ASYMPTOTIC ZERO DISTRIBUTION OF RANDOM ORTHOGONAL POLYNOMIALS

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We consider random polynomials of the form $H_n(z) = \sum_{j=0}^n \xi_j q_j(z)$ where the $\{\xi_j\}$ are i.i.d. nondegenerate complex random variables, and the $\{q_j(z)\}$ are orthonormal polynomials with respect to a compactly supported measure τ satisfying the Bernstein–Markov property on a regular compact set $K \subset \mathbb{C}$. We show that if $\mathbb{P}(|\xi_0| > e^{|z|}) = o(|z|^{-1})$, then the normalized counting measure of the zeros of H_n converges weakly in probability to the equilibrium measure of K. This is the best possible result, in the sense that the roots of $G_n(z) = \sum_{j=0}^n \xi_j z^j$ fail to converge in probability to the appropriate equilibrium measure when the above condition on the ξ_j is not satisfied.

We also consider random polynomials of the form $\sum_{k=0}^n \xi_k f_{n,k} z^k$, where the coefficients $f_{n,k}$ are complex constants satisfying certain conditions, and the random variables $\{\xi_k\}$ satisfy $\mathbb{E} \log(1 + |\xi_0|) < \infty$. In this case, we establish almost sure convergence of the normalized counting measure of the zeros to an appropriate limiting measure. Again, this is the best possible result in the same sense as above.

1. Introduction. In this paper, we will be concerned with the global distribution of the complex zeros of random polynomials.

The origin of the problems goes back to results on the *Kac ensemble* of random polynomials

$$H_n(z) = \sum_{j=0}^n \xi_j z^j,$$

where the ξ_j are i.i.d. nondegenerate complex-valued random variables. Here, a random-variable is nondegenerate if its law is supported on at least two points. The interest is in the behaviour of the zeros of $H_n(z)$ as $n \to \infty$.

The study of the global behaviour of the zeros of H_n was initiated by Hammersley [9]. Shortly thereafter, Shparo and Shur proved the first results about concentration of zeros near the unit circle [23].

The Kac ensemble has been extensively studied (see the introduction of [18] and references given there). In particular, Ibragimov and Zaporozhets [12] showed

that the condition

$$\mathbb{E}\log(1 + |\xi_0|) < \infty \tag{1}$$

is both necessary and sufficient for almost sure weak* convergence of the normalized counting measure of the zeros (i.e., $\frac{1}{D_n}\sum_{j=1}^{D_n}\delta(z_j)$ where $z_1, \ldots, z_{D_n}$ are the zeros of H_n and $D_n = \sup\{i \le n : \xi_i \ne 0\}$) to normalized Lebesgue measure on the unit circle, $\frac{1}{2\pi}d\theta$.

Shiffman and Zelditch [22] took the point of view that the functions $\{z^i\}_{i=0,1,\ldots}$ are an orthonormal basis for the polynomials in $L^2(\frac{1}{2\pi}d\theta)$. Generalizing this idea, they considered random polynomials of the form

$$H_n(z) = \sum_{j=0}^{n}\xi_j q_j(z), \tag{2}$$

where the ξ_j are complex Gaussians of mean zero and variance one, and the q_j are an orthonormal basis for the polynomials in $L^2(d\mu)$ for certain measures μ with compact support K in the complex plane. They showed that the normalized counting measure of the zeros converges almost surely to the equilibrium measure of K in the weak* topology.

The problems studied in this paper have also been studied in other contexts. For random polynomials using a basis other than orthogonal polynomials, see [11, 18]. For random holomorphic sections of a line bundle, see [3, 21]. For random polynomials in several variables, see [2, 4–6].

In this paper, we will primarily be concerned with finding the weakest possible conditions on the i.i.d. coefficients that will result in the same limiting behaviour of the zeros. Results on this problem may be found in [18], and on specializing to the one-variable case, in [2, 5, 6].

In Theorem 5.3, we establish convergence in probability for the zeros of random polynomials of the form (2) with i.i.d. coefficients satisfying

$$\mathbb{P}(|\xi| \ge e^{|z|}) = o(|z|^{-1}). \tag{3}$$

This is a best possible result in the sense that if (3) is not satisfied for the Kac ensemble, then the normalized counting measure of the zeros does not converge in probability to equilibrium measure on the unit circle. We prove this in Theorem 5.6.

As a corollary of Theorem 5.3, we resolve a conjecture of Pristker and Ramachandran ([18], Conjecture 2.5). Very roughly, they asked if there exist i.i.d. random variables $\{\xi_i\}_{i\in\mathbb{N}}$ and a sequence $\{q_i\}_{i\in\mathbb{N}}$ of orthonormal polynomials with respect to a measure τ on the unit circle such that the normalized counting measures of the zeros of

$$H_n(z) = \sum_{i=0}^{n}\xi_i q_i(z)$$

converge almost surely along a subsequence $\{n_i\}$, but not almost surely along the whole sequence. In Remark 5.4, we use Theorem 5.3 to construct random polynomials $H_n(z)$ with this property.

The basic strategy of the proofs is to prove convergence of the normalized logarithmic potential

$$(4) \qquad \frac{1}{n} \log |H_n(z)|$$

to the Green's function V_K of the compact set K. This implies that the normalized counting measure of the zeros of H_n converges to the equilibrium measure in the weak* topology.

The main difficulty here is in establishing lower bounds on the normalized logarithmic potential (4). This is accomplished in Theorem 5.2. To do this, we use the Kolmogorov–Rogozin inequality. This inequality has been previously used to establish lower bounds of this type in [13]. Unlike in that paper, our arguments dispense with the need for circular symmetry of the polynomials when applying the inequality.

In addition to the above results on convergence in probability, we also consider almost sure convergence of the normalized zero counting measure of random polynomials of the form

$$(5) \qquad G_n(z) = \sum_{i=0}^{n} \xi_i f_{n,i} z^i.$$

Here, the ξ_i are nondegenerate i.i.d. complex random variables satisfying

$$(6) \qquad \mathbb{E} \log(1 + |\xi_0|) < \infty$$

and the coefficients $\{f_{n,i} : 0 \le i \le n, n \in \mathbb{N}\}$ satisfy

$$\lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{k=0}^{n} |f_{n,k}| r^k \right) = V(r).$$

Here, $V(r)$ is a continuous function and the convergence above is locally uniform. We will also assume that sufficiently many of the coefficients $f_{n,k}$ are large enough. This assumption will be made precise in Section 6.

The conditions on the coefficients $f_{n,k}$ are quite general and, therefore, the ensembles of the form (5) include many examples of random polynomials. For example, random polynomials of the form (2), where the measure μ is rotationally symmetric, satisfy these conditions. Random polynomials formed from an array of orthogonal polynomials induced by a rotationally symmetric measure and a rotationally symmetric weight function also fit into this category (see Section 6 for details).

Random polynomials with slightly stronger restrictions on the coefficients $f_{n,k}$ were analyzed by Kabluchko and Zaporozhets in [13]. In that paper, the authors

proved that the normalized counting measure of the zeros of $G_n(z)$ converges in probability to the appropriate limiting measure, and asked when this could be extended to almost sure convergence. They showed almost sure convergence for a few particular arrays using ad hoc methods.

In Theorem 6.5, we prove almost sure convergence of the normalized counting measure of the zeros for random polynomials of the form (5) with the above conditions imposed on the sequence $\{f_{n,k}\}$, answering the question of Kabluchko and Zaporozhets. The measures converge to a measure ν that is equal as a distribution to $\frac{1}{2\pi}\Delta V(|z|)$.

Note that again, the condition $\mathbb{E}\log(1+|\xi_0|) < \infty$ on the random variables is the best possible, in the sense that if this condition fails, then almost sure convergence fails for the Kac ensemble (see [12] for details).

Kabluchko and Zaporozhets also considered random analytic functions of a similar form. Our methods can be easily extended to include this case, but we choose to only address random polynomials in this paper for ease of exposition.

Again, our method for almost sure convergence is based on proving convergence of the normalized logarithmic potential (4). The main obstacle is again in obtaining a lower bound on $\frac{1}{n}\log|G_n(z)|$. For proving almost sure convergence, the Kolmogorov–Rogozin inequality is too weak, so a stronger concentration inequality is needed.

For this, we use a small ball probability theorem of Nguyen and Vu [16]. This gives a stronger concentration estimate than the Kolmogorov–Rogozin inequality for sums of the form $\sum_{i=1}^{n}\xi_i a_i$, where the a_is are fixed and the ξ_is are i.i.d. random variables. This stronger estimate requires that the coefficients a_i are sufficiently spread out in the plane.

While we do not treat the multivariable case in this paper, our methods are flexible enough to still be applied in that setting. In particular, our proof of convergence in probability goes through with only minor modifications for the case of multivariable random orthogonal polynomials. Our proof of almost sure convergence can be adapted to fit specific multivariable random polynomial ensembles, that is, the multivariable Kac ensemble.

Further work. In a follow-up paper [7], the second author uses the small ball probability techniques of Section 6 to prove almost sure convergence of the normalized counting measure of the zeros for general random orthogonal polynomials of the form (2) under the condition (6). Necessity of this condition (as well as the condition $\mathbb{P}(|\xi| > e^{|z|}) = o(|z|^{-1})$ for convergence in probability) is also proven in [7].

2. Preliminaries. In this section, we recall some basic results in potential theory.

Let $D \subset \mathbb{C}$ be an open set. A function u on D is *subharmonic* if it:

(i) takes values in $[-\infty, +\infty)$.

(ii) is upper semicontinuous.

(iii) satisfies the submean inequality. That is, given $w \in D$, there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u\big(w + re^{it}\big)\, dt \quad (0 \leq r < \rho).$$

We denote by $sh(D)$ the collection of subharmonic functions on D. Note that if f is analytic on D then $\log|f| \in sh(D)$.

A set $E \subset \mathbb{C}$ is *polar* if there is a nonconstant subharmonic function u with $E \subset \{u = -\infty\}$. Subharmonic functions are locally integrable and thus polar sets are of Lebesgue planar measure zero.

For a function f on D, we denote by f^* its upper semicontinuous regularization given by

$$f^*(z) := \limsup_{w \to z} f(w).$$

Let $\mathcal{P}_n$ denote the space of polynomials of degree $\leq n$. Let

$$\mathcal{L}(\mathbb{C}) = \big\{ u \in sh(\mathbb{C}) \,\big|\, u(z) - \log|z| \text{ is bounded above as } |z| \to +\infty \big\}.$$

If $p \in \mathcal{P}_n$ is a nonconstant polynomial, then $\frac{1}{\deg(p)} \log|p| \in \mathcal{L}(\mathbb{C})$.

For a compact set $K \subset \mathbb{C}$, the Green's function of K is given by

(7)
$$V_K(z) := \sup\bigg\{ \frac{1}{\deg(p)} \log\big|p(z)\big| \,\big|\, $$
$$p \text{ is a nonconstant polynomial, and } \|p\|_K \leq 1 \bigg\}.$$

Whenever K is nonpolar, $V_K^* \in \mathcal{L}(\mathbb{C})$. Note that $V_K = V_{\tilde{K}}$ where

$$\tilde{K} := \big\{ z : \big|p(z)\big| \leq \|p\|_K \text{ for all polynomials } p \big\}$$

is the *polynomially convex hull* of K. Also, V_K is harmonic on $\mathbb{C} \setminus \tilde{K}$.

We say that K is *regular* when V_K is continuous, that is, $V_K = V_K^*$. This is equivalent to the unbounded component of the complement of K being regular for the Dirichlet problem. Any regular set is nonpolar and for K regular, $V_K = 0$ for all $z \in \tilde{K}$.

For a general nonpolar compact set K, the logarithmic capacity of K may be defined as $e^{-\rho}$ where the Green's function has the asymptotic expansion

$$V_K(z) = \log|z| - \rho + o(1)$$

as $|z| \to \infty$. For equivalent characterizations of logarithmic capacity, see [19], Chapter 5. In this paper, we will restrict to regular sets.

EXAMPLE 2.1. Let $K = \{z : |z| = 1\}$ be the unit circle in the plane. Then K is regular and $V_K(z) = \max(0, \log|z|)$. The polynomially convex hull of K is given by $\tilde{K} = \{z : |z| \leq 1\}$.

The following theorem is from [19] (Theorems 3.4.2 and 3.4.3).

THEOREM 2.2. *Let $D \subset \mathbb{C}$ be open. Let $\{\psi_n(z)\}_{n=1,2,\ldots}$ be a sequence in $sh(D)$ which is locally bounded above. Then*

$$w(z) := \left(\limsup_n \psi_n(z) \right)^*$$

and

$$w_1(z) := \left(\sup_n \psi_n(z) \right)^*$$

are subharmonic on D. Furthermore, $w(z) = \limsup_n \psi_n(z)$ outside of a polar set, and $w_1(z) = \sup_n \psi_n(z)$ outside a polar set.

We present the following simple result without proof.

LEMMA 2.3. *Let f be upper semicontinuous and g continuous on D with $f \leq g$. Suppose that $f = g$ at a dense set of points in D. Then $f = g$ on D.*

Let $L^1_{\text{loc}}(D)$ denote the space of locally integrable functions on D. The next theorem gives conditions for a sequence of subharmonic functions to converge in $L^1_{\text{loc}}(D)$.

THEOREM 2.4 (see also [5], Proposition 4.4). *Let $D \subset \mathbb{C}$ be open. Let $\{\psi_n(z)\}_{n=1,2,\ldots}$ be a sequence in $sh(D)$ which is locally bounded above and let $w(z) \geq (\limsup_n \psi_n(z))^*$. Suppose that w is continuous and that there is a countable dense set of points*
$\{z_i\}_{i \in \mathbb{N}} \subset D$ *such that*

$$\lim_{n \to \infty} \psi_n(z_i) = w(z_i) \quad \text{for all } i \in \mathbb{N}.$$

Then $\psi_n \to w$ in $L^1_{\text{loc}}(D)$.

PROOF. The first step is to show that for any subsequence $J \subset \mathbb{N}$ and any $z \in D$, that

$$\left(\limsup_{n \in J} \psi_n(z) \right)^* = w(z).$$

To this end, let $J \subset \mathbb{N}$ be a subsequence. By Theorem 2.2,

$$w_J := \left(\limsup_{n \in J} \psi_n(z) \right)^*$$

is subharmonic on D. By Lemma 2.3, $w_J = w$ on D. This completes the first step.

Next, we proceed by contradiction to prove the theorem. Suppose that the conclusion of the theorem does not hold. Then there exists a closed ball $B \subset D$ and $\epsilon > 0$ such that for some subsequence $J_1 \subset \mathbb{N}$ we have, for $n \in J_1$,

$$\|\psi_n - w\|_{L^1(B)} \geq \epsilon. \tag{8}$$

However, appealing to Theorem 3.2.12 of [10], there is a subsequence $J_2 \subset J_1$ and $g \in L^1(B)$ with $\lim_{n \in J_2} \psi_n = g$ in $L^1(B)$. It follows from standard measure theory that there is a further subsequence $J_3 \subset J_2$ with $\lim_{n \in J_3} \psi_n(z) = g(z)$ for a.e. $z \in B$ so that $g(z) = w_J(z) = w(z)$ a.e. in B. This contradicts (8). □

We remark that $L^1_{\text{loc}}(D)$ may be endowed with a metric as follows.

REMARK 2.5. Let $L^1_{\text{loc}}(D)$ denote the space of functions locally in L^1 on an open set $D \subset \mathbb{C}$. The space $L^1_{\text{loc}}(D)$ is a metric space as follows: let $X_1, X_2, \ldots$ be a sequence of compact subsets of D with $\bigcup_{i=1}^{\infty} X_i = D$, $X_i \subset X_{i+1}$ for all i. For $f, g \in L^1_{\text{loc}}(D)$ set

$$\rho(f, g) := \sum_{i=1}^{\infty} 2^{-i} \min\left[1, \|f - g\|_{L^1(X_i)}\right].$$

3. Construction of random polynomials. We will construct random polynomials by "randomizing" linear combinations of orthogonal polynomials. We consider random polynomials of the form

$$H_n(z) := \sum_{j=0}^{n} \xi_j q_j(z), \tag{9}$$

where the ξ_j are i.i.d. complex-valued random variables, and the $\{q_j(z)\}$ are orthonormal polynomials constructed below.

To emphasize the randomness, we will sometimes use the notation $H_n(z, \omega)$ where $\omega \in \Omega$ and the i.i.d. random variables ξ_i are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

REMARK 3.1. As discussed in the beginning of the Introduction, H_n is of degree $D_n = \sup\{i \leq n : \xi_i \neq 0\}$. The nondegeneracy of ξ_i guarantees that $\lim_{n \to \infty} D_n/n = 1$ almost surely, and no parts of this paper are affected by the discrepancy between D_n and n. To simplify the exposition, we therefore assume $D_n = n$ throughout the paper.

Let K be a compact and regular subset of $\mathbb{C}$. We construct the polynomials $q_j(z)$ as follows:

Let τ be a finite measure on K. Apply the Gram–Schmidt orthogonalization procedure to the monomials $\{z^j\}$ in $L^2(\tau)$ for $j = 0, 1, \ldots$ to obtain a sequence of

polynomials $\{q_0, q_1, \ldots\}$. Assume that τ satisfies the Bernstein–Markov property (see [6]). That is, for all $\epsilon > 0$, there is an $M > 0$ such that for all $p \in \mathcal{P}_n$ we have

$$(10) \qquad \|p\|_K \leq Me^{\epsilon n}\|p\|_{L^2(\tau)}.$$

Then we have the following convergence result for every $z \in \mathbb{C}$ (from [6]):

$$(11) \qquad V_K(z) = \lim_{n \to \infty} \frac{1}{2n} \log \sum_{j=0}^{n} |q_j(z)|^2.$$

Furthermore, since K is regular, (11) holds locally uniformly. It follows from (10) that given $\epsilon > 0$ there is an $M > 0$ such that, for all $j \in \mathbb{N}$, we have that

$$\|q_j(z)\|_K \leq Me^{\epsilon j},$$

and so

$$(12) \qquad |q_j(z)| \leq Me^{n(V_K(z)+\epsilon)} \quad \text{for all } j \in \{0, 1, \ldots, n\}, z \in \mathbb{C}.$$

We consider random variables $\xi_0, \xi_1, \ldots$ satisfying

$$(13) \qquad \mathbb{P}(|\xi_0| > e^{|z|}) = o(|z|^{-1}).$$

Our next goal is to establish versions of Theorem 2.4 that are specific to random polynomials. We first prove two lemmas which give upper bounds on logarithmic potentials.

LEMMA 3.2. *Let* $\xi_0, \xi_1, \ldots$ *be i.i.d. random variables satisfying* (13), *and let* $H_n(z, \omega)$ *be the random polynomials given by* (9).

For any subsequence $Y \subset \mathbb{N}$, *there is a further subsequence* $Y_0 \subset Y$ *such that almost surely, the family* $\{\frac{1}{n}\log|H_n| : n \in Y_0\}$ *is locally bounded above, and such that for all* $z \in \mathbb{C}$,

$$(14) \qquad \limsup_{n \in Y_0} \frac{1}{n} \log|H_n(z, \omega)| \leq V_K(z).$$

PROOF. It follows from (13) that for every $\epsilon > 0$ we have

$$(15) \qquad \mathbb{P}(|\xi_0| > e^{\epsilon|z|}) = o(|z|^{-1}).$$

Letting $\Omega_{n,\epsilon} = \{\omega \in \Omega : |\xi_i(\omega)| \leq e^{\epsilon n} \text{ for } i = 0, \ldots, n\}$, the asymptotics in (15) imply that

$$\mathbb{P}(\Omega_{n,\epsilon}^c) \to 0 \quad \text{as } n \to \infty.$$

Thus, given a subsequence $Y \subset \mathbb{N}$ there is a further subsequence $Y_0 \subset Y$, $Y_0 = \{n_1, n_2, \ldots\}$ such that

$$\sum_{s=0}^{\infty} \mathbb{P}(\Omega_{n_s,\epsilon}^c) < \infty.$$

Therefore by the Borel–Cantelli lemma, for every $\epsilon > 0$ we can almost surely find an $s_0 \in \mathbb{N}$ such that for every $s \geq s_0$, we have

$$|\xi_i(\omega)| \leq e^{\epsilon n_s} \quad \text{for every } i \in \{0, \ldots, n_s\}.$$

Therefore, for any $\epsilon > 0$, (12) implies that, almost surely,

$$\limsup_{s \to \infty} \frac{1}{n_s} \log\left(\sum_{i=0}^{n_s} |\xi_i q_i(z)|\right) \leq V_K(z) + 2\epsilon \quad \text{for every } z \in \mathbb{C}.$$

Letting ϵ tend to 0, and observing that $|H_{n_s}(z, \omega)| \leq \sum_{i=0}^{n_s} |\xi_i q_i(z)|$ completes the proof of (14). The fact that the sequence $\{\frac{1}{n} \log |H_n| : n \in Y_0\}$ is locally bounded above follows since the convergence in (11) is locally uniform. $\square$

By imposing a stronger condition on the ξ_i, we can get an almost sure upper bound on $\frac{1}{n} \log |H_n|$. We state this lemma in slightly greater generality here so that it fits into the framework that we adopt in Section 6.

LEMMA 3.3.   *Let $\xi_0, \xi_1, \ldots$ be nondegenerate i.i.d. random variables satisfying*

$$(16) \qquad\qquad\qquad \mathbb{E} \log(1 + |\xi_0|) < \infty,$$

and let $H_n(z) = \sum_{i=0}^{n} \xi_i q_{n,i}(z)$ be a sequence of random polynomials, where each $q_{n,i}$ is a polynomial of degree i. Suppose that there exists a continuous function V such that

$$\lim_{n \to \infty} \frac{1}{n} \log\left(\sum_{i=0}^{n} |q_{n,i}|\right) = V$$

locally uniformly on $\mathbb{C}$. Then almost surely, the family $\{\frac{1}{n} \log |H_n|\}_{n \in \mathbb{N}}$ is locally bounded above, and

$$\limsup_{n \to \infty} \frac{1}{n} \log |H_n(z, \omega)| \leq V(z) \quad \text{for all } z \in \mathbb{C}.$$

PROOF.   Condition (16) implies that

$$\sum_{n=0}^{\infty} \mathbb{P}(|\xi_n| > e^{\epsilon n}) < \infty \quad \text{for every } \epsilon > 0.$$

Therefore, for any $\epsilon > 0$, the Borel–Cantelli lemma implies that there exists a random constant C such that almost surely

$$|\xi_n| < C e^{\epsilon n} \quad \text{for all } n.$$

The lemma then follows by similar reasoning to that used in the proof of Lemma 3.2. $\square$

We will also use the following lemma, presented without proof.

LEMMA 3.4. *Let $a_0, a_1, \ldots$ be a sequence of nonzero complex numbers and $A \geq 0$. The following equations are equivalent*:

$$\text{(i)} \quad \lim_{n \to \infty} \frac{1}{n} \log \left(\max_{i=0}^{n} |a_i| \right) = A,$$

$$\text{(ii)} \quad \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{i=0}^{n} |a_i| \right) = A,$$

$$\text{(iii)} \quad \lim_{n \to \infty} \frac{1}{2n} \log \left(\sum_{i=0}^{n} |a_i|^2 \right) = A.$$

Note if we set $a_j = q_j(z)$ at any point z where none of the polynomials $q_j(z)$ are zero and $A = V_K(z)$, then (11) shows that each of the above conditions are met.

4. Zeros of random polynomials. We are concerned with the zeros of random polynomials of the form (9). We will prove, under appropriate circumstances, the weak* convergence as n approaches infinity of the normalized counting measure of the zeros to the equilibrium measure of K.

Given a compact nonpolar set $K \subset \mathbb{C}$, the equilibrium measure μ_K is defined as the unique probability measure which minimizes over probability measures μ on K, the functional ([20], Theorem I.3)

$$\tag{17} \iint \log \frac{1}{|z - t|} \, d\mu(z) \, d\mu(t).$$

It may also be characterized by (see [20], Appendix B, Lemma 2.4)

$$\tag{18} \mu_K = \frac{1}{2\pi} \Delta V_K^*,$$

where Δ denotes the Laplacian and the equation is in the sense of distributions.

Now, if p_n is a polynomial of degree n, the normalized counting measure of its zeros (counting multiplicity) is given by

$$\tag{19} \frac{1}{2\pi} \Delta \left(\frac{1}{n} \log |p_n| \right) = \frac{1}{n} \sum_{j=1}^{n} \delta(z_j),$$

where $z_1, z_2, \ldots, z_n$ are the zeros of p_n and $\delta(z)$ denotes the Dirac-delta measure at z.

We will use the notation Z_{H_n} to denote the normalized counting measure of the zeros of the random polynomial H_n. That is,

$$\tag{20} Z_{H_n} = \frac{1}{2\pi} \Delta \left(\frac{1}{n} \log |H_n| \right).$$

We can now state conditions under which Z_{H_n} converges almost surely and in probability. We first state the following application of Theorem 2.4 to zero measures of polynomials.

THEOREM 4.1.   *Let p_n be a sequence of degree-n polynomials and let V be a continuous, subharmonic function on $\mathbb{C}$. Suppose that the following hypotheses are satisfied*:

  (i) *The sequence $\{\frac{1}{n}\log|p_n| : n \in \mathbb{N}\}$ is locally bounded above.*
  (ii) *For every $z \in \mathbb{C}$, we have*

$$\limsup_n \frac{1}{n}\log|p_n(z)| \leq V(z).$$

  (iii) *There exists a countable dense set of points $\{z_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ such that*

$$\lim_{n \to \infty} \frac{1}{n}\log|p_n(z_i)| = V(z_i) \quad \text{for all } i \in \mathbb{N}.$$

Then Z_{p_n} converges to $\frac{1}{2\pi}\Delta V$ in the weak* topology (here we are thinking of $\frac{1}{2\pi}\Delta V$ as a measure).

PROOF.   By Theorem 2.4, $\frac{1}{n}\log|p_n|$ converges to V_K in $L^1_{\text{loc}}(\mathbb{C})$. Applying $\frac{1}{2\pi}\Delta$ to both $\frac{1}{n}\log|p_n|$ and V_K gives that Z_{p_n} converges to $\frac{1}{2\pi}\Delta V$ as distributions and, therefore, in the weak* topology.   □

We have the following consequence of Theorem 4.1 that we will use to prove convergence in probability in Section 5.

THEOREM 4.2.   *Let H_n be a sequence of random polynomials of the form* (9) *with coefficients ξ_i satisfying* (13). *Suppose that there exists a countable dense set of points $\{z_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ such that for every z_i, we have*

$$\lim_{n \to \infty} \frac{1}{n}\log|H_n(z_i, \omega)| = V_K(z_i) \quad \text{in probability.}$$

Then

$$Z_{H_n} \xrightarrow[n \to \infty]{} \mu_K \quad \text{in probability}$$

in the weak* topology on probability measures on $\mathbb{C}$. That is, for any open set U in the weak* topology containing μ_K, we have that

$$\mathbb{P}(Z_{H_n} \in U) \to 1 \quad \text{as } n \to \infty.$$

PROOF. First, we recall that (see [14], Lemma 3.2) a sequence of random elements with values in a metric space converges in probability to some limit if and only if every subsequence of those random elements contains a further subsequence which converges almost surely to the same limit.

Let $H_n(z, \omega)$ be a sequence of random polynomials satisfying the hypotheses of the theorem. Let $Y \subset \mathbb{N}$ be a subsequence. We will find a further subsequence $Y^* \subset Y$ such that

$$(21) \qquad \lim_{n \in Y^*} Z_{H_n} = \mu_K \quad \text{weak* almost surely.}$$

By condition (13) on the coefficients and Lemma 3.2, we can find a subsequence $Y_0 \subset Y$ such that the random polynomials $\{H_n(z, \omega) : n \in Y_0\}$ satisfy conditions (i) and (ii) of Theorem 4.1 almost surely. Using the hypothesis of the theorem, there is a subsequence $Y_1 \subset Y_0$ and a subset $B_1 \subset \Omega$ with $\mathbb{P}(B_1) = 1$ such that

$$(22) \qquad \lim_{n \in Y_1} \frac{1}{n} \log|H_n(z_1, \omega)| = V_K(z_1),$$

for all $\omega \in B_1$. Next, consider the point z_2. Repeating the above procedure, we obtain a subsequence $Y_2 \subset Y_1$ and a subset $B_2 \subset \Omega$ with $\mathbb{P}(B_2) = 1$ such that (22) holds on B_2 with Y_2 in place of Y_1 and z_2 in place of z_1. In this way, we proceed through the points $\{z_i\}_{i \in \mathbb{N}}$ to get a sequence of nested subsequences $Y_1 \supset Y_2 \supset Y_3 \cdots$ and sets $\{B_i\}_{i \in \mathbb{N}}$ of probability one so that (22) holds on B_i with Y_i and z_i in place of Y_1 and z_1.

We now use the Cantor diagonalization procedure to construct a sequence $Y^* = \{m_1 < m_2 < \cdots\}$. To do this, pick $m_1 \in Y_1$, and then recursively select $m_i \in Y_i \setminus \{1, \ldots, m_{i-1}\}$ for each $i \geq 2$. Since the Y_is form a nested sequence, $Y^* \setminus Y_i$ is finite for all i. In particular, this implies that for all ω in the full measure set $B = \bigcap_{n=1}^{\infty} B_n$ and $i \in \mathbb{N}$, that

$$\lim_{n \in Y^*} \frac{1}{n} \log|H_n(z_i, \omega)| = V_K(z_i).$$

Thus all the conditions of Theorem 4.1 hold almost surely for the sequence of random polynomials $\{H_n : n \in Y^*\}$; applying that theorem proves (21). $\qquad \square$

REMARK 4.3. Theorem 4.2 and the input from Lemma 3.2 don't rely very much on the fact that the sequence $H_n = \sum_{i=0}^{n} \xi_i q_i$ is constructed from an orthonormal sequence of polynomials. In particular, the proofs of Theorem 4.2 and Lemma 3.2 go through analogously in the case where $H_n = \sum_{i=0}^{n} \xi_i f_{n,i}$ for an array of degree-i polynomials $\{f_{n,i} : i \leq n \in \mathbb{N}\}$ with the property

$$\lim_{n \to \infty} \frac{1}{n} \log\left(\sum_{i=0}^{n} |f_{n,i}|\right) = V.$$

Here, the convergence above is locally uniform on $\mathbb{C}$ and V is a continuous, subharmonic function with the property that $V(z) - \log|z|$ is bounded as $|z| \to \infty$. In this case, the limit of the zero measures Z_{H_n} is $\frac{1}{2\pi} \Delta V$.

5. Convergence in probability. We wish to use Theorem 4.2 to prove convergence in probability for random polynomials of the form (9). To do this, we need to prove convergence in probability of the normalized logarithmic potential at a countable dense set of points. We will in fact prove such convergence at all but countably many points.

Pointwise convergence in probability is consequence of Theorem 5.2. Equation (26) gives the upper bound. The lower bound uses the Kolmogorov–Rogozin inequality. To complete the convergence in probability of the zeros (Theorem 5.3), we need only point out that Theorem 5.2 is applicable at all but countably many points.

For a complex random variable X and a positive real number r, define the concentration function

$$\mathcal{Q}(X; r) = \sup_{x \in \mathbb{C}} \mathbb{P}(X \in B(x, r)).$$

Here, $B(x, r)$ is the open ball of radius r centred at x.

THEOREM 5.1 (Kolmogorov–Rogozin inequality; see [8], Corollary 1 on page 304). *There is a constant C such that for any independent random variables $X_1, \ldots, X_n$ and for any $r > 0$, we have*

$$\mathcal{Q}\left(\sum_{i=1}^{n} X_i; r\right) \leq \frac{C}{\sqrt{\sum_{i=1}^{n}[1 - \mathcal{Q}(X_i; r)]}}.$$

The concentration function $\mathcal{Q}$ also has the following elementary properties. First, rescaling a complex random variable X by any $a \in \mathbb{C} \setminus \{0\}$ gives that

$$(23) \qquad\qquad \mathcal{Q}(aX; r) = \mathcal{Q}\left(X; \frac{r}{|a|}\right).$$

Also, if X and Y are independent random variables, then

$$(24) \qquad\qquad \mathcal{Q}(X + Y; r) \leq \mathcal{Q}(X; r).$$

Theorem 5.2 uses the Kolmogorov–Rogozin inequality to establish a pointwise lower bound on functions of the form $\frac{1}{n} \log |\sum \xi_i a_i|$.

THEOREM 5.2. *Let $a_0, a_1, \ldots$ be a sequence of nonzero complex numbers satisfying any of the equivalent conditions of Lemma 3.4.*

Let $\xi_0, \xi_1, \ldots$ be a sequence of i.i.d. nondegenerate complex random variables such that (13) holds. Then

$$(25) \qquad\qquad \lim_{n \to \infty} \frac{1}{n} \log \left|\sum_{i=0}^{n} \xi_i a_i\right| = A$$

in probability.

PROOF. We first show that for any $\epsilon > 0$,

$$(26) \qquad \mathbb{P}\left(\frac{1}{n}\log\left|\sum_{i=0}^{n}\xi_i a_i\right| > A + \epsilon\right) \to 0 \quad \text{as } n \to \infty.$$

Recall from the proof of Lemma 3.2 that condition (13) implies that

$$(27) \qquad \mathbb{P}\left(|\xi_i| \le e^{\epsilon n/2} \text{ for } i = 0, \dots, n\right) \to 1 \quad \text{as } n \to \infty.$$

On the event in the above probability, we have that

$$\frac{1}{n}\log\left|\sum_{i=0}^{n}\xi_i a_i\right| \le \frac{1}{n}\log\left(\max_{i=0}^{n}|\xi_i|\right) + \frac{1}{n}\log\left(\sum_{i=0}^{n}|a_i|\right) \le \frac{\epsilon}{2} + \frac{1}{n}\log\left(\sum_{i=0}^{n}|a_i|\right).$$

By Lemma 3.4(ii), for all large enough n the right-hand side above is at most $A + \epsilon$. Combining this with the convergence in (27) implies (26).

We now show that for any $\epsilon > 0$, we have

$$(28) \qquad \mathbb{P}\left(\frac{1}{n}\log\left|\sum_{i=0}^{n}\xi_i a_i\right| < A - \epsilon\right) \to 0 \quad \text{as } n \to \infty.$$

We bound the above probability in terms of the concentration function for the sum and then apply the Kolmogorov–Rogozin inequality. This gives

$$\mathbb{P}\left(\frac{1}{n}\log\left|\sum_{i=0}^{n}\xi_i a_i\right| < A - \epsilon\right) \le \mathcal{Q}\left(\sum_{i=0}^{n}\xi_i a_i; e^{n(A-\epsilon)}\right)$$

$$\le C\left(\sum_{i=0}^{n}(1 - \mathcal{Q}(\xi_i a_i; e^{n(A-\epsilon)}))\right)^{-1/2}.$$

To complete the proof of (28), it suffices to prove that the sum

$$(29) \qquad \sum_{i=0}^{n}(1 - \mathcal{Q}(\xi_i a_i; e^{n(A-\epsilon)})) = \sum_{i=0}^{n}\left(1 - \mathcal{Q}\left(\xi_i; \frac{e^{n(A-\epsilon)}}{|a_i|}\right)\right)$$

approaches infinity as $n \to \infty$. Here, the equality follows from rescaling (equation (23)).

For this, first observe that by the nondegeneracy of ξ_0, we can find positive numbers D_1, D_2 such that for all $d \le D_1$, we have that $\mathcal{Q}(\xi_0; d) \le D_2 < 1$. Therefore, to show that the sum in (29) approaches infinity as $n \to \infty$, it is enough to show that $|J_n| \to \infty$ as $n \to \infty$, where

$$J_n = \left\{i \le n : \frac{e^{n(A-\epsilon)}}{|a_i|} \le D_1\right\}.$$

We note that in Lemma 3.4, $A \ge 0$. We will consider two cases, $A = 0$ and $A > 0$, and first deal with the case $A = 0$. In this case, for all i we have that

$$\frac{e^{n(A-\epsilon)}}{|a_i|} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Therefore, for all i there exists a number N_i such that $i \in J_n$ for all $n \geq N_i$. Therefore, $|J_n| \to \infty$ as $n \to \infty$.

Now, suppose $A > 0$. We may assume that $0 < \epsilon < A$. Fix $\delta \in (0, \epsilon/2)$. By Condition (i) of Lemma 3.4, there exists an integer N such that for all $n \geq N$, we have that

$$\frac{1}{n} \log\left(\max_{k=0}^{n} |a_k|\right) \in [A - \delta, A + \delta].$$

This implies that for all $k \geq N$, we have $|a_i| \leq e^{k(A+\delta)}$ for $i \leq k$. Also, for all $m \geq N$ there exists a minimal $k(m) \leq m$ such that $|a_{k(m)}| \geq e^{m(A-\delta)}$. Fixing $k \geq N$ we consider those m such that $k = k(m)$. The conditions above guarantee that $m \geq k$ and that $m(A - \delta) \leq k(A + \delta)$. This yields

(30) $$\left|\{m \in \mathbb{N} : k = k(m)\}\right| \leq \frac{2\delta k}{A - \delta} + 1.$$

Now, observe that for all $n \geq \frac{N(A-\delta)}{A-\epsilon/2}$, for every $m \in [\frac{n(A-\epsilon/2)}{A-\delta}, n]$, we have that

$$\frac{e^{n(A-\epsilon)}}{|a_{k(m)}|} \leq e^{-\epsilon n/2}.$$

Choosing n large enough so that $e^{-\epsilon n/2} < D_1$, we then have that

$$|J_n| \geq \left|\left\{k \leq n : k = k(m) \text{ for some } m \in \left[\frac{n(A - \epsilon/2)}{A - \delta}, n\right]\right\}\right|.$$

By (30), for all n large enough, the right-hand side above can be bounded below by

$$\frac{\frac{n(\epsilon/2-\delta)}{A-\delta}}{1 + \frac{2\delta n}{A-\delta}} \geq \frac{(\epsilon/2 - \delta)}{3\delta}.$$

Since δ can be taken arbitrarily small, this implies that $|J_n| \to \infty$ as $n \to \infty$, completing the proof of (28). $\square$

We now have all the ingredients to prove convergence in probability of the normalized counting measure of the zeros for random orthonormal polynomials.

THEOREM 5.3.   *Let $K \subset \mathbb{C}$ be a regular, compact set and let μ_K be the equilibrium measure of K (see (17) and (18)). Let $\xi_0, \xi_1, \ldots$ be a sequence of nondegenerate i.i.d. complex random variables satisfying (13). Consider the random polynomials*

$$H_n(z) := \sum_{j=0}^{n} \xi_i q_j(z),$$

where $\{q_j(z)\}$ are the orthonormal polynomials with respect to a measure on K satisfying the Bernstein–Markov property defined in Section 3.

Then Z_{H_n} converges in probability to μ_K in the weak* topology on probability measures on $\mathbb{C}$.

PROOF. By (11), Lemma 3.4 is satisfied with $a_i = q_i(z)$ and $A = V_K(z)$ at all points z where no $q_i(z) = 0$, and thus at all but countably many points in the plane. Therefore, Theorem 5.2 applies and we have the convergence in probability

$$\frac{1}{n}\log|H_n(z, \omega)| \to V_K(z)$$

at all but at most countably many points in the plane. Applying Theorem 4.2, the result follows. □

REMARK 5.4. In [12], Ibragimov and Zaporozhets showed that the condition

$$(31) \qquad \mathbb{E}\log(1 + |\xi_0|) < \infty$$

is equivalent to the weak* almost sure convergence of $Z_{H_n} \to \frac{1}{2\pi}\,d\theta$ in the Kac ensemble case when $q_j(z) = z^i$ (see Theorem [12], Theorem 1).

Motivated by this and theorems of a similar flavour in [18], Pritsker and Ramachandran (Conjecture 2.5, [18]) asked if there exists a measure τ on the unit circle $\{z = 1\}$ and a sequence of i.i.d. random variables $\{\xi_i\}_{i \in \mathbb{N}}$ such that (31) does not hold, and such that for the basis of orthogonal polynomials $\{q_n(z)\}_{n \in \mathbb{N}}$ constructed with respect to τ, a subsequence $\{Z_{H_{n_i}}\}$ of the normalized counting measures of the zeros of the polynomials

$$H_n(z) = \sum_{i=0}^{n} \xi_i q_i(z)$$

still converges weak* almost surely to $\frac{1}{2\pi}\,d\theta$.

Theorem 5.3 shows that for any sequence of i.i.d. random variables $\{\xi_i\}_{i \in \mathbb{N}}$ such that

$$\mathbb{E}\log(1 + |\xi_0|) = \infty \quad \text{and} \quad \mathbb{P}(|\xi_0| > e^{|z|}) = o(|z|^{-1}),$$

and any sequence of orthonormal polynomials $\{q_n(z)\}_{n \in \mathbb{N}}$ constructed with respect to a Bernstein–Markov measure τ on $\{|z| = 1\}$, that $Z_{H_n} \to \frac{1}{2\pi}\,d\theta$ in probability, and hence any subsequence $\{Z_{H_{n_i}}\}$ has a further subsequence which converges almost surely. This resolves Pritsker and Ramachandran's conjecture.

We now show that Theorem 5.3 is the best possible result for general orthogonal ensembles, by showing that condition (13) is both necessary and sufficient for the Kac ensemble. To do this, we first need a lemma about random variables.

LEMMA 5.5. *Let X be a nonnegative real random variable. Suppose that*

$$(32) \qquad \limsup_{x \to \infty} x\mathbb{P}(X > x) > 0.$$

Then there exists a function $f : [0, \infty) \to [0, \infty)$ such that:

(i) $C(f) := \limsup_{n\to\infty} n\mathbb{P}(X > f(n)) \in (0, \infty)$. *Here, the limsup is taken over $n \in \mathbb{N}$, hence the use of n instead of x.*

(ii) *For every $x, y \in [0, \infty)$, we have that $f(x) + y \leq f(x + y)$.*

PROOF. In the case when $\limsup_{n\to\infty} n\mathbb{P}(X > 2n) \in (0, \infty)$, then the function $f(x) = 2x$ works. Therefore, noting that this limsup is positive by (32), we can assume that

$$(33) \qquad \limsup_{n\to\infty} n\mathbb{P}(X > 2n) = \infty.$$

Define a function $g : \{0, 1, \ldots\} \to [0, \infty)$ so that $g(0) = 0$, and for $n \geq 1$,

$$(34) \qquad \mathbb{P}(X > g(n)) \leq \frac{1}{n} \quad \text{and} \quad \mathbb{P}(X \geq g(n)) \geq \frac{1}{n}.$$

Now, for each $x \in [0, \infty)$, define

$$f(x) = \max\{g(n) + x - n : n \in \{0, \ldots, \lfloor x \rfloor\}\}.$$

We check that f satisfies the conditions of the lemma. First, fix $x < y \in [0, \infty)$. For some $n \in \{0, \ldots, \lfloor x \rfloor\}$, we have that $f(x) = g(n) + x - n$. By the definition of $f(y)$, we have that $f(y) \geq g(n) + y - n = f(x) + y - x$. Thus f satisfies (i).

Now, there must be infinitely many values of $n \in \mathbb{N}$ such that $f(n) = g(n)$. To see this, note that if there are only finitely many such values, then there exists an $m \in \mathbb{N}$ such that for all $x \geq m$, we have that $f(x) = g(m) + x - m$. Hence $g(n) \leq f(n) \leq 2n$ for all large enough n, and so by the first inequality in (34),

$$\limsup_{n\to\infty} n\mathbb{P}(X > 2n) \leq \limsup_{n\to\infty} n\mathbb{P}(X > g(n)) \leq 1.$$

This contradicts (33). Now let $\{n_i \in \mathbb{N}\}$ be a subsequence so that $f(n_i) = g(n_i)$ for all i. Note that $f(n_i - 1) < f(n_i) = g(n_i)$ since f is strictly increasing. Therefore, we have that

$$(35) \qquad \lim_{n_i \to\infty} (n_i - 1)\mathbb{P}(X > f(n_i - 1)) \geq \lim_{n_i \to\infty} (n_i - 1)\mathbb{P}(X \geq g(n_i)) \geq 1.$$

The final inequality follows from (34). Moreover, since $f(n) \geq g(n)$ for all $n \in \mathbb{N}$, we also have that

$$\limsup_{n\to\infty} n\mathbb{P}(X > f(n)) \leq \limsup_{n\to\infty} n\mathbb{P}(X > g(n)) = 1.$$

Here, the last inequality again follows from (34). Combining this with (35) implies that $C(f) = 1$. $\square$

THEOREM 5.6. *Consider the random polynomials*

$$H_n(z) := \sum_{i=0}^{n} \xi_i z^i,$$

where $\xi_0, \xi_1, \ldots$ is a sequence nondegenerate i.i.d. complex-valued random variables. Then Z_{H_n} converges weak* in probability to $\frac{1}{2\pi} d\theta$ if and only if the random variables ξ_i satisfy (13).

PROOF. The "if" statement is a consequence of Theorem 5.3, with $K = \{z : |z| = 1\}$ and $\tau = \frac{1}{2\pi}\,d\theta$ (see Example 2.1). Now suppose that the random variables ξ_i fail to satisfy (13).

Recall that Z_{H_n} is a random variable on the space $\mathcal{M}$ of probability measures on $\mathbb{C}$ with the weak* topology. To show that Z_{H_n} does not converge in probability to $\frac{1}{2\pi}\,d\theta$, it suffices to find an open set $U \subset \mathcal{M}$ that contains $\frac{1}{2\pi}\,d\theta$, and such that

$$\text{(36)} \qquad \liminf_{n \to \infty} \mathbb{P}(Z_{H_n} \in U) < 1.$$

Let

$$\mathcal{O} = \{z \in \mathbb{C} : 1/2 < |z| < 3/2\} \quad \text{and let} \quad U = \{\mu \in \mathcal{M} : \mu(\mathcal{O}) > 1/2\}.$$

U is open in $\mathcal{M}$ by the Portmanteau theorem, and contains $\frac{1}{2\pi}\,d\theta$. We will show that

$$\text{(37)} \qquad \limsup_{n \to \infty} \mathbb{P}(Z_{H_n}(\mathcal{O}) = 0) > 0,$$

which in turn proves (36), showing that Z_{H_n} does not converge in probability to $\frac{1}{2\pi}\,d\theta$. To prove (37), we show that

$$\text{(38)} \qquad \begin{aligned} &\limsup_{n \to \infty} \mathbb{P}\big(\text{There exists } m \in \{0, \dots, n\} \text{ such that} \\ &\quad |\xi_m| \geq e^n |\xi_j| \text{ for all } j \in \{0, \dots, n\},\ j \neq m\big) > 0. \end{aligned}$$

To see why this is sufficient, observe that on the event in (38), for all large enough n, we have that

$$|\xi_m| |z|^m > \sum_{\substack{j \in [0,n] \\ j \neq m}} |\xi_j| |z|^j \geq \left| \sum_{\substack{j \in [0,n] \\ j \neq m}} \xi_j z^j \right|$$

for all $z \in \mathcal{O}$. Therefore, H_n has no zeroes in $\mathcal{O}$ on this event. We now prove (38). For a function $f : [0, \infty) \to [0, \infty)$ and $n \in \mathbb{N}$, define

$$D_n(f) := n\mathbb{P}(|\xi_0| > f(n)) \quad \text{and} \quad D(f) := \limsup_{n \to \infty} D_n(f).$$

Since (13) is not satisfied by the ξ_i, we can apply Lemma 5.5 to the random variable $\log|\xi_0|$ getting a function f satisfying properties (i) and (ii) of that lemma. Letting $g = e^f$, we then have that:

(i) $D(g) \in (0, \infty)$.
(ii) For every $x, y \in [0, \infty)$, we have that $g(x + y) \geq e^y g(x)$.

For $\alpha \in (0, \infty)$, define $g_\alpha(x) := g(\alpha x)$. Observe that $\alpha D(g_\alpha) = D(g)$. Now define

$$B_{n,\alpha} = \big|\{i \leq n : |\xi_i| > g_\alpha(n)\}\big|.$$

For each α, $B_{n,\alpha}$ is a binomial random variable with n trials and mean $D_n(g_\alpha)$. Now for any $\alpha > 1$, there exists a subsequence $\{n_i\}$ such that

$$\lim_{n_i \to \infty} \mathbb{E}B_{n_i,\alpha} = \frac{D(g)}{\alpha} \quad \text{whereas} \quad \limsup_{n_i \to \infty} \mathbb{E}B_{n_i,\alpha-1} \le \frac{D(g)}{\alpha - 1}.$$

Poisson convergence for binomial random variables implies that

$$\liminf_{n_i \to \infty} \mathbb{P}(B_{n_i,\alpha} = 1) - \mathbb{P}(B_{n_i,\alpha-1} \ge 2) \ge \mathbb{P}(Y_{D(g)/\alpha} = 1) - \mathbb{P}(Y_{D(g)/(\alpha-1)} \ge 2),$$

where the random variables Y_z are Poisson with mean z. For large enough α, the right-hand side above is strictly positive. By property (ii) of the function g, this implies (38). $\square$

6. Almost sure convergence.

In this section, we consider almost sure convergence of zeros for random polynomial ensembles. The types of random polynomial ensembles that we consider are slightly different from the random orthogonal polynomial ensembles considered in Section 5, but all the tools that we have developed so far are still applicable in this setting.

Our main tool for proving almost sure convergence is a small ball probability theorem of Nguyen and Vu ([16], Corollary 2.10). In [16], this theorem is stated for real-valued random variables ξ satisfying the condition

$$\mathbb{P}(1 \le |\xi_1 - \xi_2| \le C) \ge 1/2$$

for some value of C, where ξ_1, ξ_2 are independent copies of ξ. However, the proof can easily be extended to all nondegenerate real random variables, which satisfy

$$\mathbb{P}(b_1 \le |\xi_1 - \xi_2| \le b_2) > 0$$

for some $b_1, b_2 > 0$ at the expense of changing some of the constants (this version of their theorem is stated in [17]). The proof can also be extended to accommodate complex-valued random variables by making a few other minor modifications.

The result we state and use here is weaker than the result from [16], since we do not need to use information about the arithmetic structure of the coefficient set $\mathcal{A}$.

THEOREM 6.1. *Let $0 < \epsilon < 1$, $C > 0$ be arbitrary constants, and $\beta > 0$ a parameter that may depend on n. Suppose that $\mathcal{A} = \{a_0, a_1, a_2, \ldots, a_n\}$ is a (multi)-subset of $\mathbb{C}$ such that $\sum_{i=0}^{n} |a_i|^2 = 1$, and let $\xi_0, \xi_1, \ldots, \xi_n$ be i.i.d. nondegenerate complex random variables. Suppose additionally that*

$$\mathcal{Q}\left(\sum_{i=0}^{n} \xi_i a_i; \beta\right) \ge n^{-C}.$$

Then there exists a constant D depending only on ξ_0 and ϵ such that for any number $n' \in (n^\epsilon, n)$, at least $n - n'$ elements of $\mathcal{A}$ can be covered by a union of $\max(\frac{Dn^C}{\sqrt{n'}}, 1)$ balls of radius β.

We translate this into a lemma that can be applied to prove almost sure convergence of random polynomial zeros.

LEMMA 6.2. *Let* $\{a_{n,i} : 0 \le i \le n, i, n \in \{0, 1, \ldots\}\}$ *be a triangular array of complex numbers such that*

$$(39) \qquad \lim_{n \to \infty} \frac{1}{2n} \log \left(\sum_{i=0}^{n} |a_{n,i}|^2 \right) = A.$$

Let $\|a^{(n)}\|$ *be the Euclidean norm of* $(a_{0,n}, \ldots, a_{n,n})$, *and let* $w_{n,i} = a_{n,i}/\|a^{(n)}\|$. *Suppose that for any* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that for all large enough* n, *the set*

$$\mathcal{W}_n = \{w_{n,i} : 0 \le i \le n\}$$

cannot be covered by a union of $n^{2/3+\delta}$ *balls of radius* $e^{-\epsilon n}$. *If* $\{\xi_0, \xi_1, \ldots\}$ *is a sequence of nondegenerate i.i.d. complex random variables, then*

$$\liminf_{n \to \infty} \frac{1}{n} \log \left| \sum_{i=0}^{n} \xi_i a_{n,i} \right| \ge A \quad \text{almost surely.}$$

PROOF. For any $\epsilon > 0$, we have that

$$\mathbb{P}\left(\frac{1}{n} \log \left| \sum_{i=0}^{n} \xi_i a_{n,i} \right| < A - 2\epsilon \right) \le \mathcal{Q}\left(\sum_{i=0}^{n} \xi_i a_{n,i}; e^{n(A-2\epsilon)} \right).$$

Therefore, by the Borel–Cantelli lemma, to prove the lemma it is enough to show that for every $\epsilon > 0$, we have

$$(40) \qquad \sum_{i=0}^{\infty} \mathcal{Q}\left(\sum_{i=0}^{n} \xi_i a_{n,i}; e^{n(A-2\epsilon)} \right) < \infty.$$

For all large enough n, (39) guarantees that

$$\left(\sum_{i=0}^{n} |a_{n,i}|^2 \right)^{1/2} \in \left[e^{n(A-\epsilon)}, e^{n(A+\epsilon)} \right].$$

Therefore, for such n, the rescaling property of $\mathcal{Q}$ (equation (23)) implies that

$$(41) \qquad \mathcal{Q}\left(\sum_{i=0}^{n} \xi_i a_{n,i}; e^{n(A-2\epsilon)} \right) \le \mathcal{Q}\left(\sum_{i=0}^{n} \xi_i w_{n,i}; e^{-\epsilon n} \right).$$

Now let δ be as in the statement of the lemma for the above value of ϵ. Take $n' = n^{2/3}$ in Theorem 6.1. By that theorem, there exists a constant D independent of n such that if the right-hand side of (41) is greater than $n^{-1-\delta/2}$, then at least $n - n^{2/3}$ elements of $\mathcal{W}_n = (w_{0,n}, \ldots w_{n,n})$ can be covered by a union of $Dn^{2/3+\delta/2}$ balls

of radius $e^{-\epsilon n}$. This implies that all elements of $\mathcal{W}_n$ can be covered by a union of $(1 + D)n^{2/3+\delta/2}$ balls of radius $e^{-\epsilon n}$. By the assumption on the array $\{w_{n,i}\}$, this can only occur for finitely many n.

Therefore, the right-hand side side of (41) is less than $n^{-1-\delta/2}$ for all large enough n, and so it is summable in n; hence so is the left-hand side, proving (40).

$\square$

We can now check that certain sequences of random polynomials satisfy the conditions of Lemma 6.2 for almost every value of z. The setting is as follows. Consider coefficients

$$\{f_{n,k} \in \mathbb{C} : k \in \{0, \ldots, n\}, n \in \{0, 1, \ldots\}\}$$

satisfying the following two conditions:

(i) There exists a continuous function $V : \mathbb{C} \to \mathbb{R}$ such that for every $z \in \mathbb{C}$, we have that

$$(42) \qquad \lim_{n \to \infty} \frac{1}{n} \log\left(\sum_{k=0}^{n} |f_{n,k}||z|^k\right) = V(z).$$

Moreover, this convergence is locally uniform. We assume that $V(z)$ is subharmonic, and that $V(z) - \log(|z|)$ is bounded as $z \to \infty$. This growth condition ensures that $\frac{1}{2\pi}\Delta V(z)$ is a probability measure.

(ii) There exists a set $\mathcal{D} \subset \mathbb{C}$ whose complement has Lebesgue measure zero, such that for every $z \in \mathcal{D}$, the following holds. For any $\epsilon > 0$, there exists an $n_0(\epsilon, z) \in \mathbb{N}$ and $\delta(\epsilon, z) > 0$ such that for all $n \geq n_0(\epsilon, z)$, we have that

$$(43) \qquad \left|\{k \in [0, n] : |f_{n,k}||z|^k \geq e^{n(V(z)-\epsilon)}\}\right| \geq n^{2/3+\delta(\epsilon,z)}.$$

Now consider a sequence of i.i.d. nondegenerate complex-valued random variables $\{\xi_0, \xi_1, \ldots\}$ with $\mathbb{E}\log(1 + |\xi_0|) < \infty$, and define the random polynomials

$$(44) \qquad G_n(z, \omega) = \sum_{k=0}^{n} \xi_k f_{n,k} z^k.$$

Then we have the following theorem. Again, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which the random variables ξ_i are defined.

THEOREM 6.3.   *Under conditions* (i) *and* (ii) *above and condition* (16), *the following statements hold*:

(I) *For almost every* $\omega \in \Omega$, *the sequence* $\{\frac{1}{n}\log|G_n| : n \in \mathbb{N}\}$ *is locally bounded above, and*

$$(45) \qquad \limsup_{n \to \infty} \frac{1}{n} \log|G_n(z, \omega)| \leq V(z) \quad \text{for every } z \in \mathbb{C}.$$

(II) *For a countable dense set of points $\{z_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$, we have that*

$$\lim_{n \to \infty} \frac{1}{n} \log|G_n(z_i, \omega)| = V(z_i) \quad \text{for almost every } \omega \in \Omega.$$

To prove the above theorem, we need a simple lemma bounding the Lebesgue measure of the set where a polynomial can take small values. Here and throughout the remainder of this section, $\mathfrak{m}$ is Lebesgue measure in $\mathbb{C}$.

LEMMA 6.4. *Let P_n be a degree n polynomial with leading coefficient c. Then for any $r > 0$,*

$$\mathfrak{m}\{z : |P_n(z)| \le r^n\} \le \pi n r^2 |c|^{-2/n}.$$

PROOF. Let $z_1, \ldots, z_n$ be the roots of P_n and let $\overline{B}(z_i, r)$ be the closed ball of radius r around z. For any $r > 0$, if $z \notin \overline{B}(z_i, r|c|^{-1/n})$ for all $i \in \{1, \ldots, n\}$, then $|P_n(z)| > r^n$. The measure of $\bigcup_{i=1}^n \overline{B}(z_i, r)$ is at most $\pi n r^2 |c|^{-2/n}$. $\square$

Note that the factor of n in Lemma 6.4 can be improved upon by Cartan's estimate (see [15], Lecture 11). We do not need this level of precision here.

PROOF OF THEOREM 6.3. Conclusion (I) holds by Lemma 3.3. We now prove (II). We will in fact show that for almost every $z \in \mathbb{C}$, we have

(46) $$\liminf_{n \to \infty} \frac{1}{n} \log|G_n(z)| \ge V(z) \quad \text{almost surely.}$$

Together with (I) this suffices to prove the result. We want to apply Lemma 6.2 to prove (46). Fix $\epsilon > 0$, let $z_0 \in \mathcal{D}$, and define

$$J_n(z_0, \epsilon) = \{k \in [0, n] : e^{n(V(z_0) - \epsilon)} \le |f_{n,k}||z_0|^k \le e^{n(V(z_0) + \epsilon)}\}.$$

By condition (ii) on the coefficients $f_{n,k}$, there exists a $\delta > 0$ such that for all large enough n, the lower bound above holds for at least $n^{2/3+\delta}$ values of k. Moreover, for large enough n, the upper bound holds for all k by condition (i). Therefore, $|J_n(z_0, \epsilon)| \ge n^{2/3+\delta}$ for large enough n.

Now, for $z \in \mathbb{C}$, let $\|(f, z)_n\|$ be the Euclidean norm of the vector $(f_{n,0}, f_{n,1}z, \ldots, f_{n,n}z^n)$. Let

$$w_{n,k}(z) = \frac{f_{n,k}z^k}{\|(f, z)_n\|} \quad \text{and define} \quad \mathcal{W}_n(z) = \{w_{n,k}(z) : k \in \{0, 1, \ldots, n\}\}.$$

Since $z_0 \ne 0$ (note that $0 \notin \mathcal{D}$), if $z \in B(z_0, |z_0|(1 - e^{-\epsilon}))$, then

$$\frac{w_{n,k}(z)}{w_{n,k}(z_0)} = \frac{|f_{n,k}z^k|}{|f_{n,k}z_0^k|} \ge e^{-\epsilon n}$$

for any k, n. Therefore, for large enough n, if $k \in J_n(z_0, \epsilon)$ and $z \in B(z_0, |z_0|(1 - e^{-\epsilon}))$, then $|w_{n,k}(z)| \geq e^{-3\epsilon n}$. Here, we have estimated $\|(f, z)_n\|$ using condition (i) and Lemma 3.4.

Now suppose that $z \in B(z_0, |z_0|(1 - e^{-\epsilon}))$ is such that $\mathcal{W}_n(z)$ can be covered by $n^{2/3+\delta/2}$ balls of radius $e^{-7\epsilon n}$. As long as n is large enough, there must exist $k_1 < k_2 \in J_n(z_0, \epsilon)$ such that $|k_1 - k_2| \leq n^{1-\delta/2}$ and $|w_{n,k_1}(z) - w_{n,k_2}(z)| \leq e^{-6\epsilon n}$. We can write

$$
\begin{aligned}
\left| w_{n,k_1}(z) - w_{n,k_2}(z) \right| &= \left| w_{n,k_1}(z) \right| \left| \frac{f_{n,k_2}}{f_{n,k_1}} z^{k_2-k_1} - 1 \right| \\
&\geq e^{-3\epsilon n} \left| \frac{f_{n,k_2}}{f_{n,k_1}} z^{k_2-k_1} - 1 \right|.
\end{aligned}
$$

(47)

Since both $k_1, k_2 \in J_n(z_0, \epsilon)$, we have that $|f_{n,k_2}/f_{n,k_1}| \geq e^{-2\epsilon n} |z_0|^{k_1-k_2}$. Using this, we can apply Lemma 6.4 to (47) to get

$$
\mathfrak{m}\{ z \in B(z_0, |z_0|(1 - e^{-\epsilon})) : |w_{n,k_1}(z) - w_{n,k_2}(z)| < e^{-6\epsilon n} \} \leq \pi n |z_0|^2 e^{-10\epsilon n^{\delta/2}}.
$$

Therefore, by a union bound,

$$
L_n := \mathfrak{m}\{ z \in B(z_0, |z_0|(1 - e^{-\epsilon})) :
$$
$$
\mathcal{W}_n(z) \text{ can be covered by } n^{2/3+\delta/2} \text{ balls of radius } e^{-7\epsilon n} \}
$$

is at most $n^3 \pi |z_0|^2 e^{-10\epsilon n^{\delta/2}}$ for all large enough n. The sequence L_n is summable in n, so by the Borel–Cantelli lemma, there exists a $\delta > 0$ such that for almost every $z \in B(z_0, |z_0|(1 - e^{-\epsilon}))$, the set $\mathcal{W}_n(z)$ can be covered by $n^{2/3+\delta}$ balls of radius $e^{-7\epsilon n}$ for at most finitely many n.

This holds for every $z_0 \in \mathcal{D}$ for some δ. Therefore, we can extend this result to get that for almost every $z \in \mathbb{C}$, there exists a $\delta > 0$ such that the set $\mathcal{W}_n(z)$ can be covered by $n^{2/3+\delta}$ balls of radius $e^{-7\epsilon n}$ for at most finitely many n.

Since $\epsilon > 0$ was arbitrary, this implies that for almost every $z \in \mathbb{C}$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that the set $\mathcal{W}_n(z)$ can be covered by $n^{2/3+\delta}$ balls of radius $e^{-7\epsilon n}$ for at most finitely many n. By Lemma 6.2, this implies (46) for almost every $z \in \mathbb{C}$. $\quad \square$

We can now use Theorem 6.3 to prove almost sure convergence of the normalized counting measure of the zeros for G_n.

THEOREM 6.5. *Let $G_n(z, \omega)$ be as in (44), where the coefficients $f_{n,k}$ satisfy conditions* (i) *and* (ii). *Let $Z_{G_n}(\omega)$ be the normalized counting measure of the zeros of G_n. Then for almost every $\omega \in \Omega$, we have that*

$$
Z_{G_n}(\omega) \to \frac{1}{2\pi} \Delta V(z) \quad \text{in the weak* topology.}
$$

PROOF. This follows immediately from Theorem 6.3 and Theorem 4.1 (note that the limit $V(z)$ in Theorem 6.3 is subharmonic). $\square$

COROLLARY 6.6. *Let $G_n(z, \omega)$ be as in* (44), *where the coefficients $f_{n,k}$ satisfy conditions* (i) *and* (ii). *Suppose the ξ_k are nondegenerate i.i.d. complex valued random variables satisfying* (13). *Then*

$$Z_{G_n}(\omega) \to \frac{1}{2\pi} \Delta V(z)$$

in probability in the weak* topology.

PROOF. We want to apply Theorem 4.2 (or more precisely, Remark 4.3). We just need to check the condition that for a countable dense set $\{z_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$, that

$$\lim_{n \to \infty} \frac{1}{n} \log |G_n(z_i, \omega)| = V(z_i)$$

in probability. The pointwise upper bound on the left-hand side above is established analogously to (26). The lower bound follows from the proof of Theorem 6.3 (note that this bound only requires the nondegeneracy of the ξ_ks). $\square$

Special cases of Theorem 6.5. We can consider the following types of coefficients, first considered by Kabluchko and Zaporozhets in [13].

Assume that there is a function $f : [0, 1] \to [0, \infty)$ satisfying the following conditions:

(i) $f(t)$ is positive and continuous for all t.
(ii) $\lim_{n \to \infty} \sup_{k \in [0, n]} ||f_{n,k}|^{1/n} - f(k/n)| = 0$.

It is easy to check that if the coefficients $f_{n,k}$ satisfy the above properties, then they satisfy the conditions required for Theorem 6.3. This gives us the following corollary.

COROLLARY 6.7. *Let $G_n(z) = \sum_{k=0}^{n} \xi_k f_{n,k} z^k$ be the random polynomial with coefficients $f_{n,k}$ as above, where the ξ_is are nondegenerate i.i.d. complex random variables satisfying $\mathbb{E} \log(1 + |\xi_0|) < \infty$. Let Z_{G_n} be the normalized counting measure of the zeros of G_n. For each z, the function $V(z)$ of* (42) *is given by*

$$V(z) = \sup_{t \in [0, 1]} \log |z|^t f(t).$$

Then for almost every $\omega \in \Omega$, we have that

$$Z_{G_n}(\omega) \to \frac{1}{2\pi} \Delta V(z) \quad \text{in the weak* topology.}$$

We can also use Theorem 6.5 to look at the roots of certain random orthogonal polynomial ensembles. Since Theorem 6.5 allows for an array of coefficients $\{f_{n,k}\}$ rather than just a sequence $\{f_n\}$, we can consider orthogonal polynomials with respect to both a weight function and a measure. Random polynomial ensembles of this form have been previously studied in [1, 2] and [5]. We give an example where condition (ii) on the coefficients $f_{n,k}$ can be directly verified.

Let $K \subset \mathbb{C}$ be compact, and let $S : K \to \mathbb{R}$ be a real-valued continuous function. Define the weighted Green's function

$$V_{K,S}(z) = \sup\left\{\frac{1}{\deg(p)}\log|p(z)|\right|$$

$$p \text{ is a nonconstant polynomial and } \|pe^{-nS}\|_K \leq 1\biggr\}$$

$$= \sup\{u \mid u \in \mathcal{L}(\mathbb{C}), u \leq S \text{ on } K\}.$$

We denote by $V_{K,S}^*$ the upper semicontinuous regularization of $V_{K,S}$. The distribution $\frac{1}{2\pi}\Delta V_{K,S}^*$ is a probability measure on K.

We say that K is *locally regular* if for all $a \in K$ and $r > 0$, the function $V_{K \cap \overline{B}(a,r)}$ is continuous at a, where $\overline{B}(a,r)$ denotes the closed disk centred at a with radius r. If K is locally regular and S is continuous, then $V_{K,S}$ is continuous, and so $V_{K,S} = V_{K,S}^*$.

Let τ be a finite, positive, Borel measure on K. We say that τ satisfies the strong Bernstein–Markov property if for all S continuous and $\epsilon > 0$ there is a constant $C = C(S, \epsilon)$ such that for every n, we have that

$$(48) \qquad \|pe^{-nS}\|_K \leq Ce^{\epsilon n}\|p\|_{L^2(e^{-2nS}\tau)}$$

for all $p \in \mathcal{P}_n$.

Now consider a locally regular nonpolar compact set K, a continuous function S, and a finite measure τ on K satisfying the strong Bernstein–Markov property. For each n, we define orthonormal polynomials $\{q_0^{(n)}, \ldots, q_n^{(n)}\}$ by applying the Gram–Schmidt procedure to the monomials $\{1, z, \ldots, z^n\}$ in $L^2(e^{-2nS}\tau)$. For every $z \in \mathbb{C}$, we have that

$$(49) \qquad V_{K,S}(z) = \lim_{n \to \infty}\frac{1}{2n}\log\left(\sum_{j=0}^{n}|q_j^{(n)}(z)|^2\right),$$

where the convergence is uniform on compact subsets of $\mathbb{C}$ (see [5]).

We say that K is circularly symmetric if for every $z \in \mathbb{C}$, we have that $z \in K$ if and only $|z| \in K$. S is circularly symmetric if $S(z) = S(|z|)$, and τ is circularly symmetric if for any rotation R, the pushforward measure $R_*\tau$ is equal to τ.

If K, τ and S are all circularly symmetric, then the polynomials $q_j^{(n)}$ are of the form $f_{n,j}z^j$. In this case, we can apply Theorem 6.5 to random polynomials formed from the set $\{q_j^{(n)}\}$.

COROLLARY 6.8. *Suppose that $K \subset \mathbb{C}$ is a locally regular nonpolar compact set, S is a continuous function on K, and τ is a finite measure on K satisfying the strong Bernstein–Markov property. Suppose additionally that K, τ, and S are all circularly symmetric, and let*

$$\{q_j^{(n)} : j \in \{0, 1, \ldots, n\}, n \in \mathbb{N}\}$$

be as constructed above. Define

(50)
$$G_n(z, \omega) = \sum_{j=0}^{n} \xi_j q_j^{(n)}(z),$$

where $\{\xi_0, \xi_1, \ldots\}$ is a sequence of nondegenerate i.i.d. complex random variables such that $\mathbb{E} \log(1 + |\xi_0|) < \infty$. Let Z_{G_n} be the normalized counting measure of the zeros of G_n. Then for almost every $\omega \in \Omega$, we have that

$$Z_{G_n} \to \frac{1}{2\pi} \Delta V_{K,S} \quad \text{in the weak* topology on probability measures on } \mathbb{C}.$$

PROOF. As mentioned above, each of the polynomials $q_j^{(n)}(z)$ is of the form $f_{n,j} z^j$ for some real number $f_{n,j}$. The coefficients $f_{n,j}$ satisfy condition (i) on the coefficients in Theorem 6.3 by (49).

We now show that the weights $f_{n,j}$ satisfy condition (ii). For any j, we have that

$$f_{n,j} = \left(\int_K |z|^{2j} e^{-2nS(z)} \, d\tau(z) \right)^{-1/2}.$$

By this formula, we get that

$$f_{n,0}^{1/n} \to \inf_{z \in \text{supp}(\tau)} e^{S(z)} \quad \text{and} \quad f_{n,1}^{1/n} \to \inf_{z \in \text{supp}(\tau) \setminus \{0\}} e^{S(z)} \quad \text{as } n \to \infty.$$

To prove this first formula, set $S_0 = \inf_{z \in \text{supp}(\tau)} S(z)$. From the formula for $f_{n,0}$, for all $n \in \mathbb{N}$ and $\epsilon > 0$, we have

$$e^{nS_0} \leq f_{n,0} \leq C_\epsilon^{-1} e^{n(S_0 + \epsilon)},$$

where $C_\epsilon = \tau\{z : S(z) \leq S_0 + \epsilon\}$. The continuity of S guarantees that $C_\epsilon > 0$ for all $\epsilon > 0$. Hence raising the above inequality to the power of $1/n$, and taking $n \to \infty$ and $\epsilon \to 0$ shows the desired convergence. The second formula can be shown in a similar fashion.

Therefore, for any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that for all $n \geq N$, we have that

$$f_{n,0} \leq e^{\epsilon n} f_{n,1},$$

so for any fixed $z \in \mathbb{C} \setminus \{0\}$ and $\epsilon > 0$, for all large enough n we have that

(51) $$\max\{f_{n,j} |z|^j : j \in \{0, \ldots, n\}\} \leq e^{\epsilon n} \max\{f_{n,j} |z|^j : j \in \{1, \ldots, n\}\}.$$

Therefore, to prove condition (ii) required for Theorem 6.3 it is enough to show that for any $\epsilon > 0$, there exists an N such that for all $n \geq N$, if $j_1, j_2 \in \{1, \ldots, n\}$ and $|j_1 - j_2| \leq n^{3/4}$, then

$$(52) \qquad \frac{f_{n,j_2}}{f_{n,j_1}} \geq e^{-\epsilon n}.$$

Combined with (51), this shows that for any $z \in \mathbb{C} \setminus \{0\}$, for all large enough n, at least $n^{3/4}$ values of $f_{n,j}|z|^j$ are close to the maximum value, which must itself be close to $e^{nV(z)}$ by (49) and Lemma 3.4. This gives condition (ii). To prove (52), fix $\epsilon > 0$, and choose $\delta > 0$ small enough so that

$$\max_{z,w \in B(0,\delta)} |S(z) - S(w)| \leq \frac{\epsilon}{4}.$$

Then for any n, j_1, j_2, we have that

$$(53) \qquad e^{-\epsilon n} M \leq \frac{f_{n,j_2}^2}{f_{n,j_1}^2} \quad \text{where}$$

$$M = \frac{\int_{B(0,\delta)} |z|^{2j_1} e^{-2nS(0)} d\tau(z) + \int_{K \setminus B(0,\delta)} |z|^{2j_1} e^{-2nS(z)} d\tau(z)}{\int_{B(0,\delta)} |z|^{2j_2} e^{-2nS(0)} d\tau(z) + \int_{K \setminus B(0,\delta)} |z|^{2j_2} e^{-2nS(z)} d\tau(z)}.$$

Now let $R > 1$ be large enough so that $K \subset B(0, R)$. For all large enough n, whenever $|j_1 - j_2| \leq n^{3/4}$, we have that

$$(54) \qquad \frac{\int_{K \setminus B(0,\delta)} |z|^{2j_1} e^{-2nS(z)} d\tau(z)}{\int_{K \setminus B(0,\delta)} |z|^{2j_2} e^{-2nS(z)} d\tau(z)} \geq R^{-|j_1 - j_2|} \geq e^{-\epsilon n}.$$

If $\tau(B(0, \delta) \setminus \{0\}) = 0$, then this proves (52). If not, then there exists a $\gamma > 0$ such that for every $j \in \{1, 2, \ldots\}$, we have that

$$\int_{B(0,\delta)} |z|^j d\tau(z) \leq 2 \int_{B(0,\delta) \setminus B(0,\gamma)} |z|^j d\tau(z).$$

Therefore, as in (54), for all large enough n, whenever $|j_1 - j_2| \leq n^{3/4}$, we have that

$$(55) \qquad \frac{\int_{B(0,\delta)} |z|^{2j_1} e^{-2nS(0)} d\tau(z)}{\int_{B(0,\delta)} |z|^{2j_2} e^{-2nS(0)} d\tau(z)} \geq \frac{\int_{B(0,\delta)} |z|^{2j_1} d\tau(z)}{2 \int_{B(0,\delta) \setminus B(0,\gamma)} |z|^{2j_2} d\tau(z)} \geq e^{-\epsilon n}.$$

Combining this with (53) implies (52). $\quad\square$

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