The Annals of Probability
2019, Vol. 47, No. 4, 2230–2256
https://doi.org/10.1214/18-AOP1307

THOULESS–ANDERSON–PALMER EQUATIONS FOR GENERIC *p*-SPIN GLASSES

BY ANTONIO AUFFINGER[1] AND AUKOSH JAGANNATH[2]

Northwestern University and Harvard University

We study the Thouless–Anderson–Palmer (TAP) equations for spin glasses on the hypercube. First, using a random, approximately ultrametric decomposition of the hypercube, we decompose the Gibbs measure, $\langle\cdot\rangle_N$, into a mixture of conditional laws, $\langle\cdot\rangle_{\alpha,N}$. We show that the TAP equations hold for the spin at any site with respect to $\langle\cdot\rangle_{\alpha,N}$ simultaneously for all α. This result holds for generic models provided that the Parisi measure of the model has a jump at the top of its support.

1. Introduction. The Thouless–Anderson–Palmer (TAP) equations were introduced by Thouless, Anderson and Palmer [15] as the mean field equations for the Sherrington–Kirkpatrick (SK) model of spin glasses. These equations can be stated informally as follows. For each $\sigma \in \Sigma_N = \{-1, 1\}^N$, let

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j$$

be the Hamiltonian for the SK model. Here, g_{ij} are i.i.d. standard Gaussian random variables for $1 \le i \le j \le N$ and $g_{ij} = g_{ji}$. Let

$$\mu_N(\{\sigma\}) = \frac{e^{-\beta H_N(\sigma)+h\sum_{i=1}^{N}\sigma_i}}{Z_N}$$

be the Gibbs measure of this system at inverse temperature, β, and external field, h. Here, β and h are nonnegative real numbers, and Z_N is chosen such that μ_N is a probability measure on Σ_N. We denote integration of a quantity, say σ_i, against μ_N as $\langle\sigma_i\rangle$. The TAP equations state that in the limit that $N \to \infty$, we have that

$$\langle\sigma_i\rangle_\alpha \approx \tanh\biggl(h + \biggl\langle \frac{1}{\sqrt{N}}\beta \sum_j (g_{ij} + g_{ji})\sigma_j \biggr\rangle_\alpha - \beta^2(1 - q_*)\langle\sigma_i\rangle_\alpha\biggr), \tag{1.1}$$

for some $q_* \in [0, 1]$ and for some random measure for which integration is denoted by $\langle\cdot\rangle_\alpha$.

Received May 2017; revised August 2018.
[1]Supported by NSF Grant DMS-1597864 and NSF CAREER Grant DMS-1653552.
[2]Supported by NSF Grant OISE-1604232.
MSC2010 subject classifications. 82D30, 60G15.
Key words and phrases. Spin glasses, TAP, random measures, ultrametricity, cluster decomposition.

There have been two approaches to proving the TAP equations rigorously. The first approach is to take $\langle\cdot\rangle_\alpha$ as integration with respect to the Gibbs measure. This has been done by Talagrand [13] and Chatterjee [7] at sufficiently high temperature for the SK model where they establish (1.1) under this interpretation. A second approach, introduced by Bolthausen [6], is to interpret $\langle\sigma_i\rangle_\alpha$ as a vector in high dimensions and to understand (1.1) through a fixed point iteration scheme. There, he showed that this iteration converges to a unique solution of (1.1) in the entire predicted high temperature regime. At low temperature, as far as we know, there is no rigorous proof of (1.1). In this regime it is expected that there are many distinct measures, $\mu_{\alpha,N}$, called "pure states," whose convex combination is μ_N and each of which satisfies (1.1).

The first goal of this paper is to study (1.1) for generic mixed p-spin glasses without an assumption on the temperature. These models are defined as follows. Consider the mixed p-spin glass Hamiltonian, $H_N(\sigma)$, which is the centered Gaussian process on $\Sigma_N = \{-1, 1\}^N$ with covariance

$$\mathbb{E}H_N(\sigma^1)H_N(\sigma^2) = N\xi(R_{12}),$$

where $R_{12} = \frac{1}{N}\sum\sigma_i^1\sigma_i^2$ is called the overlap and $\xi(t) = \sum_{p\geq 2}\beta_p t^p$ is called the model. We let μ_N denote the corresponding Gibbs measure and $\langle\cdot\rangle$ expectation under products of μ_N. The SK model corresponds to $\xi(t) = \beta_2 t^2$. A mixed p-spin glass model is called *generic*, if the set $\{t^p : \beta_p > 0\}$ is total in $(C([-1, 1]), \sup|\cdot|)$.

Denote by ζ_N the distribution of the overlap under the measure $\mathbb{E}\mu_N^{\otimes 2}$, that is,

$$\zeta_N(A) = \mathbb{E}\langle\mathbf{1}(R_{12} \in A)\rangle$$

for any measurable $A \subset [-1, 1]$. It is known that ζ_N converges to ζ, where ζ is the unique minimizer of the Parisi formula [4, 11]. It is also known that generic models satisfy the Ghirlanda–Guerra identities in the limit [9, 11]. As a result their asymptotic Gibbs measures [2] are known to have ultrametric support by Panchenko's ultrametricity theorem [10]. We assume that ζ has a jump at the top of its support. That is, if $q_* := \sup\operatorname{supp}(\zeta)$, we assume that

$$\zeta(\{q_*\}) > 0. \tag{1.2}$$

This assumption is expected to hold in a wide range of models at all temperatures. For more on this, see Remark 1.4.

This ultrametric structure is the starting point for our study of the analogue of (1.1) for generic models. It was shown in [8] that, as a consequence of Panchenko's ultrametricity theorem, Σ_N can be decomposed as the disjoint union of a collection of clusters, $\{C_{\alpha,N}\}_{\alpha\in\mathbb{N}}$, which satisfy certain ultrametric-type properties. Heuristically, these clusters are essentially balls of overlap q_*. Within a cluster the points are at overlap roughly q_*; between clusters the points have overlap less than $q_* - o_N(1)$ with high μ_N probability. We recall the precise definition of these sets in the Appendix. A similar decomposition was obtained by Talagrand in [14].

For each of these clusters, $C_{\alpha,N}$, we define

$$\mu_{\alpha,N}(\cdot) := \mu_N(\cdot|C_{\alpha,N}). \tag{1.3}$$

That is, $\mu_{\alpha,N}$ is the Gibbs measure conditioned on the set $C_{\alpha,N}$ with the convention that if $C_{\alpha,N} = \varnothing$, then $\mu_{\alpha,N} = \delta_{(1,\dots,1)}$. This yields a decomposition of the Gibbs measure μ_N as

$$\mu_N(\cdot) = \sum_\alpha \mu_{\alpha,N}(\cdot)\mu_N(C_{\alpha,N}) + o_N(1). \tag{1.4}$$

Here, $o_N(1)$ means that $\mu_N((\bigcup_\alpha C_{\alpha,N})^c)$ goes to zero in probability as N goes to infinity. The sets $C_{\alpha,N}$ are also ordered with respect to their Gibbs masses, that is,

$$\mu_N(C_{1,N}) \geq \mu_N(C_{2,N}) \geq \mu_N(C_{3,N}) \geq \cdots.$$

Integration with respect to the conditional measure $\mu_{\alpha,N}$ will be denoted by $\langle\cdot\rangle_{\alpha,N}$.

We now state our main theorem which is the equivalent of (1.1) for generic models. For $\sigma \in \Sigma_N$, let

$$y_N(\sigma) = \sum_{p\geq 2} \frac{\beta_p}{N^{\frac{p-1}{2}}} \sum_{2\leq i_2,\dots,i_p\leq N} J_{i_2\cdots i_p}\sigma_{i_2}\cdots\sigma_{i_p}, \tag{1.5}$$

with $J_{i_2,\dots,i_p} = g_{1i_2,\dots,i_p} + g_{i_2 1,\dots,i_p} + \cdots + g_{i_2,\dots,i_p 1}$, where $g_{i_1 i_2,\dots,i_p}$, $1 \leq i_1, \dots, i_p \leq N$ are i.i.d. standard Gaussian random variables. We call σ_1 the spin of the first particle and y_N the local field on the first particle. Note that y_N is a centered Gaussian process on Σ_{N-1} with covariance given by

$$\mathbb{E}y_N(\sigma^1)y_N(\sigma^2) = \xi'(N^{-1}(\sigma^1,\sigma^2)). \tag{1.6}$$

For more on y_N, see Lemma A.3. We also note here that the choice of the first spin as opposed to any fixed i will be irrelevant by site symmetry.

Our main result is that the TAP equation for a spin holds for the measures $\langle\cdot\rangle_{\alpha,N}$.

THEOREM 1.1. *Assume that $\zeta(q_*) > 0$. We have that*

$$\bigl(\langle\sigma_1\rangle_{\alpha,N} - \tanh\bigl[\langle y_N\rangle_{\alpha,N} + h - \bigl(\xi'(1) - \xi'(q_*)\bigr)\langle\sigma_1\rangle_{\alpha,N}\bigr]\bigr)_{\alpha\in\mathbb{N}} \to 0 \tag{1.7}$$

in distribution.

The proof of Theorem 1.1 has several steps, and along the way we pick up results that are of independent interest. We will outline the proof of Theorem 1.1 in the next section. We conclude this section with the following remarks.

REMARK 1.2. At high temperature and with $h = 0$, the Parisi measure $\zeta = \delta_0$, and the decomposition $C_{\alpha,N}$ is given by $C_{1,N} = \Sigma_N$, $C_{\alpha,N} = \varnothing$, $\alpha > 1$. The conditional measure $\mu_{1,N}$ is now identical to the Gibbs measure μ_N, and one recovers the result of Talagrand [13] for a single spin.

REMARK 1.3. Theorem 1.1 establishes the TAP equations for a single spin. The TAP equations are also predicted to hold for all spins $\sigma_1, \ldots, \sigma_N$ simultaneously, and have been shown at high temperature by Talagrand [13] and Chatterjee [7].

REMARK 1.4. The assumption that the Parisi measure has a jump at the top of its support, $\zeta(q_*) > 0$, is believed to be true for a large collection of (if not all) generic models at all temperatures. Results in this direction were obtained by Auffinger–Chen (see Theorem 4 in [3]). If there is no jump at the top of the support, then it is unclear the extent to which a true pure state decomposition will hold in such systems [12]. In a follow-up paper [5], we will show that at infinite particle number, (1.1) holds without this assumption. In fact we will show a multiscale generalization of these equations.

REMARK 1.5. Since the statement of Theorem 1.1 depends on the construction of the measures $\langle\cdot\rangle_{\alpha,N}$, one may wonder what would happen if one takes a different decomposition. In Section 4 we show that the decomposition (1.3) is essentially unique in the following sense. Any other collection of subsets $X_{\alpha,N}$ that satisfy the same properties as $C_{\alpha,N}$ must also satisfy $\mu_N(X_{\alpha,N}\Delta C_{\alpha,N}) \to 0$.

1.1. *Outline of the proof of Theorem* 1.1. Theorem 1.1 relates the quantities

$$\langle\sigma_1\rangle_{\alpha,N} = \frac{1}{\mu_N(C_{\alpha,N})}\int_{C_{\alpha,N}} \sigma_1 \, d\mu_N \quad \text{and}$$

$$\langle y_N\rangle_{\alpha,N} = \frac{1}{\mu_N(C_{\alpha,N})}\int_{C_{\alpha,N}} y_N(\sigma) \, d\mu_N.$$

Put differently, we are interested in the relation between σ_1 and y within a cluster, C_α. Heuristically, for large N there is little difference between a fixed coordinate and a "cavity coordinate." By a cavity coordinate we mean that we study the law of $(s_{\alpha,N}, y_{\alpha,N})$, which are distributed like $(\epsilon, y_N(\sigma))$ drawn from the tilted measure on Σ_{N+1},

$$d\mu_N^{\mathsf{T}}(\epsilon,\sigma) = \frac{e^{\epsilon y_N(\sigma)}\, d\epsilon\, d\mu_N(\sigma)}{\int 2\cosh(y_N(\sigma))\, d\mu_N},$$

conditioned on the event $\{\sigma \in C_{\alpha,N}\}$, where $d\epsilon$ denotes the counting measure on Σ_1. Call this conditional measure $\mu^{\mathsf{T}}_{\alpha,N}$. Here, we assume that y_N is independent of μ_N and satisfies

$$\mathbb{E} y_N(\sigma^1) y_N(\sigma^2) = \xi'(R_{12}) + o_N(1).$$

As a result to study convergence of $(s_{\alpha,N}, y_{\alpha,N})$ for a fixed α, it suffices to study convergence of statistics of the form

$$\mathbb{E}\prod_i \int \phi_i(\epsilon, y_N)\, d\mu^{\mathsf{T}}_{\alpha,N}$$

for any finite family of reasonable ϕ_i. These statistics, as we will find, are continuous functionals of the law of the overlap array of i.i.d. draws from $\mu_{\alpha,N}$. The $\mu_{\alpha,N}$ are asymptotically *replica symmetric*, that is, their overlap array converges to the matrix which is 1 on the diagonal and $q_* = \sup \operatorname{supp}\{\zeta\}$ on the off diagonal. This implies that the law of $(s_{\alpha,N}, y_{\alpha,N})$ converges to the law of a stochastic process, (s, y) which can be described as follows. Let h_α be a centered Gaussian with variance $\xi'(q_*)$. Then, (s, y) are the random variables with conditional density

$$p(s, y; h_\alpha) \propto e^{-\frac{(y-h_\alpha)^2}{2(\xi'(1)-\xi'(q_*))}} e^{sy}, \tag{1.8}$$

with respect to the product of the counting measure on Σ_1 and Lebesgue measure on $\mathbb{R}$. It is an elementary calculation to show that this satisfies the TAP equation,

$$\langle s \rangle_\alpha = \tanh\bigl(\langle y \rangle_\alpha - \bigl(\xi'(1) - \xi'(q_*)\bigr)\langle s \rangle_\alpha\bigr), \tag{1.9}$$

conditionally on h_α. Indeed, once making this reduction, this is similar in spirit to the high temperature setting as in [7]. (This is stated and proved in a slightly more general setting in [7].) This step is shown in Section 2.

The final question is then, "To what extent can we treat a fixed coordinate as a cavity coordinate?" The answer comes by first showing that the collection $C_{\alpha,N} \times \{\pm 1\}$ preserves most of the ultrametric properties after a (random) reshuffling. This is done in Sections 3 and 4. We then use the replica symmetric structure of the conditional measures to deal with the dependence of y_N on both the clusters and the Gibbs measure. This ends the proof of the theorem in Section 5.

2. Convergence of spins and local fields for a cavity coordinate. In this section we study the joint law of a spin and the local field on that spin for a cavity coordinate. As a consequence of this, we find that (1.7) holds for a cavity coordinate.

Note: In the remainder of this paper, we take $h = 0$. This does not change the arguments, however, it simplifies the notation.

Let $(H'(\sigma))$ be a centered Gaussian process on Σ_N with covariance

$$\mathbb{E} H'(\sigma^1) \cdot H'(\sigma^2) = N\xi(R_{12}) + o(N), \tag{2.1}$$

where, by the term $o(N)$, we mean a function of the overlap that vanishes uniformly as N tends to infinity. Let ν_N denote the Gibbs measure on Σ_N corresponding to H'. Let $(y(\sigma))$ be a centered Gaussian process on Σ_N that is independent of H' and satisfies

$$\mathbb{E} y(\sigma^1) y(\sigma^2) = \xi'(R_{12}) + o_N(1), \tag{2.2}$$

where again the $o_N(1)$ term is a function of the overlap.

Corresponding to y, we define a random tilt of ν_N, which we denote by ν_N^{T}, as the measure

$$\nu_N^{\mathsf{T}} = T(\sigma)\, d\nu_N, \tag{2.3}$$

where T is given by

$$T(\sigma)=\exp\Big(\log\big(\cosh\big(y(\sigma)\big)\big)-\log\Big(\int_{\Sigma_N}\cosh\big(y(\sigma)\big)\,d\nu_N\Big)\Big). \tag{2.4}$$

Observe that since $\cosh(x)\geq 1$, these measures are mutually absolutely continuous.

Assume that for H', the limiting overlap distribution satisfies $\zeta(q_*)>0$. As ξ is generic, there is a collection of sets, $\{X_{\alpha,N}\}\subset\Sigma_N$, that satisfies items 1–5. of Theorem A.1 with respect to the measure ν_N. We drop the N dependence in the notation of $X_{\alpha,N}$ and write X_α. For each $\alpha\in\mathbb{N}$, we define the measure

$$\nu_{\alpha,N}=\nu_N(\cdot|X_\alpha),$$

when X_α is nonnull and, on the event that it is null, let this be $\delta_{(1,\dots,1)}$. Finally, we let $\nu_{\alpha,N}^{\mathsf{T}}$ be the measure on $\{-1,1\}\times\Sigma_N$ such that for ϕ continuous and bounded,

$$\int\phi(s,\sigma)\,d\nu_{\alpha,N}^{\mathsf{T}}=\frac{\int_{X_\alpha}\int_{\Sigma_1}\phi(s,\sigma)e^{sy(\sigma)}\,ds\,d\nu_N(\sigma)}{\int_{X_\alpha}2\cosh(y(\sigma))\,d\nu_N}, \tag{2.5}$$

where ds denotes the counting measure on Σ_1. For the purposes of this section, let $\langle\cdot\rangle_{\alpha,N}$ denote integration with respect to $\nu_{\alpha,N}$, and $\langle\cdot\rangle_{\alpha,N}^{\mathsf{T}}$ to denote integration with respect to $\nu_{\alpha,N}^{\mathsf{T}}$.

Let $((s_{\alpha,N}^i,\sigma_{\alpha,N}^i))_{i\geq 1}$ then be i.i.d. draws from $\nu_{\alpha,N}^{\mathsf{T}}$, and let $y_{\alpha,N}^i=y(\sigma_{\alpha,N}^i)$. The goal of this section is to study the convergence of the joint law of $((s_{\alpha,N}^i, y_{\alpha,N}^i))_{i\geq 1}$. In particular let $h_\alpha\sim\mathcal{N}(0,\xi'(q_*))$, and let ν_α denote the measure on $\{\pm 1\}\times\mathbb{R}$ with density, $p(s,y;h_\alpha)$ from (1.8). Finally, let $((s^i,y^i))_i$ be i.i.d. draws from ν_α. Here and in the following, we of course mean that these draws are i.i.d. conditionally on the law of h_α. The main theorem of this section is the following.

THEOREM 2.1. *Assume that for H', the limiting overlap distribution satisfies $\zeta(q_*)>0$. For each $\alpha\in\mathbb{N}$,*

$$\big((s_{\alpha,N}^i,y_{\alpha,N}^i)\big)_i\to\big((s^i,y^i)\big)_i$$

in distribution.

Recall now that (s^i,y^i) satisfies (1.9). As a consequence we have the following corollary.

COROLLARY 2.2. *In the setting of Theorem 2.1, we have that*

$$\big(\langle s\rangle_{\alpha,N}^{\mathsf{T}}-\tanh\big(\langle y\rangle_{\alpha,N}^{\mathsf{T}}-\big(\xi'(1)-\xi'(q_*)\big)\langle s\rangle_{\alpha,N}^{\mathsf{T}}\big)\big)_{\alpha\in\mathbb{N}}\to 0$$

in distribution.

The goal of this section is to prove these two results. We begin by proving that the overlap distribution for $\nu_{\alpha,N}$ has a simple limit, namely that it is "replica symmetric" in the terminology of spin glasses. We then prove Portmanteau type theorems for $(s_{\alpha,N}, y_{\alpha,N})$. These results allow us to conclude that statistics of $(s_{\alpha,N}, y_{\alpha N})$ are continuous functionals of the overlap distribution of $\nu_{\alpha,N}$ (not $\nu^{\mathsf{T}}_{\alpha,N}$). Since the latter converges, we then conclude Theorem 2.1. The proof of Corollary 2.2 is then immediate.

2.1. *Convergence of overlaps within a cluster.* We now prove that the $\nu_{\alpha,N}$ are replica symmetric. Fix $\alpha \in \mathbb{N}$. Let $(\sigma^i)_{i=1}^\infty$ be drawn from $\nu_{\alpha,N}^{\otimes\infty}$, and consider R_N to be the doubly infinite overlap array defined by

$$R_N = \big(R(\sigma^i, \sigma^j)\big).$$

Finally, let Q be the deterministic matrix which is doubly infinite, all 1 on the diagonal and q_* on the off-diagonal. We then have the following theorem.

THEOREM 2.3. *We have that*

$$R_N \xrightarrow{(d)} Q.$$

PROOF. By standard properties of product spaces, it suffices to show that for any k,

$$\mathbb{E}\int_{X_\alpha^k} F(R_N^k)\, d\nu_{\alpha,N} \to F(Q^k). \tag{2.6}$$

Here, F is some smooth function on $[-1,1]^{k^2}$ and by R_N^k and Q^k are the overlap matrix for k i.i.d. draws from $\nu_{\alpha,N}$ and the first k-by-k entries of Q respectively. It suffices to work on the event that X_α is nonempty. Since F is smooth, observe that it suffices to show that

$$\mathbb{E}\int_{X_\alpha^k} \|R_N^k - Q^k\|_1\, d\nu_{\alpha,N}^{\otimes k} = o_N(1).$$

To this end, observe that

$$\int_{X_\alpha^k} \|R_N^k - Q^k\|_1\, d\nu_{\alpha,N}^{\otimes k} = k\cdot(k-1)\frac{\int_{X_\alpha^2} |R_{12} - q_*|\, d\mu_{\alpha,N}^{\otimes 2}}{\mu_N(X_\alpha)^2},$$

where R_{12} is the overlap of two replica from $\nu_{\alpha,N}$ and the diagonal terms cancelled. This goes to zero in probability by Theorem A.1 items 4 and 5. □

2.2. *Continuity and Portmanteau-type results*. We now collect some continuity and Portmanteau-type theorems which will be useful in the following.

LEMMA 2.4. *For each α, the convergence*

$$((s^i_{\alpha,N}, y^i_{\alpha,N})) \xrightarrow{(d)} ((s^i, y^i))$$

holds if any only if for every k, $d : [k] \to \{0, 1\}$, and family of continuous bounded functions $\{\phi_i\}$,

$$\mathbb{E} \prod_{i\in[k]} (s^i_{\alpha,N})^{d(i)} \phi_i(y^i_{\alpha,N}) \to \mathbb{E} \prod_{i\in[k]} (s^i)^{d(i)} \phi_i(y^i). \tag{2.7}$$

Furthermore, it is necessary and sufficient to take ϕ of polynomial growth.

This result is a standard consequence of the fact that $s_{\alpha,N}$ are $\{\pm 1\}$ valued and $\{y_{\alpha,N}\}$ have uniformly bounded sub-Gaussian tails (see Lemma A.2), so we omit its proof.

Finally, we note the following continuity result which is a consequence of Theorem 2.3. In the following we let $Y_t = W_{\xi'(t)}$, where W_t denotes a standard Brownian motion.

LEMMA 2.5. *For any $k, \ell \geq 1$ and any family of continuous bounded functions $\{\phi_i\}_{i\in[\ell]}$, we have that*

$$\begin{aligned} &\mathbb{E} \int_{X_\alpha^{k+\ell}} \prod_{i\in[\ell]} \phi_i(y(\sigma^i)) \prod_{j=\ell+1}^{\ell+k} \cosh(y(\sigma^j))\, d\nu_{\alpha,N}^{\otimes \ell+k} \\ &\quad \to \mathbb{E}\Bigl[\prod_{i\in[\ell]} \mathbb{E}(\phi_i(Y_1)|Y_{q_*}) \cdot \mathbb{E}(\cosh(Y_1)|Y_{q_*})^k\Bigr]. \end{aligned} \tag{2.8}$$

PROOF. Observe that for (σ^i) fixed, then

$$F((\sigma^i)) = \mathbb{E} \prod_{i\in[\ell]} \phi_i(y(\sigma^i)) \prod_{j=\ell+1}^{\ell+k} \cosh(y(\sigma^j))$$

is a continuous, bounded function of the overlap array R. In particular we may view it as a function of the form $F = F(\xi'(R) + o_N(1))$, where, by $\xi'(R) + o_N(1)$, we mean that we apply a function f to R coordinate wise that satisfies the estimate $f = \xi' + o_N(1)$.

Now, recall from (2.2) that y is independent of H' by construction. Thus, it is independent of $\nu_{\alpha,N}$ and X_α. We may then integrate the left-hand side of (2.8), first in y, to obtain

$$\mathbb{E} \int_{X_\alpha^{k+\ell}} F(\xi'(R) + o_N(1))\, d\nu_{\alpha,N}^{\otimes k+\ell}.$$

By a mollification argument it suffices to study the convergence of

$$\mathbb{E}\int_{X_\alpha^{k+\ell}} F\big(\xi'(R)\big)\, d\nu_{\alpha,N}^{\otimes k+\ell},$$

where this is the same function F as above. By Theorem 2.3, this converges to $F(\xi'(Q))$. It remains to understand $F(\xi'(Q))$. By the definition of the matrix Q,

$$F(\xi'(Q)) = \mathbb{E}\Big[\Big(\prod_{i\in[\ell]} \mathbb{E}\big(\phi_i(Y_1)|Y_{q_*}\big)\Big)\mathbb{E}\big(\cosh(Y_1)|Y_{q_*}\big)^k\Big],$$

as desired. □

2.3. *Proofs of main theorems*. We can now turn to the proofs of the main results. If E is a measurable set and $f \in L^1(\mu)$, then we denote $⨍_E f\,d\mu = \frac{1}{\mu(E)}\int f\,d\mu$ with the convention that this is zero if $\mu(E) = 0$.

PROOF OF THEOREM 2.1. Fix α. It suffices to work on the event that X_α is nonempty. By Lemma 2.4 it suffices to prove (2.7) for each n, $d : [n] \to \{0, 1\}$ and family of continuous bounded $\{\phi_i\}$. Furthermore, we claim that it suffices to prove

$$\mathbb{E}\prod_{i\in[n]} \langle\phi_i(y)\rangle_{\alpha,N}^{\mathsf{T}} \to \mathbb{E}\prod_{i\in[n]} \phi_i\big(y_\alpha^i\big). \tag{2.9}$$

To see this, simply note that

$$\begin{aligned}\mathbb{E}\prod\big(s_{\alpha,N}^i\big)^{d(i)}\phi_i(y_{\alpha,N}) &= \mathbb{E}\prod\big\langle s^{d(i)}\phi_i(y)\big\rangle_{\alpha,N}^{\mathsf{T}} \\ &= \mathbb{E}\prod \frac{\int_{X_\alpha}\int_{\Sigma_1} \phi_i(y(\sigma)) s^{d(i)} e^{sy(\sigma)}\, ds\, d\nu_N}{\int_{X_\alpha} 2\cosh(y)\, d\nu_N} \\ &= \mathbb{E}\prod\big\langle f_{d(i)}(y)\phi_i(y)\big\rangle_{\alpha,N}^{\mathsf{T}},\end{aligned}$$

where $f_d(x) = \tanh(x)$ if $d = 1$ and 1 if $d = 0$.

With this claim in hand, we now prove (2.9). To this end, fix ϕ_i as above. By (2.5),

$$\mathbb{E}\prod_i \langle\phi_i(y)\rangle_{\alpha,N}^{\mathsf{T}} = \mathbb{E}\prod_i \frac{⨍_{X_\alpha} \phi_i(y(\sigma))\cosh(y(\sigma))\, d\nu_N}{⨍_{X_\alpha} \cosh(y(\sigma))\, d\nu_N}.$$

Observe that $Z_\alpha = ⨍_{X_\alpha}\cosh(y)\, d\nu_N$ satisfies $Z_\alpha \geq 1$. By Lemma A.2,

$$P(Z_\alpha \geq L) \leq \frac{C(\xi)}{L}$$

uniformly in N. Thus, by a standard approximation argument, we can approximate $1/Z_\alpha^n$ by polynomials in Z_α in the above expectations. In particular, it suffices to study limits of integrals of the form

$$\mathbb{E}\prod_i ⨍_{X_\alpha} \phi_i\big(y(\sigma)\big)\cosh\big(y(\sigma)\big)\, d\nu_N \cdot \Big(⨍_{X_\alpha} \cosh\big(y(\sigma)\big)\, d\nu_N\Big)^l.$$

This is exactly of the form (2.8) with $k = l$, $\ell = 1$ and the family $\{\phi_i(y) \cdot \cosh(y)\}_{i\in[n]}$ by Fubini's theorem. Thus, by Lemma 2.5,

$$\mathbb{E}\prod_i \langle \phi_i(y)\rangle^{\mathsf{T}}_{\alpha,N} \to \mathbb{E}\prod_i \frac{\mathbb{E}(\phi_i(Y_1)\cosh(Y_1)|Y_{q_*})}{\mathbb{E}(\cosh(Y_1)|Y_{q_*})}.$$

It remains to recognize the right-hand side of the above display as an average with respect to ν_α. Observe that

$$\begin{aligned}\frac{\mathbb{E}(\phi(Y_1)\cosh(Y_1)|Y_{q_*})}{\mathbb{E}(\cosh(Y_1)|Y_{q_*})} &= \mathbb{E}\big(\phi(Y_1)e^{\log\cosh(Y_1)-\log\cosh(Y_{q*})-\frac{1}{2}(\xi'(1)-\xi'(q_*))}|Y_{q_*}\big) \\ &\overset{(d)}{=} \int \phi(y)\,d\nu_\alpha,\end{aligned}$$

where the last equality is by definition. Thus,

$$\mathbb{E}\prod \langle \phi_i(y)\rangle^{\mathsf{T}}_{\alpha,N} \to \mathbb{E}\int \prod \phi_i(y^i)\,d\nu_\alpha^{\otimes n}$$

as desired. □

PROOF OF COROLLARY 2.2. Let $m_{\alpha,N} = \langle s\rangle^{\mathsf{T}}_{\alpha,N}$ and $h_{\alpha,N} = \langle y\rangle^{\mathsf{T}}_{\alpha,N}$. It suffices to show that for each $\alpha \in \mathbb{N}$,

$$(m_{\alpha,N}, h_{\alpha,N}) \xrightarrow{(d)} \big(\langle s\rangle_\alpha, \langle y\rangle_\alpha\big).$$

Suppose first that this claim is true. Then, the result immediately follows from (1.9).

We now turn to the claim. Observe that by Lemma A.2 these random variables have sub-Gaussian tails. Thus, it suffices to prove convergence of the moments

$$\mathbb{E}m_{\alpha,N}^{k_1} h_{\alpha,N}^{k_2}.$$

To this end, let $k = k_1 + k_2$, and let $\{\psi_j\}_{j\in[k]}$ satisfy $\psi_j = 1$, if $i \le k_1$ and $\psi_j(x) = x$ if $j > k_1$. Finally, let $d : [k] \to \{0, 1\}$ be such that $d(i) = 1$, if $i \le k_1$ and $d(i) = 0$ otherwise. Then, by Lemma 2.4 and Theorem 2.1 we have that

$$\mathbb{E}m_{\alpha,N}^{k_1} h_{\alpha,N}^{k_2} = \mathbb{E}\prod_j \big(s^j_{\alpha,N}\big)^{d(j)}\psi_j\big(y^j_{\alpha,N}\big) \to \mathbb{E}\prod_j \big(s^j_\alpha\big)^{d(j)}\psi_j\big(y^j_\alpha\big) = \mathbb{E}\langle s\rangle_\alpha^{k_1}\langle y\rangle_\alpha^{k_2}$$

as desired. □

3. Stability of clusters under lifts. In this section we show that important properties of the pure states are carried over after lifting in one coordinate. We start with the following construction. For $\sigma = (\sigma_1,\dots,\sigma_N) \in \Sigma_N$ let $\rho(\sigma) = (\sigma_2,\dots,\sigma_N) \in \Sigma_{N-1}$. For any mixed p-spin glass model the Hamiltonian, H_N, decomposes into a sum of three Gaussian processes:

$$H_N(\sigma) = \tilde{H}_N\big(\rho(\sigma)\big) + \sigma_1 y_N\big(\rho(\sigma)\big) + r_N\big(\sigma_1, \rho(\sigma)\big). \tag{3.1}$$

Properties of these Gaussian processes are described in Lemma A.3. For $\sigma \in \Sigma_{N-1}$ set

$$H'_N(\sigma) := \tilde{H}_N(\sigma) + r_N(1, \sigma),$$

and let μ'_N be the Gibbs measure corresponding to the Hamiltonian H'_N. This Hamiltonian, and thus μ'_N, is independent of y_N. We are thus in the setting of Section 2, where H'_N satisfies (2.1) and y_N satisfies (2.2).

Let $\tilde{W}_{\alpha,N-1}$, $\alpha \in \mathbb{N}$ be the subsets of Σ_{N-1} constructed via Theorem A.1 relative to the measure μ'_N. Set

$$W^{\dagger}_{\alpha,N} = \Sigma_1 \times \tilde{W}_{\alpha,N-1} \subset \Sigma_N. \tag{3.2}$$

Order the sets $W^{\dagger}_{\alpha,N}$ with respect to their μ_N masses. That is, define subsets $W_{\alpha,N} \subset \Sigma_N$ for $\alpha \in \mathbb{N}$, such that

$$\mu_N(W_{1,N}) \geq \mu_N(W_{2,N}) \geq \cdots, \tag{3.3}$$

and so that

$$W_{\alpha,N} = W^{\dagger}_{\pi_N(\alpha),N},$$

for some (random) automorphism $\pi_N : \mathbb{N} \to \mathbb{N}$.

REMARK 3.1. Note that there is not a unique way to define the projection π_N since there are possibly ties $W_\alpha = W_\beta$. Note, however, this only introduces a finite indeterminacy as there are only finitely many such sets that are nonempty by construction. The reader can take any tie-breaking rule.

The goal of this section is to show that the collection $(W_{\alpha,N})_{\alpha\in\mathbb{N}}$ also satisfies items 1–5. from Theorem A.1. (For the rest of section, we drop the subscript N of our notation.) The main idea is that at the level of overlaps, the measure μ on the sets W_α will essentially be the same as the measure $(\mu')^{\mathsf{T}}$ on the sets $\tilde{W}_{\pi(\alpha)}$. Since on Σ_{N-1}, $(\mu')^{\mathsf{T}} \gg \mu'$, overlap events that are rare for μ' will still be rare for $(\mu')^{\mathsf{T}}$. We begin by recording the following lemma which is a quantification of this observation.

Recall the local field $y = y_N$ from (1.5) and the function T from (2.4). Let

$$\tilde{K}(\mu') = \left(\int \cosh(2y)\, d\mu'\right)^{1/2}. \tag{3.4}$$

LEMMA 3.2 (Tilting lemma). *There are constants $C, c > 0$ such that with probability at least $1 - \frac{1}{c}e^{-cN}$,*

$$\left(1 - \frac{C}{\sqrt{N}}\right)\int_A T\, d\mu' \leq \mu(\Sigma_1 \times A) \leq \left(1 + \frac{C}{\sqrt{N}}\right)\int_A T\mu', \quad \forall A \subset \Sigma_{N-1}.$$

In particular,

$$\mu(\Sigma_1 \times A) \le \tilde{K}(\mu')\Big(1 + \frac{C}{\sqrt{N}}\Big)\sqrt{\mu'(A)}.$$

PROOF. This result immediately follows from Lemma A.3. Observe that if we let

$$\Delta = 2 \max_{\sigma \in \Sigma_{N-1}} \big|r(1,\sigma) - r(-1,\sigma)\big|,$$

then

$$\mu(\Sigma_1 \times A) = \frac{\int_A \int_{\Sigma_1} e^{\tilde{H}(\sigma)+\epsilon y(\sigma)+r(\epsilon,\sigma)}\, d\epsilon\, d\sigma}{\int_{\Sigma_{N-1}} \int_{\Sigma_1} e^{\tilde{H}(\sigma)+\epsilon y(\sigma)+r(\epsilon,\sigma)}\, d\epsilon\, d\sigma} \le \int_A T(\sigma)\, d\mu' e^{\Delta}.$$

Similarly,

$$\mu(\Sigma_1 \times A) \ge \int_A T(\sigma)\, d\mu' e^{-\Delta}.$$

The first result then follows by Lemma A.3, and the second result follows from the first and the Cauchy–Schwarz inequality. □

We now start by proving the properties mentioned above.

LEMMA 3.3. *Let* $q'_N = q_{N-1}, a'_N = a_{N-1}, b'_N = b_N^{1/4}$, *and* $\epsilon'_N = \epsilon_N^{1/4}$. *Then, the sets* $\{W_\alpha\}_{\alpha \in [m_N]}$ *satisfy items* 1–4 *Theorem* A.1 *with probability* $1 - o_N(1)$, *where the sequences* $q'_N, a'_N, b'_N, \epsilon'_N$ *and* m_N *satisfy those conditions.*

PROOF. Since the sets $\tilde{W}_\alpha$ are disjoint, $W_\alpha^\dagger$ and W_α are as well and satisfy

$$\Big(\bigcup_\alpha W_\alpha\Big)^c = \Big(\bigcup_\alpha W_\alpha^\dagger\Big)^c = \Big(\bigcup_\alpha \Sigma_1 \times \tilde{W}_\alpha\Big)^c = \Sigma_1 \times \Big(\bigcup_\alpha \tilde{W}_\alpha\Big)^c.$$

Thus, by the tilting lemma (Lemma 3.2) and item 1 of Theorem A.1, we have that, with high probability,

$$\mu\Big(\Big(\bigcup_\alpha W_\alpha\Big)^c\Big) \le \Big(1 + \frac{C}{\sqrt{N}}\Big)\tilde{K}(\mu') \cdot \sqrt{\epsilon_N}. \tag{3.5}$$

Furthermore, by the tilting lemma and item 2 of Theorem A.1, we obtain for $\beta = \pi^{-1}(\alpha)$

$$\begin{aligned}
&\mu^{\otimes 2}(\sigma^1, \sigma^2 \in W_\alpha : R_{12} \le q_{N-1} - 2a_{N-1}) \\
&\quad \le \tilde{K}(\mu')^2\Big(1 + \frac{C}{\sqrt{N}}\Big)\sqrt{(\mu')^{\otimes 2}\Big(\sigma^1, \sigma^2 \in \tilde{W}_\beta : R_{12} \le q_{N-1} - 2a_{N-1} + \frac{1}{N}\Big)} \\
&\quad \le \tilde{K}(\mu')^2\Big(1 + \frac{C}{\sqrt{N}}\Big)\sqrt{b_N},
\end{aligned}$$

where we used the fact that we may take $a_{N-1} \geq \frac{1}{N}$. Argue similarly to get that for $\alpha_1 \neq \alpha_2$,

$$\mu^{\otimes 2}(\sigma^1 \in W_{\alpha_1}, \sigma^2 \in W_{\alpha_2} : R_{12} \geq q_{N-1} + 2a_{N-1}) \leq \tilde{K}(\mu')^2\left(1 + \frac{C}{\sqrt{N}}\right)\sqrt{b_N}.$$

Observe that by Lemma A.2, with probability tending to 1, $\tilde{K}(\mu') \leq b_N^{-\gamma} \vee \epsilon_N^{-1/4}$. This yields the desired result after observing that since $\zeta_N[q_N + a_N, 1] \geq \zeta\{q_*\} - b_N$, for N sufficiently large, the same is true for q'_N, a'_N and b'_N, and that item 4 in Theorem A.1 is implied by this fact regarding ζ_N and items 2 and 3. □

It remains to show that the weights $\mu(W_\alpha)$ converge to a Poisson–Dirichlet process.

LEMMA 3.4. *We have that*

$$\left(\mu(W_\alpha)\right)_{\alpha \in \mathbb{N}} \to (v_\alpha)_{\alpha \in \mathbb{N}}$$

in distribution on the space of mass partitions $\mathcal{P}_m$.

PROOF. Recall that $\{\mu_N\}$ satisfy the approximate Ghirlanda–Guerra identities, since H_N is a generic model. Let $U_{12} = U(\sigma^1, \sigma^2)$ be

$$U_{12} = \mathbb{1}\{\exists \alpha \in \mathbb{N} : \sigma^1, \sigma^2 \in W_\alpha\},$$

and let $L_N = \{\sigma^1, \sigma^2 \in \bigcup_\alpha W_\alpha\}$. Then, by the arguments of [8], Section 6, in order to prove that this sequence converges, it suffices to prove that for some $\phi_{\kappa,\lambda}$, which satisfies

$$\phi_{\kappa,\lambda}(x) = \begin{cases} 0 & x \leq q_* - \kappa, \\ 1 & x \geq q_* - \lambda, \end{cases}$$

and interpolates between the two values for $x \in [q_* - \kappa, q_* - \lambda]$, we have

$$\lim_{\kappa,\lambda \to 0} \overline{\lim_{N \to \infty}} \mathbb{E}\langle |U_{12} - \phi_{\kappa,\lambda}|\rangle = 0.$$

To see this, if we denote $|U_{12} - \phi_{\kappa,\lambda}| = A$, then

$$\mathbb{E}\langle A\rangle_\mu \leq \mathbb{E}\langle A\mathbb{1}\{L_N\}\rangle + o_N(1),$$

where the fact that the second term is $o_N(1)$ follows from (3.5). Now,

$$\begin{aligned}\mathbb{E}\langle AL_N\rangle &= \mathbb{E}\langle A\mathbb{1}\{L_N, R_{12} \geq q_* - \lambda\}U_{12}\rangle + \mathbb{E}\langle A\mathbb{1}\{L_N, R_{12} \leq q_* - \lambda\}U_{12}\rangle \\ &\quad + \mathbb{E}\langle A\mathbb{1}\{L_N, R_{12} \geq q_* - \kappa\}(1 - U_{12})\rangle \\ &\quad + \mathbb{E}\langle A\mathbb{1}\{L_N, R_{12} \leq q_* - \kappa\}(1 - U_{12})\rangle \\ &= I + II + III + IV.\end{aligned}$$

Note that $I = IV = 0$ identically. It remains to estimate II and III.

We start with II. Observe that

$$II \leq 2\mathbb{E}\langle U_{12}(\mathbb{1}\{R_{12} \leq q_* - 2a_{N-1}\})\rangle$$

for N large enough which is bounded by b'_N by Lemma 3.3.

Now, to estimate III, note that for N sufficiently large,

$$\mathbb{E}\langle A(1 - U_{12})\big(\mathbb{1}\{R_{12} \geq q_{N-1} + 2a_{N-1}\} + \mathbb{1}\{R_{12} \in [q_* - \kappa, q_N + 2a_{N-1})\}\big)\rangle_\mu$$
$$\leq b'_N + (a).$$

By the tilting lemma

$$(a) \leq \|\tilde{K}\|_4^2 \cdot \big(\mathbb{E}\mu^{\otimes 2}[q_* - 2\kappa, q_{N-1} + a_{N-1})\big)^{1/2}.$$

By the choice of q_N and a_N (see the first display in Theorem A.1), we have that

$$\overline{\lim}\, \zeta_N[q_* - 2\kappa, q_{N-1} + a_{N-1}) = \overline{\lim}\big(\zeta_N[q_* - 2\kappa, 1] - \zeta_N[q_{N-1} + a_{N-1}, 1]\big) = 0.$$

Thus, combining these estimates and Lemma A.2 we see that sending $N \to \infty$, $\lambda \to 0$, and then $\kappa \to 0$ yields the result. $\square$

4. Essential uniqueness of clusters. In this section we show that sets that satisfy the properties from Theorem A.1 with respect to μ are asymptotically unique.

Let $\{C_\alpha\}$ be constructed as in Theorem A.1 for the measure μ_N. Recall that they are labelled in decreasing order, that is,

$$\mu_N(C_\alpha) \geq \mu_N(C_{\alpha+1}).$$

Let a_N, b_N, m_N, $q_N \to q_*$, and ϵ_N be as in that theorem. Let $\{X_\alpha\}_{\alpha\in[m_N]}$ be another collection of sets that satisfies items 1–5 of Theorem A.1 with constants q'_N, a'_N, b'_N and ϵ'_N as in that theorem.

The main goal of this section is to prove that the pure states C_α and the sets X_α are effectively the same as far as μ is concerned.

THEOREM 4.1 (Essential uniqueness). *Suppose that we have*

$$\zeta_N\big[(q'_N - a'_N), (q_N + a_N)\big] + \zeta_N\big[(q_N - a_N), (q'_N + a'_N)\big] \to 0. \tag{4.1}$$

Then, for each $\alpha \in \mathbb{N}$, we have that

$$\mu_N(C_\alpha \Delta X_\alpha) \to 0 \tag{4.2}$$

in probability, where Δ denotes the symmetric difference.

As a corollary of this we get the following.

COROLLARY 4.2. *Let W_α be as in Lemma 3.3. Then, (4.1) holds. In particular*

$$\mu_N(C_\alpha \Delta W_\alpha) \to 0 \tag{4.3}$$

in probability.

PROOF. This follows by Lemma 3.3 and Theorem 4.1 after recalling that

$$\zeta_N[q'_N + a'_N, 1] \geq \zeta[q_*] - o_N(1),$$
$$\zeta_N[q_N + a_N, 1] \geq \zeta[q_*] - o_N(1).$$

Indeed, this implies that

$$\zeta_N[q'_N - a'_N, q_N + a_N] = \zeta_N[q_{N-1} - a_{N-1}, q_N + a_N] \to 0.$$

The same argument holds for the second limit. □

The idea of the proof Theorem 4.1 is that the overlap properties of the sets (X_α) and (C_α) from items 1–4 of Theorem A.1 will imply that each of the first n (X_α)'s will be supported by one the first M (C_α)'s for some M large but fixed and *vice versa*. The ranking of the states and basic properties of the Poisson–Dirichlet process will then imply that, in fact, for each α the sets X_α and C_α are actually supported by each other.

For this we will need the following three lemmas. Their proofs are deferred to the end of this section and follow from properties of the Poisson–Dirichlet process. The first lemma says that there is not much mass in the the tail of the collections X_α and C_α.

LEMMA 4.3. *For every $\epsilon > 0$, there is an $N_0(\epsilon)$ and $M(\epsilon)$ such that if*

$$E_N(\epsilon) = \left\{\mu_N\left(\bigcup_{\alpha \geq M(\epsilon)} X_\alpha\right) > \frac{\epsilon}{2}\right\} \cup \left\{\mu_N\left(\bigcup_{\alpha \geq M(\epsilon)} C_\alpha\right) > \frac{\epsilon}{2}\right\}$$

then for $N \geq N_0(\epsilon)$,

$$\mathbb{P}[E_N(\epsilon)] \leq \epsilon.$$

The second lemma says that, for any fixed n, the first n states (C_k) and (X_k) must have nonnegligible μ_N mass as N goes to infinity.

LEMMA 4.4. *Fix $n \geq 1$ and $\delta > 0$. Let $F_N(n, \delta)$ be the event that*

$$\mu_N(X_1) > \cdots > \mu_N(X_n) > \delta,$$
$$\mu_N(C_1) > \cdots > \mu_N(C_n) > \delta,$$

then there is a function $f_{1,n}$ satisfying $\lim_{\delta \to 0} f_{1,n}(\delta) = 0$ and an $N_1(n, \delta)$ such that for $N \geq N_1(n, \delta)$,

$$\mathbb{P}[F_N(n, \delta)] \geq 1 - f_1(\delta).$$

The last lemma concerns the gap between the masses of states.

LEMMA 4.5. *Fix $\eta > 0$ and $n \geq 1$. Let*

$$I_N(\eta, n) = \{\mu_N(C_i) - \mu_N(C_{i+1}) > \eta \; \forall i \in [n-1]\} \\ \cap \{\mu_N(X_i) - \mu_N(X_{i+1}) > \eta \; \forall i \in [n-1]\}.$$

Then there is a function $f_2(\eta, n)$ and an $N_2(\eta, n)$ such that for $N \geq N_2(\eta, n)$,

$$\mathbb{P}(I_N(\eta, n)) \geq 1 - f_2(\eta, n),$$

where for each n, $f_2(\eta, n) \to 0$ as $\eta \to 0$.

Given $\varepsilon > 0$, choose δ, ϵ and η by combining Lemma 4.3–4.5, such that if

$$\mathcal{E}_N(\epsilon, \delta, n, \eta) := E_N^c(\epsilon) \cap F_N(n, \delta) \cap I_N(\eta, n) \cap J_N,$$

where J_N is the event that the conclusions of Theorem A.1 hold, then

$$\mathbb{P}[\mathcal{E}_N] > 1 - \varepsilon, \tag{4.4}$$

for all $N \geq N_0(\varepsilon)$.

PROOF OF THEOREM 4.1. We want to show that for each $\rho > 0$, $\varepsilon > 0$ and α,

$$\mathbb{P}(\mu_N(C_\alpha \Delta X_\alpha) > \rho) \leq \varepsilon. \tag{4.5}$$

Fix ρ, ε and α. Let $n > \alpha$. Let $N \geq N_0(\varepsilon/2)$, where N_0 is defined as in (4.4). By (4.1) and Markov's inequality there is a $c_N \to 0$ such that with probability $1 - o_N(1)$,

$$\mu_N^{\otimes 2}(R_{12} \in [q'_N - a'_N, q_N + a_N]) \leq c_N. \tag{4.6}$$

Choose N sufficiently large that

$$\frac{2M(\epsilon)}{\epsilon}(m_N(b_N + b'_N) + c_N) + \epsilon_N < \frac{\rho \wedge \eta \wedge \epsilon}{2},$$

where ϵ, η are defined as above We can do this since, by assumption,

$$(b'_N + b_N) \cdot m_N = o_N(1).$$

We will prove shortly that on $\mathcal{E}_N$, for

$$\iota_N = \frac{2M(\epsilon)}{\epsilon}(m_N(b_N + b'_N) + c_N) + \epsilon_N,$$

we have that

$$\begin{aligned} \mu_N(C_\alpha \backslash X_\alpha) &\leq \iota_N, \\ \mu_N(X_\alpha \backslash C_\alpha) &\leq \iota_N. \end{aligned} \tag{4.7}$$

Note that (4.7) immediately implies (4.5) as desired. □

PROOF OF (4.7). We begin by defining two maps $\pi_1, \pi_2 : [n] \to [M(\epsilon)]$. On the event $\mathcal{E}_N$, for each i we let $\pi_1(i)$ be the first $j \in [M(\epsilon)]$ such that

$$\mu_N(X_i \cap C_{\pi_1(i)}) \geq \frac{\epsilon}{2 \cdot M(\epsilon)}$$

holds and let $\pi_2(i)$ be the first $j \in [M(\epsilon)]$ such that

$$\mu_N(X_{\pi_2(i)} \cap C_i) \geq \frac{\epsilon}{2 \cdot M(\epsilon)}$$

holds. That such j exist follows by definition of $\mathcal{E}_N$. On $\mathcal{E}_N^c$, let $\pi_1 = \pi_2 = \mathrm{Id}$. This provides two random maps $\pi_i : [n] \to [M(\epsilon)]$, $i = 1, 2$.

Suppose for the moment that on $\mathcal{E}_N$,

$$\begin{aligned} \mu_N(X_i \cap C_{\pi_1(i)}) &\geq \mu_N(X_i) - \iota_N, \\ \mu_N(C_i \cap X_{\pi_2(i)}) &\geq \mu_N(C_i) - \iota_N. \end{aligned} \tag{4.8}$$

The inequality, (4.7), provided that $\pi_1 = \pi_2 = \mathrm{Id}$ on $\mathcal{E}_N$. Let us first show that these maps are the identity map given (4.8). We then prove (4.8).

The proof that these maps are the identity map is by induction. Suppose first that $\pi_2(1) = 1$. If $\pi_1(1) > 1$, then, by (4.8),

$$\begin{aligned} \mu_N(C_1) &\leq \mu_N(X_1) + \iota_N \\ &\leq \mu_N(C_{\pi_1(1)}) + 2\iota_N \leq \mu_N(C_2) + 2\iota_N. \end{aligned}$$

This implies that

$$\mu_N(C_1) - \mu_N(C_2) \leq 2\iota_N.$$

Since $\iota_N \to 0$, this contradicts the definition of $\mathcal{E}_N$. By symmetry the same argument works if $\pi_1(1) = 1$ and $\pi_2(1) > 1$.

Now, assume that $\pi_2(1) > 1$ and $\pi_1(1) > 1$. By the ordering of these sets,

$$\begin{aligned} \mu_N(C_1) &\leq \mu_N(X_{\pi_2(1)}) + \iota_N \leq \mu_N(X_1) + \iota_N \\ &\leq \mu_N(C_{\pi_1(1)}) + 2\iota_N. \end{aligned}$$

This is, again, a contradiction. Thus $\pi_1(1) = 1 = \pi_2(1)$.

Assume now that $\pi_1(i) = \pi_2(i) = i$ for all $i \in [k-1]$. By the same reasoning, as in the base case, if $\pi_2(k) \neq k$, then it must be that $\pi_2(k) < k$. This, however, implies that

$$\mu_N(C_k) \leq \mu_N(X_{\pi_2(k)} \backslash C_{\pi_2(k)}) + \iota_N.$$

But

$$\mu_N(X_{\pi_2(k)} \backslash C_{\pi_2(k)}) \leq \mu_N(X_{\pi_2(k)}) - \mu_N(X_{\pi_2(k)} \cap C_{\pi_2(k)}) \leq \iota_N,$$

where we used the induction hypothesis in the last inequality. This implies that eventually $\mu C_\alpha \leq 2\iota_N$. This is, again, a contradiction since on $\mathcal{E}_N$, $\mu C_\alpha > \epsilon$. Thus, assuming (4.8), we have that $\pi_1 = \pi_2 = \mathrm{Id}$ by induction.

We now prove (4.8) on the event $\mathcal{E}_N$. Fix $\alpha \in [n]$. We know that, on this event,

$$\mu_N(X_\alpha \cap C_{\pi_1(\alpha)}) \geq \frac{\epsilon}{2M(\epsilon)}.$$

Now, let $\ell \neq \pi_1(\alpha)$. Write

$$\mu_N(X_\alpha \cap C_\ell) = \frac{1}{\mu_N(C_{\pi_1(\alpha)} \cap X_\alpha)} \mu_N^{\otimes 2}(\sigma^1 \in C_{\pi_1(\alpha)} \cap X_\alpha, \sigma^2 \in C_\ell \cap X_\alpha).$$

Write the event $\{R_{12} \in [-1, 1]\}$ as

$$\begin{aligned} &\{R_{12} \leq q_N' - a_N'\} \cup \{R_{12} \geq q_N + a_N\} \cup \{q_N' - a_N' < R_{12} < q_N + a_N\} \\ &\quad = I \cup II \cup III. \end{aligned}$$

Note that since we are in the event J_N,

$$\mu_N^{\otimes 2}(\sigma^1 \in C_{\pi_1(\alpha)} \cap X_\alpha, \sigma^2 \in C_\ell \cap X_\alpha, I) \leq \mu_N^{\otimes 2}(\sigma^1, \sigma^2 \in X_\alpha, I) \leq b_N',$$

while

$$\mu_N^{\otimes 2}(\sigma^1 \in C_{\pi_1(\alpha)} \cap X_\alpha, \sigma^2 \in C_\ell \cap X_\alpha, II) \leq b_N.$$

Summing on ℓ and using (4.6), we see that

$$\begin{aligned} \sum_{\ell \neq \alpha} \mu_N(X_\alpha \cap C_\ell) &\leq \frac{1}{\mu(C_{\pi_1(\alpha)} \cap X_\alpha)} (m_N(b_N' + b_N) + c_N) \\ &\leq \frac{2M}{\epsilon} \cdot (m_N(b_N' + b_N) + c_N). \end{aligned}$$

This implies the first inequality of (4.8) after recalling that $\{C_\ell\}$ (almost) partitions Σ_N and that

$$\mu_N\left(X_\alpha \cap \left(\bigcup_\alpha C_\alpha\right)^c\right) \leq \epsilon_N$$

by assumption. By symmetry the same argument shows the second inequality holds as well. □

4.1. *Propositions regarding the Poisson–Dirichlet process.* The proofs of Lemmas 4.3–4.5 follow by elementary applications of the Portmanteau lemma combined with basic properties of the Poisson–Dirichlet process. For the reader's convenience we prove Lemma 4.3. The proofs of Lemma 4.4 and Lemma 4.5 are omitted.

PROOF OF LEMMA 4.3. Fix $\epsilon > 0$. Let (v_n) be $\mathrm{PD}(1 - \zeta(q_*))$. Let $M(\epsilon)$ be such that

$$\mathbb{P}\left(\sum_{\alpha \geq M(\epsilon)} v_\alpha \geq \frac{\epsilon}{2}\right) \leq \frac{\epsilon}{4}.$$

Recall that $(\mu_N(X_\alpha)) \to (v_n)$ by Lemma 3.4. For (v_α^N), this event is contained in the closed event (in the topology of mass partitions)

$$\left\{\sum_{\alpha \le M} v_\alpha^N \le 1 - \epsilon/2\right\},$$

and for v_α these events are equal. Thus, we have that for N sufficiently large

$$\mathbb{P}\left(\mu_N\left(\bigcup_{\alpha > M(\epsilon)} X_\alpha\right) \ge \frac{\epsilon}{2}\right) \le \frac{\epsilon}{2},$$

by the Portmanteau theorem. The same argument applies to the C_α. Intersecting these events yields the result by the inclusion-exclusion principle. □

5. TAP equation for a fixed coordinate. In this section we turn to the proof of Theorem 1.1. For the reader's convenience let us briefly recap where we are and our plan of attack. Recall the construction of the states C_α from Theorem A.1 and the definition of $\langle\cdot\rangle_\alpha$. In Section 3 we constructed another collection of pure states $W_\alpha \subseteq \Sigma_N$ for the measure μ_N. As shown in Section 4, the sets C_α and W_α are essentially the same in each other. The advantage of working with W_α lies in the fact that they are rearrangements of lifts of pure states of the measure μ'_{N-1}. This will allow us to avoid the first obstruction explained in Section 1.1. The measure μ'_{N-1} is now independent of the local field y_N. The rearrangement, however, is not independent of y_N. In particular the correlation between W_α and y is through the map π_N which takes $W_\alpha^\dagger$ to $W_{\pi(\alpha)}^\dagger = W_\alpha$.

To circumvent this obstruction, we make the following observation. The measure μ, conditioned on the set $W_\alpha^\dagger$, is essentially the measure $(\mu')^\intercal$ conditioned on $\tilde{W}_\alpha$. This will allow us to conclude that (1.7) holds by an application of Corollary 2.2, provided the rearrangement map π_N is not too wild. In particular, provided the map $\mu' \mapsto (\mu')^\intercal$ does not "charge the dust at infinity," the result will follow as a consequence of the following basic fact.

LEMMA 5.1. *Let X^N be a sequence of $[-2,2]^{\mathbb{N}}$-valued random variables such that*

$$X^N \xrightarrow{(d)} 0.$$

Let p_N be a sequence of S_∞-valued random variables that satisfy the tightness criterion

$$\overline{\lim_{M\to\infty}}\ \overline{\lim_{N\to\infty}}\ P\big(p_N(n) \ge M\big) = 0 \quad \forall n. \tag{5.1}$$

Then, if $Y^N = (X^N_{p_N(n)})$, we have

$$Y^N \xrightarrow{(d)} 0.$$

We begin this section by proving the tightness of the sequence π_N. The main result will then essentially be immediate and is proved in the following subsection.

5.1. *Tightness of the reshuffling*. We begin this section by studying the random permutation $\pi : \mathbb{N} \to \mathbb{N}$ as defined in Section 3 by

$$W_{\alpha,N} = W^{\dagger}_{\pi_N(\beta),N}.$$

We recall its dependence on N by writing π_N instead of just π. We now show tightness for the sequence π_N.

LEMMA 5.2 (Tightness). *We have that, for each $n \in \mathbb{N}$,*

$$\overline{\lim_{M\to\infty}}\ \overline{\lim_{N\to\infty}}\ P\big(\pi_N(n) \geq M\big) = 0.$$

PROOF. Take N sufficiently large that $n \leq m_N$. Now, observe that

$$\begin{aligned} P\big(\pi_N(n) \geq M\big) &= P\big(\exists k \geq M : \pi_N(k) = n\big) \\ &= P\big(\exists l \leq n, k \geq M : \mu(W_l^{\dagger}) \leq \mu(W_k^{\dagger})\big) \\ &\leq \sum_{l=1}^{n} P\big(\exists k \geq M : \mu(W_l^{\dagger}) \leq \mu(W_k^{\dagger})\big) \\ &\leq \sum_{l=1}^{n} P\bigg(\mu(W_l^{\dagger}) \leq \mu\bigg(\bigcup_{k\geq M} W_k^{\dagger}\bigg)\bigg). \end{aligned}$$

It thus suffices to prove this limit for each summand.

Now, observe that for each $l \in [n]$ and each $\epsilon > 0$, the summand satisfies the inequality

$$P\bigg(\mu(W_l^{\dagger}) \leq \mu\bigg(\bigcup_{k\geq M} W_k^{\dagger}\bigg)\bigg) \leq P\big(\mu(W_l^{\dagger}) \leq \epsilon\big) + P\bigg(\mu\bigg(\bigcup_{k\geq M} W_k^{\dagger}\bigg) \geq \epsilon\bigg) = I + II.$$

We now bound I. Observe that by Lemma 3.2,

$$\mu(W_l^{\dagger})\langle\cosh(y)\rangle' \geq \bigg(1 - \frac{C}{\sqrt{N}}\bigg)\mu'(\tilde{W}_l)$$

with high probability. Thus I is bounded by

$$\begin{aligned} I &\leq P\big(\mu'(\tilde{W}_l) \leq 2\epsilon \cdot L\big) + P\big(\langle\cosh(y)\rangle' \geq L\big) + o_N(1) \\ &\leq P\big(\mu'(\tilde{W}_l) \leq 2\epsilon \cdot L\big) + \frac{C(\xi')}{L} + o_N(1) \end{aligned}$$

for each $L \geq 1$, where we have applied the localization lemma (Lemma A.2) in the second inequality.

We now turn to II. Observe that again by Lemma 3.2 with high probability,

$$\mu\bigg(\bigcup_{k\geq M} W_k^{\dagger}\bigg) \leq \bigg(1 + \frac{C}{\sqrt{N}}\bigg)\tilde{K}(\mu')\sqrt{\mu'\bigg(\bigcup_{k\geq M} \tilde{W}_k\bigg)}.$$

Thus, for N sufficiently large,

$$
\begin{aligned}
II &\le P\biggl(2\tilde{K}(\mu')\sqrt{\mu'\biggl(\bigcup_{k\ge M}\tilde{W}_k\biggr)} \ge \epsilon\biggr) + o_N(1) \\
&\le P\biggl(\mu'\biggl(\bigcup_{k\ge M}\tilde{W}_k\biggr) \ge \frac{\epsilon^2}{4L^2}\biggr) + P\bigl(\tilde{K}(\mu') \ge L\bigr) + o_N(1) \\
&= P\biggl(\mu'\biggl(\bigcup_{k\ge M}\tilde{W}_k\biggr) \ge \frac{\epsilon^2}{4L^2}\biggr) + \frac{C(\xi')}{L} + o_N(1),
\end{aligned}
$$

where again in the last step we used Lemma A.2. Denoting

$$\mu'(\tilde{W}_k) = v_k^N,$$

we can write the above as

$$I + II \le P\bigl(v_l^N \le 2\epsilon \cdot L\bigr) + P\biggl(\sum_{k\le M} v_k^N \le 1 - \frac{\epsilon^2}{4L^2}\biggr) + \frac{C}{L} + o_N(1).$$

Observe that the sets in the first two terms are closed in $\mathcal{P}_m$. Thus, by the Portmanteau theorem and the fact that $(v_l^N) \to (v_l)$ in law on $\mathcal{P}_m$ where (v_l) are PD(θ) with $\theta = 1 - \zeta(\{q_*\})$, we have that

$$\overline{\lim_N}\, I + II \le P(v_l \le 2\epsilon \cdot L) + P\biggl(\sum_{k\ge M} v_k \ge \frac{\epsilon^2}{4L^2}\biggr) + \frac{C}{L}.$$

We used here that for the Poisson–Dirichlet distribution $\sum v_k = 1$.

The Poisson–Dirichlet distribution satisfies

$$\mathbb{E}\sum_{k\ge M} v_k \le f(M,\theta),$$

where $f \to 0$ as $M \to \infty$. In particular, by Markov's inequality, we have

$$P\biggl(\sum_{k\ge M} v_k \ge \frac{\epsilon^2}{4L^2}\biggr) \le \frac{4L^2}{\epsilon^2} f(M,\theta).$$

Thus, combining the above we have that

$$\overline{\lim_{M\to\infty}}\,\overline{\lim_{N\to\infty}}\, P\bigl(\pi_N(n) \ge M\bigr) \le nP(v_n \le 2\epsilon \cdot L) + n\frac{C}{L},$$

where we have used here that $v_n < v_k$ for $k < n$. Sending $\epsilon \to 0$ and then $L \to \infty$ and using the fact that $P(v_n = 0) = 0$, yields the result. □

5.2. *Proof of Theorem* 1.1. Recall the notation

$$\langle \cdot \rangle_{\alpha,N}^{\mathsf{T}} = (\mu')_N^{\mathsf{T}}(\cdot|\tilde{W}_\alpha)$$

from Section 2, and recall that $\langle \cdot \rangle_{\alpha,N} = \mu_N(\cdot|C_\alpha)$. We begin by stating the following two lemmas whose proofs we will defer to the end of the section.

LEMMA 5.3. *For every $\alpha \in \mathbb{N}$,*

$$|\langle \sigma_1 \rangle_{\alpha,N} - \langle \sigma_1 \rangle_{\pi_N(\alpha),N}^{\mathsf{T}}| \to 0$$

in probability as $N \to \infty$.

LEMMA 5.4. *For every $\alpha \in \mathbb{N}$,*

$$\begin{aligned} &\big|\tanh\big(\langle y \rangle_{\alpha,N} - (\xi'(1) - \xi'(q_*))\langle \sigma_1 \rangle_{\alpha,N}\big) \\ &\quad - \tanh\big(\langle y \rangle_{\pi_N(\alpha),N}^{\mathsf{T}} - (\xi'(1) - \xi'(q_*))\langle \sigma_1 \rangle_{\pi_N(\alpha),N}^{\mathsf{T}}\big)\big| \to 0 \end{aligned} \tag{5.2}$$

in probability as $N \to \infty$.

PROOF OF THEOREM 1.1. By the above two lemmas it suffices to prove (1.7) with $\langle \cdot \rangle_{\pi_N(\alpha),N}^{\mathsf{T}}$ replacing $\langle \cdot \rangle_{\alpha,N}$.

Now, let $Y_\alpha^N = \langle \sigma_1 \rangle_{\pi_N(\alpha),N}^{\mathsf{T}} - \tanh(\langle y \rangle_{\pi_N(\alpha),N}^{\mathsf{T}} - (\xi'(1) - \xi'(q_*))\langle \sigma_1 \rangle_{\pi_N(\alpha),N}^{\mathsf{T}})$. Note that Y_α^N can be written as $Y_\alpha^N = X_{\pi(\alpha)}^N$, where

$$X_\alpha^N := \langle \sigma_1 \rangle_{\alpha,N}^{\mathsf{T}} - \tanh\big(\langle y \rangle_{\alpha,N}^{\mathsf{T}} - (\xi'(1) - \xi'(q_*))\langle \sigma_1 \rangle_{\alpha,N}^{\mathsf{T}}\big).$$

By Lemma 5.1 and Lemma 5.2 it thus suffices to prove convergence of X_α^N to zero.

Observe that for X_α^N, this is a statement about a cavity coordinate with the local field independent of the measure μ'. Indeed, the Hamiltonian H' satisfies (2.1), and y satisfies (2.2). Thus, X_α^N goes to zero in probability by Corollary 2.2, and Theorem 1.1 follows. □

We now turn to the proofs of the lemmas. Set

$$\tilde{\langle \cdot \rangle}_{\alpha,N} = \mu_N(\cdot|W_\alpha).$$

PROOF OF LEMMA 5.3. We begin by observing that

$$|\langle \sigma_1 \rangle_{\alpha,N} - \tilde{\langle \sigma_1 \rangle}_{\alpha,N}| = \left| \fint_{C_\alpha} \sigma_1 \, d\mu_N - \fint_{W_\alpha} \sigma_1 \, d\mu_N \right| \le \frac{2\mu_N(W_\alpha \Delta C_\alpha)}{\mu_N(C_\alpha)}$$

on the event that W_α and C_α both have positive mass. Since $\mu_N(W_\alpha \Delta C_\alpha) \to 0$ in probability by the essentially uniqueness theorem (Corollary 4.2) and $\mu_N(C_\alpha)$ and $\mu_N(W_\alpha)$ converge in law to a random variable that is almost surely positive, this goes to zero in probability. Then, note that by the tilting lemma,

$$|\tilde{\langle \sigma_1 \rangle}_{\alpha,N} - \langle \sigma_1 \rangle_{\pi_N(\alpha)}^{\mathsf{T}}| \le \frac{C'}{\sqrt{N}}$$

with high probability, so that this too goes to zero in probability. The result then follows by the triangle inequality. □

PROOF OF LEMMA 5.4. As $\tanh(x)$ is 1-Lipschitz, and we know from Lemma 5.3 that $\langle\sigma_1\rangle_\alpha - \langle\tilde{\sigma_1}\rangle^{\mathsf{T}}_{\pi_N(\alpha)} \to 0$ in probability, it suffices to show that

$$\langle y\rangle_{\alpha,N} - \langle y\rangle^{\mathsf{T}}_{\pi_N(\alpha),N} \to 0$$

in probability. Observe that

$$\left|\fint_{C_\alpha} y\,d\mu_N - \fint_{W_\alpha} y\,d\mu_N\right| \le \frac{1}{\mu_N(C_\alpha)}\|y\|_{L^2(\mu)}\sqrt{\mu_N(W_\alpha \Delta C_\alpha)}\left(1+\frac{1}{\sqrt{\mu_N(W_\alpha)}}\right),$$

and that with probability tending to 1, $(1+\frac{1}{\sqrt{\mu_N(W_\alpha)}}) \vee \frac{1}{\mu_N(C_\alpha)}$ will be finite. Furthermore, $\mu_N(W_\alpha \Delta C_\alpha) \to 0$ in probability by the quasi-uniqueness theorem (Theorem 4.1), and $\mathbb{E}||y||_2 \le C$ uniformly in N by item 3 of Lemma A.2. Thus, this tends to zero in probability as before. Similarly,

$$|\langle\tilde{y}\rangle_{\alpha,N} - \langle y\rangle^{\mathsf{T}}_{\pi_N(\alpha)}| \le \frac{C'}{\sqrt{N}}\|y\|_{L^2(\mu')}$$

which goes to zero in probability by the same argument. □

APPENDIX

A.1. The clusters $C_{\alpha,N}$ and approximate ultrametricity. In this short section we summarize the properties of the clusters $C_{\alpha,N}$ used to construct the measures $\langle\cdot\rangle_{\alpha,N}$. These properties are described in the following theorem, which is a rephrasing of the main results in [8], specifically as in Section 9, Proposition 9.5–6 and Corollary 9.7 of that paper.

THEOREM A.1 ([8]). *Assume that $\zeta(\{q_*\}) > 0$. Then, there are sequences $q_N \uparrow q_*$, ϵ_N, a_N, b_N all converging monotonically to 0, and $m_N \to \infty$, such that $m_N \cdot b_N^\gamma \to 0$ for some $\gamma \le 1$, $q_N + a_N < q_*$ and*

$$\zeta_N[q_N + a_N, 1] \ge \zeta(\{q_*\}) - b_N$$

for N sufficiently large and such that with probability $1 - o_N(1)$, there exist disjoint random sets $\{C_{\alpha,N}\}_{\alpha\in\mathbb{N}}$ of Σ_N:

1. *The collection $C_{\alpha,N}$ exhaust the set Σ_N:*

$$\sum_\alpha \mu_N(C_{\alpha,N}) \ge 1 - \epsilon_N.$$

2. *For any α points are uniformly close:*

$$\mu_N^{\otimes 2}(\sigma^1, \sigma^2 \in C_{\alpha,N} : R(\sigma^1, \sigma^2) \le q_N - a_N) \le b_N.$$

3. *For any* $\alpha \neq \beta$,

$$\mu_N^{\otimes 2}\big(\sigma^1 \in C_{\alpha,N}, \sigma^2 \in C_{\beta,N} : R(\sigma^1,\sigma^2) \geq q_N + a_N\big) \leq b_N.$$

4. *Uniformly, in* α *we have*

$$\int_{C_{\alpha,N}^{\otimes 2}} |R_{12} - q_*| \, d\mu_N^2 < o_N(1).$$

5. *The weights* $(\mu_N(C_{\alpha,N}))$ *are labeled in decreasing order of mass and converge to the weights of a Poisson–Dirichlet process of parameter* $1 - \zeta(\{q_*\})$.

Note: We may always take $\alpha_{N-1} \geq N^{-1}$ in the above by monotonicity. That we can, take $m_N \cdot b_N^\gamma \to 0$, follows by adding a constant to the definition of n_0 in Lemma 5.2 of [8].

A.2. Tail bounds for some Gibbs averages.

LEMMA A.2 (Localization lemma). *Recall* $\tilde{K}(\mu')$ *from* (3.4), $y_{\alpha,N}$ *from Section* 2 *and* y_N *from* (1.5). *For any* $L > 0$, *we have the following estimates*:

1. *For any* $\alpha \in \mathbb{N}$,

$$P\big(|y_{\alpha,N}| > L\big) \leq C_1(\xi') \cdot e^{-C_2(\xi')L^2}.$$

2. *For any* $\alpha \in \mathbb{N}$,

$$P\bigg(\int_{X_\alpha} \cosh(y_{\alpha,N}) \, d\nu_N > L\bigg) \leq C(\xi')/L.$$

3. *We have that*

$$P\big(\tilde{K}(\mu') \geq L\big) \leq C(\xi')/L.$$

4. *We have that*

$$\mathbb{E}\bigg(\int_{\Sigma_N} y_N^2 \, d\mu_N\bigg)^{1/2} \leq C(\xi').$$

PROOF. In the following, we will drop the index α of our notation without any loss. To see the first item, note that $y_{\alpha,N}$ has finite moment generating function. Fix $\lambda \geq 1$. We have

$$\begin{aligned}
\mathbb{E}e^{\lambda y_N} &= \mathbb{E}\bigg[\frac{\int_{X_\alpha} e^{\lambda y_N(\sigma)} \cosh(y_N(\sigma)) \, d\nu_N}{\int_{X_\alpha} 2\cosh(y_N(\sigma)) \, d\nu_N}\bigg] + P(X_\alpha = \varnothing) \\
&\leq \mathbb{E}\bigg[\fint_{X_\alpha} e^{\lambda y_N(\sigma)} \cosh\big(y_N(\sigma)\big) \, d\nu_N\bigg] + 1 \\
&\leq \mathbb{E}\fint_{X_\alpha} \mathbb{E}\big(\exp(\lambda y_N)\cosh(y_N)|\nu_N\big) \, d\nu_N + 1 \\
&= \frac{1}{2}\big(e^{(1+\lambda)^2 \xi'(1)}\big) + 1.
\end{aligned} \tag{A.1}$$

Then, by Markov's inequality, we have

$$P(y_N \geq L) \leq \mathbb{E}e^{\lambda y_N - \lambda L} \leq \mathbb{E}e^{(1+\lambda)^2 \xi'(1) - \lambda L} \leq C_1(\xi') \cdot e^{-C_2(\xi')L^2},$$

for L sufficiently large by choosing $\lambda = L/2$, for instance. Increasing the value of $C_1(\xi')$ if necessary, we obtain the result for all $L > 0$. Similarly, for $-y_N$.

The second item holds by Markov's inequality, conditioning on ν_N and using the Gaussian bound of item 1. For the third item note that using Lemma A.3, conditioning on μ' and letting Z be a Gaussian random variable with variance $\xi'(1)$, we have

$$P(\tilde{K}(\mu') \geq L) \leq C(\xi')\mathbb{E}[e^{-4Z} \cosh Z]\mu'_N(\Sigma_N) \leq L^{-1}C(\xi'),$$

as desired.

We prove the last item as follows. To see this, observe that it suffices to bound $\mathbb{E}\int y^2\, d\mu$. To estimate this, observe that if $\Delta = \max|r(1,\sigma) - r(-1,\sigma)|$, then,

$$\begin{aligned}\mathbb{E}\int y^2\, d\mu &\leq \mathbb{E}e^{2\Delta} \frac{\int_{\Sigma_{N-1}} y^2 \cosh(y)\, d\mu'}{\int_{\Sigma_{N-1}} \cosh(y)\, d\mu'} \\ &\leq (\mathbb{E}e^{4\Delta})^{1/2}\Big(\mathbb{E}\int y^4 \cosh(y)\, d\mu'\Big)^{1/2},\end{aligned}$$

where in the last inequality we use Cauchy–Schwarz and the fact that $\cosh(x) \geq 1$. Observe that the first term is bounded by (A.3). Since y is independent of μ', we can integrate in y to find that the second term is also uniformly bounded. □

A.3. Decomposition and regularity of mixed *p*-spin Hamiltonians. In this section we present some basic properties of mixed p-spin Hamiltonians. Recall that for $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N$, $\rho(\sigma) = (\sigma_2, \ldots, \sigma_N) \in \Sigma_{N-1}$. Now, observe that for any mixed p-spin glass model, the Hamiltonian has the following decomposition:

$$H_N(\sigma) = \tilde{H}_N(\rho(\sigma)) + \sigma_1 y_N(\rho(\sigma)) + r_N(\sigma_1, \rho(\sigma)), \tag{A.2}$$

where the processes come from the following lemma.

LEMMA A.3. *There exist centered Gaussian processes $\tilde{H}_N, y_N, r_N$ such that* (3.1) *holds and*

$$\begin{aligned}\mathbb{E}\tilde{H}_N(\sigma^1)\tilde{H}_N(\sigma^2) &= N\xi\Big(\frac{N-1}{N}R_{12}\Big), \\ \mathbb{E}y_N(\sigma^1)y_N(\sigma^2) &= \xi'(R_{12}) + o_N(1), \\ \mathbb{E}r_N(\sigma^1)r_N(\sigma^2) &= O(N^{-1}).\end{aligned}$$

Furthermore, there exist positive constant C_1 *and* C_2 *so that with probability at least* $1 - e^{-C_1 N}$,

$$\max_{\sigma \in \Sigma_{N-1}} \bigl| r_N(1,\sigma) - r_N(-1,\sigma) \bigr| \le \frac{C_2}{\sqrt{N}},$$

and a positive constant C_3 *so that*

$$\mathbb{E} \exp\Bigl(2 \max_{\sigma \in \Sigma_{N-1}} \bigl| r_N(1,\sigma) - r_N(-1,\sigma) \bigr|\Bigr) \le C_3. \tag{A.3}$$

PROOF. The lemma is a standard computation on Gaussian processes. To simplify the exposition, we will consider the pure p-spin model. The general case follows by linearity. Here, we set

$$\tilde{H}_N\bigl(\rho(\sigma)\bigr) = N^{-\frac{p-1}{2}} \sum_{2 \le i_1,\dots,i_p \le N} g_{i_1,\dots,i_p} \sigma_{i_1} \cdots \sigma_{i_p},$$

$$y_N\bigl(\rho(\sigma)\bigr) = N^{-\frac{p-1}{2}} \sum_{k=1}^{p} \sum_{\substack{2 \le i_1,\dots,i_p \le N \\ i_k = 1}} g_{i_1,\dots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}$$

and

$$r_N\bigl(\sigma_1, \rho(\sigma)\bigr) = N^{-\frac{p-1}{2}} \sum_{l=2}^{p} \sigma_1^{\ell} \sum_{2 \le i_1,\dots,i_{p-\ell} \le N} J_{i_1,\dots,i_{p-\ell}} \sigma_{i_1} \cdots \sigma_{i_{p-\ell}},$$

where $J_{i_1,\dots,i_{p-\ell}}$ are centered Gaussian random variables with variance equal to $\binom{p}{\ell}$. $J_{i_1,\dots,i_{p-\ell}}$ is the sum of the $g_{i_1,\dots,i_p}$, where the index 1 appears exactly ℓ times. Computing the variance of these three Gaussian processes, give us the the first three statements of the Lemma. For the second to last and last statement, note that for any $\sigma \in \Sigma_{N-1}$, $r(1,\sigma) - r(-1,\sigma)$ is a centered Gaussian process with variance equal to

$$\frac{4}{N^{p-1}} \sum_{\ell=3, \ell \text{ odd}}^{p} \binom{p}{\ell} (N-1)^{p-\ell} \le \frac{C_p}{N^2}$$

for some constant C_p. A standard application of Borell's inequality (Theorem 2.1.1 in [1]), the tail estimate for the maximum of a Gaussian process (equation (2.1.4) in [1]) and Sudakov–Fernique inequality (Theorem 2.2.3 in [1]), gives us the desired result. □

Acknowledgments. We thank an anonymous referee for a careful reading of this manuscript which led to numerous helpful comments and suggestions that greatly improved the presentation of this paper. The authors thank Dmitry Panchenko for numerous comments, suggestions and several discussions on a first version of this project which dramatically improved the results of this paper. We also thank Gérard Ben Arous and Ian Tobasco for several fruitful discussions. A. Auffinger would also thank Louis-Pierre Arguin, Wei-Kuo Chen and Nicola Kistler for helpful and broad discussions about TAP. A. Jagannath thanks Northwestern University for their hospitality.

REFERENCES

[1] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. Springer, New York. MR2319516

[2] ARGUIN, L.-P. and AIZENMAN, M. (2009). On the structure of quasi-stationary competing particle systems. *Ann*. *Probab*. **37** 1080–1113. MR2537550

[3] AUFFINGER, A. and CHEN, W.-K. (2015). On properties of Parisi measures. *Probab*. *Theory Related Fields* **161** 817–850. MR3334282

[4] AUFFINGER, A. and CHEN, W.-K. (2015). The Parisi formula has a unique minimizer. *Comm*. *Math*. *Phys*. **335** 1429–1444. MR3320318

[5] AUFFINGER, A. and JAGANNATH, A. (2019). On spin distributions for generic p-spin models. *J*. *Stat*. *Phys*. **174** 316–332. MR3910895

[6] BOLTHAUSEN, E. (2014). An iterative construction of solutions of the TAP equations for the Sherrington–Kirkpatrick model. *Comm*. *Math*. *Phys*. **325** 333–366. MR3147441

[7] CHATTERJEE, S. (2010). Spin glasses and Stein's method. *Probab*. *Theory Related Fields* **148** 567–600. MR2678899

[8] JAGANNATH, A. (2017). Approximate ultrametricity for random measures and applications to spin glasses. *Comm*. *Pure Appl*. *Math*. **70** 611–664. MR3628881

[9] PANCHENKO, D. (2010). The Ghirlanda–Guerra identities for mixed p-spin model. *C*. *R*. *Math*. *Acad*. *Sci*. *Paris* **348** 189–192. MR2600075

[10] PANCHENKO, D. (2013). The Parisi ultrametricity conjecture. *Ann*. *of Math*. (2) **177** 383–393. MR2999044

[11] PANCHENKO, D. (2013). *The Sherrington–Kirkpatrick Model*. *Springer Monographs in Mathematics*. Springer, New York. MR3052333

[12] PANCHENKO, D. (2015). Hierarchical exchangeability of pure states in mean field spin glass models. *Probab*. *Theory Related Fields* **161** 619–650. MR3334277

[13] TALAGRAND, M. (2003). *Spin Glasses*: *A Challenge for Mathematicians*: *Cavity and Mean Field Models*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete*. 3. *Folge*. *A Series of Modern Surveys in Mathematics* [*Results in Mathematics and Related Areas*. 3*rd Series*. *A Series of Modern Surveys in Mathematics*] **46**. Springer, Berlin. MR1993891

[14] TALAGRAND, M. (2010). Construction of pure states in mean field models for spin glasses. *Probab*. *Theory Related Fields* **148** 601–643. MR2678900

[15] THOULESS, D. J., ANDERSON, P. W. and PALMER, R. G. (1977). Solution of 'solvable model of a spin glass'. *Philos*. *Mag*. **35** 593–601.

DEPARTMENT OF MATHEMATICS
NORTHWESTERN UNIVERSITY
2033 N SHERIDAN RD
EVANSTON, ILLINOIS 60613
USA
E-MAIL: auffing@math.northwestern.edu

DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
UNIVERSITY ONE OXFORD STREET
CAMBRIDGE, MASSACHUSETTS 02138
USA
E-MAIL: aukosh@math.harvard.edu