

GAUSSIAN FREE FIELD LIGHT CONES AND $SLE_\kappa(\rho)$

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Let h be an instance of the GFF. Fix $\kappa \in (0, 4)$ and $\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2$. Recall that an *imaginary geometry ray* is a flow line of $e^{i(h/\chi + \theta)}$ that looks locally like SLE_κ . The *light cone* with parameter $\theta \in [0, \pi]$ is the set of points reachable from the origin by a sequence of rays with angles in $[-\theta/2, \theta/2]$. It is known that when $\theta = 0$, the light cone looks like SLE_κ , and when $\theta = \pi$ it looks like the range of an $SLE_{16/\kappa}$ *counterflow line*. We find that when $\theta \in (0, \pi)$ the light cones are either fractal carpets with a dense set of holes or space-filling regions with no holes. We show that every non-space-filling light cone agrees in law with the range of an $SLE_\kappa(\rho)$ process with $\rho \in ((-2 - \kappa/2) \vee (\kappa/2 - 4), -2)$. Conversely, the range of any such $SLE_\kappa(\rho)$ process agrees in law with a non-space-filling light cone. As a consequence of our analysis, we obtain the first proof that these $SLE_\kappa(\rho)$ processes are a.s. continuous curves and show that they can be constructed as natural path-valued functions of the GFF.

1. Introduction. The $SLE_\kappa(\rho)$ processes are an important variant of the Schramm–Loewner evolution (SLE) [12]. They were first introduced by Lawler, Schramm and Werner in [3], Section 8.3. Like ordinary SLE_κ , $SLE_\kappa(\rho)$ is defined using the Loewner equation and a *driving function* W that looks (at least locally) like $\sqrt{\kappa}$ times a Brownian motion. However, in addition to the driving function W , one keeps track of a so-called *force point* process V , which itself evolves according to Loewner evolution, and which exerts a *drift* on W proportional to $\rho/(W - V)$. When $\rho > 0$ (resp., $\rho < 0$), the drift pushes W away from (resp., toward) the force point V , and the case $\rho = 0$ corresponds to ordinary SLE_κ . The difference $V - W$ evolves as a positive multiple of a Bessel process of dimension $\delta(\kappa, \rho) = 1 + \frac{2(\rho+2)}{\kappa}$ (provided the force point is to the right of the origin). See Section 2 for a formal definition of $SLE_\kappa(\rho)$. Various flavors of $SLE_\kappa(\rho)$ have been discussed in the literature, but in this paper we generally assume that the processes are *chordal* (so they grow from 0 to ∞ in the upper half-plane \mathbf{H}), *one-sided* (so that all excursions of $V - W$ away from zero have the same sign) and *origin seeded* (meaning that $V_0 = W_0 = 0$).

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The time evolution of W and V is straightforward to define during intervals of time in which $W_t \neq V_t$, but to continue the evolution after W and V collide, one has to work out precisely how these processes “bounce off” one another. In the original construction in [3], Section 8.3, and in most of the later work on $SLE_\kappa(\rho)$ processes, this is only done for $\rho > -2$. The threshold -2 corresponds to $\delta(\kappa, \rho) = 1$, which is the critical threshold below which Bessel processes fail to be semimartingales [10], Chapter 11. This is related to the fact that $\delta > 1$ is necessary in order for the integral $\int_0^T (V_t - W_t)^{-1} dt$ to be a.s. finite for all T , which in turn ensures that the cumulative amount of drift exerted on W (up to any finite time) is a.s. finite.

To define $SLE_\kappa(\rho)$ when $\rho < -2$, it is necessary to introduce a local time Lévy compensation to keep the accumulated drift from sending W off to ∞ in finite time. As we recall in Section 2 (citing [16], Section 3.2), there is a natural scale-invariant way to do this if and only if $\rho > -2 - \frac{\kappa}{2}$ so that $\delta > 0$. As detailed in [16], Section 3, if one parameterizes W by the local time associated to $\{t : W_t = V_t\}$ one obtains a skew stable Lévy process, so that the classification of general $SLE_\kappa(\rho)$ processes is closely related to the classification of skew stable Lévy processes.³

The continuity and reversibility properties of $SLE_\kappa(\rho)$ with $\rho > -2$ are established in [5–8], which exhibit and make use of explicit couplings between these processes and the Gaussian free field (GFF) [2, 5, 8, 13, 14] (see also [1, 21] for the reversibility of $SLE_\kappa(\rho)$ for $\kappa \in (0, 4)$ and $\rho \geq \frac{\kappa}{2} - 2$). When $\rho > -2$, the range of an $SLE_\kappa(\rho)$ process looks locally like the range of an ordinary SLE_κ , except where the path hits the boundary.

When $\rho \leq -2$, however, one obtains interesting and qualitatively different processes. The Bessel dimension interval $\delta \in (0, 1)$ corresponds to $\rho \in (-2 - \frac{\kappa}{2}, -2)$. In this article, we focus on the set $\mathcal{T} = \{(\kappa, \rho) : (-2 - \frac{\kappa}{2}) \vee (\frac{\kappa}{2} - 4) < \rho < -2\}$, which corresponds to the yellow *light cone* region depicted in Figure 1. The *loops on trunk* regions shown in Figure 1 are studied in detail in [9].⁴ We will find that

³In the account in [16], there is a parameter β such that each $V - W$ excursion away from zero is (independently of all others) assigned a positive sign with probability $(1 + \beta)/2$ and a negative sign otherwise. When $\rho = -2$, it is necessary to take $\beta = 0$ to obtain a canonical, scale-invariant and nontrivial process, and there is an additional free parameter μ in that case. We will not consider the $\rho = -2$ setting here, except to say that in some limiting sense $\beta = 1$ and $\rho = -2$ corresponds to a trivial boundary tracing path. As mentioned above, this paper treats only the “one-sided” case $\beta = 1$, and our main results assume $\rho < -2$.

⁴In the loops-on-trunk regime explored in [9], each excursion of $V - W$ away from zero describes a loop, and it is important and relevant to consider *non-one-sided* $SLE_\kappa(\rho)$, which can be written $SLE_\kappa^\beta(\rho)$ for $\beta \in [-1, 1]$, and which correspond to different types of CLE explorations. These explorations are useful for understanding CLE percolation and the continuum FK correspondence, among other things. In general, $SLE_\kappa^\beta(\rho)$ can be defined for all $\beta \in [-1, 1]$ whenever $\rho \in (-2 - \kappa/2, \kappa/2 - 2) \setminus \{-2\}$, so that $\delta \in (0, 2) \setminus \{1\}$, and [9], Section 10.1.3, briefly describes how to interpret and prove continuity results for these processes for general β in the case $\kappa > 4$. When $\kappa \leq 4$, it remains an open problem to prove continuity for $SLE_\kappa^\beta(\rho)$ when $\beta \in (-1, 1)$ and $\rho \in ((-2 - \kappa/2) \vee (\kappa/2 - 4), \kappa/2 - 2) \setminus \{-2\}$, that is, in the light cone region and (the boundary-

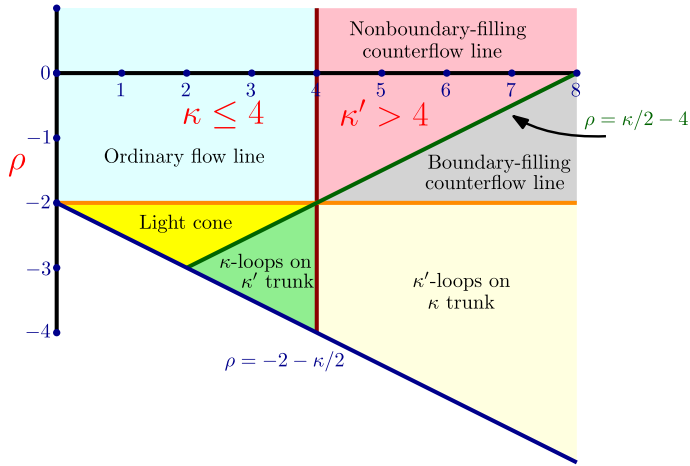


FIG. 1. Phase diagram for the behavior of $SLE_\kappa(\rho)$ for ρ above the minimal value $-2 - \frac{\kappa}{2}$ for which such a process is defined. The present paper is focused on the light cone regime (yellow triangle) where $\kappa \in (0, 4)$ and $\rho \in ((-2 - \frac{\kappa}{2}) \vee (\frac{\kappa}{2} - 4), -2)$. The other two $\rho < -2$ regimes are studied in [9] and the $\rho > -2$ cases are treated in [5, 8].

$SLE_\kappa(\rho)$ with $(\kappa, \rho) \in \mathcal{T}$ can be naturally coupled with an instance of the GFF, and that in this coupling the field a.s. determines the path. This will be accomplished by showing that such a process can be realized as an *ordered light cone* of angle-varying flow lines of the (formal) vector-field $e^{ih/\chi}$,

$$(1.1) \quad \chi := \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2},$$

where h is a GFF. We remark that for $\kappa' > 4$, we have $\frac{\kappa'}{2} - 4 > -2$ so $SLE_{\kappa'}(\rho)$ with this range of ρ values falls under the scope of [5–8]. At $\rho = \frac{\kappa}{2} - 4$, $SLE_\kappa(\rho)$ for $\kappa \in (2, 4)$ has a phase transition from the light cone regime described in this article to the loop-making/trunk regime studied by the authors together with Werner in [9]. (In fact, as we will explain here and have also mentioned in [9], the law of the range of an $SLE_\kappa(\frac{\kappa}{2} - 4)$ process is the same as the law of the range of an $SLE_{\kappa'}(\frac{\kappa'}{2} - 4)$ process, where $\kappa \in (0, 4)$ and $\kappa' = 16/\kappa > 4$.) See Table 1 and Figure 1 for a summary of the phases of $SLE_\kappa(\rho)$.

Our first main result concerns continuity and transience.

intersecting part of) the ordinary flow line region in Figure 1. We remark that in these regions, each excursion of $V - W$ away from zero should (assuming continuity of the overall path) describe a *chord* (i.e., a simple path segment starting and ending at different points) and we are not aware of a natural interpretation of an overall path that alternates between left and right going chords. As mentioned earlier, we treat only the case $\beta = 1$ in this paper. (The case $\beta = -1$ is equivalent by symmetry.)

TABLE 1

Phases of ρ values and corresponding $\delta(\kappa, \rho)$ (driving Bessel process dimension) values for SLE $_{\kappa}(\rho)$ processes with a single boundary force point of weight ρ , assuming $\kappa \in (2, 4)$. When $\kappa \in (0, 2]$, the phases are the same except that the second and third are replaced by a single “light cone” phase with $\rho \in (-2 - \frac{\kappa}{2}, -2)$ and $\delta \in (0, 1)$. The symbol “ ∂ ” should be translated as “boundary” and “rev.” stands for “reversible.” The statements in the reversible column are only applicable when the force point is located immediately to the left or to the right of the seed of the process

ρ	$\delta(\kappa, \rho)$	$\dim_{\mathcal{H}}(\text{Range})$	Process type	Simple	Rev.
$(-\infty, -2 - \frac{\kappa}{2}]$	$(-\infty, 0]$	–	Not defined	–	–
$(-2 - \frac{\kappa}{2}, \frac{\kappa}{2} - 4]$	$(0, 2 - \frac{4}{\kappa}]$	$1 + \frac{2}{\kappa}$	Trunk plus loops	X	X
$(\frac{\kappa}{2} - 4, -2)$	$(2 - \frac{4}{\kappa}, 1)$	$\frac{(\kappa - 2(2 + \rho))(\kappa + 2(6 + \rho))}{8\kappa}$	Light cone	X	X
-2	1	1	∂ tracing	✓	✓
$(-2, \frac{\kappa}{2} - 2)$	(1, 2)	$1 + \frac{\kappa}{8}$	∂ hitting	✓	✓
$[\frac{\kappa}{2} - 2, \infty)$	$[2, \infty)$	$1 + \frac{\kappa}{8}$	∂ avoiding	✓	✓

THEOREM 1.1. *The SLE $_{\kappa}(\rho)$ processes for $\kappa \in (0, 4)$, $\rho \in [\frac{\kappa}{2} - 4, -2)$, and $\rho > -2 - \frac{\kappa}{2}$ are a.s. continuous and transient. That is, if $D \subseteq \mathbf{C}$ is a Jordan domain, $x, y \in \partial D$ are distinct, and $\eta: [0, \infty) \rightarrow D$ is an SLE $_{\kappa}(\rho)$ in D from x to y then η is a.s. continuous and $\lim_{t \rightarrow \infty} \eta(t) = y$ a.s.*

The continuity of ordinary SLE was first proved by Rohde and Schramm in [11]. The main idea is to estimate the moments of the derivative of the reverse Loewner flow evaluated near the inverse image of the tip of the path. By the Girsanov theorem, during a time interval in which $V_t \neq W_t$, the evolution of an SLE $_{\kappa}(\rho)$ is absolutely continuous with respect to the evolution of ordinary SLE $_{\kappa}$. Consequently, the almost sure continuity of the process during such intervals of time can be easily derived from [11]. From this one can see immediately that SLE $_{\kappa}(\rho)$ is a.s. continuous when $\rho \geq \frac{\kappa}{2} - 2$ so that $\delta \geq 2$. A more general statement is [5], Theorem 1.3, which states that SLE $_{\kappa}(\rho)$ is a.s. continuous for all κ and all $\rho > -2$. The idea of that proof is to extract the continuity from the nonboundary-intersecting case and a conditioning trick which involves multiple SLE paths coupled together using the GFF. Theorem 1.1 extends this further to the case that $\rho \geq \frac{\kappa}{2} - 4$ and $\rho > -2 - \frac{\kappa}{2}$. Its proof is also based on GFF arguments, though the method is rather different than that of [5], Theorem 1.3. Continuity in the case that $\rho \in (-2 - \frac{\kappa}{2}, \frac{\kappa}{2} - 4]$ was established in [9], also using GFF based arguments. Combining these works, we have SLE $_{\kappa}(\rho)$ continuity for all of the regions shown in Figure 1.

Suppose that $D \subseteq \mathbf{C}$ is a Jordan domain, $x \in \partial D$ and h is a GFF on D with given boundary conditions. Fix angles $\theta_1 \leq \theta_2 \leq \theta_1 + \pi$. The SLE $_{\kappa}$ light cone $\mathbf{L}_x(\theta_1, \theta_2)$ of h starting from x with angle range $[\theta_1, \theta_2]$ is a random set in D generated from the flow lines of $e^{ih/x}$ (hereafter, we will refer to these simply as “flow lines of h ”). It is explicitly given by the closure of the set of points accessible by the flow lines

of h starting from x with angles which are either rational and contained in $[\theta_1, \theta_2]$ or equal to θ_1 or θ_2 and which change angles a finite number of times and only at positive rational times. These objects were first introduced in [5]. We call $\theta_2 - \theta_1$ the *opening angle* of $\mathbf{L}_x(\theta_1, \theta_2)$. For $\theta \in [0, \pi]$, we let $\mathbf{L}_x(\theta) = \mathbf{L}_x(-\frac{\theta}{2}, \frac{\theta}{2})$. It is shown in [5], Theorem 1.4, that a light cone with opening angle π starting from x is equal to the range of a form of $\text{SLE}_{16/\kappa}$, which is called a *counterflow line* targeted at x . More generally, if A is a segment of ∂D , we let $\mathbf{L}_A(\theta_1, \theta_2)$ be the set points accessible by flow lines of h starting from a countable dense subset of A with angles which are either rational and contained in $[\theta_1, \theta_2]$ or equal to θ_1 or θ_2 which change angles only a finite number of times and only at positive rational times. Our next result states that $\mathbf{L}_{\mathbf{R}_-}(0, \theta)$ for the intermediate values of $\theta \in (0, \pi)$ is equal to the range of an $\text{SLE}_\kappa(\rho)$ process provided the boundary data of h is chosen appropriately.

Let

$$(1.2) \quad \lambda := \frac{\pi}{\sqrt{\kappa}}.$$

THEOREM 1.2. *Fix $\kappa \in (0, 4)$, $\rho \in [\frac{\kappa}{2} - 4, -2)$ and $\rho > -2 - \frac{\kappa}{2}$, and suppose that h is a GFF on \mathbf{H} whose boundary data is given by $-\lambda$ on \mathbf{R}_- and $\lambda(1 + \rho)$ on \mathbf{R}_+ . Let η be an $\text{SLE}_\kappa(\rho)$ process on \mathbf{H} from 0 to ∞ where its force point is located at 0^+ . For each $t \geq 0$, let K_t denote the closure of the complement of the unbounded connected component of $\mathbf{H} \setminus \eta([0, t])$, let $g_t : \mathbf{H} \setminus K_t \rightarrow \mathbf{H}$ be the unique conformal transformation with $\lim_{z \rightarrow \infty} |g_t(z) - z| = 0$ and let (W, V) be the Loewner driving pair for η . There exists a unique coupling of h and η such that the following is true. For each η -stopping time τ , the conditional law of*

$$h \circ g_\tau^{-1} - \chi \arg(g_\tau^{-1})'$$

given $\eta|_{[0, \tau]}$ is that of a GFF on \mathbf{H} with boundary conditions given by

$$h|_{(-\infty, W_\tau]} \equiv -\lambda, \quad h|_{(W_\tau, V_\tau]} \equiv \lambda \quad \text{and} \quad h|_{(V_\tau, \infty)} \equiv \lambda(1 + \rho).$$

Moreover, in the coupling (h, η) , η is a.s. determined by h . Finally, let

$$(1.3) \quad \theta = \theta_\rho = \pi \left(\frac{\rho + 2}{\kappa/2 - 2} \right).$$

Then the range of η is a.s. equal to $\mathbf{L}_{\mathbf{R}_-}(0, \theta)$.

We remark that the existence statement in Theorem 1.2 takes the same form as that for $\text{SLE}_\kappa(\rho)$ when $\rho > -2$, for example, [5], Theorem 1.1. The proof that we give here, however, is quite different. The difference between the different regimes of ρ values is in the way that the coupling is interpreted. In particular, we interpret the process when $\rho > -2$ as being a flow line of the (formal) vector field $e^{ih/\chi}$ (see the introductions to [5, 17] for further explanation) while we interpret

the process when $\rho \in [\frac{\kappa}{2} - 4, -2)$ and $\rho > -2 - \frac{\kappa}{2}$ as an ordered light cone of flow lines of $e^{ih/\chi}$. The method that we use to prove existence in Theorem 1.2 is also very different from the existence proof given in [2, 13, 14, 17] for $\rho > -2$. Indeed, in these works existence is shown by proving that a sample of the GFF can be produced by first sampling the path according to its marginal distribution and then sampling a GFF on the complement of the range of the path with appropriate boundary conditions. That the marginal law of the field is a GFF is proved using tools from stochastic calculus. We emphasize, however, that these previously established existence results will be of crucial importance in the present work. Indeed, we will make use of the flow line interaction theory from [5–8] and the local set theory from [13] to show directly that the path which arises by visiting the points of a light cone with a particular order evolves as an SLE $_{\kappa}(\rho)$. The final statement of Theorem 1.2 generalizes [5], Theorem 1.4, to the setting of SLE $_{\kappa}(\rho)$ for $\rho \in [\frac{\kappa}{2} - 4, -2)$ and $\rho > -2 - \frac{\kappa}{2}$. In the case of the former, the result followed by studying the manner in which flow and counterflow lines coupled together with the GFF interact with each other. The proof of Theorem 1.2 is different. We will extract the latter from the corresponding result for SLE $_{\kappa'}(\rho)$ processes with $\rho > -2$ proved in [5], Theorem 1.4.

Let $\dim_{\mathcal{H}}(A)$ denote the Hausdorff dimension of a set A . The almost sure value of $\dim_{\mathcal{H}}(\mathbf{L}_x(\theta))$ is computed in [4], Theorem 1.1. (That $\dim_{\mathcal{H}}(\mathbf{L}_{\mathbf{R}_-}(\theta))$ is a.s. equal to $\dim_{\mathcal{H}}(\mathbf{L}_x(\theta))$ follows from the argument explained in the caption of Figure 7, as the dimension of the light cone is at least as large as that of the SLE $_{\kappa}$ -type paths which are used in its construction.)

Combining this with Theorem 1.2 gives that if η is an SLE $_{\kappa}(\rho)$ process with $\kappa \in (0, 4)$, $\rho \in [\frac{\kappa}{2} - 4, -2)$ and $\rho > -2 - \frac{\kappa}{2}$, then

$$(1.4) \quad \dim_{\mathcal{H}}(\eta) = \frac{(\kappa - 2(2 + \rho))(\kappa + 2(6 + \rho))}{8\kappa} \text{ a.s.}$$

This result is stated as [4], Theorem 1.3.

The decomposition of the range of SLE $_{\kappa}(\rho)$ into a light cone of angle-varying flow lines is related to the notion of duality for SLE $_{\kappa}$. The principle of duality states that the outer boundary of an SLE $_{\kappa'}$ process can be described by a form of SLE $_{\kappa}$ for $\kappa \in (0, 4)$ and $\kappa' = 16/\kappa \in (4, \infty)$, [1, 5, 8, 19, 20]. Since the range of an SLE $_{\kappa'}$ process can be described in terms of a light cone with opening angle π , it thus follows from Theorem 1.2 that the law of the range of an SLE $_{\kappa}(\frac{\kappa}{2} - 4)$ is the same as that of a form of SLE $_{\kappa'}$ (specifically, an SLE $_{\kappa'}(\frac{\kappa'}{2} - 4)$). It turns out, however, that the two processes visit the points in their range using a different order. This is explained in more detail in Section 4 as well as in [9]. Our final result is the continuity of the law of an SLE $_{\kappa}(\rho)$ process as a function of ρ with ρ in the light cone regime.

THEOREM 1.3. *Fix $\kappa \in (0, 4)$, let $D \subseteq \mathbf{C}$ be a bounded Jordan domain, and fix $x, y \in \partial D$ distinct. The law of the trajectory of an SLE $_{\kappa}(\rho)$ process from x to*

y in D is continuous with respect to the weak topology induced by the topology of uniform convergence modulo time parameterization as ρ varies in the range $[\frac{\kappa}{2} - 4, -2) \cap (-2 - \frac{\kappa}{2}, -2)$.

Outline. The remainder of this article is structured as follows. In Section 2, we will give some preliminaries. In Section 3, we will prove Theorem 1.2 and then use it to derive Theorem 1.1 and Theorem 1.3. Finally, in Section 4 we will explain why the law of the range of an $SLE_\kappa(\frac{\kappa}{2} - 4)$ process for $\kappa \in (2, 4)$, which is at the boundary of the light cone regime, is equal to the law of a range of an $SLE_{\kappa'}(\frac{\kappa'}{2} - 4)$ process, but the processes visit their range in a different order.

2. Preliminaries. In this section, we are going to give an overview of the chordal $SLE_\kappa(\rho)$ processes, focusing on the particular case that $\rho \in (-2 - \frac{\kappa}{2}, -2)$, as well as summarize some of the basics of imaginary geometry [5–8] which is relevant for this work.

2.1. $SLE_\kappa(\rho)$ processes. In this subsection, we are going to give an overview of the $SLE_\kappa(\rho)$ processes. These are variants of SLE first introduced in [3], Section 8.3. They are defined in the same way as ordinary SLE, except they are driven by a multiple of a Bessel process in place of a Brownian motion. The treatment that we give here will parallel that from [16], Section 3.2 and Section 3.3.

For the convenience of the reader, we will now review a few basic facts about Bessel processes. (We refer the reader to [10], Chapter 11, for a more in-depth introduction.) The starting point for the construction of the law of a *Bessel process of dimension δ* (BES^δ) is the so-called *square Bessel process of dimension δ* ($BESQ^\delta$). For a fixed value of $\delta \in \mathbf{R}$, the law of a $BESQ^\delta$ is described in terms of the SDE

$$(2.1) \quad dZ_t = \delta dt + 2\sqrt{Z_t} dB_t, \quad Z_0 = z_0 > 0,$$

where B is a standard Brownian motion. Standard results for SDEs imply that there is a unique strong solution to (2.1), at least up until the first time that Z hits 0. When $\delta > 0$, there in fact exists a unique strong solution for all $t \geq 0$ which is nonnegative for all times.

A process X has the BES^δ law if it admits the expression $X = \sqrt{Z}$ where Z is a $BESQ^\delta$. Itô’s formula implies that X solves the SDE

$$(2.2) \quad dX_t = \frac{\delta - 1}{2} \cdot \frac{1}{X_t} dt + dB_t, \quad X_0 = x_0,$$

at least up until the first time that X hits 0. Using that $X_t^{2-\delta}$ is a continuous local martingale, it is straightforward to check that a BES^δ process a.s. hits 0 if $\delta < 2$ and a.s. does not hit 0 if $\delta \geq 2$. When $\delta > 1$, a BES^δ process solves (2.2) in integrated form for all $t \geq 0$, even when it is bouncing off 0. In particular, such processes are

semimartingales. A BES¹ process X is equal in distribution to $|B|$ where B is a standard Brownian motion, hence in this case, by the Itô–Tanaka formula, X solves a version of (2.2) with an extra correction coming from the local time of X at 0. Thus BES¹ processes are also semimartingales. However, X_t^{-1} is not integrable in this case. When $\delta \in (0, 1)$, it turns out that a BES $^{\delta}$ process is not a semimartingale. In order to make sense of it as a solution to (2.2) in integrated form, one needs to make a so-called principal value correction. Namely, X satisfies the integral equation

$$(2.3) \quad X_t = x_0 + \frac{\delta - 1}{2} \text{P.V.} \int_0^t \frac{1}{X_s} ds + B_t.$$

As explained in [10], Chapter 11, the principal value correction can be understood in terms of an integral of the local time process of X at 0. We will not discuss the details of this here since the properties and definition of the principal value correction will not play much of a role in this work.

The Bessel processes that we have discussed so far are always nonnegative. We remark that it is also natural in certain contexts to consider Bessel processes which can take on both positive and negative values. These processes can be constructed by starting off with a Bessel process which is always nonnegative and then assigning a random sign to each excursion the process makes from 0 as a result of the flip of an independent coin toss. These processes give rise to so-called side-swapping SLE $_{\kappa}(\rho)$ processes, which we will not discuss in the present article.

As mentioned just above, the BES $^{\delta}$ processes are the starting point for constructing the so-called SLE $_{\kappa}(\rho)$ processes. Fix $\kappa > 0$, $\rho > -2 - \frac{\kappa}{2}$, and let

$$\delta = 1 + \frac{2(\rho + 2)}{\kappa}.$$

Note that $\delta > 0$. Let X_t be a BES $^{\delta}$ and let

$$V_t = \frac{2}{\sqrt{\kappa}} \text{P.V.} \int_0^t \frac{1}{X_s} ds \quad \text{and} \quad W_t = V_t - \sqrt{\kappa} X_t.$$

Then the chordal Loewner chain (g_t) driven by W , that is, the solution to the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

is an SLE $_{\kappa}(\rho)$ process. The point $g_t^{-1}(V_t)$ gives the location of the so-called *force point* of the SLE $_{\kappa}(\rho)$ process at time t .

Let us make a few comments about this definition. In the case that $\rho > -2$ so that $\delta > 1$, the principal value integral is the same as the usual integral. This implies that V_t is equal to the image under g_t of the rightmost intersection point of the corresponding hull at time t with \mathbf{R} . Equivalently, the force point at each time t is located at the rightmost intersection of the hull with \mathbf{R} . The continuity of the processes in this case were established in [5], building off the continuity of SLE $_{\kappa}$

proved in [11]. In the case that $\rho \in (-2 - \frac{\kappa}{2}, -2)$ so that $\delta \in (0, 1)$, the force point of an $SLE_\kappa(\rho)$ process *does not* stay in \mathbf{R} , as a consequence of the principal value correction which is necessary for its definition. In the case that $\rho \in (-2 - \frac{\kappa}{2}, \frac{\kappa}{2} - 4]$ and $\kappa \in (2, 4)$, the continuity of these processes was proved in [9] using couplings of these processes with the GFF and as a consequence of the continuity of so-called space-filling SLE established in [8]. In the present work, we will prove the continuity of these processes for $\rho \in ((-2 - \frac{\kappa}{2}) \vee (\frac{\kappa}{2} - 4), -2)$, also using the GFF and the continuity of $SLE_{\kappa'}$, thus covering all possible cases.

The $SLE_\kappa(\rho)$ processes with $\rho \in (-2 - \frac{\kappa}{2}, -2)$ admit certain approximations which are described in [16], Section 6. The reader might find the description contained there helpful for understanding why the principal value correction leads to the force point of the process not always being on the domain boundary.

We finish this subsection by collecting the following technical result, which we will use in Section 3 in conjunction with [5], Theorem 2.4, to construct couplings between the $SLE_\kappa(\rho)$ processes with $\rho \in ((-2 - \frac{\kappa}{2}) \vee (\frac{\kappa}{2} - 4), -2)$ and the GFF.

PROPOSITION 2.1. *Suppose that X is a BES $^\delta$ with $\delta \in (0, 1)$ and that U is a continuous process coupled with X such that (X, U) is strong Markov and possesses the following properties:*

- (i) (X, U) satisfies Brownian scaling: $t \mapsto (X_{\alpha t}, U_{\alpha t}) \stackrel{d}{=} t \mapsto \sqrt{\alpha}(X_t, U_t)$ for each $\alpha \geq 0$,
- (ii) for each $t \geq 0$ such that $X_t \neq 0$, we have $\frac{d}{dt}U_t = X_t^{-1}$ and
- (iii) if τ is any stopping time for X such that $X_\tau = 0$, then $t \mapsto (X_{t+\tau}, U_{t+\tau} - U_\tau)$ is independent of $\sigma((X_s, U_s) : s \leq \tau)$.

Then

$$(2.4) \quad U_t = \text{P.V.} \int_0^t \frac{1}{X_s} ds \quad \text{for all } t \geq 0 \text{ a.s.}$$

PROOF. We note that the choice of U given by (2.4) satisfies the hypotheses of the proposition. Therefore, it suffices to show that it is the only choice which satisfies the hypotheses. Suppose that U, \tilde{U} are two processes which satisfy the properties above and are coupled with X such that U, \tilde{U} are independent given X and let $\bar{U} = U - \tilde{U}$. Let ℓ denote the local time for X at 0 and, for each $s \geq 0$, let $t(s) = \inf\{t \geq 0 : \ell_t = s\}$. Note that $\frac{d}{dt}\bar{U}_t = 0$ for $t \geq 0$ such that $X_t \neq 0$. This implies that $s \mapsto \bar{U}_{t(s)}$ is a continuous process. Indeed, if $u \uparrow s$ then $t(u) \uparrow t(s)$ so that $\bar{U}_{t(u)} \rightarrow \bar{U}_{t(s)}$. Let r be the limit of $t(u)$ as $u \downarrow s$. Then ℓ is constant on $(t(s), r)$ hence $\bar{U}_{t(s)} = \bar{U}_r$ and, since \bar{U} is continuous, $\lim_{u \downarrow s} \bar{U}_{t(u)} = \bar{U}_r$. Therefore, $\lim_{u \rightarrow s} \bar{U}_{t(u)} = \bar{U}_{t(s)}$, which proves the desired continuity. By the strong Markov property and (iii), we also know that $\bar{U}_{t(s)}$ has stationary, independent increments. This implies that there exists a standard

Brownian motion B and constants $c_1, c_2 \in \mathbf{R}$ such that $\overline{U}_{t(s)} = c_1 B_s + c_2 s$. Equivalently, $\overline{U}_t = c_1 B_{\ell_t} + c_2 \ell_t$. Note that \overline{U} and B satisfy Brownian scaling and ℓ satisfies the scaling property $\ell_{ct} \stackrel{d}{=} c^\alpha \ell_t$ with $\alpha = (2 - \delta)/2$. As $\delta \in (0, 1)$, we emphasize that $\alpha \in (1/2, 1)$. It therefore follows that for each $\epsilon > 0$ there exists $C_0 > 1$ such that $\mathbf{P}[C_0^{-1} t^\alpha \leq \ell_t \leq C_0 t^\alpha] \geq 1 - \epsilon/2$. There similarly exists $C_1 > 1$ such that $\mathbf{P}[C_1^{-1} t^{1/2} \leq \sup_{s \leq t} |B_s| \leq C_1 t^{1/2}] \geq 1 - \epsilon/2$. Combining, we have for $C > 0$ sufficiently large that

$$\mathbf{P}\left[\sup_{s \leq t} |B_{\ell_s}| \leq C t^{\alpha/2}\right] \geq \mathbf{P}\left[\sup_{s \leq C_0 t^\alpha} |B_s| \leq C t^{\alpha/2}, \ell_t \leq C_0 t^\alpha\right] \geq 1 - \epsilon.$$

If $c_2 \neq 0$, then this implies that ℓ_t is the dominant term in \overline{U}_t when t is large, hence $\sup_{s \leq t} |\overline{U}_s|$ is of order t^α . As $\alpha > 1/2$, this cannot be the case since \overline{U} satisfies Brownian scaling. Therefore, $c_2 = 0$. A similar argument implies that if $c_1 \neq 0$, then $\sup_{s \leq t} |\overline{U}_s|$ must be of order $t^{\alpha/2}$. As $\alpha/2 < 1/2$, this cannot be the case since \overline{U} satisfies Brownian scaling. Therefore, $c_1 = 0$. This implies that there exists at most one process U which satisfies the hypotheses of the proposition. \square

2.2. Imaginary geometry review. We assume in this work that the reader is familiar with the GFF and with imaginary geometry. We direct the reader to [15] for a more in depth introduction to the GFF and to [5] for a basic introduction to imaginary geometry. In the present section, we will remind the reader of a few facts which are established in [5, 8] about the manner in which flow lines interact with each other.

We begin with a review of the coupling of chordal SLE $_{\kappa}(\rho)$ with $\rho > -2$ with the GFF. Throughout, we assume that $\kappa \in (0, 4)$, $\kappa' = 16/\kappa \in (4, \infty)$, and let

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}, \quad \lambda = \frac{\pi}{\sqrt{\kappa}} \quad \text{and} \quad \lambda' = \frac{\pi}{\sqrt{\kappa'}} = \lambda - \frac{\pi}{2} \chi.$$

(These are the same values as in (1.1) and (1.2).)

We suppose that $\rho > -2$ is fixed and that h is an instance of the GFF on \mathbf{H} with boundary conditions $\lambda(1 + \rho)$ (resp., $-\lambda$) on \mathbf{R}_+ (resp., \mathbf{R}_-). Then it is shown in [5], Theorem 1.1, that there exists a unique coupling (h, η) of h with an SLE $_{\kappa}(\rho)$ process η in \mathbf{H} from 0 to ∞ with a single boundary force point located at 0^+ such that the following is true. Suppose that (W, V) is the Loewner driving pair for η , (g_t) the corresponding family of conformal maps, and that τ is an η -stopping time. Then the conditional law of $h \circ g_\tau^{-1} - \chi \arg(g_\tau^{-1})'$ given $\eta|_{[0, \tau]}$ is that of a GFF on \mathbf{H} with boundary conditions given by

$$h|_{(-\infty, W_\tau]} \equiv -\lambda, \quad h|_{(W_\tau, V_\tau]} \equiv \lambda \quad \text{and} \quad h|_{(V_\tau, \infty)} \equiv \lambda(1 + \rho).$$

Equivalently, the conditional law of h given $\eta|_{[0, \tau]}$ restricted to the unbounded component \mathbf{H}_τ of $\mathbf{H} \setminus \eta([0, \tau])$ is that of a GFF with the same boundary conditions

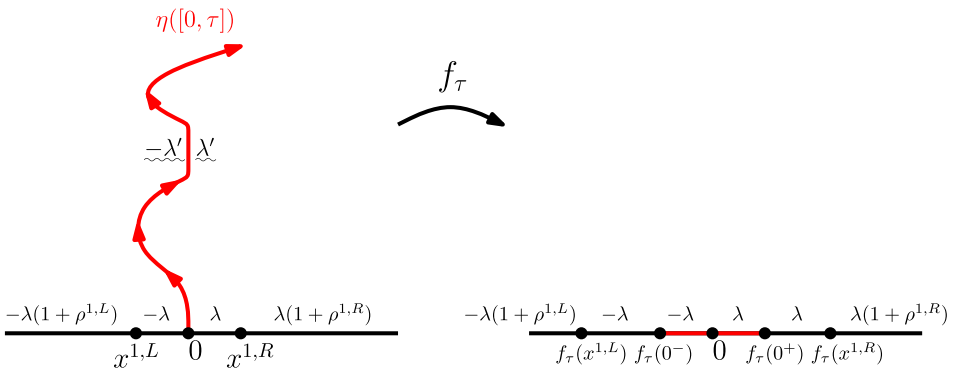


FIG. 2. Suppose that h is a GFF on \mathbf{H} with the boundary data depicted above. Then the flow line η of h starting from 0 is an $\text{SLE}_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ curve in \mathbf{H} where $|\underline{\rho}^L| = |\underline{\rho}^R| = 1$ (i.e., there is only one force point to the left of the origin and only one which is to the right of the origin). For any η stopping time τ , the law of h given $\eta|_{[0, \tau]}$ is equal in distribution to a GFF on $\mathbf{H} \setminus \eta([0, \tau])$ with the boundary data depicted above (the notation a is explained in [5], Figure 1.10). It is also possible to couple $\eta' \sim \text{SLE}_{\kappa'}(\underline{\rho}^L; \underline{\rho}^R)$ for $\kappa' > 4$ with h and the boundary data takes on the same form (with $-\lambda', \lambda' := \frac{\pi}{\sqrt{\kappa'}}$, in place of $\lambda := \frac{\pi}{\sqrt{\kappa}}$). The difference is in the interpretation. The (a.s. self-intersecting) path η' is not a flow line of h , but for each η' stopping time τ' the left and right boundaries of $\eta'([0, \tau'])$ are SLE_κ flow lines, where $\kappa = 16/\kappa'$, angled in opposite directions. The union of the left boundaries—over a collection of τ' values—is a tree of merging flow lines, while the union of the right boundaries is a corresponding dual tree whose branches do not cross those of the primal tree.

as h on $\partial\mathbf{H} \cap \partial\mathbf{H}_\tau$ and with boundary conditions which are given by $-\lambda'$ (resp., λ') plus χ times the winding of η along $\eta|_{[0, \tau]}$. Since η is not a smooth curve, its winding is not well-defined along the curve itself; however, the harmonic extension of its winding is defined. We will indicate this type of boundary data in the figures that follow using the notation introduced in [5], Figure 1.10. It is shown in [5], Theorem 1.2, that η is a.s. determined by h , which is not an obvious statement from how the coupling is constructed.

The path η has the interpretation of being a flow line of the vector field $e^{ih/\chi}$. Similar statements hold in the presence of more general piecewise constant boundary data. In the more general setting, the flow line is an $\text{SLE}_\kappa(\underline{\rho})$ process where the number of force points is equal to the number of jumps in the boundary data for h . See Figure 2 for an illustration in the case of two force points. Similar statements also hold for the existence of a unique coupling of an $\text{SLE}_{\kappa'}$ process η' with the GFF, except the interpretation is different. We refer to η' as a *counterflow line* of h because η' can be realized as a light cone of flow lines which travel in the opposite direction of η' .

We refer to a path coupled as a flow line with $h + \theta\chi$ as the flow line of h with angle θ . This is because such a path has the interpretation of being the flow line of the vector field $e^{i(h/\chi + \theta)}$, that is, the field which arises by taking all of the arrows in $e^{ih/\chi}$ and then rotating them by the angle θ . The manner in which flow lines

with different angles interact is established in [5], Theorem 1.5, as well as [8], Theorem 1.7. Specifically, if η_{θ_1} (resp., η_{θ_2}) are the flow lines of a GFF h on \mathbf{H} starting from $x_1 \leq x_2$, then the following holds. If $\theta_1 > \theta_2$, then η_{θ_1} stays to the left of (but may bounce off) η_{θ_2} . If $\theta_1 = \theta_2$, then η_{θ_1} and η_{θ_2} merge upon intersecting and do not subsequently separate. Finally, if $\theta_2 - \pi < \theta_1 < \theta_2$, then η_{θ_1} and η_{θ_2} cross upon intersecting for the first time. After crossing, the paths may continue to bounce off each other but do not cross again.

One can also consider couplings of SLE with the GFF on domains other than \mathbf{H} . Specifically, suppose that $D \subseteq \mathbf{C}$ is a simply connected domain and $x, y \in \partial D$ are distinct. Then to construct a coupling of an SLE $_{\kappa}(\rho)$ process η in D from x to y with a GFF h on D , one starts with such a coupling $(\tilde{h}, \tilde{\eta})$ on \mathbf{H} and then takes

$$(2.5) \quad h = \tilde{h} \circ \varphi^{-1} - \chi \arg(\varphi^{-1})' \quad \text{and} \quad \eta = \varphi(\tilde{\eta}),$$

where $\varphi: \mathbf{H} \rightarrow D$ is a conformal transformation which takes 0 to x and ∞ to y . We note that this change of coordinates formula is the same as the one which corresponds to the flow lines of $e^{ih/\chi}$ in the setting that h is a continuous function.

One can also consider a counterflow line of angle θ . As in the case of flow lines, this refers to the path coupled with the field $h + \theta\chi$. We note that the left/right boundaries of the path are given by flow lines with angles $\theta \pm \pi/2$.

Flow lines of the GFF starting from interior points were constructed and studied in [8]. The interaction rules for these paths are the same as in the setting of paths which start on the domain boundary; see [8], Theorem 1.7. In [8], these paths were used to construct so-called *space-filling* SLE $_{\kappa'}$, which is a form of ordinary SLE $_{\kappa'}$ except whenever it cuts off a component, it branches in and fills it up before continuing. Specifically, we suppose that h is a GFF on \mathbf{H} with boundary conditions given by λ' (resp., $-\lambda'$) on \mathbf{R}_- (resp., \mathbf{R}_+). (These are the boundary conditions so that the counterflow line of h from 0 to ∞ is a SLE $_{\kappa'}$ process.) Fix a countable dense set (w_n) in \mathbf{H} and, for each n , we let η_n be the flow line of h starting from w_n with angle $\pi/2$. Then we say that w_n comes before w_m if η_n merges with η_m on the right side of η_m (see, e.g., [8], Figure 1.16). This defines an ordering on the (w_n) and space-filling SLE $_{\kappa'}$ is a noncrossing random path which fills all of \mathbf{H} and visits the (w_n) in this order. It is shown in [8], Section 4.3, that this path is in fact continuous and it does not depend on the particular choice of countable dense set (w_n) .

It turns out that if we target a space-filling SLE $_{\kappa'}$ process at a given point z (i.e., parameterize it according to capacity as seen from that point), then we obtain exactly the counterflow line of the GFF targeted at z . Therefore, the aforementioned ordering also determines the order in which a counterflow line visits the points in its range.

The space-filling SLE $_{\kappa'}(\rho)$ processes are defined in an analogous way by starting with a GFF with different boundary data. One can similarly order space using flow lines of any given angle θ rather than the angle $\pi/2$ and obtain a continuous, space-filling path.

In this article, we will frequently consider local sets (see [13] for the basic definitions) of the GFF which arise as *limits* of local sets which are constructed using a finite union of flow lines. We will now formalize this idea here. Suppose that we have a GFF h on \mathbf{H} with piecewise constant boundary data which changes values at most a finite number of times and let L be an interval of $\partial\mathbf{H}$. Fix a countable set (x_n) of L and suppose that we have fixed a finite collection $\theta_1, \dots, \theta_k$ of angles. For each n , let η_n be the flow line of h starting from x_n with angle given by one of the θ_j for $1 \leq j \leq k$. By [5], Proposition 3.7, $A_n = \eta_1 \cup \dots \cup \eta_n$ is a local set for h for each $n \in \mathbf{N}$. Let \mathcal{F}_n be the σ -algebra generated by the values of h in an infinitesimal neighborhood of A_n (in the topology defined by the metric $d(z, w) = |\phi(z) - \phi(w)|$ where $\phi: \mathbf{H} \rightarrow \mathbf{D}$ is the unique conformal map with $\phi(i) = 0$ and $\phi'(i) > 0$) and let \mathfrak{h}_n be the \mathcal{F}_n -measurable distribution associated with A_n , which is harmonic in $\mathbf{H} \setminus A_n$ and so that we can write $h = \mathfrak{h}_n + h_n$ where, given \mathcal{F}_n , h_n is a GFF in $\mathbf{H} \setminus A_n$ with zero boundary values. The boundary conditions for \mathfrak{h}_n are as illustrated in Figure 2 along each η_j for $1 \leq j \leq n$. Note that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for each n because if we observe the values of h in a neighborhood of A_{n+1} then we can determine $\eta_1, \dots, \eta_{n+1}$, in particular A_n , as the flow lines are locally determined by the field [5], Theorem 1.3. Let A be the closure of $\bigcup_n A_n$ and let \mathcal{F} be the σ -algebra generated by the values of h in an infinitesimal neighborhood of A . Then $\mathcal{F} = \sigma(\mathcal{F}_n : n \in \mathbf{N})$ and A is a local set for h (this can be checked using the first characterization of local sets from [13], Lemma 3.9).

Suppose that U is a component of $\mathbf{H} \setminus A$. We will prove just below that ∂U can be divided into a finite number of arcs along which the boundary data for the GFF in U given \mathcal{F} is as it would be if the given arc were a genuine flow line of the GFF with one of the angles θ_j or, along a segment of $\partial U \cap \partial\mathbf{H}$, it is the same as that of h . Equivalently, if $\varphi: U \rightarrow \mathbf{H}$ is a conformal map then the boundary data of $h \circ \varphi^{-1} - \chi \arg(\varphi^{-1})'$ is piecewise constant and changes values at most a finite number of times. A *limit of flow lines* (in the Hausdorff topology using the metric defined above) is an interval of ∂U along which the boundary data for h given \mathcal{F} is as it would be if it were a genuine flow line of h .

To prove the assertion made in the paragraph just above, we fix $z \in \mathbf{H}$ and for each n let U_n be the component of $\mathbf{H} \setminus A_n$ which contains z . Let φ_n be the unique conformal map $U_n \rightarrow \mathbf{H}$ with $\varphi_n(z) = i$, $\varphi_n'(z) > 0$. Then we can write \mathfrak{h}_n as $\mathfrak{g}_n \circ \varphi_n - \chi \arg \varphi_n'$. Here, \mathfrak{g}_n is harmonic on \mathbf{H} , has piecewise constant boundary data, and the boundary values of \mathfrak{g}_n in a given interval can be read off depending on whether the interval is the image of the left or right side of a flow line (and its angle) or the image of part of $\partial\mathbf{H}$. We first claim that there exists a constant M depending on the number k of angles and the number N of intervals I_1, \dots, I_N on which the boundary data of h is constant such that the number of intervals on which the boundary data of \mathfrak{g}_n is constant is at most M . We emphasize that M does not depend on n (as we will see, one can in fact take M to be $3^k N$). It is also important here that we are conditioning on entire flow lines, drawn up until they either reach ∞ or get stuck in the boundary. We will prove this by induction on k . In fact,

upon proving it for $k = 1$ we will have proved the claim for general values of k because if we first condition on all of the flow lines with angle θ_1 then the induction hypothesis implies that the field restricted to the complementary component which contains z will have, after mapping to \mathbf{H} , piecewise constant boundary data which changes values at most a finite number of times. Conditioning further on the flow lines with angle $\theta_2, \dots, \theta_k$ then corresponds to iterating this procedure.

Assume that all of the x_n 's are in I_j for some $1 \leq j \leq N$. There are three possibilities:

- All of the η_n 's pass to the left of z .
- All of the η_n 's pass to the right of z .
- Some of the η_n 's pass to the left of z and some pass to the right of z .

Since flow lines with the same angle merge upon intersecting [5], Theorem 1.5, it follows that if we draw any number of the η_n 's and we are in the first possibility, then the number of marked boundary segments of the complementary component containing z is at most $N + 1$ and the boundary data along the new segment will be given by that of the right side of a flow line with angle θ_1 . Similarly, if we are in the second possibility, then the number of marked boundary segments will also be at most $N + 1$ and the boundary data along the new segment will be given by that of the left side of a flow line with angle θ_1 . Finally, in the final possibility, the number of marked boundary segments will be at most $N + 2$ and the boundary data along the additional segments will be given by that of the left and right sides of a flow line with angle θ_1 . This proves the case that $k = 1$ and all of the x_n 's are contained in a single I_j . If we are in the more general setting in which the x_n 's are in different such intervals, then we can apply the argument above to the x_n 's which fall into each such interval one by one. This completes the proof of the case that $k = 1$. As explained above, the result for general k follows from the $k = 1$ case.

We have that $h_n(z) \rightarrow h(z)$ as $n \rightarrow \infty$ for each $z \in \mathbf{H}$ with $z \notin A$. This allows us to read off the boundary data of h and see that it is the same as if the boundaries of the components of $\mathbf{H} \setminus A$ were given by genuine flow lines of the field with angles θ_j for $1 \leq j \leq k$. Indeed, if U is the component of $\mathbf{H} \setminus A$ which contains z and we let φ be the unique conformal map $U \rightarrow \mathbf{H}$ with $\varphi(z) = i$ and $\varphi'(z) > 0$, then we have that $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$. Since $h_n \rightarrow h$, we conclude that $g_n = h_n \circ \varphi_n^{-1} - \chi \arg(\varphi_n^{-1})'$ converges to a harmonic function g on \mathbf{H} which has piecewise constant boundary data of the same form and $h = g \circ \varphi - \chi \arg \varphi'$. Therefore, h has the claimed form.

Suppose that U is a component of $\mathbf{H} \setminus A$ and that η is one of the limits of flow lines which makes up ∂U . That is, the boundary data for h given \mathcal{F} along η is as it would be if η were a genuine flow line of the field. We will now prove that it is a.s. the case that for each $r > 0$ there exists n so that $\eta \setminus \eta_n$ is contained in $B(x, r)$ for some $x \in \partial \mathbf{H}$. This will in particular imply that limits of flow lines are continuous curves and interact with each other and the domain boundary in the same manner as genuine flow lines of the field (as then arbitrarily large segments of a limit of

flow lines correspond to a segment of a genuine flow line, so the interaction of limits of flow lines reduces to the interaction of genuine flow lines). This will be a consequence of the following fact. We may assume without loss of generality that all of the flow lines have the same angle. Indeed, in the setting in which we have k angles we can write $A = A_1 \cup \dots \cup A_k$ where A_j is the closure of the union of flow lines with angle θ_j . If h has the same boundary data on a segment η of ∂U as a flow line with angle θ_j , then this part of ∂U must be contained in a segment η^j of ∂U_j , U_j the component of $\mathbf{H} \setminus A_j$ which contains U . The case in which $k = 1$ implies that there exists n and $x \in \partial \mathbf{H}$ so that $\eta^j \setminus \eta_n \subseteq B(x, r)$ and, therefore, $\eta \setminus \eta_n \subseteq B(x, r)$. Fix $x \in \partial \mathbf{H}$ and $r > 0$ so that the boundary data of h in $B(x, r) \cap \partial \mathbf{H}$ changes at most once and let (n_j) be such that $x_{n_j} \in B(x, r/2) \cap \partial \mathbf{H}$ for every j . For each j , let τ_{n_j} be the first time that η_{n_j} leaves $B(x, r)$. Then (see the next paragraph for the proof) for every $\epsilon > 0$ there exists a J such the probability that for every j simultaneously, η_{n_j} has merged with one of $\eta_{n_1}, \dots, \eta_{n_J}$ before time τ_{n_j} , is at least $1 - \epsilon$. Equivalently, the cardinality of the set $\{\eta_{n_j}(\tau_{n_j})\}$ is a.s. finite. Informally, this means that there are only a finite number of distinct strands which correspond to the η_{n_j} which make it macroscopically far away from x .

To check this property, by the basic absolute continuity properties of the GFF (and performing a coordinate change), we may assume without loss of generality that h is a GFF on \mathbf{H} with boundary data which is constant on \mathbf{R}_- and on \mathbf{R}_+ , $x = 0$, and $r = 2$. We may also assume that all of the η_{n_j} start from a point in $[0, 1]$ since we can apply the same argument to $[-1, 0]$. Let η' be the space-filling counterflow line of h from ∞ to 0 chosen so that the left boundary of η' stopped upon hitting a point $x \in \mathbf{R}_+$ is equal to the flow line starting from x with angle 0. In particular, η_{n_j} is equal to the left boundary of η' stopped upon hitting x_{n_j} . Recall that η' visits the points of each η_{n_j} in reverse chronological order. Moreover, if $0 < x_{n_j} < x_{n_{j'}}$ then η' visits all of the range of $\eta_{n_{j'}}$ before visiting the part of η_{n_j} before it merges with $\eta_{n_{j'}}$. It therefore follows that the cardinality of the set $\{\eta_{n_j}(\tau_{n_j})\}$ is bounded from above by the number of excursions that η' makes from $\partial B(0, 1)$ to $\partial B(0, 2)$ which is a.s. finite since η' is a.s. continuous [8]. This completes the proof of the claim.

3. GFF couplings. In this section, we are going to prove Theorem 1.1 and Theorem 1.2 simultaneously and then explain how to extract Theorem 1.3 from these results. We will begin in Section 3.1 by proving several results about the structure of the complementary components (“pockets”) of light cones and then in Section 3.2 we will explain how we can use an $\text{SLE}_{\kappa'}$, $\kappa' = 16/\kappa \in (4, \infty)$, counterflow line to generate a continuous path which explores the range of a light cone. In both of these sections, we will restrict ourselves to the case in which the light cone starts from a single boundary point (rather than a continuum) so that we can work in a unified framework. We will then explain in Section 3.3 that these results also hold in the setting in which the light cone starts from a continuum of

boundary points using a conditioning argument and then make the connection to $SLE_{\kappa}(\rho)$ processes with $\rho \in [\frac{\kappa}{2} - 4, -2)$ and $\rho > -2 - \frac{\kappa}{2}$.

Throughout, unless explicitly stated otherwise, we shall assume that h is a GFF on \mathbf{D} which is given by a conformal coordinate change as in (2.5) of a GFF on \mathbf{H} with piecewise constant boundary data which changes values at most a finite number of times. The reason for this is that it will be more convenient to work on a bounded Jordan domain rather than \mathbf{H} because then $SLE_{\kappa'}$ is uniformly continuous. We also let

$$(3.1) \quad \theta_c = \frac{\pi\kappa}{4 - \kappa}.$$

This is the so-called *critical angle*—the angle difference below which GFF flow lines can intersect each other and at or above which they cannot (see [5], Theorem 1.5, and [8], Theorem 1.7). It is shown in [4] that the almost sure dimension of a light cone with opening angle $\theta \in [0, \theta_c \wedge \pi)$ is contained in $[1, 2)$ and that the dimension is equal to 2 for $\theta \in [\theta_c \wedge \pi, \pi]$. Note that $\theta_c \leq \pi$ if and only if $\kappa \leq 2$, which is closely connected with the fact that ordinary $SLE_{\kappa'}$ is space-filling if and only if $\kappa' \geq 8$ [11].

3.1. Pocket structure. Fix $\theta_1 \leq \theta_2$ with $\theta_2 - \theta_1 \leq \pi$. For each $n \in \mathbf{N}$, let $\mathbf{L}_n(\theta_1, \theta_2)$ be the closure of the set of points accessible by angle-varying flow lines of h starting from $-i$ which travel either with angle θ_1 or θ_2 , change directions at most n times, and only change directions at positive rational times. The *light cone* $\mathbf{L}(\theta_1, \theta_2) = \overline{\bigcup_n \mathbf{L}_n(\theta_1, \theta_2)}$ of h (starting from $-i$) with angle range $[\theta_1, \theta_2]$ is the closure of the set of points accessible by flow lines of h starting from $-i$ with angle-varying trajectories with angle either equal to θ_1 or θ_2 and which change directions a finite number of times and only at positive rational times. Note that this definition is slightly different than that given in the Introduction because we only allow the paths to travel with the extremal angles θ_1 and θ_2 (and do not allow the intermediate angles). This definition will be more convenient for us to work with and we will shortly show that it and the one given in the Introduction a.s. agree. For $\theta \in [0, \pi]$, we also let $\mathbf{L}(\theta) = \mathbf{L}(-\frac{\theta}{2}, \frac{\theta}{2})$.

We note that $\mathbf{L}_1(\theta_1, \theta_2)$ consists of the two flow lines of h starting from $-i$ with angles θ_1, θ_2 . However, $\mathbf{L}_2(\theta_1, \theta_2)$ consists of limits of flow lines with angle θ_1, θ_2 which can change angle at most once at a positive rational time. If we think of first conditioning on $\mathbf{L}_1(\theta_1, \theta_2)$, then $\mathbf{L}_2(\theta_1, \theta_2)$ consists of limits of flow lines in $\mathbf{D} \setminus \mathbf{L}_1(\theta_1, \theta_2)$ with angle either θ_1 or θ_2 and which does not change. More generally, we have that $\mathbf{L}_{n+1}(\theta_1, \theta_2)$ consists of limits of flow lines in $\mathbf{D} \setminus \mathbf{L}_n(\theta_1, \theta_2)$ with angle either θ_1 or θ_2 which does not change. Therefore the discussion at the end of Section 2.2 applies to this setting to describe how these paths interact with each other.

For each $z \in \mathbf{D}$ and $n \in \mathbf{N}$, let $P_n(z)$ be the complementary component of $\mathbf{L}_n(\theta_1, \theta_2)$ which contains z and let $P(z)$ be the complementary component of

$\mathbf{L}(\theta_1, \theta_2)$ which contains z . Throughout, we will refer to such complementary components as (complementary) *pockets* of $\mathbf{L}(\theta_1, \theta_2)$. We are next going to describe the boundary data of h given $\mathbf{L}(\theta_1, \theta_2)$ on $\partial P(z)$. It is a consequence of the main result of [4] that $P(z) \neq \emptyset$ a.s. provided $\theta_2 - \theta_1 < \theta_c$ and $\theta_2 - \theta_1 \leq \pi$.

LEMMA 3.1. *Suppose that $\theta_2 - \theta_1 < \theta_c$ and $\theta_2 - \theta_1 \leq \pi$ and that the flow lines of h with angles θ_1, θ_2 do not hit the continuation threshold before reaching i . Fix $z \in \mathbf{D}$ and let $E(z)$ be the event that $\mathbf{L}(\theta_1, \theta_2)$ disconnects z from $\partial \mathbf{D}$. On $E(z)$, let $\varphi: P(z) \rightarrow \mathbf{D}$ be the unique conformal transformation with $\varphi(z) = 0$ and $\varphi'(z) > 0$. Then the boundary data for $\tilde{h} = h \circ \varphi^{-1} - \chi \arg(\varphi^{-1})'$ is as described in the left side of Figure 4. In particular, there exists two distinct marked points $x, y \in \partial P(z)$ such that the boundary behavior of h along the clockwise (resp., counterclockwise) boundary segment $S_1(z)$ (resp., $S_2(z)$) of $\partial P(z)$ from x to y is the same as that of the right (resp., left) side of a flow line with angle θ_1 (resp., θ_2).*

PROOF. Assume that we are working on $E(z)$. Then the left and right boundaries of $\mathbf{L}(\theta_1, \theta_2)$, that is, the flow lines of h starting from $-i$ with angles θ_1, θ_2 , separate z from $\partial \mathbf{D}$. For each n , let $\varphi_n: P_n(z) \rightarrow \mathbf{D}$ be the unique conformal transformation with $\varphi_n(z) = 0$ and $\varphi_n'(z) > 0$. Let $\tilde{h}_n = h \circ \varphi_n^{-1} - \chi \arg(\varphi_n^{-1})'$ be the GFF on \mathbf{D} given by conformally mapping $P_n(z)$ to \mathbf{D} using φ_n and applying the coordinate change formula (2.5).

We claim that, as shown in Figure 3 (in the case that $n = 2$), the boundary data for \tilde{h}_n has four (possibly degenerate) marked points. Indeed, this can be seen by induction on n . It is clearly true for $n = 1$. Suppose that the result holds for some $n \in \mathbf{N}$. We will prove that it also holds for $n + 1$. Consider $P_n(z)$. The four marked points divide $\partial \mathbf{D}$ into the images $L_n^{\theta_1}$ and $L_n^{\theta_2}$ of the pocket boundary formed by the left sides of (limits of) flow lines with angles θ_1 and θ_2 , respectively, and the images $R_n^{\theta_1}$ and $R_n^{\theta_2}$ of the pocket boundary formed by the right sides of flow lines with angles θ_1 and θ_2 , respectively. Let x_1 (resp., x_2) be the first point along $R_n^{\theta_2}$ (resp., $L_n^{\theta_1}$) from which the flow line of \tilde{h}_n of angle θ_1 (resp., θ_2) has z on its right (resp., left) side. Note that there do exist such points x_1, x_2 because a flow line of angle θ_1 starting from a point on $R_n^{\theta_2}$ terminates in $R_n^{\theta_1}$ (resp., $L_n^{\theta_1}$) with probability tending to 1 as it gets closer and closer to $R_n^{\theta_1}$ (resp., $L_n^{\theta_1}$) and the analogous fact holds for the flow lines of angle θ_2 starting from $L_n^{\theta_1}$. Note that there are two paths, say γ_1^1, γ_2^1 , which emanate from x_1 : the limits of the flow lines of \tilde{h}_n with angle θ_1 starting from points before or after x_1 along $R_n^{\theta_2}$. We assume that γ_1^1 is to the left of γ_2^1 . We note that z is by definition to the right of γ_1^1 . We also have that z is to the left of γ_2^1 since γ_2^1 is the limit of paths which start to the right of x_1 and all such paths have z on their left-hand side. Moreover, z is a.s. not contained in γ_1^1 or γ_2^1 as z is a.s. not contained in the entire light cone. Note that each of γ_1^1, γ_2^1 can terminate in $R_n^{\theta_1}$ or $L_n^{\theta_1}$ or in $L_n^{\theta_2}$. There are similarly two paths γ_1^2, γ_2^2 which start

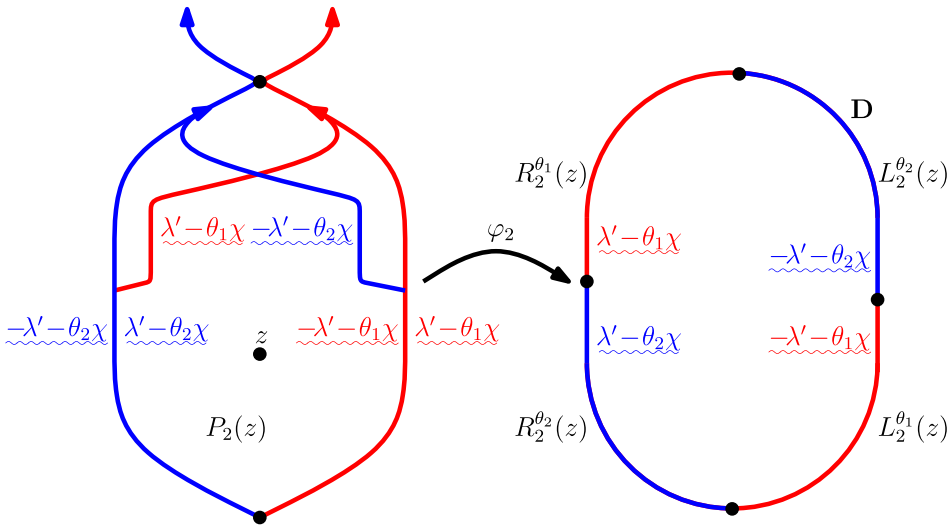


FIG. 3. Shown on the left is the pocket $P_2(z)$ of $\mathbf{L}_2(\theta_1, \theta_2)$ containing z on the event that $P_2(z)$ separates z from $\partial\mathbf{D}$. We let $\varphi_2: P_2(z) \rightarrow \mathbf{D}$ be the unique conformal transformation with $\varphi_2(z) = 0$ and $\varphi_2'(z) > 0$. Shown on the right is the boundary data of the GFF $h \circ \varphi^{-1} - \chi \arg(\varphi^{-1})'$ on $\partial\mathbf{D}$. The reason that \mathbf{D} on the right side appears not to be perfectly round is so that we can use our notation to label the boundary data.

from x_2 (but this time with angle θ_2) and we take the convention that γ_1^2 is to the left of γ_2^2 . Then we likewise have that each of γ_1^2, γ_2^2 can terminate in $R_n^{\theta_2}$ or $L_n^{\theta_2}$ or in $R_n^{\theta_1}$. We also note that the interaction between $\gamma_1^1, \gamma_2^1, \gamma_1^2, \gamma_2^2$ is as described in the discussion at the end of Section 2.2.

We are now going to consider three possibilities for γ_1^1 . We will argue that, in each of these three possibilities, the boundary of the component U_1^1 of $P_n(z) \setminus \gamma_1^1$ containing z can be decomposed into a union of four arcs (possibly degenerate) of the same form as the boundary of $P_n(z)$. Upon showing this, we can then apply the same reasoning to γ_2^1 (viewed as a path in U_1^1) to see that the boundary of the component U_2^1 of $P_n(z) \setminus (\gamma_1^1 \cup \gamma_2^1)$ containing z again can be decomposed into a union of four (possibly degenerate) arcs. We will explain just below why x_2 must be on the boundary of U_1^1 and by the same reasoning also U_2^1 . Applying this reasoning two more times to γ_1^2, γ_2^2 , we see that the boundary of the component of $P_n(z) \setminus (\gamma_1^1 \cup \gamma_2^1 \cup \gamma_1^2 \cup \gamma_2^2)$ containing z can also be decomposed into a union of four (possibly degenerate) arcs. Note that this component is exactly equal to $P_{n+1}(z)$, so this will complete the proof of the inductive step.

The following are the three possibilities for γ_1^1 :

- (i) γ_1^1 terminates in $R_n^{\theta_1}$. Then the pocket formed containing z has four marked boundary arcs corresponding to γ_1^1 together with part of $R_n^{\theta_1}$ (right side of θ_1 -angle

flow line), $L_n^{\theta_2}$ (left side of θ_2 -angle flow line), $L_n^{\theta_1}$ (left side of θ_1 -angle flow line) and part of $R_n^{\theta_2}$ (right side of θ_2 -angle flow line).

(ii) γ_1^1 terminates in $L_n^{\theta_2}$. Then the pocket formed containing z has four marked boundary arcs corresponding to γ_1^1 (right side of θ_1 -angle flow line), part of $L_n^{\theta_2}$ (left side of θ_2 -angle flow line), $L_n^{\theta_1}$ (left side of θ_1 -angle flow line) and part of $R_n^{\theta_2}$ (right side of θ_2 -angle flow line).

(iii) γ_1^1 terminates in $L_n^{\theta_1}$. Then the pocket formed containing z has three marked boundary arcs corresponding to γ_1^1 (right side of θ_1 -angle flow line), part of $L_n^{\theta_1}$ (left side of θ_1 -angle flow line) and part of $R_n^{\theta_2}$ (right side of θ_2 -angle flow line).

That is, the pocket containing z formed upon drawing in γ_1^1 has exactly the same type of boundary data as before drawing in γ_1^1 . Namely, it consists of at most four marked boundary arcs along which the boundary data of the field is that of the left or right side of a flow line of angle θ_1 or θ_2 , as is illustrated in Figure 3. (Note also that x_2 must be on the boundary of this pocket because otherwise it is to the left of γ_1^1 . This would then prevent the (limits of) flow lines γ_1^2, γ_2^2 of angle θ_2 starting from x_2 from forming a pocket around z .) The same analysis applies when we draw in γ_2^1 and then finally γ_1^2, γ_2^2 .

We have thus dealt with all possibilities, which completes the proof of the inductive step.

Note that $\tilde{h} = \lim_n \tilde{h}_n$. Consequently, the boundary data for \tilde{h} takes the same form. Let $L^{\theta_1}, L^{\theta_2}, R^{\theta_1}$ and R^{θ_2} be the four marked boundary segments for the boundary data of \tilde{h} . If $L^{\theta_1} \neq \emptyset$ or $R^{\theta_2} \neq \emptyset$, then $\text{diam}(L_n^{\theta_1})$ or $\text{diam}(R_n^{\theta_2})$ is bounded from below for arbitrarily large values of n . This is a contradiction because it is easy to see that on this event, the conformal radius of $P_{n+1}(z)$ as seen from z decreases by a uniformly positive amount with uniformly positive probability. Consequently, $L^{\theta_1} = \emptyset$ and $R^{\theta_2} = \emptyset$ a.s. That is, the boundary data for \tilde{h} is in fact as illustrated in the left-hand side of Figure 4, as desired. \square

Throughout, we shall refer to the point x in the statement of Lemma 3.1 as the *opening point* of $P(z)$. If we want to emphasize the dependency of x on z , we will write $x(z)$ for x . For a generic pocket P , we will write $x(P)$ for the opening point of P . Similarly, we will refer to the point y in the statement of Lemma 3.1 as the *closing point* of $P(z)$. As before, we will write $y(z)$ if we want to emphasize the dependency on z and write $y(P)$ for the closing point of a generic pocket P . We will also use the notation $S_j(z)$ introduced in the statement of Lemma 3.1 to indicate the θ_j -angle side of $\partial P(z)$ for $j = 1, 2$ and write $S_j(P)$ to indicate the same for a generic pocket P . If P or z is understood from the context, then we will simply write S_j for $j = 1, 2$. Finally, we note that $S_j(z)$ is equal to a limit $\eta_j(z)$ of flow lines of h with angle θ_j . It starts from $x(z)$ and is stopped upon hitting $y(z)$. We will write $\eta_j(P)$ to indicate these paths for a generic pocket P and write η_j

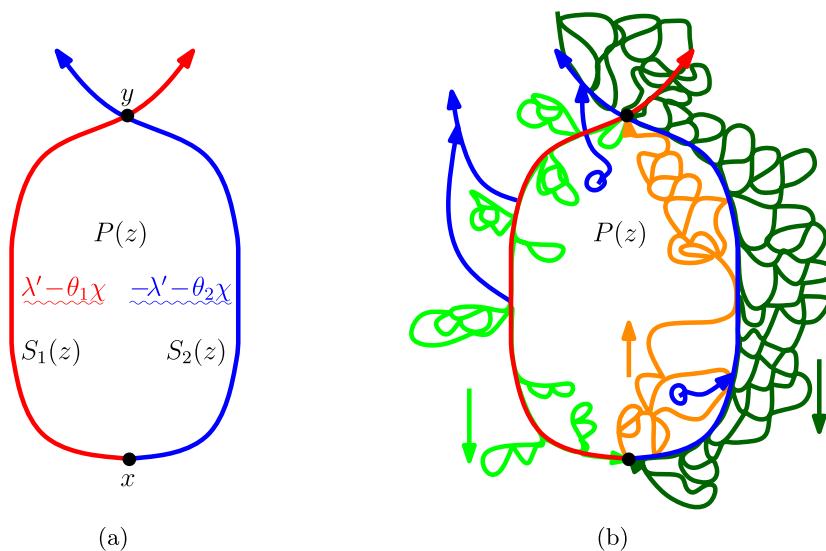


FIG. 4. Shown on the left is a pocket $P(z)$ of $\mathbf{L}(\theta_1, \theta_2)$ containing a given point z and the boundary data for the conditional law of h given $\mathbf{L}(\theta_1, \theta_2)$ on $\partial P(z)$. Note that it is not possible to draw θ_1 -angle (resp., θ_2 -angle) flow lines of h contained in $P(z)$ which start from points on $S_2(z)$ (resp., $S_1(z)$). On the right side, the extra θ_2 -angle flow lines have been drawn in blue to indicate how the paths are ordered using an $SLE_{\kappa'}$ counterflow line η' . The dark green path indicates the part of η' that fills the right side of $S_2(z)$, the orange path indicates the part of η' which travels from the opening point x to the closing point y of $P(z)$, and the light green path indicates the part of η' after it has hit y . The colored arrows indicate the direction in which the different segments of η' are traveling. In particular, η' fills the right side of $S_2(z)$ before entering (the interior of) $P(z)$. Since it has to hit the points on $S_1(z)$ in the reverse order in which they are drawn by $\eta_1(z)$, after reaching x , η' enters into the interior of $P(z)$ and then travels to y . As it travels up to y , it visits point on the left side of $S_2(z)$, does not hit $S_1(z)$, and does not leave $\overline{P(z)}$. After reaching y , it then visits the points of $S_1(z)$ in the reverse order in which they are drawn by $\eta_1(z)$. While it does so, it makes excursions both into and outside of $P(z)$.

if either P or z is understood from the context. We will sometimes refer to these paths as flow lines, even though they are only limits of flow lines because they start from special points of the GFF. We emphasize that the boundary data for the GFF along these paths is the same as if they were actual flow lines. The interaction rules of these paths with genuine flow lines of the field is also the same as if they were genuine flow lines themselves. We will now use Lemma 3.1 to show that the definition of the light cone introduced in this section agrees with the one given in the Introduction.

LEMMA 3.2. Fix $\theta_1 \leq \theta_2$ with $\theta_2 - \theta_1 \leq \pi$. Let $\mathbf{L}(\theta_1, \theta_2)$ be as defined in the beginning of the subsection and let $\widehat{\mathbf{L}}(\theta_1, \theta_2)$ be the closure of the set of points accessible by angle-varying trajectories of h starting from $-i$ with angles which are rational and contained in $[\theta_1, \theta_2]$ or equal to θ_1 or θ_2 and which change angles

at most a finite number of times and only at positive rational times. (This is the definition of the light cone given in the *Introduction*.) Then $\mathbf{L}(\theta_1, \theta_2) = \widehat{\mathbf{L}}(\theta_1, \theta_2)$ a.s.

PROOF. We may assume without loss of generality that $\theta_1 < \theta_2$ since if $\theta_1 = \theta_2$ then the result is trivially true because both $\mathbf{L}(\theta_1, \theta_2)$ and $\widehat{\mathbf{L}}(\theta_1, \theta_2)$ are equal to the flow line of h starting from $-i$ with angle $\theta_1 = \theta_2$. It is clear from the definition that $\mathbf{L}(\theta_1, \theta_2) \subseteq \widehat{\mathbf{L}}(\theta_1, \theta_2)$ a.s., so we just need to prove the reverse inclusion. We first suppose that $\theta_2 - \theta_1 < \theta_c$. In this case, the result follows because, for each fixed $z \in \mathbf{D}$, the flow line interaction rules [8], Theorem 1.7, and Lemma 3.1 imply that an angle-varying trajectory with angles which are rational and contained in $[\theta_1, \theta_2]$ or equal to θ_1 or θ_2 which changes angles at most a finite number of times cannot enter the pocket $P(z)$ of $\mathbf{L}(\theta_1, \theta_2)$ which contains z . Indeed, a flow line of angle θ_2 cannot cross a flow line of angle θ_1 from left to right since $\theta_2 > \theta_1$ and likewise a flow line of angle θ_1 cannot cross a flow line of angle θ_2 from right to left. Also, such a path cannot enter $P(z)$ at the opening point $x(z)$ and then stay in $P(z)$. The reason for this is that if it did, then its conditional law inside of $P(z)$ before exiting is that of an $\text{SLE}_\kappa(\rho_1; \rho_2)$ process with $\rho_1 = -2$ and $\rho_2 < -2$ (if traveling with angle θ_1) or $\rho_1 < -2$ and $\rho_2 = -2$ (if traveling with angle θ_2). The case that $\theta_2 - \theta_1 \geq \theta_c$ follows since for these values we know that both $\mathbf{L}(\theta_1, \theta_2)$ and $\widehat{\mathbf{L}}(\theta_1, \theta_2)$ are equal to the set of points which lie between their left and right boundaries. (This can be seen, e.g., from the argument illustrated in [4], Figure 3.2.) \square

Fix angles $\theta_1 < \theta_2$ with $\theta_2 - \theta_1 < \theta_c$ and $\theta_2 - \theta_1 \leq \pi$. Assume that the boundary data of h is such that the flow lines η_1, η_2 starting from $-i$ with angles θ_1, θ_2 a.s. do not hit the continuation threshold (as defined just before the statement of [5], Theorem 1.1). That is, they both connect $-i$ to i . Let η' be the counterflow line of $h + (\theta_2 - \frac{\pi}{2})\chi$ starting from i and targeted at $-i$. Let also η'_z be the counterflow line of the same field starting from i but targeted at z . Then η'_z a.s. hits z as η' does not hit the continuation threshold hence hits $-i$. Moreover, η' and η'_z a.s. agree until they separate z from i . The part of the boundary of η'_z which is to the left stopped upon hitting a point $z \in \mathbf{D}$ is equal to the flow line starting from z with angle θ_2 . We are now going to use the flow line interaction rules [8], Theorem 1.7, to explain how η' interacts with a pocket $P(z)$ of $\mathbf{L}(\theta_1, \theta_2)$. See Figure 4 for an illustration. If we start a flow line η_w with angle θ_2 from a point w inside of $P(z)$, then it has to merge with $\eta_2(z)$ on its left side. Indeed, this is obviously true for topological reasons if η_w merges with $\eta_2(z)$ before leaving $P(z)$. If η_w first leaves $P(z)$ before merging into $\eta_2(z)$, then it necessarily crosses $\eta_1(z)$ from the right to the left. If η_w were to subsequently wrap around and merge with $\eta_2(z)$ on its right side, then it would be forced to cross $\eta_1(z)$ a second time (at $y(z)$ or above it), which is a contradiction to [8], Theorem 1.7. This proves the claim since flow lines with the same angle a.s. merge. Similarly, if we start a flow line of angle θ_2

from a point w on $S_1(z)$ then it merges with $\eta_2(z)$ on its left side. Consequently, it follows from [8], Theorem 1.13, that:

1. η' enters (the interior of) $P(z)$ at $x(z)$ after filling the right side of $S_2(z)$.
2. Upon entering $P(z)$, η' visits some points on the left side of $S_2(z)$ as it travels from $x(z)$ to $y(z)$. It does not touch $S_1(z)$ until hitting $y(z)$.
3. Upon hitting $y(z)$, it visits all of the points of $S_1(z)$ in the reverse order in which they are drawn by $\eta_1(z)$ and, while doing so, η' makes excursions both into and out of $P(z)$.

We emphasize that how η' interacts with $P(z)$ does not depend on whether $P(z)$ hits the boundary, because the interaction rules for flow lines is the same even if they are hitting the boundary.

We are now going to extract from this and the continuity of $SLE_{\kappa'}$ the local finiteness of the pockets of the light cone.

LEMMA 3.3. *Suppose that we have the setup described just above (in particular, the boundary data of h is such that the left and right boundaries η_1, η_2 of $\mathbf{L}(\theta_1, \theta_2)$ a.s. do not hit the continuation threshold before hitting i). The pockets of $\mathbf{L}(\theta_1, \theta_2)$ are a.s. locally finite: that is, for each $\epsilon > 0$, the number of pockets of $\mathbf{L}(\theta_1, \theta_2)$ with diameter at least ϵ is finite a.s.*

PROOF. The result trivially holds for $\theta_2 - \theta_1 \geq \theta_c$ because then $\mathbf{L}(\theta_1, \theta_2)$ is space-filling hence does not have pockets which lie between η_1 and η_2 . The pockets which are not surrounded by η_1 and η_2 are locally finite because η_1 and η_2 are continuous paths. We now suppose that $\theta_2 - \theta_1 < \theta_c$ so that $\mathbf{L}(\theta_1, \theta_2)$ has pockets which lie between η_1 and η_2 . Since the components of $\mathbf{D} \setminus (\eta_1 \cup \eta_2)$ are locally finite, it suffices to show that the pockets of $\mathbf{L}(\theta_1, \theta_2)$ which are contained in a given component are locally finite.

Fix such a component C and let η' be the space-filling SLE starting from y , the last point on ∂C hit by η_1 and η_2 , and targeted at x , the first point on ∂C hit by η_1 and η_2 . We choose η' so that, stopped upon hitting x , the part of its boundary which is to the left is equal to the part of η_2 which draws the left side of ∂C . Then η' interacts with a pocket $P(z)$ of $\mathbf{L}(\theta_1, \theta_2)$ for $z \in C$ in the same manner as the counterflow line described before the statement of the lemma except that it completely fills $S_2(z)$ while traveling from $x(z)$ to $y(z)$. Note that for disjoint pockets $P(z)$ and $P(w)$ of $\mathbf{L}(\theta_1, \theta_2)$ contained in C , the time-interval I_z in which η' travels from $x(z)$ to $y(z)$ is disjoint from the time-interval I_w in which it travels from $x(w)$ to $y(w)$. Moreover, for each $z \in \mathbf{D}$, $\eta'(I_z)$ contains $S_2(z)$. Consequently, it follows from the continuity of $SLE_{\kappa'}$ that the number of pockets P such that $\text{diam}(S_2(P)) \geq \epsilon$ is finite a.s. The same is also true for the number of pockets P such that $\text{diam}(S_1(P)) \geq \epsilon$ because we can take the space-filling SLE so that, stopped upon hitting x , its boundary which is to the right is equal to the part of ∂C drawn by η_1 and then apply the same analysis. This completes the proof since the

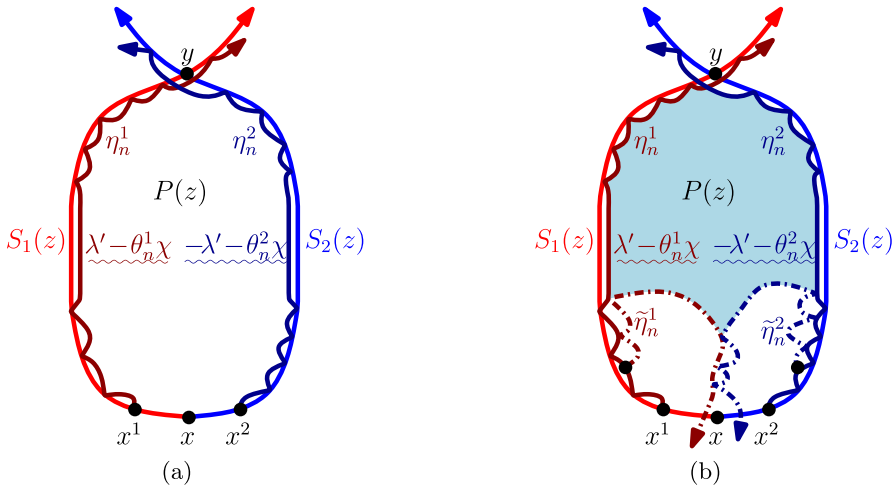


FIG. 5. Shown on the left side is the pocket $P(z)$ of $\mathbf{L}(\theta_1, \theta_2)$ containing z . Its opening (resp., closing) point is x (resp., y). Suppose that (θ_n^1) (resp., (θ_n^2)) is a sequence of angles which increase to θ_1 (resp., decrease to θ_2). We take η_n^1 (resp., η_n^2) to be a flow line of angle θ_n^1 (resp., θ_n^2) starting from $x^1 \in \partial P(z)$ (resp., $x^2 \in \partial P(z)$). As $n \rightarrow \infty$, η_n^1 and η_n^2 converge in the Hausdorff topology to the segments of $S_1(z)$ and $S_2(z)$, respectively, which connect x^1 and x^2 to y . The right is the same as the left except we have drawn dual paths $\tilde{\eta}_n^1, \tilde{\eta}_n^2$ starting from points on η_n^1, η_n^2 , respectively. Explicitly, $\tilde{\eta}_n^1$ (resp., $\tilde{\eta}_n^2$) has angle $\theta_n^1 - \pi$ (resp., $\theta_n^2 + \pi$). These paths will intersect and bounce off each other as shown. By the flow line interaction rules, $\tilde{\eta}_n^1$ cannot cross either η_n^1 or η_n^2 but can cross out of $P(z)$ through the clockwise segment of $S_1(z)$ from x to x^1 and the symmetric fact holds for $\tilde{\eta}_n^2$. Since an angle varying flow line with angles contained in $[\theta_n^1, \theta_n^2]$ cannot cross from the right to the left (resp., left to the right) of $\tilde{\eta}_n^1$ (resp., $\tilde{\eta}_n^2$), it follows that the pocket of $\mathbf{L}(\theta_n^1, \theta_n^2)$ which contains z a.s. contains the light blue region on the right. This allows us to prove the continuity of the law of $\mathbf{L}(\theta_1, \theta_2)$ in θ_1, θ_2 with respect to the Hausdorff topology because the Hausdorff distance between $P(z)$ and the blue region will with probability tending to 1 decrease to 0 as we take a limit first as $n \rightarrow \infty$ and then as $x^1, x^2 \rightarrow x$, and then finally the starting points of $\tilde{\eta}_n^1, \tilde{\eta}_n^2$ to x as well.

triangle inequality implies that $\text{diam}(P) \leq \text{diam}(S_1(P)) + \text{diam}(S_2(P))$ for any pocket P . \square

We are now going to establish the continuity of the law of $\mathbf{L}(\theta_1, \theta_2)$ in $\theta_1 \leq \theta_2$ with $\theta_2 - \theta_1 \leq \pi$ with respect to the Hausdorff topology. See Figure 5 for an illustration of the setup and the proof.

PROPOSITION 3.4. *Suppose that we have the same setup as in Lemma 3.3 and that $\theta_1 \leq \theta_2$ are angles with $\theta_2 - \theta_1 \leq \pi$. Let $(\theta_n^1), (\theta_n^2)$ be sequences of angles with $\theta_n^1 \leq \theta_n^2$ and $\theta_n^2 - \theta_n^1 \leq \pi$ for all $n \in \mathbf{N}$ such that $\theta_n^j \rightarrow \theta_j$ as $n \rightarrow \infty$ for $j = 1, 2$. Then $\mathbf{L}(\theta_n^1, \theta_n^2) \rightarrow \mathbf{L}(\theta_1, \theta_2)$ as $n \rightarrow \infty$ a.s. with respect to the Hausdorff topology.*

REMARK 3.5. Proposition 3.4 implies that for a fixed choice of $\theta_1 \leq \theta_2$, we have that $\mathbf{L}(\theta_n^1, \theta_n^2) \rightarrow \mathbf{L}(\theta_1, \theta_2)$ a.s. It does not imply that $(\theta_1, \theta_2) \mapsto \mathbf{L}(\theta_1, \theta_2)$ is a

continuous function with respect to the Hausdorff topology for a fixed realization of h . Indeed, this statement is not true because the left boundary of $\mathbf{L}(\theta_1, \theta_2)$ is the flow line of h with angle θ_1 and for a fixed realization of h the map which takes an angle to the flow line of h starting from that angle is not continuous with respect to the Hausdorff topology. Indeed, if this were true then the *fan* defined in [5] would a.s. have positive Lebesgue measure but it is shown in [5] that the Lebesgue measure is zero a.s. In fact, it is shown in [4] that the dimension of the fan is the same as the dimension of a single SLE $_{\kappa}$ path.

PROOF OF PROPOSITION 3.4. We are going to give the proof in the case that (θ_n^1) increases to θ_1 and (θ_n^2) decreases to θ_2 . We will also assume that $\theta_2 - \theta_1 < \theta_c$. The proof in the other possible cases is similar. By Lemma 3.3, we know that the pockets of $\mathbf{L}(\theta_1, \theta_2)$ are locally finite. Fix $\epsilon > 0$ and let P_1, \dots, P_n be the pockets of $\mathbf{L}(\theta_1, \theta_2)$ which have diameter at least ϵ . For each j , we let $x_j = x(P_j)$ (resp., $y_j = y(P_j)$) be the opening (resp., closing) point of P_j . Fix $\delta \in (0, \epsilon)$ and, for each j , let x_j^1 (resp., x_j^2) the first point on $S_1(P_j)$ (resp., $S_2(P_j)$) with distance at least δ from x_j . Let $\eta_{j,n}^1$ (resp., $\eta_{j,n}^2$) be the flow line of h starting from x_j^1 (resp., x_j^2) with angle θ_n^1 (resp., θ_n^2). As $n \rightarrow \infty$, these paths stopped upon exiting \bar{P}_j a.s. converge in the Hausdorff topology to the segments of $S_1(P_j)$ and $S_2(P_j)$ which start from x_j^1 and x_j^2 , respectively, and terminate at y_j . Indeed, this follows for $\eta_{j,n}^1$ because it is an SLE $_{\kappa}(\rho^L; \rho_1^R, \rho_2^R)$ process in P_j with force points located at $(x_j^1)^-, (x_j^1)^+$, and x_j^2 and $\rho^L \downarrow -2$ as $n \rightarrow \infty$. This follows for $\eta_{j,n}^2$ for an analogous reason.

Fix $\delta_1 \in (2\delta, \epsilon)$; we will shortly send $\delta \downarrow 0$ while leaving δ_1 and ϵ fixed. For each n , let $x_{n,j}^1$ be a point on $\eta_{n,j}^1$ which has distance δ_1 from x_j and let $\tilde{\eta}_{n,j}^1$ be the flow line of h starting from $x_{n,j}^1$ with angle $\theta_n^1 - \pi$ (the angle dual to that of $\eta_{n,j}^1$). We define $x_{n,j}^2$ and $\tilde{\eta}_{n,j}^2$ similarly (the angle of $\tilde{\eta}_{n,j}^2$ is $\theta_n^2 + \pi$). Let C_j be the component of $P_j \setminus (\eta_{n,j}^1 \cup \eta_{n,j}^2)$ which $\tilde{\eta}_{n,j}^1, \tilde{\eta}_{n,j}^2$ enter immediately upon getting started (there exists such a component with probability tending to 1 as $n \rightarrow \infty$). Then the joint law of $\tilde{\eta}_{n,j}^1, \tilde{\eta}_{n,j}^2$ in C_j stopped upon hitting $B(x_j, 2\delta)$ is absolutely continuous with respect to that of the pair of paths $(\hat{\eta}_{n,j}^1, \hat{\eta}_{n,j}^2)$ which are distributed as in the case that the boundary data along C_j takes the same form as if it were a pocket of a light cone with angle range $[\theta_n^1, \theta_n^2]$. Moreover, by [4], Lemma 2.1, the Radon–Nikodym derivative is bounded from above and below by universal finite and positive constants which do not depend on n . By the flow line interaction rules [5], Theorem 1.5, $\hat{\eta}_{n,j}^1, \hat{\eta}_{n,j}^2$ a.s. intersect before exiting C_j . Consequently, sending first $n \rightarrow \infty$, then $\delta \downarrow 0$, we see that the probability that $\tilde{\eta}_{n,j}^1$ intersects $\tilde{\eta}_{n,j}^2$ before hitting $B(x_j, 2\delta)$ tends to 1. Moreover, the diameter of the paths up until intersecting a.s. tends to zero upon taking another limit as $\delta_1 \downarrow 0$. The desired result follows because, on the high probability event that z is in the correct component of C_j , the pocket of $\mathbf{L}(\theta_n^1, \theta_n^2)$ which contains z is contained in

$P(z)$ and contains the component of $C_j \setminus (\tilde{\eta}_{n,j}^1 \cup \tilde{\eta}_{n,j}^2)$ containing z on the event that $\tilde{\eta}_{n,j}^1$ and $\tilde{\eta}_{n,j}^2$ intersect before leaving $B(x_j, 2\delta)$ provided δ is small enough. See the caption of Figure 5 for further explanation of this final point. \square

PROPOSITION 3.6. *Suppose that we have the same setup as in Lemma 3.3 and that $\theta_1 \leq \theta_2$ are angles with $\theta_2 - \theta_1 < \theta_c$ and $\theta_2 - \theta_1 \leq \pi$. Let $(\theta_n^1), (\theta_n^2)$ be sequences of angles with $\theta_n^1 \leq \theta_n^2$ and $\theta_n^2 - \theta_n^1 \leq \pi$ for all $n \in \mathbf{N}$ such that $\theta_n^j \rightarrow \theta_j$ as $n \rightarrow \infty$ for $j = 1, 2$. For each $z \in \mathbf{D}$ and $n \in \mathbf{N}$, let $\eta_j^n(z)$ be the flow line which forms the θ_j^n -angle boundary of the pocket of $\mathbf{L}(\theta_n^1, \theta_n^2)$ which contains z . Then $\eta_j^n(z) \rightarrow \eta_j(z)$ for $j = 1, 2$ a.s. as $n \rightarrow \infty$ with respect to the uniform topology modulo parameterization.*

PROOF. This follows from the same argument used to prove Proposition 3.4. \square

3.2. Explorations and continuity. We assume that $\theta_1 < \theta_2$ are angles with $\theta_2 - \theta_1 < \theta_c$ and $\theta_2 - \theta_1 \leq \pi$. We also assume that the boundary data of h is such that the flow lines η_1, η_2 with angles θ_1, θ_2 starting from $-i$, respectively, a.s. reach i before hitting the continuation threshold. Let η' be the counterflow line of $h + (\theta_2 - \frac{\pi}{2})\chi$ starting from i and targeted at $-i$. If $z \in D$ and we let η'_z be the counterflow line of the same field but targeted at z , then η' and η'_z a.s. agree up until the first time that z is disconnected from $-i$. Moreover, the part of the boundary of η'_z which is to the left is equal to the flow line of angle θ_2 starting from z . We will use the path η' to order the points on $\mathbf{L}(\theta)$ then use the continuity of η' to show that there exists a continuous, noncrossing path whose range is equal to $\mathbf{L}(\theta_1, \theta_2)$ and which visits the points of $\mathbf{L}(\theta_1, \theta_2)$ in this order. We will then show that the path has a continuous chordal Loewner driving function and, in certain special cases, yields a local set for h when drawn up to any stopping time. In the next section, we will use these facts to complete the proof of Theorem 1.2 by showing that the corresponding path (in a slightly modified setup) evolves as the appropriate $\text{SLE}_\kappa(\rho)$ process and is coupled with and determined by the field in the desired manner. This will also give Theorem 1.1. The path which traverses $\mathbf{L}(\theta_1, \theta_2)$ is constructed in the following manner:

1. Suppose that $z, w \in \mathbf{D}$ are distinct. We say that $P(z)$ comes before $P(w)$ if η' visits $x(z)$ before $x(w)$. Equivalently, $P(z)$ comes before $P(w)$ if the flow line $\eta_2(z)$ of h starting from $x(z)$ with angle θ_2 merges with the flow line $\eta_2(w)$ of angle θ_2 starting from $x(w)$ on its right side (i.e., the right side of the latter flow line).
2. We take η to be the concatenation of the paths $\eta_1(z)$ (oriented from $x(z)$ to $y(z)$) using the same ordering as for the pockets $P(z)$, parameterized by capacity as seen from $-i$.

It is not a priori obvious that η is continuous and can be parameterized by capacity. We will now use the continuity of η' to deduce these facts about η .

LEMMA 3.7. *The trajectory η from i to $-i$ in $\bar{\mathbf{D}}$ described above is a.s. continuous and admits a continuous chordal Loewner driving function.*

PROOF. Let η_1, η_2 be the flow lines of h starting from $-i$ with angles θ_1, θ_2 , respectively, as before, and let η' be the counterflow line of $h + (\theta_2 - \frac{\pi}{2})\chi$ starting from i and targeted at $-i$. From [5], Theorem 1.3, we know that η' is a.s. continuous. We are going to prove the continuity of η in two steps. First, we will construct an intermediate path by starting with η' and then excising the excursions that it makes out of $\mathbf{L}(\theta_1, \theta_2)$. Second, we will modify this intermediate path to get η .

Let $I = [0, \infty) \setminus (\eta')^{-1}(\mathbf{L}(\theta_1, \theta_2))$. Since η' is continuous, $I \subseteq [0, \infty)$ is open, hence we can write $I = \bigcup_j (s_j, t_j)$ as a countable, disjoint union of open intervals. Note that for each j there exists $z \in \mathbf{D}$ such that $\eta'((s_j, t_j)) \subseteq P(z)$. Suppose that $\eta'(s_j) \in S_1(z)$. Since $S_1(z)$ is contained in the range of η' and η' visits all of the points of $S_1(z)$ in the reverse chronological order in which they are drawn by $\eta_1(z)$, it must be that $\eta'(s_j) = \eta'(t_j)$. Consequently, letting $\tilde{\eta}|_{[0, \infty) \setminus I} = \eta'|_{[0, \infty) \setminus I}$ and $\tilde{\eta}|_{(s_j, t_j)} = \eta'(s_j) = \eta'(t_j)$ for each $j \in \mathbf{N}$ such that $\eta'(s_j) \in S_1(z)$, we see that $\tilde{\eta}$ is a.s. continuous. Note that after filling the right side of $S_2(z)$ for a pocket $P(z)$ and then after hitting $x(z)$ for the first time, $\tilde{\eta}$ travels inside $P(z)$ starting from $x(z)$ until reaching $y(z)$ while bouncing off the left side of $S_2(z)$ and does not hit the right side of $S_1(z)$. The amount of time that this takes is equal to the amount of time it takes η' to travel from $x(z)$ to $y(z)$. Next, $\tilde{\eta}$ fills $S_1(z)$ until reaching $x(z)$. While filling $S_1(z)$, it makes excursions out of $P(z)$ but never into (the interior of) $P(z)$.

Recall from Lemma 3.3 that the pockets of $\mathbf{L}(\theta_1, \theta_2)$ are a.s. locally finite. Let (P_n) be an ordering of the pockets of $\mathbf{L}(\theta_1, \theta_2)$ such that $\text{diam}(P_n) \geq \text{diam}(P_{n+1})$ for all n . (For example, we can order the pockets by diameter and then break ties using a fixed ordering of the rationals.) For each j , we let $\tilde{\eta}_j$ be the path which agrees with $\tilde{\eta}$ in P_m for $m \geq j + 1$ and, for $1 \leq m \leq j$, follows $\eta_1(P_m)$ rather than $\tilde{\eta}$ while traveling from $x(z)$ to $y(z)$ (but in the same interval of time). The local finiteness of the (P_j) implies that the sequence $(\tilde{\eta}_j)$ is Cauchy with respect to the uniform topology. Therefore, the sequence $(\tilde{\eta}_j)$ has a continuous limit $\hat{\eta}$.

Note that the range of $\hat{\eta}$ is equal to the closure of the collection of paths $\eta_1(z)$ used in the definition of η . Moreover, $\hat{\eta}$ traces these paths in the same order used in the definition of η .

We apply a conformal change of coordinates $\mathbf{D} \rightarrow \mathbf{H}$ which sends i to 0 and $-i$ to ∞ so that we may assume without loss of generality that we are working on \mathbf{H} . We abuse notation and also call the image of $\hat{\eta}$ under this map $\hat{\eta}$ and likewise the image of η' under this map η' . To complete the proof of the lemma, we just need to explain why reparameterizing $\hat{\eta}$ by capacity yields a continuous curve.

The continuity of $\widehat{\eta}$ implies that the capacity $\phi(t)$ of $\widehat{\eta}([0, t])$ is continuous in t . Suppose that we have $0 < s < t$ such that $\phi(s) = \phi(t)$. Note that $\widehat{\eta}$ does not complete drawing a pocket between times s and t (for otherwise $\phi(s) < \phi(t)$). If $\widehat{\eta}$ is drawing the same pocket at times s and t , then it follows from the construction that $\widehat{\eta}(s) = \widehat{\eta}(t)$. Combining, we see that if ϕ^{-1} denotes the right-continuous inverse of ϕ then $\widehat{\eta} \circ \phi^{-1}$ is continuous. To finish checking that $\widehat{\eta} \circ \phi^{-1}$ has a continuous Loewner driving function, we will apply [18], Proposition 2.1. For each $t \geq 0$, we let K_t be complement of the unbounded connected component of $\mathbf{H} \setminus \widehat{\eta} \circ \phi^{-1}([0, t])$. Then we need to check that for all $T > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \leq T$ there exists a connected set $S \subseteq \mathbf{H} \setminus K_t$ with diameter at most ϵ which disconnects $K_{t+\delta} \setminus K_t$ from ∞ in $\mathbf{H} \setminus K_t$. We take $\delta = \omega_T(\epsilon/2)$ where ω_T is the modulus of continuity for $\widehat{\eta} \circ \phi^{-1}$ on $[0, T]$ and we take S to be the arc of $\partial B(\widehat{\eta} \circ \phi^{-1}(t), \epsilon) \setminus K_t$ which separates $\widehat{\eta} \circ \phi^{-1}(t)$ from ∞ in $\mathbf{H} \setminus K_t$. We note that S is well-defined because when $\widehat{\eta} \circ \phi^{-1}(t)$ corresponds to multiple prime ends on $\partial(\mathbf{H} \setminus K_t)$, only one of them is on the boundary of the unbounded component. \square

We are next going to argue that the path η together with the left and right boundaries η_2 and η_1 , respectively, of $\mathbf{L}(\theta_1, \theta_2)$ is local (in the sense of [13]) for and a.s. determined by h .

PROPOSITION 3.8. *For each $t \geq 0$, let \mathcal{F}_t be the σ -algebra generated by $\eta|_{[0,t]}$ and the left and right boundaries η_2 and η_1 , respectively, of $\mathbf{L}(\theta_1, \theta_2)$. For each (\mathcal{F}_t) -stopping time τ , $\eta([0, \tau]) \cup \eta_1 \cup \eta_2$ is a local set for and a.s. determined by h .*

Let η' be the counterflow line of $h + (\theta_2 - \frac{\pi}{2})\chi$ starting from i and targeted at $-i$. For each z , we also let η'_z be the counterflow line of the same field targeted at z . Then the boundary of η'_z , stopped upon hitting a point z , which is to the left is equal to the flow line starting from z with angle θ_2 . To prove Proposition 3.8, we are going to describe a “local” construction of η from η' (one which will only require us first to observe the left and right boundaries η_2 and η_1 , respectively, of $\mathbf{L}(\theta_1, \theta_2)$ but not all of $\mathbf{L}(\theta_1, \theta_2)$).

We begin by using η' to define paths as follows. Fix $\epsilon > 0$. For each $k \in \mathbf{N}$ and $t > 0$ let F_t^k be the closure of the union of all flow lines of h with angle θ_1 starting from a point with rational coordinates in the dyadic square with side length 2^{-k} and corners in $(2^{-k}\mathbf{Z})^2$ containing $\eta'(t)$ drawn up until the first time that they hit $\eta'([0, t])$. Let $\tau_{\epsilon,k,1}$ be the first time t that there exists a component of $\mathbf{D} \setminus (F_t^k \cup \eta'([0, t]))$ part of whose boundary is given by a segment of the part of the boundary of $\eta'([0, t])$ which is to its left and has diameter at least ϵ . Note that the times $\tau_{\epsilon,k,1}$ are nondecreasing as k increases. We then let $\tau_{\epsilon,1} = \sup_{k \in \mathbf{N}} \tau_{\epsilon,k,1}$. Note that $\bigcap_{k \in \mathbf{N}} F_{\tau_{\epsilon,1}}^k$ consists of a single path $\eta_{\epsilon,1}$ starting from $\eta'(\tau_{\epsilon,1})$ which is stopped upon hitting the part of the boundary of $\eta'([0, \tau_{\epsilon,1}])$ which is to its left. Note that the pocket thus formed will have diameter at least ϵ if either:

1. $\eta_{\epsilon,1}$ has diameter at least ϵ or
2. $\eta_{\epsilon,1}$ has diameter less than ϵ hence closes the pocket before leaving the ϵ -neighborhood of $\eta'([0, \tau_{\epsilon,1}])$.

In particular, each of the two possibilities can be determined by observing the values of h in an ϵ -neighborhood of $\eta'([0, \tau_{\epsilon,1}])$. Moreover, we note that since $\eta_{\epsilon,1}$ crosses the part of the boundary of $\eta'([0, \tau_{\epsilon,1}])$ which is to the left we have that $\eta'|_{[\tau_{\epsilon,1}, \infty)}$ immediately enters the complementary component of $\eta'([0, \tau_{\epsilon,1}]) \cup \eta_{\epsilon,1}$ part of whose boundary consists of the right side of $\eta_{\epsilon,1}$. This behavior of $\eta'|_{[\tau_{\epsilon,1}, \infty)}$ is also determined by observing the values of h in an ϵ -neighborhood of $\eta'([0, \tau_{\epsilon,1}])$.

We then let $\eta'_{\epsilon,1}$ be the path which agrees with η' until time $\tau_{\epsilon,1}$ and then follows $\eta_{\epsilon,1}$ until hitting the boundary of $\eta'([0, \tau_{\epsilon,1}])$. Let $P_{\epsilon,1}$ be the pocket thus formed by $\eta_{\epsilon,1}$ and the part of the boundary of $\eta'([0, \tau_{\epsilon,1}])$ which is to its left. Note that $\partial P_{\epsilon,1}$ consists of the right side $\eta_{\epsilon,1}$ and the left side of a flow line starting from $\eta'(\tau_{\epsilon,1})$ with angle θ_2 . In other words, $\partial P_{\epsilon,1}$ has the same structure as a pocket of $\mathbf{L}(\theta_1, \theta_2)$; recall Lemma 3.1. We let $x_{\epsilon,1} = \eta'(\tau_{\epsilon,1})$ be the opening point of $P_{\epsilon,1}$ and let $y_{\epsilon,1}$ be the closing point of $P_{\epsilon,1}$. Explicitly, $y_{\epsilon,1}$ is the point at which $\eta_{\epsilon,1}$ crosses into $\eta'([0, \tau_{\epsilon,1}])$. Moreover, $\eta'|_{[\tau_{\epsilon,1}, \infty)}$ interacts with $P_{\epsilon,1}$ in the same manner that η' interacts with a pocket of $\mathbf{L}(\theta_1, \theta_2)$ as described in Figure 4 and Figure 6. In particular, $\eta'|_{[\tau_{\epsilon,1}, \infty)}$ enters (the interior of) $P_{\epsilon,1}$ at $x_{\epsilon,1}$ and does not

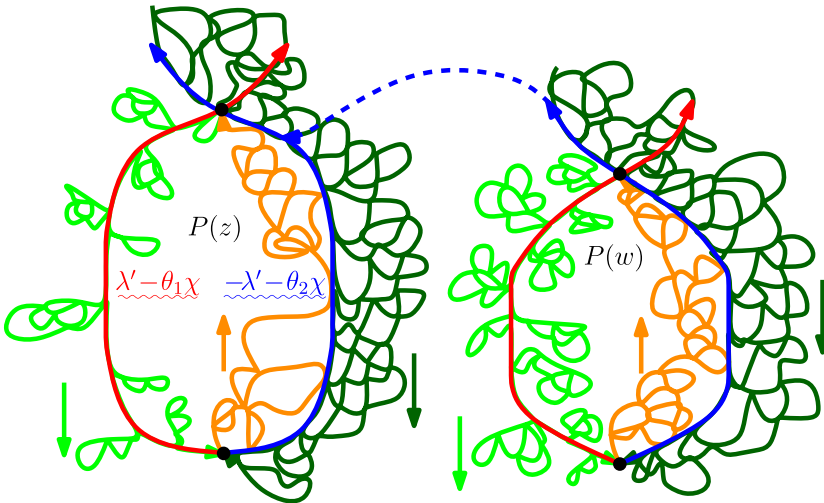


FIG. 6. (Continuation of Figure 4.) Shown is a second pocket $P(w)$ of $\mathbf{L}(\theta_1, \theta_2)$. If the θ_2 -angle flow line $\eta_2(w)$ which generates $S_2(w)$ merges into the right side of the θ_2 -angle flow line $\eta_2(z)$ which generates $S_2(z)$ of $\partial P(z)$ as illustrated, then the counterflow line η' visits a point in the interior of $P(w)$ before visiting a point in the interior of $P(z)$. This determines (and is the same as) the order in which the trajectory we consider which explores $\mathbf{L}(\theta_1, \theta_2)$ visits $P(z)$ and $P(w)$. The same color scheme for the segments of η' as it visits the points of $\partial P(w)$ is used as in Figure 4.

leave $\overline{P}_{\epsilon,1}$ or hit $\eta_{\epsilon,1}$ until hitting $y_{\epsilon,1}$ for the first time, say at time $\sigma_{\epsilon,1}$. After hitting $y_{\epsilon,1}$ it visits the points on $\eta_{\epsilon,1}$ in the reverse order in which they are drawn by $\eta_{\epsilon,1}$. In particular, $\eta'|_{[\sigma_{\epsilon,1},\infty)}$ makes excursions both into and out of $P_{\epsilon,1}$ and each such excursion starts and ends at the same point on $\eta_{\epsilon,1}$ (different excursions, however, are rooted at different points on $\eta_{\epsilon,1}$). We take the part of $\eta'_{\epsilon,1}$ after it has finished drawing $\eta_{\epsilon,1}$ to be given by $\eta'|_{[\sigma_{\epsilon,1},\infty)}$ with those excursions of η' from $\eta_{\epsilon,1}$ into $P_{\epsilon,1}$ excised (we leave the excursions out of $P_{\epsilon,1}$ alone).

Suppose that $k \geq 1$ and that paths $\eta'_{\epsilon,1}, \eta_{\epsilon,1}, \dots, \eta'_{\epsilon,k}, \eta_{\epsilon,k}$, stopping times $\tau_{\epsilon,1}, \sigma_{\epsilon,1}, \dots, \tau_{\epsilon,k}, \sigma_{\epsilon,k}$ and pockets $P_{\epsilon,1}, \dots, P_{\epsilon,k}$ with opening and closing points $x_{\epsilon,1}, y_{\epsilon,1}, \dots, x_{\epsilon,k}, y_{\epsilon,k}$ have been defined. We then let $\tau_{\epsilon,k+1}$ be defined in exactly the same way as $\tau_{\epsilon,k+1}$ except in terms of times $t \geq \sigma_{\epsilon,k}$ and in terms of the path $\eta'_{\epsilon,k}([0, t])$. We then take $\eta'_{\epsilon,k+1}$ to be the path constructed from $\eta'_{\epsilon,k}$ in the same manner that we constructed $\eta'_{\epsilon,1}$ from η' and let $\sigma_{\epsilon,k+1}$ (resp., $P_{\epsilon,k+1}$) be the corresponding stopping time (resp., pocket). Finally, we let $x_{\epsilon,k+1}$ (resp., $y_{\epsilon,k+1}$) be the opening (resp., closing) point of $P_{\epsilon,k+1}$.

For each $\epsilon > 0$, we let $\mathcal{P}_\epsilon(\theta_1, \theta_2)$ consist of those pockets of $\mathbf{L}(\theta_1, \theta_2)$ which have diameter at least ϵ ; recall from Lemma 3.7 that $\mathcal{P}_\epsilon(\theta_1, \theta_2)$ is finite a.s. Let $\mathbf{L}_\epsilon(\theta_1, \theta_2) = \{S_1(P) : P \in \mathcal{P}_\epsilon(\theta_1, \theta_2)\}$. Let $J_\epsilon = \sup\{j \geq 1 : \tau_{\epsilon,j} < \infty\}$ and let $\mathcal{R}_\epsilon = \{\eta_{\epsilon,j} : 1 \leq j \leq J_\epsilon\}$ consist of the θ_1 -angle boundary segments of the pockets $P_{\epsilon,j}$ (we will explain below that $J_\epsilon < \infty$ almost surely).

We are now going to collect several observations about the exploration procedure that we have just defined.

LEMMA 3.9. *Fix $\epsilon > 0$. The following are true:*

- (i) *Suppose that ζ is a stopping time for $\eta'_{\epsilon,j}$. Then the ϵ -neighborhood of $\eta'_{\epsilon,j}([0, \zeta])$ is a local set for h .*
- (ii) *For each $j \geq i$, a.s. $\eta'_{\epsilon,j}$ does not enter (the interior of) $P_{\epsilon,i}$.*
- (iii) *Almost surely, $J_\epsilon < \infty$.*
- (iv) *For each $1 \leq j \leq J_\epsilon$ such that $P_{\epsilon,j}$ lies between the left and right boundaries of $\mathbf{L}(\theta_1, \theta_2)$ there a.s. exists $P \in \mathcal{P}_\epsilon(\theta_1, \theta_2)$ such that $P_{\epsilon,j} \subseteq P$ and $\eta_{\epsilon,j}$ emanates from a point on $S_2(P)$.*
- (v) *Almost surely, $\mathbf{L}_\epsilon(\theta_1, \theta_2)$ is equal to the set which consists of those elements $\eta_{\epsilon,j}$ of \mathcal{R}_ϵ for $1 \leq j \leq J_\epsilon$ which lie between the left and right boundaries of $\mathbf{L}(\theta_1, \theta_2)$.*

PROOF. To prove part (i), we will use the characterization of local sets given in the first part of [13], Lemma 3.9. We are first going to explain the proof in the case that $j = 1$. Fix $B \subseteq \mathbf{D}$ open and let $\tau_{\epsilon,B}$ be the first time t that $\text{dist}(\eta'(t), B) \leq \epsilon$. Let h_B be the projection of h onto the subspace of functions which are harmonic on B . Then [5], Theorem 1.2, implies that $\eta'|_{[0, \tau_{\epsilon,B}]}$ is a.s. determined by h_B . Note that the event $\tau_{\epsilon,1} \leq \tau_{\epsilon,B}$ is also a.s. determined by h_B because the set of all flow lines with angle θ_1 starting from points in $\mathbf{D} \setminus B$ and stopped upon exiting $\mathbf{D} \setminus B$

is (simultaneously) a.s. determined by h_B . In particular, we only need to observe these flow lines in an ϵ -neighborhood of $\eta'|_{[0, \tau_{\epsilon, B}]}$ to see if $\tau_{\epsilon, 1} \leq \tau_{\epsilon, B}$; recall the discussion after the statement of Proposition 3.8. Assume that we are working on the event $\tau_{\epsilon, 1} \leq \tau_{\epsilon, B}$. Then $\eta'|_{[0, \tau_{\epsilon, 1}]}$ is a.s. determined by h_B for the same reason. Let $\tau_{\epsilon, 1, B}$ be the first time t that $\text{dist}(\eta_{\epsilon, 1}(t), B) \leq \epsilon$. Then $\eta_{\epsilon, 1}|_{[0, \tau_{\epsilon, 1, B}]}$ is also a.s. determined by h_B , again for the same reason. Finally, on the event that $\eta_{\epsilon, 1}$ terminates in $\eta'([0, \tau_{\epsilon, 1}])$ before time $\tau_{\epsilon, 1, B}$, it is easy to see that $\eta'_{\epsilon, 1}|_{[\sigma_{\epsilon, 1}, \infty)}$ stopped upon getting within distance ϵ of B is a.s. determined by h_B because it is given by the counterflow line of h starting from the terminal point of $\eta_{\epsilon, 1}$ with its excursions into $P_{\epsilon, 1}$ excised. In particular, this is the same as the excised counterflow line of the conditional GFF h given $\eta'|_{[0, \tau_{\epsilon, 1}]}$ and $\eta_{\epsilon, 1}$ starting from the terminal point of $\eta_{\epsilon, 1}$. This proves part (i) for $j = 1$. The result for $j \geq 2$ follows using a similar argument and induction on j .

Part (ii) follows because, by our construction, after drawing a pocket we excise all of the excursions that the counterflow line makes into that pocket and the flow line interaction rules imply that a flow line of angle θ_1 (i.e., one of the $\eta_{\epsilon, j}$) cannot cross into the interior of such a pocket.

We turn to part (iii). For each $k \geq 1$, consider the path $\tilde{\eta}'_{\epsilon, k}$ which is given by starting with η' and then excising the excursions that η' makes into the interior of each $P_{\epsilon, j}$ for $1 \leq j \leq k$. Then each path $\tilde{\eta}'_{\epsilon, k}$ is continuous and has the same range as $\eta'_{\epsilon, k}$ by the argument described after Lemma 3.2 (as the same analysis leads to a description of how η' interacts with the pockets $P_{\epsilon, j}$). (We emphasize that $\tilde{\eta}'_{\epsilon, k}$ is not the same as the path $\eta'_{\epsilon, k}$ described above because as $\tilde{\eta}'_{\epsilon, k}$ travels from $y(P)$ to $x(P)$ for a given pocket P , it makes excursions out of P but not into P . During these excursions out of P , it may visit another complementary pocket.) As $k \geq 1$ increases, more and more excursions are excised in order to generate $\tilde{\eta}'_{\epsilon, k}$. Thus arguing as in the proof of Lemma 3.7, this implies that the limit $\tilde{\eta}'_{\epsilon}$ of $\tilde{\eta}'_{\epsilon, k}$ as $k \rightarrow \infty$ exists as a uniform limit of continuous paths on a compact interval and $\tilde{\eta}'_{\epsilon}$ is continuous and non-self-crossing. Moreover, the complement of the range of $\tilde{\eta}'_{\epsilon}$ can only have a finite number of components of diameter larger than $\epsilon > 0$. Indeed, for otherwise the range of $\tilde{\eta}'_{\epsilon}$ would not be locally connected which in turn would contradict continuity. This gives part (iii).

We are now going to explain the proof of part (iv). We first condition h on the ϵ -neighborhood of $\eta'_{\epsilon, j}|_{[0, \sigma_{\epsilon, j}]}$ for some j . Note that $\partial P_{\epsilon, j}$ consists of the right side of a flow line with angle θ_1 and the left side of a flow line with angle θ_2 . Consequently, an angle-varying flow line with angles contained in $[\theta_1, \theta_2]$ which changes angles only a finite number of times and at positive rational times cannot enter (the interior of) $P_{\epsilon, j}$ by the flow line interaction rules. Thus if $P_{\epsilon, j}$ for $1 \leq j \leq J_{\epsilon}$ is between the left and right boundaries of $\mathbf{L}(\theta_1, \theta_2)$ then it is a subset of some element in $\mathcal{P}_{\epsilon}(\theta_1, \theta_2)$. This gives the first part of part (iv). To establish the second part of part (iv), we first condition on $\mathbf{L}(\theta_1, \theta_2)$. Note that η' enters the interior of a pocket of $\mathbf{L}(\theta_1, \theta_2)$ at its opening point. Thus, $\eta'_{\epsilon, 1}(\tau_{\epsilon, 1})$ must be on the boundary of such a

pocket, say $P \in \mathcal{P}_\epsilon(\theta_1, \theta_2)$. Indeed, for otherwise the exploration used to generate $\eta'_{\epsilon,1}$ would have skipped following $\eta_1(P)$. Iterating this proves the claim for $j \geq 1$.

We turn to part (v). We fix $P \in \mathcal{P}_\epsilon(\theta_1, \theta_2)$. We claim that either $\eta_{\epsilon,1} = \eta_1(P)$ or, if not, cannot merge into $\eta_1(P)$. To see that this is the case, we assume that $\eta_{\epsilon,1}$ is not equal to $\eta_1(P)$. If $\eta_{\epsilon,1}$ did merge into $\eta_1(P)$, then η' would visit the left side of $\eta_1(P)$ before hitting $x(P)$ because the path would have to visit the left side of $\eta_{\epsilon,1}$ before hitting $x(P)$. (This follows because whenever η' hits the opening point of a pocket, the flow line interaction rules imply that it immediately enters and then exits at the closing point of the pocket. Once it exits at the closing point, it immediately starts filling the θ_1 -angle boundary segment.) This, in turn, would contradict the ordering because η' would hit the left side of $S_1(P)$ before hitting $x(P)$. Iterating this argument implies that $\eta_{\epsilon,j}$ is either equal to $\eta_1(P)$ where P is the pocket of $\mathbf{L}(\theta_1, \theta_2)$ which contains $P_{\epsilon,j}$ or does not merge with $\eta_1(P)$. If $\eta_1(P)$ for some $P \in \mathcal{P}_\epsilon(\theta_1, \theta_2)$ was not equal to one of the $\eta_{\epsilon,j}$ for $1 \leq j \leq J_\epsilon$, then η'_ϵ would have to visit $x(P)$. This is a contradiction since exploring $\eta_1(P)$ upon hitting $x(P)$ would lead to a pocket with diameter at least ϵ (since the interior of P would have been explored by η'_ϵ before the path hits the opening point). This proves that $\mathbf{L}_\epsilon(\theta_1, \theta_2) \subseteq \mathcal{R}_\epsilon$ a.s.

We are now going to prove that the set which consists of those elements $\eta_{\epsilon,j}$ of \mathcal{R}_ϵ for $1 \leq j \leq J_\epsilon$ which lie between the left and right boundaries of $\mathbf{L}(\theta_1, \theta_2)$ is contained in $\mathbf{L}_\epsilon(\theta_1, \theta_2)$ a.s. Fix $1 \leq j \leq J_\epsilon$. Suppose that $P_{\epsilon,j}$ is strictly contained in the pocket P of $\mathcal{P}_\epsilon(\theta_1, \theta_2)$ which contains $P_{\epsilon,j}$. Upon hitting the opening point $x_{\epsilon,j}$ of $P_{\epsilon,j}$, η' has to enter into the interior of $P_{\epsilon,j}$ hence the interior of P as explained above. If $x_{\epsilon,j}$ is not equal to $x(P)$, then this implies that η' enters the interior of P before hitting $x(P)$. This is a contradiction, therefore, $P_{\epsilon,j} = P$ as desired. This proves part (v). \square

PROOF OF PROPOSITION 3.8. As in the proof of Lemma 3.9, we let $\eta'_\epsilon = \eta'_{\epsilon,J_\epsilon}$. By the construction and part (v) of Lemma 3.9, η'_ϵ visits the elements of $\mathcal{P}_\epsilon(\theta_1, \theta_2)$ in the same order as η defined just before Lemma 3.7. Therefore, it follows from the proof of Lemma 3.7 that η'_ϵ with its excursions outside of the region between η_1 and η_2 excised converges uniformly modulo parameterization to η as $\epsilon \rightarrow 0$. Therefore, Lemma 3.9 implies that $\eta([0, t]) \cup \eta_1 \cup \eta_2$ is a local set for h for each rational time t . Combining this with the characterization of local sets given in the first part of [13], Lemma 3.9, implies that $\eta([0, \tau]) \cup \eta_1 \cup \eta_2$ is local for each η -stopping time τ . \square

We are now going to show that the law of the exploration path is continuous in the angles of the light cone. This, in turn, will be used in Section 3.3 to establish the continuity of the law of $\text{SLE}_\kappa(\rho)$ as the value of ρ varies between $(-2 - \frac{\kappa}{2}) \vee (\frac{\kappa}{2} - 4)$ and -2 .

PROPOSITION 3.10. *Suppose that $\theta_1 \leq \theta_2$ are angles with $\theta_2 - \theta_1 < \theta_c$ and $\theta_2 - \theta_1 \leq \pi$ and that $(\theta_n^1), (\theta_n^2)$ are sequences of angles such that $\theta_n^1 \leq \theta_n^2$ and $\theta_n^2 - \theta_n^1 < \theta_c$ and $\theta_n^2 - \theta_n^1 \leq \pi$ for each $n \in \mathbf{N}$ and $\theta_n^j \rightarrow \theta_j$ as $n \rightarrow \infty$ for $j = 1, 2$. For each $n \in \mathbf{N}$, let η_n be the path described above which visits the points of $\mathbf{L}(\theta_n^1, \theta_n^2)$ and let η be the path associated with $\mathbf{L}(\theta_1, \theta_2)$. Then $\eta_n \rightarrow \eta$ as $n \rightarrow \infty$ a.s. with respect to the uniform topology modulo reparameterization.*

REMARK 3.11. Proposition 3.10 does not imply that the map which takes a pair of angles (θ_1, θ_2) to the exploration path of $\mathbf{L}(\theta_1, \theta_2)$ is a continuous function into the space of paths equipped with the uniform topology modulo parameterization for a fixed realization of h . This follows from the same reasoning as in Remark 3.5 in which it was explained that $(\theta_1, \theta_2) \mapsto \mathbf{L}(\theta_1, \theta_2)$ is not a continuous function into the space of closed sets equipped with the Hausdorff topology for a fixed realization of h . Proposition 3.10 does, however, imply that the map which takes a pair of angles (θ_1, θ_2) to the law of the exploration path of $\mathbf{L}(\theta_1, \theta_2)$ is continuous with respect to the weak topology.

PROOF OF PROPOSITION 3.10. For each $n \in \mathbf{N}$, let η'_n (resp., η') be the counterflow line of h which orders $\mathbf{L}(\theta_n^1, \theta_n^2)$ (resp., $\mathbf{L}(\theta_1, \theta_2)$) to generate the light cone exploration path η_n (resp., η). Then we know that $\eta'_n \rightarrow \eta'$ a.s. as $n \rightarrow \infty$ with respect to the uniform topology.⁵ We also know from Proposition 3.4 that $\mathbf{L}(\theta_n^1, \theta_n^2) \rightarrow \mathbf{L}(\theta_1, \theta_2)$ a.s. as $n \rightarrow \infty$ with respect to the Hausdorff topology. Fix an ordering (r_j) of the points in \mathbf{D} with rational coordinates. For each $n \in \mathbf{N}$, let (P_j^n) be the ordering of the pockets of $\mathbf{L}(\theta_n^1, \theta_n^2)$ according to diameter in which ties are broken according to which pocket contains the element of (r_j) with the smallest index and let (P_j) be the ordering of the pockets of $\mathbf{L}(\theta_1, \theta_2)$ defined in the same way. For each $j, n \in \mathbf{N}$, we also let I_j^n (resp., I_j) be the interval of time in which η'_n (resp., η') travels from the opening to the closing point of P_j^n (resp., P_j). Note that I_j^n (resp., I_j) is also the interval of time in which η_n (resp., η) travels from the opening to the closing point of P_j^n (resp., P_j) along $S_1(P_j^n)$ (resp., $S_1(P_j)$). Let $\eta_{1,j}^n = \eta_n|_{I_j^n}$ and $\eta_{1,j} = \eta|_{I_j}$. It follows from Proposition 3.4 and Proposition 3.6 that $\eta_{1,j}^n \rightarrow \eta_{1,j}$ a.s. as $n \rightarrow \infty$ with respect to the uniform topology modulo parameterization. Combining all of the above, we can see that for each k there exists $n_0 \in \mathbf{N}$ such that the following is true with high probability. We have that $n \geq n_0$ implies that:

⁵This follows because if we fix any finite collection of points $z_1, \dots, z_k \in \mathbf{D}$, the “cells” generated by the flow and dual flow lines corresponding to η'_n starting from these points will converge to those of η' as $n \rightarrow \infty$. If we fix enough points, then w.h.p. the maximal diameter of the cells will be smaller than a fixed choice of $\epsilon > 0$. The claim follows by reparameterizing η'_n so that it spends the same amount of time in a given cell as η' does. Note that this time change converges to the identity as $n \rightarrow \infty$ since asymptotically the area of the cells converge, too.

1. the uniform distance modulo parameterization between $\eta_{1,j}^n$ and $\eta_{1,j}$ is at most ϵ for each $1 \leq j \leq k$,
2. $\text{diam}(P_j^n) \leq \epsilon$ for all $j > k$ and
3. $\|\eta'_n - \eta'\|_\infty \leq \epsilon$.

Reparameterizing the time of η'_n and η_n so that $I_j^n = I_j$ for each $1 \leq j \leq k$, it thus follows that, after possibly reparameterizing the time of η'_n and η_n within each I_j , with $\mathcal{I} = \bigcup_{1 \leq j \leq k} I_j$ we have that

$$(3.2) \quad \|\eta_n|_{\mathcal{I}} - \eta|_{\mathcal{I}}\|_\infty \leq \epsilon.$$

Let $\mathcal{J} = (\eta'_n)^{-1}(\bigcup_{j>k} P_j^n) = (\eta')^{-1}(\bigcup_{j>k} P_j)$. By the way that we have defined the light cone exploration path, we also have that

$$(3.3) \quad \|\eta'_n|_{\mathcal{J}} - \eta_n|_{\mathcal{J}}\|_\infty \leq \epsilon \quad \text{and} \quad \|\eta'|_{\mathcal{J}} - \eta|_{\mathcal{J}}\|_\infty \leq \epsilon.$$

Note that η_n (resp., η) is determined by its values on $\mathcal{I} \cup \mathcal{J}$ since the times in $[0, \infty) \setminus \overline{\mathcal{I} \cup \mathcal{J}}$ correspond to those times in which η'_n (resp., η') makes an excursion from $S_1(P_j^n)$ (resp., $S_1(P_j)$) into P_j^n (resp., P_j) for some $1 \leq j \leq k$. In particular, η_n (resp., η) is piecewise constant in $[0, \infty) \setminus \overline{\mathcal{I} \cup \mathcal{J}}$. Combining (3.2) and (3.3) implies that

$$\begin{aligned} \|\eta_n - \eta\|_\infty &\leq \|\eta_n|_{\mathcal{I}} - \eta|_{\mathcal{I}}\|_\infty + \|\eta_n|_{\mathcal{J}} - \eta|_{\mathcal{J}}\|_\infty \\ &\leq \epsilon + \|\eta_n|_{\mathcal{J}} - \eta'_n|_{\mathcal{J}}\|_\infty + \|\eta'_n|_{\mathcal{J}} - \eta'|_{\mathcal{J}}\|_\infty + \|\eta'|_{\mathcal{J}} - \eta|_{\mathcal{J}}\|_\infty \\ &\leq 3\epsilon + \|\eta'_n - \eta'\|_\infty \leq 4\epsilon, \end{aligned}$$

which gives the desired result. \square

3.3. Law of the exploration path. It will be more convenient for us to work on \mathbf{H} in this section. Throughout, we fix $\rho \in [\frac{\kappa}{2} - 4, -2)$ and suppose that h is a GFF on \mathbf{H} with boundary conditions given by $-\lambda$ on \mathbf{R}_- and $\lambda(1 + \rho)$ on \mathbf{R}_+ , as shown in Figure 8. Let θ_ρ be as in (1.3). Let η' be the counterflow line starting from the origin whose left boundary stopped upon hitting a point z is equal to the flow line with angle θ_ρ starting from z . Explicitly, η' is the counterflow line of $h + (\frac{\pi}{2} + \theta_\rho)\chi$ starting from the origin. Note that this is the “same” as the corresponding counterflow line starting from ∞ because the path starting from ∞ will trace along \mathbf{R}_+ and does not enter (the interior of) \mathbf{H} until hitting the origin. Using exactly the same analysis as in Section 3.1 and Section 3.2, we can construct from η' a path η which explores $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$. This path is continuous, has a continuous chordal Loewner driving function, and is a.s. determined by h . Moreover, the path drawn up to any stopping time is local for h (in contrast to Proposition 3.8, it is not necessary also to condition on the outer boundary of the light cone). That these properties hold follows from the results of the previous subsections and the conditioning argument explained in Figure 7.

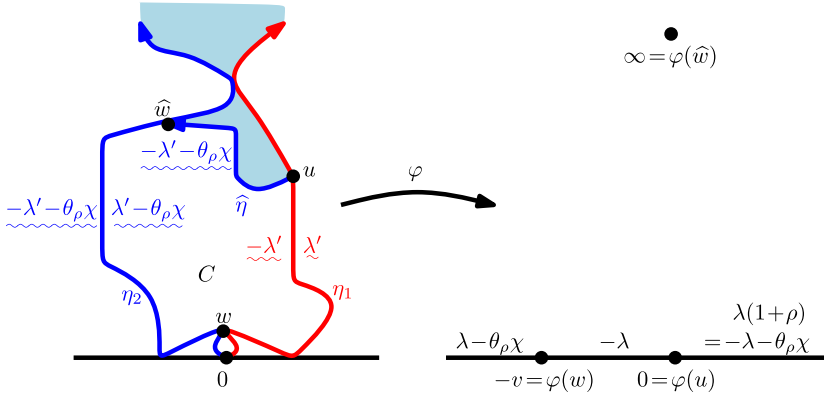


FIG. 7. Suppose that h is a GFF on \mathbf{H} with piecewise constant boundary data which changes values at most a finite number of times. Shown on the left side are the flow lines η_1, η_2 with angles $0, \theta_\rho$, respectively, of h starting from 0 , both of which we assume reach ∞ before hitting the continuation threshold, and the flow line $\hat{\eta}$ of angle θ_ρ starting from a point u on the boundary of a component of $\mathbf{H} \setminus (\eta_1 \cup \eta_2)$ which is between η_1 and η_2 . The outer boundary of $\mathbf{L}(0, \theta_\rho)$ is given by $\eta_1 \cup \eta_2$. The exploration path η of $\mathbf{L}(0, \theta_\rho)$ starts from ∞ and its outer boundary stopped upon hitting u is equal to the union of $\hat{\eta}$ and the part of η_1 (resp., η_2) after it hits u (resp., \hat{w}). The light blue region indicates the hull of η stopped upon hitting u . Let C be the component surrounded by η_1, η_2 and $\hat{\eta}$ as shown and let $\varphi: C \rightarrow \mathbf{H}$ be the conformal map which takes u to $0, \hat{w}$, the point where η_2 and $\hat{\eta}$ merge, to ∞ , and w , the point on ∂C where η_1, η_2 first intersect, to $-v$. The boundary data for $\tilde{h} = h \circ \varphi^{-1} - \chi \arg(\varphi^{-1})'$ is as shown on the right. The image of the part of $\mathbf{L}(0, \theta_\rho)$ contained in \bar{C} under φ is equal to the light cone $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ of \tilde{h} starting from 0 and the image of the part of η when it is in \bar{C} gives the corresponding exploration path from 0 to $-v$. Sending $v \rightarrow \infty, \mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ converges to the corresponding light cone of a field whose boundary conditions are given by $-\lambda$ (resp., $\lambda(1 + \rho)$) on \mathbf{R}_- (resp., \mathbf{R}_+).

We will now determine the law of η . This, in turn, will lead to the proofs of Theorem 1.1 and Theorem 1.2 (it does not quite imply Theorem 1.3 because the boundary data is different for different ρ values). For each $t \geq 0$, let K_t be the closure of the complement of the unbounded connected component \mathbf{H}_t of $\mathbf{H} \setminus \eta([0, t])$. For each $t \geq 0$, such that η is drawing a segment of $S_1(P)$ where P is a pocket of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ in an open interval of time containing t , let P_t be the corresponding pocket and let X_t be its opening point. For other values of t , we take $P_t = \emptyset$ and let X_t be the limit of X_s as $s \downarrow t$ where the times s are restricted to those in which η is drawing a segment of $S_1(P)$ for a pocket P of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$. The main step in determining the law of η is the following, which gives the conditional law of h given η drawn up to a fixed stopping time.

LEMMA 3.12. Suppose that τ is an a.s. finite stopping time for η . Then the conditional law of h given $\eta|_{[0, \tau]}$ is independently that of a GFF in each of the components of $\mathbf{H} \setminus \eta([0, \tau])$. The boundary conditions in each of the bounded com-

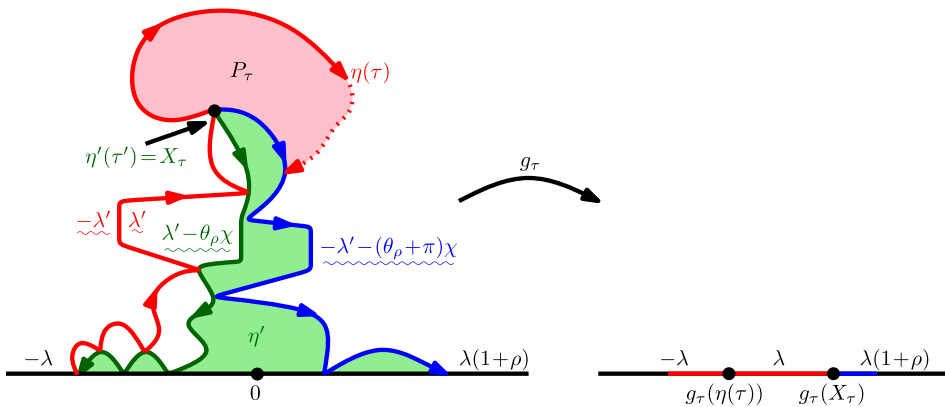


FIG. 8. Setup for the proof of Lemma 3.12. Suppose that h is a GFF on \mathbf{H} with the illustrated boundary data where $\rho \in [\frac{\kappa}{2} - 4, -2)$ and that η is the exploration path associated with $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ where $\theta_\rho = \pi(\rho + 2)/(\frac{\kappa}{2} - 2)$. Suppose that τ is a stopping time for η . Then we can describe the boundary behavior of the conditional law of h given $\eta|_{[0, \tau]}$ restricted to the unbounded connected component of $\mathbf{H} \setminus \eta([0, \tau])$ by relating the outer boundary of $\eta([0, \tau])$ (the union of the red and blue paths in the illustration) to the outer boundary of the counterflow line η' (the hull of which is indicated in light green; the blue and green paths are its left and right boundaries) stopped at the first time τ' that it hits X_τ , the opening point of the pocket whose boundary is being drawn by η at time τ , and the 0-angle flow line starting from the leftmost point of $\eta'([0, \tau']) \cap \mathbf{R}$ (red). The region P_τ bounded by the solid red, dashed red, and blue paths is the pocket of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ whose 0-angle boundary is being drawn by η at time τ . We emphasize that the red path is not a flow line of h by itself.

ponents agrees with that of h given $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ in the corresponding component (recall Lemma 3.1). On $\partial\mathbf{H}_\tau$, the boundary conditions are given by:

- (i) the left side of a 0-angle flow line on the segment of $\partial\mathbf{H}_\tau$ which is to the left of $\eta(\tau)$ (left side of the red path in Figure 8),
- (ii) the right side of a 0-angle flow line on the right side of the segment of $\partial\mathbf{H}_\tau$ from $\eta(\tau)$ to X_τ (right side of red path in Figure 8) and
- (iii) the left side of a θ_ρ -angle flow line on the segment from X_τ to \mathbf{R}_+ (left side of blue path in Figure 8).

PROOF. Let τ be any a.s. finite stopping time for η such that $\eta(\tau)$ is contained in the interior of a 0-angle boundary segment of a pocket of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$. It suffices to show that the conditional law of h given $\eta|_{[0, \tau]}$ is as described in the statement of the proposition for stopping times τ of this form. Indeed, we know that stopping times of this form are dense in $[0, \infty)$ by the proof of Lemma 3.7 and, by Proposition 3.8, we know that $\eta([0, \sigma])$ is a local set for h for every η -stopping time σ , so we can use the continuity result for local sets proved in [5], Proposition 6.5. The statement regarding the conditional law of h restricted to the components which

are surrounded by $\eta([0, \tau])$ follows from [5], Proposition 3.8, by comparing to $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$.

We are now going to describe the boundary behavior for h on $\partial\mathbf{H}_\tau$ using [5], Proposition 3.8, and a construction involving η' and some auxiliary paths. See Figure 8 for an illustration of the setup of the proof. Let τ' be the first time that η' hits X_τ . It follows from the way that we constructed the ordering of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ that the left boundary of $\eta'([0, \tau'])$ is contained in $\eta([0, \tau])$ and is in fact equal to the segment of $\partial\mathbf{H}_\tau$ which connects X_τ to \mathbf{R}_+ (left side of blue path in Figure 8). Suppose that $t \in \mathbf{Q}_+$. On the event $\{t < \tau'\}$, we can use [5], Proposition 3.8, to get that the boundary behavior of h given $\eta|_{[0, \tau]}$ on the segment of $\partial\mathbf{H}_\tau$ which is to the right of X_τ and contained in $\eta'([0, t])$ is as claimed in (iii). This proves the boundary behavior claimed in (iii) because by continuity and because this holds for all $t \in \mathbf{Q}_+$ simultaneously a.s.

For each $s \in \mathbf{Q}_+$, we let $A_s = \eta'([0, s]) \cup \eta_s$ where η_s is the 0-angle flow line of the conditional GFF h given $\eta'|_{[0, s]}$ starting from the leftmost point of $\eta'([0, s]) \cap \mathbf{R}$. Note that η_s reflects off the right boundary of $\eta'([0, s])$. We are now going to establish the boundary behavior claimed in (i) by showing that there a.s. exists $s \in \mathbf{Q}_+$ such that the segment of $\partial\mathbf{H}_\tau$ which is to the left of $\eta(\tau)$ is contained in η_s . This will also give (ii). Indeed, this suffices since we can use [5], Proposition 3.8, to compare the boundary behavior of h given $\eta|_{[0, \tau]}$ to that of h given A_s .

We are now going to show that η_s is equal to the closure C_s of the 0-angle boundaries of the pockets of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ which intersect the right boundary R'_s of $\eta'([0, s])$ (dark green path in Figure 8). We will first show that η_s is (nonstrictly) to the left of C_s . Fix a countable, dense set D in R'_s . If $z \in D$ then [5], Theorem 1.5, implies that η_s is to the left of the 0-angle flow line of h given $\eta'|_{[0, s]}$ starting from z . Since D is countable, this holds for all $z \in D$ simultaneously a.s. Moreover, it is easy to see that $S_1(P)$ for a pocket P of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ which intersects R'_s can be written as a limit of 0-angle flow lines starting from points in D by taking starting points contained $P \cap R'_s$ which get progressively closer to $x(P)$ and which are much closer to $S_1(P)$ than to $S_2(P)$. Indeed, this follows since such a flow line will merge with $S_1(P)$ upon intersecting it by [5], Theorem 1.5. This proves that η_s is (nonstrictly) to the left of C_s . We will next argue that η_s is (nonstrictly) to the right of (and hence equal to) C_s . To see this, we will use the flow line interaction rules [5], Theorem 1.5, to show that an angle-varying flow line with angles contained in $[0, \theta_\rho]$ cannot enter into a pocket formed by η_s and $\eta'([0, s])$. Suppose that we have such a pocket P . An angle-varying flow line with angles contained in $[0, \theta_\rho]$ cannot cross into P through its left boundary (i.e., the part of ∂P formed by η_s) as η_s has a zero angle and such a path necessarily has angle which is at least 0. Similarly, such an angle-varying flow line cannot intersect the right boundary of P (i.e., the part of ∂P which is drawn by η'), including at its opening and closing points, because the height difference between these two paths upon intersecting

would be equal to $(\theta - \theta_\rho)\chi - \pi\chi \leq -\pi\chi$. Here, θ is the angle at which the angle-varying flow line is traveling at the intersection time. This proves the assertion and hence the claim that $\eta_s = C_s$.

Take $s \in \mathbf{Q}_+$ with $s > \tau$ such that $\eta'([0, s])$ has not hit the closing point of the pocket of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ whose opening point is given by X_τ . Note that η visits a pocket P of $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ before time τ if and only if η' visits the interior of P before time τ' . Consequently, it is easy to see that the boundary segments referred to in (i) and (ii) are contained in η_s . This proves the desired result by invoking [5], Proposition 3.8. \square

Now that we have determined by the boundary behavior for the conditional law of h given $\eta|_{[0, \tau]}$ up to any stopping time τ , we can now give the law of η .

LEMMA 3.13. *The law of η is given by that of an $\text{SLE}_\kappa(\rho)$ process in \mathbf{H} from 0 to ∞ where*

$$(3.4) \quad \rho = \frac{\theta_\rho}{\pi} \left(\frac{\kappa}{2} - 2 \right) - 2.$$

PROOF. The martingale characterization of the $\text{SLE}_\kappa(\rho)$ processes given in [5], Theorem 2.4, combined with Lemma 3.12 implies that η evolves as an $\text{SLE}_\kappa(\rho)$ process with the value of ρ determined by θ_ρ as given in (3.4) in those time intervals in which η is not intersecting the past of its range, that is, those times t such that $\eta(t) \notin \eta([0, t])$. For each t , let $Z_t = g_t(X_t)$. This implies that $Z - W$ evolves as $\sqrt{\kappa}$ times a Bessel process of dimension $d(\kappa, \rho) = 1 + \frac{2(\rho+2)}{\kappa}$ during these times. By Lemma 3.7, we know that η has a continuous Loewner driving function W . As it is parameterized by capacity, it follows that the Lebesgue measure of the set of t such that $\eta(t) \in \eta([0, t])$ is equal to 0. Indeed, to see this for each $s \geq 0$ we let $A_s = \{t > s : \eta(t) \in \eta([0, s])\}$, then observe that $\{t : \eta(t) \in \eta([0, t])\} = \bigcup_{s \in \mathbf{Q}_+} A_s$, and then finally note that each A_s for $s \in \mathbf{Q}_+$ a.s. has zero Lebesgue measure by mapping back and applying [5], Lemma 2.5. Here, we are using that it is a.s. the case that for each $s \in \mathbf{Q}_+$ we have that $g_s(\eta(s + \cdot))$ is a continuous curve. To see this, fix $s \in \mathbf{Q}_+$ and note that Lemma 3.12 implies that $g_s(\eta(s + \cdot))$ evolves as an $\text{SLE}_\kappa(\rho)$ process up until the first time σ that it collides with its force point and is therefore continuous in $[0, \sigma]$ by [5], Theorem 1.3. Another application of Lemma 3.12 implies that $g_{s+\sigma}(\eta(s + \sigma + \cdot))$ has the same law as η (up to a horizontal translation), hence is continuous. Since $g_s(\eta(s + \cdot))|_{[0, \sigma]}$ is continuous, it follows that the conformal transformation $g_s \circ g_{s+\sigma}^{-1}$ from \mathbf{H} to the unbounded component of $\mathbf{H} \setminus g_s(\eta(s + [0, \sigma]))$ extends continuously to the boundary. Hence $g_s(\eta(s + \sigma + \cdot))$ is continuous as the continuous image of a continuous curve. Therefore, $g_s(\eta(s + \cdot))$ is continuous as a concatenation of continuous curves.

Since $Z_t - W_t = 0$ implies that $\eta(t) \in \eta([0, t])$, it therefore follows that the set of t such that $Z_t - W_t = 0$ has Lebesgue measure zero. That is, $Z - W$ is

instantaneously reflecting at 0. To see that $Z - W$ is continuous, let B be a standard complex Brownian motion independent of everything else, let \mathbf{P}^z be the law under which $B_0 = z$ (with everything else fixed), and for each $t > 0$ let $\tau(t)$ be the first time that B leaves $\mathbf{H} \setminus \eta([0, t])$. Also let $L(t)$ (resp., $R(t)$) be the left (resp., right) side of the range of η between when it has hit X_t and $\eta(t)$. Then we have that $Z_t - W_t = \lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy}[B_{\tau(t)} \in R(t)]$. Fix $T > 0$ and suppose that $0 \leq a \leq s < t \leq b \leq T$ are such that $Z_r - W_r > 0$ for all $r \in [a, b]$. Then we have

$$\begin{aligned} & |(Z_t - W_t) - (Z_s - W_s)| \\ &= \lim_{y \rightarrow \infty} \pi y |\mathbf{P}^{iy}[B_{\tau(t)} \in R(t)] - \mathbf{P}^{iy}[B_{\tau(s)} \in R(s)]| \\ &= \lim_{y \rightarrow \infty} \pi y |\mathbf{P}^{iy}[B_{\tau(t)} \in R(t)] \\ &\quad - \mathbf{P}^{iy}[B_{\tau(s)} \in R(s), B_{\tau(t)} \in (R(t) \cup (L(t) \setminus L(s)))]| \\ &\leq \lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy}[B_{\tau(t)} \in R(t), B_{\tau(s)} \notin R(s)] + \lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy}[B_{\tau(t)} \in L(t) \setminus L(s)] \\ &\leq \lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy}[B_{\tau(t)} \in R(t) \setminus R(s)] + \lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy}[B_{\tau(t)} \in L(t) \setminus L(s)] \\ &= \lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy}[B_{\tau(t)} \in \eta([s, t])]. \end{aligned}$$

Note that the probability that B starting from iy (with y very large) hits $\partial B(0, 2 \text{diam}(\eta([0, T])))$ before hitting $\partial \mathbf{H}$ is at most a constant times $\text{diam}(\eta([0, T]))/y$. Given that this happens, the Beurling estimate implies that the probability that B hits $\eta([s, t])$ before time $\tau(t)$ is at most a constant (depending on $\text{diam}(\eta([0, T]))$) times $\text{diam}(\eta([s, t]))^{1/2}$. Combining, we see that there exists a constant $c > 0$ depending only on $\text{diam}(\eta([0, T]))$ such that $\lim_{y \rightarrow \infty} \pi y \mathbf{P}^{iy}[B_{\tau(t)} \in \eta([s, t])] \leq c \text{diam}(\eta([s, t]))^{1/2}$. This bounds the modulus of continuity of $Z - W$ in $[a, b]$ as we know that η is continuous. If $0 \leq s < t \leq T$ and $Z - W$ hits 0 in (s, t) , we can reduce to the setting considered above as follows. Let q (resp., r) be the first after (resp., largest time before) s (resp., t) that $Z - W$ is equal to 0. As $Z_q - W_q = Z_r - W_r = 0$, we have that

$$\begin{aligned} & |(Z_t - W_t) - (Z_s - W_s)| \\ &\leq |(Z_t - W_t) - (Z_r - W_r)| + |(Z_q - W_q) - (Z_s - W_s)| \\ &\leq c(\text{diam}(\eta([r, t]))^{1/2} + \text{diam}(\eta([s, q]))^{1/2}) \\ &\leq 2c \text{diam}(\eta([s, t]))^{1/2}. \end{aligned}$$

Therefore, $Z - W$ is continuous. Combining, we have that $Z - W$ evolves as $\sqrt{\kappa}$ times a Bessel process of dimension $d(\kappa, \rho)$ for all $t \geq 0$.

We will now complete the proof by checking that the conditions of Proposition 2.1 hold for the pair $(Z - W, Z)$. We first note that $Z = (Z - W) + W$ is

continuous as we have written it as a sum of continuous processes (i.e., the sum of the Bessel process $Z - W$ and W). We next note that the GFF instance h is scale invariant (as the GFF is conformally invariant). Moreover, we know that η is a.s. determined by h since its range is a.s. determined by h (as each of the flow lines which makes up its range is a.s. determined by h) and the order in which η hits the points of its range is a.s. determined by the field (as the counterflow line which determines this ordering is a.s. determined by h). Consequently, it follows that η is also scale-invariant. That condition (i) holds for $(Z - W, Z)$ follows since capacity scales as the square of the diameter. Condition (ii) holds since $Z_t = g_t(X_t)$ evolves according to the Loewner flow for those t at which $Z_t - W_t \neq 0$. Finally, condition (iii) holds as a consequence of Lemma 3.12 and that η is a.s. determined by h . \square

PROOF OF THEOREM 1.1 AND THEOREM 1.2. By Lemma 3.13, we know that η is an $SLE_\kappa(\rho)$ process with the desired value of ρ and by Lemma 3.12 we know that η is coupled with the field as described in Theorem 1.2. Finally, that η is a.s. determined by the field is explained in the proof of Lemma 3.13. \square

Now that we have proved Theorem 1.1 and Theorem 1.2, it is left to prove Theorem 1.3. The result does not immediately follow from Proposition 3.10 because that result describes what happens to the light cone path when we change the angles of the light cone but leave the GFF fixed. In the present setting, we are changing the angles of the light cone *and* the boundary data of the GFF.

PROOF OF THEOREM 1.3. See Figure 9 for an illustration of the argument. We are going to extract the result in two steps by first applying Proposition 3.10 and then using a conditioning argument. (This is similar in spirit to our proof of the continuity of the $SLE_\kappa(\rho)$ processes for $\rho > -2$ given in [5].) Let $\Psi : \mathbf{H} \rightarrow \mathbf{D}$ be a conformal transformation with $\Psi(0) = -i$ and $\Psi(\infty) = i$. Fix $\rho \in [\frac{\kappa}{2} - 4, -2)$ with $\rho > -2 - \frac{\kappa}{2}$ and suppose that h is a GFF on \mathbf{H} with boundary conditions which are given by $-\lambda$ on \mathbf{R}_- and $\lambda(1 + \rho)$ on \mathbf{R}_+ . Then h is compatible with a coupling with an $SLE_\kappa(\rho)$ process η from 0 to ∞ as in Theorem 1.2. Moreover, η is equal to the light cone exploration path associated with $\mathbf{L}_{\mathbf{R}_-}(0, \theta_\rho)$ where $\theta_\rho = \frac{\pi(\rho+2)}{\kappa/2-2}$. For each $\theta \geq \theta_\rho$, let η^θ be the light cone exploration path associated with $\mathbf{L}_{\mathbf{R}_-}(0, \theta)$. By Proposition 3.10, we know that $\Psi(\eta^\theta) \rightarrow \Psi(\eta)$ locally uniformly (modulo reparameterization) as $\theta \downarrow \theta_\rho$. For each $\theta \geq \theta_\rho$, we let η_θ be the flow line of h with angle θ starting from 0. (For $\theta = \theta_\rho$, we take η_θ to be equal to \mathbf{R}_+ .) Then we know that η_θ a.s. converges to η_{θ_ρ} locally uniformly as $\theta \downarrow \theta_\rho$. Let φ_θ be the conformal transformation which takes the component \mathbf{H}_θ of $\mathbf{H} \setminus \eta_\theta$ which is to the left of η_θ to \mathbf{H} fixing 0, -1 and ∞ . Then $\Psi \circ \varphi_\theta \circ \Psi^{-1}$ converges uniformly to the identity on \mathbf{D} a.s. as $\theta \downarrow \theta_\rho$. Note that the boundary conditions for the GFF $h_\theta = h \circ \varphi_\theta^{-1} - \chi \arg(\varphi_\theta^{-1})'$ are given by $-\lambda$ on \mathbf{R}_- and by $-\lambda - \theta\chi$ on \mathbf{R}_+ . Since $\varphi_\theta(\eta^\theta)$ is the light cone path associated with the light cone with angle range $[0, \theta]$

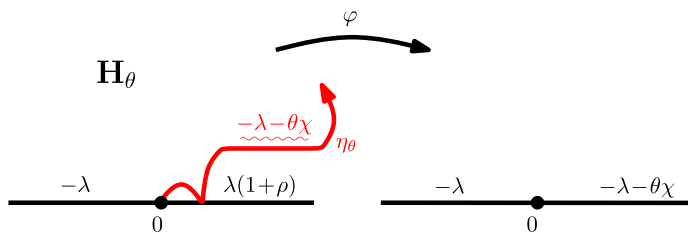


FIG. 9. Illustration of the idea of the proof of Theorem 1.3. Suppose that h is a GFF on \mathbf{H} with the illustrated boundary data. Then h is compatible with a coupling with an SLE $_{\kappa}(\rho)$ process η starting from 0. Fix $\theta > 0$ and let η_{θ} be the flow line of h starting from 0 with angle θ and let \mathbf{H}_{θ} be the component of $\mathbf{H} \setminus \eta_{\theta}$ which is to the left of η_{θ} . With $\varphi: \mathbf{H}_{\theta} \rightarrow \mathbf{H}$ a conformal transformation which fixes 0 and ∞ , $h \circ \varphi^{-1} - \chi \arg(\varphi^{-1})'$ is a GFF on \mathbf{H} with the boundary data shown on the right. Since the law of η_{θ} is continuous in θ and the light cone exploration path is continuous in its angles, we get the desired interpolation result for SLE $_{\kappa}(\rho)$.

of h_{θ} , we know that $\varphi_{\theta}(\eta^{\theta})$ is an SLE $_{\kappa}(\rho_{\theta})$ process where $\rho_{\theta} = \frac{\theta}{\pi}(\frac{\kappa}{2} - 2) - 2$. The desired result follows since combining everything implies that $\Psi(\varphi_{\theta}(\eta^{\theta})) \rightarrow \Psi(\eta)$ a.s. as $\theta \downarrow \theta_{\rho}$. The continuity when $\theta \uparrow \theta_{\rho}$ is proved similarly. \square

4. Behavior at the boundary of the light cone regime. We are now going to describe the behavior of SLE $_{\kappa}(\rho)$ at the threshold $\rho = \frac{\kappa}{2} - 4$ which lies between the light cone and trunk regimes. When $\rho = \frac{\kappa}{2} - 4$, the opening angle for the light cone is equal to π . Note that $\pi < \theta_c$ if and only if $\kappa \in (2, 4)$. As we mentioned earlier, this is closely connected with the fact that an SLE $_{\kappa'}$ process is space-filling if and only if $\kappa' \geq 8$. In analogy with [5], Theorem 1.4, in this case, the range of the path is equal to that of a form of an SLE $_{\kappa'}$ process as stated in the following proposition.

PROPOSITION 4.1. *Suppose that $\kappa \in (2, 4)$ (so that $\pi < \theta_c$) and let η be an SLE $_{\kappa}(\frac{\kappa}{2} - 4)$ process in \mathbf{H} from 0 to ∞ with a single force point located at 0^+ . Then the range of η is equal in law to that of an SLE $_{\kappa'}(\frac{\kappa'}{2} - 4)$ process η' in \mathbf{H} from 0 to ∞ where the force point is located at 0^- .*

REMARK 4.2. We emphasize that the statement of Proposition 4.1 is that the law of the range of η is equal to the law of the range of η' . As explained in Figure 10 and Figure 11, the order in which the paths visit the points in their common range is different.

PROOF OF PROPOSITION 4.1. Suppose that h is a GFF on \mathbf{H} with boundary data given by $-\lambda$ (resp., $-\lambda - \pi\chi$) on \mathbf{R}_- (resp., \mathbf{R}_+) and let η be the SLE $_{\kappa}(\frac{\kappa}{2} - 4)$ process coupled with h as the light cone path from 0 to ∞ as in Theorem 1.2. Note that

$$-\lambda = -\lambda' - \frac{\pi\chi}{2} \quad \text{and} \quad -\lambda - \pi\chi = -\lambda' - \frac{3\pi\chi}{2}.$$

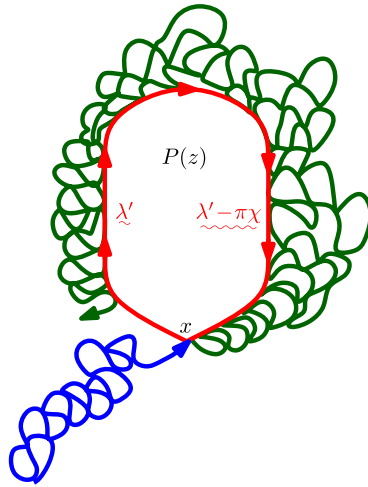


FIG. 10. Illustration of the order in which an $SLE_{\kappa}(\frac{\kappa}{2} - 4)$ process η visits the points in its range. Shown is a pocket $P(z)$ of η with opening point x and a clockwise orientation. Note that $\partial P(z)$ is given by a 0-angle flow line loop starting from x . The blue path indicates η up until hitting x . Upon hitting x , η immediately traces $\partial P(z)$ in the clockwise direction. The green path indicates the range of η after it finishes drawing $\partial P(z)$. This part of the path will crawl along $\partial P(z)$ in the counterclockwise direction. In contrast, the $SLE_{\kappa'}(\frac{\kappa'}{2} - 4)$ counterflow line η' whose range is equal to η (see Proposition 4.1) will draw $\partial P(z)$ in the opposite (counterclockwise) direction and, while doing so, visits the pockets in its range which intersect $\partial P(z)$. (We emphasize that $\partial P(z)$ is given by $\eta_1(z)$ and only item 3 from the description of the interaction between η and η' given just before Lemma 3.3 η' occurs.)

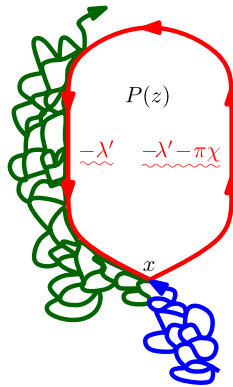


FIG. 11. (Continuation of Figure 10.) Shown is a pocket $P(z)$ of η with opening point which has a counterclockwise orientation. Note that $\partial P(z)$ is given by a π -angle flow line loop starting from x . The blue path segment indicates the part of η up until it hits x and the green path segment indicates part of η as it draws $\partial P(z)$. In contrast to the case of a clockwise loop, as considered Figure 10, η visits the points on $\partial P(z)$ in the same order as η' . Moreover, as it does so, it draws the boundaries of the pockets which intersect $\partial P(z)$. (We emphasize that $\partial P(z)$ is given by $\eta_2(z)$ and only item 1 from the description of the interaction between η and η' given just before Lemma 3.3 occurs.)

Let η' be the counterflow line of $h + 3\pi\chi/2$ starting from 0. Then η' is an $SLE_{\kappa'}(\frac{\kappa'}{2} - 4)$ process where the force point is located at 0^- . By [5], Theorem 1.4, we have that the range of η' stopped upon hitting any point in $x \in \mathbf{R}_-$ (recall that it fills \mathbf{R}_-) is equal to the set of points accessible by angle-varying flow lines with angles in $[0, \pi]$ starting from x . Since this holds for every $x \in \mathbf{R}_-$, it therefore follows that the range of η' is equal to the range of η . \square

We finish by recording two immediate consequences of Proposition 4.1.

COROLLARY 4.3. *Suppose that $\kappa \in (2, 4)$ and let η be an $SLE_{\kappa}(\frac{\kappa}{2} - 4)$ process in \mathbf{H} from 0 to ∞ with a single boundary force point located at 0^+ . Then \mathbf{R}_- is a.s. contained in the range of η .*

PROOF. This follows from Proposition 4.1 and the fact that $\frac{\kappa'}{2} - 4$ is the critical value of ρ at or below which a counterflow line is boundary filling. In particular, with η' as in the statement of Proposition 4.1, we have that \mathbf{R}_- is contained in the range of η' . \square

COROLLARY 4.4. *Suppose that $\kappa \in (2, 4)$ and let η be an $SLE_{\kappa}(\frac{\kappa}{2} - 4)$ process in \mathbf{H} from 0 to ∞ with a single boundary force point located at 0^+ coupled with a GFF h on \mathbf{H} with boundary data equal to $-\lambda$ (resp., $-\lambda - \pi\chi$) on \mathbf{R}_- (resp., \mathbf{R}_+). If η separates z from $\partial\mathbf{H}$, then $\partial P(z)$ is equal to the flow line of h with angle 0 (resp., π) starting from $x(z)$ if η traverses $\partial P(z)$ with a clockwise (resp., counterclockwise) orientation. In particular, the boundaries of the pockets of η have only one side.*

PROOF. This follows from Proposition 4.1 since the same is true for the counterflow line η' (see, e.g., [8], Theorem 1.13). \square

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