DYNKIN ISOMORPHISM AND MERMIN–WAGNER THEOREMS FOR HYPERBOLIC SIGMA MODELS AND RECURRENCE OF THE TWO-DIMENSIONAL VERTEX-REINFORCED JUMP PROCESS

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We prove the vertex-reinforced jump process (VRJP) is recurrent in two dimensions for any translation invariant finite-range initial rates. Our proof has two main ingredients. The first is a direct connection between the VRJP and sigma models whose target space is a hyperbolic space $\mathbb{H}^n$ or its super-symmetric counterpart $\mathbb{H}^{2|2}$. These results are analogues of well-known relations between the Gaussian free field and the local times of simple random walk. The second ingredient is a Mermin–Wagner theorem for these sigma models. This result is of intrinsic interest for the sigma models and also implies our main theorem on the VRJP. Surprisingly, our Mermin–Wagner theorem applies even though the symmetry groups of $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$ are nonamenable.

1. Introduction and results.

1.1. *Introduction.* Our results have motivation from two different perspectives, that of sigma models with hyperbolic symmetry and their relevance for the Anderson transition, and that of a model of reinforced random walks known as the *vertex-reinforced jump process* (VRJP).

The VRJP was originally introduced by Werner and has attracted a great deal of attention recently [6–8, 23, 24]. The VRJP on a vertex set Λ is a continuous-time random walk that jumps from a vertex i to a neighbouring vertex j at time t with rate $\beta_{ij}(1 + L_t^j)$, where L_t^j is the local time of j at time t and $\beta_{ij} \geq 0$ are the initial rates. One should view Λ as the vertex set of an undirected graph with edge set $E = \{\langle ij \rangle \mid \beta_{ij} > 0\}$. The dependence of the jump rates on the local time leads the VRJP to be attracted to itself.

One of our new results is the following theorem.

THEOREM 1.1. *Consider a vertex-reinforced jump process (X_t) on the vertex set $\mathbb{Z}^d$ with initial rates β that are finite-range and translation invariant. If $d = 1, 2$*

then (X_t) is recurrent in the sense that the expected time (X_t) spends at the origin is infinite.

As the VRJP is not a Markov process, different notions of recurrence are not a priori equivalent. For example, another natural notion of recurrence would be to ask if the VRJP visits the origin infinitely often almost surely. For non-Markovian processes, neither of these definitions of recurrence implies the other: there may be infinitely many visits to the origin with the increments of the local time being summable. To the best of our knowledge, neither implication is known for the VRJP.

For sufficiently small initial rates, recurrence results for the VRJP have previously been established [1, 9, 23]. These results are for recurrence in the sense of visiting the origin infinitely often almost surely. See [1] for a discussion and precise statements. It has also been shown that the linearly edge-reinforced random walk (ERRW) with constant initial weights is recurrent in two dimensions [19, 24], but the recurrence of the VRJP for all initial rates was an open problem until the present work. The relation between the ERRW and VRJP is discussed below.

Theorem 1.1 is in fact a consequence of our proof of a Mermin–Wagner theorem for hyperbolic sigma models and a new and very direct relation between VRJPs and hyperbolic sigma models that parallels the well-known relationship between simple random walks and Gaussian free fields (the BFS–Dynkin isomorphism theorem).

Before giving precise definitions of our models and stating our results, we briefly indicate the motivations behind hyperbolic sigma models, and their relations with reinforced random walks. We also explain some consequences of our results for hyperbolic sigma models. Readers primarily interested in the VRJP may wish to skip ahead to Section 1.2.

Hyperbolic sigma models were introduced as effective models to understand the Anderson transition [10, 27–29, 32]. In Efetov's supersymmetric method [13], the expected absolute value squared of the resolvent of random band matrices, that is, $\mathbb{E}|(H - z)^{-1}(i, j)|^2$ where $z \in \mathbb{C}_+$ and H is a random band matrix, can be expressed as a correlation function of a supersymmetric spin model. The spins of this model are invariant under the hyperbolic symmetry $O\mathrm{Sp}(2, 1|2)$. Extended states correspond to spontaneous breaking of this noncompact symmetry. The supersymmetric hyperbolic sigma model, or $\mathbb{H}^{2|2}$ model, was introduced by Zirnbauer [32] and first studied by Disertori, Spencer and Zirnbauer [10]. It is an approximation of the random band matrix model above where radial fluctuations are neglected. This is similar to how the $O(n)$ model is an approximation of models of $\mathbb{R}^n$-valued spins with rotational symmetry such as $|\varphi|^4$-theories. More detailed motivation for hyperbolic spin models is given in [27, 29].

The $\mathbb{H}^{2|2}$ model is believed to capture the physics of the Anderson transition. As is expected for the Anderson model, it was proved in [10] that the $O\mathrm{Sp}(2, 1|2)$ symmetry of the $\mathbb{H}^{2|2}$ model is spontaneously broken in $d \geq 3$ for sufficiently

small disorder—consistent with the existence of extended states. Furthermore, it was proved [9] that for sufficiently large disorder this is not the case—consistent with Anderson localisation. In dimension $d \leq 2$, it is conjectured that extended states do not exist for any disorder strength. Equation (16) below is the corresponding statement for the $\mathbb{H}^{2|2}$ model, and we have thus completed the expected qualitative picture for the phase diagram of the $\mathbb{H}^{2|2}$ model; see Remark 1.9 for a discussion of the conjectured optimal bounds. Equation (16) can be considered as a version of the Mermin–Wagner theorem. For recent and extremely precise results in dimension one, see [26].

Based on the similarity of certain explicit formulas, it was suggested that there is a connection between the $\mathbb{H}^{2|2}$ model and linearly edge-reinforced random walks [10]. This connection was first confirmed in [23] by relating marginals of the $\mathbb{H}^{2|2}$ model to the limiting local time profile of a time change of the VRJP. It was also shown there that the linearly edge reinforced walk is obtained from the VRJP when averaging over random initial rates. Further marginals of the $\mathbb{H}^{2|2}$ model were explored in [7]. For a discussion of the history of the VRJP, see [23].

Our hyperbolic analogue of the BFS–Dynkin isomorphism theorem, Theorem 1.2 below, is a different relation between the $\mathbb{H}^{2|2}$ model and the VRJP than was found in [23], and it provides a more direct relation between the correlation structures of the models. Moreover, our statement also applies without supersymmetry, that is, when the spins take values in $\mathbb{H}^n$. We will explain further extensions of Theorem 1.2 in the case of $\mathbb{H}^n$, for example, to multipoint correlations, in a forthcoming publication.

1.2. *Model definitions.* We now define the VRJP and the hyperbolic sigma models. The walk and the sigma models are both defined in terms of a set Λ of vertices and nonnegative edge weights $\beta = (\beta_{ij})_{i,j \in \Lambda}$, where by edge weights we mean that $\beta_{ij} = \beta_{ji}$. For our Mermin–Wagner theorem, we will make use of two assumptions on β. We call β *finite-range* if for each $i \in \Lambda$ we have $\beta_{ij} = 0$ for all but finitely many j. If $\Lambda = \mathbb{Z}^d$ we call β *translation invariant* if $\beta_{ij} = \beta_{T(i)T(j)}$ for all translations T of $\mathbb{Z}^d$.

1.2.1. *Vertex-reinforced jump process.* Let Λ be a finite or countable set. The VRJP is a history-dependent continuous-time random walk (X_t) on Λ that takes jumps from vertex i to vertex j with rate $\beta_{ij}(1 + L_t^j)$, where

$$(1) \qquad L_t^j \equiv \int_0^t 1_{X_s = j} \, ds.$$

L_t^j is called the *local time* of the walk at vertex j up to time t. We will write $L_t \equiv (L_t^i)_{i \in \Lambda}$ for the collection of local times. It will also be useful to consider the joint process (X_t, L_t), which is a Markov process with generator $\mathcal{L}$ acting on

sufficiently nice functions $g \colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ by

$$(2) \quad \mathcal{L}^\beta g(i, \ell) = \sum_j \beta_{ij}(1 + \ell_j)\big(g(j, \ell) - g(i, \ell)\big) + \frac{\partial}{\partial \ell_i} g(i, \ell), \quad i \in \Lambda, \ell \in \mathbb{R}^\Lambda.$$

We denote by $\mathbb{E}_{i,\ell}^\beta$ the expectation of the process (X_t, L_t) with initial condition $X_0 = i$ and $L_0 = \ell$. The VRJP is the marginal of X_t in the special case $L_0 = 0$; by a slight abuse of terminology we call (X_t, L_t) the VRJP as well.

1.2.2. *Hyperbolic sigma models.* Let $\mathbb{R}^{n,1}$ denote $(n + 1)$-dimensional Minkowski space. Its elements are vectors $u = (x, y^1, \ldots, y^{n-1}, z)$, and it is equipped with the indefinite inner product $u \cdot u = x^2 + (y^1)^2 + \cdots + (y^{n-1})^2 - z^2$. Note that although x plays the same role as the y^i, we distinguish it in our notation for later convenience. Recall that n-dimensional hyperbolic space $\mathbb{H}^n$ can be realized as

$$(3) \quad \mathbb{H}^n \equiv \big\{ u \in \mathbb{R}^{n,1} \mid u \cdot u = -1, z > 0 \big\}.$$

Suppose Λ is finite and $h > 0$. To each vertex $i \in \Lambda$ we associate a spin $u_i \in \mathbb{H}^n$. The energy of a spin configuration $u = (u_i)_{i \in \Lambda} \in (\mathbb{H}^n)^\Lambda$ is

$$(4) \quad H(u) = H_{\beta,h}(u) \equiv \sum_{\langle ij \rangle} \beta_{ij}(-u_i \cdot u_j - 1) + h \sum_j (z_j - 1),$$

where the sum is over edges $\langle ij \rangle$; since the summands are symmetric in i and j this notation will not cause any confusion. The $\mathbb{H}^n$ sigma model is the measure with density proportional to $e^{-H(u)}$ with respect to the $|\Lambda|$-fold product of the measure μ on $\mathbb{H}^n$ induced by the Minkowski metric (see (23) and (25) for explicit expressions), and we let $\langle \cdot \rangle_{\mathbb{H}^n}$ denote the expectation associated to this model:

$$(5) \quad \langle F(u) \rangle_{\mathbb{H}^n} \equiv \frac{\int_{(\mathbb{H}^n)^\Lambda} F(u) e^{-H(u)} \mu^{\otimes \Lambda}(du)}{\int_{(\mathbb{H}^n)^\Lambda} e^{-H(u)} \mu^{\otimes \Lambda}(du)}.$$

The energy (4) favours spin alignment because $u \cdot v \leq -1$ for $u, v \in \mathbb{H}^n$ with equality if and only if $u = v$.

1.2.3. *Supersymmetric hyperbolic sigma model.* In this section, we will introduce a probability measure which enables the computation of a special class of observables of the full supersymmetric $\mathbb{H}^{2|2}$ model. These restricted observables will suffice for a description of a special, but interesting, case of our results. Our most general results use the full supersymmetric formalism.

As will be explained further in Section 2, at each vertex $i \in \Lambda$ there is a super-spin $u_i = (x_i, y_i, z_i, \xi_i, \eta_i) \in \mathbb{H}^{2|2}$ where ξ_i and η_i are Grassmann variables. For the moment, all that is needed is that the expectation of a function $F(y)$ of the $y \equiv (y_i)_{i \in \Lambda}$ coordinates can be written as

$$(6) \quad \langle F(y) \rangle_{\mathbb{H}^{2|2}} = \int_{(\mathbb{R}^2)^\Lambda} F(e^t s) e^{-\tilde{H}(s,t)} \, dt \, ds,$$

where $dt\,ds \equiv \prod_i dt_i\,ds_i$, $e^t s \equiv (e^{t_i} s_i)_{i\in\Lambda}$,

$$\widetilde{H}(s,t) = \widetilde{H}_{\beta,h}(s,t) \equiv \sum_{\langle ij\rangle} \beta_{ij}\left(\cosh(t_i - t_j) - 1 + \frac{1}{2}(s_i - s_j)^2 e^{t_i+t_j}\right)$$

(7)
$$+ h\sum_i \left(\cosh(t_i) - 1 + \frac{1}{2}s_i^2 e^{t_i}\right)$$

$$+ \sum_i (t_i + \log(2\pi)) - \log\det D_{\beta,h}(t),$$

and the matrix $D_{\beta,h}(t)$ on $\mathbb{R}^\Lambda$ is defined by the quadratic form

$$(8)\qquad (v, D_{\beta,h}(t)v) \equiv \sum_{\langle ij\rangle} \beta_{ij} e^{t_i+t_j}(v_i - v_j)^2 + h\sum_i e^{t_i} v_i^2, \quad v\in\mathbb{R}^\Lambda.$$

The determinant $\det D_{\beta,h}(t)$ does not depend on the s variables and it is positive since $D_{\beta,h}(t)$ is positive definite. Thus $e^{-\widetilde{H}(s,t)}\,dt\,ds$ is a positive measure, and we will show in Section 2 that it is in fact a probability measure, that is, $\langle 1\rangle_{\mathbb{H}^{2|2}} = 1$.

1.3. *Results.* We now state our main results and show how Theorem 1.1 is a consequence.

1.3.1. *Hyperbolic BFS–Dynkin isomorphism.* The following theorem is a hyperbolic analogue of the Dynkin isomorphism theorem, which relates the local times of a simple random walk to the square of a Gaussian free field. As the Dynkin isomorphism theorem was proved by Brydges–Fröhlich–Spencer in [4], Theorem 2.2, and later expressed in a better form by Dynkin [12], we prefer to call it the BFS–Dynkin isomorphism. The general idea of relating Gaussian fields to simple random walks is due to Symanzik [30]. For recent discussions of these ideas, see [16, 31]. Supersymmetric versions of these results for simple random walks go back to Luttinger and Le Jan [15, 17].

Note that while we have not yet defined the meaning of $\langle g\rangle_{\mathbb{H}^{2|2}}$ for a general function g, we have given a meaning in the case that g is identically one by (6). It is this case of g identically one that will be most relevant for the VRJP.

THEOREM 1.2. *Suppose Λ is finite and β is a collection of nonnegative edge weights. Let $h > 0$, let $g\colon \Lambda\times\mathbb{R}^\Lambda \to \mathbb{R}$ be any bounded smooth function, and let $a, b\in\Lambda$. Consider the $\mathbb{H}^n$ model, $n\geq 2$, let $y = (y_i)_{i\in\Lambda} = (y_i^r)_{i\in\Lambda}$ for some $r = 1,\ldots,n-1$, and $z = (z_i)_{i\in\Lambda}$. Then*

$$(9)\qquad \sum_b \langle y_a y_b g(b, z-1)\rangle_{\mathbb{H}^n} = \left\langle z_a \int_0^\infty \mathbb{E}_{a,z-1}^\beta(g(X_t, L_t))e^{-ht}\,dt\right\rangle_{\mathbb{H}^n}.$$

For the $\mathbb{H}^{2|2}$ model, we have

$$(10)\qquad \sum_b \langle y_a y_b g(b, z-1)\rangle_{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{E}_{a,0}^\beta(g(X_t, L_t))e^{-ht}\,dt.$$

REMARK 1.3. Theorem 1.2 also holds for the $\mathbb{H}^1$ model, but as the proof requires slightly different considerations we have not included it here.

Taking the function g to be identically one in (10) implies that

$$(11) \qquad \langle y_a y_b \rangle_{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{E}_{a,0}^\beta (1_{X_t=b}) e^{-ht} \, dt.$$

The right-hand side can be interpreted as the two-point function of the VRJP with a uniform killing rate h.

REMARK 1.4. Theorem 1.2 can be extended in a straightforward way to the case in which $h = (h_i)_{i \in \Lambda}$ is nonconstant, provided $h_i \geq 0$ and at least one value is strictly positive.

1.3.2. *Hyperbolic Mermin–Wagner theorem.* In this section, we assume that $\Lambda = \Lambda_L$ is the discrete d-dimensional torus $\mathbb{Z}^d/(L\mathbb{Z})^d$ of side length $L \in \mathbb{N}$, and that β is translation invariant and finite-range. We will write $\langle \cdot \rangle = \langle \cdot \rangle_{\beta,h}$ in place of $\langle \cdot \rangle_{\mathbb{H}^n}$ and $\langle \cdot \rangle_{\mathbb{H}^{2|2}}$. Denote

$$(12) \qquad \lambda(p) \equiv \sum_{j \in \Lambda} \beta_{0j}(1 - \cos(p \cdot j)), \quad p \in \Lambda^\star,$$

where here $\cdot$ is the Euclidean inner product on $\mathbb{R}^d$ and $\Lambda^\star$ is the Fourier dual of the discrete torus Λ. Denote the two-point function and its Fourier transform by

$$G_{\beta,h}(j) = G_{\beta,h}^L(j) \equiv \langle y_0 y_j \rangle_{\beta,h},$$
$$(13)$$
$$\hat{G}_{\beta,h}(p) = \hat{G}_{\beta,h}^L(p) = \sum_{j \in \Lambda} G_{\beta,h}(j) e^{i(p \cdot j)}.$$

The following theorem is an analogue of the Mermin–Wagner theorem for the $O(n)$ model, in the form presented in [14].

THEOREM 1.5. *Let $\Lambda = \mathbb{Z}^d/(L\mathbb{Z})^d$, $L \in \mathbb{N}$. For the $\mathbb{H}^n$ model, $n \geq 2$, with magnetic field $h > 0$,*

$$(14) \qquad \hat{G}_{\beta,h}(p) \geq \frac{1}{(1 + (n+1)G_{\beta,h}(0))\lambda(p) + h}.$$

Similarly, for the $\mathbb{H}^{2|2}$ model with $h > 0$,

$$(15) \qquad \hat{G}_{\beta,h}(p) \geq \frac{1}{(1 + G_{\beta,h}(0))\lambda(p) + h}.$$

REMARK 1.6. By (11) the two-point function $G_{\beta,h}$ equals that of the VRJP in the case of the $\mathbb{H}^{2|2}$ model, and hence the two-point function of the VRJP satisfies (15) as well.

REMARK 1.7. For $d \geq 3$, the bound (15) shows that $\tilde{f}$ can be replaced by f in [10], Theorem 3, using the upper bound proved there for $G_{\beta,h}(0)$.

COROLLARY 1.8. *Under the assumptions of Theorem* 1.5, *for* $d = 1, 2$,

$$(16) \qquad \lim_{h \downarrow 0} \lim_{L \to \infty} G_{\beta,h}(0) = \infty.$$

PROOF. Since $(2\pi L)^{-d} \sum_{p \in \Lambda^*} e^{i(p \cdot j)} = 1_{j=0}$, summing the bounds (14) and (15) over $p \in \Lambda^\star$ and interchanging sums implies (with $n = 0$ for $\mathbb{H}^{2|2}$)

$$(17) \qquad G_{\beta,h}(0) \geq \frac{1}{(2\pi L)^d} \sum_{p \in \Lambda^\star} \frac{1}{(1 + (n+1)G_{\beta,h}(0))\lambda(p) + h}.$$

The assumption of β being finite-range and nonnegative implies $\lambda(p) \leq C(\beta)|p|^2$. If $d \leq 2$ it follows that

$$(18) \qquad \lim_{L \to \infty} \frac{1}{(2\pi L)^d} \sum_{p \in \Lambda^\star} \frac{1}{\lambda(p) + h} \uparrow \infty \quad \text{as } h \downarrow 0,$$

and, as $G_{\beta,h} \geq 0$, this implies (16). $\square$

REMARK 1.9. In fact, the proof shows $G_{\beta,h}(0) \geq c_\beta / \sqrt{\log h}$ with $c_\beta > 0$ when $h > 0$ is small. For the $\mathbb{H}^{2|2}$ model, we conjecture that the optimal bound is $G_{\beta,h}(0) \asymp c_\beta / h$ for h small, with $c_\beta > 0$ exponentially small as β becomes large. This is consistent with Anderson localisation. On the other hand, for the $\mathbb{H}^n$ model with $n \geq 2$, localisation is not expected, that is, $G_{\beta,h}(0) \ll 1/h$.

1.3.3. *Consequences for the vertex-reinforced jump process.* In contrast to Corollary 1.8, it has been proven [10, 29] that when $d \geq 3$ and $\beta_{ij} = \beta 1_{|i-j|=1}$,

$$(19) \qquad \lim_{h \downarrow 0} \lim_{L \to \infty} G_{\beta,h}(0) < \infty$$

for all $\beta > 0$ in the case of $\mathbb{H}^2$ and for all sufficiently large $\beta > 0$ for $\mathbb{H}^{2|2}$. In the $\mathbb{H}^{2|2}$ case, (19) corresponds to transience of the VRJP (in the sense of bounded expected local time, see Corollary 1.10 below) and to the uniform boundedness (in the spectral parameter $z \in \mathbb{C}_+$) of the expected square of the absolute value of the resolvent for random band matrices in the sigma model approximation [27] (recall Section 1.1). It also implies that the hyperbolic symmetry is spontaneously broken.

Due to the nonamenability of hyperbolic group actions, the question of spontaneous symmetry breaking for hyperbolic sigma models is, in general, subtle. The usual formulations of the Mermin–Wagner theorem for models with compact symmetries cannot hold in the nonamenable case [25], and, in fact, spontaneous symmetry breaking appears to occur in all dimensions [11, 22]. Nonetheless, (16) and (19) show that the two-point function—the observable of interest for the VRJP

and the random matrix problem—does undergo a transition analogous to that occurring in systems with compact symmetries.

PROOF OF THEOREM 1.1. We must prove that for any translation invariant finite-range β

$$\int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})\,dt = \infty, \tag{20}$$

where the expectation refers to that of the VRJP on $\mathbb{Z}^d$ and $d = 1, 2$. This is true since, for any finite-range β, one has

$$
\begin{aligned}
\int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})\,dt &= \lim_{h\downarrow 0}\int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})e^{-ht}\,dt \\
&= \lim_{h\downarrow 0}\lim_{L\to\infty}\int_0^\infty \mathbb{E}_{0,0}^{\beta,\Lambda_L}(1_{X_t=0})e^{-ht}\,dt = \infty.
\end{aligned}
\tag{21}
$$

The first equality is by monotone convergence, and the final equality is obtained by combining (16) for the $\mathbb{H}^{2|2}$ model and (11).

For the second equality, it suffices, by using the tail of the exponential e^{-ht}, to verify that the integrand converges for $t \le T$ for any bounded T. Since the jump rate $1 + L_t^i$ is bounded by $1 + T$, the walk is exponentially unlikely to take more than $O(T^3)$ jumps to new vertices up to time T. VRJPs on Λ_L and $\mathbb{Z}^d$ can be coupled to be the same until they exit a ball of radius less than $\frac{1}{2}L$, an event which requires at least L/R jumps to occur, where R is the radius of the finite-range step distribution. This completes the proof. □

The analogue of Theorem 1.1 for the ERRW with constant initial weights was established in [19, 24], but not for the VRJP. Mermin–Wagner-type theorems have also been proven for the ERRW in one and two dimensions [18, 19]. The techniques used deal directly with ERRWs, and hence are rather different from those employed in this paper.

Our relation between the two-point functions of the $\mathbb{H}^{2|2}$ model and the VRJP also yields a transience result.

COROLLARY 1.10. *The vertex-reinforced jump process (X_t) on $\mathbb{Z}^d$, $d \ge 3$, with initial rates $\beta_{ij} = \beta 1_{|i-j|=1}$ and β sufficiently large is transient, in the sense that the expected time (X_t) spends at the origin is finite.*

PROOF. The argument mirrors the proof of Theorem 1.1, using (19) in place of (16). □

Transience in the sense of visiting the origin finitely often almost surely when β is sufficiently large was established in [23], Corollary 4; this result also makes use of [10]. As with recurrence, see the discussion following the statement of Theorem 1.1, there is in general no relation between the two notions of transience.

2. Supersymmetry and horospherical coordinates. In this section, we define horospherical coordinates for $\mathbb{H}^n$ and then define the supersymmetric $\mathbb{H}^{2|2}$ model precisely. We also collect Ward identities and relations between derivatives that will be used in the proofs of Theorems 1.2 and 1.5.

2.1. *Horospherical coordinates*. As observed in [29, 32], the hyperbolic spaces $\mathbb{H}^n$ are naturally parametrised by horospherical coordinates that are useful for the analysis of the corresponding sigma models. For $\mathbb{H}^n$, these are global coordinates $t \in \mathbb{R}$, $\tilde{s} \in \mathbb{R}^{n-1}$, in terms of which

(22)
$$x = \sinh t - \frac{1}{2}|\tilde{s}|^2 e^t, \qquad y^i = e^t s^i \quad (i = 1, \ldots, n-1),$$
$$z = \cosh t + \frac{1}{2}|\tilde{s}|^2 e^t.$$

Both x, z are scalars while $\tilde{y} = (y^1, \ldots, y^{n-1})$ and $\tilde{s} = (s^1, \ldots, s^{n-1}) \in \mathbb{R}^{n-1}$ are $n-1$ dimensional vectors and $|\tilde{s}|^2 = \sum_{i=1}^{n-1}(s^i)^2$. By this change of variables, one has (see the Appendix)

(23)
$$\int_{(\mathbb{H}^n)^\Lambda} F(u)\mu^{\otimes \Lambda}(du) = \int_{(\mathbb{R}^n)^\Lambda} F(u(\tilde{s}, t)) \prod_i e^{(n-1)t_i} \, dt_i \, d\tilde{s}_i.$$

By a short calculation,

(24) $\quad -u_i \cdot u_j = \cosh(t_i - t_j) + \frac{1}{2}|\tilde{s}_i - \tilde{s}_j|^2 e^{t_i + t_j}, \qquad z_i = \cosh t_i + \frac{1}{2}|\tilde{s}_i|^2 e^{t_i}.$

Thus in horospherical coordinates,

(25)
$$H(\tilde{s}, t) = \sum_{\langle ij \rangle} \beta_{ij} \left(\cosh(t_i - t_j) - 1 + \frac{1}{2}|\tilde{s}_i - \tilde{s}_j|^2 e^{t_i + t_j} \right)$$
$$+ h \sum_i \left(\cosh(t_i) - 1 + \frac{1}{2}|\tilde{s}_i|^2 e^{t_i} \right),$$

where by a slight abuse of notation we have re-used the symbol H. Moreover, the following relations, in which we set $s_i = s_i^r$ and $y_i = y_i^r$ for some fixed $r = 1, \ldots, n-1$, hold:

(26) $\quad \dfrac{\partial z_i}{\partial s_i} = y_i, \qquad \dfrac{\partial y_i}{\partial s_i} = x_i + z_i, \qquad \dfrac{\partial (u_i \cdot u_j)}{\partial s_i} = y_j(x_i + z_i) - y_i(x_j + z_j).$

Furthermore,

$$\frac{\partial^2}{\partial s_j^2} z_j = e^{t_j} = x_j + z_j,$$

(27)
$$\frac{\partial^2}{\partial s_i \, \partial s_l}(-1 - u_j \cdot u_l) = \begin{cases} -e^{t_j + t_l} = -(x_j + z_j)(x_l + z_l) & i = j, \\ +e^{t_j + t_l} = +(x_j + z_j)(x_l + z_l) & i = l, \\ 0 & \text{else.} \end{cases}$$

2.2. *Supersymmetry.* Let Λ be a finite set. We will define an algebra Ω_Λ of forms (which generalise random variables) that constitute the observables on the superspace $(\mathbb{R}^{2|2})^\Lambda$. The superspace itself only has meaning through this algebra of observables. We also define an integral associated to this algebra. We then introduce the supersymmetry generator and the localisation lemma. For a more detailed introduction to the mathematics of supersymmetry, see, for example, [3, 5, 10].

2.2.1. *Supersymmetric integration.* For each vertex $i \in \Lambda$, let x_i, y_i be real variables and ξ_i, η_i be two Grassmann variables. Thus by definition all of the x_i and y_i commute with each other and with all of the ξ_i and η_i and all of the ξ_i and η_i anti-commute. The way in which the anti-commutation relations are realized is unimportant, but concretely, we can define an algebra of $4^{|\Lambda|} \times 4^{|\Lambda|}$ matrices ξ_i and η_i realising the required anti-commutation relations for the Grassmann variables. To fix signs in forthcoming expressions, fix an arbitrary order $i_1, \ldots, i_{|\Lambda|}$ of the vertices in Λ.

We define the algebra Ω_Λ to be the algebra of smooth functions on $(\mathbb{R}^2)^\Lambda$ with values in the algebra of $4^{|\Lambda|} \times 4^{|\Lambda|}$ matrices that have the form

$$(28) \qquad F = \sum_{I, J \subset \Lambda} F_{I,J}(x, y)(\eta \xi)_{I,J},$$

where the coefficients $F_{I,J}$ are smooth functions on $(\mathbb{R}^2)^\Lambda$, and $(\eta \xi)_{I,J}$ is given by the ordered product $\prod_{i \in I \cap J} \eta_i \xi_i \prod_{i \in I \setminus J} \xi_i \prod_{j \in J \setminus I} \eta_j$. This ordering has been chosen so that $(\eta \xi)_{\Lambda, \Lambda}$ is $\eta_1 \xi_1 \ldots \eta_\Lambda \xi_\Lambda$. We call elements of Ω_Λ forms because the forms of differential geometry are instances [5, 15]. The integral (sometimes called a superintegral) of a form $F \in \Omega_\Lambda$ is defined by

$$(29) \qquad \int_{(\mathbb{R}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^2)^\Lambda} F_{\Lambda, \Lambda}(x, y) \prod_{i \in \Lambda} \frac{dx_i \, dy_i}{2\pi},$$

where $\mathbb{R}^{2|2}$ refers to the number of commuting and anti-commuting variables.

The *degree* of a coefficient $F_{I,J}$ is $|I| + |J|$. Thus the integral of a form F is a constant multiple of the usual Lebesgue integral of the top degree part of F. A form $F \in \Omega_\Lambda$ is *even* if the degree of all nonvanishing coefficients $F_{I,J}$ is even in (28). Even forms commute. For even forms $F^1, \ldots, F^p$ and a smooth function $g \in C^\infty(\mathbb{R}^p)$, the form $g(F^1, \ldots, F^p) \in \Omega_\Lambda$ is defined by formally Taylor expanding g about the degree-0 part $(F^1_{\varnothing, \varnothing}(x, y), \ldots, F^p_{\varnothing, \varnothing}(x, y))$. This is well-defined as there is no ambiguity in the ordering if the F^i are all even, and the anti-commutation relations satisfied by the ξ_i and η_i imply the expansion is finite.

2.2.2. *Localisation.* Temporarily set $x = x_i$, $y = y_i$, $\xi = \xi_i$, and $\eta = \eta_i$. Define an operator $\partial_\eta \colon \Omega_\Lambda \to \Omega_\Lambda$ by linearity, $\partial_\eta(\eta F) = F$, and $\partial_\eta F = 0$ if F does not contain a factor η. Define ∂_ξ in the same manner. Define Q_i by its action on forms F by

$$(30) \qquad Q_i F \equiv \xi \partial_x F + \eta \partial_y F + x \partial_\eta F - y \partial_\xi F.$$

The *supersymmetry generator* Q acts on a form $F \in \Omega_\Lambda$ by $QF \equiv \sum_{i \in \Lambda} Q_i F$.

DEFINITION 2.1. $F \in \Omega_\Lambda$ is *supersymmetric* if $QF = 0$.

The supersymmetry generator acts as an anti-derivation on the algebra of forms, see, for example, [5], Section 6. This implies that the forms

$$(31) \qquad \tau_{ji} = \tau_{ij} \equiv x_i x_j + y_i y_j + \xi_i \eta_j - \eta_i \xi_j, \quad i, j \in \Lambda,$$

are supersymmetric. Moreover, any smooth function of the τ_{ij} is supersymmetric as Q obeys a chain rule, see [5], equation (6.5). The following *localisation lemma* is fundamental. For a proof, see [10], Lemma 16.

LEMMA 2.2 (Localisation lemma).   *Let $F \in \Omega_\Lambda$ be a smooth form with sufficient decay that is supersymmetric, that is, satisfies $QF = 0$. Then*

$$(32) \qquad \int_{(\mathbb{R}^{2|2})^\Lambda} F = F_{\varnothing,\varnothing}(0,0).$$

2.3. *The $\mathbb{H}^{2|2}$ model.*   We can now define the $\mathbb{H}^{2|2}$ sigma model and justify our earlier claim that its y marginal is the probability measure (6). Given $(x_i, y_i, \xi_i, \eta_i)$ as above define an even variable z_i by

$$(33) \qquad z_i \equiv \sqrt{1 + x_i^2 + y_i^2 + 2\xi_i \eta_i} = \sqrt{1 + x_i^2 + y_i^2} + \frac{\xi_i \eta_i}{\sqrt{1 + x_i^2 + y_i^2}},$$

where the equality is by the definition of a function of a form. We will write $u_i = (x_i, y_i, z_i, \xi_i, \eta_i)$. Define the "inner product"

$$(34) \qquad u_i \cdot u_j \equiv x_i x_j + y_i y_j - z_i z_j + \xi_i \eta_j - \eta_i \xi_j,$$

generalising the Minkowski inner product above (3); we have written "inner product" as this is only terminology, since (34) is not a quadratic form in the classical sense. Then by a short calculation

$$(35) \qquad u_i \cdot u_i = -1,$$

which we interpret as meaning that u_i is in the supermanifold $\mathbb{H}^{2|2}$. Since $z_i = \sqrt{1 + \tau_{ii}}$ and $u_i \cdot u_j = \tau_{ij} - z_i z_j$, the forms $u_i \cdot u_j$ and z_i are supersymmetric for all $i, j \in \Lambda$.

The $\mathbb{H}^{2|2}$ integral of a form $F \in \Omega_\Lambda$ is defined by

$$(36) \qquad \int_{(\mathbb{H}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^{2|2})^\Lambda} F \prod_{i \in \Lambda} \frac{1}{z_i},$$

and the $\mathbb{H}^{2|2}$ model is defined by the following action (which is now a form in Ω_Λ)

$$(37) \qquad H \equiv H_{\beta,h} = \sum_{\langle ij \rangle} \beta_{ij}(-u_i \cdot u_j - 1) + h \sum_i (z_i - 1) \in \Omega_\Lambda.$$

Lastly, we define the superexpectation of an observable $F \in \Omega_\Lambda$ in the $\mathbb{H}^{2|2}$ model by

$$(38) \qquad \langle F \rangle_{\mathbb{H}^{2|2}} \equiv \int_{(\mathbb{H}^{2|2})^\Lambda} F e^{-H}.$$

Lemma 2.2 implies that $\langle 1 \rangle_{\mathbb{H}^{2|2}} = 1$, as promised in Section 1.2.3.

2.4. *Supersymmetric horospherical coordinates.* The $\mathbb{H}^{2|2}$ model can also be reparametrised in a supersymmetric version of horospherical coordinates [10], Section 2.2. For the convenience of the reader, the explicit change of variables is computed in the Appendix. In this parametrisation, t and s are two real variables and $\bar{\psi}$ and ψ are two Grassmann variables. As in the previous section, we denote the algebra of such forms by $\widetilde{\Omega}_\Lambda$. The tilde refers to horospherical coordinates. We write

$$x = \sinh t - e^t \left(\frac{1}{2} s^2 + \bar{\psi}\psi \right), \qquad y = e^t s,$$

$$(39) \qquad z = \cosh t + e^t \left(\frac{1}{2} s^2 + \bar{\psi}\psi \right),$$

$$\xi = e^t \bar{\psi}, \qquad \eta = e^t \psi.$$

There is a generalisation of the change of variables formula from standard integration to superintegration. We only require the following special case given in [10], Section 2.2, and Appendix. Forms $F \in \Omega_\Lambda$ are in correspondence with forms $\widetilde{F} \in \widetilde{\Omega}_\Lambda$ obtained by substituting the relations (39) into (28) using the definition of functions of forms. Moreover, expanding

$$(40) \qquad \widetilde{F} = \sum_{I,J \subset \Lambda} \widetilde{F}_{I,J}(t,s)(\psi\bar{\psi})_{I,J}$$

the superintegral over F can expressed as

$$(41) \qquad \int_{(\mathbb{H}^{2|2})^\Lambda} F = \int_{(\mathbb{R}^2)^\Lambda} \widetilde{F}_{\Lambda,\Lambda}(t,s) \prod_i e^{-t_i} \frac{dt_i \, ds_i}{2\pi}.$$

If a function $F(y)$ depends only on the y coordinates then F has degree 0, and a computation (see [10], Section 2.2, and Appendix) shows that

$$\langle F(y) \rangle_{\mathbb{H}^{2|2}} = \int_{(\mathbb{H}^{2|2})^\Lambda} F(y) e^{-H} = \int_{(\mathbb{R}^2)^\Lambda} F(e^t s)(e^{-H})_{\Lambda,\Lambda} \prod_i e^{-t_i} \frac{dt_i \, ds_i}{2\pi}$$

$$(42) \qquad = \int_{(\mathbb{R}^2)^\Lambda} F(e^t s) e^{-\widetilde{H}(t,s)} \prod_i dt_i \, ds_i,$$

with the function $\widetilde{H}$ given by (6).

Analogously to (24) a calculation gives the expressions

$$
\begin{aligned}
(43) \qquad -u_i \cdot u_j &= \cosh(t_i - t_j) + \frac{1}{2}(s_i - s_j)^2 e^{t_i + t_j} \\
&\quad + (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) e^{t_i + t_j}
\end{aligned}
$$

$$
(44) \qquad z_i = \cosh t_i + \left(\frac{1}{2}s_i^2 + \bar{\psi}_i \psi_i\right) e^{t_i}.
$$

We again check that

$$
(45) \quad \frac{\partial z_i}{\partial s_i} = y_i, \qquad \frac{\partial y_i}{\partial s_i} = x_i + z_i, \qquad \frac{\partial (u_i \cdot u_j)}{\partial s_i} = y_j(x_i + z_i) - y_i(x_j + z_j)
$$

and

$$
\frac{\partial^2}{\partial s_j^2} z_j = e^{t_j} = x_j + z_j,
$$

$$
(46) \qquad \frac{\partial^2}{\partial s_i \partial s_l}(-1 - u_j \cdot u_l) = \begin{cases} -e^{t_j + t_l} = -(x_j + z_j)(x_l + z_l) & i = j, \\ +e^{t_j + t_l} = +(x_j + z_j)(x_l + z_l) & i = l, \\ 0 & \text{else.} \end{cases}
$$

2.5. *Ward identities.* In this section, we establish some useful Ward identities. These Ward identities are a reflection of the underlying symmetries of the target spaces $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$; see [10], Appendix B. Note that these identities are most easily seen in the ambient coordinates $(x, y^1, \ldots, y^{n-1}, z)$.

2.5.1. $\mathbb{H}^n$. For the $\mathbb{H}^n$ model, we have the identities

$$
(47) \qquad \langle x_j g(z) \rangle_{\mathbb{H}^n} = 0
$$

for any smooth function g. This identity follows simply from the invariance of the measure under $x \mapsto -x$ (see (4)–(5)). Moreover, by rotational symmetry, we have $\langle g(y^r) \rangle_{\mathbb{H}^n} = \langle g(x) \rangle_{\mathbb{H}^n}$ for $r = 1, \ldots, n - 1$.

2.5.2. $\mathbb{H}^{2|2}$. For the $\mathbb{H}^{2|2}$ model, we have identities analogous to (47):

$$
(48) \qquad \langle x_j g(z) \rangle_{\mathbb{H}^{2|2}} = 0
$$

for any smooth function g. This identity again follows from the symmetry $x \mapsto -x$ (see (37)–(38)). We also have $\langle g(x) \rangle_{\mathbb{H}^{2|2}} = \langle g(y) \rangle_{\mathbb{H}^{2|2}}$ by rotational symmetry. The following identities arise from (48):

$$
\begin{aligned}
(49) \qquad \langle e^{t_j + t_l} \rangle_{\mathbb{H}^{2|2}} &= \langle (x_j + z_j)(x_l + z_l) \rangle_{\mathbb{H}^{2|2}} = \langle x_j x_l + z_j z_l \rangle_{\mathbb{H}^{2|2}}, \\
\langle e^{t_j} \rangle_{\mathbb{H}^{2|2}} &= \langle x_j + z_j \rangle_{\mathbb{H}^{2|2}}
\end{aligned}
$$

and hence by supersymmetry and rotational invariance

$$\langle e^{t_j+t_l}\rangle_{\mathbb{H}^{2|2}} = 1 + \langle y_j y_l\rangle_{\mathbb{H}^{2|2}},$$
$$\langle e^{t_j}\rangle_{\mathbb{H}^{2|2}} = 1.$$
(50)

Indeed, the evaluations $\langle z_i z_j\rangle_{\mathbb{H}^{2|2}} = \langle z_i\rangle_{\mathbb{H}^{2|2}} = 1$ are by Lemma 2.2, which implies more generally that for any smooth function g with rapid decay,

$$\int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,0}} g(z) = g(1).$$
(51)

3. Proof of Theorem 1.2. In this section, for the $\mathbb{H}^n$ model, we will let y_a denote the component y_a^1 of $u_a \in \mathbb{H}^n$ and s_a the corresponding component s_a^1 in horospherical coordinates. By symmetry (recall Section 2.5), the results of this section are valid if we replace y_a^1 by any of the first $n-1$ components of u_a.

We will prove that for the $\mathbb{H}^n$ model, $n \geq 2$,

$$\sum_b \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,h}} y_a y_b g(b, z-1)$$
$$= \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,h}} z_a \int_0^\infty \mathbb{E}^\beta_{a,z-1}(g(X_t, L_t)) e^{-ht}\, dt.$$
(52)

In (52), and in the rest of this section, we omit the measure $\mu^{\otimes\Lambda}(du)$ for integrals over $(\mathbb{H}^n)^\Lambda$ from the notation. For the $\mathbb{H}^{2|2}$ model, we prove that

$$\sum_b \int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,h}} y_a y_b g(b, z-1) = \int_0^\infty \mathbb{E}^\beta_{a,0}(g(X_t, L_t)) e^{-ht}\, dt.$$
(53)

Theorem 1.2 in the case of $\mathbb{H}^{2|2}$ is precisely (53), and Theorem 1.2 in the case of $\mathbb{H}^n$ follows by normalising (52). The identities (52) and (53) are a result of the following integration by parts formulas. Recall that $\mathcal{L}^\beta$ denotes the generator (2) of the joint position and local time process (X_t, L_t) of the VRJP.

LEMMA 3.1. *Let Λ be finite, let $a \in \Lambda$ and let $g: \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay. For the $\mathbb{H}^n$ model, $n \geq 2$,*

$$-\sum_b \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,0}} y_a y_b \mathcal{L}^\beta g(b, z-1) = \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,0}} z_a g(a, z-1).$$
(54)

For the $\mathbb{H}^{2|2}$ model,

$$-\sum_b \int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,0}} y_a y_b \mathcal{L}^\beta g(b, z-1) = g(a, 0).$$
(55)

PROOF.   The proofs are essentially the same for $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$, so we carry them out in parallel.

We write $\mathcal{L}$ for $\mathcal{L}^\beta$, H for $H_{\beta,0}$, and the integral $\int$ for $\int_{(\mathbb{H}^n)^\wedge}$ and, respectively, $\int_{(\mathbb{H}^{2|2})^\wedge}$. By (26) (resp., (45)) we have $y_b \frac{\partial}{\partial \ell_b} g(b, z-1) = \frac{\partial}{\partial s_b} g(b, z-1)$ where $\frac{\partial}{\partial \ell_b}$ denotes the derivative with respect to the bth component of the second argument. Therefore,

$$
\sum_b \int e^{-H} y_a y_b \mathcal{L} g(b, z-1)
$$

(56)
$$
= \int e^{-H} y_a \left(\sum_{b,c} \beta_{bc} y_b z_c \big(g(c, z-1) - g(b, z-1) \big) \right.
$$

$$
\left. + \sum_b \frac{\partial}{\partial s_b} g(b, z-1) \right).
$$

Recall (23) (resp., (41)) and integrate the second term in the equation above by parts. This produces two terms; by the rapid decay of g there are no boundary terms. For the first term produced by the integration by parts, using (26) (resp., (45)) again,

$$
\sum_b \int e^{-H} y_a \left(-\frac{\partial H}{\partial s_b} \right) g(b, z-1)
$$

(57)
$$
= \sum_b \int e^{-H} y_a \left(\sum_c \beta_{bc} \frac{\partial (u_b \cdot u_c)}{\partial s_b} \right) g(b, z-1)
$$

$$
= \sum_{b,c} \int e^{-H} y_a \beta_{bc} y_b z_c \big(g(c, z-1) - g(b, z-1) \big).
$$

This term cancels the first term on the right-hand side of (56). For the second term produced by the integration by parts, we use that $\int x_a e^{-H} g(b, z) = 0$ by (47) (resp., (48)):

(58)
$$
\int e^{-H} \frac{\partial y_a}{\partial s_b} g(b, z-1) = \delta_{ab} \int e^{-H} (x_a + z_a) g(b, z-1)
$$

$$
= \delta_{ab} \int e^{-H} z_a g(a, z-1).
$$

In the supersymmetric case, the localisation lemma in the special case (51) further implies that the last right-hand side can be evaluated as

(59)
$$
\delta_{ab} \int e^{-H} z_a g(a, z-1) = \delta_{ab} g(a, 0).
$$

Altogether, we have shown (54) (resp., (55)).   □

PROOF OF THEOREM 1.2.   It suffices to show (52) and (53) with $h = 0$, by replacing $g(b, z - 1)$ by $g(b, z - 1)e^{-h(z-1)}$. Therefore, from now on, assume $h = 0$. To get (53) from (55), we apply (55) with $g(i, \ell)$ replaced by $g_t(i, \ell) = \mathbb{E}_{i,\ell}(g(X_t, L_t))$. By the definition of the generator, we have $\mathcal{L}g_t(i, \ell) = \frac{\partial}{\partial t}g_t(i, \ell)$, so (55) gives

$$(60) \qquad \mathbb{E}_{a,0}\big(g(X_t, L_t)\big) = -\frac{\partial}{\partial t}\bigg(\sum_b \int e^{-H} y_a y_b g_t(b, z - 1)\bigg).$$

Note that the process (X_t, L_t) is transient even if the marginal (X_t) is recurrent because $\sum_i L_t^i \to \infty$ as $t \to \infty$. Therefore, integrating both sides over t and using that $g_t(x, \ell) \to 0$ as $t \to \infty$, which follows from the transience of (X_t, L_t) and the rapid decay of $g = g_0$, we get

$$(61) \qquad \int_0^\infty \mathbb{E}_{a,0}\big(g(X_t, L_t)\big)\, dt = \sum_b \int e^{-H} y_a y_b g(b, z - 1).$$

The proof of (52) from (54) is entirely analogous.   □

4. Proof of Theorem 1.5.   The proof of the hyperbolic Mermin–Wagner follows that of the usual Mermin–Wagner theorem closely [20, 21]; see also the presentation in [14]. We begin with the nonsupersymmetric case. Due to the noncompact target space, differences occur in the bound of the term $\langle |DH|^2\rangle$ and in the role of the coordinate in the direction of the magnetic field. As in the previous section, we write H for $H_{\beta,h}$. We will write $\bar{A}$ to denote the complex conjugate of A.

PROOF OF (14).   As in the previous section, we write y_j for y_j^1. We also write $\langle\cdot\rangle$ for $\langle\cdot\rangle_{\mathbb{H}^n}$, and we use horospherical coordinates throughout the proof. Throughout the proof, H will denote the energy of a spin configuration in horospherical coordinates; recall (25).

Let

$$(62) \qquad S(p) = \frac{1}{\sqrt{|\Lambda|}}\sum_j e^{i(p\cdot j)} y_j, \qquad D = \frac{1}{\sqrt{|\Lambda|}}\sum_j e^{-i(p\cdot j)}\frac{\partial}{\partial s_j}.$$

By the Cauchy–Schwarz inequality,

$$(63) \qquad \langle |S(p)|^2\rangle \geq \frac{|\langle S(p)DH\rangle|^2}{\langle |DH|^2\rangle}.$$

In the following, we compute the terms on the left- and right-hand sides of the above inequality. Note that we have the integration by parts identity $\langle FDH\rangle = \langle DF\rangle$ for any smooth $F\colon (\mathbb{H}^n)^\Lambda \to \mathbb{R}$ that does not grow too fast; the vanishing of boundary terms can be seen by looking at the expression for H (i.e., by (25)).

By the assumed translation invariance of β,

$$
\begin{aligned}
\langle |S(p)|^2 \rangle &= \frac{1}{|\Lambda|} \sum_{j,l} e^{ip\cdot(j-l)} \langle y_j y_l \rangle = \frac{1}{|\Lambda|} \sum_{j,l} e^{ip\cdot(j-l)} \langle y_0 y_{j-l} \rangle \\
&= \sum_j e^{i(p\cdot j)} \langle y_0 y_j \rangle,
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
\langle S(p) DH \rangle = \langle DS(p) \rangle &= \frac{1}{|\Lambda|} \sum_{j,l} e^{ip\cdot(j-l)} \left\langle \frac{\partial y_j}{\partial s_l} \right\rangle = \frac{1}{|\Lambda|} \sum_j \langle x_j + z_j \rangle \\
&= \langle z_0 \rangle,
\end{aligned}
\tag{65}
$$

$$
\langle |DH|^2 \rangle = \langle D\bar{D}H \rangle = \frac{1}{|\Lambda|} \sum_{j,l} e^{ip\cdot(j-l)} \left\langle \frac{\partial^2 H}{\partial s_j \partial s_l} \right\rangle.
\tag{66}
$$

In (65), we have used $\langle x_j \rangle = 0$; recall Section 2.5. By $\langle x_j z_k \rangle = 0$, Cauchy–Schwarz, translation invariance, that $\langle x_0^2 \rangle = \langle y_0^2 \rangle$ (recall the symmetries from Section 2.5.1), and the constraint $u_0 \cdot u_0 = -1$, observe that

$$
\langle (x_j + z_j)(x_l + z_l) \rangle = \langle x_j x_l + z_j z_l \rangle \leq \langle x_0^2 \rangle + \langle z_0^2 \rangle = 1 + (n+1)\langle y_0^2 \rangle.
\tag{67}
$$

Thus, using (27) and $\langle x_j \rangle = 0$ once more, (66) can be rewritten and bounded above by

$$
\begin{aligned}
\langle |DH|^2 \rangle &= \frac{1}{|\Lambda|} \sum_{j,l} \beta_{jl} \langle (x_j + z_j)(x_l + z_l) \rangle (1 - e^{ip\cdot(j-l)}) \\
&\quad + \frac{h}{|\Lambda|} \sum_j \langle x_j + z_j \rangle \\
&\leq \frac{1}{|\Lambda|} \sum_{j,l} \beta_{jl} (1 + (n+1)\langle y_0^2 \rangle)(1 - \cos(p\cdot(j-l))) + h\langle z_0 \rangle.
\end{aligned}
\tag{68}
$$

In summary, we have shown (recall (12))

$$
\langle |DH|^2 \rangle \leq (1 + (n+1)\langle y_0^2 \rangle)\lambda(p) + h\langle z_0 \rangle.
\tag{69}
$$

Using (64) and substituting the above bounds into (63) gives

$$
\begin{aligned}
\sum_j e^{i(p\cdot j)} \langle y_0 y_j \rangle &\geq \frac{|\langle S(p) DH \rangle|^2}{\langle |DH|^2 \rangle} \geq \frac{\langle z_0 \rangle^2}{(1 + (n+1)\langle y_0^2 \rangle)\lambda(p) + h\langle z_0 \rangle} \\
&\geq \frac{1}{(1 + (n+1)\langle y_0^2 \rangle)\lambda(p) + h}.
\end{aligned}
\tag{70}
$$

The last inequality follows from $h \geq 0$ and $1 \leq \langle z_0 \rangle$, which holds by the definition of $\mathbb{H}^n$. $\quad \square$

PROOF OF (15). We use that the expectation of a function $F(y)$ can be written using horospherical coordinates in terms of the *probability measure* (6). Throughout this proof, we denote the expectation with respect to this probability measure by $\langle \cdot \rangle$. By the Cauchy–Schwarz inequality, and since $S(p)$ is a function of the y,

$$\langle |S(p)|^2 \rangle_{\mathbb{H}^{2|2}} = \langle |S(p)|^2 \rangle \geq \frac{|\langle S(p) D\widetilde{H} \rangle|^2}{\langle |D\widetilde{H}|^2 \rangle}. \tag{71}$$

The probability measure $\langle \cdot \rangle$ obeys the integration by parts $\langle F D\widetilde{H} \rangle = \langle DF \rangle$ identity for any function $F = F(s, t)$ that does not grow too fast. Therefore, by translation invariance we find that, as in the case of $\mathbb{H}^n$,

$$\langle |S(p)|^2 \rangle = \frac{1}{|\Lambda|} \sum_{j,l} e^{ip \cdot (j-l)} \langle y_j y_l \rangle = \frac{1}{|\Lambda|} \sum_{j,l} e^{ip \cdot (j-l)} \langle y_0 y_{j-l} \rangle$$

$$\tag{72}$$

$$= \sum_j e^{i(p \cdot j)} \langle y_0 y_j \rangle,$$

$$\langle S(p) D\widetilde{H} \rangle = \langle DS(p) \rangle = \frac{1}{|\Lambda|} \sum_{j,l} e^{ip \cdot (j-l)} \left\langle \frac{\partial y_j}{\partial s_l} \right\rangle = \frac{1}{|\Lambda|} \sum_j \langle e^{t_j} \rangle = 1, \tag{73}$$

where the last identity uses (50). By (50), Cauchy–Schwarz, and translation invariance we have

$$\langle e^{t_j + t_l} \rangle = 1 + \langle y_j y_l \rangle \leq 1 + \langle y_0^2 \rangle. \tag{74}$$

Using (74) and the integration by parts identity, it follows that

$$\langle |D\widetilde{H}|^2 \rangle = \langle D\bar{D}\widetilde{H} \rangle = \frac{1}{|\Lambda|} \sum_{j,l} \beta_{jl} \langle e^{t_j + t_l} \rangle (1 - \cos(p \cdot (j-l))) + \frac{h}{|\Lambda|} \sum_j \langle e^{t_j} \rangle$$

$$\tag{75} \qquad \leq \frac{1}{|\Lambda|} \sum_{j,l} \beta_{jl} (1 + \langle y_0^2 \rangle)(1 - \cos(p \cdot (j-l))) + h$$

$$= (1 + \langle y_0^2 \rangle) \lambda(p) + h.$$

In summary, we have proved

$$\sum_j e^{i(p \cdot j)} \langle y_0 y_j \rangle = \langle |S(p)|^2 \rangle \geq \frac{|\langle S(p) D\widetilde{H} \rangle|^2}{\langle |D\widetilde{H}|^2 \rangle} \geq \frac{1}{(1 + \langle y_0^2 \rangle) \lambda(p) + h} \tag{76}$$

as claimed. $\quad \square$

APPENDIX: HOROSPHERICAL COORDINATES

A.1. $\mathbb{H}^n$. Under the change of variables,

$$(77) \qquad x = \sinh t - \frac{1}{2}|\tilde{s}|^2 e^t, \qquad y^i = e^t s^i, \qquad z = \cosh t + \frac{1}{2}|\tilde{s}|^2 e^t,$$

the measure transforms as

$$(78) \qquad \frac{1}{z} dx \wedge dy^1 \wedge \cdots \wedge dy^{n-1} \mapsto \frac{\det J}{\cosh t + \frac{1}{2}|\tilde{s}|^2 e^t} dt \wedge ds^1 \wedge \cdots \wedge ds^{n-1},$$

where the Jacobian matrix in block form is

$$(79) \qquad J = \begin{bmatrix} A_{1 \times 1} & B_{1 \times n-1} \\ C_{n-1 \times 1} & D_{n-1 \times n-1} \end{bmatrix}$$

with

$$(80) \qquad A = \frac{\partial x}{\partial t} = \cosh t - \frac{1}{2}|\tilde{s}|^2 e^t, \qquad B_j = \frac{\partial x}{\partial s^j} = -s_j e^t,$$

$$(81) \qquad C_i = \frac{\partial y^i}{\partial t} = s^i e^t, \qquad D_{ij} = \frac{\partial y^i}{\partial s^j} = \delta_{ij} e^t.$$

Noting that $D = e^t I$, the determinant is easily computed using the Schur complement formula,

$$\det J = (\det D) \det (A - B D^{-1} C)$$

$$(82) \qquad = e^{(n-1)t} \left(\cosh t - \frac{1}{2}|\tilde{s}^2|e^t - \sum_{i=1}^{n-1} (-s_i e^t) e^{-t} (s_i e^t) \right)$$

$$= e^{(n-1)t} \left(\cosh t + \frac{1}{2}|\tilde{s}|^2 e^t \right),$$

giving the transformed measure as

$$(83) \qquad \frac{\det J}{\cosh t + \frac{1}{2}|\tilde{s}|^2 e^t} dt \wedge ds^1 \wedge \cdots \wedge ds^{n-1} = e^{(n-1)t} dt \wedge ds^1 \wedge \cdots \wedge ds^{n-1}.$$

A.2. $\mathbb{H}^{2|2}$. The calculation for $\mathbb{H}^{2|2}$ is similar to the previous case, but the Jacobian is replaced by the Berezinian. The notation in (29) corresponds to the following notation in [10] (resp., [2]):

$$(84) \qquad \int_{\mathbb{R}^{2|2}} F = \int dx \wedge dy \circ \partial_\xi \partial_\eta F = \int F \, d_\eta d_\xi \, dx \, dy.$$

Applying [2], Theorem 2.1, to the change of variables

$$(85) \qquad \begin{aligned} x &= \sinh t - \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t, & y &= se^t, \\ z &= \cosh t + \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t, & \eta &= \psi e^t, \xi = \bar{\psi} e^t, \end{aligned}$$

the Berezin measure transforms as

$$(86) \qquad \frac{1}{z} d_\eta d_\xi \, dx \, dy \mapsto \frac{\mathrm{sdet}\, M}{\cosh t + \frac{1}{2}(s^2 + 2\bar\psi\psi)e^t} d_\psi d_{\bar\psi} \, dt \, ds,$$

where M is the Berezinian supermatrix

$$(87) \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial \eta}{\partial t} & \frac{\partial \xi}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial \eta}{\partial s} & \frac{\partial \xi}{\partial s} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial \eta} & \frac{\partial \psi}{\partial \xi} \\ \frac{\partial \bar\psi}{\partial x} & \frac{\partial \bar\psi}{\partial y} & \frac{\partial \bar\psi}{\partial \eta} & \frac{\partial \bar\psi}{\partial \xi} \end{bmatrix},$$

and $\mathrm{sdet}\, M = (\det D)^{-1} \det(A - BD^{-1}C)$ is its Berezinian (superdeterminant). The four blocks are then

$$(88) \qquad A = \begin{bmatrix} \cosh t - \frac{1}{2}(s^2 + 2\bar\psi\psi)e^t & se^t \\ -se^t & e^t \end{bmatrix}, \qquad B = \begin{bmatrix} \psi e^t & \bar\psi e^t \\ 0 & 0 \end{bmatrix},$$

$$(89) \qquad C = \begin{bmatrix} \bar\psi e^t & 0 \\ -\psi e^t & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.$$

The first term in the Berezinian is simply $(\det D)^{-1} = e^{-2t}$, whilst the second is

$$(90) \qquad \begin{aligned} &\det(A - BD^{-1}C) \\ &= \det\left(\begin{bmatrix} \cosh t - \frac{1}{2}(s^2 + 2\bar\psi\psi)e^t & se^t \\ -se^t & e^t \end{bmatrix} + \begin{bmatrix} 2\bar\psi\psi e^t & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= e^t\left(\cosh t + \frac{1}{2}(s^2 + 2\bar\psi\psi)e^t \right), \end{aligned}$$

giving the transformed Berezin measure as

$$(91) \qquad \frac{\mathrm{sdet}\, M}{\cosh t + \frac{1}{2}(s^2 + 2\bar\psi\psi)e^t} d_\psi d_{\bar\psi} \, dt \, ds = e^{-t} d_\psi d_{\bar\psi} \, dt \, ds,$$

which corresponds to (41).

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