SAMPLE PATH LARGE DEVIATIONS FOR LÉVY PROCESSES AND RANDOM WALKS WITH REGULARLY VARYING INCREMENTS

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Let X be a Lévy process with regularly varying Lévy measure ν. We obtain sample-path large deviations for scaled processes $\bar{X}_n(t) \triangleq X(nt)/n$ and obtain a similar result for random walks with regularly varying increments. Our results yield detailed asymptotic estimates in scenarios where multiple big jumps in the increment are required to make a rare event happen; we illustrate this through detailed conditional limit theorems. In addition, we investigate connections with the classical large deviations framework. In that setting, we show that a weak large deviation principle (with logarithmic speed) holds, but a full large deviation principle does not hold.

CONTENTS

1. Introduction. In this paper, we develop sample-path large deviations for one-dimensional Lévy processes and random walks, assuming the jump sizes are heavy-tailed. Specifically, let $X(t), t \geq 0$, be a centered Lévy process with regularly varying Lévy measure ν. Assume that $\mathbf{P}(X(1) > x)$ is regularly varying with index $-\alpha$, and that $\mathbf{P}(X(1) < -x)$ is regularly varying with index $-\beta$, that is, there exist slowly varying functions L_+ and L_- such that

$$(1.1) \qquad \mathbf{P}\big(X(1) > x\big) = L_+(x)x^{-\alpha}, \quad \mathbf{P}\big(X(1) < -x\big) = L_-(x)x^{-\beta}.$$

Throughout the paper, we assume $\alpha, \beta > 1$. We also consider spectrally one-sided processes; in that case only α plays a role. Define $\bar{X}_n = \{\bar{X}_n(t), t \in [0, 1]\}$, with $\bar{X}_n(t) = X(nt)/n, t \geq 0$. We are interested in large deviations of $\bar{X}_n$.

This topic fits well in a branch of limit theory that has a long history, has intimate connections to point processes and extreme value theory, and is still a subject of intense activity. The investigation of tail estimates of the one-dimensional distributions of $\bar{X}_n$ (or random walks with heavy-tailed step size distribution) was initiated in Nagaev (1969, 1977). The state of the art of such results is well summarized in Borovkov and Borovkov (2008), Denisov, Dieker and Shneer (2008), Embrechts, Klüppelberg and Mikosch (1997), Foss, Korshunov and Zachary (2011). In particular, Denisov, Dieker and Shneer (2008) describe in detail how fast x needs to grow with n for the asymptotic relation

$$(1.2) \qquad\qquad \mathbf{P}\big(X(n) > x\big) = n\mathbf{P}\big(X(1) > x\big)\big(1 + o(1)\big)$$

to hold, as $n \to \infty$, in settings that go beyond (1.1). If (1.2) is valid, the so-called *principle of one big jump* is said to hold. A functional version of this insight has been derived in Hult et al. (2005). A significant number of studies investigate the question of if and how the principle of a single big jump is affected by the impact of (various forms of) dependence, and cover stable processes, autoregressive processes, modulated processes and stochastic differential equations; see Buraczewski et al. (2013), Foss, Konstantopoulos and Zachary (2007), Hult and Lindskog (2007), Konstantinides and Mikosch (2005), Mikosch and Samorodnitsky (2000), Mikosch and Wintenberger (2013, 2016), Samorodnitsky (2004).

The problem we investigate in this paper is markedly different from all of these works. Our aim is to develop asymptotic estimates of $\mathbf{P}(\bar{X}_n \in A)$ for a sufficiently general collection of sets A, so that it is possible to study continuous functionals of $\bar{X}_n$ in a systematic manner. For many such functionals, and many sets A, the associated rare event will not be caused by a single big jump, but multiple jumps. The results in this domain (e.g., Blanchet and Shi (2012), Foss and Korshunov (2012), Zwart, Borst and Mandjes (2004)) are few, each with an ad hoc approach. As in large deviations theory for light tails, it is desirable to have more general tools available.

Another aspect of heavy-tailed large deviations we aim to clarify in this paper is the connection with the standard large-deviations approach, which has not been

touched upon in any of the above-mentioned references. In our setting, the goal would be to obtain a function I such that

$$
\begin{aligned}
(1.3) \quad -\inf_{\xi \in A^\circ} I(\xi) &\le \liminf_{n \to \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{\log n} \\
&\le \limsup_{n \to \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{\log n} \le -\inf_{\xi \in \bar{A}} I(\xi),
\end{aligned}
$$

where A° and $\bar{A}$ are the interior and closure of A; all of our large deviations results are derived in the Skorokhod space $\mathbb{D} = \mathbb{D}([0, 1], \mathbb{R})$—the space of real-valued RCLL functions on $[0, 1]$—and w.r.t. Skorokhod J_1 topology. Equation (1.3) is a classical large deviations principle (LDP) with speed $\log n$ (cf. Dembo and Zeitouni (2010)). Using existing results in the literature (e.g., Denisov, Dieker and Shneer (2008)), it is not difficult to show that $X(n)/n = \bar{X}_n(1)$ satisfies an LDP with rate function $I_1 = I_1(x)$ which is 0 at 0, equal to $(\alpha - 1)$ if $x > 0$, and $(\beta - 1)$ if $x < 0$. This is a lower-semicontinuous function whose level sets are not compact. Thus, in large-deviations terminology, I_1 is a rate function, but is not a good one. This implies that techniques such as the projective limit approach cannot be applied. In fact, in Section 4.4, we show that there does not exist an LDP of the form (1.3) for general sets A, by giving a counterexample. A version of (1.3) for compact sets is derived in Section 4.3, as a corollary of our main results. A result similar to (1.3) for random walks with semiexponential (Weibullian) tails has been derived in Gantert (1998) (see also Gantert (2000), Gantert, Ramanan and Rembart (2014) for related results). Though an LDP for finite-dimensional distributions can be derived, lack of exponential tightness also persists at the sample-path level. To make the rate function good (i.e., to have compact level sets), a topology chosen in Gantert (1998) is considerably weaker than any of the Skorokhod topologies (but sufficient for the application that is central in that work).

The approach followed in the present paper is based on recent developments in the theory of regular variation. In particular, in Lindskog, Resnick and Roy (2014), the classical notion of regular variation is redefined through a new convergence concept called $\mathbb{M}$-convergence (this is in itself a refinement of other reformulations of regular variation in function spaces; see de Haan and Lin (2001), Mikosch and Wintenberger (2005, 2006)). In Section 2, we further investigate the $\mathbb{M}$-convergence framework by deriving a number of general results that facilitate the development of our proofs.

This paves the way toward our main large deviations results, which are presented in Section 3. We actually obtain estimates that are sharper than (1.3), though we impose a condition on A. For one-sided Lévy processes with Lévy measure ν,

our result takes the form

$$
\begin{aligned}
C_{\mathcal{J}(A)}(A^\circ) &\leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \\
&\leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \leq C_{\mathcal{J}(A)}(\bar{A}).
\end{aligned}
$$

(1.4)

Precise definitions can be found in Section 3.1; for now we just mention that C_j is a measure on the Skorokhod space, and $\mathcal{J}(\cdot)$ is an integer-valued set function acting on $\mathbb{D}$ defined as $\mathcal{J}(A) = \inf_{\xi \in A \cap \mathbb{D}_s^\uparrow} \mathcal{D}_+(\xi)$, where $\mathcal{D}_+(\xi)$ is the number of discontinuities of ξ, and $\mathbb{D}_s^\uparrow$ is the set of all nonincreasing step functions vanishing at the origin. Throughout the paper, we adopt the convention that the infimum over an empty set is ∞. Letting $\mathbb{D}_j$ and $\mathbb{D}_{<j}$ be the sets of step functions vanishing at the origin with precisely j and at most $j-1$ steps, respectively, we note that the measure C_j, defined on $\mathbb{D} \setminus \mathbb{D}_{<j}$ has its support on $\mathbb{D}_j$. A crucial assumption for (1.4) to hold is that the Skorokhod J_1 distance between the sets A and $\mathbb{D}_{<\mathcal{J}(A)}$ is strictly positive. For A such that $\mathcal{J}(A) = 1$, this result corresponds to the one shown in Hult et al. (2005). (Note that Hult et al. (2005) deals with multivariate regular variation whereas we focus on one-dimensional regular variation in this paper.) The interpretation of the "rate function" $\mathcal{J}(A)$ is that it provides the number of jumps in the Lévy process that are necessary to make the event A happen. This can be seen as an extension of the principle of a single big jump to multiple jumps. A rigorous statement on when (1.4) holds can be found in Theorem 3.2, which is the first main result of the paper.

The result that comes closest to (1.4) is Theorem 5.1 in Lindskog, Resnick and Roy (2014) which considers the $\mathbb{M}$-convergence of $\nu[n,\infty)^{-j}\mathbf{P}(X/n \in A)$. This result could be used as a starting point to investigate rare events that happen on a timescale of $O(1)$. However, in the large-deviations scaling we consider, rare events happen on a timescale of $O(n)$. Controlling the Lévy process on this larger timescale requires more delicate estimates, eventually leading to an additional factor n^j in the asymptotic results. We further show that the choice $j = \mathcal{J}(A)$ is the only choice that leads to a nontrivial limit. One useful notion that we develop and rely on in our setting is a form of asymptotic equivalence, which can best be compared with exponential equivalence in classical large deviations theory.

In Section 3.2, we present sample-path large deviations for two-sided Lévy processes. Our main results in this case are Theorems 3.3–3.5. In the two-sided case, determining the most likely path requires resolving significant combinatorial issues which do not appear in the one-sided case. The polynomial rate of decay for $\mathbf{P}(\bar{X}_n \in A)$, which was described by the function $\mathcal{J}(A)$ in the one-sided case, has a more complicated description: the corresponding polynomial rate in the two-sided

case is

$$(1.5) \qquad \inf_{\xi, \zeta \in \mathbb{D}_s^\uparrow; \xi - \zeta \in A} (\alpha - 1)\mathcal{D}_+(\xi) + (\beta - 1)\mathcal{D}_+(\zeta).$$

Note that this is a result that one could expect from the result for one-sided Lévy processes and a heuristic application of the contraction principle. A rigorous treatment of the two-sided case requires a more delicate argument compared to the one-sided case: in the one-sided case, the argument simplifies since if one takes j largest jumps away from $\bar{X}_n$, then the probability that the residual process is of significant size is $o((n\nu[n, \infty))^j)$ so that it does not contribute in (1.4), while in two-sided case, taking j largest upward jumps and k largest downward jumps from $\bar{X}_n$ does not guarantee that the residual process remains small with high enough probability, that is, the probability that the residual process is of significant size cannot be bounded by $o((n\nu[n, \infty))^j (n\nu(-\infty, -n])^k)$. In addition, it may be the case that multiple pairs (j, k) of jumps lead to optimal solutions of (1.5). To overcome such difficulties, we first develop general tools—Lemma 2.2 and 2.3—that establish a suitable notion of $\mathbb{M}$-convergence on product spaces. Using these results, we prove in Theorem 5.1 the suitable $\mathbb{M}$-convergence for multiple Lévy processes in the associated product space. Viewing the two-sided Lévy process as a superposition of one-sided Lévy processes, we then apply the continuous mapping principle for $\mathbb{M}$-convergence to Theorem 5.1 to establish our main results. Although no further implications are discussed in this paper, we believe that Theorem 5.1 itself is of independent interest as well because it can be applied to generate large deviations results for a general class of functionals of multiple Lévy processes.

We derive analogous results for random walks in Section 4.1. Random walks cannot be decomposed into independent components with small jumps and large jumps as easily as Lévy processes, making the analysis of random walks more technical if done directly. However, it is possible to follow an indirect approach. Given a random walk $S_k, k \geq 0$, one can study a subordinated version $S_{N(t)}, t \geq 0$ with $N(t), t \geq 0$ an independent unit rate Poisson process. The Skorokhod J_1 distance between the scaled versions of $S_k, k \geq 0$ and $S_{N(t)}, t \geq 0$ can then be bounded essentially in terms of the deviations of $N(t)$ from t, which have been studied thoroughly.

In Section 4.2, we provide conditional limit theorems which give a precise description of the limit behavior of $\bar{X}_n$ given that $\bar{X}_n \in A$ as $n \to \infty$. An early result of this type is given in Durrett (1980), which focuses on regularly varying random walks with finite variance conditioned on the event $A = \{\bar{X}_n(1) > a\}$. Using the recent results that we have discussed (e.g., Hult et al. (2005)), more general conditional limit theorems can be derived for single-jump events.

We prove an LDP of the form (1.3) in Section 4.3, where the upper bound requires a compactness assumption. We construct a counterexample showing that the compactness assumption cannot be totally removed, and thus, a full LDP does not hold. Essentially, if a rare event is caused by j big jumps, then the framework

developed in this paper applies if each of these jumps is bounded away from below by a strictly positive constant. Our counterexample in Section 4.4 indicates that it is not trivial to remove this condition.

As one may expect, it is not possible to apply classical variational methods to derive an expression for the exponent $\mathcal{J}(A)$, as is often the case in large deviations for light tails. Nevertheless, there seems to be a generic connection with a class of control problems called impulse control problems. Equation (1.5) is a specific deterministic impulse-control problem, which is related to Barles (1985). We expect that techniques similar to those in Barles (1985) will be useful in characterizing the optimal solutions for problems like (1.5). The latter challenge is not taken up in the present study and will be addressed elsewhere. Instead, in Section 6 of the extended version of the current paper (Rhee, Blanchet and Zwart (2016)) available online, we analyze (1.5) directly in several examples; see also Chen et al. (2018). In each case, a condition needs to be checked to see whether our framework is applicable. We provide a general result that essentially states that we only need to verify this condition for step functions in A, which makes this check rather straightforward.

In summary, this paper is organized as follows. After developing some preliminary results in Section 2, we present our main results in Section 3. Applications to random walks and connections with classical large deviations theory are investigated in Section 4. Section 5 is devoted to proofs. We collect some useful bounds in the Appendix.

2. $\mathbb{M}$-convergence. This section reviews and develops general concepts and tools that are useful in deriving our large deviations results. The proofs of the lemmas and corollaries stated throughout this section are provided in Section 5.1. We start with briefly reviewing the notion of $\mathbb{M}$-convergence, introduced in Lindskog, Resnick and Roy (2014).

Let $(\mathbb{S}, d)$ be a complete separable metric space, and $\mathscr{S}$ be the Borel σ-algebra on $\mathbb{S}$. Given a closed subset $\mathbb{C}$ of $\mathbb{S}$, let $\mathbb{S} \setminus \mathbb{C}$ be equipped with the relative topology as a subspace of $\mathbb{S}$, and consider the associated sub σ-algebra $\mathscr{S}_{\mathbb{S}\setminus\mathbb{C}} \triangleq \{A : A \subseteq \mathbb{S} \setminus \mathbb{C}, A \in \mathscr{S}\}$ on it. Define $\mathbb{C}^r \triangleq \{x \in \mathbb{S} : d(x, \mathbb{C}) < r\}$ for $r > 0$, and let $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ be the class of measures defined on $\mathscr{S}_{\mathbb{S}\setminus\mathbb{C}}$ whose restrictions to $\mathbb{S} \setminus \mathbb{C}^r$ are finite for all $r > 0$. Topologize $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ with a sub-basis $\{\{\nu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C}) : \nu(f) \in G\}: f \in \mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}, G \text{ open in } \mathbb{R}_+\}$ where $\mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}$ is the set of real-valued, nonnegative, bounded, continuous functions whose support is bounded away from $\mathbb{C}$ (i.e., $f(\mathbb{C}^r) = \{0\}$ for some $r > 0$). A sequence of measures $\mu_n \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ converges to $\mu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ if $\mu_n(f) \to \mu(f)$ for each $f \in \mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}$. Note that this notion of convergence in $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ coincides with the classical notion of weak convergence of measures (Billingsley (1968)) if $\mathbb{C}$ is an empty set. We say that a set $A \subseteq \mathbb{S}$ is bounded away from another set $B \subseteq \mathbb{S}$ if $\inf_{x \in A, y \in B} d(x, y) > 0$. An important characterization of $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$-convergence is as follows.

THEOREM 2.1 (Theorem 2.1 of Lindskog, Resnick and Roy (2014)). *Let* $\mu, \mu_n \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$. *Then* $\mu_n \to \mu$ *in* $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ *as* $n \to \infty$ *if and only if*

$$(2.1) \qquad \limsup_{n \to \infty} \mu_n(F) \leq \mu(F)$$

for all closed $F \in \mathscr{S}_{\mathbb{S} \setminus \mathbb{C}}$ *bounded away from* $\mathbb{C}$ *and*

$$(2.2) \qquad \liminf_{n \to \infty} \mu_n(G) \geq \mu(G)$$

for all open $G \in \mathscr{S}_{\mathbb{S} \setminus \mathbb{C}}$ *bounded away from* $\mathbb{C}$.

We now introduce a new notion of equivalence between two families of random objects, which will prove to be useful in Section 3.1 and Section 4.1. Let $F_\delta \triangleq \{x \in \mathbb{S} : d(x, F) \leq \delta\}$ and $G^{-\delta} \triangleq ((G^c)_\delta)^c$. (Compare these notation to $\mathbb{C}^r$; note that we are using the convention that superscript implies open sets and subscript implies closed sets.)

DEFINITION 1. Suppose that X_n and Y_n are random elements taking values in a complete separable metric space $(\mathbb{S}, d)$, and ϵ_n is a sequence of positive real numbers. Y_n is said to be asymptotically equivalent to X_n with respect to ϵ_n if for each $\delta > 0$,

$$\limsup_{n \to \infty} \epsilon_n^{-1} \mathbf{P}\big(d(X_n, Y_n) \geq \delta\big) = 0.$$

The usefulness of this notion of equivalence comes from the following lemma, which states that if Y_n is asymptotically equivalent to X_n, and X_n satisfies a limit theorem, then Y_n satisfies the same limit theorem. Moreover, it also allows one to extend the lower and upper bounds to more general sets in case there are asymptotically equivalent distributions that are supported on a subspace $\mathbb{S}_0$ of $\mathbb{S}$:

LEMMA 2.1. *Suppose that* $\epsilon_n^{-1} \mathbf{P}(X_n \in \cdot) \to \mu(\cdot)$ *in* $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ *for some sequence* ϵ_n *and a closed set* $\mathbb{C}$. *In addition, suppose that* $\mu(\mathbb{S} \setminus \mathbb{S}_0) = 0$ *and* $\mathbf{P}(X_n \in \mathbb{S}_0) = 1$ *for each n. If* Y_n *is asymptotically equivalent to* X_n *with respect to* ϵ_n, *then*

$$\liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in G) \geq \mu(G)$$

if G is open and $G \cap \mathbb{S}_0$ *is bounded away from* $\mathbb{C}$;

$$\limsup_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in F) \leq \mu(F)$$

if F is closed and there is a $\delta > 0$ *such that* $F_\delta \cap \mathbb{S}_0$ *is bounded away from* $\mathbb{C}$.

This lemma is particularly useful when we work in Skorokhod space, and $\mathbb{S}_0$ is the class of step functions. Taking $\mathbb{S}_0 = \mathbb{S}$, a simpler version of Lemma 2.1 follows immediately.

COROLLARY 2.1. *Suppose that $\epsilon_n^{-1}\mathbf{P}(X_n \in \cdot) \to \mu(\cdot)$ in $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ for some sequence ϵ_n. If Y_n is asymptotically equivalent to X_n with respect to ϵ_n, then the law of Y_n has the same (normalized) limit, that is, $\epsilon_n^{-1}\mathbf{P}(Y_n \in \cdot) \to \mu(\cdot)$ in $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$.*

Next, we discuss the $\mathbb{M}$-convergence in a product space as a result of the $\mathbb{M}$-convergences on each space.

LEMMA 2.2. *Suppose that $\mathbb{S}_1, \ldots, \mathbb{S}_d$ are separable metric spaces, $\mathbb{C}_1, \ldots, \mathbb{C}_d$ are closed subsets of $\mathbb{S}_1, \ldots, \mathbb{S}_d$, respectively. If $\mu_n^{(i)}(\cdot) \to \mu^{(i)}(\cdot)$ in $\mathbb{M}(\mathbb{S}_i \setminus \mathbb{C}_i)$ for each $i = 1, \ldots, d$, then*

$$(2.3) \qquad \mu_n^{(1)} \times \cdots \times \mu_n^{(d)}(\cdot) \to \mu^{(1)} \times \cdots \times \mu^{(d)}(\cdot)$$

in $\mathbb{M}((\prod_{i=1}^d \mathbb{S}_i) \setminus \bigcup_{i=1}^d ((\prod_{j=1}^{i-1} \mathbb{S}_j) \times \mathbb{C}_i \times (\prod_{j=i+1}^d \mathbb{S}_j)))$.

It should be noted that Lemma 2.2 itself is not exactly "right" in the sense that the set we take away is unnecessarily large, and hence, has limited applicability. More specifically, the $\mathbb{M}$-convergence in (2.3) applies only to the sets that are contained in a "rectangular" domain $\prod_{i=1}^d (\mathbb{S}_i \setminus \mathbb{C}_i)$. Our next observation allows one to combine multiple instances of $\mathbb{M}$-convergences to establish a more refined one so that (2.3) applies to a class of sets that are not confined to a rectangular domain. In particular, we will see later in Theorem 3.3 and Theorem 5.1 that in combination with Lemma 2.2, the following lemma produces the "right" $\mathbb{M}$-convergence for two-sided Lévy processes and random walks.

LEMMA 2.3. *Consider a family of measures $\{\mu^{(i)}\}_{i=0,1,\ldots,m}$ and a family of closed subsets $\{\mathbb{C}(i)\}_{i=0,1,\ldots,m}$ of $\mathbb{S}$ such that $\frac{1}{\epsilon_n(i)}\mathbf{P}(X_n \in \cdot) \to \mu^{(i)}(\cdot)$ in $\mathbb{M}(\mathbb{S} \setminus \mathbb{C}(i))$ for $i = 0, \ldots, m$ where $\{\{\epsilon_n(i) : n \geq 1\}\}_{i=0,1,\ldots,m}$ is the family of associated normalizing sequences. Let $\bar{\mu}^{(0)}(\cdot) \triangleq \mu^{(0)}(\cdot \setminus \mathbb{C}(0))$, and suppose that $\bar{\mu}^{(0)} \in \mathbb{M}(\mathbb{S} \setminus \bigcap_{i=0}^m \mathbb{C}(i))$; $\limsup_{n\to\infty} \frac{\epsilon_n(i)}{\epsilon_n(0)} = 0$ for $i = 1, \ldots, m$; and for each $r > 0$, there exist positive numbers $r_0, \ldots, r_m$ such that $\bigcap_{i=0}^m \mathbb{C}(i)^{r_i} \subseteq (\bigcap_{i=0}^m \mathbb{C}(i))^r$. Then*

$$\frac{1}{\epsilon_n(0)}\mathbf{P}(X_n \in \cdot) \to \bar{\mu}^{(0)}$$

in $\mathbb{M}(\mathbb{S} \setminus \bigcap_{i=0}^m \mathbb{C}(i))$.

A version of the continuous mapping principle is satisfied by $\mathbb{M}$-convergence. Let $(\mathbb{S}', d')$ be a complete separable metric space, and let $\mathbb{C}'$ be a closed subset of $\mathbb{S}'$.

THEOREM 2.2 (Mapping theorem; Theorem 2.3 of Lindskog, Resnick and Roy (2014)). *Let $h : (\mathbb{S} \setminus \mathbb{C}, \mathscr{S}_{\mathbb{S} \setminus \mathbb{C}}) \to (\mathbb{S}' \setminus \mathbb{C}', \mathscr{S}_{\mathbb{S}' \setminus \mathbb{C}'})$ be a measurable mapping such*

that $h^{-1}(A')$ is bounded away from $\mathbb{C}$ for any $A' \in \mathscr{S}_{\mathbb{S}' \setminus \mathbb{C}'}$ bounded away from $\mathbb{C}'$. Then $\hat{h} : \mathbb{M}(\mathbb{S} \setminus \mathbb{C}) \to \mathbb{M}(\mathbb{S}' \setminus \mathbb{C}')$ defined by $\hat{h}(\nu) = \nu \circ h^{-1}$ is continuous at μ provided $\mu(D_h) = 0$, where D_h is the set of discontinuity points of h.

For our purpose, the following slight extension will prove to be useful in developing rigorous arguments.

LEMMA 2.4.   *Let $\mathbb{S}_0$ be a measurable subset of $\mathbb{S}$, and $h : (\mathbb{S}_0, \mathscr{S}_{\mathbb{S}_0}) \to (\mathbb{S}' \setminus \mathbb{C}', \mathscr{S}_{\mathbb{S}' \setminus \mathbb{C}'})$ be a measurable mapping such that $h^{-1}(A')$ is bounded away from $\mathbb{C}$ for any $A' \in \mathscr{S}_{\mathbb{S}' \setminus \mathbb{C}'}$ bounded away from $\mathbb{C}'$. Then $\hat{h} : \mathbb{M}(\mathbb{S} \setminus \mathbb{C}) \to \mathbb{M}(\mathbb{S}' \setminus \mathbb{C}')$ defined by $\hat{h}(\nu) = \nu \circ h^{-1}$ is continuous at μ provided that $\mu(\partial \mathbb{S}_0 \setminus \mathbb{C}^r) = 0$ and $\mu(D_h \setminus \mathbb{C}^r) = 0$ for all $r > 0$, where D_h is the set of discontinuity points of h.*

When we focus on Lévy processes, we are specifically interested in the case where $\mathbb{S}$ is $\mathbb{R}_+^{\infty \downarrow} \times [0, 1]^\infty$, where $\mathbb{R}_+^{\infty \downarrow} \triangleq \{x \in \mathbb{R}_+^\infty : x_1 \geq x_2 \geq \cdots\}$, and $\mathbb{S}'$ is the Skorokhod space $\mathbb{D}$. We use the usual product metrics $d_{\mathbb{R}_+^{\infty \downarrow}}(x, y) = \sum_{i=1}^\infty \frac{|x_i - y_i| \wedge 1}{2^i}$ and $d_{[0,1]^\infty}(x, y) = \sum_{i=1}^\infty \frac{|x_i - y_i|}{2^i}$ for $\mathbb{R}_+^{\infty \downarrow}$ and $[0, 1]^\infty$, respectively. For the finite product of metric spaces, we use the maximum metric; that is, we use $d_{\mathbb{S}_1 \times \cdots \times \mathbb{S}_d}((x_1, \ldots, x_d), (y_1, \ldots, y_d)) \triangleq \max_{i=1,\ldots,d} d_{\mathbb{S}_i}(x_i, y_i)$ for the product $\mathbb{S}_1 \times \cdots \times \mathbb{S}_d$ of metric spaces $(\mathbb{S}_i, d_{\mathbb{S}_i})$. For $\mathbb{D}$, we use the usual Skorokhod J_1 metric $d(x, y) \triangleq \inf_{\lambda \in \Lambda} \|\lambda - e\| \vee \|x \circ \lambda - y\|$, where Λ denotes the set of all nondecreasing homeomorphisms from $[0, 1]$ onto itself, e denotes the identity and $\| \cdot \|$ denotes the supremum norm. Let

$$S_j \triangleq \{(x, u) \in \mathbb{R}_+^{\infty \downarrow} \times [0, 1]^\infty : 0, 1, u_1, \ldots, u_j \text{ are all distinct}\}.$$

This set will play the role of $\mathbb{S}_0$ of Lemma 2.4. Define $T_j : S_j \to \mathbb{D}$ to be $T_j(x, u) = \sum_{i=1}^j x_i 1_{[u_i, 1]}$. Let $\mathbb{D}_j$ be the subspaces of the Skorokhod space consisting of nondecreasing step functions, vanishing at the origin, with exactly j jumps, and $\mathbb{D}_{\leq j} \triangleq \bigcup_{0 \leq i \leq j} \mathbb{D}_i$, that is, nondecreasing step functions vanishing at the origin with at most j jumps. Similarly, let $\mathbb{D}_{<j} \triangleq \bigcup_{0 \leq i < j} \mathbb{D}_i$. Define $\mathbb{H}_j \triangleq \{x \in \mathbb{R}_+^{\infty \downarrow} : x_j > 0, x_{j+1} = 0\}$, and $\mathbb{H}_{<j} \triangleq \{x \in \mathbb{R}_+^{\infty \downarrow} : x_j = 0\}$. The continuous mapping principle applies to T_j, as we can see in the following result.

LEMMA 2.5 (Lemma 5.3 and Lemma 5.4 of Lindskog, Resnick and Roy (2014)).   *Suppose $A \subset \mathbb{D}$ is bounded away from $\mathbb{D}_{<j}$. Then $T_j^{-1}(A)$ is bounded away from $\mathbb{H}_{<j} \times [0, 1]^\infty$. Moreover, $T_j : S_j \to \mathbb{D}$ is continuous.*

A consequence of Lemma 2.5 and Lemma 2.4 along with the observation that S_j is open is that one can derive a limit theorem in a path space from a limit theorem for jump sizes.

COROLLARY 2.2.   *If $\mu_n \to \mu$ in $\mathbb{M}((\mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty) \setminus (\mathbb{H}_{<j} \times [0,1]^\infty))$, and $\mu(S_j^c \setminus (\mathbb{H}_{<j} \times [0,1]^\infty)^r) = 0$ for all $r > 0$, then $\mu_n \circ T_j^{-1} \to \mu \circ T_j^{-1}$ in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j})$.*

To obtain the large deviations for two-sided Lévy measures, we will first establish the large deviations for independent spectrally positive Lévy processes, and then apply Lemma 2.4 with $h(\xi, \zeta) = \xi - \zeta$. The next lemma verifies two important conditions of Lemma 2.4 for such h. Let $\mathbb{D}_{l,m}$ denote the subspace of the Skorokhod space consisting of step functions vanishing at the origin with exactly l upward jumps and m downward jumps. Given $\alpha, \beta > 1$, let $\mathbb{D}_{<j,k} \triangleq \bigcup_{(l,m) \in \mathbb{I}_{<j,k}} \mathbb{D}_{l,m}$ and $\mathbb{D}_{<(j,k)} \triangleq \bigcup_{(l,m) \in \mathbb{I}_{<j,k}} \mathbb{D}_l \times \mathbb{D}_m$, where $\mathbb{I}_{<j,k} \triangleq \{(l,m) \in \mathbb{Z}_+^2 \setminus \{(j,k)\} : (\alpha - 1)l + (\beta - 1)m \leq (\alpha - 1)j + (\beta - 1)k\}$ and $\mathbb{Z}_+$ denotes the set of nonnegative integers. Note that in the definition of $\mathbb{I}_{<j,k}$, the inequality is not strict; however, we choose to use the strict inequality in our notation to emphasize that (j, k) is not included in $\mathbb{I}_{<j,k}$.

LEMMA 2.6.   *Let $h : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ be defined as $h(\xi, \zeta) \triangleq \xi - \zeta$. Then h is continuous at $(\xi, \zeta) \in \mathbb{D} \times \mathbb{D}$ such that $(\xi(t) - \xi(t-))(\zeta(t) - \zeta(t-)) = 0$ for all $t \in (0, 1]$. Moreover, $h^{-1}(A) \subseteq \mathbb{D} \times \mathbb{D}$ is bounded away from $\mathbb{D}_{<(j,k)}$ for any $A \subseteq \mathbb{D}$ bounded away from $\mathbb{D}_{<j,k}$.*

We next characterize convergence-determining classes for convergence in $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$.

LEMMA 2.7.   *Suppose that* (i) *$\mathcal{A}_p$ is a π-system;* (ii) *each open set $G \subseteq \mathbb{S}$ bounded away from $\mathbb{C}$ is a countable union of sets in $\mathcal{A}_p$; and* (iii) *for each closed set $F \subseteq \mathbb{S}$ bounded away from $\mathbb{C}$, there is a set $A \in \mathcal{A}_p$ bounded away from $\mathbb{C}$ such that $F \subseteq A^\circ$ and $\mu(A \setminus A^\circ) = 0$. If, in addition, $\mu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ and $\mu_n(A) \to \mu(A)$ for every $A \in \mathcal{A}_p$ such that A is bounded away from $\mathbb{C}$, then $\mu_n \to \mu$ in $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$.*

REMARK 1.   Since $\mathbb{S}$ is a separable metric space, the Lindelöf property holds. Therefore, a sufficient condition for assumption (ii) of Lemma 2.7 is that for every $x \in \mathbb{S} \setminus \mathbb{C}$ and $\epsilon > 0$, there is $A \in \mathcal{A}_p$ such that $x \in A^\circ \subseteq B(x, \epsilon)$. To see that this implies assumption (ii), note that for any given open set G, one can construct a cover $\{(A_x)^\circ : x \in G\}$ of G by choosing A_x so that $x \in (A_x)^\circ \subseteq G$ and then extract a countable subcover (due to the Lindelöf property) whose union is equal to G. Note also that if A in assumption (iii) is open, then $\mu(A \setminus A^\circ) = \mu(\varnothing) = 0$ automatically.

3. Sample-path large deviations.   In this section, we present large-deviations results for scaled Lévy processes with heavy-tailed Lévy measures. Section 3.1 studies a special case, where the Lévy measure is concentrated on the positive

part of the real line, and Section 3.2 extends this result to Lévy processes with two-sided Lévy measures. In both cases, let $X_n(t) \triangleq X(nt)$ be a scaled process of X, where X is a Lévy process with a Lévy measure ν. Recall that X_n has Itô representation (see, e.g., Section 2 of Kyprianou (2014)):

$$X_n(s) = nsa + B(ns)$$

$$+ \int_{|x| \leq 1} x [N([0, ns] \times dx) - ns\nu(dx)] + \int_{|x| > 1} x N([0, ns] \times dx),$$

with a a drift parameter, B a Brownian motion, and N a Poisson random measure with mean measure Leb $\times \nu$ on $[0, n] \times (0, \infty)$; Leb denotes the Lebesgue measure.

3.1. *One-sided large deviations.* Let X be a Lévy process with Lévy measure ν. In this section, we assume that ν is a regularly varying (at infinity, with index $-\alpha < -1$) Lévy measure concentrated on $(0, \infty)$. Consider a centered and scaled process

(3.1) $$\bar{X}_n(s) \triangleq \frac{1}{n} X_n(s) - sa - \mu_1^+ \nu_1^+ s,$$

where $\mu_1^+ \triangleq \frac{1}{\nu_1^+} \int_{[1,\infty)} x\nu(dx)$, and $\nu_1^+ \triangleq \nu[1, \infty)$. For each constant $\gamma > 1$, let $\nu_\gamma(x, \infty) \triangleq x^{-\gamma}$, and let ν_γ^j denote the restriction (to $\mathbb{R}_+^{j\downarrow}$) of the j-fold product measure of ν_γ. Let $C_0(\cdot) \triangleq \delta_{\mathbf{0}}(\cdot)$ be the Dirac measure concentrated on the zero function. Additionally, for each $j \geq 1$, define a measure $C_j \in \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j})$ concentrated on $\mathbb{D}_j$ as

$$C_j(\cdot) \triangleq \mathbf{E}\left[\nu_\alpha^j \left\{ y \in (0, \infty)^j : \sum_{i=1}^j y_i 1_{[U_i, 1]} \in \cdot \right\} \right],$$

where the random variables $U_i, i \geq 1$ are i.i.d. uniform on $[0, 1]$.

The proof of the main result of this section hinges critically on the following limit theorem.

THEOREM 3.1. *For each $j \geq 0$,*

(3.2) $$(n\nu[n, \infty))^{-j} \mathbf{P}(\bar{X}_n \in \cdot) \to C_j(\cdot),$$

in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j})$, as $n \to \infty$. Moreover, $\bar{X}_n$ is asymptotically equivalent to a process that assumes values in $\mathbb{D}_{\leqslant \mathcal{J}(A)}$ almost surely.

PROOF SKETCH. The proof of Theorem 3.1 is based on establishing the asymptotic equivalence of $\bar{X}_n$ and the process obtained by just keeping its j biggest jumps, which we will denote by $\hat{J}_n^{\leqslant j}$ in Section 5. Such an equivalence is established via Proposition 5.1, and Proposition 5.2. Then Proposition 5.3 identifies the limit of $\hat{J}_n^{\leqslant j}$, which coincides with the limit in (3.2). The full proof of Theorem 3.1 is provided in Section 5.2. □

Recall that $\mathbb{D}_s^\uparrow$ denotes the subset of $\mathbb{D}$ consisting of nondecreasing step functions vanishing at the origin, and $\mathcal{D}_+(\xi)$ denotes the number of upward jumps of an element ξ in $\mathbb{D}$. Finally, set

$$(3.3) \qquad \mathcal{J}(A) \triangleq \inf_{\xi \in \mathbb{D}_s^\uparrow \cap A} \mathcal{D}_+(\xi).$$

Now we are ready to present the main result of this section, which is the following large-deviations theorem for $\bar{X}_n$.

THEOREM 3.2. *Suppose that A is a measurable set. If $\mathcal{J}(A) < \infty$, and if $A_\delta \cap \mathbb{D}_{\leqslant \mathcal{J}(A)}$ is bounded away from $\mathbb{D}_{< \mathcal{J}(A)}$ for some $\delta > 0$, then*

$$(3.4)$$
$$C_{\mathcal{J}(A)}(A^\circ) \leq \liminf_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}}$$

$$\leq \limsup_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \leq C_{\mathcal{J}(A)}(\bar{A}).$$

If $\mathcal{J}(A) = \infty$, and $A_\delta \cap \mathbb{D}_{\leqslant i+1}$ is bounded away from $\mathbb{D}_{\leqslant i}$ for some $\delta > 0$ and $i \geq 0$, then

$$(3.5) \qquad \lim_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^i} = 0.$$

In particular, in case $\mathcal{J}(A) < \infty$, (3.4) holds if A is bounded away from $\mathbb{D}_{< \mathcal{J}(A)}$; in case $\mathcal{J}(A) = \infty$, (3.5) holds if A is bounded away from $\mathbb{D}_{\leqslant i}$.

PROOF. We first consider the case $\mathcal{J}(A) < \infty$. Note that $\mathcal{J}(A^\circ) > \mathcal{J}(A)$ implies that A° does not contain any element of $\mathbb{D}_{\leqslant \mathcal{J}(A)}$. Since $C_{\mathcal{J}(A)}$ is supported on $\mathbb{D}_{\leqslant \mathcal{J}(A)}$, A° is a $C_{\mathcal{J}(A)}$-null set. Therefore, the lower bound holds trivially if $\mathcal{J}(A^\circ) > \mathcal{J}(A)$. On the other hand, $\mathcal{J}(A) = \mathcal{J}(\bar{A})$. To see this, suppose not, that is, $\mathcal{J}(\bar{A}) < \mathcal{J}(A)$. Then there exists $\zeta \in \mathbb{D}_s^\uparrow \cap \bar{A}$ such that $\zeta \in \mathbb{D}_{< \mathcal{J}(A)}$. This implies that $\zeta \in A_\delta \cap \mathbb{D}_{\leqslant \mathcal{J}(A)}$ for any $\delta > 0$, which is contradictory to the assumption that $A_\delta \cap \mathbb{D}_{\leqslant J(A)}$ is bounded away from $\mathbb{D}_{< \mathcal{J}(A)}$ for some $\delta > 0$. In view of these observations, we can assume w.l.o.g. that $\mathcal{J}(A^\circ) = \mathcal{J}(A) = \mathcal{J}(\bar{A})$. Now, from Theorem 3.1 with $j = \mathcal{J}(A^\circ)$ along with the lower bound of Lemma 2.1 with $\mathbb{S}_0 = \mathbb{D}_{\leqslant \mathcal{J}(A)}$,

$$C_{\mathcal{J}(A)}(A^\circ) = C_{\mathcal{J}(A^\circ)}(A^\circ)$$

$$\leq \liminf_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A^\circ)}{(n\nu[n,\infty))^{\mathcal{J}(A^\circ)}} \leq \liminf_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}}.$$

Similarly, from Theorem 3.1 with $j = \mathcal{J}(\bar{A})$ along with the upper bound of Lemma 2.1,

$$\limsup_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \leq \limsup_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in \bar{A})}{(n\nu[n,\infty))^{\mathcal{J}(\bar{A})}} \leq C_{\mathcal{J}(\bar{A})}(\bar{A}) = C_{\mathcal{J}(A)}(\bar{A}).$$

In case $\mathcal{J}(A) = \infty$, we reach the conclusion by applying Theorem 3.1 with $j = i$ along with noting that $C_i(\bar{A}) = 0$. $\square$

Theorem 3.2 dictates the "right" choice of j in Theorem 3.1 for which (3.2) can lead to a limit in $(0, \infty)$. We conclude this section with an investigation of a sufficient condition for C_j-continuity, that is, we provide a sufficient condition on A which guarantees $C_j(\partial A) = 0$. Such a property implies

$$(3.6) \qquad C_j(A^\circ) = C_j(A) = C_j(\bar{A}),$$

implying that the liminf and limsup in our asymptotic estimates yield the same value. Assume that A is a subset of $\mathbb{D}_j$ bounded away from $\mathbb{D}_{<j}$, that is, $d(A, \mathbb{D}_{<j}) > \gamma$ for some $\gamma > 0$. Consider a path $\xi \in A$. Note that every $\xi \in \mathbb{D}_j$ is determined by the pair of jump sizes and jump times $(x, u) \in (0, \infty)^j \times [0, 1]^j$, that is, $\xi(t) = \sum_{i=1}^{j} x_i 1_{[u_i, 1]}(t)$. Formally, we define a mapping $\hat{T}_j : \hat{S}_j \to \mathbb{D}_j$ by $\hat{T}_j(x, u) = \sum_{i=1}^{j} x_i 1_{[u_i, 1]}$, where $\hat{S}_j \triangleq \{(x, u) \in \mathbb{R}_+^{j\downarrow} \times [0, 1]^j : 0, 1, u_1, \ldots, u_j$ are all distinct$\}$. Since $d(A, \mathbb{D}_{<j}) > \gamma$, we know that $\hat{T}_j(x, u) \in A$ implies $x \in (\gamma, \infty)^j$; see Lemma 5.4 (b). In view of this, we can see that (3.6) holds if the Lebesgue measure of $\hat{T}_j^{-1}(\partial A)$ is 0 since $C_j(A) = \int_{(x,u) \in \hat{T}_j^{-1}(A)} du \, d\nu_\alpha^j(x)$. One of the typical settings that arises in applications is that the set A can be written as a finite combination of unions and intersections of $\phi_1^{-1}(A_1), \ldots, \phi_m^{-1}(A_m)$, where each $\phi_i : \mathbb{D} \to \mathbb{S}_i$ is a continuous function, and all sets A_i are subsets of a general topological space $\mathbb{S}_i$. If we denote this operation of taking unions and intersections by Ψ (i.e., $A = \Psi(\phi_1^{-1}(A_1), \ldots, \phi_m^{-1}(A_m))$), then

$$\Psi(\phi_1^{-1}(A_1^\circ), \ldots, \phi_m^{-1}(A_m^\circ)) \subseteq A^\circ \subseteq A \subseteq \bar{A} \subseteq \Psi(\phi_1^{-1}(\bar{A}_1), \ldots, \phi_m^{-1}(\bar{A}_m)).$$

Therefore, (3.6) holds if $\hat{T}_j^{-1}(\Psi(\phi_1^{-1}(\bar{A}_1), \ldots, \phi_m^{-1}(\bar{A}_m))) \setminus \hat{T}_j^{-1}(\Psi(\phi_1^{-1}(A_1^\circ), \ldots, \phi_m^{-1}(A_m^\circ)))$ has Lebesgue measure zero. A similar principle holds for the limit measures $C_{j,k}$, defined in the next section where we deal with two-sided Lévy processes.

3.2. *Two-sided large deviations.* Consider a two-sided Lévy measure ν for which $\nu[x, \infty)$ is regularly varying with index $-\alpha$ and $\nu(-\infty, -x]$ is regularly varying with index $-\beta$. Let

$$\bar{X}_n(s) \triangleq \frac{1}{n} X_n(s) - sa - (\mu_1^+ \nu_1^+ - \mu_1^- \nu_1^-)s,$$

where

$$\mu_1^+ \triangleq \frac{1}{\nu_1^+} \int_{[1, \infty)} x \nu(dx), \quad \nu_1^+ \triangleq \nu[1, \infty),$$

$$\mu_1^- \triangleq \frac{-1}{\nu_1^-} \int_{(-\infty, -1]} x \nu(dx), \quad \nu_1^- \triangleq \nu(-\infty, -1].$$

Recall the definition of $\mathbb{D}_{j,k}$ given below Corollary 2.2, and the definition of v_α^j and v_β^k as given below (3.1). Let $C_{0,0}(\cdot) \triangleq \delta_{\mathbf{0}}(\cdot)$ be the Dirac measure concentrated on the zero function. For each $(j,k) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$, define a measure $C_{j,k} \in \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$ concentrated on $\mathbb{D}_{j,k}$ as

$$C_{j,k}(\cdot)$$

$$\triangleq \mathbf{E}\left[v_\alpha^j \times v_\beta^k \left\{(x,y) \in (0,\infty)^j \times (0,\infty)^k : \sum_{i=1}^{j} x_i 1_{[U_i,1]} - \sum_{i=1}^{k} y_i 1_{[V_i,1]} \in \cdot\right\}\right],$$

where U_i's and V_i's are i.i.d. uniform on $[0,1]$. Recall that $\mathbb{D}_{<j,k} = \bigcup_{(l,m)\in\mathbb{I}<j,k} \mathbb{D}_{l,m}$ and $\mathbb{I}< j,k = \{(l,m) \in \mathbb{Z}_+^2 \setminus \{(j,k)\} : (\alpha-1)l + (\beta-1)m \leq (\alpha-1)j + (\beta-1)k\}$.

As in the one-sided case, the proof of the main theorem of this section hinges on the following limit theorem.

THEOREM 3.3.   *For each $(j,k) \in \mathbb{Z}_+^2$,*

(3.7) $$\left(nv[n,\infty)\right)^{-j}\left(nv(-\infty,-n]\right)^{-k}\mathbf{P}(\bar{X}_n \in \cdot) \to C_{j,k}(\cdot)$$

in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$ as $n \to \infty$.

The proof of Theorem 3.3 builds on Theorem 3.1, using Lemma 2.2, Lemma 2.3, Lemma 2.6 and Theorem 5.1. We provide the full proof in Section 5.2.

Let $\mathcal{I}(j,k) \triangleq (\alpha-1)j + (\beta-1)k$, and consider a pair of integers $(\mathcal{J}(A), \mathcal{K}(A))$ such that

(3.8) $$\left(\mathcal{J}(A), \mathcal{K}(A)\right) \in \operatorname*{arg\,min}_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A\neq\varnothing}} \mathcal{I}(j,k).$$

The next theorem is the first main result of this section.

THEOREM 3.4.   *Suppose that A is a measurable set. If the argument minimum in (3.8) is nonempty and A is bounded away from $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$, then the argument minimum is unique and*

(3.9)
$$\liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(nv[n,\infty))^{\mathcal{J}(A)}(nv(-\infty,-n])^{\mathcal{K}(A)}} \geq C_{\mathcal{J}(A),\mathcal{K}(A)}(A^\circ),$$

$$\limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(nv[n,\infty))^{\mathcal{J}(A)}(nv(-\infty,-n])^{\mathcal{K}(A)}} \leq C_{\mathcal{J}(A),\mathcal{K}(A)}(\bar{A}).$$

Moreover, if the argument minimum in (3.8) is empty and A is bounded away from $\mathbb{D}_{<l,m} \cup \mathbb{D}_{l,m}$ for some $(l,m) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$, then

(3.10) $$\lim_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(nv[n,\infty))^l(nv(-\infty,-n])^m} = 0.$$

The proof of the theorem is provided below as a consequence of the following lemma.

LEMMA 3.1. *Suppose that a sequence of $\mathbb{D}$-valued random elements Y_n satisfies* (3.7) *(with $\bar{X}_n$ replaced with Y_n) for each $(j, k) \in \mathbb{Z}_+^2$. Then* (3.9) *(with $\bar{X}_n$ replaced with Y_n) holds if A is a measurable set for which the argument minimum in* (3.8) *is nonempty, and A is bounded away from $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$. Moreover,* (3.10) *(with $\bar{X}_n$ replaced with Y_n) holds if the argument minimum in* (3.8) *is empty and A is bounded away from $\mathbb{D}_{<l,m} \cup \mathbb{D}_{l,m}$ for some $(l, m) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$.*

The proof of this lemma is provided in Section 5.2.

PROOF OF THEOREM 3.4. The uniqueness of the argument minimum is immediate from the assumption that A is bounded-away from $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$. Since $\bar{X}_n$ satisfies (3.7) by Theorem 3.3, the conclusion of the theorem follows from applying Lemma 3.1 with $Y_n = \bar{X}_n$. $\square$

In case one is interested in a set for which the arg min of $\mathcal{I}$ in (3.8) is not unique, a natural approach is to partition A into smaller sets and analyze each element separately. In the next theorem, we show that this strategy can be successfully employed with a minimal requirement on A. However, due to the presence of two different slowly varying functions $n^\alpha \nu[n, \infty)$ and $n^\beta \nu(-\infty, -n]$, the limit behavior may not be dominated by a single $\mathbb{D}_{l,m}$.

To deal with this case, let $\mathbb{I}_{=j,k} \triangleq \{(l, m) : (\alpha - 1)l + (\beta - 1)m = (\alpha - 1)j + (\beta - 1)k\}$, $\mathbb{I}_{\ll j,k} \triangleq \{(l, m) : (\alpha - 1)l + (\beta - 1)m < (\alpha - 1)j + (\beta - 1)k\}$, $\mathbb{D}_{=j,k} \triangleq \bigcup_{(l,m) \in \mathbb{I}_{=j,k}} \mathbb{D}_{l,m}$, and $\mathbb{D}_{\ll j,k} \triangleq \bigcup_{(l,m) \in \mathbb{I}_{\ll j,k}} \mathbb{D}_{l,m}$. Denote the slowly varying functions $n^\alpha \nu[n, \infty)$ and $n^\beta \nu(-\infty, -n]$ with $L_+(n)$ and $L_-(n)$, respectively.

THEOREM 3.5. *Let A be a measurable set and suppose that the argument minimum in* (3.8) *is nonempty and contains a pair of integers $(\mathcal{J}(A), \mathcal{K}(A))$. If $A_\delta \cap \mathbb{D}_{=\mathcal{J}(A),\mathcal{K}(A)}$ is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$ for some $\delta > 0$, then for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

(3.11)
$$\mathbf{P}(\bar{X}_n \in A) \geq \frac{\sum_{(l,m)} (C_{l,m}(A^\circ) - \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}},$$

$$\mathbf{P}(\bar{X}_n \in A) \leq \frac{\sum_{(l,m)} (C_{l,m}(\bar{A}) + \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}},$$

for all $n \geq N$, where the summations are over the pairs $(l, m) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$. In particular, (3.11) *holds if A is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$.*

PROOF.   Note first that from Lemma 5.5 (i), there exists a $\delta' > 0$ such that $\mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}$ is bounded away from $A \cap (\mathbb{D}_{l,m})_{\delta'}$ for all $(l, m) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}$. Moreover, applying Lemma 5.5 (ii) to each $A \cap (\mathbb{D}_{l,m})_{\delta'}$, we conclude that there exists $\rho > 0$ such that $A \cap (\mathbb{D}_{l,m})_\rho$ is bounded away from $(\mathbb{D}_{j,k})_\rho$ for any two distinct pairs $(l, m), (j, k) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}$. This means that $A \cap (\mathbb{D}_{l,m})_\rho$'s are all disjoint and bounded away from $\mathbb{D}_{<l,m}$.

To derive the lower bound, we apply Theorem 3.4 to $A^\circ \cap (\mathbb{D}_{l,m})^\rho$ to obtain

$$
\begin{aligned}
C_{l,m}(A^\circ) &= C_{l,m}(A^\circ \cap \mathbb{D}_{l,m}) \\
&= C_{l,m}(A^\circ \cap \mathbb{D}_{l,m} \cap (\mathbb{D}_{l,m})^\rho) \\
&= C_{l,m}(A^\circ \cap (\mathbb{D}_{l,m})^\rho) \\
&\leq \liminf_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A^\circ \cap (\mathbb{D}_{l,m})^\rho)}{(n\nu[n, \infty))^l (n\nu(-\infty, -n])^m} \\
&\leq \liminf_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A \cap (\mathbb{D}_{l,m})^\rho)}{(n\nu[n, \infty))^l (n\nu(-\infty, -n])^m},
\end{aligned}
$$

for each $(l, m) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}$. That is, for any given $\epsilon > 0$, there exists an $N_{l,m} \in \mathbb{N}$ such that

$$
(3.12) \qquad \frac{(C_{l,m}(A^\circ) - \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)l + (\beta-1)m}} \leq \mathbf{P}(\bar{X}_n \in A \cap (\mathbb{D}_{l,m})^\rho),
$$

for all $n \geq N_{l,m}$. Meanwhile, the following obvious bound holds:

$$
(3.13) \qquad 0 \leq \mathbf{P}\left(\bar{X}_n \in A \setminus \bigcup_{(l,m) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho \right).
$$

Since $(\alpha - 1)l + (\beta - 1)m = (\alpha - 1)\mathcal{J}(A) + (\beta - 1)\mathcal{K}(A)$ for $(l, m) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}$, summing (3.12) over $(l, m) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}$ together with (3.13), we arrive at the lower bound of the theorem, with $N = \max_{(l,m) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}} N_{l,m}$.

Turning to the upper bound, we apply Theorem 3.4 to $\bar{A} \cap (\mathbb{D}_{l,m})_\rho$ to get

$$
\limsup_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in \bar{A} \cap (\mathbb{D}_{l,m})_\rho)}{(n\nu[n, \infty))^l (n\nu(-\infty, -n])^m} \leq C_{l,m}(\bar{A} \cap (\mathbb{D}_{l,m})_\rho) = C_{l,m}(\bar{A})
$$

for each $(l, m) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}$. That is, for any given $\epsilon > 0$, there exists $N'_{l,m} \in \mathbb{N}$ such that

$$
(3.14) \qquad \mathbf{P}(\bar{X}_n \in A \cap (\mathbb{D}_{l,m})_\rho) \leq \frac{(C_{l,m}(\bar{A}) + \epsilon/2) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)l + (\beta-1)m}},
$$

for all $n \geq N'_{l,m}$. On the other hand, since $\bar{A} \setminus \bigcup_{(l,m) \in \mathbb{I}_{= \mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho$ is closed and bounded away from $\mathbb{D}_{< \mathcal{J}(A), \mathcal{K}(A)}$,

$$
\limsup_{n \to \infty} \frac{\mathbf{P}(\bar{X}_n \in A \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\rho)}{(n\nu[n, \infty))^{\mathcal{J}(A)} (n\nu(-\infty, -n])^{\mathcal{K}(A)}} \leq C_{\mathcal{J}(A), \mathcal{K}(A)}\left(\bar{A} \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\rho \right),
$$

where the union is over the pairs $(l, m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$. Therefore, there exists N' such that

$$
\mathbf{P}\left(\bar{X}_n \in A \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\rho\right)
$$

$$
(3.15) \qquad \leq \frac{(C_{\mathcal{J}(A), \mathcal{K}(A)}(\bar{A} \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\rho) + \epsilon/2) L_+^{\mathcal{J}(A)}(n) L_-^{\mathcal{K}(A)}(n)}{n^{(\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)}}
$$

$$
= \frac{(\epsilon/2) L_+^{\mathcal{J}(A)}(n) L_-^{\mathcal{K}(A)}(n)}{n^{(\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)}},
$$

for $n \geq N'$ since $\bar{A} \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\rho$ is disjoint from the support of $C_{\mathcal{J}(A), \mathcal{K}(A)}$. Summing (3.14) over $(l, m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$ and (3.15),

$$
(3.16) \qquad \mathbf{P}(\bar{X}_n \in A) \leq \frac{\sum_{(l,m)} (C_{l,m}(\bar{A}) + \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)}},
$$

for $n \geq N$, where $N = N' \vee \max_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} N'_{l,m}$. $\square$

4. Implications.

This section explores the implications of the large-deviations results in Section 3, and is organized as follows. Section 4.1 proves a result similar to Theorem 3.4, now focusing on random walks with regularly varying increments. Section 4.2 illustrates that conditional limit theorems can easily be studied by means of the limit theorems established in Section 3. Section 4.3 develops a weak large deviation priciple (LDP) of the form (1.3) for the scaled Lévy processes. Finally, Section 4.4 shows that the weak LDP proved in Section 4.3 is the best one can hope for in the presence of regularly varying tails, by showing that a full LDP of the form (1.3) does not exist.

4.1. *Random walks.* Let $S_k, k \geq 0$, be a mean-zero random walk, set $\bar{S}_n(t) = S_{[nt]}/n, t \geq 0$, and define $\bar{S}_n = \{\bar{S}_n(t), t \in [0, 1]\}$. Let $N(t), t \geq 0$, be an independent unit rate Poisson process. Define the Lévy process $X(t) \triangleq S_{N(t)}, t \geq 0$, and set $\bar{X}_n(t) \triangleq X(nt)/n, t \geq 0$. The goal is to prove an analogue of Theorem 3.4 for the scaled random walk $\bar{S}_n$. Let $\mathcal{J}(\cdot)$, $\mathcal{K}(\cdot)$, and $C_{j,k}(\cdot)$ be defined as in Section 3.2.

THEOREM 4.1. *Suppose that* $\mathbf{P}(S_1 \geq x)$ *is regularly varying with index* $-\alpha$ *and* $\mathbf{P}(S_1 \leq -x)$ *is regularly varying with index* $-\beta$. *Let* A *be a measurable set bounded away from* $\mathbb{D}_{<\mathcal{J}(A), \mathcal{K}(A)}$. *Then*

$$
(4.1) \qquad
\begin{aligned}
\liminf_{n \to \infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{(n\mathbf{P}(S_1 \geq n))^{\mathcal{J}(A)} (n\mathbf{P}(S_1 \leq -n))^{\mathcal{K}(A)}} &\geq C_{\mathcal{J}(A), \mathcal{K}(A)}(A^\circ), \\
\limsup_{n \to \infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{(n\mathbf{P}(S_1 \geq n))^{\mathcal{J}(A)} (n\mathbf{P}(S_1 \leq -n))^{\mathcal{K}(A)}} &\leq C_{\mathcal{J}(A), \mathcal{K}(A)}(\bar{A}).
\end{aligned}
$$

PROOF.   The idea is to show that $\bar{S}_n$ and $\bar{X}_n$ has the same large deviations limit. First, we derive the asymptotic behavior of the Lévy measure of the constructed Lévy process $\bar{X}_n$. From Example A3.17 in Embrechts, Klüppelberg and Mikosch (1997), we obtain $\mathbf{P}(X(1) \geq x) \sim \mathbf{P}(S_1 \geq x)$. Moreover, Embrechts, Goldie and Veraverbeke (1979) implies that $\nu[x, \infty) \sim \mathbf{P}(S_1 \geq x)$. Similarly, it follows that $\nu(-\infty, -x] \sim \mathbf{P}(S_1 \leq -x)$.

Now, from Lemma 3.1, (4.1) is proved if (3.7) holds for $\bar{S}_n$. To show that (3.7) holds, we first claim that

(4.2)
$$\limsup_{n \to \infty} \frac{\mathbf{P}(d(\bar{X}_n, \mathbb{D}_{<j,k}) \geq \gamma, d(\bar{X}_n, \bar{S}_n) \geq \delta)}{(n\nu[n, \infty))^j (n\nu(-\infty, -n])^k}$$
$$= \limsup_{n \to \infty} \frac{\mathbf{P}(d(\bar{S}_n, \mathbb{D}_{<j,k}) \geq \gamma, d(\bar{X}_n, \bar{S}_n) \geq \delta)}{(n\nu[n, \infty))^j (n\nu(-\infty, -n])^k} = 0.$$

To prove this claim, we consider $\bar{X}_n$ on a longer time horizon $[0, 2]$. Let $\bar{X}_n^{[0,2]}$ denote the stochastic process $\{\bar{X}_n(t), t \in [0, 2]\}$, and $\mathbb{D}_{<j,k}^{[0,2]}$ denote the space of step functions on $[0, 2]$ that corresponds to $\mathbb{D}_{<j,k}$. Let $d^{[0,2]}$ denote the Skokhod J_1 metric on $\mathbb{D}([0, 2], \mathbb{R})$. Note that $d(\bar{X}_n, \mathbb{D}_{<j,k}) \geq \gamma$ implies that $d^{[0,2]}(\bar{X}_n^{[0,2]}, \mathbb{D}_{<j,k}^{[0,2]}) \geq \gamma$, and $d(\bar{S}_n, \mathbb{D}_{<j,k}) \geq \gamma$ implies that either $d^{[0,2]}(\bar{X}_n^{[0,2]}, \mathbb{D}_{<j,k}^{[0,2]}) \geq \gamma$ or $N(2n) \leq n$. Therefore, we see that (4.2) is implied by

(4.3)
$$\limsup_{n \to \infty} \frac{\mathbf{P}(d^{[0,2]}(\bar{X}_n^{[0,2]}, \mathbb{D}_{<j,k}^{[0,2]}) \geq \gamma, d(\bar{X}_n, \bar{S}_n) \geq \delta)}{(n\nu[n, \infty))^j (n\nu(-\infty, -n])^k} = 0.$$

To prove (4.3), we construct a piecewise linear nondecreasing homeomorphism $\bar{\lambda}_n$ as follows. Let $t_0 \triangleq 0$ and t_i be ith jump time of $N(n\cdot)$. Let $L \triangleq (n-1) \wedge N(n)$. Define $\bar{\lambda}_n$ in such a way that $\bar{\lambda}_n(t) = \frac{1}{n}N(nt)$ on $t_0, \ldots, t_L$, $\bar{\lambda}_n(1) = 1$, and $\bar{\lambda}_n$ is a linear interpolation in between. For such $\bar{\lambda}_n$, $\bar{S}_n(\bar{\lambda}_n(t)) = \bar{X}_n(t)$ at all $t \in [0, t_L]$, and hence, $d(\bar{S}_n, \bar{X}_n) \leq \|\bar{\lambda}_n - e\|_\infty \vee \|\bar{S}_n \circ \bar{\lambda}_n - \bar{X}_n\|_\infty = \|\bar{\lambda}_n - e\|_\infty \vee \sup_{t \in [t_L, 1]} |\bar{S}_n \circ \bar{\lambda}_n(t) - \bar{X}_n(t)|$. Note that by the construction of $\bar{\lambda}_n$, $\sup_{t \in [t_L, 1]} |\bar{S}_n \circ \bar{\lambda}_n(t) - \bar{X}_n(t)| \leq \sup_{s,t \in [1-\epsilon, 1+\epsilon]} |\bar{X}_n(t) - \bar{X}_n(s)|$ on $N(n - \epsilon n) + 1 \leq n \leq N(n + \epsilon n)$, and hence,

$$\mathbf{P}\big(d(\bar{X}_n^{[0,2]}, \mathbb{D}_{<j,k}^{[0,2]}) \geq \gamma, d(\bar{X}_n, \bar{S}_n) \geq \delta\big)$$

(4.4)
$$\leq \mathbf{P}\Big(d(\bar{X}_n^{[0,2]}, \mathbb{D}_{<j,k}^{[0,2]}) \geq \gamma, \sup_{s,t \in [1-\epsilon, 1+\epsilon]} |\bar{X}_n(t) - \bar{X}_n(s)| \geq \delta\Big)$$
$$+ \mathbf{P}\big(\{N(n - \epsilon n) + 1 \leq n \leq N(n + \epsilon n)\}^c\big) + \mathbf{P}\big(\|\bar{\lambda}_n - e\|_\infty \geq \delta\big).$$

The last two terms of (4.4) decay at geometric rates and are negligible asymptotically. Applying Theorem 3.4 on $\mathbb{D}[0, 2]$ to the first term of (4.4) and then letting $\epsilon \to 0$, we arrive at (4.3). This proves the claim (4.2).

Now we are ready to prove (3.7) for $\bar{S}_n$ and conclude the proof of Theorem 4.1. Let G be an open set bounded away from $\mathbb{D}_{<j,k}$ so that there exists a $\gamma > 0$ such that $\bar{X}_n \in G$ implies $d(\bar{X}_n, \mathbb{D}_{<j,k}) > \gamma$. Since $\mathbf{P}(\bar{S}_n \in G) \geq \mathbf{P}(\bar{X}_n \in G^{-\delta}, d(\bar{X}_n, \bar{S}_n) < \delta) = \mathbf{P}(\bar{X}_n \in G^{-\delta}) - \mathbf{P}(\bar{X}_n \in G^{-\delta}, d(\bar{X}_n, \bar{S}_n) \geq \delta) \geq \mathbf{P}(\bar{X}_n \in G^{-\delta}) - \mathbf{P}(d(\bar{X}_n, \mathbb{D}_{<j,k}) \geq \gamma, d(\bar{X}_n, \bar{S}_n) \geq \delta)$, (4.2) implies that

$$\liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in G)}{(n\nu[n,\infty))^j (n\nu(-\infty,-n])^k} \geq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in G^{-\delta})}{(n\nu[n,\infty))^j (n\nu(-\infty,-n])^k}$$
$$\geq C_{j,k}(G^{-\delta}).$$

We arrive at the lower bound for (3.7) by taking $\delta \to 0$. Turning to the upper bound for (3.7), consider a closed set F bounded away from $\mathbb{D}_{<j,k}$. Then there exists a $\gamma > 0$ such that $\bar{S}_n \in F$ implies $d(\bar{S}_n, \mathbb{D}_{<j,k}) > \gamma$. Since for any sufficiently small $\delta > 0$ such that F_δ is bounded away from $\mathbb{C}$, $\mathbf{P}(\bar{S}_n \in F) = \mathbf{P}(\bar{S}_n \in F, d(\bar{X}_n, \bar{S}_n) < \delta) + \mathbf{P}(\bar{S}_n \in F, d(\bar{X}_n, \bar{S}_n) \geq \delta) \leq \mathbf{P}(\bar{X}_n \in F_\delta) + \mathbf{P}(d(\bar{S}_n, \mathbb{D}_{<j,k}) \geq \gamma, d(\bar{X}_n, \bar{S}_n) \geq \delta)\mathbf{P}(\bar{X}_n \in F_\delta)$, from (4.2),

$$\limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in F)}{(n\nu[n,\infty))^j (n\nu(-\infty,-n])^k} \leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in F_\delta)}{(n\nu[n,\infty))^j (n\nu(-\infty,-n])^k}$$
$$\leq C_{j,k}(F_\delta).$$

Taking $\delta \to 0$, we arrive at the upper bound for (3.7). This concludes the proof of Theorem 4.1. $\square$

4.2. Conditional limit theorems.

As before, $\bar{X}_n$ denotes the scaled Lévy process defined as in Section 3.1 for the one-sided case and Section 3.2 for the two-sided case, respectively. In this section, we present conditional limit theorems which give a precise description of the limit law of $\bar{X}_n$ conditional on $\bar{X}_n \in A$.

The next result, for the one-sided case, follows immediately from the definition of weak convergence and Theorem 3.2.

COROLLARY 4.1. *Suppose that a measurable subset B of $\mathbb{D}$ satisfies the same conditions as those that A is required to satisfy in Theorem 3.2 and that $C_{\mathcal{J}(B)}(B^\circ) = C_{\mathcal{J}(B)}(B) = C_{\mathcal{J}(B)}(\bar{B}) > 0$. Let $\bar{X}_n^{|B}$ be a process having the conditional law of $\bar{X}_n$ given that $\bar{X}_n \in B$, then there exists a process $\bar{X}_\infty^{|B}$ such that*

$$\bar{X}_n^{|B} \Rightarrow \bar{X}_\infty^{|B}$$

in $\mathbb{D}$. Moreover, if $\mathbf{P}^{|B}(\cdot)$ is the law of $\bar{X}_\infty^{|B}$, then

$$\mathbf{P}^{|B}\big(\bar{X}_\infty^{|B} \in \cdot\big) := \frac{C_{\mathcal{J}(B)}(\cdot \cap B)}{C_{\mathcal{J}(B)}(B)}.$$

Let us provide a more direct probabilistic description of the process $\bar{X}_\infty^{|B}$. Directly from the definition of $\mathbf{P}^{|B}$ we have that

$$\bar{X}_\infty^{|B}(t) = \sum_{n=1}^{\mathcal{J}(B)} \chi_n 1_{[U_n, 1]}(t),$$

where $U_1, \ldots, U_{\mathcal{J}(B)}$ are i.i.d. uniform random variables on $[0, 1]$ and

$$\mathbf{P}^{|B}(\chi_1 \in dx_1, \ldots, \chi_{\mathcal{J}(B)} \in dx_{\mathcal{J}(B)})$$
$$= \frac{\prod_{i=1}^{\mathcal{J}(B)} (\alpha x_i^{-\alpha-1} dx_i) \mathbb{I}(x_{\mathcal{J}(B)} > \cdots > x_1 > 0) \mathbf{P}(\sum_{n=1}^{\mathcal{J}(B)} x_n 1_{[U_n, 1]}(\cdot) \in B)}{C_{\mathcal{J}(B)}(B)}.$$

An easier to interpret description of $\mathbf{P}^{|B}$ can be obtained by using the fact that $\delta_B := d(B, \mathbb{D}_{\leqslant \mathcal{J}(B)-1}) > 0$. Define an auxiliary probability measure, $\mathbf{P}_\#^{|B}$, under which, not only $U_1, \ldots, U_{\mathcal{J}(B)}$ are i.i.d. Uniform$(0, 1)$, but also $\chi_1, \ldots, \chi_{\mathcal{J}(B)}$ are i.i.d. distributed Pareto(α, δ_B) and independent of the U_i's; that is,

$$\mathbf{P}_\#^{|B}(\chi_1 \in dx_1, \ldots, \chi_{\mathcal{J}(B)} \in dx_{\mathcal{J}(B)})$$
$$= (\alpha/\delta_B)^{\mathcal{J}(B)} \prod_{i=1}^{\mathcal{J}(B)} (x_i/\delta_B)^{-\alpha-1} dx_i \mathbb{I}(x_i \geq \delta_B).$$

Then, we have that

(4.5) $$\mathbf{P}^{|B}(\bar{X}_\infty^{|B} \in \cdot) = \mathbf{P}_\#^{|B}(\bar{X}_\infty^{|B} \in \cdot \mid \bar{X}_\infty^{|B} \in B).$$

Moreover, note that

(4.6) $$\mathbf{P}_\#^{|B}(\bar{X}_\infty^{|B} \in B) = \delta_B^{-\mathcal{J}(B)(\alpha+2)} C_{\mathcal{J}(B)}(B) > 0.$$

In view of (4.5) and (4.6) one can say, at least qualitatively, that the most likely way in which the event $\bar{X}_n \in B$ is seen to occur is by means of $\mathcal{J}(B)$ i.i.d. jumps which are suitably Pareto distributed and occurring uniformly throughout the time interval $[0, 1]$.

We now are ready to provide the corresponding conditional limit theorem for the two-sided case, building on Theorem 3.4. The proof is again immediate, using the definition of weak convergence.

COROLLARY 4.2. *Suppose that a subset B of $\mathbb{D}$ satisfies the conditions in Theorem* 3.4 *and that*

$$C_{\mathcal{J}(B),\mathcal{K}(B)}(B^\circ) = C_{\mathcal{J}(B),\mathcal{K}(B)}(B) = C_{\mathcal{J}(B),\mathcal{K}(B)}(\bar{B}) > 0.$$

Let $\bar{X}_n^{|B}$ be a process having the conditional law of $\bar{X}_n$ given that $\bar{X}_n \in B$, then there exists a process $\bar{X}_\infty^{|B}$ such that

$$\bar{X}_n^{|B} \Rightarrow \bar{X}_\infty^{|B}$$

in $\mathbb{D}$. *Moreover, if* $\mathbf{P}^{|B}(\cdot)$ *is the law of* $\bar{X}_\infty^{|B}$, *then*

$$\mathbf{P}^{|B}\big(\bar{X}_\infty^{|B} \in \cdot\big) := \frac{C_{\mathcal{J}(B),\mathcal{K}(B)}(\cdot \cap B)}{C_{\mathcal{J}(B),\mathcal{K}(B)}(B)}.$$

A probabilistic description, completely analogous to that given for the one-sided case, can also be provided in this case. Define $\delta_B = d(B, \mathbb{D}_{<\mathcal{J}(B),\mathcal{K}(B)}) > 0$ and introduce a probability measure $\mathbf{P}_\#^{|B}$ under which we have the following: First, $U_1, \ldots, U_{\mathcal{J}(B)}, V_1, \ldots, V_{\mathcal{K}(B)}$ are i.i.d. $U(0,1)$; second, $\chi_1, \ldots, \chi_{\mathcal{J}(B)}$ are i.i.d. Pareto(α, δ_B), and, finally $\varrho_1, \ldots, \varrho_{\mathcal{K}(B)}$ are i.i.d. Pareto(β, δ_B) random variables (all of these random variables are mutually independent). Then write

$$\bar{X}_\infty^{|B}(t) = \sum_{n=1}^{\mathcal{J}(B)} \chi_n 1_{[U_n,1]}(t) - \sum_{n=1}^{\mathcal{K}(B)} \varrho_n 1_{[V_n,1]}(t).$$

Applying the same reasoning as in the one-sided case we have that

$$\mathbf{P}^{|B}\big(\bar{X}_\infty^{|B} \in \cdot\big) = \mathbf{P}_\#^{|B}\big(\bar{X}_\infty^{|B} \in \cdot \mid \bar{X}_\infty^{|B} \in B\big)$$

and

$$\mathbf{P}_\#^{|B}\big(\bar{X}_\infty^{|B} \in B\big) = \delta_B^{-\mathcal{J}(B)(\alpha+2)-\mathcal{K}(B)(\beta+2)} C_{\mathcal{J}(B),\mathcal{K}(B)}(B) > 0.$$

We note that these results also hold for random walks, and thus forms a significant extension of Theorem 3.1 in Durrett (1980), where it is assumed that $\alpha > 2$ and $B = \{\bar{X}_n(1) \geq a\}$.

4.3. *Large deviation principle.* In this section, we show that $\bar{X}_n$ satisfies a weak large deviation principle with speed $\log n$, and a rate function which is piece-wise constant in the number of discontinuities. More specifically, define

(4.7)
$$I(\xi) \triangleq \begin{cases} (\alpha-1)\mathcal{D}_+(\xi) + (\beta-1)\mathcal{D}_-(\xi) \\ \quad \text{if } \xi \text{ is a step function } \& \xi(0) = 0; \\ \infty \quad \text{otherwise,} \end{cases}$$

where $\mathcal{D}_-(\xi)$ denotes the number of downward jumps in ξ.

THEOREM 4.2. *The scaled process* $\bar{X}_n$ *satisfies the weak large deviation principle with rate function* I *and speed* $\log n$, *that is,*

(4.8)
$$-\inf_{x \in G} I(x) \leq \liminf_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in G)}{\log n}$$

for every open set G, *and*

(4.9)
$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in K)}{\log n} \leq -\inf_{x \in K} I(x)$$

for every compact set K.

The proof of Theorem 4.2 is provided in Section 5.3. It is based on Theorem 3.4, and a reduction of the case of general A to open neighborhoods; reminiscent of arguments made in the proof of Cramérs theorem Dembo and Zeitouni (2010).

4.4. *Nonexistence of strong large deviation principle.* We conclude the current section by showing that the weak LDP presented in the previous section is the best one can hope for from the classical LDP framework in our heavy-tailed setting, in the sense that for any Lévy process X with a regularly varying Lévy measure, $\bar{X}_n$ cannot satisfy a strong LDP, that is, (4.9) in Theorem 4.2 cannot be extended to all closed sets.

Consider a mapping $\pi : \mathbb{D} \to \mathbb{R}_+^2$ that maps paths in $\mathbb{D}$ to their largest jump sizes, that is,

$$\pi(\xi) \triangleq \Big(\sup_{t \in (0,1]} \big(\xi(t) - \xi(t-)\big), \sup_{t \in (0,1]} \big(\xi(t-) - \xi(t)\big)\Big).$$

The supremum in the above definition should be attained since the number of jumps whose sizes are greater than ϵ should be finite for any fixed $\epsilon > 0$. Note that π is continuous, since each coordinate is continuous. For example, if the first coordinate (the largest upward jump size) of $\pi(\xi)$ and $\pi(\zeta)$ differ by ϵ, then $d(\xi, \zeta) \geq \epsilon/2$, which implies that the first coordinate is continuous. Now, to derive a contradiction, suppose that $\bar{X}_n$ satisfies a strong LDP. In particular, suppose (4.9) in Theorem 4.2 is true for all closed sets rather than just compact sets. Since π is continuous w.r.t. the J_1 metric, $\pi(\bar{X}_n)$ has to satisfy a strong LDP with rate function $I'(y) = \inf\{I(\xi) : \xi \in \mathbb{D}, y = \pi(x)\}$ by the contraction principle, in case I' is a rate function. (Since I is not a good rate function, I' is not automatically guaranteed to be a rate function per se; see, e.g., Theorem 4.2.1 and the subsequent remarks of Dembo and Zeitouni (2010).) From the exact form of I', given by

$$I'(y_1, y_2) = (\alpha - 1)\mathbb{I}(y_1 > 0) + (\beta - 1)\mathbb{I}(y_2 > 0),$$

one can check that I' indeed happens to be a rate function. For the sake of simplicity, suppose that $\alpha = \beta = 2$, and $\nu[x, \infty) = \nu(-\infty, -x] = x^{-2}$. Let $\hat{J}_n^{\leqslant 1} \triangleq \frac{1}{n} Q_n^{\leftarrow}(\Gamma_1) 1_{[U_1, 1]}$ and $\hat{K}_n^{\leqslant 1} \triangleq \frac{1}{n} R_n^{\leftarrow}(\Delta_1) 1_{[V_1, 1]}$ where $Q_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu[s, \infty) < y\} = (n/y)^{1/2}$ and $R_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu(-\infty, -s] < y\} = (n/y)^{1/2}$. The random variables Γ_1 and Δ_1 are standard exponential, and U_1, V_1 uniform $[0, 1]$ (see also Section 5 for similar and more general notational conventions). Note that $\bar{Y}_n \triangleq (\hat{J}_n^{\leqslant 1}, \hat{K}_n^{\leqslant 1})$ is exponentially equivalent to $\pi(\bar{X}_n)$ if we couple $\pi(\bar{X}_n)$ and $(\hat{J}_n^{\leqslant 1}, \hat{K}_n^{\leqslant 1})$, using the representation of $\bar{X}_n$ as in (5.4): for any $\delta > 0$, $\mathbf{P}(|\bar{Y}_n - \pi(\bar{X}_n)| > \delta) \leq \mathbf{P}(\bar{Y}_n \neq \pi(\bar{X}_n)) = \mathbf{P}(Q_n^{\leftarrow}(\Gamma_1) \leq 1 \text{ or } R_n^{\leftarrow}(\Delta_1) \leq$

1), which decays at an exponential rate. Hence,

$$\frac{\log \mathbf{P}(|\bar{Y}_n - \pi(\bar{X}_n)| > \delta)}{\log n} \to -\infty,$$

as $n \to \infty$, where $|\cdot|$ is the Euclidean distance. As a result, $\bar{Y}_n$ should satisfy the same (strong) LDP as $\pi(\bar{X}_n)$. Now, consider the set $A \triangleq \bigcup_{k=2}^{\infty} [\log k, \infty) \times [k^{-1/2}, \infty)$. Then, since $[\log k, \infty) \times [k^{-1/2}, \infty) \subseteq A$ for $k \geq 2$,

$$\mathbf{P}(\bar{Y}_n \in A) \geq \mathbf{P}((\hat{J}_n^{\leqslant 1}, \hat{K}_n^{\leqslant 1}) \in [\log n, \infty) \times [n^{-1/2}, \infty))$$

$$= \mathbf{P}(Q_n^{\leftarrow}(\Gamma_1) > n \log n, R_n^{\leftarrow}(\Delta_1) > n^{1/2})$$

$$= \mathbf{P}\left(\left(\frac{n}{\Gamma_1}\right)^{1/2} > n \log n, \left(\frac{n}{\Delta_1}\right)^{1/2} > n^{1/2}\right)$$

$$= \mathbf{P}\left(\Gamma_1 < \frac{1}{n(\log n)^2}\right)\mathbf{P}(\Delta_1 < 1)$$

$$= \left(1 - e^{-\frac{1}{n(\log n)^2}}\right)\left(1 - e^{-1}\right).$$

Thus,

$$\limsup_{n \to \infty} \mathbf{P}(\bar{Y}_n \in A) \geq \limsup_{n \to \infty} \frac{\log(1 - e^{-\frac{1}{n(\log n)^2}})(1 - e^{-1})}{\log n}$$

$$\geq \limsup_{n \to \infty} \frac{\log \frac{1}{n(\log n)^2}(1 - \frac{1}{2n(\log n)^2})(1 - e^{-1})}{\log n}$$

$$= -1.$$

On the other hand, since $A \subseteq (0, \infty) \times (0, \infty)$,

(4.10) $$- \inf_{(y_1, y_2) \in A} I'(y_1, y_2) = -2.$$

Noting that A is a closed (but not compact) set, we arrive at a contradiction to the large deviation upper bound for $\bar{Y}_n$. This, in turn, proves that $\bar{X}_n$ cannot satisfy a full LDP.

5. Proofs. Section 5.1, Section 5.2 and Section 5.3 provide proofs of the results in Section 2, Section 3 and Section 4, respectively.

5.1. *Proofs of Section* 2. Recall that $F_\delta = \{x \in \mathbb{S} : d(x, F) \leq \delta\}$ and $G^{-\delta} = ((G^c)_\delta)^c$.

PROOF OF LEMMA 2.1. Let G be an open set such that $G \cap \mathbb{S}_0$ is bounded away from $\mathbb{C}$. For a given $\delta > 0$, due to the assumed asymptotic equivalence,

$\mathbf{P}(X_n \in G^{-\delta}, d(X_n, Y_n) \geq \delta) = o(\epsilon_n)$. Therefore,

$$\liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in G)$$

$$\geq \liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta}, d(X_n, Y_n) < \delta)$$

(5.1)

$$= \liminf_{n \to \infty} \epsilon_n^{-1} \{ \mathbf{P}(X_n \in G^{-\delta}) - \mathbf{P}(X_n \in G^{-\delta}, d(X_n, Y_n) \geq \delta) \}$$

$$= \liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta}).$$

Pick $r > 0$ such that $G^{-\delta} \cap \mathbb{S}_0 \cap \mathbb{C}_r = 0$ and note that $G^{-\delta} \cap (\mathbb{C}_r)^c$ is an open set bounded away from $\mathbb{C}$. Then

$$\liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta}) = \liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta} \cap \mathbb{S}_0)$$

$$= \liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta} \cap \mathbb{S}_0 \cap (\mathbb{C}_r)^c)$$

$$= \liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta} \cap (\mathbb{C}_r)^c)$$

$$\geq \mu(G^{-\delta} \cap (\mathbb{C}_r)^c)$$

$$= \mu(G^{-\delta} \cap (\mathbb{C}_r)^c \cap \mathbb{S}_0)$$

$$= \mu(G^{-\delta} \cap \mathbb{S}_0)$$

$$= \mu(G^{-\delta}).$$

Since G is an open set, $G = \bigcup_{\delta > 0} G^{-\delta}$. Due to the continuity of measures, $\lim_{\delta \to 0} \mu(G^{-\delta}) = \mu(G)$, and hence, we arrive at the lower bound

$$\liminf_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in G) \geq \mu(G)$$

by taking $\delta \to 0$.

Now, turning to the upper bound, consider a closed set F such that $F_\delta \cap \mathbb{S}_0$ is bounded away from $\mathbb{C}$. Given a $\delta > 0$, by the equivalence assumption, $\mathbf{P}(Y_n \in F, d(X_n, Y_n) \geq \delta) = o(\epsilon_n)$. Therefore,

$$\limsup_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in F)$$

$$= \limsup_{n \to \infty} \epsilon_n^{-1} \{ \mathbf{P}(Y_n \in F, d(X_n, Y_n) < \delta)$$

$$+ \mathbf{P}(Y_n \in F, d(X_n, Y_n) \geq \delta) \}$$

$$= \limsup_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in F_\delta)$$

(5.2)

$$= \limsup_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in F_\delta \cap \mathbb{S}_0)$$

$$\leq \limsup_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in \overline{F_\delta \cap \mathbb{S}_0})$$

$$\leq \mu(\overline{F_\delta \cap \mathbb{S}_0}) = \mu(\overline{F_\delta \cap \mathbb{S}_0} \cap \mathbb{S}_0)$$

$$\leq \mu(\bar{F}_\delta \cap \mathbb{S}_0) = \mu(\bar{F}_\delta) = \mu(F_\delta).$$

Note that $\{F_\delta\}$ is a decreasing sequence of sets, $F = \bigcap_{\delta>0} F_\delta$ (since F is closed), and $\mu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ (and hence, μ is a finite measure on $\mathbb{S} \setminus \mathbb{C}^r$ for some $r > 0$ such that $F_\delta \subseteq \mathbb{S} \setminus \mathbb{C}^r$ for some $\delta > 0$). Due to the continuity (from above) of finite measures, $\lim_{\delta \to 0} \mu(F_\delta) = \mu(F)$. Therefore, we arrive at the upper bound

$$\limsup_{n \to \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in F) \leq \mu(F)$$

by taking $\delta \to 0$. $\quad \square$

For a measure μ on a measurable space $\mathbb{S}$, denote the restriction of μ to a subspace $\mathbb{O} \subseteq \mathbb{S}$ with $\mu_{|\mathbb{O}}$.

PROOF OF LEMMA 2.2. We provide a proof for $d = 2$ which suffices for the application in this article. The extension to general d is straightforward, and hence, omitted. In view of the Portmanteau theorem for $\mathbb{M}$-convergence—in particular item (v) of Theorem 2.1 of Lindskog, Resnick and Roy (2014)—it is enough to show that for all but countably many $r > 0$,

$$\left(\mu_n^{(1)} \times \mu_n^{(2)}\right)_{|(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r}(\cdot)$$

converges to

$$\left(\mu^{(1)} \times \mu^{(2)}\right)_{|(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r}(\cdot)$$

weakly on $(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r$, which is equipped with the relative topology as a subspace of $\mathbb{S}_1 \times \mathbb{S}_2$. From the assumptions of the lemma and again by Portmanteau theorem for $\mathbb{M}$-convergence, we note that $\mu_n^{(1)}{}_{|\mathbb{S}_1 \setminus \mathbb{C}_1^r}$ converges to $\mu^{(1)}{}_{|\mathbb{S}_1 \setminus \mathbb{C}_1^r}$ weakly in $\mathbb{S}_1 \setminus \mathbb{C}_1^r$, and $\mu_n^{(2)}{}_{|\mathbb{S}_2 \setminus \mathbb{C}_2^r}$ converges to $\mu^{(2)}{}_{|\mathbb{S}_2 \setminus \mathbb{C}_2^r}$ weakly in $\mathbb{S}_2 \setminus \mathbb{C}_2^r$ for all but countably many $r > 0$. For such r's, $\mu_n^{(1)}{}_{|\mathbb{S}_1 \setminus \mathbb{C}_1^r} \times \mu_n^{(2)}{}_{|\mathbb{S}_2 \setminus \mathbb{C}_2^r}$ converges weakly to $\mu^{(1)}{}_{|\mathbb{S}_1 \setminus \mathbb{C}_1^r} \times \mu^{(2)}{}_{|\mathbb{S}_2 \setminus \mathbb{C}_2^r}$ in $(\mathbb{S}_1 \setminus \mathbb{C}_1^r) \times (\mathbb{S}_2 \setminus \mathbb{C}_2^r)$. Noting that $(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r$ coincides with $(\mathbb{S}_1 \setminus \mathbb{C}_1^r) \times (\mathbb{S}_2 \setminus \mathbb{C}_2^r)$, and $\mu^{(1)}{}_{|\mathbb{S}_1 \setminus \mathbb{C}_1^r} \times \mu^{(2)}{}_{|\mathbb{S}_2 \setminus \mathbb{C}_2^r}$ and $\mu_n^{(1)}{}_{|\mathbb{S}_1 \setminus \mathbb{C}_1} \times \mu_n^{(2)}{}_{|\mathbb{S}_2 \setminus \mathbb{C}_2}$ coincide with $(\mu^{(1)} \times \mu^{(2)})_{|(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r}$ and $(\mu_n^{(1)} \times \mu_n^{(2)})_{|(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r}$, respectively, we reach the conclusion. $\quad \square$

PROOF OF LEMMA 2.3. Starting with the upper bound, suppose that F is a closed set bounded away from $\bigcap_{i=0}^m \mathbb{C}(i)$. From the assumption, there exist

$r_0, \ldots, r_m$ such that $F \subseteq \bigcup_{i=0}^m (\mathbb{S} \setminus \mathbb{C}(i)^{r_i})$, and hence,

$$
\begin{aligned}
\limsup_{n \to \infty} \frac{\mathbf{P}(X_n \in F)}{\epsilon_n(0)} &\le \limsup_{n \to \infty} \sum_{i=0}^m \frac{\mathbf{P}(X_n \in F \cap (\mathbb{S} \setminus \mathbb{C}(i)^{r_i}))}{\epsilon_n(i)} \frac{\epsilon_n(i)}{\epsilon_n(0)} \\
&\le \limsup_{n \to \infty} \sum_{i=0}^m \frac{\mathbf{P}(X_n \in F \setminus \mathbb{C}(i)^{r_i})}{\epsilon_n(i)} \frac{\epsilon_n(i)}{\epsilon_n(0)} \\
&= \limsup_{n \to \infty} \frac{\mathbf{P}(X_n \in F \setminus \mathbb{C}(0)^{r_0})}{\epsilon_n(0)} \\
&\le \mu^{(0)}(F \setminus \mathbb{C}(0)^{r_0}) \le \bar{\mu}^{(0)}(F).
\end{aligned}
$$

Turning to the lower bound, if G is an open set bounded away from $\bigcap_{i=0}^m \mathbb{C}(i)$,

$$
\liminf_{n \to \infty} \frac{\mathbf{P}(X_n \in G)}{\epsilon_n(0)} \ge \liminf_{n \to \infty} \frac{\mathbf{P}(X_n \in G \setminus \mathbb{C}(0)_r)}{\epsilon_n(0)} \ge \mu^{(0)}(G \setminus \mathbb{C}(0)_r).
$$

Taking $r \to 0$ yields the lower bound. $\square$

PROOF OF LEMMA 2.4. Suppose that $\mu_n \to \mu$ in $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$, and $\mu(D_h \setminus \mathbb{C}^r) = 0$ and $\mu(\partial \mathbb{S}_0 \setminus \mathbb{C}^r) = 0$ for each $r > 0$. Note that $\partial h^{-1}(A') \subseteq \mathbb{S} \setminus \mathbb{C}^r$ for some $r > 0$ due to the assumption, and $\partial h^{-1}(A') \subseteq h^{-1}(\partial A') \cup D_h \cup \partial \mathbb{S}_0$. Therefore, $\mu(\partial h^{-1}(A')) \le \mu \circ h^{-1}(\partial A') + \mu(D_h \setminus \mathbb{C}^r) + \mu(\partial \mathbb{S}_0 \setminus \mathbb{C}^r) = 0$. Applying Theorem 2.1 (iv) of Lindskog, Resnick and Roy (2014) for $h^{-1}(A')$, we conclude that $\mu_n(h^{-1}(A')) \to \mu(h^{-1}(A'))$. Again, by Theorem 2.1(iv) of Lindskog, Resnick and Roy (2014), this means that $\mu_n \circ h^{-1} \to \mu \circ h^{-1}$ in $\mathbb{M}(\mathbb{S}' \setminus \mathbb{C}')$, and hence, $\hat{h}$ is continuous at μ. $\square$

PROOF OF LEMMA 2.6. The continuity of h is well known; see, for example, Section 3.5 of Whitt (1980). For the second claim, it is enough to prove that for each j and k, $h^{-1}(A) \subseteq \mathbb{D} \times \mathbb{D}$ is bounded away from $\mathbb{D}_j \times \mathbb{D}_k$ whenever $A \subseteq \mathbb{D}$ is bounded away from $\mathbb{D}_{j,k}$. Given j and k, let $A \subseteq \mathbb{D}$ be bounded away from $\mathbb{D}_{j,k}$. To prove that $h^{-1}(A)$ is bounded away from $\mathbb{D}_j \times \mathbb{D}_k$ by contradiction, suppose that it is not. Then, for any given $\epsilon > 0$, one can find $\xi \in \mathbb{D}$ and $\zeta \in \mathbb{D}$ such that $d(\xi, \mathbb{D}_j) < \epsilon/2$, $d(\zeta, \mathbb{D}_k) < \epsilon/2$, and $\xi - \zeta \in A$. Since a continuous time-change of a step function does not change the number of jumps and jump-sizes, there exist $\xi' \in \mathbb{D}_j$ and $\zeta' \in \mathbb{D}_k$ such that $\|\xi - \xi'\|_\infty < \epsilon/2$ and $\|\zeta - \zeta'\|_\infty < \epsilon/2$. Therefore, $d(\xi - \zeta, \xi' - \zeta') \le \|(\xi - \zeta) - (\xi' - \zeta')\|_\infty \le \|\xi - \xi'\|_\infty + \|\zeta - \zeta'\|_\infty < \epsilon$. From this along with the property $d(\xi' - \zeta', \mathbb{D}_{j,k}) = 0$, we conclude that $d(\xi - \zeta, \mathbb{D}_{j,k}) < \epsilon$. Taking $\epsilon \to 0$, we arrive at $d(A, \mathbb{D}_j \times \mathbb{D}_k) = 0$ which is contradictory to the assumption. $\square$

PROOF OF LEMMA 2.7. From (i) and the inclusion-exclusion formula, $\mu_n(\bigcup_{i=1}^m A_i) \to \mu(\bigcup_{i=1}^m A_i)$ as $n \to \infty$ for any finite m if $A_i \in \mathcal{A}_p$ is bounded

away from $\mathbb{C}$ for $i = 1, \ldots, m$. If G is open and bounded away from $\mathbb{C}$, there is a sequence of sets $A_i, i \geq 1$ in $\mathcal{A}_p$ such that $G = \bigcup_{i=1}^{\infty} A_i$; note that since G is bounded away from $\mathbb{C}$, A_i's are also bounded away from $\mathbb{C}$. For any $\epsilon > 0$, one can find M_ϵ such that $\mu(\bigcup_{i=1}^{M_\epsilon} A_i) \geq \mu(G) - \epsilon$, and hence,

$$\liminf_{n\to\infty} \mu_n(G) \geq \liminf_{n\to\infty} \mu_n\left(\bigcup_{i=1}^{M_\epsilon} A_i\right) = \mu\left(\bigcup_{i=1}^{M_\epsilon} A_i\right) \geq \mu(G) - \epsilon.$$

Taking $\epsilon \to 0$, we arrive at the lower bound (2.2). Turning to the upper bound, given a closed set F, we pick $A \in \mathcal{A}_p$ bounded away from $\mathbb{C}$ such that $F \subseteq A^\circ$. Then

$$\begin{aligned}
\mu(A) - \limsup_{n\to\infty} \mu_n(F) &= \lim_{n\to\infty} \mu_n(A) + \liminf_{n\to\infty}\bigl(-\mu_n(F)\bigr) \\
&= \liminf_{n\to\infty}\bigl(\mu_n(A) - \mu_n(F)\bigr) \\
&= \liminf_{n\to\infty} \mu_n(A \setminus F) \\
&\geq \liminf_{n\to\infty} \mu_n\bigl(A^\circ \setminus F\bigr) \\
&\geq \mu\bigl(A^\circ \setminus F\bigr) = \mu(A) - \mu(F).
\end{aligned}$$

Note that $\mu(A) < \infty$ since A is bounded away from $\mathbb{C}$, which together with the above inequality establishes the upper bound (2.2). $\square$

5.2. *Proofs of Section* 3. This section provides the proofs for the limit theorems (Theorem 3.1, Theorem 3.3) presented in Section 3. The proof of Theorem 3.1 is based on:

1. The asymptotic equivalence between the target object $\bar{X}_n$ and the process obtained by keeping its j largest jumps, which will be denoted as $\hat{J}_n^{\leqslant j}$: Proposition 5.1 and Proposition 5.2 prove such asymptotic equivalences. Two technical lemmas (Lemma 5.1 and Lemma 5.2) play key roles in Proposition 5.2.

2. $\mathbb{M}$-convergence of $\hat{J}_n^{\leqslant j}$: Lemma 5.3 identifies the convergence of jump size sequences, and Proposition 5.3 deduces the convergence of $\hat{J}_n^{\leqslant j}$ from the convergence of the jump size sequences via the mapping theorem established in Section 2.

For Theorem 3.3, we first establish a general result (Theorem 5.1) for the $\mathbb{M}$-convergence of multiple Lévy processes in the associated product space using Lemmas 2.2 and 2.3. We then apply Lemma 2.6 to prove Theorem 3.3.

Recall that $X_n(t) \triangleq X(nt)$ is a scaled process of X, where X is a Lévy process with a Lévy measure ν supported on $(0, \infty)$. Also recall that X_n has Itô represen-

tation

$$X_n(s) = nsa + B(ns) + \int_{|x|\le 1} x\big[N([0, ns] \times dx) - ns\nu(dx)\big]$$

(5.3)

$$+ \int_{|x|>1} x N([0, ns] \times dx),$$

where N is the Poisson random measure with mean measure Leb $\times \nu$ on $[0, n] \times (0, \infty)$ and Leb denotes the Lebesgue measure. It is easy to see that

$$J_n(s) \triangleq \sum_{l=1}^{\tilde{N}_n} Q_n^{\leftarrow}(\Gamma_l) 1_{[U_l, 1]}(s) \stackrel{\mathcal{D}}{=} \int_{|x|>1} x N([0, ns] \times dx),$$

where $\Gamma_l = E_1 + E_2 + \cdots + E_l$; E_i's are i.i.d. and standard exponential random variables; U_l's are i.i.d. and uniform variables in $[0, 1]$; $\tilde{N}_n = N_n([0, 1] \times [1, \infty))$; $N_n = \sum_{l=1}^{\infty} \delta_{(U_l, Q_n^{\leftarrow}(\Gamma_l))}$, where $\delta_{(x,y)}$ is the Dirac measure concentrated on (x, y); $Q_n(x) \triangleq n\nu[x, \infty)$, $Q_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu[s, \infty) < y\}$. Note that $\tilde{N}_n$ is the number of Γ_l's such that $\Gamma_l \le n\nu_1^+$, where $\nu_1^+ \triangleq \nu[1, \infty)$, and hence, $\tilde{N}_n \sim$ Poisson($n\nu_1^+$). Throughout the rest of this section, we use the following representation for the centered and scaled process $\bar{X}_n \triangleq \frac{1}{n} X_n$:

$$\bar{X}_n(s) \stackrel{\mathcal{D}}{=} \frac{1}{n} J_n(s) + \frac{1}{n} B(ns)$$

(5.4)

$$+ \frac{1}{n} \int_{|x|\le 1} x\big[N([0, ns] \times dx) - ns\nu(dx)\big] - (\mu_1^+ \nu_1^+)s.$$

PROOF OF THEOREM 3.1.   We decompose $\bar{X}_n$ into a centered compound Poisson process $\bar{J}_n$, a centered Lévy process with small jumps and continuous increments $\bar{Y}_n$, and a residual process that arises due to centering $\bar{Z}_n$. After that, we will show that the compound Poisson process determines the limit. More specifically, consider the following decomposition:

$$\bar{X}_n(s) \stackrel{\mathcal{D}}{=} \bar{Y}_n(s) + \bar{J}_n(s) + \bar{Z}_n(s),$$

$$\bar{Y}_n(s) \triangleq \frac{1}{n} B(ns) + \frac{1}{n} \int_{|x|\le 1} x\big[N([0, ns] \times dx) - ns\nu(dx)\big],$$

(5.5) $$\bar{J}_n(s) \triangleq \frac{1}{n} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+) 1_{[U_l, 1]}(s),$$

$$\bar{Z}_n(s) \triangleq \frac{1}{n} \sum_{l=1}^{\tilde{N}_n} \mu_1^+ 1_{[U_l, 1]}(s) - \mu_1^+ \nu_1^+ s,$$

where $\mu_1^+ \triangleq \frac{1}{v_1^+} \int_{[1,\infty)} x v(dx)$. Let $\hat{J}_n^{\leqslant j} \triangleq \frac{1}{n} \sum_{l=1}^j Q_n^{\leftarrow}(\Gamma_l) 1_{[U_l,1]}$ be, roughly speaking, the process obtained by just keeping the j largest (uncentered) jumps of $\bar{J}_n$. In view of Corollary 2.1 and Proposition 5.3, it suffices to show that $\bar{X}_n$ and $\hat{J}_n^{\leqslant j}$ are asymptotically equivalent. Proposition 5.1 along with Proposition 5.2 prove the desired asymptotic equivalence, and hence, conclude the proof of the Theorem 3.1. $\square$

PROPOSITION 5.1. *Let $\bar{X}_n$ and $\bar{J}_n$ be as in the proof of Theorem 3.1. Then $\bar{X}_n$ and $\bar{J}_n$ are asymptotically equivalent w.r.t. $(nv[n,\infty))^j$ for any $j \geq 0$.*

PROOF. In view of the decomposition (5.5), we are done if we show that $\mathbf{P}(\|\bar{Y}_n\| > \delta) = o((nv[n,\infty))^{-j})$ and $\mathbf{P}(\|\bar{Z}_n\| > \delta) = o((nv[n,\infty))^{-j})$. For the tail probability of $\|\bar{Y}_n\|$,

$$\mathbf{P}\Big[\sup_{t \in [0,1]} |\bar{Y}_n(t)| > \delta\Big] \leq \mathbf{P}\Big[\sup_{t \in [0,n]} |B(t)| > n\delta/2\Big]$$
$$+ \mathbf{P}\Big[\sup_{t \in [0,n]} \Big|\int_{|x| \leq 1} x[N((0,t] \times dx) - tv(dx)]\Big| > n\delta/2\Big].$$

We have an explicit expression for the first term by the reflection principle, and in particular, it decays at a geometric rate w.r.t. n. For the second term, let $Y'(t) \triangleq \int_{|x| \leq 1} x[N((0,t] \times dx) - tv(dx)]$. Using Etemadi's bound for Lévy processes (see Lemma A.4), we obtain

$$\mathbf{P}\Big[\sup_{t \in [0,n]} \Big|\int_{|x| \leq 1} x[N([0,t] \times dx) - tv(dx)]\Big| > n\delta/2\Big]$$

$$\leq 3 \sup_{t \in [0,n]} \mathbf{P}[|Y'(t)| > n\delta/6]$$

$$\leq 3 \sup_{t \in [0,n]} \{\mathbf{P}[|Y'(\lfloor t \rfloor)| > n\delta/12] + \mathbf{P}[|Y'(t) - Y'(\lfloor t \rfloor)| > n\delta/12]\}$$

$$\leq 3 \sup_{t \in [0,n]} \mathbf{P}[|Y'(\lfloor t \rfloor)| > n\delta/12] + 3 \sup_{t \in [0,n]} \mathbf{P}[|Y'(t) - Y'(\lfloor t \rfloor)| > n\delta/12]$$

$$= 3 \sup_{1 \leq k \leq n} \mathbf{P}[|Y'(k)| > n\delta/12] + 3 \sup_{t \in [0,1]} \mathbf{P}[|Y'(t)| > n\delta/12]$$

$$\leq 3 \sup_{1 \leq k \leq n} \mathbf{P}\Big[\Big|\sum_{i=1}^k \{Y'(i) - Y'(i-1)\}\Big| > n\delta/12\Big]$$
$$+ 3\mathbf{P}\Big[\sup_{t \in [0,1]} |Y'(t)|^m > (n\delta/12)^m\Big].$$

Since $Y'(i) - Y'(i-1)$ are i.i.d. with $Y'(i) - Y'(i-1) \stackrel{\mathcal{D}}{=} Y'(1) = \int_{|x| \leq 1} x[N((0, 1] \times dx) - v(dx)]$ and $Y'(1)$ has exponential moments, the first term decreases at

a geometric rate w.r.t. n due to the Chernoff bound; on the other hand, since $Y'(t)$ is a martingale, the second term is bounded by $3\frac{\mathbf{E}|Y'(1)|^m}{n^m(\delta/12)^m}$ for any m by Doob's submartingale maximal inequality. Therefore, by choosing m large enough, this term can be made negligible. For the tail probability of $\|\bar{Z}_n\|$, note that $\bar{Z}_n$ is a mean zero Lévy process with the same distribution as $\mu_1^+(N(ns)/n - \nu_1^+ s)$, where N is the Poisson process with rate ν_1^+. Therefore, again from the continuous-time version of Etemadi's bound, we see that $\mathbf{P}(\|\bar{Z}_n\| > \delta)$ decays at a geometric rate w.r.t. n for any $\delta > 0$. $\quad\square$

PROPOSITION 5.2. *For each $j \geq 0$, let $\bar{J}_n$ and $\hat{J}_n^{\leqslant j}$ be defined as in the proof of Theorem* 3.1. *Then $\bar{J}_n$ and $\hat{J}_n^{\leqslant j}$ are asymptotically equivalent w.r.t. $(n\nu[n,\infty))^j$.*

PROOF. With the convention that the summation is 0 in case the superscript is strictly smaller than the subscript, consider the following decomposition of $\bar{J}_n$:

$$\hat{J}_n^{\leqslant j} \triangleq \frac{1}{n} \sum_{l=1}^{j} Q_n^{\leftarrow}(\Gamma_l) 1_{[U_l, 1]},$$

$$\bar{J}_n^{>j} \triangleq \frac{1}{n} \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+) 1_{[U_l, 1]},$$

$$\check{J}_n^{\leqslant j} \triangleq \frac{1}{n} \sum_{l=1}^{j} -\mu_1^+ 1_{[U_l, 1]},$$

$$\bar{R}_n \triangleq \frac{1}{n} \mathbb{I}(\tilde{N}_n < j) \sum_{l=\tilde{N}_n+1}^{j} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+) 1_{[U_l, 1]},$$

so that

$$\bar{J}_n = \hat{J}_n^{\leqslant j} + \check{J}_n^{\leqslant j} + \bar{J}_n^{>j} - \bar{R}_n.$$

Note that $\mathbf{P}(\|\check{J}_n^{\leqslant j}\| \geq \delta) = 0$ for sufficiently large n since $\|\check{J}_n^{\leqslant j}\| = j\mu_1/n$. On the other hand, $\mathbf{P}(\|\bar{R}_n\| \geq \delta)$ decays at a geometric rate since $\{\|\bar{R}_n\| \geq \delta\} \subseteq \{\tilde{N}_n < j\}$ and $\mathbf{P}(\tilde{N}_n < j)$ decays at a geometric rate. Since $\mathbf{P}(\|\bar{J}_n^{>j}\| \geq \delta) \leq \mathbf{P}(\|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma) + \mathbf{P}(\|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma)$, Lemma 5.1 and Lemma 5.2 given below imply $\mathbf{P}(\|\bar{J}_n^{>j}\| \geq \delta) = o((n\nu[n,\infty))^j)$ by choosing γ small enough. Therefore, $\hat{J}_n^{\leqslant j}$ and $\bar{J}_n$ are asymptotically equivalent w.r.t. $(n\nu[n,\infty))^j$. $\quad\square$

Define a measure $\mu_\alpha^{(j)}$ on $\mathbb{R}_+^{\infty\downarrow}$ by

$$\mu_\alpha^{(j)}(dx_1, dx_2, \ldots) \triangleq \prod_{i=1}^{j} \nu_\alpha(dx_i) \mathbb{I}_{[x_1 \geq x_2 \geq \cdots \geq x_j > 0]} \prod_{i=j+1}^{\infty} \delta_0(dx_i),$$

where $\nu_\alpha(x, \infty) = x^{-\alpha}$, and δ_0 is the Dirac measure concentrated at 0.

PROPOSITION 5.3. *For each $j \geq 0$,*

$$(n\nu[n,\infty))^{-j}\mathbf{P}(\hat{J}_n^{\leqslant j} \in \cdot) \to C_j(\cdot)$$

in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j})$ as $n \to \infty$.

PROOF. Noting that $(\mu_\alpha^{(j)} \times \text{Leb}) \circ T_j^{-1} = C_j$ and $\mathbf{P}(\hat{J}_n^{\leqslant j} \in \cdot) = \mathbf{P}(((Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1), (U_l, l \geq 1)) \in T_j^{-1}(\cdot))$, Lemma 5.3 and Corollary 2.2 prove the proposition. $\square$

LEMMA 5.1. *For any fixed $\gamma > 0$, $\delta > 0$, and $j \geq 0$,*

(5.6) $$\mathbf{P}\{\|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma\} = o((n\nu[n,\infty))^j).$$

PROOF. (Throughout the proof of this lemma, we use μ_1 and ν_1 in place of μ_1^+ and ν_1^+, respectively.) We start with the following decomposition of $\bar{J}_n^{>j}$: for any fixed $\lambda \in (0, \frac{\delta}{3\nu_1\mu_1})$,

$$\begin{aligned}
\bar{J}_n^{>j} &= \frac{1}{n} \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]} \\
&= \tilde{J}_n^{[j+1,n\nu_1(1+\lambda)]} - \tilde{J}_n^{[\tilde{N}_n+1,n\nu_1(1+\lambda)]}\mathbb{I}(\tilde{N}_n < n\nu_1(1+\lambda)) \\
&\quad + \tilde{J}_n^{[n\nu_1(1+\lambda)+1,\tilde{N}_n]}\mathbb{I}(\tilde{N}_n > n\nu_1(1+\lambda)),
\end{aligned}$$

where

$$\tilde{J}_n^{[a,b]} \triangleq \frac{1}{n} \sum_{l=\lceil a \rceil}^{\lfloor b \rfloor} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}.$$

Therefore,

$$\begin{aligned}
&\mathbf{P}\{\|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma\} \\
&\quad \leq \mathbf{P}(\|\tilde{J}_n^{[j+1,n\nu_1(1+\lambda)]}\| \geq \delta/3, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma) \\
&\qquad + \mathbf{P}(\|\tilde{J}_n^{[\tilde{N}_n+1,n\nu_1(1+\lambda)]}\| \geq \delta/3) + \mathbf{P}(\tilde{N}_n > n\nu_1(1+\lambda)) \\
&\quad = \text{(i)} + \text{(ii)} + \text{(iii)}.
\end{aligned}$$

Noting that $\|\tilde{J}_n^{[\tilde{N}_n+1,n\nu_1(1+\lambda)]}\| \leq (\nu_1(1+\lambda) - \tilde{N}_n/n)\mu_1$—recall that $\tilde{N}_n$ is defined to be the number of l's such that $Q_n^{\leftarrow}(\Gamma_l) \geq 1$, and hence, $0 \leq Q_n^{\leftarrow}(\Gamma_l) < 1$ for $l > \tilde{N}_n$—we see that (ii) is bounded by

$$\mathbf{P}((\nu_1(1+\lambda) - \tilde{N}_n/n)\mu_1 \geq \delta/3) = \mathbf{P}\left(\frac{\tilde{N}_n}{n\nu_1} \leq 1 + \lambda - \frac{\delta}{3\nu_1\mu_1}\right),$$

which decays at a geometric rate w.r.t. n since $\tilde{N}_n$ is Poisson with rate $n\nu_1$. For the same reason, (iii) decays at a geometric rate w.r.t. n. We are done if we prove that (i) is $o((n\nu[n, \infty))^j)$. Note that $Q_n^\leftarrow(\Gamma_j) \geq n\gamma$ implies $Q_n(n\gamma) \geq \Gamma_j$, and hence,

$$\sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^\leftarrow(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1)1_{[U_l,1]}$$

$$\leq \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^\leftarrow(\Gamma_l) - \mu_1)1_{[U_l,1]}$$

$$\leq \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^\leftarrow(\Gamma_l - \Gamma_j) - \mu_1)1_{[U_l,1]}.$$

Therefore, if we define

$$A_n \triangleq \{Q_n^\leftarrow(\Gamma_j) \geq n\gamma\},$$

$$B_n' \triangleq \left\{ \sup_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^\leftarrow(\Gamma_l - \Gamma_j) - \mu_1)1_{[U_l,1]}(t) \geq n\delta \right\},$$

$$B_n'' \triangleq \left\{ \inf_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^\leftarrow(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1)1_{[U_l,1]}(t) \leq -n\delta \right\},$$

then we have that

$$(i) \leq \mathbf{P}(A_n \cap (B_n' \cup B_n''))$$

$$\leq \mathbf{P}(A_n \cap B_n') + \mathbf{P}(A_n \cap B_n'')$$

$$= \mathbf{P}(A_n)(\mathbf{P}(B_n') + \mathbf{P}(B_n'')),$$

where the last equality is from the independence of A_n and B_n' as well as of A_n and B_n'' (which is, in turn, due to the independence of Γ_j and $\Gamma_l - \Gamma_j$). From Lemma 5.4(c) and Proposition 5.3, $\mathbf{P}(A_n) = \mathbf{P}(\hat{J}_n^{\leq j} \in (\mathbb{D} \setminus \mathbb{D}_{<j})^{-\gamma/2}) = O((n\nu[n, \infty))^j)$, and hence, it suffices to show that the probabilities of the complements of B_n' and B_n'' converge to 1, that is, for any fixed $\gamma > 0$,

$$(5.7) \qquad \mathbf{P}\left\{ \sup_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^\leftarrow(\Gamma_l - \Gamma_j) - \mu_1)1_{[U_l,1]}(t) < n\delta \right\} \to 1$$

and

$$(5.8) \quad \mathbf{P}\left\{ \inf_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^\leftarrow(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1)1_{[U_l,1]}(t) > -n\delta \right\} \to 1.$$

Starting with (5.7)

$$\mathbf{P}\left\{\sup_{t\in[0,1]}\sum_{l=j+1}^{(1+\lambda)n\nu_1}(Q_n^{\leftarrow}(\Gamma_l-\Gamma_j)-\mu_1)1_{[U_l,1]}(t)<n\delta\right\}$$

$$=\mathbf{P}\left\{\sup_{t\in[0,1]}\sum_{l=1}^{(1+\lambda)n\nu_1-j}(Q_n^{\leftarrow}(\Gamma_l)-\mu_1)1_{[U_l,1]}(t)<n\delta\right\}$$

$$\geq\mathbf{P}\left\{\sup_{t\in[0,1]}\sum_{l=1}^{(1+\lambda)n\nu_1-j}(Q_n^{\leftarrow}(\Gamma_l)-\mu_1)1_{[U_l,1]}(t)<n\delta,\tilde{N}_n\leq(1+\lambda)n\nu_1-j\right\}$$

$$\geq\mathbf{P}\left\{\sup_{t\in[0,1]}\sum_{l=1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l)-\mu_1)1_{[U_l,1]}(t)<n\delta,\tilde{N}_n\leq(1+\lambda)n\nu_1-j\right\}$$

$$\geq\mathbf{P}\left\{\sup_{t\in[0,1]}\sum_{l=1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l)-\mu_1)1_{[U_l,1]}(t)<n\delta)\right\}$$

$$-\mathbf{P}\{\tilde{N}_n>(1+\lambda)n\nu_1-j\}.$$

The second inequality is due to the definition of $Q_n^{\leftarrow}$ and that $\mu_1\geq1$ (and hence, $Q_n^{\leftarrow}(\Gamma_l)-\mu_1\leq0$ on $l\geq\tilde{N}_n$), while the last inequality comes from the generic inequality $\mathbf{P}(A\cap B)\geq\mathbf{P}(A)-\mathbf{P}(B^c)$. The second probability converges to 0 since $\tilde{N}$ is Poisson with rate $n\nu_1$. Moving on to the first probability in the last expression, observe that $\sum_{l=1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l)-\mu_1)1_{[U_l,1]}(\cdot)$ has the same distribution as the compound Poisson process $\sum_{i=1}^{J(n\cdot)}(D_i-\mu_1)$, where J is a Poisson process with rate ν_1 and D_i's are i.i.d. random variables with the distribution ν conditioned (and normalized) on $[1,\infty)$, that is, $\mathbf{P}\{D_i\geq s\}=1\wedge(\nu[s,\infty)/\nu[1,\infty))$. Using this, we obtain

$$\mathbf{P}\left\{\sup_{t\in[0,1]}\sum_{l=1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l)-\mu_1)1_{[U_l,1]}(t)<n\delta\right\}$$

(5.9)
$$=\mathbf{P}\left\{\sup_{1\leq m\leq J(n)}\sum_{l=1}^{m}(D_l-\mu_1)<n\delta\right\}$$

$$\geq\mathbf{P}\left\{\sup_{1\leq m\leq2n\nu_1}\sum_{l=1}^{m}(D_l-\mu_1)<n\delta,J(n)\leq2n\nu_1\right\}$$

$$\geq\mathbf{P}\left\{\sup_{1\leq m\leq2n\nu_1}\sum_{l=1}^{m}(D_l-\mu_1)<n\delta\right\}-\mathbf{P}\{J(n)>2n\nu_1\}.$$

The second probability vanishes at a geometric rate w.r.t. n because $J(n)$ is Poisson with rate $n\nu_1$. The first term can be investigated by the generalized Kolmogorov

inequality; cf. Lemma in page 335 of Shneer and Wachtel (2011) (given as Result A.1 in the Appendix):

$$\mathbf{P}\left(\max_{1 \leq m \leq 2n\nu_1} \sum_{l=1}^{m} (D_l - \mu_1) \geq n\delta/2\right) \leq C\frac{2n\nu_1 V(n\delta/2)}{(n\delta/2)^2},$$

where $V(x) = \mathbf{E}[(D_l - \mu_1)^2; \mu_1 - x \leq D_l \leq \mu_1 + x] \leq \mu_1^2 + \mathbf{E}[D_l^2; D_l \leq \mu_1 + x]$. Note that

$$\mathbf{E}[D_l^2; D_l \leq \mu_1 + x] = \int_0^1 2s\,ds + \int_1^{\mu_1+x} 2s\frac{\nu[s,\infty)}{\nu[1,\infty)}\,ds$$

$$= 1 + \frac{2}{\nu_1}(\mu_1 + x)^{2-\alpha}L(\mu_1 + x),$$

for some slowly varying L. Hence,

$$\mathbf{P}\left(\max_{1 \leq m \leq 2n\nu_1} \sum_{l=1}^{m} (D_l - \mu_1) < n\delta\right)$$

$$\geq 1 - \mathbf{P}\left(\max_{1 \leq m \leq 2n\nu_1} \sum_{l=1}^{m} (D_l - \mu_1) \geq n\delta/2\right) \to 1,$$

as $n \to \infty$.

Now, turning to (5.8), let $\gamma_n \triangleq Q_n(n\gamma)$.

$$\mathbf{P}\left\{\inf_{t \in [0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1)1_{[U_l,1]}(t) > -n\delta\right\}$$

$$= \mathbf{P}\left\{\inf_{t \in [0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1-j} (Q_n^{\leftarrow}(\Gamma_l + \gamma_n) - \mu_1)1_{[U_l,1]}(t) > -n\delta\right\}$$

$$\geq \mathbf{P}\left\{\inf_{t \in [0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1-j} (Q_n^{\leftarrow}(\Gamma_l + \gamma_n) - \mu_1)1_{[U_l,1]}(t) > -n\delta, E_0 \geq \gamma_n\right\}$$

$$\geq \mathbf{P}\left\{\inf_{t \in [0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1-j} (Q_n^{\leftarrow}(\Gamma_l + E_0) - \mu_1)1_{[U_l,1]}(t) > -n\delta, E_0 \geq \gamma_n\right\}$$

$$= \mathbf{P}\left\{\inf_{t \in [0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) > -n\delta, \Gamma_1 \geq \gamma_n\right\}$$

$$\geq \mathbf{P}\left\{\inf_{t \in [0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) > -n\delta\right\} - \mathbf{P}\{\Gamma_1 < \gamma_n\}$$

$$= (A) - (B),$$

where E_0 is a standard exponential random variable. (Recall that $\Gamma_l \triangleq E_1 + E_2 + \cdots + E_l$, and hence, $(\Gamma_l + E_0, U_l) \overset{\mathcal{D}}{=} (\Gamma_{l+1}, U_l) \overset{\mathcal{D}}{=} (\Gamma_{l+1}, U_{l+1})$.) Since $(B) = \mathbf{P}\{\Gamma_1 < \gamma_n\} \to 0$ (recall that $\gamma_n = n\nu[n\gamma, \infty)$ and ν is regularly varying with index $-\alpha < -1$), we focus on proving that the first term (A) converges to 1:

$$(A) = \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) > -n\delta\right\}$$

$$\geq \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) > -n\delta,\right.$$

$$\left. \tilde{N}_n \leq (1+\lambda)n\nu_1 - j + 1\right\}$$

$$\geq \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) \geq -n\delta/3,\right.$$

$$\inf_{t\in[0,1]} -(Q_n^{\leftarrow}(\Gamma_1) - \mu_1)1_{[U_1,1]}(t) > -n\delta/3,$$

$$\inf_{t\in[0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) \geq -n\delta/3,$$

$$\left. \tilde{N}_n \leq (1+\lambda)n\nu_1 - j + 1\right\}$$

$$\geq \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) \geq -n\delta/3\right\}$$

$$+ \mathbf{P}\left\{Q_n^{\leftarrow}(\Gamma_1) - \mu_1 < n\delta/3\right\}$$

$$+ \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) \geq -n\delta/3\right\}$$

$$+ \mathbf{P}\left\{\tilde{N}_n \leq (1+\lambda)n\nu_1 - j + 1\right\} - 3$$

$$= (AI) + (AII) + (AIII) + (AIV) - 3.$$

The third inequality comes from applying the generic inequality $\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1$ three times. Since $\tilde{N}_n$ is Poisson with rate $n\nu_1$,

$$(AIV) = \mathbf{P}\{\tilde{N}_n \leq (1+\lambda)n\nu_1 - j + 1\} = \mathbf{P}\left\{\frac{\tilde{N}_n}{n\nu_1} \leq 1 + \lambda - \frac{j-1}{n\nu_1}\right\} \to 1.$$

For the first term (AI),

$$(AI) = \mathbf{P}\left\{ \inf_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l, 1]}(t) \geq -n\delta/3 \right\}$$

$$= \mathbf{P}\left\{ \sup_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (\mu_1 - Q_n^{\leftarrow}(\Gamma_l)) 1_{[U_l, 1]}(t) \leq n\delta/3 \right\}$$

$$= \mathbf{P}\left\{ \sup_{1 \leq m \leq J(n)} \sum_{l=1}^{m} (\mu_1 - D_l) \leq n\delta/3 \right\},$$

where D_i is defined as before. Note that this is of exactly same form as (5.9) except for the sign of D_l, and hence, we can proceed exactly the same way using the generalized Kolmogorov inequality to prove that this quantity converges to 1— recall that the formula only involves the square of the increments, and hence, the change of the sign has no effect. For the second term (AII),

$$(AII) \geq \mathbf{P}\{Q_n^{\leftarrow}(\Gamma_1) \leq n\delta/3\} \geq \mathbf{P}\{\Gamma_1 > Q_n(n\delta/3)\} \to 1,$$

since $Q_n(n\delta/3) \to 0$. For the third term (AIII),

$$(AIII) = \mathbf{P}\left\{ \inf_{t \in [0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l, 1]}(t) \geq -n\delta/3 \right\}$$

$$\geq \mathbf{P}\left\{ \inf_{t \in [0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} -\mu_1 1_{[U_l, 1]}(t) \geq -n\delta/3 \right\}$$

$$\geq \mathbf{P}\left\{ \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} \mu_1 \leq n\delta/3 \right\}$$

$$\geq \mathbf{P}\{\mu_1((1+\lambda)n\nu_1 - j - \tilde{N}_n + 1) \leq n\delta/3\}$$

$$\geq \mathbf{P}\left\{ 1 + \lambda - \frac{\delta}{3\nu_1\mu_1} \leq \frac{\tilde{N}_n}{n\nu_1} + \frac{j-1}{n\nu_1} \right\}$$

$$\to 1,$$

since $\lambda < \frac{\delta}{3\nu_1\mu_1}$. This concludes the proof of the lemma. $\square$

LEMMA 5.2. *For any $j \geq 0$, $\delta > 0$, and $m < \infty$, there is $\gamma_0 > 0$ such that*

$$\mathbf{P}\{\|\bar{J}_n^{>j}\| > \delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma_0\} = o(n^{-m}).$$

PROOF. (Throughout the proof of this lemma, we use μ_1 and ν_1 in place of μ_1^+ and ν_1^+, respectively, for the sake of notational simplicity.) Note first that

$Q_n^{\leftarrow}(\Gamma_j) = \infty$ if $j = 0$, and hence, the claim of the lemma is trivial. Therefore, we assume $j \geq 1$ throughout the rest of the proof. Since for any $\lambda > 0$,

(5.10)
$$\mathbf{P}\{\|\bar{J}_n^{>j}\| > \delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma\}$$
$$\leq \mathbf{P}\left\{\left\|\sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}\right\| > n\delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma,\right.$$
$$\left.\frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1 + \lambda\right]\right\}$$
$$+ \mathbf{P}\left\{\frac{\tilde{N}_n}{n\nu_1} \notin \left[\frac{j}{n\nu_1}, 1 + \lambda\right]\right\},$$

and $\mathbf{P}\{\frac{\tilde{N}_n}{n\nu_1} \notin [\frac{j}{n\nu_1}, 1 + \lambda]\}$ decays at a geometric rate w.r.t. n, it suffices to show that (5.10) is $o(n^{-m})$ for small enough $\gamma > 0$. First, recall that by the definition of $Q_n^{\leftarrow}(\cdot)$,

$$Q_n^{\leftarrow}(x) \geq s \iff x \leq Q_n(s),$$

and

$$n\nu(Q_n^{\leftarrow}(x), \infty) \leq x \leq n\nu[Q_n^{\leftarrow}(x), \infty).$$

Let L be a random variable conditionally (on $\tilde{N}_n$) independent of everything else and uniformly sampled on $\{j + 1, j + 2, \ldots, \tilde{N}_n\}$. Recall that given $\tilde{N}_n$ and Γ_j, the distribution of $\{\Gamma_{j+1}, \Gamma_{j+2}, \ldots, \Gamma_{\tilde{N}_n}\}$ is the same as that of the order statistics of $\tilde{N}_n - j$ uniform random variables on $[\Gamma_j, n\nu[1, \infty)]$. Let $D_l, l \geq 1$, be i.i.d. random variables whose conditional distribution is the same as the conditional distribution of $Q_n^{\leftarrow}(\Gamma_L)$ given $\tilde{N}_n$ and Γ_j. Then the conditional distribution of $\sum_{l=j+1}^{\tilde{N}_n}(Q_n(\Gamma_l) - \mu_1)1_{[U_l,1]}$ is the same as that of $\sum_{l=1}^{\tilde{N}_n-j}(D_l - \mu_1)1_{[U_l,1]}$. Therefore, the conditional distribution of $\|\sum_{l=j+1}^{\tilde{N}_n}(Q_n(\Gamma_l) - \mu_1)1_{[U_l,1]}\|_\infty$ is the same as the corresponding conditional distribution of $\sup_{1 \leq m \leq \tilde{N}_n - j} |\sum_{l=1}^m (D_l - \mu_1)|$. To make use of this in the analysis that follows, we make a few observations on the conditional distribution of $Q_n^{\leftarrow}(\Gamma_L)$ given Γ_j and $\tilde{N}_n$.

(a) *The conditional distribution of $Q_n^{\leftarrow}(\Gamma_L)$:*

Let $q \triangleq Q_n^{\leftarrow}(\Gamma_j)$. Since Γ_L is uniformly distributed on $[\Gamma_j, Q_n(1)] = [\Gamma_j, n\nu[1, \infty)]$, the tail probability is

$$\mathbf{P}\{Q_n^{\leftarrow}(\Gamma_L) \geq s|\Gamma_j, \tilde{N}_n\} = \mathbf{P}\{\Gamma_L \leq Q_n(s)|\Gamma_j, \tilde{N}_n\}$$
$$= \mathbf{P}\{\Gamma_L \leq n\nu[s, \infty)|\Gamma_j, \tilde{N}_n\}$$

$$= \mathbf{P}\left\{\frac{\Gamma_L - \Gamma_j}{n\nu[1, \infty) - \Gamma_j} \le \frac{n\nu[s, \infty) - \Gamma_j}{n\nu[1, \infty) - \Gamma_j}\Big|\Gamma_j, \tilde{N}_n\right\}$$

$$= \frac{n\nu[s, \infty) - \Gamma_j}{n\nu[1, \infty) - \Gamma_j}$$

for $s \in [1, q]$; since this is nonincreasing w.r.t. Γ_j and $n\nu(q, \infty) \le \Gamma_j \le n\nu[q, \infty)$, we have that

$$\frac{\nu[s, q)}{\nu[1, q)} \le \mathbf{P}\{Q_n^{\leftarrow}(\Gamma_L) \ge s | \Gamma_j, \tilde{N}_n\} \le \frac{\nu[s, q]}{\nu[1, q]}.$$

(b) *Difference in mean between conditional and unconditional distribution:* From (a), we obtain

$$\tilde{\mu}_n \triangleq \mathbf{E}[Q_n^{\leftarrow}(\Gamma_L)|\Gamma_j, \tilde{N}_n] \in \left[1 + \int_1^q \frac{\nu[s, q)}{\nu[1, q)}\, ds, 1 + \int_1^q \frac{\nu[s, q]}{\nu[1, q]}\, ds\right],$$

and hence,

$$|\mu_1 - \tilde{\mu}_n| \le \left|\frac{\nu[1, q)\int_1^\infty \nu[s, \infty)\, ds - \nu[1, \infty)\int_1^q \nu[s, q)\, ds}{\nu[1, \infty)\nu[1, q)}\right|$$

$$\vee \left|\frac{\nu[1, q]\int_1^\infty \nu[s, \infty)\, ds - \nu[1, \infty)\int_1^q \nu[s, q]\, ds}{\nu[1, \infty)\nu[1, q]}\right|.$$

Since

$$\frac{\nu[1, q)\int_1^\infty \nu[s, \infty)\, ds - \nu[1, \infty)\int_1^q \nu[s, q)\, ds}{\nu[1, \infty)\nu[1, q)}$$

$$= \frac{\nu[q, \infty)}{\nu[1, q)}(q - 1) + \frac{\int_q^\infty \nu[s, \infty)\, ds}{\nu[1, \infty)} - \frac{\nu[q, \infty)\int_1^q \nu[s, \infty)\, ds}{\nu[1, \infty)\nu[1, q)},$$

and

$$\frac{\nu[1, q)\int_1^\infty \nu[s, \infty)\, ds - \nu[1, \infty)\int_1^q \nu[s, q)\, ds}{\nu[1, \infty)\nu[1, q)}$$

$$- \frac{\nu[1, q]\int_1^\infty \nu[s, \infty)\, ds - \nu[1, \infty)\int_1^q \nu[s, q]\, ds}{\nu[1, \infty)\nu[1, q]}$$

$$= \frac{\nu\{q\}((q - 1)\nu[1, \infty) + \int_q^\infty \nu[s, \infty)\, ds + \int_1^q \nu[s, \infty)\, ds)}{\nu[1, \infty)(\nu[1, q) + \nu\{q\})},$$

we see that $|\mu_1 - \tilde{\mu}_n|$ is bounded by a regularly varying function with index $1 - \alpha$ (w.r.t. q) from Karamata's theorem.

(c) *Variance of $Q_n^{\leftarrow}(\Gamma_L)$:* Turning to the variance, we observe that, if $\alpha \leq 2$,

$$\mathbf{E}\big[Q_n^{\leftarrow}(\Gamma_L)^2 | \Gamma_j, \tilde{N}_n\big] \leq \int_0^1 2s\, ds + 2\int_1^q s\frac{v[s,q]}{v[1,q]}\, ds$$

(5.11)
$$\leq 1 + \frac{2}{v[1,q]}\int_1^q sv[s,\infty)\, ds$$

$$= 1 + q^{2-\alpha}L(q)$$

for some slowly varying function $L(\cdot)$. If $\alpha > 2$, the variance is bounded w.r.t. q.

Now, with (b) and (c) in hand, we can proceed with an explicit bound since all the randomness is contained in q. Namely, we infer

$$\mathbf{P}\Bigg(\bigg\|\sum_{l=j+1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}\bigg\|_\infty > n\delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma, \frac{\tilde{N}_n}{n\nu_1} \in \bigg[\frac{j}{n\nu_1}, 1+\lambda\bigg]\Bigg)$$

$$= \mathbf{P}\Bigg(\bigg\|\sum_{l=j+1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}\bigg\|_\infty > n\delta, \Gamma_j \geq Q_n(n\gamma),$$

$$\frac{\tilde{N}_n}{n\nu_1} \in \bigg[\frac{j}{n\nu_1}, 1+\lambda\bigg]\Bigg)$$

$$= \mathbf{E}\Bigg[\mathbf{P}\bigg(\bigg\|\sum_{l=j+1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}\bigg\|_\infty > n\delta \bigg| \Gamma_j, \tilde{N}_n\bigg); \Gamma_j \geq Q_n(n\gamma),$$

$$\frac{\tilde{N}_n}{n\nu_1} \in \bigg[\frac{j}{n\nu_1}, 1+\lambda\bigg]\Bigg]$$

$$= \mathbf{E}\Bigg[\mathbf{P}\bigg(\max_{1 \leq m \leq \tilde{N}_n - j}\bigg|\sum_{l=1}^m(D_l - \mu_1)\bigg| > n\delta \bigg| \Gamma_j, \tilde{N}_n\bigg); \Gamma_j \geq Q_n(n\gamma),$$

$$\frac{\tilde{N}_n}{n\nu_1} \in \bigg[\frac{j}{n\nu_1}, 1+\lambda\bigg]\Bigg].$$

By Etemadi's bound (Result A.2 in Appendix),

$$\mathbf{P}\bigg(\max_{1 \leq m \leq \tilde{N}_n - j}\bigg|\sum_{l=1}^m(D_l - \mu_1)\bigg| \geq n\delta \bigg| \Gamma_j, \tilde{N}_n\bigg)$$

(5.12)
$$\leq 3\max_{1 \leq m \leq \tilde{N}_n}\mathbf{P}\bigg(\bigg|\sum_{l=1}^m(D_l - \mu_1)\bigg| \geq n\delta \bigg| \Gamma_j, \tilde{N}_n\bigg)$$

$$\leq 3 \max_{1 \leq m \leq \tilde{N}_n} \left\{ \mathbf{P}\left(\sum_{l=1}^{m}(D_l - \mu_1) \geq n\delta \Big| \Gamma_j, \tilde{N}_n \right) \right.$$

$$\left. + \mathbf{P}\left(\sum_{l=1}^{m}(\mu_1 - D_l) \geq n\delta \Big| \Gamma_j, \tilde{N}_n \right) \right\}$$

and as $|D_l - \tilde{\mu}_n|$ is bounded by q, we can apply Prokhorov's bound (Result A.3 in Appendix) to get

$$\mathbf{P}\left(\sum_{l=1}^{m}(\mu_1 - D_l) \geq n\delta \Big| \Gamma_j, \tilde{N}_n \right)$$

$$= \mathbf{P}\left(\sum_{l=1}^{m}(\tilde{\mu}_n - D_l) \geq n\delta - m(\mu_1 - \tilde{\mu}_n) \Big| \Gamma_j, \tilde{N}_n \right)$$

$$\leq \mathbf{P}\left(\sum_{l=1}^{m}(\tilde{\mu}_n - D_l) \geq n\delta - n\nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n) \Big| \Gamma_j, \tilde{N}_n \right)$$

$$\leq \left(\frac{qn(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{m\,\mathbf{var}(Q_n^{\leftarrow}(\Gamma_L))} \right)^{-\frac{n(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{2q}}$$

$$\leq \left(\frac{n\nu_1(1+\lambda)\,\mathbf{var}(Q_n^{\leftarrow}(\Gamma_L))}{qn(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))} \right)^{\frac{n(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{2q}}$$

$$= \begin{cases} \left(\dfrac{\nu_1(1+\lambda)(1 + q^{2-\alpha}L_1(q))}{q(\delta - \nu_1(1+\lambda)q^{1-\alpha}L_2(q))} \right)^{\frac{n(\delta - \nu_1(1+\lambda)q^{1-\alpha}L_2(q))}{2q}} & \text{if } \alpha \leq 2, \\[2em] \left(\dfrac{\nu_1(1+\lambda)C}{q(\delta - \nu_1(1+\lambda)q^{1-\alpha}L_2(q))} \right)^{\frac{n(\delta - \nu_1(1+\lambda)q^{1-\alpha}L_2(q))}{2q}} & \text{otherwise,} \end{cases}$$

for some $C > 0$ if $m \leq (1+\lambda)n\nu_1$. Therefore, there exist constants M and c such that $q \geq M$ (i.e., $\Gamma_j \leq Q_n(M)$) implies

$$\mathbf{P}\left(\sum_{l=1}^{m}(\mu_1 - D_l) \geq n\delta \Big| \Gamma_j \right) \leq c\left(q^{1-\alpha\wedge 2}\right)^{\frac{n\delta}{8q}},$$

and since we are conditioning on $q = Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma$,

$$c\left(q^{1-\alpha\wedge 2}\right)^{\frac{n\delta}{8q}} \leq c\left(q^{1-\alpha\wedge 2}\right)^{\frac{\delta}{8\gamma}}.$$

Hence,

$$\mathbf{P}\left(\sum_{l=1}^{m}(\mu_1 - D_l) \geq n\delta \Big| \Gamma_j \right) \leq c\left(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\right)^{\frac{\delta}{8\gamma}}.$$

With the same argument, we also get

$$\mathbf{P}\left(\sum_{l=1}^{m}(D_l - \mu_1) \geq n\delta \Big| \Gamma_j\right) \leq c\big(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\big)^{\frac{\delta}{8\gamma}}.$$

Combining (5.12) with the two previous estimates, we obtain

$$\mathbf{P}\left(\max_{1\leq m\leq \tilde{N}_n-j}\left|\sum_{l=1}^{m}(D_l - \mu_1)\right| \geq n\delta \Big| \Gamma_j, \tilde{N}_n\right) \leq 6c\big(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\big)^{\frac{\delta}{8\gamma}},$$

on $\Gamma_j \geq Q_n(n\gamma)$, $\tilde{N}_n - j \leq n\nu_1(1+\lambda)$, and $\Gamma_j \leq Q_n(M)$. Now,

$$\mathbf{E}\left[\mathbf{P}\left(\max_{1\leq m\leq \tilde{N}_n-j}\left|\sum_{l=1}^{m}(D_l - \mu_1)\right| > n\delta \Big| \Gamma_j, \tilde{N}_n\right); \Gamma_j \geq Q_n(n\gamma); \right.$$
$$\left. \frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1+\lambda\right]\right]$$

$$\leq \mathbf{E}\left[\mathbf{P}\left(\max_{1\leq m\leq \tilde{N}_n-j}\left|\sum_{l=1}^{m}(D_l - \mu_1)\right| > n\delta \Big| \Gamma_j, \tilde{N}_n\right); \Gamma_j \geq Q_n(n\gamma); \right.$$
$$\left. \frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1+\lambda\right]; \Gamma_j \leq Q_n(M)\right]$$

$$+ \mathbf{P}(\Gamma_j > Q_n(M))$$

$$\leq \mathbf{E}\big[6c\big(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\big)^{\frac{\delta}{8\gamma}}\big] + \mathbf{P}(\Gamma_j > Q_n(M))$$

$$\leq \mathbf{E}\big[6c\big(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\big)^{\frac{\delta}{8\gamma}}; Q_n^{\leftarrow}(\Gamma_j) \geq n^{\beta}\big] + \mathbf{P}\big(Q_n^{\leftarrow}(\Gamma_j) < n^{\beta}\big)$$
$$+ \mathbf{P}(\Gamma_j > Q_n(M))$$

$$\leq 6c\big(n^{\beta(1-\alpha\wedge 2)}\big)^{\frac{\delta}{8\gamma}} + \mathbf{P}\big(\Gamma_j > Q_n(n^{\beta})\big) + \mathbf{P}(\Gamma_j > Q_n(M))$$

$$\leq 6c\big(n^{\beta(1-\alpha\wedge 2)}\big)^{\frac{\delta}{8\gamma}} + \mathbf{P}\big(\Gamma_j > \big(n^{1-\alpha\beta}L(n)\big)\big) + \mathbf{P}(\Gamma_j > Q_n(M)),$$

for any $\beta > 0$. If one chooses β so that $1 - \alpha\beta > 0$ (e.g., $\beta = \frac{1}{2\alpha}$), the second and third terms vanish at a geometric rate w.r.t. n. On the other hand, we can pick γ small enough compared to δ, so that the first term is decreasing at an arbitrarily fast polynomial rate. This concludes the proof of the lemma. $\square$

Recall that we denote the Lebesgue measure on $[0,1]^{\infty}$ with Leb and defined measures $\mu_{\alpha}^{(j)}$ and $\mu_{\beta}^{(j)}$ on $\mathbb{R}_+^{\infty\downarrow}$ as

$$\mu_{\alpha}^{(j)}(dx_1, dx_2, \ldots) \triangleq \prod_{i=1}^{j} \nu_{\alpha}(dx_i) \mathbb{I}_{[x_1 \geq x_2 \geq \cdots \geq x_j > 0]} \prod_{i=j+1}^{\infty} \delta_0(dx_i),$$

and $\nu_{\alpha}(x, \infty) = x^{-\alpha}$, where δ_0 is the Dirac measure concentrated at 0.

LEMMA 5.3. *For each $j \geq 0$,*

$$(n\nu[n, \infty))^{-j} \mathbf{P}\big[\big((Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1), (U_l, l \geq 1)\big) \in \cdot\big] \to (\mu_\alpha^{(j)} \times \mathrm{Leb})(\cdot)$$

in $\mathbb{M}((\mathbb{R}_+^{\infty\downarrow} \times [0, 1]^\infty) \setminus (\mathbb{H}_{<j} \times [0, 1]^\infty))$ *as* $n \to \infty$.

PROOF. We first prove that

$$(5.13) \qquad (n\nu[n, \infty))^{-j} \mathbf{P}\big[(Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1) \in \cdot\big] \to \mu_\alpha^{(j)}(\cdot)$$

in $\mathbb{M}(\mathbb{R}_+^{\infty\downarrow} \setminus \mathbb{H}_{<j})$ as $n \to \infty$. To show this, we only need to check that

$$(5.14) \qquad (n\nu[n, \infty))^{-j} \mathbf{P}\big[(Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1) \in A\big] \to \mu_\alpha^{(j)}(A)$$

for A's that belong to the convergence-determining class $\mathcal{A}_j \triangleq \{\{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1, \ldots, x_l \leq z_l\} : l \geq j, x_1 \geq \cdots \geq x_l > 0\}$. To see that $\mathcal{A}_j$ is a convergence-determining class for $\mathbb{M}(\mathbb{R}_+^{\infty\downarrow} \setminus \mathbb{H}_{<j})$-convergence, note that $\mathcal{A}_j' \triangleq \{\{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1 < y_1, \ldots, x_l \leq z_l < y_l\} : l \geq j, x_1, \ldots, x_l \in (0, \infty), y_1, \ldots, y_l \in (0, \infty]\}$ satisfies conditions (i), (ii) and (iii) of Lemma 2.7, and hence, is a convergence-determining class. Now define $\mathcal{A}_j(i)$'s recursively as $\mathcal{A}_j(i+1) \triangleq \{B \setminus A : A, B \in \mathcal{A}_j(i), A \subseteq B\}$ for $i \geq 0$, and $\mathcal{A}_j(0) = \mathcal{A}_j'' \triangleq \{\{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1, \ldots, x_l \leq z_l\} : l \geq j, x_1, \ldots, x_l > 0\}$. Since we restrict the set-difference operation between nested sets, the limit associated with the sets in $\mathcal{A}_j(i+1)$ is determined by the sets in $\mathcal{A}_j(i)$, and eventually, $\mathcal{A}_j''$. Noting that $\mathcal{A}_j' \subseteq \bigcup_{i=0}^\infty \mathcal{A}_j(i)$, we see that $\mathcal{A}_j''$ is a convergence-determining class. Now, since both $\mathbf{P}[(Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1) \in \cdot]$ and $\mu_\alpha^{(j)}(\cdot)$ are supported on $\mathbb{R}_+^{\infty\downarrow}$, one can further reduce the convergence determining class from $\mathcal{A}_j''$ to $\mathcal{A}_j$.

To check the desired convergence for the sets in $\mathcal{A}_j$, we first characterize the limit measure. Let $l \geq j$ and $x_1 \geq \cdots \geq x_l > 0$. By the change of variables $v_i = x_i^\alpha y_i^{-\alpha}$ for $i = 1, \ldots, j$,

$$\mu_\alpha^{(j)}\big(\{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1, \ldots, x_l \leq z_l\}\big)$$

$$= \mathbb{I}(j = l) \cdot \int_{x_j}^\infty \cdots \int_{x_1}^\infty \mathbb{I}(y_1 \geq \cdots \geq y_j) \, d\nu_\alpha(y_1) \cdots d\nu_\alpha(y_j)$$

$$= \mathbb{I}(j = l) \cdot \left(\prod_{i=1}^j x_i\right)^{-\alpha} \cdot \int_0^1 \cdots \int_0^1 \mathbb{I}(x_1^{-\alpha} v_1 \leq \cdots \leq x_j^{-\alpha} v_j) \, dv_1 \cdots dv_j.$$

Next, we find a similar representation for the distribution of $\Gamma_1, \ldots, \Gamma_l$. Let $U_{(1)}, \ldots, U_{(l-1)}$ be the order statistics of $l - 1$ i.i.d. uniform random variables on $[0, 1]$. Recall first that the conditional distribution of $(\Gamma_1/\Gamma_l, \ldots, \Gamma_{l-1}/\Gamma_l)$ given Γ_l does not depend on Γ_l and coincides with the distribution of $(U_{(1)}, \ldots, U_{(l-1)})$;

see, for example, Section 4 of Pyke (1965). Suppose that $l \geq j$ and $0 \leq y_1 \leq \cdots \leq y_l$. By the change of variables $u_i = \gamma^{-1} y_i v_i$ for $i = 1, \ldots, l-1$, and $\gamma = y_l v_l$,

$$\mathbf{P}(\Gamma_1 \leq y_1, \ldots, \Gamma_l \leq y_l)$$

$$= \mathbf{E}\big[\mathbf{P}(\Gamma_1/\Gamma_l \leq y_1/\Gamma_l, \ldots, \Gamma_{l-1}/\Gamma_l \leq y_{l-1}/\Gamma_l | \Gamma_l) \cdot \mathbb{I}(\Gamma_l \leq y_l)\big]$$

$$= \int_0^{y_l} \mathbf{P}(U_{(1)} \leq y_1/\gamma, \ldots, U_{(l-1)} \leq y_{l-1}/\gamma) \frac{e^{-\gamma} \gamma^{l-1}}{(l-1)!} \, d\gamma$$

$$= \int_0^{y_l} e^{-\gamma} \gamma^{l-1} \int_0^{y_{l-1}/\gamma} \cdots \int_0^{y_1/\gamma} \mathbb{I}(u_1 \leq \cdots \leq u_{l-1} \leq 1) \, du_1 \cdots du_{l-1} \, d\gamma$$

$$= \left(\prod_{i=1}^{l-1} y_i\right) \int_0^{y_l} e^{-\gamma} \int_0^1 \cdots \int_0^1 \mathbb{I}(y_1 v_1 \leq \cdots \leq y_{l-1} v_{l-1} \leq \gamma) \, dv_1 \cdots dv_{l-1} \, d\gamma$$

$$= \left(\prod_{i=1}^{l} y_i\right) \cdot \int_0^1 \cdots \int_0^1 e^{-y_l v_l} \mathbb{I}(y_1 v_1 \leq \cdots \leq y_l v_l) \, dv_1 \cdots dv_l.$$

Since $0 \leq Q_n(n x_1) \leq \cdots \leq Q_n(n x_l)$ for $x_1 \geq \cdots \geq x_l > 0$,

$$\big(n v[n, \infty)\big)^{-j} \mathbf{P}\big[Q_n^{\leftarrow}(\Gamma_1)/n \geq x_1, \ldots, Q_n^{\leftarrow}(\Gamma_l) \geq x_l\big]$$

$$= \big(n v[n, \infty)\big)^{-j} \mathbf{P}\big[\Gamma_1 \leq Q_n(n x_1), \ldots, \Gamma_l \leq Q_n(n x_l)\big]$$

$$= \big(n v[n, \infty)\big)^{-j} \cdot \left(\prod_{i=1}^{l} Q_n(n x_i)\right)$$

$$\cdot \int_0^1 \cdots \int_0^1 e^{-Q_n(n x_l) v_l} \mathbb{I}(Q_n(n x_1) v_1 \leq \cdots \leq Q_n(n x_l) v_l) \, dv_1 \cdots dv_l$$

$$= \left(\prod_{i=1}^{j} \frac{Q_n(n x_i)}{n v[n, \infty)}\right) \cdot \left(\prod_{i=j+1}^{l} Q_n(n x_i)\right)$$

$$\cdot \int_0^1 \cdots \int_0^1 e^{-Q_n(n x_l) v_l} \mathbb{I}\left(\frac{Q_n(n x_i)}{n v[n, \infty)} v_1 \leq \cdots \leq \frac{Q_n(n x_i)}{n v[n, \infty)} v_l\right) dv_1 \cdots dv_l.$$

Note that $Q_n(n x_i) \to 0$ and $\frac{Q_n(n x_i)}{n v[n, \infty)} \to x_i^{-\alpha}$ as $n \to \infty$ for each $i = 1, \ldots, l$. Therefore, by bounded convergence,

$$\big(n v[n, \infty)\big)^{-j} \mathbf{P}\big[Q_n^{\leftarrow}(\Gamma_1)/n \geq x_1, \ldots, Q_n^{\leftarrow}(\Gamma_l) \geq x_l\big]$$

$$\to \mathbb{I}(j = l) \left(\prod_{i=1}^{j} x_i\right)^{-\alpha} \cdot \int_0^1 \cdots \int_0^1 \mathbb{I}(x_1^{-\alpha} v_1 \leq \cdots \leq x_j^{-\alpha} v_j) \, dv_1 \cdots dv_j$$

$$= \mu_\alpha^{(j)}\big(\{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1, \ldots, x_l \leq z_l\}\big),$$

which concludes the proof of (5.13). The conclusion of the lemma follows from the independence of $(Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1)$ and $(U_l, l \geq 1)$ and Lemma 2.2. $\quad\square$

LEMMA 5.4. *Suppose that $x_1 \geq \cdots \geq x_j \geq 0$; $u_i \in (0, 1)$ for $i = 1, \ldots, j$; $y_1 \geq \cdots \geq y_k \geq 0$; $v_i \in (0, 1)$ for $i = 1, \ldots, k$; $u_1, \ldots, u_j, v_1, \ldots, v_k$ are all distinct.*

(a) *For any $\epsilon > 0$,*
$$\{x \in G : d(x, y) < (1 + \epsilon)\delta \text{ implies } y \in G\}$$
$$\subseteq G^{-\delta}$$
$$\subseteq \{x \in G : d(x, y) < \delta \text{ implies } y \in G\}.$$

Also, $(A \cap B)_\delta \subseteq A_\delta \cap B_\delta$ and $A^{-\delta} \cup B^{-\delta} \subseteq (A \cup B)^{-\delta}$ for any A and B.

(b) $\sum_{i=1}^{j} x_i 1_{[u_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{<j})^{-\delta}$ *implies $x_j \geq \delta$.*

(c) $\sum_{i=1}^{j} x_i 1_{[u_i, 1]} \notin (\mathbb{D} \setminus \mathbb{D}_{<j})^{-\delta}$ *implies $x_j \leq 2\delta$.*

(d) $\sum_{i=1}^{j} x_i 1_{[u_i, 1]} - \sum_{i=1}^{k} y_i 1_{[v_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\delta}$ *implies $x_j \geq \delta$ and $y_k \geq \delta$.*

(e) *Suppose that $\xi \in \mathbb{D}_{j,k}$. If $l < j$ or $m < k$, then ξ is bounded away from $\mathbb{D}_{l,m}$.*

(f) *If $I(\xi) > (\alpha - 1)j + (\beta - 1)k$, then ξ is bounded away from $\mathbb{D}_{<j,k} \cup \mathbb{D}_{j,k}$.*

PROOF. (a) Immediate consequences of the definition.

(b) From (a), we see that $\sum_{i=1}^{j} x_i 1_{[u_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{<j})^{-\delta}$ and $\sum_{i=1}^{j-1} x_i 1_{[u_i, 1]} \in \mathbb{D}_{<j}$ implies $d(\sum_{i=1}^{j} x_i 1_{[u_i, 1]}, \sum_{i=1}^{j-1} x_i 1_{[u_i, 1]}) \geq \delta$, which is not possible if $x_j < \delta$.

(c) We prove that for any $\epsilon > 0$, $\sum_{i=1}^{j} x_i 1_{[u_i, 1]} \notin (\mathbb{D} \setminus \mathbb{D}_{<j})^{-\delta}$ implies $x_j \leq (2 + \epsilon)\delta$. To show this, in turn, we work with the contrapositive. Suppose that $x_j > (2 + \epsilon)\delta$. If $d(\sum_{i=1}^{j} x_i 1_{[u_i, 1]}, \zeta) < (1 + \epsilon/2)\delta$, by the definition of the Skorokhod metric, there exists a nondecreasing homeomorphism ϕ of $[0, 1]$ onto itself such that $\|\sum_{i=1}^{j} x_i 1_{[u_i, 1]} - \zeta \circ \phi\|_\infty < (1 + \epsilon/2)\delta$. Note that at each discontinuity point of $\sum_{i=1}^{j} x_i 1_{[y_i, 1]}$, $\zeta \circ \phi$ should also be discontinuous. Otherwise, the supremum distance between $\sum_{i=1}^{j} x_i 1_{[u_i, 1]}$ and $\zeta \circ \phi$ has to be greater than $(1 + \epsilon/2)\delta$, since the smallest jump size of $\sum_{i=1}^{j} x_i 1_{[u_i, 1]}$ is greater than $(2 + \epsilon)\delta$. Hence, there has to be at least j discontinuities in the path of ζ; that is, $\zeta \in \mathbb{D} \setminus \mathbb{D}_{<j}$. We have shown that $d(\sum_{i=1}^{j} x_i 1_{[u_i, 1]}, \zeta) < (1 + \epsilon/2)\delta$ implies $\zeta \in \mathbb{D} \setminus \mathbb{D}_{<j}$, which in turn, along with (a), shows that $\sum_{i=1}^{j} x_i 1_{[u_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{<j})^{-\delta}$.

(d) Suppose that $\sum_{i=1}^{j} x_i 1_{[u_i, 1]} - \sum_{i=1}^{k} y_i 1_{[v_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\delta}$. Since $\sum_{i=1}^{j-1} x_i 1_{[u_i, 1]} - \sum_{i=1}^{k} y_i 1_{[v_i, 1]} \notin \mathbb{D} \setminus \mathbb{D}_{<j,k}$,

$$x_j \geq d\left(\sum_{i=1}^{j} x_i 1_{[u_i, 1]} - \sum_{i=1}^{k} y_i 1_{[v_i, 1]}, \sum_{i=1}^{j-1} x_i 1_{[u_i, 1]} - \sum_{i=1}^{k} y_i 1_{[v_i, 1]}\right) \geq \delta.$$

Similarly, we get $y_k \geq \delta$.

(e) Let $\xi = \sum_{i=1}^{j} x_i 1_{[u_i, 1]} - \sum_{i=1}^{k} y_i 1_{[v_i, 1]}$. First, we claim that $d(\zeta, \xi) \geq x_j/2$ for any $\zeta \in \mathbb{D}_{l,m}$ with $l < j$. Suppose not, that is, $d(\zeta, \xi) < x_j/2$. Then there exists a nondecreasing homeomorphism ϕ of $[0, 1]$ onto itself such that $\| \sum_{i=1}^{j} x_i 1_{[u_i, 1]} - \zeta \circ \phi \|_\infty < x_j/2$. Note that this implies that at each discontinuity point s of ξ, $\zeta \circ \phi$ should also be discontinuous. Otherwise, $|\zeta \circ \phi(s) - \xi(s)| + |\zeta \circ \phi(s-) - \xi(s-)| \geq |\xi(s) - \xi(s-)| \geq x_j$, and hence, it is contradictory to the bound on the supremum distance between ξ and $\zeta \circ \phi$. However, this implies that ζ has j upward jumps, and hence, contradictory to the assumption $\zeta \in \mathbb{D}_{l,m}$, proving the claim. Likewise, $d(\zeta, \xi) \geq y_k/2$ for any $\xi \in \mathbb{D}_{l,m}$ with $m < k$.

(f) Note that in case $I(\xi)$ is finite, $\mathcal{D}_+(\xi) > j$ or $\mathcal{D}_-(\xi) > k$. In this case, the conclusion is immediate from (e). In case $I(\xi) = \infty$, either $\mathcal{D}_+(\zeta) = \infty$, $\mathcal{D}_-(\zeta) = \infty$, $\xi(0) \neq 0$, or ξ contains a continuous nonconstant piece. By containing a continuous nonconstant piece, we refer to the case that there exist t_1 and t_2 such that $t_1 < t_2$, $\xi(t_1) \neq \xi(t_2-)$ and ξ is continuous on (t_1, t_2). For the first two cases where the number of jumps is infinite, the conclusion is an immediate consequence of (e). The case $\xi(0) \neq 0$ is also obvious. Now we are left with dealing with the last case, where ξ has a continuous nonconstant piece. To discuss this case, assume w.l.o.g. that $\xi(t_1) < \xi(t_2-)$. We claim that $d(\xi, \mathbb{D}_{j,k}) \geq \frac{\xi(t_2-) - \xi(t_1)}{2(j+1)}$. Note that for any step function ζ,

$$
\begin{aligned}
\| \xi - \zeta \| &\geq \left| \xi(t_2-) - \zeta(t_2-) \right| \vee \left| \xi(t_1) - \zeta(t_1) \right| \\
&\geq \left(\xi(t_2-) - \zeta(t_2-) \right) \vee \left(\zeta(t_1) - \xi(t_1) \right) \\
&\geq \frac{1}{2} \left\{ \left(\xi(t_2-) - \xi(t_1) \right) - \left(\zeta(t_2-) - \zeta(t_1) \right) \right\} \\
&\geq \frac{1}{2} \left\{ \left(\xi(t_2-) - \xi(t_1) \right) - \sum_{t \in (t_1, t_2)} \left(\zeta(t) - \zeta(t-) \right) \right\} \\
&\geq \frac{1}{2} \left\{ \left(\xi(t_2-) - \xi(t_1) \right) - 2\mathcal{D}_+(\zeta) \| \xi - \zeta \| \right\},
\end{aligned}
$$

where the fourth inequality is due to the fact that $\| \xi - \zeta \| \geq \frac{\zeta(t) - \zeta(t-)}{2}$ for all $t \in (t_1, t_2)$. From this, we get

$$
\| \xi - \zeta \| \geq \frac{\xi(t_2-) - \xi(t_1)}{2(\mathcal{D}_+(\zeta) + 1)} \geq \frac{\xi(t_2-) - \xi(t_1)}{2(j+1)},
$$

for $\zeta \in \mathbb{D}_{j,k}$. Now, suppose that $\zeta \in \mathbb{D}_{j,k}$. Since $\zeta \circ \phi$ is again in $\mathbb{D}_{j,k}$ for any nondecreasing homeomorphism ϕ of $[0, 1]$ onto itself,

$$
d(\xi, \zeta) \geq \frac{\xi(t_2-) - \xi(t_1)}{2(j+1)},
$$

which proves the claim. $\square$

Now we move on to the proof of Theorem 3.3. We first establish Theorem 5.1, which plays a key role in the proof. Recall that $\mathbb{D}_{<j} = \bigcup_{0 \le l < j} \mathbb{D}_l$ and let $\mathbb{D}_{<(j_1,\ldots,j_d)} \triangleq \bigcup_{(l_1,\ldots,l_d) \in \mathbb{I}_{<(j_1,\ldots,j_d)}} \prod_{i=1}^d \mathbb{D}_{l_i}$ where $\mathbb{I}_{<(j_1,\ldots,j_d)} \triangleq \{(l_1,\ldots,l_d) \in \mathbb{Z}_+^d \setminus \{(j_1,\ldots,j_d)\}) : (\alpha_1 - 1)l_1 + \cdots + (\alpha_d - 1)l_d \le (\alpha_1 - 1)j_1 + \cdots + (\alpha_d - 1)j_d\}$. For each $l \in \mathbb{Z}_+$ and $i = 1,\ldots,d$, let $C_l^{(i)}(\cdot) \triangleq \mathbf{E}[\nu_{\alpha_i}^l \{x \in (0,\infty)^l : \sum_{j=1}^l x_j 1_{[U_j,1]} \in \cdot\}]$ where $U_1,\ldots,U_l$ are i.i.d. uniform on $[0,1]$, and $\nu_{\alpha_i}^l$ is as defined right below (3.1).

THEOREM 5.1.   *Consider independent one-dimensional Lévy processes $X^{(1)}$, $\ldots$, $X^{(d)}$ with spectrally positive Lévy measures $\nu_1(\cdot),\ldots,\nu_d(\cdot)$, respectively. Suppose that each ν_i is regularly varying (at infinity) with index $-\alpha_i < -1$, and let $\bar{X}_n^{(i)}$ be centered and scaled scaled version of $X^{(i)}$ for each $i = 1,\ldots,d$. Then, for each $(j_1,\ldots,j_d) \in \mathbb{Z}_+^d$,*

$$\frac{\mathbf{P}((\bar{X}_n^{(1)},\ldots,\bar{X}_n^{(d)}) \in \cdot)}{\prod_{i=1}^d (n\nu_i[n,\infty))^{j_i}} \to C_{j_1}^{(1)} \times \cdots \times C_{j_d}^{(d)}(\cdot)$$

in $\mathbb{M}(\prod_{i=1}^d \mathbb{D} \setminus \mathbb{D}_{<(j_1,\ldots,j_d)})$.

PROOF.   From Theorem 3.1, we know that $(n\nu_i[n,\infty))^{-j}\mathbf{P}(\bar{X}_n^{(i)} \in \cdot) \to C_j(\cdot)$ in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j})$ for $i = 1,\ldots,d$ and any $j \ge 0$. This along with Lemma 2.2, for each $(l_1,\ldots,l_d) \in \mathbb{Z}_+^d$ we obtain

$$\prod_{i=1}^d (n\nu_i[n,\infty))^{-l_i}\mathbf{P}((\bar{X}_n^{(1)},\ldots,\bar{X}_n^{(d)}) \in \cdot) \to C_{l_1}^{(1)} \times \cdots \times C_{l_d}^{(d)}(\cdot)$$

in $\mathbb{M}(\prod_{i=1}^d \mathbb{D} \setminus \mathbb{C}_{(l_1,\ldots,l_d)})$ where $\mathbb{C}_{(l_1,\ldots,l_d)} \triangleq \bigcup_{i=1}^d (\mathbb{D}^{i-1} \times \mathbb{D}_{<l_i} \times \mathbb{D}^{d-i})$. Since $\mathbb{D}_{<(j_1,\ldots,j_d)} = \bigcap_{(l_1,\ldots,l_d) \notin \mathbb{I}_{<(j_1,\ldots,j_d)}} \mathbb{C}_{(l_1,\ldots,l_d)}$, our strategy is to proceed with Lemma 2.3 to obtain the desired $\mathbb{M}(\prod_{i=1}^d \mathbb{D} \setminus \mathbb{D}_{<(j_1,\ldots,j_d)})$-convergence by combining the $\mathbb{M}(\prod_{i=1}^d \mathbb{D} \setminus \mathbb{C}_{(l_1,\ldots,l_d)})$-convergences for $(l_1,\ldots,l_d) \notin \mathbb{I}_{<(j_1,\ldots,j_d)}$. We first rewrite the infinite intersection over $\mathbb{Z}_+^d \setminus \mathbb{I}_{<(j_1,\ldots,j_d)}$ as a finite one to facilitate the application of the lemma. Consider a partial order $\prec$ on $\mathbb{Z}_+^d$ such that $(l_1,\ldots,l_d) \prec (m_1,\ldots,m_d)$ if and only if $\mathbb{C}_{(l_1,\ldots,l_d)} \subsetneq \mathbb{C}_{(m_1,\ldots,m_d)}$. Note that this is equivalent to $l_i \le m_i$ for $i = 1,\ldots,d$ and $l_i < m_i$ for at least one $i = 1,\ldots,d$. Let $\mathbb{J}_{j_1,\ldots,j_d}$ be the subset of $\mathbb{Z}_+^d$ consisting of the minimal elements of $\mathbb{Z}_+^d \setminus \mathbb{I}_{<(j_1,\ldots,j_d)}$, that is, $\mathbb{J}_{j_1,\ldots,j_d} \triangleq \{(l_1,\ldots,l_d) \in \mathbb{Z}_+^d \setminus \mathbb{I}_{<(j_1,\ldots,j_d)} : (m_1,\ldots,m_d) \prec (l_1,\ldots,l_d)$ implies $(m_1,\ldots,m_d) \in \mathbb{I}_{<(j_1,\ldots,j_d)}\}$. Figure 1 illustrates how the sets $\mathbb{I}_{<(j_1,\ldots,j_d)}$ and $\mathbb{J}_{j_1,\ldots,j_d}$ look when $d = 2$, $j_1 = 2$, $j_2 = 2$, $\alpha_1 = 2$, $\alpha_2 = 3$. It is straightforward to show that $|\mathbb{J}_{j_1,\ldots,j_d}| < \infty$, and that $(m_1,\ldots,m_d) \notin \mathbb{I}_{<(j_1,\ldots,j_d)}$ implies $\mathbb{C}_{(l_1,\ldots,l_d)} \subseteq \mathbb{C}_{(m_1,\ldots,m_d)}$ for some $(l_1,\ldots,l_d) \in \mathbb{J}_{j_1,\ldots,j_d}$; therefore, $\mathbb{D}_{<(j_1,\ldots,j_d)} = \bigcap_{(l_1,\ldots,l_d) \in \mathbb{J}_{j_1,\ldots,j_d}} \mathbb{C}_{(l_1,\ldots,l_d)}$. In view
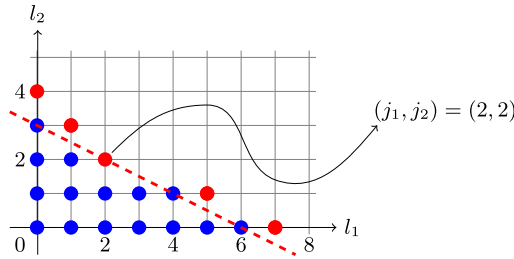
FIG. 1. *An example of $\mathbb{I}_{<(j_1,...,j_d)}$ and $\mathbb{J}_{j_1,...,j_d}$ where $d=2$, $j_1=2$, $j_2=2$, $\alpha_1=2$ and $\alpha_2=3$. The blue dots represent the elements of $\mathbb{I}_{<(j_1,j_2)}$, and the red dots represent the elements of $\mathbb{J}_{j_1,j_2}$. The dashed red line represents (l_1,l_2) such that $(\alpha_1-1)l_1+(\alpha_2-1)l_2=(\alpha_1-1)j_1+(\alpha_2-1)j_2$.*

of this and the fact that $\limsup \frac{\prod_{i=1}^d (n\nu_i[n,\infty))^{-l_i}}{\prod_{i=1}^d (n\nu_i[n,\infty))^{-j_i}} \to 0$ for $(l_1,\ldots,l_d) \in \mathbb{J}_{j_1,...,j_d} \setminus \{(j_1,\ldots,j_d)\}$, the conclusion of the theorem follows from Lemma 2.3 if we show that for each $r > 0$, $\xi \triangleq (\xi_1,\ldots,\xi_d) \notin (\bigcup_{(l_1,...,l_d)\in\mathbb{I}_{<j_1,...,j_d}} \prod_{i=1}^d \mathbb{D}_{l_i})^r$ implies $\xi \notin (\mathbb{C}_{(l_1,...,l_d)})^r$ for some $(l_1,\ldots,l_d) \in \mathbb{J}_{j_1,...,j_d}$. To see that this is the case, suppose that ξ is bounded away from $\bigcup_{(l_1,...,l_d)\in\mathbb{I}_{<j_1,...,j_d}} \prod_{i=1}^d \mathbb{D}_{l_i}$ by $r > 0$. Let $m_i \triangleq \inf\{k \geq 0 : \xi_i \in (\mathbb{D}_{\leqslant k})^r\}$. In case $m_i = \infty$ for some i, one can pick a large enough $M \in \mathbb{Z}_+$ such that $M\mathbf{e}_i \notin \mathbb{I}_{<(j_1,...,j_d)}$ where $\mathbf{e}_i$ is the unit vector with 0 entries except for the ith coordinate. Letting $(l_1,\ldots,l_d) \in \mathbb{J}_{j_1,...,j_d}$ be an index such that $\mathbb{C}_{(l_1,...,l_d)} \subseteq \mathbb{C}_{M\mathbf{e}_i}$, we find that $\xi \notin (\mathbb{C}_{(l_1,...,l_j)})^r \subseteq (\mathbb{C}_{M\mathbf{e}_i})^r$ verifying the premise. If $\max_{i=1,...,d} m_i < \infty$, $\xi \in (\prod_{i=1}^d \mathbb{D}_{m_i})^r$, and hence, $(m_1,\ldots,m_d) \notin \mathbb{I}_{<(j_1,...,j_d)}$, which, in turn, implies that there exists $(l_1,\ldots,l_d) \in \mathbb{J}_{j_1,...,j_d}$ such that $\mathbb{C}_{(l_1,...,l_d)} \subseteq \mathbb{C}_{(m_1,...,m_d)}$. However, due to the construction of m_i's, each ξ_i is bounded away from $\mathbb{D}_{<m_i}$ by r, and hence, ξ is bounded away from $\mathbb{D}^{i-1} \times \mathbb{D}_{<m_i} \times \mathbb{D}^{d-i}$ by r for each i. Therefore, $\xi \notin (\mathbb{C}_{(l_1,...,l_j)})^r \subseteq (\mathbb{C}_{(m_1,...,m_j)})^r$, and hence, the premise is verified. Now we can apply Lemma 2.3 to reach the conclusion of the theorem. $\square$

PROOF OF THEOREM 3.3. Let $X^{(+)}$ and $X^{(-)}$ be Lèvy processes with spectrally positive Lèvy measures ν_+ and ν_-, respectively, where $\nu_+[x,\infty) = \nu[x,\infty)$ and $\nu_-[x,\infty) = \nu(-\infty,-x]$ for each $x > 0$, and denote the corresponding scaled processes as $\bar{X}_n^{(+)}(\cdot) \triangleq X^{(+)}(n\cdot)/n$ and $\bar{X}_n^{(-)}(\cdot) \triangleq X^{(-)}(n\cdot)/n$. More specifically, let

$$\bar{X}_n^{(+)}(s) = sa + B(ns)/n + \frac{1}{n}\int_{|x|\leq 1} x\big[N([0,ns]\times dx) - ns\nu(dx)\big]$$

$$+ \frac{1}{n}\int_{x>1} xN([0,ns]\times dx),$$

$$\bar{X}_n^{(-)}(s) = \frac{1}{n}\int_{x<-1} xN([0,ns]\times dx).$$

From Theorem 5.1, we know that $(n\nu[n,\infty))^{-j}(n\nu(-\infty,-n])^{-k}\mathbf{P}((\bar{X}_n^{(+)},$ $\bar{X}_n^{(-)}) \in \cdot) \to C_j^+ \times C_k^-(\cdot)$ in $\mathbb{M}((\mathbb{D}\times\mathbb{D})\setminus\mathbb{D}_{<(j,k)})$ where $C_j^+(\cdot) \triangleq \mathbf{E}[\nu_\alpha^j\{x \in (0,\infty)^j : \sum_{i=1}^j x_i 1_{[U_i,1]} \in \cdot\}]$ and $C_k^-(\cdot) \triangleq \mathbf{E}[\nu_\beta^k\{y \in (0,\infty)^k : \sum_{i=1}^k y_i 1_{[U_i,1]} \in \cdot\}]$. In view of Lemma 2.6 and that $C_j^+ \times C_k^- \{(\xi,\zeta) \in \mathbb{D}\times\mathbb{D} : (\xi(t)-\xi(t-))(\zeta(t)-\zeta(t-)) \neq 0$ for some $t \in (0,1]\} = 0$, we can apply Lemma 2.4 for $h(\xi,\zeta) = \xi - \zeta$. Noting that $C_{j,k}(\cdot) = (C_j^+ \times C_k^-) \circ h^{-1}(\cdot)$, we conclude that $(n\nu[n,\infty))^{-j} \times (n\nu(-\infty,-n])^{-k}\mathbf{P}(\bar{X}_n^{(+)} - \bar{X}_n^{(-)} \in \cdot) \to C_{j,k}(\cdot)$ in $\mathbb{M}(\mathbb{D}\setminus\mathbb{D}_{<j,k})$. Since $\bar{X}_n$ has the same distribution as $\bar{X}_n^{(+)} - \bar{X}_n^{(-)}$, the desired $\mathbb{M}(\mathbb{D}\setminus\mathbb{D}_{\ll j,k})$-convergence for $\bar{X}_n$ follows. $\square$

PROOF OF LEMMA 3.1. In general,

$$\min_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap\bar{A}\neq\varnothing}} \mathcal{I}(j,k) \leq \mathcal{I}(\mathcal{J}(A),\mathcal{K}(A)) \leq \min_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A^\circ\neq\varnothing}} \mathcal{I}(j,k),$$

and the left inequality cannot be strict since A is bounded away from $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$. On the other hand, in case the right inequality is strict, then $\mathbb{D}_{\mathcal{J}(A),\mathcal{K}(A)} \cap A^\circ = \varnothing$, which in turn implies $C_{\mathcal{J}(A),\mathcal{K}(A)}(A^\circ) = 0$ since $C_{\mathcal{J}(A),\mathcal{K}(A)}$ is supported on $\mathbb{D}_{\mathcal{J}(A),\mathcal{K}(A)}$. Therefore, the lower bound is trivial if the right inequality is strict. In view of these observations, we can assume w.l.o.g. that $(\mathcal{J}(A),\mathcal{K}(A))$ is also in both

$$\underset{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A^\circ\neq\varnothing}}{\arg\min} \mathcal{I}(j,k) \quad \text{and} \quad \underset{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap\bar{A}\neq\varnothing}}{\arg\min} \mathcal{I}(j,k).$$

Since A° and $\bar{A}$ are also bounded-away from $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$, the upper bound of (3.9) is obtained from (2.1) and Theorem 3.3 for $\bar{A}$, $j = \mathcal{J}(\bar{A}) = \mathcal{J}(A)$, and $k = \mathcal{K}(\bar{A}) = \mathcal{K}(A)$; the lower bound of (3.9) is obtained from (2.2) and Theorem 3.3 for A°, $j = \mathcal{J}(A^\circ) = \mathcal{J}(A)$, and $k = \mathcal{K}(A^\circ) = \mathcal{K}(A)$. Finally, we obtain (3.10) from Theorem 3.3 and (2.1) with $j = l$, $k = m$, $F = \bar{A}$ along with the fact that $C_{l,m}(\bar{A}) = 0$ since A is bounded away from $\mathbb{D}_{l,m}$. $\square$

LEMMA 5.5. *Let A be a measurable set and suppose that the argument minimum in (3.8) is nonempty and contains a pair of integers $(\mathcal{J}(A),\mathcal{K}(A))$. Let $(l,m) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$.*

(i) *If $A_\delta \cap \mathbb{D}_{l,m}$ is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$ for some $\delta > 0$, then $A \cap (\mathbb{D}_{l,m})_\gamma$ is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$ for some $\gamma > 0$.*

(ii) *If A is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$, then there exists $\delta > 0$ such that $A \cap (\mathbb{D}_{l,m})_\delta$ is bounded away from $\mathbb{D}_{j,k}$ for any $(j,k) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)} \setminus \{(l,m)\}$.*

PROOF. For (i), we prove that if $d(A_{2\delta} \cap \mathbb{D}_{l,m}, \mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}) > 3\delta$ then $d(A \cap (\mathbb{D}_{l,m})_\delta, \mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}) \geq \delta$. Suppose that $d(A \cap (\mathbb{D}_{l,m})_\delta, \mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}) < \delta$. Then

there exists $\xi \in A \cap (\mathbb{D}_{l,m})_\delta$ and $\zeta \in \mathbb{D}_{\ll \mathcal{J}(A),\mathcal{K}(A)}$ such that $d(\xi,\zeta) < \delta$. Note that we can find $\xi' \in \mathbb{D}_{l,m}$ such that $d(\xi,\xi') \leq 2\delta$, which means that $\xi' \in A_{2\delta} \cap \mathbb{D}_{l,m}$. Therefore, $d(A_{2\delta} \cap \mathbb{D}_{l,m}, \mathbb{D}_{\ll \mathcal{J}(A),\mathcal{K}(A)}) \leq d(\xi',\zeta) \leq d(\xi',\xi) + d(\xi,\zeta) \leq 2\delta + \delta \leq 3\delta$.

For (ii), suppose that $d(A, \mathbb{D}_{\ll \mathcal{J}(A),\mathcal{K}(A)}) > \gamma$ for some $\gamma > 0$ and (l,m) and (j,k) are two distinct pairs that belong to $\mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$. Assume w.l.o.g. that $j < l$. (If $j > l$, it should be the case that $k < m$, and hence, one can proceed in the same way by switching the roles of upward jumps and downward jumps in the following argument.) Let c be a positive number such that $c > 8(l - j) + 2$ and set $\delta = \gamma/c$. We will show that $A \cap (\mathbb{D}_{l,m})_\delta$ and $(\mathbb{D}_{j,k})_\delta$ are bounded away from each other. Let ξ be an arbitrary element of $A \cap (\mathbb{D}_{l,m})_\delta$. Then there exists a $\zeta \in \mathbb{D}_{l,m}$ such that $d(\zeta,\xi) \leq 2\delta$. Note that $d(\zeta, \mathbb{D}_{\ll \mathcal{J}(A),\mathcal{K}(A)}) \geq (c-2)\delta$; in particular, $d(\zeta, \mathbb{D}_{j,m}) \geq (c-2)\delta$. If we write $\zeta \triangleq \sum_{i=1}^{l} x_i 1_{[u_i,1]} - \sum_{i=1}^{m} y_i 1_{[v_i,1]}$, this implies that $x_{j+1} \geq \frac{(c-2)\delta}{l-j}$. Otherwise, $(c-2)\delta > \sum_{i=j+1}^{l} x_i = \|\zeta - \zeta'\| \geq d(\zeta,\zeta')$, where $\zeta' \triangleq \zeta - \sum_{i=j+1}^{l} x_i 1_{[u_i,1]} \in \mathbb{D}_{j,m}$. Therefore, $d(\zeta, \mathbb{D}_{j,k}) \geq \frac{(c-2)\delta}{2(l-j)}$, which in turn implies $d(\xi, \mathbb{D}_{j,k}) \geq \frac{(c-2)\delta}{2(l-j)} - 2\delta > 2\delta$. Since ξ was arbitrary, we conclude that $A \cap (\mathbb{D}_{l,m})_\delta$ bounded away from $(\mathbb{D}_{j,k})_\delta$. $\quad\square$

5.3. *Proofs for Section* 4. Recall that

$$I(\xi) \triangleq \begin{cases} (\alpha - 1)\mathcal{D}_+(\xi) + (\beta - 1)\mathcal{D}_-(\xi) & \text{if } \xi \text{ is a step function with } \xi(0) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

PROOF OF THEOREM 4.2. Observe first that $I(\cdot)$ is a rate function. The level sets $\{\xi : I(\xi) \leq x\}$ equal $\bigcup_{\substack{(l,m)\in\mathbb{Z}_+^2 \\ (\alpha-1)l+(\beta-1)m\leq\lfloor x\rfloor}} \mathbb{D}_{l,m}$ and are therefore closed—note the level sets are not compact so $I(\cdot)$ is not a good rate function (see, e.g., Dembo and Zeitouni (2010) for the definition and properties of good rate functions).

Starting with the lower bound, suppose that G is an open set. We assume w.l.o.g. that $\inf_{\xi \in G} I(\xi) < \infty$, since the inequality is trivial otherwise. Due to the discrete nature of $I(\cdot)$, there exists a $\xi^* \in G$ such that $I(\xi^*) = \inf_{\xi \in G} I(\xi)$. Set $j \triangleq \mathcal{D}_+(\xi^*)$ and $k \triangleq \mathcal{D}_-(\xi^*)$. Let $u_1^+, \ldots, u_j^+$ be the sorted (from the earliest to the latest) upward jump times of ξ^*; $x_1^+, \ldots, x_j^+$ be the sorted (from the largest to the smallest) upward jump sizes of ξ^*; $u_1^-, \ldots, u_k^-$ be the sorted downward jump times of ξ^*; $x_1^-, \ldots, x_k^-$ be the sorted downward jump sizes of ξ^*. Also, let $x_{j+1}^+ = x_{k+1}^- = 0$, $u_0^+ = u_0^- = 0$, and $u_{j+1}^+ = u_{k+1}^- = 1$. Note that if $\zeta \in \mathbb{D}_{l,m}$ for $l < j$, then $d(\xi^*,\zeta) \geq x_j^+/2$ since at least one of the j upward jumps of ξ^* cannot be matched by ξ. Likewise, if $\zeta \in \mathbb{D}_{l,m}$ for $m < k$, then $d(\xi^*,\zeta) \geq x_k^-/2$. Therefore, $d(\mathbb{D}_{<j,k},\xi^*) \geq (x_j^+ \wedge x_k^-)/2$. On the other hand, since G is an open set, we can pick $\delta_0 > 0$ so that the open ball $B_{\xi^*,\delta_0} \triangleq \{\zeta \in \mathbb{D} : d(\zeta,\xi) < \delta_0\}$ centered at

ξ^* with radius δ_0 is a subset of G, that is, $B_{\xi^*, \delta_0} \subset G$. Let $\delta = (\delta_0 \wedge x_j^+ \wedge x_k^-)/4$. If $j = k = 0$, then $\xi^* \equiv 0$, and hence, $\{\bar{X}_n \in G\}$ contains $\{\|\bar{X}_n\| \le \delta\}$ which is a subset of $B_{\xi^*, \delta}$. One can apply Lemma A.4 to show that $\mathbf{P}(X_n \in G)$ converges to 1, which, in turn, proves the inequality. Now, suppose that either $j \ge 1$ or $k \ge 1$. Then $d(B_{\xi^*, \delta}, \mathbb{D}_{<j,k}) \ge \delta$. As $d(B_{\xi^*, \delta}, \mathbb{D}_{<j,k}) > 0$ and $B_{\xi^*, \delta}$ is open, we see from our sharp asymptotics (Theorem 3.1) that

$$C_{j,k}(B_{\xi^*, \delta}) \le \liminf_{n \to \infty} (n\nu[n, \infty))^{-j} (n\nu(-\infty, -n])^{-k} P(\bar{X}_n \in B_{\xi^*, \delta}).$$

From the definition of $C_{j,k}$, it follows that $C_j(B_{\xi^*, \delta}) > 0$. To see this, note first that we can assume w.l.o.g. that $x_i^{\pm}$'s are all distinct since G is open (because, if some of the jump sizes are identical, we can pick ϵ such that $B_{\xi^*, \epsilon} \subseteq G$, and then perturb those jump sizes by ϵ to get a new ξ^* which still belongs to G while whose jump sizes are all distinct). Suppose that $\xi^* = \sum_{l=1}^{j} x_{i_l^+}^+ 1_{[u_{i_l^+}, 1]} - \sum_{l=1}^{k} x_{i_l^-}^- 1_{[u_{i_l^-}, 1]}$, where $\{i_1^{\pm}, \ldots, i_j^{\pm}\}$ are permutations of $\{1, \ldots, j\}$. Let $2\delta' \triangleq \delta \wedge \underline{\Delta}_u^+ \wedge \underline{\Delta}_x^+ \wedge \underline{\Delta}_u^- \wedge \underline{\Delta}_x^-$, where $\underline{\Delta}_u^+ = \min_{i=1,\ldots,j+1}(u_i^+ - u_{i-1}^+)$, $\underline{\Delta}_x^+ = \min_{i=1,\ldots,j}(x_{i-1}^+ - x_i^+)$, $\underline{\Delta}_u^- = \min_{i=1,\ldots,k+1}(u_i^- - u_{i-1}^-)$, and $\underline{\Delta}_x^- = \min_{i=1,\ldots,k}(x_{i-1}^- - x_i^-)$. Consider a subset B' of $B_{\xi^*, \delta}$:

$$B' \triangleq \left\{ \sum_{l=1}^{j} y_{i_l^+}^+ 1_{[v_{i_l^+}, 1]} - \sum_{l=1}^{k} y_{i_l^-}^- 1_{[v_{i_l^-}, 1]} : \right.$$
$$v_i^+ \in (u_i^+ - \delta', u_i^+ + \delta'), y_i^+ \in (x_i^+ - \delta', x_i^+ + \delta'), i = 1, \ldots, j;$$
$$\left. v_i^- \in (u_i^- - \delta', u_i^- + \delta'), y_i^- \in (x_i^- - \delta', x_i^- + \delta'), i = 1, \ldots, k \right\}.$$

Then

$$C_{j,k}(B_{\xi^*, \delta}) \ge C_{j,k}(B')$$
$$= \int_{(u_1^+ - \delta', u_1^+ + \delta') \times \cdots \times (u_j^+ - \delta', u_j^+ + \delta')} d\mathrm{Leb}$$
$$\cdot \int_{(x_1^+ - \delta', x_1^+ + \delta') \times \cdots \times (x_j^+ - \delta', x_j^+ + \delta')} d\nu_\alpha$$
$$\cdot \int_{(u_1^- - \delta', u_1^- + \delta') \times \cdots \times (u_k^- - \delta', u_k^- + \delta')} d\mathrm{Leb}$$
$$\cdot \int_{(x_1^- - \delta', x_1^- + \delta') \times \cdots \times (x_k^- - \delta', x_k^- + \delta')} d\nu_\beta$$
$$\ge (2\delta')^j (2\delta'(x_1^+)^\alpha)^j (2\delta')^k (2\delta'(x_1^-)^\beta)^k > 0.$$

We conclude that

$$
\liminf_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in G)}{\log n}
$$

$$
\geq \liminf_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in B_{\xi^*,\delta})}{\log n}
$$

(5.15)

$$
\geq \liminf_{n\to\infty} \frac{\log(C_{j,k}(B_{\xi^*,\delta})(n\nu[n,\infty))^j (n\nu(-\infty,-n])^k (1+o(1)))}{\log n}
$$

$$
= -((\alpha - 1)j + (\beta - 1)k),
$$

which is the lower bound. Turning to the upper bound, suppose that K is a compact set. We first consider the case where $\inf_{\xi \in K} I(\xi) < \infty$. Pick ξ^*, j and k as in the lower bound, that is, $I(\xi^*) \triangleq \inf_{\xi \in K} I(\xi)$, $j \triangleq \mathcal{D}_+(\xi^*)$, and $k \triangleq \mathcal{D}_-(\xi^*)$. Here, we can assume w.l.o.g. either $j \geq 1$ or $k \geq 1$ since the inequality is trivial in case $j = k = 0$. For each $\zeta \in K$, either $I(\zeta) > I(\xi^*)$, or $I(\zeta) = I(\xi^*)$. We construct an open cover of K by considering these two cases separately:

- If $I(\zeta) > I(\xi^*)$, ζ is bounded away from $\mathbb{D}_{<j,k} \cup \mathbb{D}_{j,k}$ (Lemma 5.4(f)). For each such ζ's, pick a $\delta_\zeta > 0$ in such a way that $d(\zeta, \mathbb{D}_{<j,k} \cup \mathbb{D}_{j,k}) > \delta_\zeta$. Set $j_\zeta \triangleq j$ and $k_\zeta \triangleq k$. Note that in this case $C_{j_\zeta,k_\zeta}(\bar{B}_{\zeta,\delta_\zeta}) = 0$.
- If $I(\zeta) = I(\xi^*)$, set $j_\zeta \triangleq \mathcal{D}_+(\zeta)$ and $k_\zeta \triangleq \mathcal{D}_-(\zeta)$. Since they are bounded away from $\mathbb{D}_{<j_\zeta,k_\zeta}$ (Lemma 5.4(e)), we can choose $\delta_\zeta > 0$ such that $d(\zeta, \mathbb{D}_{<j_\zeta,k_\zeta}) > \delta_\zeta$ and $C_{j_\zeta,k_\zeta}(\bar{B}_{\zeta,\delta_\zeta}) < \infty$.

Consider an open cover $\{B_{\zeta;\delta_\zeta} : \zeta \in K\}$ of K and its finite subcover $\{B_{\zeta_i;\delta_{\zeta_i}}\}_{i=1,\dots,m}$. For each ζ_i, we apply the sharp asymptotics (Theorem 3.3) to $\bar{B}_{\zeta_i;\delta_{\zeta_i}}$ to get

(5.16)
$$
\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in \bar{B}_{\zeta_i;\delta_{\zeta_i}})}{\log n} \leq (\alpha - 1)j_{\zeta_i} + (\beta - 1)k_{\zeta_i} = -I(\xi^*).
$$

Therefore,

$$
\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in \bar{F})}{\log n} \leq \limsup_{n\to\infty} \frac{\log \sum_{i=1}^{m} \mathbf{P}(\bar{X}_n \in \bar{B}_{\zeta_i;\delta_{\zeta_i}})}{\log n}
$$

(5.17)

$$
= \max_{i=1,\dots,m} \limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in \bar{B}_{\zeta_i;\delta_{\zeta_i}})}{\log n}
$$

$$
\leq -I(\xi^*) = -\inf_{\xi \in K} I(\xi),
$$

completing the proof of the upper bound in case the right-hand side is finite.

Now, turning to the case $\inf_{\xi \in K} I(\xi) = \infty$, fix an arbitrary positive integer l. Since $\mathbb{D}_{<l,l}$ is closed and disjoint with a compact set K, it is also bounded away

from each $\zeta \in K$. Now picking $\delta_\zeta > 0$ so that $\bar{B}_{\zeta,\delta_\zeta}$ is disjoint with K for each ζ, one can construct an open cover $\{B_{\zeta;\delta_\zeta} : \zeta \in K\}$ of K. Let $\{B_{\zeta_i;\delta_{\zeta_i}}\}_{i=1,\ldots,m}$ its finite subcover, then from the same calculation as (5.16) and (5.17),

$$\limsup_{n \to \infty} \frac{\log \mathbf{P}(\bar{X}_n \in K)}{\log n} \leq -(\alpha + \beta - 2)m.$$

Taking $m \to \infty$, we arrive at the desired upper bound. $\quad\square$

APPENDIX: INEQUALITIES

LEMMA A.1 (Generalized Kolmogorov inequality; Lemma in p. 335 of Shneer and Wachtel (2011)). *Let $S_n = X_1 + \cdots + X_n$ be a random walk with mean zero increments, that is, $\mathbf{E}X_i = 0$. Then*

$$\mathbf{P}\Big(\max_{k \leq n} S_k \geq x\Big) \leq C \frac{n V(x)}{x^2},$$

where $V(x) = \mathbf{E}(X_1^2; |X_1| \leq x)$, for all $x > 0$.

LEMMA A.2 (Etemadi's inequality). *Let $X_1, \ldots, X_n$ be independent real-valued random variables defined on some common probability space, and let $x \geq 0$. Let S_k denote the partial sum $S_k = X_1 + \cdots + X_k$. Then*

$$\mathbb{P}\Big(\max_{1 \leq k \leq n} |S_k| \geq 3x\Big) \leq 3 \max_{1 \leq k \leq n} \mathbb{P}(|S_k| \geq x).$$

LEMMA A.3 (Prokhorov's inequality; Prokhorov (1959)). *Suppose that ξ_i, $i = 1, \ldots, n$ are independent, zero-mean random variables such that there exists a constant c for which $|\xi_i| \leq c$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n \mathbf{var}\,\xi_i < \infty$. Then*

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i > x\right) \leq \exp\left\{-\frac{x}{2c} \operatorname{arcsinh} \frac{xc}{2\sum_{i=1}^n \mathbf{var}\,\xi_i}\right\},$$

which, in turn, implies

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i > x\right) \leq \left(\frac{cx}{\sum_{i=1}^n \mathbf{var}\,\xi_i}\right)^{-\frac{x}{2c}}.$$

We extend the Etemadi's inequality to Lévy processes in the following lemma.

LEMMA A.4. *Let Z be a Lévy process. Then*

$$\mathbb{P}\Big(\sup_{t \in [0,n]} |Z(t)| \geq x\Big) \leq 3 \sup_{t \in [0,n]} \mathbf{P}(|Z(t)| \geq x/3).$$

PROOF.   Since Z (and hence, $|Z|$ also) is in $\mathbb{D}$, $\sup_{0 \leq k \leq 2^m} |Z(\frac{nk}{2^m})|$ converges to $\sup_{t \in [0,n]} |Z(t)|$ almost surely as $m \to \infty$. To see this, note that one can choose t_i's such that $|Z(t_i)| \geq \sup_{t \in [0,n]} |Z(t)| - i^{-1}$. Since $\{t_i\}$'s are in a compact set $[0, n]$, there is a subsequence, say, t_i', such that $t_i' \to t_0$ for some $t_0 \in [0, n]$. The supremum has to be achieved at either t_0^- or t_0. Either way, with large enough m, $\sup_{0 \leq k \leq 2^m} |Z(\frac{nk}{2^m})|$ becomes arbitrarily close to the supremum. Now, by bounded convergence,

$$
\begin{aligned}
&\mathbf{P}\Big\{ \sup_{t \in [0,n]} |Z(t)| > x \Big\} \\
&\quad = \lim_{m \to \infty} \mathbf{P}\Big\{ \sup_{0 \leq k \leq 2^m} \Big| Z\Big(\frac{nk}{2^m}\Big) \Big| > x \Big\} \\
&\quad = \lim_{m \to \infty} \mathbf{P}\Big\{ \sup_{0 \leq k \leq 2^m} \Big| \sum_{i=0}^{k} \Big(Z\Big(\frac{ni}{2^m}\Big) - Z\Big(\frac{n(i-1)}{2^m}\Big) \Big) \Big| > x \Big\} \\
&\quad \leq \lim_{m \to \infty} 3 \sup_{0 \leq k \leq 2^m} \mathbf{P}\Big\{ \Big| \sum_{i=0}^{k} \Big(Z\Big(\frac{ni}{2^m}\Big) - Z\Big(\frac{n(i-1)}{2^m}\Big) \Big) \Big| > x/3 \Big\} \\
&\quad = \lim_{m \to \infty} 3 \sup_{0 \leq k \leq 2^m} \mathbf{P}\Big\{ \Big| Z\Big(\frac{nk}{2^m}\Big) \Big| > x/3 \Big\} \\
&\quad \leq 3 \sup_{t \in [0,n]} \mathbf{P}\big\{ |Z(t)| > x/3 \big\},
\end{aligned}
$$

where $Z(t) \triangleq 0$ for $t < 0$. $\square$

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