THE SCALING LIMIT OF THE MEMBRANE MODEL

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On the integer lattice, we consider the discrete membrane model, a random interface in which the field has Laplacian interaction. We prove that, under appropriate rescaling, the discrete membrane model converges to the continuum membrane model in $d \geq 2$. Namely, it is shown that the scaling limit in $d = 2, 3$ is a Hölder continuous random field, while in $d \geq 4$ the membrane model converges to a random distribution. As a by-product of the proof in $d = 2, 3$, we obtain the scaling limit of the maximum. This work complements the analogous results of Caravenna and Deuschel (*Ann. Probab.* **37** (2009) 903–945) in $d = 1$.

1. Introduction. The main object of study in this article is the membrane model (MM), also known as discrete bi-Laplacian model (see Figure 1 for a simulation). The membrane model is a special instance of a more general class of interface models in which the interaction of the system is governed by the exponential of an Hamiltonian function $H : \mathbb{R}^{\mathbb{Z}^d} \to [0, \infty)$. More specifically, random interfaces are fields $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$, whose distribution is determined by the probability measure on $\mathbb{R}^{\mathbb{Z}^d}$, $d \geq 1$, with density

$$\mathbf{P}_W(\mathrm{d}\varphi) := \frac{\mathrm{e}^{-H(\varphi)}}{Z_W} \prod_{x \in W} \mathrm{d}\varphi_x \prod_{x \in \mathbb{Z}^d \setminus W} \delta_0(\mathrm{d}\varphi_x),$$

where $W \Subset \mathbb{Z}^d$ is a finite subset, $\mathrm{d}\varphi_x$ is the 1-dimensional Lebesgue measure on $\mathbb{R}$, δ_0 is the Dirac measure at 0 and Z_W is a normalising constant. We are imposing zero boundary conditions, that is, almost surely $\varphi_x = 0$ for all $x \in \mathbb{Z}^d \setminus W$, but the definition holds for more general boundary conditions. A relevant example is where the Hamiltonian is driven by a convex function of the gradient, that is, $H(\varphi) = \sum_{x \sim y} V(\varphi_y - \varphi_x)$, $V : \mathbb{R} \to \mathbb{R}$ convex, and the sum being over nearest
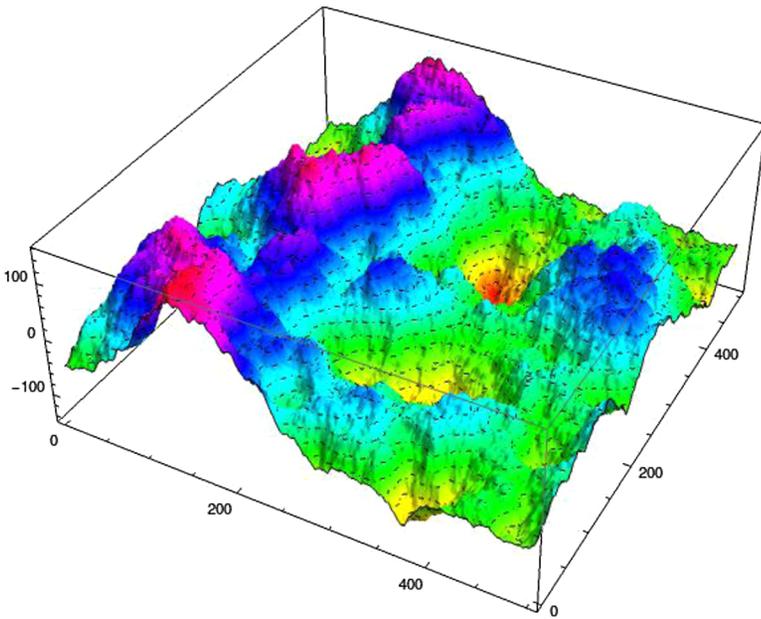
FIG. 1.   *A sample of the MM in d* = 2 *on a box of side-length* 500.

neighbors. The most well known among these interfaces is the discrete Gaussian free field (DGFF) when $V(x) \propto x^2/2$. The quadratic potential allows one to have various tools at one's disposal, like the random walk representation of covariances and inequalities like FKG. These tools can be generalized to (strictly) convex potentials in the form of the Brascamp–Lieb inequality and the Helffer–Sjöstrand random walk representation. We refer to Funaki (2005), Giacomin, Olla and Spohn (2001), Naddaf and Spencer (1997), Velenik (2006) for an overview. Outside the convex regime, the nonconvex regime was recently studied, for example, in Biskup and Spohn (2011), Cotar, Deuschel and Müller (2009).

A very natural probabilistic question one can ask oneself is: "What happens to a random interface when one rescales it suitably?" In $d = 1$ in the example of the DGFF, the scaling limit is the Brownian bridge. In $d \geq 2$, the limit, the continuum Gaussian free field, is not a random variable and can only be interpreted in the language of distribution theory (see, e.g., Biskup (2011), Sheffield (2007)). The importance of the continuum Gaussian free field in $d = 2$ relies on its universality property due to conformal invariance, and links it to other stochastic processes like SLE, CLE and Liouville quantum gravity. The recent developments concerning the extreme value theory of DGFF (and, more generally, log-correlated fields) have shown impressive connections also to number theory, branching processes and random matrices.

In comparison to the DGFF, the membrane model has received slightly less attention, mainly due to the technical challenges intrinsic of the model. It is the

Gaussian interface for which

$$(1.1) \qquad H(\varphi) := \frac{1}{2} \sum_{x \in \mathbb{Z}^d} |\Delta_1 \varphi_x|^2$$

and Δ_1 is the discrete Laplacian defined by

$$\Delta_1 f(x) = \frac{1}{2d} \sum_{y \sim x} (f(y) - f(x)), \qquad f : \mathbb{Z}^d \to \mathbb{R}, \quad x \in \mathbb{Z}^d.$$

In case $W = V_N := [-N, N]^d \cap \mathbb{Z}^d$, we will denote the measure $\mathbf{P}_{V_N}$ with Hamiltonian (1.1) by $\mathbf{P}_N$. Introduced by Sakagawa (2003) in the probabilistic literature, the MM looks for certain aspects very similar to the DGFF: it is log-correlated in $d = 4$, has a supercritical regime in $d \geq 5$ and is subcritical in $d \leq 3$. In particular in $d = 2, 3, 4$, there is no thermodynamic limit of the measures $\mathbf{P}_N$ as $N \uparrow \infty$. The MM displays however certain crucial difficulties in that, for example, it exhibits no random walk representation, and several correlation inequalities are lacking. Nonetheless, it is possible, via analytic and numerical methods, to obtain sharp results on its behavior. Examples are the study of the entropic repulsion and pinning effects (Adams, Kister and Weber (2016), Bolthausen, Cipriani and Kurt (2017), Caravenna and Deuschel (2008), Kurt (2007, 2009b)), extreme value theory (Chiarini, Cipriani and Hazra (2016)) and connections to other statistical mechanics models (Cipriani, Hazra and Ruszel (2018)). In this framework we present our work which aims at determining the scaling limit of the bi-Laplacian model. The answer in $d = 1$ was given by Caravenna and Deuschel (2009), who also look at the situation in which a pinning force is added to the model. We complement their work by determining the scaling limit in all $d \geq 2$. We also mention Hryniv and Velenik (2009), who consider general semiflexible membranes as well with a different scaling approach. Their results are derived using an integrated random walk representation which is difficult to adapt in higher dimensions. The main contributions of this article are as follows:

♠ in $d = 2, 3$, we consider the discrete membrane model on a box of side-length $2N$ and interpolate it in a continuous way. We show that the process converges to a real-valued process with continuous trajectories and the convergence takes place in the space of continuous functions (see Theorem 2.1). The utility of this type of convergence is that it yields the scaling limit of the discrete maximum exploiting the continuous mapping theorem (Corollary 2.2). While the limiting maximum of the discrete membrane model in $d \geq 5$ was derived by Chiarini, Cipriani and Hazra (2016), in $d = 4$ the problem remains open as far as the authors know (tightness can be derived from Ding, Roy and Zeitouni (2017)). The limit field also turns to be Hölder continuous with exponent less than 1 in $d = 2$ and less than $1/2$ in $d = 3$.

   The proof of the above facts is based on two basic steps: tightness and finite dimensional convergence. Tightness depends on the gradient estimates of

the discrete Green's functions which were very recently derived in Müller and Schweiger (2019); finite dimensional convergence follows from the convergence of the Green's function.

♠ In $d \geq 4$, the limiting process on a sufficiently nice domain D will be a fractional Gaussian field with Hurst parameter $H := s - d/2$ on D. The theory of fractional Gaussian fields was surveyed recently in Lodhia et al. (2016). The authors there construct the continuum membrane model using characteristic functionals. We take here a bit different route and give a representation using the eigenvalues of the biharmonic operator in the continuum. We remark however that these eigenvalues differ from the square of the Laplacian eigenvalues due to boundary conditions. The GFF theory which is based on $H_0^1(D)$ (the first-order Sobolev space) needs to be replaced by $H_0^2(D)$ (second-order Sobolev space).

Our main result is given in Theorem 3.11. Its proof is again split into two steps: finite dimensional convergence and tightness. Both steps crucially require an approximation result of PDEs given by Thomée (1964): there he gives quantitative estimates on the approximation of solutions of PDEs involving "nice" elliptic operators by their discrete counterparts. We believe that the techniques used in that article might have implications in the development of the theory of the membrane model, in particular the idea of tackling boundary values by rescaling the standard discrete Sobolev norm around the boundary. Especially in $d = 4$, this allows one to overcome the difficulty of extending estimates from the bulk up to the boundary, which is generally one stumbling block in the study of the MM.

♠ In $d \geq 5$, we also consider the infinite volume membrane model on $\mathbb{Z}^d$. We show in Lemma 4.3 that the limit is the fractional Gaussian field of Hurst parameter $H := 2 - d/2 < 0$ on $\mathbb{R}^d$ (see Lodhia et al. (2016)) and we prove in Theorem 4.4 the convergence with the help of characteristic functionals. We utilize the classical result of Fernique (1968) (recently extended in the tempered distribution setting by Biermé, Durieu and Wang (2017)) stating that convergence of tempered distributions is equivalent to that of their characteristic functionals. Technical tools useful for this scope are the explicit Fourier transform of the infinite volume Green's function and the Poisson summation formula.

We stress that, regardless of the dimension, the field is always rescaled as $N^{(d-4)/2}\varphi_{Nx}$ for $x \in N^{-1}\mathbb{Z}^d$. Heuristically, the factor N^{4-d} corresponds to the order of growth of the variance of the model in a box, which we recall here for completeness.

(i) In $d = 2, 3$, if $d(\cdot)$ denotes the distance to the boundary of V_N one has for some constant $C > 0$ (Müller and Schweiger (2019), Theorem 1.1),

$$\left|\mathbf{Cov}_N(\varphi_x, \varphi_y)\right| \leq C \min\left(d(x)^{2-d/2}d(y)^{2-d/2}, \frac{d(x)^2 d(y)^2}{(\|x - y\| + 1)^2}\right).$$

(ii) In $d = 4$, let us denote the bulk of V_N by $V_N^\delta := \{x \in V_N : d(x) > \delta N\}$ for $\delta \in (0, 1)$. Then from Cipriani ((2013), Lemma 2.1) we have: there exists a constant $C(\delta) > 0$ such that

$$\sup_{x, y \in V_N^\delta} \left| \mathbf{Cov}_N(\varphi_x, \varphi_y) - \frac{8}{\pi^2} (\log N - \log(\|x - y\| + 1)) \right| \le C(\delta).$$

Asymptotics up to the boundary are not known to the best of the authors' knowledge. The approach of Thomée (1964) allows to circumvent this lack of estimates.

(iii) In $d \ge 5$, the infinite volume covariance satisfies (Sakagawa (2003), Lemma 5.1),

$$\left| \mathbf{Cov}(\varphi_x, \varphi_y) \right| \sim C_d \|x - y\|^{4-d} \quad \text{as } \|x - y\| \to \infty.$$

Interestingly this reflects the behavior of the characteristic singular solution (fundamental solution) of the biharmonic equation, which is

$$\begin{cases} C_d \|x\|^{4-d} & d \text{ odd or } d \text{ even and } d \ge 6, \\ C_d \|x\|^{4-d} \log \|x\| & d \text{ even and } d \le 4. \end{cases}$$

The reader can consult Mayboroda and Maz'ya (2014), Mitrea and Mitrea ((2011), Section 5) and references therein for sharp pointwise estimates of the Green's function of the bi-Laplacian in general domains and for regularity properties of the biharmonic Green's function.

We would like to conclude the Introduction with a few open questions:

- Is the maximum of the discrete membrane model at the critical dimension scaling to a randomly shifted Gumbel, as predicted by Ding, Roy and Zeitouni (2017)?
- What will the scaling limit be for interfaces with mixed Hamiltonian of the form $H(\varphi) := \sum_x V_1(\nabla \varphi_x) + \sum_x V_2(\Delta \varphi_x)$, V_1, V_2 convex functions (in particular, $V_1 \equiv 0$)? Results on these models were shown in Borecki and Caravenna (2010) in $d = 1$.

Structure of the paper. In Section 2, we handle the case $d \in \{2, 3\}$, while in Section 3 we treat the finite-volume case in $d \ge 4$. In Section 4, we analyse the case of the infinite-volume model in $d \ge 5$. To keep the article self-contained in the Appendix, we discuss the results from Thomée (1964) and also deduce a quantitative version of the approximation result proved there.

NOTATION. We fix a constant $\kappa := (2d)^{-1}$ throughout the whole paper. In the following, $C > 0$ always denotes a universal constant whose value however may change in each occurrence. We will use $\xrightarrow{d}$ to denote convergence in distribution. We denote, for any $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, $d \ge 1$, the "integer part" of y as $\lfloor y \rfloor = (\lfloor y_1 \rfloor, \dots, \lfloor y_d \rfloor)$, and similarly $\{y\} = y - \lfloor y \rfloor$ is the "fractional part" of y.

2. Convergence in $d = 2, 3$.

2.1. *Description of the limiting field.* Let $V = (-1, 1)^d$ and $V_N = N\overline{V} \cap \mathbb{Z}^d$, where $N \in \mathbb{N}$. Let $(\varphi_x)_{x \in V_{N-1}}$ be the MM on V_{N-1} and let G_{N-1} be the covariance function for this model. It is known (Kurt (2009a), Section 1), to satisfy the following discrete boundary value problem for all $x \in V_{N-1}$:

$$\begin{cases} \Delta_1^2 G_{N-1}(x, y) = \delta_x(y), & y \in V_{N-1}, \\ G_{N-1}(x, y) = 0, & y \notin V_{N-1}. \end{cases}$$

First, we want to define a *continuous* interpolation Ψ_N of the discrete field to have convergence in the space of continuous functions. There are many ways to define the field $(\Psi_N(t))_{t \in \overline{V}}$. We take one of the simplest geometric ways which is akin to the interpolation of simple random walk trajectories in Donsker's invariance principle. Mind that we take the domain as a square since the recent gradient estimates and convergence of the Green's function of Müller and Schweiger (2019) can be applied easily.

Interpolation in $d = 2$. Let $t = (t_1, t_2) \in \overline{V}$. Then $p := Nt$ lies in the square box with vertices $a = \lfloor Nt \rfloor$, $b = \lfloor Nt \rfloor + e_1$, $c = \lfloor Nt \rfloor + e_1 + e_2$, $d = \lfloor Nt \rfloor + e_2$, where e_1, e_2 are the standard basis vectors of $\mathbb{R}^2$. Suppose p is a point in the triangle abc. Then we can write $p = \alpha a + \beta b + \gamma c$ with $\alpha = 1 - \{Nt_1\}$, $\beta = \{Nt_1\} - \{Nt_2\}$, $\gamma = \{Nt_2\}$. And in this case we define

$$\Psi_N(t) = \frac{\kappa}{N}[\alpha \varphi_{\lfloor Nt \rfloor} + \beta \varphi_{\lfloor Nt \rfloor + e_1} + \gamma \varphi_{\lfloor Nt \rfloor + e_1 + e_2}].$$

Similarly, if $p \in \triangle acd$ then we define

$$\Psi_N(t) = \frac{\kappa}{N}[\alpha' \varphi_{\lfloor Nt \rfloor} + \beta' \varphi_{\lfloor Nt \rfloor + e_2} + \gamma' \varphi_{\lfloor Nt \rfloor + e_1 + e_2}],$$

where

$$\alpha' = 1 - \{Nt_2\}, \qquad \beta' = \{Nt_2\} - \{Nt_1\}, \qquad \gamma' = \{Nt_1\}.$$

Thus the interpolated field $(\Psi_N(t))_{t \in \overline{V}}$ is defined by

$$\begin{aligned} \Psi_N(t) = \frac{\kappa}{N}\big[\varphi_{\lfloor Nt \rfloor} + \{Nt_i\}(\varphi_{\lfloor Nt \rfloor + e_i} - \varphi_{\lfloor Nt \rfloor}) \\ + \{Nt_j\}(\varphi_{\lfloor Nt \rfloor + e_i + e_j} - \varphi_{\lfloor Nt \rfloor + e_i})\big], \quad \text{if } \{Nt_i\} \geq \{Nt_j\}, \end{aligned}$$

where $i, j \in \{1, 2\}$, $i \neq j$.

Interpolation in $d = 3$. In $d = 3$, the interpolated field can be defined in the same way as above. We use tetrahedrons to define the interpolated field as

$$\Psi_N(t) = \frac{\kappa}{\sqrt{N}} \big[\varphi_{\lfloor Nt \rfloor} + \{Nt_i\}(\varphi_{\lfloor Nt \rfloor + e_i} - \varphi_{\lfloor Nt \rfloor})$$

$$+ \{Nt_j\}(\varphi_{\lfloor Nt \rfloor + e_i + e_j} - \varphi_{\lfloor Nt \rfloor + e_i})$$

$$+ \{Nt_k\}(\varphi_{\lfloor Nt \rfloor + e_i + e_j + e_k} - \varphi_{\lfloor Nt \rfloor + e_i + e_j}) \big],$$

where $\{Nt_i\} \geq \{Nt_j\} \geq \{Nt_k\}$, $t = (t_1, t_2, t_3) \in \overline{V}$ and $i, j, k \in \{1, 2, 3\}$ are pairwise different.

Note that in both $d = 2, 3$ we have

$$\Psi_N(t) = \kappa N^{\frac{d-4}{2}} \varphi_{Nt}, \quad t \in \frac{1}{N} \mathbb{Z}^d.$$

From the above construction, it follows that, for each N, Ψ_N is a continuous function on $\overline{V}$. This shows that Ψ_N can be considered as a random variable taking values in $(C(\overline{V}), \mathcal{C}(\overline{V}))$ where $C(\overline{V})$ is the space of continuous functions on $\overline{V}$ and $\mathcal{C}(\overline{V})$ is its Borel σ-algebra. Also recall the definition of Green's function: the Green's function for the biharmonic operator is $G_V : V \times V \to \mathbb{R}$ such that for every fixed $x \in V$, it solves the equation

$$\Delta^2 G_V(x, y) = \delta_x(y), \quad y \in V,$$

in the space $H_0^2(V)$, the completion of $C_c^\infty(V)$ with respect to the norm

$$\|f\|_{H_0^2(V)} := \|\nabla^2 f\|_{L^2(V)}.$$

In the above equations Δ^2, the continuum bi-Laplacian, acts on the y component, and ∇^2 is the Hessian. The detailed properties of such spaces are needed in $d \geq 4$ so we defer the discussions on them to Section 3. We denote the continuum Green's function by G_V to indicate the dependence on the domain V.

We are now ready to state our main result for the case $d = 2, 3$. It shows that the convergence of the above described process occurs in the space of continuous functions.

THEOREM 2.1 (Scaling limit in $d = 2, 3$). *Consider the interpolated membrane model $(\Psi_N(t))_{t \in \overline{V}}$ in $d = 2$ and 3 as above. Then there exists a centered continuous Gaussian process Ψ with covariance $G_V(\cdot, \cdot)$ on $\overline{V}$ such that Ψ_N converges in distribution to Ψ in the space of all continuous functions on $\overline{V}$. Furthermore, the process Ψ is almost surely Hölder continuous with exponent η, for every $\eta \in (0, 1)$ respectively $\eta \in (0, 1/2)$ in $d = 2$, respectively, $d = 3$.*

An immediate consequence of the continuous mapping theorem is that, as $N \to \infty$,

$$\sup_{x \in \overline{V}} \Psi_N(x) \xrightarrow{d} \sup_{x \in \overline{V}} \Psi(x).$$

It is easy to see that for any square or a cube A in the $\frac{1}{N}\mathbb{Z}^d$ lattice,

$$\sup_{x \in A} \Psi_N(x) = \kappa N^{\frac{d-4}{2}} \max\{\varphi_{Nx} : x \text{ is a vertex of A}\}.$$

Hence $\sup_{x \in \overline{V}} \Psi_N(x) = \kappa N^{\frac{d-4}{2}} \max_{x \in V_N} \varphi_x$. So combining these observations, we obtain the scaling limit of the maximum of the discrete membrane model in lower dimensions.

COROLLARY 2.2.   *Let* $d \in \{2, 3\}$ *and let* $M_N = \max_{x \in V_N} \varphi_x$. *Then as* $N \uparrow \infty$,

$$\kappa N^{\frac{d-4}{2}} M_N \xrightarrow{d} \sup_{x \in \overline{V}} \Psi(x).$$

2.2. *Proof of the scaling limit* (*Theorem* 2.1).   The proof follows the general methodology of a functional CLT; namely, we first show the tightness of the interpolated field, and second we show that the finite dimensional distributions converge. As a by-product of the proof, the limiting Gaussian process will be well defined, that is, its covariance function will be positive definite. The finite dimensional convergence follows easily from the very recent work of Müller and Schweiger (2019) where the convergence of the discrete Green's function to the continuum one is shown. Tightness also requires the crucial bounds on gradients which were derived in the same article. Since we have interpolated the field continuously and not piecewise in boxes or cubes, one of the main efforts is to deduce moment bounds from integer lattice points.

2.2.1. *Tightness and Hölder continuity.*   To derive the tightness, we need the following ingredients. The first one consists in the following bounds for the discrete Green's function and its gradients which follow from Müller and Schweiger (2019). We define the directional derivative of a function $u : \mathbb{Z}^d \to \mathbb{R}$ as

$$D_i u(x) := u(x + e_i) - u(x),$$

and the discrete gradient as

$$\nabla u(x) = \left(D_i u(x)\right)_{i=1}^d.$$

For functions of several variables, we use a subscript to indicate the variable with respect to which a derivative is taken, for example, in $D_{i,1} D_{j,2} u(x, y)$ we take the discrete derivative in the direction i in the variable x and in j in the variable y, and $\nabla_x G(x, y)$ means we are taking the gradient in the x variable. We now state some bounds on the covariance function and its gradient from Müller and Schweiger (2019), where they appear in a more general version.

LEMMA 2.3 ([Müller and Schweiger (2019)](), Theorem 1.1). *Let $d \in \{2, 3\}$.*

1. *For any $x, y \in \mathbb{Z}^d$,*

$$\left| G_N(x, y) \right| \le C N^{4-d}.$$

2. *For any $x, y \in \mathbb{Z}^d$,*

$$\left\| \nabla_x G_N(x, y) \right\| \le C N^{3-d}.$$

3. *For any $x, y \in \mathbb{Z}^d$,*

$$\left\| \nabla_x \nabla_y G_N(x, y) \right\| \le \begin{cases} C \log\left(1 + \dfrac{N^2}{(\|x - y\| + 1)^2}\right) & \text{if } d = 2, \\ C & \text{if } d = 3. \end{cases}$$

Now from the estimate (3) and the fact that

$$\mathbf{E}_N\left[(\varphi_{z+e_i} - \varphi_z)^2\right] = D_{i,2} D_{i,1} G(z, z)$$

one can observe the following fact.

FACT 2.4. *For $z \in \mathbb{Z}^d$,*

$$\mathbf{E}_N\left[(\varphi_{z+e_i} - \varphi_z)^2\right] \le \begin{cases} C \log N & \text{if } d = 2, \\ C & \text{if } d = 3. \end{cases}$$

Next, we want to show that the sequence $\{\Psi_N\}_{N \in \mathbb{N}}$ is tight in $C(\overline{V})$. We use the following theorem, whose proof follows from that of Theorem 14.9 of [Kallenberg (1997)]().

THEOREM 2.5. *Let $X^1, X^2, \ldots$ be continuous processes on $\overline{V}$ with values in a complete separable metric space (S, ρ). Assume that (X_0^n) is tight in S and that for constants $\alpha, \beta > 0$,*

$$(2.1) \qquad \mathbf{E}\left[\rho(X_s^n, X_t^n)^\alpha\right] \le C\|s - t\|^{d+\beta}, \quad s, t \in \overline{V}$$

uniformly in n. Then (X^n) is tight in $C(\overline{V}, S)$ and for every $c \in (0, \beta/\alpha)$ the limiting processes are almost surely Hölder continuous with exponent c.

Observe that the process $(\Psi_N(t))_{t \in \overline{V}}$ is Gaussian, and since from Lemma 2.3 it follows that $G_{N-1}(0, 0) \le N^{4-d}$, it is easy to see that $(\Psi_N(0))$ is tight. Again, using the properties of Gaussian laws, to show (2.1) it is enough to show the following the lemma.

LEMMA 2.6. *Let $b \in (0, 1)$ in $d = 2$ and $b = 0$ in $d = 3$. Then there exists a constant $C > 0$ (which depends on b in $d = 2$) such that*

$$(2.2) \qquad \mathbf{E}\big[\big|\Psi_N(t) - \Psi_N(s)\big|^2\big] \leq C\|t - s\|^{1+b}$$

for all $t, s \in \overline{V}$, uniformly in N.

This lemma will immediately give (2.1), and hence the Hölder continuity of the limiting field.

COROLLARY 2.7. *The field Ψ is almost surely Hölder continuous with exponent η, where $\eta \in (0, 1)$ in $d = 2$ and $\eta \in (0, 1/2)$ in $d = 3$.*

PROOF. We note that for $t, s \in \overline{V}$, the random variable $\Psi_N(t) - \Psi_N(s)$ is Gaussian. Therefore, using Lemma 2.6 we have, for any α such that $(1 + b)\alpha/2 > d$, that there is a constant C such that the following holds uniformly in N with $\beta := (1 + b)\alpha/2 - d$:

$$\mathbf{E}\big[\big|\Psi_N(t) - \Psi_N(s)\big|^\alpha\big] \leq C\|t - s\|^{d+\beta}, \quad s, t \in \overline{V}.$$

The conclusion follows then from Theorem 2.5. $\square$

Now we show the proof of the lemma.

PROOF OF LEMMA 2.6. First, we consider $d = 2$. We fix a $b \in (0, 1)$ and let $t, s \in \overline{V}$. We split the proof into a few cases.

Case 1: Suppose t, s belong to the same smallest square box in the lattice $\frac{1}{N}\mathbb{Z}^2$. First, assume $\lfloor Nt \rfloor = \lfloor Ns \rfloor$, that is, the points are in the interior and not touching the top and right boundaries. In this case, if we have $\{Nt_1\} \geq \{Nt_2\}$ and $\{Ns_1\} \geq \{Ns_2\}$, then by definition of the interpolation we have

$$\Psi_N(t) - \Psi_N(s) = \kappa\big[(t_1 - s_1)(\varphi_{\lfloor Nt \rfloor + e_1} - \varphi_{\lfloor Nt \rfloor}) \\ + (t_2 - s_2)(\varphi_{\lfloor Nt \rfloor + e_1 + e_2} - \varphi_{\lfloor Nt \rfloor + e_1})\big].$$

So from the above expression, we have

$$\mathbf{E}\big[(\Psi_N(t) - \Psi_N(s))^2\big] \leq 2\kappa^2\big[(t_1 - s_1)^2\mathbf{E}\big[(\varphi_{\lfloor Nt \rfloor + e_1} - \varphi_{\lfloor Nt \rfloor})^2\big] \\ + (t_2 - s_2)^2\mathbf{E}\big[(\varphi_{\lfloor Nt \rfloor + e_1 + e_2} - \varphi_{\lfloor Nt \rfloor + e_1})^2\big]\big].$$

Now from Fact 2.4 and $|t_1 - s_1|, |t_2 - s_2| < N^{-1}$ we obtain (2.2). The argument is similar if one has $\{Nt_1\} \leq \{Nt_2\}$ and $\{Ns_1\} \leq \{Ns_2\}$.

Again if $\{Nt_1\} \geq \{Nt_2\}$ and $\{Ns_1\} < \{Ns_2\}$, or if $\{Nt_1\} < \{Nt_2\}$ and $\{Ns_1\} \geq \{Ns_2\}$ then we consider the point u on the line segment joining t and s such that

Nu is the point of intersection of the line segment joining Nt, Ns and the diagonal joining $\lfloor Nt \rfloor$, $\lfloor Nt \rfloor + e_1 + e_2$. Then we have using the above computations

$$\mathbf{E}\big[\big|\Psi_N(t) - \Psi_N(s)\big|^2\big] \leq 2\mathbf{E}\big[\big|\Psi_N(t) - \Psi_N(u)\big|^2\big] + 2\mathbf{E}\big[\big|\Psi_N(u) - \Psi_N(s)\big|^2\big]$$
$$\leq C\big[\|t - u\|^{1+b} + \|u - s\|^{1+b}\big]$$
$$\leq C\|t - s\|^{1+b}.$$

Now the other case, that is, when $\lfloor Nt \rfloor \neq \lfloor Ns \rfloor$ follows from above by continuity.

Case 2: Suppose t, s do not belong to the same smallest square box in the lattice $\frac{1}{N}\mathbb{Z}^2$. In this case, if $\|t - s\| \leq 1/N$ then one can obtain (2.2) by the above case and a suitable point in between. So we assume $\|t - s\| > 1/N$. Depending on whether Nt and Ns belong to the discrete lattice we split the proof in two broad cases. We will use bounds on mixed discrete derivatives for a better control of finite differences of the Green's function.

Subcase 2(a). Suppose $t, s \in \frac{1}{N}\mathbb{Z}^2$. Then

$$\mathbf{E}\big[\big|\Psi_N(t) - \Psi_N(s)\big|^2\big] = \frac{\kappa^2}{N^2}\big[G_{N-1}(Nt, Nt) - G_{N-1}(Ns, Nt)$$
$$- G_{N-1}(Nt, Ns) + G_{N-1}(Ns, Ns)\big].$$

We assume without loss of generality $Ns_1 \leq Nt_1$, $Ns_2 \leq Nt_2$. Also denote $M := N(t_1 - s_1 + t_2 - s_2)$ and let $(u_i)_{i=0}^M$ be such that $u_i = s + i/Ne_1$ for $i \leq N(t_1 - s_1)$ and $u_i = s + (t_1 - s_1)e_1 + (i/N - (t_1 - s_1))e_2$ for $i > N(t_1 - s_1)$. Then

$$\mathbf{E}\big[\big|\Psi_N(t) - \Psi_N(s)\big|^2\big]$$
$$= \frac{\kappa^2}{N^2}\sum_{i=0}^{M-1}\big[G_{N-1}(Nu_{i+1}, Nt) - G_{N-1}(Nu_i, Nt)\big]$$
$$- \big[G_{N-1}(Nu_{i+1}, Ns) - G_{N-1}(Nu_i, Ns)\big]$$
$$= \frac{\kappa^2}{N^2}\sum_{i,j=0}^{M-1}\big[G_{N-1}(Nu_{i+1}, Nu_{j+1}) - G_{N-1}(Nu_{i+1}, Nu_j)$$
$$- G_{N-1}(Nu_i, Nu_{j+1}) + G_{N-1}(Nu_i, Nu_j)\big]$$
$$\leq \frac{C}{N^2}\sum_{i,j=0}^{M-1}\log\Big(1 + \frac{N^2}{(\|Nu_i - Nu_j\| + 1)^2}\Big),$$

where we have used Lemma 2.3 (3) in the last inequality and we have absorbed the constant κ^2 in the generic constant C. Now using the definition of u_i, u_j the

right-hand side above is bounded above by

$$\frac{C}{N^2} \sum_{i,j=0}^{M-1} \log\left(1 + \frac{N^2}{(\frac{|i-j|}{\sqrt{2}} + 1)^2}\right)$$

$$\leq \frac{C}{N^2} \sum_{i,j=0}^{M-1} \log\left(1 + \frac{N}{(|i-j|+1)}\right)$$

$$\leq \frac{CM}{N^2} \sum_{l=-M+1}^{M-1} \log\left(1 + \frac{N}{(|l|+1)}\right)$$

$$\leq \frac{CM}{N} \int_0^{\frac{M}{N}} \log\left(1 + \frac{1}{x}\right) dx$$

$$\leq C\left(\frac{M}{N}\right)^2 \left[1 + \log\left(1 + \frac{N}{M}\right)\right] \leq C\|t - s\|^{1+b}.$$

Subcase 2(b) Suppose at least one between t, s does not belong to $\frac{1}{N}\mathbb{Z}^2$. Then

$$\mathbf{E}\left[|\Psi_N(t) - \Psi_N(s)|^2\right]$$

$$\leq 3\mathbf{E}\left[\left|\Psi_N(t) - \Psi_N\left(\frac{\lfloor Nt \rfloor}{N}\right)\right|^2\right]$$

$$+ 3\mathbf{E}\left[\left|\Psi_N\left(\frac{\lfloor Nt \rfloor}{N}\right) - \Psi_N\left(\frac{\lfloor Ns \rfloor}{N}\right)\right|^2\right] + 3\mathbf{E}\left[\left|\Psi_N\left(\frac{\lfloor Ns \rfloor}{N}\right) - \Psi_N(s)\right|^2\right]$$

$$\leq C\left[\left\|t - \frac{\lfloor Nt \rfloor}{N}\right\|^{1+b} + \left\|\frac{\lfloor Nt \rfloor}{N} - \frac{\lfloor Ns \rfloor}{N}\right\|^{1+b} + \left\|\frac{\lfloor Ns \rfloor}{N} - s\right\|^{1+b}\right]$$

$$\leq C\|t - s\|^{1+b}.$$

Note that for the last inequality we have used our assumption $\|t - s\| > 1/N$.

Now we consider $d = 3$. Let $t, s \in \overline{V}$. We split the proof into cases similar to those of $d = 2$. We give a brief description. For Case 1, suppose t, s belong to the same smallest cube in the lattice $\frac{1}{N}\mathbb{Z}^3$. First, assume $\lfloor Nt \rfloor = \lfloor Ns \rfloor$. In this case, if $\{Nt_1\} \geq \{Nt_2\} \geq \{Nt_3\}$ and $\{Ns_1\} \geq \{Ns_2\} \geq \{Ns_3\}$, then it follows from the definition of interpolation

$$\mathbf{E}\left[(\Psi_N(t) - \Psi_N(s))^2\right] \leq 3N\kappa^2\left[(t_1 - s_1)^2\mathbf{E}\left[(\varphi_{\lfloor Nt \rfloor + e_1} - \varphi_{\lfloor Nt \rfloor})^2\right]\right.$$

$$+ (t_2 - s_2)^2\mathbf{E}\left[(\varphi_{\lfloor Nt \rfloor + e_1 + e_2} - \varphi_{\lfloor Nt \rfloor + e_1})^2\right]$$

$$\left. + (t_3 - s_3)^2\mathbf{E}\left[(\varphi_{\lfloor Nt \rfloor + e_1 + e_2 + e_3} - \varphi_{\lfloor Nt \rfloor + e_1 + e_2})^2\right]\right].$$

Now from Fact 2.4 and the fact that $|t_1 - s_1|, |t_2 - s_2|, |t_3 - s_3| < 1/N$ we have (2.2). Note that this is a particular case of t, s lying in the same tetrahedral portion

of the cube. Hence if t, s lie in the same tetrahedral portion of the cube then by similar arguments (2.2) holds. If t, s do not lie in the same tetrahedral part, then we consider points (at most 3) on the line segment joining them such that two consecutive between t, the selected points and s lie in the same tetrahedral part. Then applying the previous argument we can obtain (2.2). Now the case when $\lfloor Nt \rfloor \neq \lfloor Ns \rfloor$ follows by continuity. For Case 2, we describe subcase 2(a) which turns out to be simpler in $d = 3$. The rest of the argument is similar to that in $d = 2$. Suppose $t, s \in \frac{1}{N}\mathbb{Z}^3$ with $\|t - s\| > 1/N$. Then

$$\mathbf{E}\big[|\Psi_N(t) - \Psi_N(s)|^2\big] = \frac{\kappa^2}{N}\big[G_{N-1}(Nt, Nt) - G_{N-1}(Ns, Nt)$$
$$- G_{N-1}(Nt, Ns) + G_{N-1}(Ns, Ns)\big].$$

Without loss of generality, assume $Ns_1 \leq Nt_1$, $Ns_2 \leq Nt_2$, $Ns_3 \leq Nt_3$. Then

$$G_{N-1}(Nt, Nt) - G_{N-1}(Ns, Nt)$$

$$= \sum_{i=1}^{N(t_1-s_1)} D_{1,1}G_{N-1}\big(Ns + (i-1)e_1, Nt\big)$$

$$+ \sum_{j=1}^{N(t_2-s_2)} D_{2,1}G_{N-1}\big(Ns + N(t_1 - s_1)e_1 + (j-1)e_2, Nt\big)$$

$$+ \sum_{l=1}^{N(t_3-s_3)} D_{3,1}G_{N-1}\big(Ns + N(t_1 - s_1)e_1 + N(t_2 - s_2)e_2 + (l-1)e_3, Nt\big)$$

$$\overset{\text{Lemma 2.3(2)}}{\leq} C\big(N(t_1 - s_1) + N(t_2 - s_2) + N(t_3 - s_3)\big) \leq CN\|t - s\|.$$

Hence (2.2) follows. $\square$

2.2.2. *Finite dimensional convergence.* The main content of this subsubsection is to show

PROPOSITION 2.8. *With the notation of Theorem 2.1, for all $s, t \in \overline{V}$,*

$$\lim_{N \to \infty} \mathrm{Cov}\big(\Psi_N(t), \Psi_N(s)\big) = \mathrm{Cov}\big(\Psi(t), \Psi(s)\big).$$

PROOF. To show the finite dimensional convergence, we use Corollary 1.4 of Müller and Schweiger (2019) (in their setting the domain was $(0, 1)^d$ but the result works for V as well). We observe that for $h := 1/N$, one has $G_{N-1}(x, y) = 4d^2h^{d-4}G_h(hx, hy)$ where G_h satisfies for $x \in \mathrm{int}(V_h)$ with $V_h = [-1, 1]^d \cap h\mathbb{Z}^d$ the following boundary value problem (Δ_h is defined in the Appendix):

$$\begin{cases} \Delta_h^2 G_h(x, y) = \dfrac{1}{h^d}\delta_x(y) & y \in \mathrm{int}(V_h), \\ G_h(x, y) = 0 & y \notin \mathrm{int}(V_h). \end{cases}$$

Let Ψ be the Gaussian process on $\overline{V}$ such that $\mathbf{E}[\Psi(t)\Psi(s)] = G_V(t,s)$ for all $t,s \in \overline{V}$, where G_V is the Green's function for the biharmonic equation with homogeneous Dirichlet boundary conditions (it will be a byproduct of this proof that such a process exists). First, we consider $d = 2$. For $t \in \overline{V}$, we have

$$\Psi_N(t) = \Psi_{N,1}(t) + \Psi_{N,2}(t),$$

where $\Psi_{N,1}(t) = \frac{\kappa}{N}\varphi_{\lfloor Nt \rfloor}$ and

$$\Psi_{N,2}(t) = \frac{\kappa}{N} \sum_{i,j \in \{1,2\}, i \neq j} \mathbb{1}_{(\{Nt_i\} \geq \{Nt_j\})}(t)\big[\{Nt_i\}(\varphi_{\lfloor Nt \rfloor + e_i} - \varphi_{\lfloor Nt \rfloor})$$
$$+ \{Nt_j\}(\varphi_{\lfloor Nt \rfloor + e_i + e_j} - \varphi_{\lfloor Nt \rfloor + e_i})\big].$$

Then using Fact 2.4, we have $\mathbf{Var}(\Psi_{N,2}(t)) \leq C(\log N)N^{-2}$, and hence $\Psi_{N,2}(t)$ converges to zero in probability as N tends to infinity.

Again if $t \in V$, then

$$\mathbf{Var}(\Psi_{N,1}(t)) = \frac{\kappa^2}{N^2} G_{N-1}(\lfloor Nt \rfloor, \lfloor Nt \rfloor) = G_h(h\lfloor Nt \rfloor, h\lfloor Nt \rfloor)$$

and $G_h(h\lfloor Nt \rfloor, h\lfloor Nt \rfloor)$ converges to $G_V(t,t)$ by Corollary 1.4 of Müller and Schweiger (2019). Also if $t \in \partial V$, then $\mathbf{Var}(\Psi_{N,1}(t)) = 0 = G_V(t,t)$. Hence $\Psi_N(t) \xrightarrow{d} \Psi(t)$.

Similarly, one can show using Lemma 2.3, Fact 2.4 and Müller and Schweiger ((2019), Corollary 1.4) that for any $t,s \in \overline{V}$,

$$\mathrm{Cov}(\Psi_N(t), \Psi_N(s)) \to \mathrm{Cov}(\Psi(t), \Psi(s)).$$

Since these variables under consideration are Gaussian, the finite dimensional follows from the convergence of the covariance.

In $d = 3$, for $t \in \overline{V}$, we have

$$\Psi_N(t) = \frac{\kappa}{\sqrt{N}}\varphi_{\lfloor Nt \rfloor} + \frac{\kappa}{\sqrt{N}} \sum_{i,j,k \in \{1,2,3\}, \text{pairwise different}} \mathbb{1}_{(\{Nt_i\} \geq \{Nt_j\} \geq \{Nt_k\})}(t)$$
$$\times \big[\{Nt_i\}(\varphi_{\lfloor Nt \rfloor + e_i} - \varphi_{\lfloor Nt \rfloor}) + \{Nt_j\}(\varphi_{\lfloor Nt \rfloor + e_i + e_j} - \varphi_{\lfloor Nt \rfloor + e_i})$$
$$+ \{Nt_k\}(\varphi_{\lfloor Nt \rfloor + e_i + e_j + e_k} - \varphi_{\lfloor Nt \rfloor + e_i + e_j})\big]$$
$$=: \Psi_{N,1}(t) + \Psi_{N,2}(t).$$

By means of Fact 2.4, we have $\mathbf{Var}(\Psi_{N,2}(t)) \leq C/N$, and hence $\Psi_{N,2}(t)$ converges to zero in probability as $N \to \infty$. The rest of the proof is the same as $d = 2$ and follows from Corollary 1.4 of Müller and Schweiger (2019). $\square$

3. Convergence of finite volume measure in $d \geq 4$. In this section, D denotes a bounded domain in $\mathbb{R}^d$, $d \geq 4$, with smooth boundary.

REMARK 3.1 (Regularity of the boundary of the domain). In what follows, the assumption of smoothness of the boundary is required to obtain asymptotics of the eigenvalues of the biharmonic operator (cf. Proposition 3.8).

3.1. *Description of the limiting field.*

3.1.1. *Spectral theory for the biharmonic operator.* Let $C_c^\infty(D)$ denote the space of infinitely differentiable functions $u : D \to \mathbb{R}$ with compact support inside D. For $\alpha = (\alpha_1, \ldots, \alpha_d)$ a multi-index define

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u.$$

Suppose $f, g \in L_{\text{loc}}^1(D)$. One says that g is the αth weak partial derivative of f (written $D^\alpha f = g$) if

$$\int_D f \, D^\alpha u \, dx = (-1)^{|\alpha|} \int_D gu \, dx \quad \forall u \in C_c^\infty(D).$$

The Sobolev space $W^{k,p}$ is defined in the usual way as

$$W^{k,p} = \{ f \in L_{\text{loc}}^1(D) : D^\alpha f \in L^p(D), |\alpha| \leq k \}.$$

Denote by $H^k(D) := W^{k,2}(D)$, $k = 0, 1, \ldots$, which is a Hilbert space with norm

$$\|f\|_{H^k(D)} = \left(\sum_{|\alpha| \leq k} \int_D |D^\alpha f|^2 \, dx \right)^{1/2}.$$

It is true that if $a > b$ then $H^a(D) \subset H^b(D)$. Let us define another Hilbert space,

$$H_0^k(D) := \overline{C_c^\infty(D)}^{\|\cdot\|_{H^k(D)}}$$

and let $H^{-k}(D) = [H_0^k(D)]^*$ be its dual. In this section, we will use round brackets $(\cdot, \cdot)$ to denote the action of a dual Hilbert space on the original space, and $\langle \cdot, \cdot \rangle$ for inner products. We consider the inner product

$$\langle u, v \rangle_{H_0^2} = \int_D \Delta u \, \Delta v \, dx$$

which induces a norm on $H_0^2(D)$ equivalent to the standard Sobolev norm (Gazzola, Grunau and Sweers (2010), Corollary 2.29). We always consider $H_0^2(D)$ with this norm.

We review briefly the spectral theory for the biharmonic operator as it helps us to give an explicit construction of the continuum bi-Laplacian field. We have the following theorem, which basically says that we can construct an operator B being the inverse of the bi-Laplacian (see also Remark 3.10).

THEOREM 3.2.   *There exists a bounded linear isometry*

$$B_0 : H^{-2}(D) \to H_0^2(D)$$

such that, for all $f \in H^{-2}(D)$ and for all $v \in H_0^2(D)$,

$$(f, v) = \langle v, B_0 f \rangle_{H_0^2}.$$

Moreover, the restriction B on $L^2(D)$ of the operator $i \circ B_0 : H^{-2}(D) \to L^2(D)$ is a compact and self-adjoint operator, where $i : H_0^2(D) \hookrightarrow L^2(D)$ is the inclusion map.

PROOF.   Fix $f \in H^{-2}(D)$. By the Riesz representation theorem, there exists a unique $u_f \in H_0^2(D)$ such that for all $v \in H_0^2(D)$

$$(f, v) = \langle v, u_f \rangle_{H_0^2}.$$

We define $B_0 f := u_f$. Then by definition B_0 is a bounded linear isometry and for all $v \in H_0^2(D)$

$$(f, v) = \langle v, B_0 f \rangle_{H_0^2}.$$

We have $H_0^2(D) \hookrightarrow H_0^1(D) \hookrightarrow L^2(D)$ and the second embedding is compact. So $i : H_0^2(D) \hookrightarrow L^2(D)$ is compact, and hence the operator $i \circ B_0 : H^{-2}(D) \to L^2(D)$ is compact. This implies that the restriction B is compact. B is self-adjoint as for any $f, g \in L^2(D)$,

$$\langle Bf, g \rangle_{L^2} = (g, Bf) = \langle Bf, Bg \rangle_{H_0^2} = (f, Bg) = \langle f, Bg \rangle_{L^2}. \qquad \square$$

Consequently, we can find now an orthonormal basis of elements of $H_0^2(D)$, as the next theorem shows.

THEOREM 3.3.   *There exist $u_1, u_2, \ldots$ in $H_0^2(D)$ and numbers*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$$

such that:

- $\{u_j\}_{j \in \mathbb{N}}$ *is an orthonormal basis for $L^2(D)$*,
- $Bu_j = \lambda_j^{-1} u_j$, *where B is as in Theorem* 3.2,
- $\langle u_j, v \rangle_{H_0^2} = \lambda_j \langle u_j, v \rangle_{L^2}$ *for all $v \in H_0^2(D)$*,
- $\{\lambda_j^{-1/2} u_j\}$ *is an orthonormal basis for $H_0^2(D)$*.

PROOF.   By the spectral theorem for compact self-adjoint operators, we get an orthonormal basis of $L^2(D)$ consisting of eigenvectors of B with $Bu_j = \widetilde{\lambda}_j u_j$ and eigenvalues $\widetilde{\lambda}_j \to 0$. Note that for any $f \in L^2(D)$, $Bf = 0$ implies that

$$\langle v, f \rangle_{L^2} = \langle v, Bf \rangle_{H_0^2} = 0 \quad \forall v \in H_0^2(D),$$

and hence $\langle g, f \rangle_{L^2} = 0$ for all $g \in L^2(D)$ (since $H_0^2(D)$ is dense in $L^2(D)$) and so $f \equiv 0$. Thus 0 is not an eigenvalue of B and we have for any $j \in \mathbb{N}$

$$u_j = \frac{1}{\widetilde{\lambda}_j} B u_j = B \frac{u_j}{\widetilde{\lambda}_j} \in \text{Range}(B) \subset H_0^2(D).$$

Hence $u_j \in H_0^2(D)$. Now observe that, for any $j \in \mathbb{N}$, $\widetilde{\lambda}_j \langle u_j, v \rangle_{H_0^2} = \langle Bu_j, v \rangle_{H_0^2} = \langle u_j, v \rangle_{L^2}$ for all $v \in H_0^2(D)$. So this gives

$$\widetilde{\lambda}_j \langle u_j, u_j \rangle_{H_0^2} = \|u_j\|_{L^2} = 1.$$

But $\langle u_j, u_j \rangle_{H_0^2} > 0$, and hence $\widetilde{\lambda}_j > 0$ for all $j \in \mathbb{N}$. We define $\lambda_j := 1/\widetilde{\lambda}_j$. So we can conclude

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

Moreover, $Bu_j = \lambda_j^{-1} u_j$ and

$$(3.1) \qquad \langle u_j, v \rangle_{H_0^2} = \lambda_j \langle u_j, v \rangle_{L^2} \quad \forall v \in H_0^2(D).$$

We now show that $\{\lambda_j^{-1/2} u_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for $H_0^2(D)$. Indeed we have

$$\langle \lambda_j^{-\frac{1}{2}} u_j, \lambda_k^{-\frac{1}{2}} u_k \rangle_{H_0^2} = \lambda_j^{-\frac{1}{2}} \lambda_k^{-\frac{1}{2}} \langle u_j, u_k \rangle_{H_0^2}$$

$$= \lambda_j^{\frac{1}{2}} \lambda_k^{-\frac{1}{2}} \langle u_j, u_k \rangle_{L^2} = \delta_{jk}.$$

So $\{\lambda_j^{-1/2} u_j\}$ is an orthonormal system. But for any $v \in H_0^2(D)$, $\langle u_j, v \rangle_{H_0^2} = 0$ for all j implies that $\langle u_j, v \rangle_{L_2} = 0$ for all j which in turn implies $v = 0$. This completes the proof. $\square$

COROLLARY 3.4. *For each $j \in \mathbb{N}$ one has $u_j \in C^\infty(D)$. Moreover, u_j is an eigenfunction of Δ^2 with eigenvalue λ_j.*

PROOF. We have for all $v \in H_0^2(D)$:

$$\langle \Delta^2 u_j, v \rangle_{L^2} \overset{\text{GI}}{=} \langle u_j, v \rangle_{H_0^2} \overset{\text{Theorem 3.3}}{=} \lambda_j \langle u_j, v \rangle_{L^2},$$

where "GI" stands for Green's first identity

$$\int_D u \Delta v \, dV = - \int_D \nabla u \cdot \nabla v \, dV + \int_{\partial D} u \nabla v \cdot \mathbf{n} \, dS.$$

Thus u_j is an eigenfunction of Δ^2 with eigenvalue λ_j in the weak sense. The smoothness of u_j follows from the fact that Δ^2 is an elliptic operator with smooth coefficients and the elliptic regularity theorem (Folland (1999), Theorem 9.26). Hence u_j is an eigenfunction of Δ^2 with eigenvalue λ_j. $\square$

REMARK 3.5.   As a consequence of the above, one easily has that

$$\|f\|_{H_0^2}^2 = \sum_{j \geq 1} \lambda_j \langle f, u_j \rangle_{L^2}^2$$

for any $f \in H_0^2(D)$.

We conclude this subsection with some bounds for the derivatives of the eigen-functions u_j of Theorem 3.3.

LEMMA 3.6.   *The following bounds hold*:

(3.2)
$$\sup_{x \in \overline{D}} |u_j(x)| \leq C \lambda_j^{l_0},$$

(3.3)
$$\sum_{|\alpha| \leq 2} \sup_{x \in \overline{D}} |D^\alpha u_j(x)| \leq C \lambda_j^{l_2},$$

(3.4)
$$\sum_{|\alpha| \leq 5} \sup_{x \in \overline{D}} |D^\alpha u_j(x)| \leq C \lambda_j^{l_5},$$

where

$$l_m := \left\lceil \frac{1}{4} \left(\left\lfloor \frac{d}{2} \right\rfloor + m + 1 \right) \right\rceil, \quad m = 0, 2, 5.$$

PROOF.   Taking $l_0 = \lceil 1/4(\lfloor d/2 \rfloor + 1) \rceil$ we obtain from Evans ((2010), Chapter 5, Theorem 6(ii)) that $\sup_{x \in \overline{D}} |u_j(x)| \leq C \|u_j\|_{H^{4l_0}(D)}$. Now a repeated application of Gazzola, Grunau and Sweers ((2010), Corollary 2.21) gives

$$\sup_{x \in \overline{D}} |u_j(x)| \leq C \|u_j\|_{H^{4l_0}(D)} \leq C \lambda_j \|u_j\|_{H^{4l_0-4}(D)} \leq \cdots \leq C \lambda_j^{l_0}.$$

The other two bounds are obtained similarly. We make a passing remark that the smoothness of the boundary is needed in the results quoted above.   □

3.1.2. *Definition of the limiting field via Wiener series.*   For any $v \in C_c^\infty(D)$ and for any $s > 0$, we define

$$\|v\|_s^2 := \sum_{j \in \mathbb{N}} \lambda_j^{s/2} \langle v, u_j \rangle_{L^2}^2.$$

We define $\mathcal{H}_0^s(D)$ to be the Hilbert space completion of $C_c^\infty(D)$ with respect to the norm $\| \cdot \|_s$. Then $(\mathcal{H}_0^s(D), \| \cdot \|_s)$ is a Hilbert space for all $s > 0$.

REMARK 3.7.

• Note that for $s = 2$ we have $\mathcal{H}_0^2(D) = H_0^2(D)$ by Remark 3.5.
• $i : \mathcal{H}_0^s(D) \hookrightarrow L^2(D)$ is a continuous embedding.

Dual spaces. For $s > 0$, we define $\mathcal{H}^{-s}(D) = (\mathcal{H}_0^s(D))^*$, the dual space of $\mathcal{H}_0^s(D)$. Then we have

$$\mathcal{H}_0^s(D) \subseteq L^2(D) \subseteq \mathcal{H}^{-s}(D).$$

One can show using the Riesz representation theorem that for $s > 0$ the norm of $\mathcal{H}^{-s}(D)$ is given by

$$\|v\|_{-s}^2 := \sum_{j \in \mathbb{N}} \lambda_j^{-s/2}(v, u_j)^2, \quad v \in \mathcal{H}^{-s}(D).$$

Recall that $(\cdot, \cdot)$ denotes the action of the dual space $\mathcal{H}^{-s}(D)$ on $\mathcal{H}_0^s(D)$. Moreover, for $v \in L^2(D)$ we have

$$\|v\|_{-s}^2 := \sum_{j \in \mathbb{N}} \lambda_j^{-s/2} \langle v, u_j \rangle_{L^2}^2.$$

Before we show the definition of the continuum membrane model, we need an analog of Weyl's law for the eigenvalues of the biharmonic operator.

PROPOSITION 3.8 (Beals (1967), Theorem 5.1, Pleijel (1950)). *There exists an explicit constant c such that, as $j \uparrow +\infty$,*

$$\lambda_j \sim c^{-d/4} j^{4/d}.$$

The result we will prove now shows the well-posedness of the series expansion for ψ_D.

PROPOSITION 3.9. *Let $(\xi_j)_{j \in \mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\psi_D := \sum_{j \in \mathbb{N}} \lambda_j^{-1/2} \xi_j u_j.$$

Then $\psi_D \in \mathcal{H}^{-s}(D)$ a.s. for all $s > (d-4)/2$.

PROOF. Fix $s > (d-4)/2$. Clearly, $u_j \in L^2(D) \subseteq \mathcal{H}^{-s}(D)$. We need to show that $\|\psi_D\|_{-s} < +\infty$ almost surely. Now this boils down to showing the finiteness of the random series

$$\|\psi_D\|_{-s}^2 = \sum_{j \geq 1} \lambda_j^{-s/2} \left(\sum_{k \geq 1} \lambda_k^{-1/2} u_k \xi_k, u_j \right)^2 = \sum_{j \geq 1} \lambda_j^{-\frac{s}{2}-1} \xi_j^2,$$

where the last equality is true since $(u_j)_{j \geq 1}$ form an orthonormal basis of $L^2(D)$. Observe that the assumptions of Kolmogorov's two-series theorem are satisfied: indeed using Proposition 3.8 one has

$$\sum_{j \geq 1} \mathbf{E}(\lambda_j^{-\frac{s}{2}-1} \xi_j^2) \asymp c \sum_{j \geq 1} j^{-\frac{4}{d}(\frac{s}{2}+1)} < +\infty$$

for $s > (d-4)/2$ and

$$\sum_{j \geq 1} \mathbf{Var}(\lambda_j^{-\frac{s}{2}-1} \xi_j^2) \asymp c \sum_{j \geq 1} j^{-\frac{4}{d}(s+2)} < +\infty$$

for $s > (d-8)/4$. The result then follows. $\quad\square$

3.2. *Definition of the limiting field via abstract Wiener spaces.* We want now to connect the series representation given in Proposition 3.9 with an equivalent characterisation of ψ_D. This alternative definition can be given through the theory of abstract Wiener space (AWS). For a comprehensive overview of the theory, we refer the readers to Stroock (2011), for example. For our purposes it will suffice to recall that an abstract Wiener space is a triple $(\Theta, H, \mathcal{W})$, where:

- Θ is a separable Banach space,
- H is a Hilbert space which is continuously embedded as a dense subspace of Θ, equipped with the scalar product $\langle \cdot, \cdot \rangle_H$,
- $\mathcal{W}$ is a Gaussian probability measure on Θ defined as follows.

Let Θ^* be the dual space of Θ. Given any $x^* \in \Theta^*$ there exists a unique $h_{x^*} \in H$ such that for all $h \in H$, $(h, x^*) = \langle h, h_{x^*} \rangle_H$ where $(\cdot, x^*)$ denotes the action of x^* on Θ. The σ-algebra $\mathcal{B}(\Theta)$ on Θ is such that all the maps $\theta \mapsto (\theta, x^*)$ are measurable. $\mathcal{W}$ is a probability measure such that, for all $x^* \in \Theta^*$,

$$(3.5) \qquad \mathsf{E}_{\mathcal{W}}[\exp(\iota(\cdot, x^*))] = \exp\left(-\frac{\|h_{x^*}\|_H^2}{2}\right).$$

In other words, the variable $(\cdot, x^*)$ under $\mathcal{W}$ is a centered Gaussian with variance $\|h_{x^*}\|_H^2$. Next, we introduce the *Paley–Wiener map* $\mathcal{I}$. $\mathcal{I}$ is viewed as a mapping

$$\begin{aligned} \mathcal{I} : h_{x^*} \in H &\quad\mapsto\quad \mathcal{I}(h_{x^*}) \in L^2(\mathcal{W}), \\ \theta \in \Theta &\quad\mapsto\quad [\mathcal{I}(h_{x^*})](\theta) := (\theta, x^*). \end{aligned}$$

Since $\{h_{x^*} : x^* \in \Theta^*\}$ is dense in H, the map $h_{x^*} \mapsto I(h_{x^*})$ can be uniquely extended as a linear isometry from H to $L^2(\mathcal{W})$. Stroock ((2011), Theorem 8.2.6) yields that the family of Paley–Wiener integrals $\{\mathcal{I}(h) : h \in H\}$ is Gaussian, where each $\mathcal{I}(h)$ has mean zero and variance $\|h\|_H^2$. Given (3.5) the family $\{\mathcal{I}(u_j) : \{u_j\}_{j \in \mathbb{N}}$ orthonormal basis of $H\}$ is formed by i.i.d. standard Gaussians.

In our setting, by combining Stroock ((2011), Section 8.3.2) and the Wiener series given in Proposition 3.9, we can take $H := H_0^2(D)$ and $\mathcal{W}$ to be the law of ψ_D on $\Theta := \mathcal{H}^{-s}(D)$, for an arbitrary $s > (d-4)/2$. (the choice of Θ is not unique as explained in Stroock (2011), Corollary 8.3.2). Also by Theorem 3.2 we can index the Paley–Wiener integrals $\mathcal{I}(u)$ over $u \in \mathcal{H}_0^2(D)$ or take the maps $\mathcal{I}(B_0(f))$ over $f \in \mathcal{H}^{-2}(D)$.

REMARK 3.10. By means of integration by parts we obtain, for every $f \in C_c^\infty(D)$, that the solution u_f of the boundary value problem

$$(3.6) \qquad \begin{cases} \Delta^2 u(x) = f(x), & x \in D, \\ D^\beta u(x) = 0, & |\beta| \le 1, x \in \partial D, \end{cases}$$

is such that for all $v \in C_c^\infty(D)$

$$\int_D v(x) f(x) \, dx = \int_D v(x) \Delta^2 u_f(x) \, dx = \langle v, u_f \rangle_{H_0^2}.$$

Using the denseness of $C_c^\infty(D)$ in $H_0^2(D)$, we conclude from Theorem 3.2 that $B_0 f = u_f$. Thus we have

$$\|f\|_{-2}^2 = \int_D u_f(x) f(x) \, dx = \|u_f\|_{H_0^2}^2.$$

3.3. *Discretisation set-up.* We will use the parameter $h := 1/N$ for $N \in \mathbb{N}$. Let $D_h := \overline{D} \cap h\mathbb{Z}^d$. Let us denote by R_h the set of points ξ in D_h such that for every $i, j \in \{1, \ldots, d\}$, the points $\xi \pm h(e_i \pm e_j), \xi \pm he_i$ are all in D_h. Let $\Lambda_N = \frac{1}{h} R_h \subset \mathbb{Z}^d$ be the "blow-up" of R_h. In other words, $\Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d$ is the largest set satisfying $\partial_2 \Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d$ where $\partial_2 \Lambda_N := \{y \in \mathbb{Z}^d \setminus \Lambda_N : \text{dist}(y, \Lambda_N) \le 2\}$ is the double (outer) boundary of Λ_N of points at ℓ^1 distance at most 2 from it. Let $(\varphi_z)_{z \in \Lambda_N}$ be the membrane model on Λ_N whose covariance is denoted by G_{Λ_N}. It satisfies the following boundary value problem: for all $x \in \Lambda_N$,

$$(3.7) \qquad \begin{cases} \Delta_1^2 G_{\Lambda_N}(x, y) = \delta_x(y), & y \in \Lambda_N, \\ G_{\Lambda_N}(x, y) = 0, & y \notin \Lambda_N. \end{cases}$$

Define ψ_h by

$$(3.8) \qquad (\psi_h, f) := \kappa \sum_{x \in R_h} h^{\frac{d+4}{2}} \varphi_{x/h} f(x), \quad f \in \mathcal{H}_0^s(D).$$

We first show that $\psi_h \in \mathcal{H}^{-s}(D)$ for all $s > d/2 + \lfloor d/2 \rfloor + 1$. Clearly, ψ_h is a linear functional on $\mathcal{H}_0^s(D)$. To show ψ_h is bounded, with the aid of Lemma 3.6 we observe that

$$\sum_{j \ge 1} \lambda_j^{-\frac{s}{2}} (\psi_h, u_j)^2 = \kappa^2 h^{d+4} \sum_{j \ge 1} \lambda_j^{-\frac{s}{2}} \left(\sum_{x \in R_h} \varphi_{x/h} u_j(x) \right)^2$$

$$\overset{(3.2)}{\le} \kappa^2 h^{d+4} \left(\sum_{x \in R_h} |\varphi_{x/h}| \right)^2 \sum_{j \ge 1} \lambda_j^{-\frac{s}{2} + 2l_0}.$$

Now using Proposition 3.8 we conclude that the sum in the right-hand side in finite whenever $s > d/2 + \lfloor d/2 \rfloor + 1$. Thus we have shown that $\psi_h \in \mathcal{H}^{-s}(D)$ for all

$s > d/2 + \lfloor d/2 \rfloor + 1$ and we have

$$(3.9) \qquad \|\psi_h\|_{-s}^2 = \sum_{j \geq 1} \lambda_j^{-\frac{s}{2}} (\psi_h, u_j)^2.$$

The result we want to show is the following.

THEOREM 3.11 (Scaling limit in $d \geq 4$).   *One has that, as $h \to 0$, the field ψ_h converges in distribution to ψ_D of Proposition* 3.9 *in the topology of $\mathcal{H}^{-s}(D)$ for $s > s_d$, where*

$$s_d := \frac{d}{2} + 2\left(\left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right)\right\rceil + \left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + 6\right)\right\rceil - 1\right).$$

REMARK 3.12.   An analogous result holds in $d = 2, 3$, but we will not discuss it here as it is superseded by Theorem 2.1.

3.4. *Proof of the scaling limit* (*Theorem* 3.11).   Once again, we need to prove tightness and "convergence of marginal laws." In $d \geq 4$, however, we are concerned with a field which is not defined pointwise, so that "marginal" from now takes on the meaning of the law of (ψ_h, f), namely the action of ψ_h, seen as a distribution, on the test function f. The results are built on the approximation of the continuum Dirichlet problem for the bi-Laplacian by Thomée (1964), combined with classical embeddings for Sobolev spaces.

3.4.1. *Convergence of the marginals.*   To prove that the scaling limit is indeed ψ_D, we first have to find the marginal limiting laws. The set $C_c^\infty(D)$ is dense in $\mathcal{H}_0^s(D)$, so we can use only smooth and compactly supported functions to test the convergence.

PROPOSITION 3.13. (ψ_h, f) *converges in law to* (ψ_D, f) *as $h \to 0$ for any f smooth and compactly supported in D.*

PROOF.   Since the Gaussian field φ is centered, we shall focus on the convergence of the variance only. Note that $\mathbf{Var}(\psi_D, f) = \|f\|_{-2}^2$. Remark 3.10 tells us that we can limit ourselves to showing that

$$\lim_{h \to 0} \mathbf{Var}(\psi_h, f) = \int_D u(x) f(x) \, dx,$$

where u is the solution of (3.6). We define

$$G_{R_h}(x, y) := \mathbf{E}[\varphi_{x/h} \varphi_{y/h}], \quad x, y \in D_h.$$

Note that if Δ_h (defined in the Appendix) is the discrete Laplacian on $h\mathbb{Z}^d$ then by (3.7) we have, for all $x \in R_h$,

$$
\begin{cases}
\Delta_h^2 G_{R_h}(x, y) = \dfrac{4d^2}{h^4}\delta_x(y), & y \in R_h, \\
G_{R_h}(x, y) = 0, & y \notin R_h.
\end{cases}
$$

We have

$$
\mathbf{Var}\big[(\psi_h, f)\big] = \kappa^2 \sum_{x,y\in R_h} h^{d+4} G_{R_h}(x, y) f(x) f(y)
$$
$$
= \sum_{x\in R_h} h^d H_h(x) f(x),
$$

where $H_h(x) = \kappa^2 \sum_{y\in R_h} h^4 G_{R_h}(x, y) f(y)$, $x \in D_h$. It is immediate that H_h is the solution of the following Dirichlet problem:

$$
\begin{cases}
\Delta_h^2 H_h(x) = f(x), & x \in R_h, \\
H_h(x) = 0, & x \notin R_h.
\end{cases}
$$

It is known that the above discrete solution is close to the continuum solution. The details of the result are described in the Appendix; here, we only recall that if we define $e_h(x) := u(x) - H_h(x)$ for $x \in D_h$ and $R_h f$ is the restriction of a function f to the set R_h as in (A.2), then from Theorem A.12 we have

$$
(3.10) \qquad \|R_h e_h\|_{h,\text{grid}} \leq Ch^{1/2}.
$$

We have defined $\|f\|_{h,\text{grid}}^2 := h^d \sum_{\xi\in h\mathbb{Z}^d} f(\xi)^2$, where f is any grid function with finite support. Hence we get that

$$
\mathbf{Var}\big[(\psi_h, f)\big] = - \sum_{x\in R_h} e_h(x) f(x) h^d + \sum_{x\in R_h} u(x) f(x) h^d.
$$

Note that by Cauchy–Schwarz the first term in absolute value is bounded by $\|R_h e_h\|_{h,\text{grid}}\|f\|_{h,\text{grid}}$ and it goes to zero by (3.10) as $h \to 0$. For the second term, we have

$$
(3.11) \qquad \lim_{h\to 0} \sum_{x\in R_h} u(x) f(x) h^d = \int_D u(x) f(x)\, dx. \qquad \square
$$

3.4.2. *Tightness.* We next prove the following lemma.

LEMMA 3.14.

$$
\limsup_{h\to 0} \mathbf{E}\big[\|\psi_h\|_{-s}^2\big] < \infty \quad \forall s > s_d.
$$

PROOF.   From (3.9), we have

$$\mathbf{E}\big[\|\psi_h\|_{-s}^2\big] = \sum_{j \in \mathbb{N}} \lambda_j^{-s/2} \mathbf{E}\big[(\psi_h, u_j)^2\big].$$

Note that $u = \lambda_j^{-1} u_j$ is the unique solution of (3.6) for $f = u_j$. We therefore obtain as in the proof of Proposition 3.13 by defining $e_{h,j}$ to be the error corresponding to $f = u_j$,

$$\mathbf{E}\big[(\psi_h, u_j)^2\big] = -\sum_{x \in R_h} e_{h,j}(x) u_j(x) h^d + \sum_{x \in R_h} \lambda_j^{-1} u_j(x) u_j(x) h^d$$

$$\leq C \sup_{x \in D} |u_j(x)| \bigg(h^d \sum_{x \in R_h} e_{h,j}(x)^2\bigg)^{1/2} + C \lambda_j^{-1} \bigg(\sup_{x \in D} |u_j(x)|\bigg)^2.$$

Using Theorem A.12 along with the bounds (3.2), (3.3), (3.4), we obtain

$$\mathbf{E}\big[(\psi_h, u_j)^2\big] \leq C \lambda_j^{l_0} \big[\lambda_j^{2l_5 - 2} h^2 + h(\lambda_j^{2l_5 - 2} h^6 + \lambda_j^{2l_2 - 2})\big]^{\frac{1}{2}} + C \lambda_j^{2l_0 - 1}$$

$$\leq C \lambda_j^{l_0 + l_5 - 1}.$$

Therefore, we have

$$\mathbf{E}\big[\|\psi_h\|_{-s}^2\big] \leq C \sum_{j \in \mathbb{N}} \lambda_j^{-\frac{s}{2}} \lambda_j^{l_0 + l_5 - 1}.$$

Thus

$$\limsup_{h \to 0} \mathbf{E}\big[\|\psi_h\|_{-s}^2\big] < \infty \quad \text{if} \quad \sum_{j \in \mathbb{N}} \lambda_j^{-\frac{s}{2} + l_0 + l_5 - 1} < \infty.$$

And from Proposition 3.8, we obtain that $\sum_{j \in \mathbb{N}} \lambda_j^{-\frac{s}{2} + l_0 + l_5 - 1} < \infty$ whenever $s > s_d$. $\square$

To show tightness of ψ_h, we need the following theorem.

THEOREM 3.15.   *For $0 \leq s_1 < s_2$, $\mathcal{H}^{-s_1}(D)$ is compactly embedded in $\mathcal{H}^{-s_2}(D)$.*

PROOF.   It is enough to prove that $\mathcal{H}_0^{s_2}(D)$ is compactly embedded in $\mathcal{H}_0^{s_1}(D)$. The inclusion $\mathcal{H}_0^{s_2}(D) \hookrightarrow \mathcal{H}_0^{s_1}(D)$ is linear and continuous. To prove the inclusion to be compact, let B be the unit ball of $\mathcal{H}_0^{s_2}(D)$. Given $\epsilon > 0$, we choose $N \in \mathbb{N}$ large enough so that $N^{s_1 - s_2} < \epsilon^4$. Now we consider the subspace Z of $\mathcal{H}_0^{s_2}(D)$ defined by $Z := \{f \in \mathcal{H}_0^{s_2}(D) : \langle f, u_j \rangle_{L^2} = 0 \; \forall j < N\}$. Then for any $f \in B \cap Z$,

we have

$$
\begin{aligned}
\|f\|_{s_1}^2 &= \sum_{j \in \mathbb{N}} \lambda_j^{s_1/2} \langle f, u_j \rangle_{L^2}^2 \\
&= \sum_{j \geq N} \lambda_j^{s_1/2} \langle f, u_j \rangle_{L^2}^2 \\
&= \sum_{j \geq N} \lambda_j^{s_1/2 - s_2/2} \lambda_j^{s_2/2} \langle f, u_j \rangle_{L^2}^2 \\
&\leq N^{(s_1 - s_2)/2} \sum_{j \geq N} \lambda_j^{s_2/2} \langle f, u_j \rangle_{L^2}^2 \\
&= N^{(s_1 - s_2)/2} \|f\|_{s_2}^2 < \epsilon^2.
\end{aligned}
$$

Also note that the dimension of $\mathcal{H}_0^{s_2}(D)/Z$ is finite, so the unit ball of $\mathcal{H}_0^{s_2}(D)/Z$ is compact, and hence can be covered by finitely many balls of radius ϵ. Hence B can be covered by finitely many balls of radius 2ϵ in the $\|\cdot\|_{s_1}$-norm. Since ϵ is arbitrary, B is precompact in $\mathcal{H}_0^{s_1}(D)$. Therefore, the inclusion map is compact. $\square$

COROLLARY 3.16. *The sequence $(\psi_h)_{h = \frac{1}{N}, N \in \mathbb{N}}$ is tight in $\mathcal{H}^{-s}(D)$ for all $s > s_d$.*

PROOF. Fix $s_0 > s_d$ and let $s_d < s_1 < s_0$. By Theorem 3.15, for any $R > 0$, $\overline{B_{\mathcal{H}^{-s_1}(D)}(0, R)}$ is compact in $\mathcal{H}^{-s_0}(D)$. By Lemma 3.14, we have for some $M > 0$,

$$
\mathbf{E}\big[\|\psi_h\|_{-s_1}^2\big] \leq M \quad \forall h.
$$

Given $\epsilon > 0$, we take $R = \sqrt{2M\epsilon^{-1}}$ so that $MR^{-2} < \epsilon$. Now for all h,

$$
\mathbf{P}\big(\psi_h \notin \overline{B_{\mathcal{H}^{-s_1}(D)}(0, R)}\big) = \mathbf{P}\big(\|\psi_h\|_{-s_1} > R\big) \leq \frac{\mathbf{E}\big[\|\psi_h\|_{-s_1}^2\big]}{R^2} < \epsilon.
$$

Thus $(\psi_h)_h$ is tight in $\mathcal{H}^{-s_0}(D)$. $\square$

Having obtained tightness and convergence of the marginals, all is left to do is to combine these ideas together to show the scaling limit.

PROOF OF THEOREM 3.11. As (ψ_h) is tight in $\mathcal{H}^{-s}(D)$, it is enough to prove that every converging subsequence (ψ_{h_i}) converges in distribution to ψ_D. Let (ψ_{h_i}) be a subsequence of (ψ_h) converging in distribution to ψ in $\mathcal{H}^{-s}(D)$. Then (ψ_{h_i}, f) converges in distribution to (ψ, f) for any $f \in \mathcal{H}_0^s(D)$. But since (ψ_h, f) converges in distribution to (ψ_D, f) for all $f \in C_c^\infty(D)$, we must have $(\psi_D, f) \stackrel{d}{=} (\psi, f)$ for all $f \in C_c^\infty(D)$. Now let $g \in \mathcal{H}_0^s(D)$. Since $C_c^\infty(D)$ is dense

in $\mathcal{H}_0^s(D)$, we have a sequence (f_k) in $C_c^\infty(D)$ such that $f_k \to g$ in $\mathcal{H}_0^s(D)$. Therefore, (ψ_D, f_k) and (ψ, f_k) converge to (ψ_D, g) and (ψ, g), respectively, and hence (ψ_D, f_k) and (ψ, f_k) converge in distribution to (ψ_D, g) and (ψ, g), respectively. But since $(\psi_D, f_k) \stackrel{d}{=} (\psi, f_k)$ for all k, we have $(\psi_D, g) \stackrel{d}{=} (\psi, g)$. Thus we have $(\psi_D, f) \stackrel{d}{=} (\psi, f)$ for all $f \in \mathcal{H}_0^s(D)$. Hence $\psi_D \stackrel{d}{=} \psi$, since the fields under considerations are linear. $\square$

4. Convergence in infinite volume in $d \geq 5$.

4.1. *Description of the limiting field.*   In this section, we deal with the infinite volume membrane model defined on the whole of $\mathbb{Z}^d$ and show that the rescaled field converges to the continuum bi-Laplacian field on $\mathbb{R}^d$. Let $\mathbf{P}_N$ be the finite volume MM measure defined on V_N as mentioned in the Introduction. It is known that in $d \geq 5$ there exists $\mathbf{P}$ on $\mathbb{R}^{\mathbb{Z}^d}$ such that $\mathbf{P}_N \to \mathbf{P}$ in the weak topology of probability measures (Kurt (2009a), Proposition 1.2.3). Under $\mathbf{P}$, the canonical coordinates $(\varphi_x)_{x \in \mathbb{Z}^d}$ form a centered Gaussian process with covariance given by

$$G(x, y) = \Delta^{-2}(x, y) = \sum_{z \in \mathbb{Z}^d} \Delta^{-1}(x, z)\Delta^{-1}(z, y) = \sum_{z \in \mathbb{Z}^d} \Gamma(x, z)\Gamma(z, y),$$

where Γ denotes the covariance of the DGFF. Γ has an easy representation in terms of the simple random walk $(S_n)_{n \geq 0}$ on $\mathbb{Z}^d$ given by

$$\Gamma(x, y) = \sum_{m \geq 0} \mathrm{P}_x[S_m = y]$$

(P_x is the law of S starting at x). This entails that

$$(4.1) \qquad G(x, y) = \sum_{m \geq 0} (m + 1)\mathrm{P}_x[S_m = y] = \mathrm{E}_{x,y}\left[\sum_{\ell,m=0}^{+\infty} \mathbb{1}_{\{S_m = \tilde{S}_\ell\}} \right],$$

where S and $\tilde{S}$ are two independent simple random walks started at x and y, respectively. First, one can note from this representation that $G(\cdot, \cdot)$ is translation invariant. The existence of the infinite volume measure in $d \geq 5$ gives that $G(0, 0) < +\infty$. Using the above, one can derive the following property of the covariance:

FACT 4.1 (Sakagawa (2003), Lemma 5.1).

$$(4.2) \qquad \lim_{\|x\| \to +\infty} \frac{G(x, 0)}{\|x\|^{4-d}} = \eta_2,$$

where

$$\eta_2 = (2\pi)^{-d} \int_0^{+\infty} \int_{\mathbb{R}^d} \exp\left(\iota\langle \zeta, \theta \rangle - \frac{\|\theta\|^4 t}{4\pi^2} \right) d\theta \, dt$$

for any $\zeta \in \mathbb{S}^{d-1}$.

It is convenient to consider the convergence in the space of tempered distribution (dual of the Schwartz space on $\mathbb{R}^d$). For this, we are giving some preliminary theoretical results.

4.1.1. *Generalized random fields and limiting field.* We consider $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ to be the Schwartz space that consists of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that, for all $m \in \mathbb{N} \cup \{0\}$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$,

$$\|f\|_{m,\alpha} = \sup_{x \in \mathbb{R}^d} (1 + \|x\|^m) |D^\alpha f(x)| < \infty.$$

$\mathcal{S}$ is a linear vector space and it is equipped with the topology generated by the family of seminorms $\|\cdot\|_{m,\alpha}$, $m \in \mathbb{N} \cup \{0\}$ and $\alpha \in (\mathbb{N} \cup \{0\})^d$. The topological dual $\mathcal{S}^*$ of $\mathcal{S}$ is called the space of tempered distributions. For $F \in \mathcal{S}^*$ and $f \in \mathcal{S}$, we denote $F(f)$ by (F, f). We shall work with two topologies on $\mathcal{S}^*$, the strong topology τ_s and the weak topology τ_w. The strong topology τ_s is generated by the family of seminorms $\{e_B : B$ is a bounded subset of $\mathcal{S}\}$ where $e_B(F) = \sup_{f \in B}(F, f)$, $F \in \mathcal{S}^*$. τ_w is induced by the family of seminorms $\{|(\cdot, f)| : f \in \mathcal{S}\}$. In particular, F_n converges to F in $\mathcal{S}^*$ with respect to the weak topology when $\lim_n (F_n, f) = (F, f)$ for all $f \in \mathcal{S}$. It can be shown that the Borel σ-fields corresponding to both topologies coincide. Therefore, we shall talk about the Borel σ-field $\mathcal{B}(\mathcal{S}^*)$ of $\mathcal{S}^*$ without specifying the topology.

Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space. By a generalized random field defined on $(\Omega, \mathcal{A}, \mathrm{P})$, we refer to a random variable X with values in $(\mathcal{S}^*, \mathcal{B}(\mathcal{S}^*))$. For $(X_n)_{n \geq 1}$ and X generalized random fields with laws $(\mathbf{P}_{X_n})_{n \geq 1}$ and $\mathbf{P}_X$ respectively, we say that X_n converges in distribution to X (and write $X_n \overset{d}{\to} X$) with respect to the strong topology if

$$\lim_{n \to \infty} \int_{\mathcal{S}^*} \varphi(F) \, d\mathbf{P}_{X_n}(F) = \int_{\mathcal{S}^*} \varphi(F) \, d\mathbf{P}_X(F) \quad \forall \varphi \in C_b(\mathcal{S}^*, \tau_s),$$

where $C_b(\mathcal{S}^*, \tau_s)$ is the space of bounded continuous functions on $\mathcal{S}^*$ given the strong topology. The convergence in distribution with respect to the weak topology is defined similarly with test functions in $C_b(\mathcal{S}^*, \tau_w)$. For a generalized random field X with law $\mathbf{P}_X$, we define its characteristic functional by

$$\mathcal{L}_X(f) = \mathbf{E}(e^{\iota(X, f)}) = \int_{\mathcal{S}^*} e^{\iota(F, f)} \, d\mathbf{P}_X(F)$$

for $f \in \mathcal{S}$. Note that $\mathcal{L}_X$ is positive definite, continuous, and $\mathcal{L}_X(0) = 1$. The Bochner–Minlos theorem says that the converse is also true: if a functional $\mathcal{L} : \mathcal{S} \to \mathbb{C}$ is positive definite, continuous at 0 and satisfies $\mathcal{L}(0) = 1$ then there exists a generalized random field X defined on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ such that $\mathcal{L}_X = \mathcal{L}$. For a proof of this theorem, see for instance Hida and Si (2004), Appendix 1. Another important feature of characteristic functions is that their convergence determines convergence of generalized random fields. This is classical

result of Lévy which was generalized and proved in the nuclear space setting first by Fernique (1968). We use the version for tempered distributions which was recently proved in Biermé, Durieu and Wang (2017).

FACT 4.2 (Biermé, Durieu and Wang (2017), Corollary 2.4). Let $(X_n)_{n\geq 1}$, X be generalized random fields. The following conditions are equivalent:

1. $X_n \xrightarrow{d} X$ in the strong topology.
2. $X_n \xrightarrow{d} X$ in the weak topology.
3. $\mathcal{L}_{X_n}(f) \to \mathcal{L}_X(f)$ for all $f \in \mathcal{S}$.
4. $(X_n, f) \xrightarrow{d} (X, f)$ in $\mathbb{R}$ for all $f \in \mathcal{S}$.

For $f \in \mathcal{S}$, we define $\widehat{f} \in \mathcal{S}$ by

$$\widehat{f}(\theta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\iota\langle x,\theta\rangle} f(x)\,dx.$$

Let us define an operator $(-\Delta)^{-1} : \mathcal{S} \to L^2(\mathbb{R}^d)$ as follows (Adams and Hedberg (1996), Section 1.2.2):

$$(-\Delta)^{-1} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{\iota\langle x,\xi\rangle} \|\xi\|^{-2} \widehat{f}(\xi)\,d\xi.$$

We use now the operator $(-\Delta)^{-1}$ to define the limiting field ψ. It is the fractional Gaussian field of parameter $s := 2$ described in Lodhia et al. ((2016), Section 3.1), to which we refer for a proof of the following fact, relying on the Bochner–Minlos theorem.

LEMMA 4.3. *There exists a generalized random field ψ on $\mathcal{S}^*$ whose characteristic functional $\mathcal{L}_\psi$ is given by*

$$\mathcal{L}_\psi(f) = \exp\left(-\frac{1}{2}\|(-\Delta)^{-1} f\|^2_{L^2(\mathbb{R}^d)}\right), \quad f \in \mathcal{S}.$$

Consider $(\varphi_x)_{x\in\mathbb{Z}^d}$ to be the membrane model in $d \geq 5$. We define

$$\psi_N(x) := \kappa N^{\frac{d-4}{2}} \varphi_{Nx}, \quad x \in \frac{1}{N}\mathbb{Z}^d.$$

For $f \in \mathcal{S}$, we define

$$(4.3) \qquad (\psi_N, f) := N^{-d} \sum_{x\in\frac{1}{N}\mathbb{Z}^d} \psi_N(x) f(x).$$

The above definition makes sense since, using Mill's ratio and the uniform boundedness of $G(\cdot, \cdot)$, one can show that, as $\|x\| \to \infty$,

$$|\psi_N(x)| = O\left(N^{\frac{d-4}{2}} \sqrt{\log(1 + \|x\|)}\right) \quad \text{a.s.}$$

via a Borell–Cantelli argument. This justifies (4.3) using the fast decay of f at infinity. Also it follows that $\psi_N \in \mathcal{S}^*$ and the characteristic functional of ψ_N is given by

$$\mathcal{L}_{\psi_N}(f) := \exp\bigl(-\mathbf{Var}(\psi_N, f)/2\bigr).$$

The following theorem shows that the field ψ_N constructed above converges to ψ defined in Lemma 4.3.

THEOREM 4.4 (Scaling limit in $d \geq 5$). *Let $d \geq 5$ and ψ_N be the field on $\mathcal{S}^*$ defined by* (4.3). *Then $\psi_N \xrightarrow{d} \psi$ in the strong topology where ψ is defined in Lemma* 4.3.

4.2. *Proof of the scaling limit* (*Theorem* 4.4). The proof of our last theorem relies on the result recalled in Fact 4.2, therefore, unlike the two previous theorems it is not divided into tightness and finite dimensional convergence. The argument is based on Fourier analysis, and will be a consequence of two claims which we will show after the main proof.

PROOF OF THEOREM 4.4. We first show that for any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathbf{E}\bigl[(\psi_N, f)^2\bigr] \to \bigl\|(-\Delta)^{-1} f\bigr\|_{L^2(\mathbb{R}^d)}^2.$$

By our definition, we have for $f, g \in \mathcal{S}$,

$$\mathrm{Cov}\bigl((\psi_N, f), (\psi_N, g)\bigr) = \kappa^2 N^{-(d+4)} \sum_{x,y \in \frac{1}{N}\mathbb{Z}^d} G\bigl(0, N(y-x)\bigr) f(x) g(y).$$

Hence

$$\mathbf{E}\bigl[(\psi_N, f)^2\bigr] = \kappa^2 N^{-(d+4)} \sum_{x,y \in \frac{1}{N}\mathbb{Z}^d} G\bigl(0, N(y-x)\bigr) f(x) f(y).$$

We deduce from the Fourier inversion formula, in the same fashion of the proofs of Kurt ((2009a), Lemmas 1.2.2, 1.2.3), that

$$G(0, x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \bigl(\mu(\theta)\bigr)^{-2} e^{-\iota\langle x,\theta\rangle} \, d\theta,$$

where $\mu(\theta) = \frac{1}{d} \sum_{i=1}^{d}(1 - \cos(\theta_i)) = \frac{2}{d} \sum_{i=1}^{d} \sin^2(\frac{\theta_i}{2})$. Hence we have

$$\mathbf{E}\bigl[(\psi_N, f)^2\bigr]$$
$$= \frac{\kappa^2 N^{-(d+4)}}{(2\pi)^d} \sum_{x,y \in \frac{1}{N}\mathbb{Z}^d} \int_{[-\pi,\pi]^d} \frac{e^{-\iota\langle N(y-x),\theta\rangle}}{\mu(\theta)^2} f(x) f(y) \, d\theta$$

$$= \frac{\kappa^2 N^{-(d+4)}}{(2\pi)^d} \sum_{x,y \in \frac{1}{N}\mathbb{Z}^d} \int_{[-\pi,\pi]^d} \left(\mu(\theta)\right)^{-2} \mathrm{e}^{-\iota\langle(y-x),N\theta\rangle} f(x) f(y) \, \mathrm{d}\theta$$

$$= \frac{\kappa^2 N^{-4}}{(2\pi)^d} \int_{[-N\pi,N\pi]^d} \left(\mu\left(\frac{\theta}{N}\right)\right)^{-2} \left| N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \mathrm{e}^{-\iota\langle x,\theta\rangle} f(x) \right|^2 \mathrm{d}\theta.$$

We have used in the above Fubini's theorem, justified by the following bound (Cipriani, Hazra and Ruszel (2018), Lemma 7): there exists $C > 0$ such that for all $N \in \mathbb{N}$ and $w \in [-N\pi/2, N\pi/2]^d \setminus \{0\}$ we have

$$(4.4) \qquad \frac{1}{\|w\|^4} \le N^{-4} \left(\sum_{i=1}^{d} \sin^2\left(\frac{w_i}{N}\right)\right)^{-2} \le \left(\frac{1}{\|w\|^2} + \frac{C}{N^2}\right)^2.$$

We make two claims which will prove the convergence of variance.

CLAIM 4.5.

$$\lim_{N \to +\infty} \left| \frac{\kappa^2 N^{-4}}{(2\pi)^d} \int_{[-N\pi,N\pi]^d} \left(\mu\left(\frac{\theta}{N}\right)\right)^{-2} \left| N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \mathrm{e}^{-\iota\langle x,\theta\rangle} f(x) \right|^2 \mathrm{d}\theta \right.$$

$$\left. - \frac{1}{(2\pi)^d} \int_{[-N\pi,N\pi]^d} \|\theta\|^{-4} \left| N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \mathrm{e}^{-\iota\langle x,\theta\rangle} f(x) \right|^2 \mathrm{d}\theta \right| = 0.$$

Next, we claim the convergence of the following term.

CLAIM 4.6.

$$\lim_{N \to +\infty} \frac{1}{(2\pi)^d} \int_{[-N\pi,N\pi]^d} \|\theta\|^{-4} \left| N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \mathrm{e}^{-\iota\langle x,\theta\rangle} f(x) \right|^2 \mathrm{d}\theta$$

$$= \left\| (-\Delta)^{-1} f \right\|_{L^2(\mathbb{R}^d)}^2.$$

Claims 4.5–4.6 entail that

$$\lim_{N \to \infty} \mathcal{L}_{\psi_N}(f) = \exp\left(-\frac{1}{2} \left\| (-\Delta)^{-1} f \right\|_{L^2(\mathbb{R}^d)}^2\right).$$

Thus we have for all $f \in \mathcal{S}$

$$\mathcal{L}_{\psi_N}(f) \to \mathcal{L}_{\psi}(f).$$

Hence the conclusion follows from Fact 4.2. □

To prove the above two claims, we use crucially the following estimate for approximating Riemann sums for Schwartz functions. Since we could not find a reference, we provide a short proof of the following fact.

LEMMA 4.7. *For any $N \geq 1$ and $s > 0$, we have*

$$(4.5) \qquad \left| (2\pi)^{-d/2} N^{-d} \sum_{x \in \mathbb{Z}^d} e^{-\iota \langle \frac{x}{N}, \theta \rangle} f\left(\frac{x}{N}\right) - \widehat{f}(\theta) \right| \leq C N^{-s},$$

where C may depend on f.

PROOF. To show the above result, we use the Poisson summation formula (Stein and Weiss (1971), Chapter 7). Let us define $g(x) := (2\pi)^{-d/2} e^{-\iota \langle x, \theta \rangle} f(x)$. Using the Poisson summation formula, we get

$$N^{-d} \sum_{x \in \mathbb{Z}^d} g\left(\frac{x}{N}\right) = \sum_{x \in \mathbb{Z}^d} \widehat{f}(\theta + 2\pi x N).$$

Hence we have

$$\left| \frac{1}{(2\pi)^{d/2} N^d} \sum_{x \in \mathbb{Z}^d} e^{-\iota \langle \frac{x}{N}, \theta \rangle} f\left(\frac{x}{N}\right) - \widehat{f}(\theta) \right|$$

$$\leq \sum_{x \neq 0, x \in \mathbb{Z}^d} \left| \widehat{f}(\theta + 2\pi x N) \right|$$

$$\leq \sum_{x \neq 0, x \in \mathbb{Z}^d} \frac{C}{\|\theta + 2\pi x N\|_\infty^s},$$

where the last inequality holds for any $s \geq 0$ because $\widehat{f} \in \mathcal{S}$. But

$$\|2\pi x N\|_\infty \leq \|\theta + 2\pi x N\|_\infty + \|\theta\|_\infty \leq \|\theta + 2\pi x N\|_\infty + N\pi$$

and hence, for $s > 1$, $\|2\pi x N\|_\infty^s \leq 2^{s-1}(\|\theta + 2\pi x N\|_\infty^s + (N\pi)^s)$. Thus for any $s \geq d_0 > d$, we have

$$\left| (2\pi)^{-d/2} N^{-d} \sum_{x \in \mathbb{Z}^d} e^{-\iota \langle \frac{x}{N}, \theta \rangle} f\left(\frac{x}{N}\right) - \widehat{f}(\theta) \right|$$

$$\leq \sum_{x \neq 0, x \in \mathbb{Z}^d} \frac{C}{(N\pi)^s (2\|x\|_\infty^s - 1)}$$

$$\leq C N^{-s},$$

where the constant C depends on d_0 but not on s. Hence the result follows. $\square$

We can now begin with the proof of the two claims.

PROOF OF CLAIM 4.5.   Recall that $\kappa = 1/(2d)$. Using the bound (4.4) for $w_i = \theta_i/2$, we have

$$\left| \frac{\kappa^2 N^{-4}}{(2\pi)^d} \int_{[-N\pi, N\pi]^d} \left(\mu\left(\frac{\theta}{N}\right) \right)^{-2} \left| N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} e^{-\iota\langle x,\theta\rangle} f(x) \right|^2 d\theta \right.$$

$$\left. - \frac{1}{(2\pi)^d} \int_{[-N\pi, N\pi]^d} \|\theta\|^{-4} \left| N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} e^{-\iota\langle x,\theta\rangle} f(x) \right|^2 d\theta \right|$$

$$\leq \int_{[-N\pi, N\pi]^d} \left(2\|\theta\|^{-2} \frac{C}{N^2} + \frac{C}{N^4} \right) \left| (2\pi)^{-d/2} N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} e^{-\iota\langle x,\theta\rangle} f(x) \right|^2 d\theta.$$

Using $(a+b)^2 \leq 2(a^2 + b^2)$ after adding and subtracting $\widehat{f}(\theta)$ in the modulus above, we have the bound

$$\int_{[-N\pi, N\pi]^d} \left(2\|\theta\|^{-2} \frac{C}{N^2} + \frac{C}{N^4} \right) (CN^{-s} + |\widehat{f}(\theta)|)^2 d\theta$$

$$\leq \int_{[-N\pi, N\pi]^d} 2 \left(2\|\theta\|^{-2} \frac{C}{N^2} + \frac{C}{N^4} \right) (CN^{-2s} + |\widehat{f}(\theta)|^2) d\theta.$$

Again the last term amounts to estimating

$$CN^{-2s-2} O(N^{d-2}) + CN^{-2s-4} O(N^d)$$

$$+ CN^{-2} \int_{[-N\pi, N\pi]^d} \|\theta\|^{-2} |\widehat{f}(\theta)|^2 d\theta + CN^{-4} \int_{[-N\pi, N\pi]^d} |\widehat{f}(\theta)|^2 d\theta$$

which goes to 0 due to the fact that $f \in \mathcal{S}$. $\square$

PROOF OF CLAIM 4.6.   We have

$$\|(-\Delta)^{-1} f\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \|\theta\|^{-4} |\widehat{f}(\theta)|^2 d\theta$$

and

$$\left| \int_{[-N\pi, N\pi]^d} \|\theta\|^{-4} \left| (2\pi)^{-d/2} N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} e^{-\iota\langle x,\theta\rangle} f(x) \right|^2 - \int_{\mathbb{R}^d} \frac{\widehat{f}(\theta)|^2}{\|\theta\|^4} d\theta \right|$$

$$\leq \left| \int_{[-N\pi, N\pi]^d} \|\theta\|^{-4} \left| (2\pi)^{-d/2} N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} e^{-\iota\langle x,\theta\rangle} f(x) \right|^2 \right.$$

$$\left. - \int_{[-N\pi, N\pi]^d} \frac{\widehat{f}(\theta)|^2}{\|\theta\|^4} d\theta \right|$$

$$+ \left| \int_{[-N\pi, N\pi]^d} \frac{\widehat{f}(\theta)|^2}{\|\theta\|^4} d\theta - \int_{\mathbb{R}^d} \frac{\widehat{f}(\theta)|^2}{\|\theta\|^4} d\theta \right|.$$

Clearly, the second term goes to zero as N tends to infinity. As for the first term, we have the following bound:

$$\left| \int_{[-N\pi, N\pi]^d} \frac{1}{\|\theta\|^4 |(2\pi)^{d/2} N^d} \left| \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \mathrm{e}^{-\iota\langle x, \theta \rangle} f(x) \right|^2 - \int_{[-N\pi, N\pi]^d} \frac{|\widehat{f}(\theta)|^2}{\|\theta\|^4} \, d\theta \right|$$

$$\leq \int_{[-N\pi, N\pi]^d} \|\theta\|^{-4} \left| \left| (2\pi)^{-d/2} N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \mathrm{e}^{-\iota\langle x, \theta \rangle} f(x) \right|^2 - |\widehat{f}(\theta)|^2 \right| \, d\theta$$

$$\leq \frac{2\|f\|_{L^1} + C}{(2\pi)^{d/2} N^d} \int_{[-N\pi, N\pi]^d} \|\theta\|^{-4} \left| \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{-\iota\langle \frac{x}{N}, \theta \rangle} f\left(\frac{x}{N}\right) - \widehat{f}(\theta) \right| \, d\theta$$

$$= O(N^{d-4-s}),$$

where the bound in the second inequality is obtained using the formula $(a^2 - b^2) = (a+b)(a-b)$ and (4.5). Thus the first term also goes to zero as N tends to infinity. $\square$

APPENDIX: QUANTITATIVE ESTIMATE ON THE DISCRETE APPROXIMATION

This section is devoted to obtaining quantitative estimates on approximation of solutions of PDEs. The building block of our analysis is the paper Thomée (1964). Let V be any bounded domain in $\mathbb{R}^d$ with C^2 boundary. We denote $L := \Delta^2$ and consider the following continuum Dirichlet problem:

$$\text{(A.1)} \qquad \begin{cases} Lu(x) = f(x), & x \in V, \\ D^\beta u(x) = 0, & |\beta| \leq 1, x \in \partial V. \end{cases}$$

Let $h > 0$. We will call the points in $h\mathbb{Z}^d$ the grid points in $\mathbb{R}^d$. We consider $L_h u := \Delta_h^2 u$ to be the discrete approximation of Lu, where Δ_h is defined by

$$\Delta_h f(x) := \frac{1}{h^2} \sum_{i=1}^d (f(x + he_i) + f(x - he_i) - 2f(x))$$

and f is any function on $h\mathbb{Z}^d$. We call such a function a grid function. Thus we have, for $x \in h\mathbb{Z}^d$,

$$L_h u(x) = \frac{1}{h^2} \sum_{i=1}^d (\Delta_h u(x + he_i) + \Delta_h u(x - he_i) - 2\Delta_h u(x))$$

$$= \frac{1}{h^4} \left[\sum_{i,j=1}^d \{ u(x + h(e_i + e_j)) \right.$$

$$+ u\big(x - h(e_i + e_j)\big) + u\big(x + h(e_i - e_j)\big)$$

$$+ u\big(x - h(e_i - e_j)\big)\big\} - 4d \sum_{i=1}^{d} \{u(x + he_i) + u(x - he_i)\}$$

$$+ 4d^2 u(x) \bigg].$$

Let V_h be the set of grid points in $\overline{V}$, that is, $V_h = \overline{V} \cap h\mathbb{Z}^d$. We say that ξ is an interior grid point in V_h or $\xi \in R_h$ if for every i, j, the points $\xi \pm h(e_i \pm e_j)$, $\xi \pm he_i$ are all in V_h. We denote B_h to be $V_h \setminus R_h$. We will denote by $\mathcal{D}_h$ the set of grid functions vanishing outside R_h. For a grid function f, we define $R_h f \in \mathcal{D}_h$ by

(A.2)
$$R_h f(\xi) = \begin{cases} f(\xi) & \xi \in R_h, \\ 0 & \xi \notin R_h. \end{cases}$$

In Thomée (1964), it is crucially used that the discrete approximation of the elliptic operator is consistent. In our case, it is easy to see this using Taylor's expansion.

LEMMA A.8. *The operator L_h is consistent with the operator L, that is, if W is a neighborhood of the origin in $\mathbb{R}^d$ and $u \in C^4(W)$ then*

$$L_h u(0) = \mathrm{L}u(0) + o(1) \quad \text{as } h \to 0.$$

We will divide R_h further into R_h^* and B_h^* where R_h^* is the set of ξ in R_h such that for every i, j, the points $\xi \pm h(e_i \pm e_j)$, $\xi \pm he_i$ are all in R_h and B_h^* is the set of remaining points in R_h. Thus we have

$$V_h = B_h \cup R_h = B_h \cup B_h^* \cup R_h^*.$$

We say that the domain V has property $\mathcal{B}_2^*$ if there is a natural number K such that for all sufficiently small h, the following is valid: consider for any $\xi \in B_h^*$ all half-rays through ξ. At least one of them contains within the distance Kh from ξ two consecutive grid-points in B_h.

The following proposition shows that if the boundary of the domain is regular enough then the property $\mathcal{B}_2^*$ is true. Namely, recall the uniform exterior ball condition (UEBC) for a domain V, which states that there exists $\delta > 0$ such that for any $z \in \partial V$ there is a ball $B_\delta(c)$ of radius δ with center at some point c satisfying $\overline{B_\delta(c)} \cap \overline{V} = \{z\}$ (Gilbarg and Trudinger (1977), page 27). We show that the UEBC is a sufficient condition for $\mathcal{B}_2^*$ to hold. In particular, any domain with C^2 boundary satisfy the UEBC, and hence possesses $\mathcal{B}_2^*$.

PROPOSITION A.9. *If a bounded domain V satisfies the UEBC, then the property $\mathcal{B}_2^*$ holds.*

Since the proof of this result is purely geometric and combinatorial in nature, we discuss it in the Supplementary Material (Cipriani, Dan and Hazra (2019)). We would like to remark that property $\mathcal{B}_2^*$ is a crucial requirement in the proof of Theorem A.11. In fact, it allows us to use Thomée's theorem (Thomée (1964)), which compares the standard discrete Sobolev norm with a modified Sobolev norm weighted on boundary points.

We now define the finite difference analogue of the Dirichlet's problem (A.1). For given h, we look for a function $u(\xi)$ defined on V_h such that

$$(A.3) \qquad L_h u_h(\xi) = f(\xi), \quad \xi \in R_h$$

and

$$(A.4) \qquad u_h(\xi) = 0, \quad \xi \in B_h.$$

It follows from Lemma A.8 and Theorem 5.1 of Thomée (1964) that the finite difference Dirichlet problem (A.3) and (A.4) has exactly one solution for arbitrary f. Recall also the norm $\|f\|_{h,\mathrm{grid}}^2 := h^d \sum_{\xi \in h \mathbb{Z}^d} f(\xi)^2$. Before we prove the approximation theorem, let us cite two results from Thomée (1964) (stated, in the original article, in a slightly more general way).

LEMMA A.10 (Thomée (1964), Lemma 3.1). *There are constants $C > 0$ independent of f and h such that*

$$\|f\|_{h,\mathrm{grid}} \leq C \|D_j f\|_{h,\mathrm{grid}}, \quad j = 1, \ldots, d$$

and

$$\|f\|_{h,\mathrm{grid}} \leq C \|f\|_{h,2} := \left(\sum_{|\beta| \leq 2} \|D^\beta f\|_{h,grid}^2 \right)^{1/2}$$

for any grid function f vanishing outside R_h, where

$$D_j f(x) := \frac{1}{h}\big(f(x + he_j) - f(x)\big), \quad j = 1, \ldots, d$$

and

$$D^\beta f := D_1^{\beta_1} \cdots D_d^{\beta_d} f, \quad \beta = (\beta_1, \ldots, \beta_d), \beta_i \geq 0.$$

For the next result, we need the definition of the operator $L_{h,2}$ from Thomée (1964) as follows:

$$L_{h,2} f(x) = \begin{cases} L_h f(x) & x \in R_h^*, \\ h^2 L_h f(x) & x \in B_h^*, \\ 0 & x \notin R_h. \end{cases}$$

THEOREM A.11 (Thomée (1964), Theorem 4.2).   *There exists a constant $C >$ 0 such that for all grid functions f vanishing outside R_h*

$$\|f\|_{h,2} \leq C\|L_{h,2}f\|_{h,\text{grid}},$$

where C is independent of h as well.

We have now all the ingredients to show the following.

THEOREM A.12.   *Let $u \in C^5(\overline{V})$ be the solution of the Dirichlet's problem A.1 and u_h be the solution of the discrete problem (A.3)–(A.4). If $e_h := u - u_h$, then we have for all sufficiently small h,*

$$\|R_h e_h\|_{h,\text{grid}}^2 \leq C[M_5^2 h^2 + h(M_5^2 h^6 + M_2^2)],$$

where $M_k = \sum_{|\alpha| \leq k} \sup_{x \in V} |D^\alpha u(x)|$.

PROOF.   We denote all constants by C and they do not depend on u, f. Using Taylor's expansion, we have for all $x \in R_h$ and for small h

$$L_h u(x) = \text{L}u(x) + h^{-4}\mathcal{R}_5(x),$$

where $|\mathcal{R}_5(x)| \leq CM_5 h^5$. We obtain for $\xi \in R_h$,

$$L_h e_h(\xi) = L_h u(\xi) - L_h u_h(\xi)$$
$$= \text{L}u(\xi) + h^{-4}\mathcal{R}_5(\xi) - L_h u_h(\xi) = h^{-4}\mathcal{R}_5(\xi).$$

For $\xi \in R_h^*$, we have

$$L_{h,2} R_h e_h(\xi) = L_h R_h e_h(\xi) = L_h e_h(\xi) = h^{-4}\mathcal{R}_5(\xi).$$

For $\xi \in B_h^*$ at least one among $\xi \pm h(e_i \pm e_j), \xi \pm he_i$ is in B_h. For any $\eta \in B_h \setminus \partial V$, we consider a point $b(\eta)$ on ∂V of minimal distance to η. Note that this distance is at most $2h$. Now using Taylor expansion and the fact that the value of u and all its first-order derivatives are zero at $b(\eta)$, one sees that

$$u(\eta) = u_h(\eta) + \mathcal{R}_2(\eta),$$

where $|\mathcal{R}_2(\eta)| \leq CM_2 h^2$. For $\xi \in B_h^*$, denote by

$$S_{i,j}(\xi) = \{\eta : \eta \in B_h \setminus (B_h \cap \partial V) \cap \{\xi \pm he_i, \xi \pm h(e_i \pm e_j)\}\}.$$

Therefore, for $\xi \in B_h^*$,

$$L_{h,2} R_h e_h(\xi) = h^2 L_h R_h e_h(\xi)$$

$$= h^2 \left\{ L_h e_h(\xi) - h^{-4} \sum_{i,j=1}^{d} \sum_{\eta \in S_{i,j}(\xi)} C(\eta) e_h(\eta) \right\}$$

$$= h^{-2}\mathcal{R}_5(\xi) + h^{-2} C\mathcal{R}_2'(\xi),$$

where $C(\eta)$ is a constant depending on η and $|\mathcal{R}_2'(\xi)| \le C M_2 h^2$. Hence

$$
\begin{aligned}
\|L_{h,2} R_h e_h\|_{h,\text{grid}}^2 &= h^d \sum_{x \in R_h} (L_{h,2} R_h e_h(x))^2 \\
&= h^d \Bigg[\sum_{x \in R_h^*} (L_{h,2} R_h e_h(x))^2 + \sum_{x \in B_h^*} (L_{h,2} R_h e_h(x))^2 \Bigg] \\
&= h^d \Bigg[\sum_{x \in R_h^*} (h^{-4} \mathcal{R}_5(x))^2 + \sum_{x \in B_h^*} (h^{-2} \mathcal{R}_5(x) + h^{-2} C \mathcal{R}_2'(x))^2 \Bigg] \\
&\le h^d \Bigg[\sum_{x \in R_h^*} C M_5^2 h^2 + \sum_{x \in B_h^*} (C M_5^2 h^6 + C M_2^2) \Bigg] \\
&\le C \big[M_5^2 h^2 + h(M_5^2 h^6 + M_2^2) \big],
\end{aligned}
$$

where in the last inequality we have used that the number of points in B_h^* is $O(h^{-(d-1)})$ following from Penrose ((2003), Lemma 5.4) and the assumption of a C^2 boundary. Finally, to complete our proof we obtain

$$
\text{(A.5)} \qquad \|R_h e_h\|_{h,\text{grid}}^2 \le C \big[M_5^2 h^2 + h(M_5^2 h^6 + M_2^2) \big]
$$

using Lemma A.10 and Theorem A.11. This concludes the proof. □

SUPPLEMENTARY MATERIAL

Supplement to "The scaling limit of the membrane model" (DOI: 10.1214/19-AOP1351SUPP; .pdf). Supplementary information.

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