$; hence

$$\begin{aligned}&\langle\tilde\omega_t^{N_i},\sigma_k\cdot\nabla e_l\rangle\langle\tilde\omega_t^{N_i},\sigma_k\cdot\nabla e_m\rangle\\&\quad=2\pi^2C_{k,l}C_{k,m}\langle\tilde\omega_t^{N_i},e_ke_{-l}\rangle\langle\tilde\omega_t^{N_i},e_ke_{-m}\rangle\\&\quad=2\pi^2C_{k,l}C_{k,m}\big[\langle\tilde\omega_t^{N_i},e_ke_{-l}\rangle\langle\tilde\omega_t^{N_i},e_ke_{-m}\rangle-\delta_{l,m}\big]+2\pi^2\delta_{l,m}C_{k,l}^2.\end{aligned}$$

As a result,

$$\begin{aligned}&\mathrm{d}\langle\tilde\omega_t^{N_i},e_l\rangle\cdot\mathrm{d}\langle\tilde\omega_t^{N_i},e_m\rangle\\&\quad=16\nu\pi^2\varepsilon_{N_i}^2\sum_{k\in\Lambda_{N_i}}C_{k,l}C_{k,m}\big[\langle\tilde\omega_t^{N_i},e_ke_{-l}\rangle\langle\tilde\omega_t^{N_i},e_ke_{-m}\rangle-\delta_{l,m}\big]\,\mathrm{d}t\\&\qquad+8\nu\pi^2\delta_{l,m}|l|^2\,\mathrm{d}t,\end{aligned}$$

where in the last step we have used Lemma A.1. To simplify the notation, we denote by

$$R_{l,m}(\tilde\omega_t^{N_i})=8\nu\pi^2\sum_{k\in\Lambda_{N_i}}C_{k,l}C_{k,m}\big[\langle\tilde\omega_t^{N_i},e_ke_{-l}\rangle\langle\tilde\omega_t^{N_i},e_ke_{-m}\rangle-\delta_{l,m}\big].$$

Recall that $\tilde\omega_t^{N_i}$ is a white noise for any $t\in[0,T]$, thus by the second assertion of Proposition A.3, $R_{l,m}(\tilde\omega_t^{N_i})$ is bounded in any $L^p([0,T]\times\tilde\Theta)$, $p>1$. Finally, we get

$$\mathrm{d}\langle\tilde\omega_t^{N_i},e_l\rangle\cdot\mathrm{d}\langle\tilde\omega_t^{N_i},e_m\rangle=2\varepsilon_{N_i}^2R_{l,m}(\tilde\omega_t^{N_i})\,\mathrm{d}t+8\nu\pi^2\delta_{l,m}|l|^2\,\mathrm{d}t. \tag{2.18}$$

By the Itô formula and (2.17), (2.18),

$$\begin{aligned}\mathrm{d}F(\tilde{\omega}_t^{N_i}) &= \mathrm{d}f(\langle\tilde{\omega}_t^{N_i}, e_l\rangle; l\in\Lambda)\\ &= \sum_{l\in\Lambda} f_l(\tilde{\omega}_t^{N_i})\big[\langle\tilde{\omega}_t^{N_i}\otimes\tilde{\omega}_t^{N_i}, H_{e_l}\rangle - 4\nu\pi^2|l|^2\langle\tilde{\omega}_t^{N_i}, e_l\rangle\big]\,\mathrm{d}t\\ &\quad - 2\sqrt{2\nu}\varepsilon_{N_i}\sum_{l\in\Lambda} f_l(\tilde{\omega}_t^{N_i})\sum_{k\in\Lambda_{N_i}}\langle\tilde{\omega}_t^{N_i}, \sigma_k\cdot\nabla e_l\rangle\,\mathrm{d}\tilde{W}_t^{N_i,k}\\ &\quad + \sum_{l,m\in\Lambda} f_{l,m}(\tilde{\omega}_t^{N_i})\big[\varepsilon_{N_i}^2 R_{l,m}(\tilde{\omega}_t^{N_i}) + 4\nu\pi^2\delta_{l,m}|l|^2\big]\,\mathrm{d}t.\end{aligned}$$

Recalling the operator $\mathcal{L}_\infty$ defined in (2.14), the above formula can be rewritten as

$$\mathrm{d}F(\tilde{\omega}_t^{N_i}) = \mathcal{L}_\infty F(\tilde{\omega}_t^{N_i})\,\mathrm{d}t + \varepsilon_{N_i}^2\tilde{\zeta}_t^{N_i}\,\mathrm{d}t + \mathrm{d}\tilde{M}_t^{N_i}, \tag{2.19}$$

where

$$\tilde{\zeta}_t^{N_i} = \sum_{l,m\in\Lambda} f_{l,m}(\tilde{\omega}_t^{N_i})R_{l,m}(\tilde{\omega}_t^{N_i})$$

is bounded in $L^p([0,T]\times\tilde{\Theta})$ for any $p>1$, and the martingale part

$$\mathrm{d}\tilde{M}_t^{N_i} = -2\sqrt{2\nu}\varepsilon_{N_i}\sum_{l\in\Lambda} f_l(\tilde{\omega}_t^{N_i})\sum_{k\in\Lambda_{N_i}}\langle\tilde{\omega}_t^{N_i}, \sigma_k\cdot\nabla e_l\rangle\,\mathrm{d}\tilde{W}_t^{N_i,k}.$$

Note that $\tilde{M}_t^{N_i}$ is a martingale w.r.t. the filtration

$$\tilde{\mathcal{F}}_t^{N_i} = \sigma(\tilde{\omega}_s^{N_i}, \tilde{W}_s^{N_i} : s\le t),$$

where we denote by $\tilde{W}_s^{N_i} = \{\tilde{W}_s^{N_i,k}\}_{k\in\mathbb{Z}_0^2}$.

Next, we show that the formula (2.19) converges as $i\to\infty$ in a suitable sense. To this end, we follow the argument of [10], page 232. Fix any $0<s<t\le T$. Take a real valued, bounded and continuous function $\varphi : C([0,s], H^{-1-}\times\mathbb{R}^{\mathbb{Z}_0^2})\to\mathbb{R}$. By (2.19), we have

$$\tilde{\mathbb{E}}\bigg[\bigg(F(\tilde{\omega}_t^{N_i}) - F(\tilde{\omega}_s^{N_i}) - \int_s^t\mathcal{L}_\infty F(\tilde{\omega}_r^{N_i})\,\mathrm{d}r - \varepsilon_{N_i}^2\int_s^t\tilde{\zeta}_r^{N_i}\,\mathrm{d}r\bigg)\varphi(\tilde{\omega}_\cdot^{N_i}, \tilde{W}_\cdot^{N_i})\bigg] = 0.$$

Since $F\in\mathcal{FC}_b^2$ and $\tilde{\omega}_t^{N_i}$ is a white noise, all the terms in the bracket belong to $L^p(\tilde{\mathbb{P}})$ for any $p>1$. Recalling that, $\tilde{\mathbb{P}}$-a.s., $(\tilde{\omega}_\cdot^{N_i}, \tilde{W}_\cdot^{N_i})$ converges to $(\tilde{\omega}_\cdot, \tilde{W}_\cdot)$ in $C([0,T], H^{-1-}\times\mathbb{R}^{\mathbb{Z}_0^2})$, thus letting $i\to\infty$ in the above equality yields

$$\tilde{\mathbb{E}}\bigg[\bigg(F(\tilde{\omega}_t) - F(\tilde{\omega}_s) - \int_s^t\mathcal{L}_\infty F(\tilde{\omega}_r)\,\mathrm{d}r\bigg)\varphi(\tilde{\omega}_\cdot, \tilde{W}_\cdot)\bigg] = 0.$$

The arbitrariness of $0<s<t$ and $\varphi : C([0,s], H^{-1-}\times\mathbb{R}^{\mathbb{Z}_0^2})\to\mathbb{R}$ implies that $\tilde{M}_\cdot^F$ is a martingale with respect to the filtration $\tilde{\mathcal{G}}_t = \sigma(\tilde{\omega}_s, \tilde{W}_s : s\le t)$, $t\in[0,T]$. For any $0\le s<t\le T$, we have $\tilde{\mathcal{F}}_s\subset\tilde{\mathcal{G}}_s$, thus

$$\tilde{\mathbb{E}}(\tilde{M}_t^F|\tilde{\mathcal{F}}_s) = \tilde{\mathbb{E}}[\tilde{\mathbb{E}}(\tilde{M}_t^F|\tilde{\mathcal{G}}_s)|\tilde{\mathcal{F}}_s] = \tilde{\mathbb{E}}[\tilde{M}_s^F|\tilde{\mathcal{F}}_s] = \tilde{M}_s^F,$$

since $\tilde{M}_s^F$ is adapted to $\tilde{\mathcal{F}}_s$. □

Next, we show that $\tilde{\omega}_\cdot$ solves (2.5) in a weak sense; cf. [9], Definition 4.1.

PROPOSITION 2.10. *There exists a family of independent standard Brownian motions* $\{\tilde W_t^k : t \ge 0\}_{k\in\mathbb{Z}_0^2}$ *such that* $(\tilde\omega_\cdot, \tilde W_\cdot)$ *solves* (2.5) *in the sense of* (2.25), *where* $\tilde W_t = \sum_{k\in\mathbb{Z}_0^2} \tilde W_t^{-k} e_k \frac{k^\perp}{|k|}$.

PROOF. In order to identify the process $\tilde\omega_t$, we take some special cylinder functions F. First, let $F(\omega) = \langle \omega, e_l\rangle$ for some $l \in \mathbb{Z}_0^2$, then

$$\mathcal{L}_\infty F(\omega) = -4\nu\pi^2|l|^2\langle \omega, e_l\rangle - \langle u(\omega)\cdot\nabla\omega, e_l\rangle.$$

Thus, by Proposition 2.9, the processes

$$\tilde M_t^{(l)} := \langle\tilde\omega_t, e_l\rangle - \langle\tilde\omega_0, e_l\rangle + \int_0^t \big(4\nu\pi^2|l|^2\langle\tilde\omega_s, e_l\rangle + \langle u(\tilde\omega_s)\cdot\nabla\tilde\omega_s, e_l\rangle\big)\,\mathrm{d}s, \quad l\in\mathbb{Z}_0^2$$

are martingales. In particular,

$$\mathrm{d}\langle\tilde\omega_t, e_l\rangle = \mathrm{d}\tilde M_t^{(l)} - \big(4\nu\pi^2|l|^2\langle\tilde\omega_t, e_l\rangle + \langle u(\tilde\omega_t)\cdot\nabla\tilde\omega_t, e_l\rangle\big)\,\mathrm{d}t, \quad l\in\mathbb{Z}_0^2. \tag{2.20}$$

Therefore, for $l, m \in \mathbb{Z}_0^2$, $l \ne m$,

$$\begin{aligned}
&\mathrm{d}\big[\langle\tilde\omega_t, e_l\rangle\langle\tilde\omega_t, e_m\rangle\big]\\
&\quad= \langle\tilde\omega_t, e_m\rangle\,\mathrm{d}\tilde M_t^{(l)} - \langle\tilde\omega_t, e_m\rangle\big(4\nu\pi^2|l|^2\langle\tilde\omega_t, e_l\rangle + \langle u(\tilde\omega_t)\cdot\nabla\tilde\omega_t, e_l\rangle\big)\,\mathrm{d}t\\
&\qquad+ \langle\tilde\omega_t, e_l\rangle\,\mathrm{d}\tilde M_t^{(m)} - \langle\tilde\omega_t, e_l\rangle\big(4\nu\pi^2|m|^2\langle\tilde\omega_t, e_m\rangle + \langle u(\tilde\omega_t)\cdot\nabla\tilde\omega_t, e_m\rangle\big)\,\mathrm{d}t\\
&\qquad+ \mathrm{d}\langle\tilde M^{(l)}, \tilde M^{(m)}\rangle_t.
\end{aligned}$$

Equivalently, denoting by $\tilde M_t$ the martingale part,

$$\begin{aligned}
&\langle\tilde\omega_t, e_l\rangle\langle\tilde\omega_t, e_m\rangle\\
&\quad= \langle\tilde\omega_0, e_l\rangle\langle\tilde\omega_0, e_m\rangle + \tilde M_t - 4\nu\pi^2\big(|l|^2+|m|^2\big)\int_0^t \langle\tilde\omega_s, e_l\rangle\langle\tilde\omega_s, e_m\rangle\,\mathrm{d}s\\
&\qquad- \int_0^t \big[\langle\tilde\omega_s, e_m\rangle\langle u(\tilde\omega_s)\cdot\nabla\tilde\omega_s, e_l\rangle + \langle\tilde\omega_s, e_l\rangle\langle u(\tilde\omega_s)\cdot\nabla\tilde\omega_s, e_m\rangle\big]\,\mathrm{d}s\\
&\qquad+ \langle\tilde M^{(l)}, \tilde M^{(m)}\rangle_t.
\end{aligned} \tag{2.21}$$

On the other hand, taking $F(\omega) = \langle\omega, e_l\rangle\langle\omega, e_m\rangle$, we have

$$\begin{aligned}
\mathcal{L}_\infty F(\omega) &= \langle\omega, e_m\rangle\big(-4\nu\pi^2|l|^2\langle\omega, e_l\rangle - \langle u(\omega)\cdot\nabla\omega, e_l\rangle\big)\\
&\quad+ \langle\omega, e_l\rangle\big(-4\nu\pi^2|m|^2\langle\omega, e_m\rangle - \langle u(\omega)\cdot\nabla\omega, e_m\rangle\big)\\
&= -4\nu\pi^2\big(|l|^2+|m|^2\big)\langle\omega, e_l\rangle\langle\omega, e_m\rangle - \langle\omega, e_m\rangle\langle u(\omega)\cdot\nabla\omega, e_l\rangle\\
&\quad- \langle\omega, e_l\rangle\langle u(\omega)\cdot\nabla\omega, e_m\rangle.
\end{aligned}$$

Therefore, by (2.16), we obtain the martingale

$$\begin{aligned}
\tilde M_t^{(l,m)} &= \langle\tilde\omega_t, e_l\rangle\langle\tilde\omega_t, e_m\rangle - \langle\tilde\omega_0, e_l\rangle\langle\tilde\omega_0, e_m\rangle\\
&\quad+ 4\nu\pi^2\big(|l|^2+|m|^2\big)\int_0^t \langle\tilde\omega_s, e_l\rangle\langle\tilde\omega_s, e_m\rangle\,\mathrm{d}s\\
&\quad+ \int_0^t \big[\langle\tilde\omega_s, e_m\rangle\langle u(\tilde\omega_s)\cdot\nabla\tilde\omega_s, e_l\rangle + \langle\tilde\omega_s, e_l\rangle\langle u(\tilde\omega_s)\cdot\nabla\tilde\omega_s, e_m\rangle\big]\,\mathrm{d}s.
\end{aligned}$$

Comparing this equality with (2.21), we deduce

$$\langle \tilde{M}^{(l)}, \tilde{M}^{(m)} \rangle_t = 0, \quad l \neq m. \tag{2.22}$$

Next, by (2.20), we have

$$\begin{aligned} \mathrm{d}\big(\langle \tilde{\omega}_t, e_l \rangle^2\big) &= 2\langle \tilde{\omega}_t, e_l \rangle \big[\mathrm{d}\tilde{M}^{(l)}_t - \big(4\nu\pi^2|l|^2\langle \tilde{\omega}_t, e_l \rangle + \langle u(\tilde{\omega}_t)\cdot\nabla\tilde{\omega}_t, e_l \rangle\big)\,\mathrm{d}t\big] \\ &\quad + \mathrm{d}\langle \tilde{M}^{(l)} \rangle_t, \end{aligned}$$

which implies

$$\begin{aligned} \langle \tilde{\omega}_t, e_l \rangle^2 &= \langle \tilde{\omega}_0, e_l \rangle^2 + 2\int_0^t \langle \tilde{\omega}_s, e_l \rangle\,\mathrm{d}\tilde{M}^{(l)}_s + \langle \tilde{M}^{(l)} \rangle_t \\ &\quad - 2\int_0^t \langle \tilde{\omega}_s, e_l \rangle \big(4\nu\pi^2|l|^2\langle \tilde{\omega}_s, e_l \rangle + \langle u(\tilde{\omega}_s)\cdot\nabla\tilde{\omega}_s, e_l \rangle\big)\,\mathrm{d}s. \end{aligned} \tag{2.23}$$

Similarly, taking $F(\omega) = \langle \omega, e_l \rangle^2$, one has

$$\mathcal{L}_\infty F(\omega) = -2\langle \omega, e_l \rangle \langle u(\omega)\cdot\nabla\omega, e_l \rangle - 8\nu\pi^2|l|^2\big(\langle \omega, e_l \rangle^2 - 1\big).$$

Substituting this into (2.16) gives us the martingale

$$\begin{aligned} \tilde{M}^{(l,l)}_t &= \langle \tilde{\omega}_t, e_l \rangle^2 - \langle \tilde{\omega}_0, e_l \rangle^2 + 2\int_0^t \langle \tilde{\omega}_s, e_l \rangle \langle u(\tilde{\omega}_s)\cdot\nabla\tilde{\omega}_s, e_l \rangle\,\mathrm{d}s \\ &\quad + 8\nu\pi^2|l|^2\int_0^t \big(\langle \tilde{\omega}_s, e_l \rangle^2 - 1\big)\,\mathrm{d}s. \end{aligned}$$

Comparing this identity with (2.23) yields

$$\langle \tilde{M}^{(l)} \rangle_t = 8\nu\pi^2|l|^2 t. \tag{2.24}$$

According to the equalities (2.22) and (2.24), if we define

$$\tilde{W}^l_t = \frac{1}{2\sqrt{2\nu}\pi|l|}\tilde{M}^{(l)}_t, \quad l \in \mathbb{Z}^2_0,$$

then $\{\tilde{W}^l\}_{l\in\mathbb{Z}^2_0}$ is a family of independent standard Brownian motions. Now the formula (2.20) becomes

$$\begin{aligned} \mathrm{d}\langle \tilde{\omega}_t, e_l \rangle &= 2\sqrt{2\nu}\pi|l|\,\mathrm{d}\tilde{W}^l_t \\ &\quad - \big(4\nu\pi^2|l|^2\langle \tilde{\omega}_t, e_l \rangle + \langle u(\tilde{\omega}_t)\cdot\nabla\tilde{\omega}_t, e_l \rangle\big)\,\mathrm{d}t, \quad l \in \mathbb{Z}^2_0. \end{aligned} \tag{2.25}$$

The above equations are the component form of the equation below,

$$\mathrm{d}\tilde{\omega}_t + u(\tilde{\omega}_t)\cdot\nabla\tilde{\omega}_t\,\mathrm{d}t = \nu\Delta\tilde{\omega}_t\,\mathrm{d}t + \sqrt{2\nu}\nabla^\perp\cdot\mathrm{d}\tilde{W}_t, \tag{2.26}$$

where $\tilde{W}_t$ is the vector valued white noise defined in the statement of the proposition. Therefore, $\tilde{\omega}_t$ solves the vorticity form (2.26) of the Navier–Stokes equation driven by space-time white noise. □

We can rewrite (2.26) in the velocity variable $\tilde{u}_\cdot = u(\tilde{\omega}_\cdot)$ as follows:

$$\mathrm{d}\tilde{u} + b(\tilde{u})\,\mathrm{d}t = \nu A\tilde{u}\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}\tilde{W}. \tag{2.27}$$

Here, $b(u) = \mathcal{P}\,\mathrm{div}(u\otimes u)$ and $Au = \mathcal{P}\Delta u$, in which $\mathcal{P}$ is the orthogonal projection onto the space of divergence-free vector fields on $\mathbb{T}^2$. It is clear that $\tilde{u}$ has trajectories in $C([0,T], H^-(\mathbb{T}^2))$, that is, in $C([0,T], H^{-\delta}(\mathbb{T}^2))$ for any $\delta > 0$. As mentioned at the beginning of this paper, the above equation has been studied intensively in the last two decades.

We deduce from Lemma 2.7 and Proposition 2.10 that the process $\tilde{u}$ is a stationary solution to (2.27) in the sense of [9], Definition 4.1. Let us remark that this definition is based only on the Sobolev regularity of $\tilde{u} \in C([0,T], H^{-}(\mathbb{T}^2))$; the definition of the nonlinear part $b(\tilde{u})$ is based on the Galerkin approximation and coincides with our definition, as explained by Theorem A.12 in terms of the vorticity variable.

Similar to the arguments in [20], Section 3.5, we can prove the following.

PROPOSITION 2.11. *The uniqueness in law holds for stationary solutions to* (2.27).

PROOF. By [19], Theorem 3.14, it is sufficient to show that the pathwise uniqueness holds for stationary solutions of (2.27). Let u_i $(i=1,2)$ be two stationary solutions to the equation (2.27) in the sense of [9], Definition 4.1, which are defined on the same probability space $(\Theta, \mathcal{F}, \mathbb{P})$, with the same initial data $u_1(0) = u_2(0) = u(0)$ ($\mathbb{P}$-a.s.) and the same cylindrical Brownian motion $W(t)$, $0 \le t \le T$. Then, for $i = 1, 2$, $\mathbb{P}$-a.s.,

$$u_i(t) = u(0) - \int_0^t b\big(u_i(s)\big)\,\mathrm{d}s + \nu \int_0^t A u_i(s)\,\mathrm{d}s + \sqrt{2\nu}\, W(t), \quad 0 \le t \le T.$$

These equations can be rewritten as

$$u_i(t) = \mathrm{e}^{\nu t A} u(0) - \int_0^t \mathrm{e}^{\nu(t-s)A} b\big(u_i(s)\big)\,\mathrm{d}s + \sqrt{2\nu} \int_0^t \mathrm{e}^{\nu(t-s)A}\,\mathrm{d}W(s).$$

We extend $W(\cdot)$ to be a two-sided cylindrical Brownian motion on $\mathbb{R}$ (possibly at the price of enlarging $(\Theta, \mathcal{F}, \mathbb{P})$) and define

$$Z(t) = \sqrt{2\nu} \int_{-\infty}^t \mathrm{e}^{\nu(t-s)A}\,\mathrm{d}W(s).$$

It is well known that Z is a stationary process with paths in $C([0,T], B^{\sigma}_{p,\rho})$ for any $\sigma < 0$, $\rho \ge p \ge 2$ (cf. the last line on page 196 of [9]). Here, for any $s \in \mathbb{R}$, $B^s_{p,\rho}$ is the Besov space on $\mathbb{T}^2$. Note that

$$\sqrt{2\nu} \int_0^t \mathrm{e}^{\nu(t-s)A}\,\mathrm{d}W(s) = Z(t) - \mathrm{e}^{\nu t A} Z(0),$$

we arrive at

$$u_i(t) - Z(t) = \mathrm{e}^{\nu t A}\big(u(0) - Z(0)\big) - \int_0^t \mathrm{e}^{\nu(t-s)A} b\big(u_i(s)\big)\,\mathrm{d}s, \quad i = 1, 2. \tag{2.28}$$

As in [9], Theorem 5.2, page 196, let α, β, p, ρ, σ be such that

$$\frac{2}{p} > \alpha > -\sigma > 0, \qquad \rho = p \ge 2, \qquad \beta \ge 1, \qquad -\frac{1}{2} + \frac{1}{p} < \frac{\alpha}{2} - \frac{1}{\beta} < \frac{\sigma}{2}.$$

Using these parameters, we define the following space:

$$\mathcal{E} = L^{\beta}\big(0,T; \mathcal{B}^{\alpha}_{p,\rho}\big) \cap C\big([0,T], \mathcal{B}^{\sigma}_{p,\rho}\big).$$

Since for any $t \in [0,T]$, $u_i(t)$ is distributed as

$$\mathcal{N}\big(0, (-A)^{-1}\big) = \bigotimes_{k \in \mathbb{Z}^2_0} \mathcal{N}\big(0, 1/(4\pi^2 |k|^2)\big),$$

one has $u_i(t) \in B^{\sigma}_{p,\rho}$, $\mathbb{P}$-a.s. (see [4], Proposition 3.1). We also have $Z(0) \in B^{\sigma}_{p,\rho}$ ($\mathbb{P}$-a.s.), thus by [9], Lemma 6.1, we obtain that, $\mathbb{P}$-a.s., $[0,T] \ni t \mapsto \mathrm{e}^{\nu t A}(u(0) - Z(0)) \in \mathcal{E}$. Next, for any $\gamma \ge 1$ and $\varepsilon > 0$, since

$$\mathbb{E}\bigg(\int_0^T \big\|b\big(u_i(t)\big)\big\|^{\gamma}_{H^{-1-\varepsilon}}\,\mathrm{d}t\bigg) = \int_0^T \mathbb{E}\big(\big\|b\big(u_i(t)\big)\big\|^{\gamma}_{H^{-1-\varepsilon}}\big)\,\mathrm{d}t,$$

using estimates on the operator $b(\cdot)$ and the regularity provided by the Gaussian marginal of $u_i(\cdot)$, we can prove $b(u_i(\cdot)) \in L^\gamma(0,T;H^{-1-\varepsilon})$ ($\mathbb{P}$-a.s.); see the arguments on the top of page 197 in [9] for details. Therefore, [9], Lemma 6.2, gives us that $\int_0^t \mathrm{e}^{\nu(t-s)A} b(u_i(s))\,\mathrm{d}s \in \mathcal{E}$. Combining these discussions with the equations (2.28), we deduce that $u_i - Z \in \mathcal{E}$ ($\mathbb{P}$-a.s.) for $i = 1, 2$. By [9], Theorem 5.2, page 196 (see, in particular, the arguments on page 200 after the proof), we obtain $u_1(t) = u_2(t)$ for all $t \in [0, T]$ $\mathbb{P}$-a.s. Thus the pathwise uniqueness holds for stationary solutions to (2.27). $\square$

REMARK 2.12. Recently, R. Zhu and X. Zhu proved in [25], Section 4, the strong Feller property for the semigroup associated to the equation (2.27). Combining this result with the existence of a unique solution of (2.27) for a.e. initial condition in the Besov space $B^\sigma_{p,\rho}$ (see [9], Theorem 5.2), one can show that this equation actually admits a unique solution for all initial data in $B^\sigma_{p,\rho}$. In the proof of Proposition 2.11, we are dealing only with the pathwise uniqueness of stationary solutions of (2.27), for which we do not need the latter stronger existence result.

Recall that ω^N_t is the stationary solution of (2.3), and $\{Q^N\}_{N\ge1}$ are the laws of $(\omega^N_t)_{0\le t\le T}$. Now we can prove the main result of this paper.

THEOREM 2.13. *The whole sequence $\{Q^N\}_{N\ge1}$ converges weakly to the law of the solution to* (2.26).

PROOF. Proposition 2.11 implies that the stationary solutions to (2.26) are unique in law, thus we deduce the assertion from the tightness of the family $\{Q^N\}_{N\ge1}$. $\square$

REMARK 2.14. Denote by $\mathcal{L}_{NS}$ the Kolmogorov operator associated to (2.27). Under an explicit condition on the lower bound of ν, it is shown in [21], Corollary 1, page 572, that the operator $(\mathcal{L}_{NS}, \mathcal{F}\mathcal{C}^2_b)$ is L^1-unique, which implies that its closure $(\overline{\mathcal{L}_{NS}}, D(\overline{\mathcal{L}_{NS}}))$ generates a C_0-semigroup of contractions $\{P_t\}_{t\ge0}$ in $L^1(\mu)$ and μ is invariant for P_t. According to [24], Remark 1.2, the martingale problem associated to $(\mathcal{L}_{NS}, \mathcal{F}\mathcal{C}^2_b)$ has a unique solution.

For other weaker uniqueness results on $(\mathcal{L}_{NS}, \mathcal{F}\mathcal{C}^2_b)$ see, for example, [1, 3, 23]. A similar L^1-uniqueness result was proved in [24], Theroem 1.1, with less precise estimate on the lower bound of ν.

3. The Kolmogorov equation corresponding to (2.5)**.** The purpose of this section is to solve the Kolmogorov equation associated to the vorticity form of the Navier–Stokes equation (2.5) driven by the space-time white noise. To simplify notation, we write H^{-1-} instead of $H^{-1-}(\mathbb{T}^2)$. The main result is the following.

THEOREM 3.1. *Let $\rho_0 \in L^2(H^{-1-},\mu)$. Then there exists a measurable function $\rho \in L^\infty(0,T;L^2(H^{-1-},\mu))$ which solves*

$$\partial_t \rho_t = \mathcal{L}^*_\infty \rho_t, \qquad \rho|_{t=0} = \rho_0. \tag{3.1}$$

More precisely, for any cylindrical function $F = f \circ \Pi_\Lambda$ and $\alpha \in C^1([0,T],\mathbb{R})$ satisfying $\alpha(T) = 0$, one has

$$\begin{aligned} 0 = \alpha(0)\int_{H^{-1-}} F\rho_0\,\mathrm{d}\mu &+ \int_0^T\int_{H^{-1-}} \rho_t\big(\alpha'(t)F - \alpha(t)\langle u(\omega)\cdot\nabla\omega, DF\rangle\big)\,\mathrm{d}\mu\,\mathrm{d}t \\ &+ 4\nu\pi^2\sum_{l\in\Lambda}|l|^2\int_0^T\int_{H^{-1-}} \alpha(t)\rho_t\big[f_{l,l}(\omega) - f_l(\omega)\langle\omega, e_l\rangle\big]\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned} \tag{3.2}$$

REMARK 3.2. Unlike [15], Theorem 1.1, we do not have result on $\langle\sigma_k\cdot\nabla\omega, D\rho_t\rangle (k\in\mathbb{Z}_0^2)$; see Remark 3.6 below for details.

We can prove Theorem 3.1 by following the line of arguments in [15]. Due to a technical problem which will become clear in the proof of Theorem 3.1, as in [15], Section 4, we consider an equation slightly different from (2.3):

$$\mathrm{d}\omega_t^N + u_t^N\cdot\nabla\omega_t^N = 2\sqrt{2\nu}\tilde\varepsilon_N\sum_{k\in\Gamma_N}\sigma_k\cdot\nabla\omega_t^N\circ\mathrm{d}W_t^k, \tag{3.3}$$

where $\Gamma_N=\{k\in\mathbb{Z}_0^2:|k|\le N/3\}$ and

$$\tilde\varepsilon_N=\bigg(\sum_{k\in\Gamma_N}\frac{1}{|k|^2}\bigg)^{-1/2}. \tag{3.4}$$

The generator of (3.3) is given as below: for $F\in\mathcal{FC}_b^2$,

$$\begin{aligned}\mathcal{L}_N F(\omega) = {} & 4\nu\tilde\varepsilon_N^2\sum_{k\in\Gamma_N}\big\langle\sigma_k\cdot\nabla\omega, D\langle\sigma_k\cdot\nabla\omega, DF\rangle\big\rangle \\ & -\big\langle u(\omega)\cdot\nabla\omega, DF\big\rangle.\end{aligned} \tag{3.5}$$

For any cylindrical function $F\in\mathcal{FC}_b^2$, we denote by

$$\mathcal{L}_N^0F(\omega):=\frac{1}{2}\sum_{k\in\Gamma_N}\big\langle\sigma_k\cdot\nabla\omega, D\langle\sigma_k\cdot\nabla\omega, DF\rangle\big\rangle, \tag{3.6}$$

then

$$\mathcal{L}_NF(\omega)=8\nu\tilde\varepsilon_N^2\mathcal{L}_N^0F(\omega)-\big\langle u(\omega)\cdot\nabla\omega, DF\big\rangle.$$

Now we need the decomposition formula proved in Proposition A.3. Replacing Λ_N there by Γ_N, we obtain the following.

PROPOSITION 3.3. *For any $F\in\mathcal{FC}_b^2$, it holds that*

$$\lim_{N\to\infty}\mathcal{L}_NF=\mathcal{L}_\infty F\quad \text{in } L^2\big(H^{-1-},\mu\big).$$

With this result in hand, we will define the Galerkin approximation of the operator $\mathcal{L}_\infty$ for which we need some notation (see [15] for details). Let $H_N=\operatorname{span}\{e_k:k\in\Lambda_N\}$ and $\Pi_N:H^{-1-}(\mathbb{T}^2)\to H_N$ be the projection operator, which is an orthogonal projection when restricted to $L^2(\mathbb{T}^2)$. We project the drift term $u(\omega)\cdot\nabla\omega$ in (3.5) as follows:

$$b_N(\omega):=\Pi_N\big(u(\Pi_N\omega)\cdot\nabla(\Pi_N\omega)\big),\quad \omega\in H^{-1-}\big(\mathbb{T}^2\big),$$

where $u(\Pi_N\omega)$ is obtained from the Biot–Savart law:

$$u(\Pi_N\omega)(x)=\int_{\mathbb{T}^2}K(x-y)(\Pi_N\omega)(y)\,\mathrm{d}y.$$

We shall consider b_N as a vector field on H_N whose generic element is denoted by $\xi=\sum_{k\in\Lambda_N}\xi_ke_k$. Thus

$$b_N(\xi)=\Pi_N\big(u(\xi)\cdot\nabla\xi\big),\quad \xi\in H_N.$$

Analogously, we define the projection of the diffusion coefficient $\sigma_k\cdot\nabla\omega$ in (3.5):

$$G_N^k(\xi)=\Pi_N(\sigma_k\cdot\nabla\xi),\quad \xi\in H_N.$$

It can be shown that b_N and G_N^k are divergence-free with respect to the standard Gaussian measure μ_N on H_N. With the above preparations, we can define the Galerkin approximation of the operator $\mathcal{L}_\infty$ as

$$\tilde{\mathcal{L}}_N\phi(\xi) = 4\nu\tilde{\varepsilon}_N^2 \sum_{k\in\Gamma_N} \langle G_N^k, \nabla_N \langle G_N^k, \nabla_N\phi\rangle_{H_N}\rangle_{H_N}(\xi) - \langle b_N, \nabla_N\phi\rangle_{H_N}(\xi).$$

Consider the Kolmogorov equation on H_N:

$$\partial_t \rho_t^N = \tilde{\mathcal{L}}_N^* \rho_t^N, \qquad \rho^N|_{t=0} = \rho_0^N \in C_b^2(H_N), \tag{3.7}$$

where $\tilde{\mathcal{L}}_N^*$ is the adjoint operator of $\tilde{\mathcal{L}}_N$ with respect to μ_N. We slightly abuse the notation and denote by $\rho_t^N(\omega) = \rho_t^N(\Pi_N\omega)$, $N \geq 1$. It is easy to show that, for all $t \in [0, T]$,

$$\begin{aligned} &\|\rho_t^N\|_{L^2(\mu)}^2 + 8\nu\tilde{\varepsilon}_N^2 \sum_{k\in\Gamma_N} \int_0^t \int_{H^{-1-}} \langle \sigma_k \cdot \nabla(\Pi_N\omega), D\rho_s^N\rangle_{L^2(\mathbb{T}^2)}^2 \,\mathrm{d}\mu\,\mathrm{d}s \\ &\quad = \|\rho_0^N\|_{L^2(\mu)}^2. \end{aligned} \tag{3.8}$$

For $k \notin \Gamma_N$, we set $\langle \sigma_k \cdot \nabla(\Pi_N\omega), D\rho_s^N\rangle_{L^2(\mathbb{T}^2)} \equiv 0$. Here are two simple observations.

PROPOSITION 3.4.

(1) *The sequence* $\{\rho^N\}_{N\in\mathbb{N}}$ *of functions is bounded in* $L^\infty(0, T; L^2(H^{-1-}, \mu))$;
(2) *the family*

$$\{\tilde{\varepsilon}_N \langle \sigma_k \cdot \nabla(\Pi_N\omega), D\rho_t^N\rangle_{L^2(\mathbb{T}^2)} : (k, t, \omega) \in \mathbb{Z}_0^2 \times [0, T] \times H^{-1-}\}_{N\in\mathbb{N}}$$

is bounded in the Hilbert space $L^2(\mathbb{Z}_0^2 \times [0, T] \times H^{-1-}, \# \otimes \mathrm{d}t \otimes \mu)$, *where # is the counting measure on* $\mathbb{Z}_0^2$.

As a consequence, we obtain the following.

COROLLARY 3.5. *Assume* $\rho_0 \in L^2(H^{-1-}, \mu)$. *Then the family* $\{\rho^N\}_{N\in\mathbb{N}}$ *has a subsequence which converges weakly-∗ to some measurable function* $\rho \in L^\infty(0, T; L^2(H^{-1-}, \mu))$.

REMARK 3.6. Unlike [15], Theorem 3.2, we are unable to show that $\langle \sigma_k \cdot \nabla\omega, D\rho_t\rangle$ exists in the distributional sense, and the gradient estimate below holds:

$$\sum_{k\in\mathbb{Z}_0^2} \int_0^T \int_{H^{-1-}} \langle \sigma_k \cdot \nabla\omega, D\rho_t\rangle^2 \,\mathrm{d}\mu\,\mathrm{d}t \leq \|\rho_0\|_{L^2(\mu)}^2.$$

We repeat the proof of [15], Theorem 3.2, to see the difference. Recall that, by convention,

$$\langle \sigma_k \cdot \nabla(\Pi_N\omega), D\rho_s^N\rangle_{L^2(\mathbb{T}^2)} \equiv 0, \quad k \notin \Gamma_N.$$

By Proposition 3.4, there exists a subsequence $\{N_i\}_{i\in\mathbb{N}}$ such that:

(a) ρ^{N_i} converges weakly-∗ to some ρ in $L^\infty(0, T; L^2(H^{-1-}, \mu))$;
(b) $\tilde{\varepsilon}_{N_i} \langle \sigma_k \cdot \nabla(\Pi_{N_i}\omega), D\rho_t^{N_i}\rangle_{L^2(\mathbb{T}^2)}$ converges weakly to some $\varphi \in L^2(\mathbb{Z}_0^2 \times [0, T] \times H^{-1-}, \# \otimes \mathrm{d}t \otimes \mu)$.

Let $\alpha \in C([0,T],\mathbb{R})$ and $\beta \in L^2(\mathbb{Z}_0^2 \times H^{-1-}, \# \otimes \mu)$ such that $\beta_k \in \mathcal{FC}_b^2$ for all $k \in \mathbb{Z}_0^2$. By the assertion (b),

$$\lim_{i\to\infty} \sum_{k\in\mathbb{Z}_0^2} \int_0^T \int_{H^{-1-}} \tilde{\varepsilon}_{N_i} \langle \sigma_k \cdot \nabla(\Pi_{N_i}\omega), D\rho_t^{N_i}\rangle_{L^2(\mathbb{T}^2)} \alpha(t)\beta_k \,\mathrm{d}\mu\,\mathrm{d}t$$
$$= \sum_{k\in\mathbb{Z}_0^2} \int_0^T \int_{H^{-1-}} \varphi_k(t)\alpha(t)\beta_k(t)\,\mathrm{d}\mu\,\mathrm{d}t.$$

Fix some $k \in \mathbb{Z}_0^2$, we assume that $\beta_j \equiv 0$ for all $j \neq k$ and $\beta_k = \beta_k \circ \Pi_\Lambda$ for some $\Lambda \Subset \mathbb{Z}_0^2$. Then the above limit reduces to

$$\begin{aligned} &\lim_{i\to\infty} \tilde{\varepsilon}_{N_i} \int_0^T \int_{H^{-1-}} \langle \sigma_k \cdot \nabla(\Pi_{N_i}\omega), D\rho_t^{N_i}\rangle_{L^2(\mathbb{T}^2)} \alpha(t)\beta_k \,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad = \int_0^T \int_{H^{-1-}} \varphi_k(t)\alpha(t)\beta_k\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned} \tag{3.9}$$

For all big N_i such that $\Lambda \subset \Lambda_{N_i}$, we have

$$\begin{aligned} &\int_0^T \int_{H^{-1-}} \langle \sigma_k \cdot \nabla(\Pi_{N_i}\omega), D\rho_t^{N_i}\rangle_{L^2(\mathbb{T}^2)} \alpha(t)\beta_k(\omega)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad = \int_0^T \int_{H_{N_i}} \langle G_{N_i}^k, \nabla_{N_i}\rho_t^{N_i}\rangle_{H_{N_i}}(\xi)\alpha(t)\beta_k(\xi)\,\mathrm{d}\mu_{N_i}\,\mathrm{d}t \\ &\quad = -\int_0^T \int_{H_{N_i}} \rho_t^{N_i}(\xi)\alpha(t)\langle G_{N_i}^k, \nabla_{N_i}\beta_k\rangle_{H_{N_i}}(\xi)\,\mathrm{d}\mu_{N_i}\,\mathrm{d}t \\ &\quad = -\int_0^T \int_{H^{-1-}} \rho_t^{N_i}(\omega)\alpha(t)\langle \sigma_k \cdot \nabla(\Pi_{N_i}\omega), D\beta_k\rangle_{L^2(\mathbb{T}^2)}\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned}$$

Since $k \in \mathbb{Z}_0^2$ is fixed, if N_i is big enough, we have

$$\begin{aligned} \langle \sigma_k \cdot \nabla(\Pi_{N_i}\omega), D\beta_k\rangle_{L^2(\mathbb{T}^2)} &= -\langle \Pi_{N_i}\omega, \sigma_k \cdot \nabla(D\beta_k)\rangle \\ &= -\langle \omega, \sigma_k \cdot \nabla(D\beta_k)\rangle = \langle \sigma_k \cdot \nabla\omega, D\beta_k\rangle. \end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned} &\tilde{\varepsilon}_{N_i} \int_0^T \int_{H^{-1-}} \langle \sigma_k \cdot \nabla(\Pi_{N_i}\omega), D\rho_t^{N_i}\rangle_{L^2(\mathbb{T}^2)} \alpha(t)\beta_k\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad = -\tilde{\varepsilon}_{N_i} \int_0^T \int_{H^{-1-}} \rho_t^{N_i}\alpha(t)\langle \sigma_k \cdot \nabla\omega, D\beta_k\rangle\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad \to -0 \cdot \int_0^T \int_{H^{-1-}} \rho_t\alpha(t)\langle \sigma_k \cdot \nabla\omega, D\beta_k\rangle\,\mathrm{d}\mu\,\mathrm{d}t = 0, \end{aligned}$$

where the second step is due to (a). Combining this limit with (3.9) yields

$$\int_0^T \int_{H^{-1-}} \varphi_k(t)\alpha(t)\beta_k\,\mathrm{d}\mu\,\mathrm{d}t = 0.$$

By the arbitrariness of $\alpha \in C([0,T])$ and $\beta_k \in \mathcal{FC}_b^2$, we see that

$$\varphi_k(t) = 0 \quad \text{for all } k \in \mathbb{Z}_0^2.$$

Now we are ready to present the following.

PROOF OF THEOREM 3.1. Recall that μ_N is the standard Gaussian measure on H_N. Let $F \in \mathcal{FC}_b^2$ and $\alpha \in C^1([0,T],\mathbb{R})$ satisfying $\alpha(T)=0$. For all N large enough, we can regard F as a smooth function on H_N. Multiplying both sides of (3.7) by $\alpha(t)F$ and integrating by parts with respect to μ_N, we obtain

$$0=\alpha(0)\int_{H_N} F\rho_0^N \,\mathrm{d}\mu_N + \int_0^T\int_{H_N}\rho_s^N\big[\alpha'(s)F+\alpha(s)\tilde{\mathcal{L}}_N F\big]\,\mathrm{d}\mu_N\,\mathrm{d}s.$$

We transform the integrals to those on $H^{-1-}(\mathbb{T}^2)$ and obtain

$$\begin{aligned}0&=\alpha(0)\int_{H^{-1-}} F\rho_0^N\,\mathrm{d}\mu\\&\quad+\int_0^T\int_{H^{-1-}}\rho_s^N\big[\alpha'(s)F-\alpha(s)\langle u(\Pi_N\omega)\cdot\nabla(\Pi_N\omega),DF\rangle\big]\,\mathrm{d}\mu\,\mathrm{d}s\\&\quad+4\nu\tilde{\varepsilon}_N^2\sum_{k\in\Gamma_N}\int_0^T\int_{H^{-1-}}\rho_s^N\alpha(s)\langle\sigma_k\cdot\nabla(\Pi_N\omega),D\langle\sigma_k\cdot\nabla(\Pi_N\omega),DF\rangle\rangle\,\mathrm{d}\mu\,\mathrm{d}s.\end{aligned}\tag{3.11}$$

Assume F has the form $f\circ\Pi_\Lambda$; in this case we say that F is measurable with respect to $H_\Lambda=\mathrm{span}\{e_k:k\in\Lambda\}$, or H_Λ-measurable. Of course, F is also $H_{\Lambda'}$-measurable for any $\Lambda'\supset\Lambda$. When N is big enough, we have $\Lambda\subset\Gamma_N=\Lambda_{N/3}$. For all $k\in\Gamma_N$, analogous to (3.10),

$$\begin{aligned}\langle\sigma_k\cdot\nabla(\Pi_N\omega),DF\rangle&=-\langle\Pi_N\omega,\sigma_k\cdot\nabla(DF)\rangle\\&=-\langle\omega,\sigma_k\cdot\nabla(DF)\rangle\\&=\langle\sigma_k\cdot\nabla\omega,DF\rangle.\end{aligned}$$

We see that $\langle\sigma_k\cdot\nabla\omega,DF\rangle$ is $H_{\Lambda_{2N/3}}$-measurable. In the same way, we have

$$\begin{aligned}\langle\sigma_k\cdot\nabla(\Pi_N\omega),D\langle\sigma_k\cdot\nabla(\Pi_N\omega),DF\rangle\rangle&=\langle\sigma_k\cdot\nabla(\Pi_N\omega),D\langle\sigma_k\cdot\nabla\omega,DF\rangle\rangle\\&=\langle\sigma_k\cdot\nabla\omega,D\langle\sigma_k\cdot\nabla\omega,DF\rangle\rangle,\end{aligned}$$

which is H_{Λ_N}-measurable. Therefore, by (3.6),

$$\frac12\sum_{k\in\Gamma_N}\langle\sigma_k\cdot\nabla(\Pi_N\omega),D\langle\sigma_k\cdot\nabla(\Pi_N\omega),DF\rangle\rangle=\mathcal{L}_N^0F(\omega),$$

and (3.11) becomes

$$\begin{aligned}0&=\alpha(0)\int_{H^{-1-}}F\rho_0^N\,\mathrm{d}\mu+\int_0^T\int_{H^{-1-}}\rho_s^N\big[\alpha'(s)F-\alpha(s)\langle u(\Pi_N\omega)\cdot\nabla(\Pi_N\omega),DF\rangle\big]\,\mathrm{d}\mu\,\mathrm{d}s\\&\quad+8\nu\tilde{\varepsilon}_N^2\sum_{k\in\Gamma_N}\int_0^T\int_{H^{-1-}}\rho_s^N\alpha(s)\mathcal{L}_N^0F(\omega)\,\mathrm{d}\mu\,\mathrm{d}s.\end{aligned}$$

By Proposition A.3, changing N into N_i and letting $i\to\infty$, we arrive at

$$\begin{aligned}0&=\alpha(0)\int_{H^{-1-}}F\rho_0\,\mathrm{d}\mu+\int_0^T\int_{H^{-1-}}\rho_s\big[\alpha'(s)F-\alpha(s)\langle u(\omega)\cdot\omega,DF\rangle\big]\,\mathrm{d}\mu\,\mathrm{d}s\\&\quad+4\nu\pi^2\sum_{l\in\Lambda}|l|^2\int_0^T\int_{H^{-1-}}\alpha(s)\rho_s\big[f_{l,l}(\omega)-f_l(\omega)\langle\omega,e_l\rangle\big]\,\mathrm{d}\mu\,\mathrm{d}s.\end{aligned}$$

The proof is complete. □

APPENDIX

A.1. Decomposition of the diffusion part (A.2). For the reader's convenience, we recall some useful results which were proved in [15]. First, recall that $C_{k,l}$ is defined in (2.15) and $\Lambda_N = \{k \in \mathbb{Z}_0^2 : |k| \le N\}$. The following identity is taken from [15], Lemma 3.4.

LEMMA A.1. *It holds that*

$$\sum_{k\in\Lambda_N} C_{k,l}^2 = \frac{1}{2}\varepsilon_N^{-2}|l|^2 \quad \text{with } \varepsilon_N = \bigg(\sum_{k\in\Lambda_N}\frac{1}{|k|^2}\bigg)^{-1/2}. \tag{A.1}$$

PROOF. Denoting by $D_{k,l} = \frac{k\cdot l}{|k|^2}$, then

$$C_{k,l}^2 + D_{k,l}^2 = \frac{(k^\perp\cdot l)^2}{|k|^4} + \frac{(k\cdot l)^2}{|k|^4} = \frac{1}{|k|^2}\bigg[\bigg(\frac{k^\perp}{|k|}\cdot l\bigg)^2 + \bigg(\frac{k}{|k|}\cdot l\bigg)^2\bigg] = \frac{|l|^2}{|k|^2}.$$

The transformation $k \to k^\perp$ is 1–1 on the set $\Lambda_N = \{k \in \mathbb{Z}_0^2 : |k| \le N\}$, and preserves the norm $|\cdot|$. As a result,

$$\sum_{k\in\Lambda_N} C_{k,l}^2 = \sum_{k\in\Lambda_N}\frac{(k^\perp\cdot l)^2}{|k|^4} = \sum_{k\in\Lambda_N}\frac{((k^\perp)^\perp\cdot l)^2}{|k^\perp|^4} = \sum_{k\in\Lambda_N}\frac{(k\cdot l)^2}{|k|^4} = \sum_{k\in\Lambda_N} D_{k,l}^2.$$

Combining the above two equalities, we obtain

$$\sum_{k\in\Lambda_N} C_{k,l}^2 = \frac{1}{2}\sum_{k\in\Lambda_N}\big(C_{k,l}^2 + D_{k,l}^2\big) = \frac{1}{2}|l|^2\sum_{k\in\Lambda_N}\frac{1}{|k|^2} = \frac{1}{2}\varepsilon_N^{-2}|l|^2.$$

□

Next, we recall a decomposition formula of the operator

$$\mathcal{L}_N^0 F(\omega) = \frac{1}{2}\sum_{k\in\Lambda_N}\big\langle \sigma_k\cdot\nabla\omega, D\langle\sigma_k\cdot\nabla\omega, DF\rangle\big\rangle, \quad F \in \mathcal{FC}_b^2, \tag{A.2}$$

which was proved in [15], Proposition 4.2. To this end, we need the following simple result.

LEMMA A.2. *Assume that $F = f\circ\Pi_\Lambda$ for some finite set $\Lambda \subset \mathbb{Z}_0^2$. We have*

$$\begin{aligned}\mathcal{L}_N^0 F(\omega) = {}& \pi^2\sum_{k\in\Lambda_N}\sum_{l,m\in\Lambda} C_{k,l}C_{k,m} f_{l,m}(\omega)\langle\omega, e_k e_{-l}\rangle\langle\omega, e_k e_{-m}\rangle \\ & - \pi^2\sum_{k\in\Lambda_N}\sum_{l\in\Lambda} C_{k,l}^2 f_l(\omega)\langle\omega, e_k^2 e_l\rangle.\end{aligned} \tag{A.3}$$

PROOF. Note that $DF(\omega) = \sum_{l\in\Lambda}(\partial_l f)(\Pi_\Lambda\omega)e_l = \sum_{l\in\Lambda} f_l(\omega)e_l$; therefore,

$$\begin{aligned}\langle\sigma_k\cdot\nabla\omega, DF\rangle &= \sum_{l\in\Lambda} f_l(\omega)\langle\sigma_k\cdot\nabla\omega, e_l\rangle \\ &= -\sum_{l\in\Lambda} f_l(\omega)\langle\omega, \sigma_k\cdot\nabla e_l\rangle \\ &= -\sqrt{2}\pi\sum_{l\in\Lambda} C_{k,l} f_l(\omega)\langle\omega, e_k e_{-l}\rangle.\end{aligned}$$

Furthermore,

$$\begin{aligned}D\langle\sigma_k\cdot\nabla\omega, DF\rangle &= -\sqrt{2}\pi\sum_{l\in\Lambda}C_{k,l}\big(\langle\omega, e_k e_{-l}\rangle D\big[f_l(\omega)\big] + f_l(\omega)e_k e_{-l}\big)\\ &= -\sqrt{2}\pi\sum_{l,m\in\Lambda}C_{k,l}\langle\omega, e_k e_{-l}\rangle f_{l,m}(\omega)e_m\\ &\quad -\sqrt{2}\pi\sum_{l\in\Lambda}C_{k,l}f_l(\omega)e_k e_{-l}.\end{aligned}$$

As a result,

$$\begin{aligned}&\big\langle\sigma_k\cdot\nabla\omega, D\langle\sigma_k\cdot\nabla\omega, DF\rangle\big\rangle\\ &\quad= -\sqrt{2}\pi\sum_{l,m\in\Lambda}C_{k,l}f_{l,m}(\omega)\langle\omega, e_k e_{-l}\rangle\langle\sigma_k\cdot\nabla\omega, e_m\rangle\\ &\qquad -\sqrt{2}\pi\sum_{l\in\Lambda}C_{k,l}f_l(\omega)\langle\sigma_k\cdot\nabla\omega, e_k e_{-l}\rangle.\end{aligned}\tag{A.4}$$

We have $\langle\sigma_k\cdot\nabla\omega, e_m\rangle = -\langle\omega, \sigma_k\cdot\nabla e_m\rangle = -\sqrt{2}\pi C_{k,m}\langle\omega, e_k e_{-m}\rangle$ and

$$\langle\sigma_k\cdot\nabla\omega, e_k e_{-l}\rangle = -\big\langle\omega, \sigma_k\cdot\nabla(e_k e_{-l})\big\rangle = \sqrt{2}\pi C_{k,l}\big\langle\omega, e_k^2 e_l\big\rangle.$$

Substituting these facts into (A.4) and summing over k yield the desired result. □

Now we can rewrite $\mathcal{L}_N^0 F(\omega)$ as the sum of two parts, in which one part is convergent while the other is in general divergent.

PROPOSITION A.3. *It holds that*

$$\begin{aligned}\mathcal{L}_N^0 F(\omega) &= \pi^2\sum_{l,m\in\Lambda}f_{l,m}(\omega)\sum_{k\in\Lambda_N}C_{k,l}C_{k,m}\big(\langle\omega, e_k e_{-l}\rangle\langle\omega, e_k e_{-m}\rangle - \delta_{l,m}\big)\\ &\quad + \frac{1}{2}\pi^2\varepsilon_N^{-2}\sum_{l\in\Lambda}|l|^2\big[f_{l,l}(\omega) - f_l(\omega)\langle\omega, e_l\rangle\big].\end{aligned}\tag{A.5}$$

Moreover, for any $l, m\in\mathbb{Z}_0^2$, the quantity

$$R_{l,m}(N) = \sum_{k\in\Lambda_N}C_{k,l}C_{k,m}\big(\langle\omega, e_k e_{-l}\rangle\langle\omega, e_k e_{-m}\rangle - \delta_{l,m}\big)\tag{A.6}$$

is a Cauchy sequence in $L^p(H^{-1-}, \mu)$ for any $p>1$.

PROOF. The proof of the second assertion is quite long and can be found in the Appendix of [15]. Here, we only prove the equality (A.5). We have, by Lemma A.1,

$$\begin{aligned}&\sum_{k\in\Lambda_N}\sum_{l,m\in\Lambda}C_{k,l}C_{k,m}f_{l,m}(\omega)\langle\omega, e_k e_{-l}\rangle\langle\omega, e_k e_{-m}\rangle\\ &\quad= \sum_{l,m\in\Lambda}f_{l,m}(\omega)\sum_{k\in\Lambda_N}C_{k,l}C_{k,m}\big(\langle\omega, e_k e_{-l}\rangle\langle\omega, e_k e_{-m}\rangle - \delta_{l,m}\big)\\ &\qquad + \frac{1}{2}\varepsilon_N^{-2}\sum_{l\in\Lambda}|l|^2 f_{l,l}(\omega).\end{aligned}\tag{A.7}$$

Next, note that $C_{-k,l} = -C_{k,l}$ and $e_k^2 + e_{-k}^2 \equiv 2$ for all $k \in \mathbb{Z}_0^2$, we have

$$\sum_{k\in\Lambda_N} C_{k,l}^2 \langle \omega, e_k^2 e_l\rangle = \sum_{k\in\Lambda_N, k\in\mathbb{Z}_+^2} \big[C_{k,l}^2\langle \omega, e_k^2 e_l\rangle + C_{-k,l}^2\langle \omega, e_{-k}^2 e_l\rangle\big]$$
$$= \sum_{k\in\Lambda_N, k\in\mathbb{Z}_+^2} 2C_{k,l}^2\langle \omega, e_l\rangle = \frac{1}{2}\varepsilon_N^{-2}|l|^2\langle \omega, e_l\rangle,$$

where the last step is due to Lemma A.1. Therefore,

$$\sum_{k\in\Lambda_N}\sum_{l\in\Lambda} C_{k,l}^2 f_l(\omega)\langle \omega, e_k^2 e_l\rangle = \frac{1}{2}\varepsilon_N^{-2}\sum_{l\in\Lambda}|l|^2 f_l(\omega)\langle \omega, e_l\rangle.$$

Combining this equality with (A.3) and (A.7) leads to the identity (A.5). □

A.2. Coincidence of nonlinear parts. Our purpose in this part is to show that the nonlinear term in the vorticity form of the Euler equation defined in [13], Theorem 8, agrees with that defined by Galerkin approximation; therefore, we can freely use any of them. Although we mainly work in the real setting, it is sometimes convenient to use the canonical complex orthonormal basis $\{\tilde e_k\}_{k\in\mathbb{Z}^2}$ of $L^2(\mathbb{T}^2,\mathbb{C})$. Note that $\{\tilde e_k\otimes\tilde e_l\}_{k,l\in\mathbb{Z}^2}$ is an orthonormal basis of $L^2((\mathbb{T}^2)^2,\mathbb{C})$.

LEMMA A.4. *Assume $f\in C^\infty((\mathbb{T}^2)^2,\mathbb{R})$ is a symmetric function and $\int_{\mathbb{T}^2} f(x,x)\,\mathrm{d}x = 0$. Then*

$$\langle \omega\otimes\omega, f\rangle = \sum_{k,l\in\mathbb{Z}^2} f_{k,l}\langle \omega, \tilde e_k\rangle\langle \omega, \tilde e_l\rangle \quad \textit{holds in } L^2(H^{-1-},\mu),$$

where

$$f_{k,l} = \langle f, \tilde e_k\otimes\tilde e_l\rangle = \int_{(\mathbb{T}^2)^2} f(x,y)\tilde e_k(x)\tilde e_l(y)\,\mathrm{d}x\,\mathrm{d}y.$$

PROOF. Denote by

$$\hat\Lambda_N = \{k\in\mathbb{Z}^2 : |k|\le N\} = \Lambda_N\cup\{0\}. \tag{A.8}$$

Since $f\in C^\infty((\mathbb{T}^2)^2)$, the partial sum of the Fourier series

$$f_N(x,y) := \sum_{k,l\in\hat\Lambda_N} f_{k,l}\tilde e_k(x)\tilde e_l(y)$$

converges to f, uniformly on $(\mathbb{T}^2)^2$ and in $L^2((\mathbb{T}^2)^2)$. In particular,

$$\lim_{N\to\infty}\int_{\mathbb{T}^2} f_N(x,x)\,\mathrm{d}x = \int_{\mathbb{T}^2} f(x,x)\,\mathrm{d}x = 0. \tag{A.9}$$

It is obvious that $f_N(x,y)$ is smooth and symmetric. By [13], Corollary 6(ii), (iii),

$$\mathbb{E}_\mu\bigg[\bigg(\langle \omega\otimes\omega, f-f_N\rangle + \int_{\mathbb{T}^2} f_N(x,x)\,\mathrm{d}x\bigg)^2\bigg] = 2\int_{(\mathbb{T}^2)^2}(f-f_N)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

As a result,

$$\begin{aligned}&\mathbb{E}_\mu\big[\langle \omega\otimes\omega, f-f_N\rangle^2\big]\\ &\quad\le 4\int_{(\mathbb{T}^2)^2}(f-f_N)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y + 2\bigg[\int_{\mathbb{T}^2} f_N(x,x)\,\mathrm{d}x\bigg]^2.\end{aligned} \tag{A.10}$$

Next, note that

$$\langle \omega \otimes \omega, f_N \rangle = \sum_{k,l \in \hat{\Lambda}_N} f_{k,l} \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle.$$

Therefore, by (A.10),

$$\begin{aligned} &\mathbb{E}_\mu \bigg[\bigg(\langle \omega \otimes \omega, f \rangle - \sum_{k,l \in \hat{\Lambda}_N} f_{k,l} \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle \bigg)^2 \bigg] \\ &\quad \le 4 \int_{(\mathbb{T}^2)^2} (f - f_N)^2(x,y)\, \mathrm{d}x\, \mathrm{d}y + 2 \bigg[\int_{\mathbb{T}^2} f_N(x,x)\, \mathrm{d}x \bigg]^2. \end{aligned}$$

Thanks to (A.9), the desired result follows by letting $N \to \infty$. □

We need the following simple equality.

LEMMA A.5. *Let* $\{a_{k,l}\}_{k,l \in \hat{\Lambda}_N} \subset \mathbb{C}$ *be satisfying* $a_{k,l} = a_{l,k}$, $\overline{a_{k,l}} = a_{-k,-l}$. *Then*

$$\mathbb{E}_\mu \bigg[\bigg| \sum_{k,l \in \hat{\Lambda}_N} a_{k,l} \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle - \sum_{k \in \hat{\Lambda}_N} a_{k,-k} \bigg|^2 \bigg] = 2 \sum_{k,l \in \hat{\Lambda}_N} |a_{k,l}|^2. \tag{A.11}$$

PROOF. It is clear that $\sum_{k,l \in \hat{\Lambda}_N} a_{k,l} \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle$ is real and

$$\sum_{k \in \hat{\Lambda}_N} a_{k,-k} = \mathbb{E}_\mu \bigg(\sum_{k,l \in \hat{\Lambda}_N} a_{k,l} \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle \bigg).$$

Following the arguments of [15], Lemma 5.1, we can prove the desired equality. □

Recall the expression of H_ϕ for $\phi \in C^\infty(\mathbb{T}^2)$ in Remark 2.1. Now we can prove the intermediate result below.

PROPOSITION A.6. *For any* $j \in \mathbb{Z}_0^2$,

$$\langle \omega \otimes \omega, H_{e_j} \rangle = \sum_{k,l \in \mathbb{Z}^2} \langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle \quad \textit{holds in } L^2(H^{-1-}, \mu),$$

where e_j *is defined in* (2.1).

PROOF. Let $H_{e_j}^n$ be the smooth approximating functions of H_{e_j} constructed in [13], Remark 9, which satisfy the conditions in Lemma A.4. Recall the definition of $\hat{\Lambda}_N$ in (A.8). To simplify the notation, we introduce

$$\hat{\omega}_N = \hat{\Pi}_N \omega = \sum_{k \in \hat{\Lambda}_N} \langle \omega, \tilde{e}_k \rangle \tilde{e}_k, \quad \omega \in H^{-1-}.$$

Then

$$\langle \hat{\omega}_N \otimes \hat{\omega}_N, H_{e_j} \rangle = \sum_{k,l \in \hat{\Lambda}_N} \langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle$$

is the partial sum of the series. We have

$$\begin{aligned}
&\mathbb{E}_\mu\big[\big(\langle\omega\otimes\omega, H_{e_j}\rangle - \langle\hat\omega_N\otimes\hat\omega_N, H_{e_j}\rangle\big)^2\big] \\
&\quad\le 3\mathbb{E}_\mu\big[\langle\omega\otimes\omega, H_{e_j}-H^n_{e_j}\rangle^2\big] \\
&\qquad + 3\mathbb{E}\big[\big(\langle\omega\otimes\omega, H^n_{e_j}\rangle - \langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}\rangle\big)^2\big] \\
&\qquad + 3\mathbb{E}_\mu\big[\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}-H_{e_j}\rangle^2\big].
\end{aligned} \tag{A.12}$$

We estimate the three terms one-by-one. By the proof of [13], Theorem 8,

$$\mathbb{E}_\mu\big[\langle\omega\otimes\omega, H_{e_j}-H^n_{e_j}\rangle^2\big] \le 2\int_{(\mathbb{T}^2)^2}\big(H_{e_j}-H^n_{e_j}\big)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{A.13}$$

Next, by Lemmas A.7 and A.8 below (see also [15], Lemma 5.3), we have $\mathbb{E}_\mu\langle\hat\omega_N\otimes\hat\omega_N, H_{e_j}\rangle = 0$. Moreover, for any fixed $n\ge 1$, Lemma A.4 implies

$$\mathbb{E}_\mu\big[\big(\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}\rangle - \langle\omega\otimes\omega, H^n_{e_j}\rangle\big)^2\big] \to 0 \quad \text{as } N\to\infty. \tag{A.14}$$

It remains to deal with the last term on the right-hand side of (A.12). As a result of (A.14),

$$\lim_{N\to\infty}\mathbb{E}_\mu\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}\rangle = \mathbb{E}_\mu\langle\omega\otimes\omega, H^n_{e_j}\rangle = \int_{\mathbb{T}^2} H^n_{e_j}(x,x)\,\mathrm{d}x = 0, \tag{A.15}$$

where the second step is due to [13], Corollary 6(ii). By (A.11),

$$\begin{aligned}
&\mathbb{E}_\mu\big[\big(\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}-H_{e_j}\rangle - \mathbb{E}_\mu\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}-H_{e_j}\rangle\big)^2\big] \\
&\quad = 2\sum_{k,l\in\hat\Lambda_N}\big|\langle H^n_{e_j}-H_{e_j}, \tilde e_k\otimes\tilde e_l\rangle\big|^2 \\
&\quad \le 2\int_{(\mathbb{T}^2)^2}\big(H^n_{e_j}-H_{e_j}\big)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}_\mu\big[\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}-H_{e_j}\rangle^2\big] \\
&\quad \le 4\int_{(\mathbb{T}^2)^2}\big(H^n_{e_j}-H_{e_j}\big)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y + 2\big[\mathbb{E}_\mu\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}-H_{e_j}\rangle\big]^2 \\
&\quad = 4\int_{(\mathbb{T}^2)^2}\big(H^n_{e_j}-H_{e_j}\big)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y + 2\big[\mathbb{E}_\mu\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}\rangle\big]^2,
\end{aligned}$$

where we used again $\mathbb{E}_\mu\langle\hat\omega_N\otimes\hat\omega_N, H_{e_j}\rangle = 0$. Thanks to (A.15),

$$\limsup_{N\to\infty}\mathbb{E}\big[\langle\hat\omega_N\otimes\hat\omega_N, H^n_{e_j}-H_{e_j}\rangle^2\big] \le 4\int_{(\mathbb{T}^2)^2}\big(H^n_{e_j}-H_{e_j}\big)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Combining the above inequality with (A.12)–(A.14), first letting $N\to\infty$ in (A.12) yield

$$\begin{aligned}
&\limsup_{N\to\infty}\mathbb{E}\big[\big(\langle\omega\otimes\omega, H_{e_j}\rangle - \langle\hat\omega_N\otimes\hat\omega_N, H_{e_j}\rangle\big)^2\big] \\
&\quad \le 18\int_{(\mathbb{T}^2)^2}\big(H^n_{e_j}-H_{e_j}\big)^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

We complete the proof by sending $n\to\infty$. □

It remains to prove the following.

LEMMA A.7. *For any* $j \in \mathbb{Z}_0^2$,

$$\langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle = 0 \quad \textit{for } k = 0 \textit{ or } l = 0.$$

PROOF. We have

$$H_{e_j}(x, y) = \pi\big(e_{-j}(x) - e_{-j}(y)\big) j \cdot K(x - y), \quad (x, y) \in \mathbb{T}^2 \times \mathbb{T}^2. \tag{A.16}$$

Without loss of generality, we assume $j \in \mathbb{Z}_+^2$ thus $-j \in \mathbb{Z}_-^2$ and

$$e_{-j}(x) = \frac{1}{\sqrt{2}\mathrm{i}}\big[\tilde{e}_{-j}(x) - \tilde{e}_j(x)\big]. \tag{A.17}$$

Recall that

$$\tilde{e}_k * K = 2\pi \mathrm{i} \delta_{k \neq 0} \frac{k^\perp}{|k|^2} \tilde{e}_k \quad \text{for all } k \in \mathbb{Z}^2. \tag{A.18}$$

Case 1: $k = l = 0$. We have

$$\begin{aligned}\int_{(\mathbb{T}^2)^2} H_{e_j}(x, y)\,\mathrm{d}x\,\mathrm{d}y &= \pi j \cdot \int_{(\mathbb{T}^2)^2} \big(e_{-j}(x) - e_{-j}(y)\big) K(x - y)\,\mathrm{d}x\,\mathrm{d}y \\ &= -2\pi j \cdot \int_{\mathbb{T}^2} (e_{-j} * K)(x)\,\mathrm{d}x.\end{aligned}$$

Using (A.17) and (A.18), we obtain

$$\begin{aligned}&\int_{(\mathbb{T}^2)^2} H_{e_j}(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = -2\pi j \cdot \int_{\mathbb{T}^2} \frac{1}{\sqrt{2}\mathrm{i}} \bigg(2\pi \mathrm{i} \frac{(-j)^\perp}{|j|^2} \tilde{e}_{-j}(x) - 2\pi \mathrm{i} \frac{j^\perp}{|j|^2} \tilde{e}_j(x)\bigg)\,\mathrm{d}x = 0.\end{aligned}$$

Case 2: $k = 0$ and $l \neq 0$. Then

$$\begin{aligned}\langle H_{e_j}, \tilde{e}_0 \otimes \tilde{e}_l \rangle &= \int_{(\mathbb{T}^2)^2} H_{e_j}(x, y) \tilde{e}_l(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \pi j \cdot \int_{(\mathbb{T}^2)^2} \big(e_{-j}(x) - e_{-j}(y)\big) K(x - y) \tilde{e}_l(y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We divide the right-hand side into two terms I_1 and I_2. We have, by (A.18),

$$I_1 = \pi j \cdot \int_{\mathbb{T}^2} e_{-j}(x) (K * \tilde{e}_l)(x)\,\mathrm{d}x = 2\pi^2 \mathrm{i} \frac{j \cdot l^\perp}{|l|^2} \int_{\mathbb{T}^2} e_{-j}(x) \tilde{e}_l(x)\,\mathrm{d}x.$$

According to (A.17), it is clear that if $l \neq \pm j$, then $I_1 = 0$. On the other hand, if $l = j$ or $l = -j$, we still have $I_1 = 0$.

Next, we deal with I_2. Again by (A.17),

$$\begin{aligned}I_2 &= -\frac{\pi}{\sqrt{2}\mathrm{i}} j \cdot \int_{(\mathbb{T}^2)^2} \big[\tilde{e}_{-j}(y) - \tilde{e}_j(y)\big] K(x - y) \tilde{e}_l(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= -\frac{\pi}{\sqrt{2}\mathrm{i}} j \cdot \int_{\mathbb{T}^2} \big[(K * \tilde{e}_{l-j})(x) - (K * \tilde{e}_{l+j})(x)\big]\,\mathrm{d}x.\end{aligned} \tag{A.19}$$

If $l = j$, then by (A.18),

$$I_2 = \frac{\pi}{\sqrt{2}\mathrm{i}} j \cdot \int_{\mathbb{T}^2} 2\pi \mathrm{i} \frac{(2j)^\perp}{|2j|^2} \tilde{e}_{2j}(x)\,\mathrm{d}x = 0.$$

Similarly, $I_2 = 0$ if $l = -j$. Finally, if $l \neq \pm j$, then we deduce easily from (A.18) and (A.19) that $I_2 = 0$.

Summarizing these computations, we conclude that $\langle H_{e_j}, \tilde{e}_0 \otimes \tilde{e}_l \rangle = 0$ for all $l \in \mathbb{Z}_0^2$.

Case 3: $k \neq 0$ and $l = 0$. The arguments are similar as in the second case and we omit it here. We can also deduce the result by using the symmetry property of H_{e_j}. □

We also used the following result in the proof of Proposition A.6.

LEMMA A.8. *For all* $j, k, l \in \mathbb{Z}_0^2$,

$$\langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle = \sqrt{2}\pi^2 \left(\frac{j \cdot l^\perp}{|l|^2} + \frac{j \cdot k^\perp}{|k|^2} \right) \times \begin{cases} \delta_{j,k+l} - \delta_{j,-k-l}, & j \in \mathbb{Z}_+^2; \\ \mathrm{i}(\delta_{j,k+l} + \delta_{j,-k-l}), & j \in \mathbb{Z}_-^2. \end{cases}$$

PROOF. Assume $j \in \mathbb{Z}_+^2$. By (A.16),

$$\begin{aligned} \langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle = {} & \pi j \cdot \int_{\mathbb{T}^2} e_{-j}(x) \tilde{e}_k(x) (K * \tilde{e}_l)(x) \,\mathrm{d}x \\ & + \pi j \cdot \int_{\mathbb{T}^2} e_{-j}(y) \tilde{e}_l(y) (K * \tilde{e}_k)(y) \,\mathrm{d}y. \end{aligned}$$

We denote the two terms by J_1 and J_2. By (A.18) and (A.17),

$$J_1 = 2\pi^2 \mathrm{i} \frac{j \cdot l^\perp}{|l|^2} \int_{\mathbb{T}^2} e_{-j}(x) \tilde{e}_k(x) \tilde{e}_l(x) \,\mathrm{d}x = \sqrt{2}\pi^2 \frac{j \cdot l^\perp}{|l|^2} (\delta_{j,k+l} - \delta_{j,-k-l}).$$

Similarly,

$$J_2 = \sqrt{2}\pi^2 \frac{j \cdot k^\perp}{|k|^2} (\delta_{j,k+l} - \delta_{j,-k-l}).$$

The proof is complete. □

Recall that we have defined in Section 3 the projection

$$\omega_N = \Pi_N \omega = \sum_{k \in \Lambda_N} \langle \omega, e_k \rangle e_k = \sum_{k \in \Lambda_N} \langle \omega, \tilde{e}_k \rangle \tilde{e}_k,$$

where the last step follows by a simple computation. According to (A.8), we have $\hat{\omega}_N = \omega_N + \langle \omega, 1 \rangle$. Taking into account Lemma A.7 above, we conclude that, for any $j \in \mathbb{Z}_0^2$,

$$\langle \hat{\omega}_N \otimes \hat{\omega}_N, H_{e_j} \rangle = \langle \omega_N \otimes \omega_N, H_{e_j} \rangle \quad \text{for all } N \geq 1. \tag{A.20}$$

Now we can prove the first main result of this part.

THEOREM A.9. *For any* $j \in \mathbb{Z}_0^2$,

$$\langle \omega \otimes \omega, H_{e_j} \rangle = \lim_{N \to \infty} \langle \omega_N \otimes \omega_N, H_{e_j} \rangle \quad \textit{holds in } L^2(H^{-1-}, \mu).$$

Moreover,

$$\begin{aligned} \mathbb{E}\big[\langle \omega \otimes \omega, H_{e_j} \rangle^2\big] &= 2 \int_{(\mathbb{T}^2)^2} H_{e_j}^2(x, y) \,\mathrm{d}x \,\mathrm{d}y \\ &= 2 \sum_{k,l \in \mathbb{Z}_0^2} \big| \langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle \big|^2. \end{aligned} \tag{A.21}$$

PROOF. The first assertion follows from Proposition A.6 and (A.20). Next, by Lemma A.8 above,

$$\langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_{-k} \rangle = 0 \quad \text{for all } k \in \mathbb{Z}^2.$$

Hence, Lemma A.5 and (A.20) imply

$$\begin{aligned}\mathbb{E}\big[\langle \omega_N \otimes \omega_N, H_{e_j} \rangle^2\big] &= \mathbb{E}\bigg[\bigg| \sum_{k,l \in \Lambda_N} \langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle \bigg|^2\bigg] \\ &= 2 \sum_{k,l \in \Lambda_N} \big|\langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle\big|^2.\end{aligned}$$

Letting $N \to \infty$ yields the second result. □

In the following, we denote formally by

$$b(\omega) = u(\omega) \cdot \nabla \omega, b_N(\omega) = \Pi_N\big[u(\omega_N) \cdot \nabla \omega_N\big], \quad N \geq 1. \tag{A.22}$$

We shall prove that b is well defined as an element in $L^2(H^{-1-}, \mu; H^{-2-})$ and $b_N \to b$ w.r.t. the norm of this space as $N \to \infty$. This assertion is consistent with [9], Proposition 3.1 and [4], Proposition 3.2.

For any $j \in \mathbb{Z}_0^2$, by Theorem A.9,

$$\big\langle b(\omega), e_j \big\rangle = -\langle \omega \otimes \omega, H_{e_j} \rangle = -\sum_{k,l \in \mathbb{Z}_0^2} \langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle \langle \omega, \tilde{e}_k \rangle \langle \omega, \tilde{e}_l \rangle. \tag{A.23}$$

Lemma A.8 implies that

$$\begin{aligned}\big|\langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle\big|^2 &= 2\pi^4 \bigg(\frac{j \cdot l^\perp}{|l|^2} + \frac{j \cdot k^\perp}{|k|^2}\bigg)^2 (\delta_{j,k+l} + \delta_{j,-k-l}) \\ &= 2\pi^4 \delta_{j,\pm(k+l)} \bigg(\frac{j \cdot l^\perp}{|l|^2} + \frac{j \cdot k^\perp}{|k|^2}\bigg)^2.\end{aligned}$$

Using the fact $j = \pm(k+l)$, we obtain

$$\big|\langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle\big|^2 = 2\pi^4 \delta_{j,\pm(k+l)} (j \cdot k^\perp)^2 \bigg(\frac{1}{|l|^2} - \frac{1}{|k|^2}\bigg)^2. \tag{A.24}$$

The next estimate will play a key role in the sequel.

LEMMA A.10. *There exists $C > 0$ such that for all $|j| \geq 2$,*

$$\sum_{k,l \in \mathbb{Z}_0^2} \big|\langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle\big|^2 \leq C|j|^2 \log|j|.$$

PROOF. Thanks to (A.24), we have

$$\begin{aligned}\sum_{k,l \in \mathbb{Z}_0^2} \big|\langle H_{e_j}, \tilde{e}_k \otimes \tilde{e}_l \rangle\big|^2 &= 2\pi^4 \sum_{k \in \mathbb{Z}_0^2 \setminus \{j\}} (j \cdot k^\perp)^2 \bigg(\frac{1}{|j-k|^2} - \frac{1}{|k|^2}\bigg)^2 \\ &\quad + 2\pi^4 \sum_{k \in \mathbb{Z}_0^2 \setminus \{-j\}} (j \cdot k^\perp)^2 \bigg(\frac{1}{|j+k|^2} - \frac{1}{|k|^2}\bigg)^2\end{aligned}$$

which is easily seen to be convergent. We denote the two quantities on the r.h.s. by I_1 and I_2, respectively. Note that

$$\left(\frac{1}{|j-k|^2}-\frac{1}{|k|^2}\right)^2=\frac{(|j|^2-2j\cdot k)^2}{|j-k|^4|k|^4}\le 2\frac{|j|^4+4(j\cdot k)^2}{|j-k|^4|k|^4},$$

thus

$$\begin{aligned}I_1&\le 4\pi^4|j|^4\sum_{k\in\mathbb{Z}_0^2\setminus\{j\}}\frac{(j\cdot k^\perp)^2}{|j-k|^4|k|^4}+16\pi^4\sum_{k\in\mathbb{Z}_0^2\setminus\{j\}}\frac{(j\cdot k^\perp)^2(j\cdot k)^2}{|j-k|^4|k|^4}\\&=: I_{1,1}+I_{1,2}.\end{aligned}$$

We have

$$\begin{aligned}I_{1,1}&=4\pi^4|j|^4\sum_{k\in\mathbb{Z}_0^2\setminus\{j\}}\frac{((j-k)\cdot k^\perp)^2}{|j-k|^4|k|^4}\\&\le 4\pi^4|j|^4\sum_{k\in\mathbb{Z}_0^2\setminus\{j\}}\frac{1}{|j-k|^2|k|^2}\le C|j|^2\log|j|,\end{aligned}$$

where the last step is due to [4], Proposition A.1. Similarly,

$$\begin{aligned}I_{1,2}&=16\pi^4\sum_{k\in\mathbb{Z}_0^2\setminus\{j\}}\frac{(j\cdot(k-j)^\perp)^2(j\cdot k)^2}{|j-k|^4|k|^4}\\&\le 16\pi^4|j|^4\sum_{k\in\mathbb{Z}_0^2\setminus\{j\}}\frac{1}{|j-k|^2|k|^2}\le C|j|^2\log|j|.\end{aligned}$$

Therefore, we obtain

$$I_1\le C|j|^2\log|j|. \tag{A.25}$$

In the same way, we have $I_2\le C|j|^2\log|j|$ which, together with (A.25), implies the result. □

REMARK A.11. Recall the definition of $\mathcal{L}_\infty$. The above estimate shows that the nonlinear part in $\mathcal{L}_\infty$ is not dominated by the diffusion part. Indeed, taking $F(\omega)=\langle\omega,e_j\rangle$, $|j|\ge 2$, then by (A.23), Theorem A.9 and Lemma A.10,

$$\mathbb{E}_\mu\big[\langle b(\omega),\mathrm{DF}\rangle^2\big]=\mathbb{E}_\mu\big[\langle\omega\otimes\omega,H_{e_j}\rangle^2\big]\le C|j|^2\log|j|.$$

Note that the factor $\log|j|$ cannot be eliminated. On the other hand, regarding the diffusion part $\mathcal{L}_\infty^D$ in $\mathcal{L}_\infty$, we have

$$-\mathbb{E}_\mu\big[F\mathcal{L}_\infty^D F\big]=4\pi^2|j|^2.$$

As a result, the Lions approach does not work here to give us the uniqueness of solutions to (3.1).

Now we can prove the second main result of this part.

THEOREM A.12. *For any $\delta>0$,*

$$\lim_{N\to\infty}\mathbb{E}_\mu\big(\|b_N(\omega)-b(\omega)\|^2_{H^{-2-\delta}(\mathbb{T}^2)}\big)=0.$$

PROOF. Note that

$$\|b_N(\omega)-b(\omega)\|^2_{H^{-2-\delta}(\mathbb{T}^2)}=\sum_{j\in\mathbb{Z}_0^2}\frac{1}{|j|^{4+2\delta}}\big(\langle b_N(\omega),e_j\rangle-\langle b(\omega),e_j\rangle\big)^2$$

and by (A.22),

$$\begin{aligned}\langle b_N(\omega),e_j\rangle&=-\mathbf{1}_{\Lambda_N}(j)\langle\omega_N\otimes\omega_N,H_{e_j}\rangle\\&=-\mathbf{1}_{\Lambda_N}(j)\sum_{k,l\in\Lambda_N}\langle H_{e_j},\tilde e_k\otimes\tilde e_l\rangle\langle\omega,\tilde e_k\rangle\langle\omega,\tilde e_l\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\|b_N(\omega)-b(\omega)\|^2_{H^{-2-\delta}(\mathbb{T}^2)}=&\sum_{j\in\Lambda_N}\frac{1}{|j|^{4+2\delta}}\big(\langle b_N(\omega),e_j\rangle-\langle b(\omega),e_j\rangle\big)^2\\&+\sum_{j\in\Lambda_N^c}\frac{\langle b(\omega),e_j\rangle^2}{|j|^{4+2\delta}}.\end{aligned}$$

Denote the two quantities by $J_{1,N}$ and $J_{2,N}$, respectively.

First, by Theorem A.9 and (A.23), we have

$$\begin{aligned}\mathbb{E}J_{2,N}&=\sum_{j\in\Lambda_N^c}\frac{\mathbb{E}\langle b(\omega),e_j\rangle^2}{|j|^{4+2\delta}}\\&=2\sum_{j\in\Lambda_N^c}\frac{1}{|j|^{4+2\delta}}\sum_{k,l\in\mathbb{Z}_0^2}|\langle H_{e_j},\tilde e_k\otimes\tilde e_l\rangle|^2\\&\le C\sum_{j\in\Lambda_N^c}\frac{\log|j|}{|j|^{2+2\delta}},\end{aligned}$$

where the last inequality follows from Lemma A.10. Therefore,

$$\lim_{N\to\infty}\mathbb{E}J_{2,N}=0. \tag{A.26}$$

Recalling (A.23) and denoting by $\Lambda_{N,N}^c=(\mathbb{Z}_0^2\times\mathbb{Z}_0^2)\setminus(\Lambda_N\times\Lambda_N)$, we arrive at

$$\langle b_N(\omega),e_j\rangle-\langle b(\omega),e_j\rangle=\sum_{(k,l)\in\Lambda_{N,N}^c}\langle H_{e_j},\tilde e_k\otimes\tilde e_l\rangle\langle\omega,\tilde e_k\rangle\langle\omega,\tilde e_l\rangle,\qquad j\in\Lambda_N.$$

Analogous to (A.21),

$$\mathbb{E}\big(\langle b_N(\omega),e_j\rangle-\langle b(\omega),e_j\rangle\big)^2=2\sum_{(k,l)\in\Lambda_{N,N}^c}|\langle H_{e_j},\tilde e_k\otimes\tilde e_l\rangle|^2.$$

As a result,

$$\begin{aligned}\mathbb{E}J_{1,N}&=2\sum_{j\in\Lambda_N}\frac{1}{|j|^{4+2\delta}}\sum_{(k,l)\in\Lambda_{N,N}^c}|\langle H_{e_j},\tilde e_k\otimes\tilde e_l\rangle|^2\\&\le 2\sum_{j\in\mathbb{Z}_0^2}\frac{1}{|j|^{4+2\delta}}\sum_{(k,l)\in\Lambda_{N,N}^c}|\langle H_{e_j},\tilde e_k\otimes\tilde e_l\rangle|^2.\end{aligned}$$

By Lemma A.10 and the dominated convergence theorem, we obtain

$$\lim_{N\to\infty}\mathbb{E}J_{1,N}=0.$$

Combining this limit with (A.26), we complete the proof. □

Acknowledgments. Both authors would like to thank the two anonymous referees for careful reading of the paper and for many valuable suggestions of corrections. D. Luo is grateful to Professors Rongchan Zhu and Xiangchan Zhu for helpful discussions, and for drawing his attention to the reference [19].

The second author was supported by the National Natural Science Foundation of China (Nos. 11571347, 11688101), and the Youth Innovation Promotion Association, CAS (2017003).

REFERENCES

[1] ALBEVERIO, S., BARBU, V. and FERRARIO, B. (2008). Uniqueness of the generators of the 2D Euler and Navier–Stokes flows. *Stochastic Process. Appl.* **118** 2071–2084. MR2462289 https://doi.org/10.1016/j.spa.2007.12.003

[2] ALBEVERIO, S. and CRUZEIRO, A. B. (1990). Global flows with invariant (Gibbs) measures for Euler and Navier–Stokes two-dimensional fluids. *Comm. Math. Phys.* **129** 431–444. MR1051499

[3] ALBEVERIO, S. and FERRARIO, B. (2002). Uniqueness results for the generators of the two-dimensional Euler and Navier–Stokes flows. The case of Gaussian invariant measures. *J. Funct. Anal.* **193** 77–93. MR1923629 https://doi.org/10.1006/jfan.2001.3927

[4] ALBEVERIO, S. and FERRARIO, B. (2004). Uniqueness of solutions of the stochastic Navier–Stokes equation with invariant measure given by the enstrophy. *Ann. Probab.* **32** 1632–1649. MR2060312 https://doi.org/10.1214/009117904000000379

[5] ALBEVERIO, S. and FERRARIO, B. (2008). Some methods of infinite dimensional analysis in hydrodynamics: An introduction. In *SPDE in Hydrodynamic: Recent Progress and Prospects. Lecture Notes in Math.* **1942** 1–50. Springer, Berlin. MR2459084 https://doi.org/10.1007/978-3-540-78493-7_1

[6] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics: Probability and Statistics.* Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

[7] CAILLERIE, N. and VOVELLE, J. (2017). Diffusion-approximation for a kinetic equation with perturbed velocity redistribution process. Available at arXiv:1712.10173v1.

[8] COGHI, M. and FLANDOLI, F. (2016). Propagation of chaos for interacting particles subject to environmental noise. *Ann. Appl. Probab.* **26** 1407–1442. MR3513594 https://doi.org/10.1214/15-AAP1120

[9] DA PRATO, G. and DEBUSSCHE, A. (2002). Two-dimensional Navier–Stokes equations driven by a space-time white noise. *J. Funct. Anal.* **196** 180–210. MR1941997 https://doi.org/10.1006/jfan.2002.3919

[10] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136 https://doi.org/10.1017/CBO9780511666223

[11] DEBUSSCHE, A. (2003). The 2D-Navier–Stokes equations perturbed by a delta correlated noise. In *Probabilistic Methods in Fluids* 115–129. World Sci. Publ., River Edge, NJ. MR2083368 https://doi.org/10.1142/9789812703989_0008

[12] DEBUSSCHE, A. and VOVELLE, J. (2017). Diffusion-approximation in stochastically forced kinetic equations. Available at arXiv:1707.07874v3.

[13] FLANDOLI, F. (2018). Weak vorticity formulation of 2D Euler equations with white noise initial condition. *Comm. Partial Differential Equations* **43** 1102–1149. MR3910197 https://doi.org/10.1080/03605302.2018.1467448

[14] FLANDOLI, F. and LUO, D. (2019). ρ-white noise solution to 2D stochastic Euler equations. *Probab. Theory Related Fields.* https://doi.org/10.1007/s00440-019-00902-8

[15] FLANDOLI, F. and LUO, D. (2019). Kolmogorov equations associated to the stochastic two dimensional Euler equations. *SIAM J. Math. Anal.* **51** 1761–1791. MR3947288 https://doi.org/10.1137/18M1175756

[16] GALEATI, L. (2019). On the convergence of stochastic transport equations to a deterministic parabolic one. Available at arXiv:1902.06960.

[17] KUKSIN, S. and SHIRIKYAN, A. (2003). Some limiting properties of randomly forced two-dimensional Navier–Stokes equations. *Proc. Roy. Soc. Edinburgh Sect. A* **133** 875–891. MR2006207 https://doi.org/10.1017/S0308210500002729

[18] KUKSIN, S. and SHIRIKYAN, A. (2004). Randomly forced CGL equation: Stationary measures and the inviscid limit. *J. Phys. A* **37** 3805–3822. MR2039838 https://doi.org/10.1088/0305-4470/37/12/006

[19] KURTZ, T. G. (2007). The Yamada–Watanabe–Engelbert theorem for general stochastic equations and inequalities. *Electron. J. Probab.* **12** 951–965. MR2336594 https://doi.org/10.1214/EJP.v12-431

[20] RÖCKNER, M., ZHU, R. and ZHU, X. (2017). Restricted Markov uniqueness for the stochastic quantization of $P(\Phi)_2$ and its applications. *J. Funct. Anal.* **272** 4263–4303. MR3626040 https://doi.org/10.1016/j.jfa.2017.01.023

[21] SAUER, M. (2016). L^1-uniqueness of Kolmogorov operators associated with two-dimensional stochastic Navier–Stokes Coriolis equations with space-time white noise. *J. Theoret. Probab.* **29** 569–589. MR3500411 https://doi.org/10.1007/s10959-014-0582-8

[22] SIMON, J. (1987). Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl.* (4) **146** 65–96. MR0916688 https://doi.org/10.1007/BF01762360

[23] STANNAT, W. (2003). L^1-uniqueness of regularized 2D-Euler and stochastic Navier–Stokes equations. *J. Funct. Anal.* **200** 101–117. MR1974090 https://doi.org/10.1016/S0022-1236(02)00182-9

[24] STANNAT, W. (2007). A new a priori estimate for the Kolmogorov operator of a 2D-stochastic Navier–Stokes equation. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** 483–497. MR2376438 https://doi.org/10.1142/S0219025707002920

[25] ZHU, R. and ZHU, X. (2017). Strong-Feller property for Navier-Stokes equations driven by space-time white noise. Available at arXiv:1709.09306.