GENEALOGICAL CONSTRUCTIONS OF POPULATION MODELS

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Representations of population models in terms of countable systems of particles are constructed, in which each particle has a "type," typically recording both spatial position and genetic type, and a level. For finite intensity models, the levels are distributed on $[0, \lambda]$, whereas in the infinite intensity limit $\lambda \to \infty$, at each time t, the joint distribution of types and levels is conditionally Poisson, with mean measure $\Xi(t) \times \ell$ where ℓ denotes Lebesgue measure and $\Xi(t)$ is a measure-valued population process. The time-evolution of the levels captures the genealogies of the particles in the population.

Key forces of ecology and genetics can be captured within this common framework. Models covered incorporate both individual and event based births and deaths, one-for-one replacement, immigration, independent "thinning" and independent or exchangeable spatial motion and mutation of individuals. Since birth and death probabilities can depend on type, they also include natural selection. The primary goal of the paper is to present particle-with-level or lookdown constructions for each of these elements of a population model. Then the elements can be combined to specify the desired model. In particular, a nontrivial extension of the spatial Λ-Fleming–Viot process is constructed.

CONTENTS

1. Introduction.

1.1. *Background.* There is now a vast mathematical literature devoted to modeling the dynamics of biological populations. The models employed generally fall into one of two classes: ecological models, that aim to elucidate the interactions within and between populations, and between those populations and the environment; and models of population genetics, that aim to explain the patterns of genetic variation observed in samples from a population. Ecological models typically take into account (some of) spatial structure, competition for resources, predator–prey interactions and changing environmental conditions. Often they assume infinite populations, allowing one to concentrate on fluctuations in growth rates and ignore demographic stochasticity. Models from population genetics, by contrast, often concentrate on the demographic stochasticity (known in that context as random genetic drift) which arises from the randomness due to reproduction in a finite population and assume that the population from which one is sampling is *panmictic* (i.e., there are no group structures or mating restrictions) and of constant size. The "size," however, is not taken to be the census population size, but rather an *effective* population size, which is intended to capture the effects of things like varying population size and spatial structure. In particular, the underlying ecology is supposed to be encapsulated in this single parameter. This strategy has been surprisingly effective, but in most situations, notably when the population is geographically dispersed, the influence of different evolutionary and ecological forces on the value of the effective population size remains unresolved. To address these effects, one must combine ecological and genetical models.

Whereas in ecological models one usually asks about the existence of equilibria or the probability that a species can invade new territory, in population genetics, data on the differences between genes sampled from a finite number of individuals in the population is used to infer the "genealogical trees" that relate those genes, and so from a practical point of view, it is the distribution of these trees that one would like to describe. As a result, we require a framework for modeling populations which allows one to combine ecology and genetics in such a way that the genealogical trees relating individuals in a sample from the population are retained. Our goal in this paper is to provide just such a framework.

Mathematical population genetics is concerned with models that capture, for large populations, the key forces of evolution that are acting on the population, but which are robust to changes in the fine detail of local reproduction mechanisms. Diffusion limits lie at the heart of the theory. The prototypical example is the Wright–Fisher diffusion which arises as an approximation to the dynamics of allele frequencies in large panmictic populations of neutral genes whose dynamics can be governed by a plethora of different models. In this situation, the genealogical trees relating individuals in a sample are approximated by Kingman's coalescent, in which each pair of ancestral lineages coalesces into a common ancestor at a rate inversely proportional to the effective population size. Naïvely one obtains the Kingman coalescent as a "moment dual" to the diffusion. However, this is not sufficient to guarantee that it really approximates the genealogy of a sample from one of the individual based models. Indeed, there are examples of systems of individual based models for which the allele frequencies are approximated by a common diffusion, but for which the genealogical trees relating individuals in a sample from the limiting populations have *different* distributions [Taylor (2009)]. Whereas the structure of the genealogical trees is usually implicit in the description of individual based models, in the diffusion limit the individuals have disappeared and with them their genealogies. Our approach allows us to retain information about the genealogies as we pass to the limit.

The framework that we shall present here is very general. It will allow us to construct population models that capture the key ecological forces shaping the population as well as demographic stochasticity. Many "classical" examples will emerge as special cases. We shall use it to pass from individual based models to continuous approximations, but while retaining information about the way in which individuals in a random sample from the population are related to one another. In particular, we shall fulfill one of our primary aims when we began this project, by constructing the spatial Λ-Fleming–Viot process [that was introduced in Barton, Etheridge and Véber (2010) and Etheridge (2008)] as a high-density limit of a class of individual based models that generalize those considered by Berestycki, Etheridge and Hutzenthaler (2009) (Section 4.1). We also present a different construction, equivalent in the high-density limit to that of Véber and Wakolbinger (2015), but requiring somewhat weaker conditions. Moreover, we present a generalisation of the spatial Λ-Fleming–Viot process which incorporates fluctuations of the local population density (Section 4.2).

1.2. *Approach.* Our approach belongs to the family of "lookdown constructions." Building on the ideas of Donnelly and Kurtz (1996) and Donnelly and Kurtz (1999), a number of authors have developed constructions of population models that incorporate information about genealogical relationships. These constructions typically involve assigning each individual in the population to a (nonnegative) integer or real-valued "level," with connections between the levels determining the genealogical trees. They are generically referred to as "lookdown" constructions since, in most cases, during reproduction events, offspring inserted at a given level "look down" to individuals at lower levels to determine their parent.

Lookdown constructions are simplest if the spatial locations or types of individuals in the population do not affect the reproductive dynamics. In that setting, the "levels" can be taken to be nonnegative integer-valued. The processes are constructed in such a way that at each time t, the types, elements of an appropriate space E, of the individuals indexed by their levels $\{X_i(t)\}$ are exchangeable, that is, the joint distribution does not change if we permute the indices, and in an infinite population limit, the measure that gives the state of the limiting measure-valued process is simply the de Finetti measure of the infinite exchangeable family $\{X_i(t)\}$.

We illustrate the key idea for the simple example originally considered in Donnelly and Kurtz (1996). Consider a population of constant size N. Individuals are assigned levels $1, \ldots, N$ by choosing uniformly at random among all possible assignments. The dynamics are as follows: we attach an independent Poisson process $\pi_{(i,j)}$, of rate λ, to each pair (i, j) of levels. At a point of $\pi_{(i,j)}$, the individual with the higher of the two levels i and j dies and is replaced by a copy of the individual with the lower level. In between these replacement events, individuals (independently) accumulate mutations. Since the level of an individual has such a strong impact on its evolution, it is not at all obvious that this description gives rise to a sensible population model. To see that it does, one must show that if $\{X_i(0)\}$ is exchangeable, then for each $t > 0$, $\{X_i(t)\}$ is exchangeable, and that the probability-measure-valued process Z_N given by the empirical measure $Z_N(t) = \sum_{i=1}^N X_i(t)/N$ has the same distribution as the probability-measure-valued process $\widehat{Z}_N$ obtained from a sensible population model.

Ignoring the possibility of mutations, the generator of the process described above is

$$A^N f(x) = \sum_{1 \leq i < j \leq N} \lambda \big(f \big(\Phi_{ij}(x) \big) - f(x) \big),$$

where $\Phi_{ij}(x)$ is obtained from x by replacing x_j by x_i. A sensible population model, specifically, a simple Moran model, has generator

$$\widehat{A}^N f(x) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} \lambda \big(f \big(\Phi_{ij}(x) \big) - f(x) \big).$$

In Donnelly and Kurtz (1996), it was shown that if X^N is a solution of the martingale problem for A^N and $\widehat{X}^N$ is a solution of the martingale problem for $\widehat{A}^N$ such that $X^N(0)$ and $\widehat{X}^N(0)$ have the same exchangeable initial distribution, then Z^N and $\widehat{Z}^N$ have the same distribution as $\mathcal{P}(E)$-valued processes.

The proof in Donnelly and Kurtz (1996) is based on an explicit construction and a filtering argument. This filtering argument, along with a similar argument used in Kliemann, Koch and Marchetti (1990) in a proof of Burke's theorem in queueing theory, motivated the development of the Markov mapping theorem in Kurtz (1998), Theorem A.2 in the Appendix of this paper, which is a fundamental tool in the present work.

To apply the Markov mapping theorem in the setting of Donnelly and Kurtz (1996), for $x \in E^N$, let $z_N \in \mathcal{P}(E)$ be given by $z_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$. For $f \in B(E^N)$, the bounded, measurable functions on E^N, define

$$\alpha f(z_N) = \frac{1}{N!} \sum_\sigma f(x_{\sigma(1)}, \ldots, x_{\sigma(N)}),$$

where the sum is over all permutations of $\{1, \ldots, N\}$. In other words, we *average out* over the (uniform) distribution of the assignment of individuals to levels. We then observe that for $f \in B(E^N)$,

$$
\begin{aligned}
\alpha A^N f(z_N) &= \alpha \widehat{A}^N f(z_N) \\
&= \frac{1}{2} \sum_{1 \le i \ne j \le N} \lambda \big(\alpha f \big(z_N + N^{-1} (\delta_{x_i} - \delta_{x_j}) \big) - \alpha f(z_N) \big) \\
&\equiv C_N \alpha f(z_N)
\end{aligned}
$$

for any choice of x satisfying $z_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$. Theorem A.2 then implies that for any solution $\widetilde{Z}$ of the martingale problem for C_N there exist solutions X^N and $\widehat{X}^N$ of the martingale problems for A and $\widehat{A}$, respectively, such that Z^N and $\widehat{Z}^N$ have the same distribution as $\widetilde{Z}^N$. In other words, our model is really just the classical Moran model, but augmented with a very particular labeling of the individuals in the population. A nice property of this labeling, is that the model for a population of size N is embedded in that for a population of size M for any $M > N$, and so it is straightforward to identify what will happen in the limit as $N \to \infty$.

Finally, observe that for $f \in \bigcup_N B(E^N)$, we can define

$$Af(x) = \sum_{1 \le i < j} \lambda \big(f(\Phi_{ij}(x)) - f(x) \big),$$

that is, if $f \in B(E^N)$, $Af(x) = A^N f(x)$.

Let $\{X_i(0)\}$ be an infinite exchangeable sequence in E, and construct a process $X(t) = \{X_i(t)\}$ using independent Poisson processes $\pi(i, j)$ as above. Then

$\{X_1, \ldots, X_N\}$ is a solution of the martingale problem for A^N and hence X is a solution of the martingale problem for A. The limit $Z(t)$ of $Z^N(t)$ is the de Finetti measure for $\{X_i(t)\}$ and averaging implies that for each $f \in B(E^N)$,

$$\langle f, Z^{(N)}(t) \rangle - \langle f, Z^{(N)}(0) \rangle - \int_0^t \langle Af, Z^{(N)}(s) \rangle \, ds,$$

where $Z^{(N)}(t)$ is the N-fold product measure of $Z(t)$, is a $\{\mathcal{F}_t^Z\}$-martingale. That, in turn, implies Z is a Fleming–Viot process. These observations give an explicit construction of a process given implicitly in Dawson and Hochberg (1982).

From the construction of X, it is also a simple matter to see that the genealogical trees relating individuals in the population are governed by the Kingman coalescent, just as for the Moran model. In addition, the genealogy of a sample of size n, that is, the particles at the n lowest levels, does not change as we increase the population size since, by construction, the processes at the n lowest level are not affected by the processes at the higher levels.

In order to extend the lookdown construction to the setting in which the locations or types of individuals in the population affect their reproductive dynamics, Kurtz (2000) introduced the idea of taking random levels in $[0, \infty)$. More precisely, writing E for the space in which the population evolves, conditional on the empirical measure of the population configuration being $K(t)$ at time t, "individuals" are assigned types and levels according to a Poisson distribution on $E \times [0, \infty)$ with mean measure $K(t) \times \ell$, where ℓ is Lebesgue measure. If we "average out" over the distribution of the levels we recover $K(t)$. Under appropriate conditions, the most important of which is that the generator governing the dynamics of the labeled population respects the conditionally Poisson structure (the analogue of the exchangeability in the case of fixed levels), the Markov mapping theorem, Theorem A.2, allows us to conclude that by "removing the levels" we recover the Markov process whose generator is obtained through this process of averaging. In particular, existence of a solution to the martingale problem for the unlabeled population process is enough to guarantee existence of a solution to the martingale problem for the labeled population, from which a solution to that for the unlabeled population can be read off by averaging. Moreover, uniqueness of the solution of the labeled martingale problem guarantees that of the solution to the unlabeled martingale problem. In Kurtz (2000), this approach was used to construct measure-valued population models with spatially dependent birth and death rates: for a given spatial location, offspring can be inserted at rates that depend on the local configuration without destroying the conditionally Poisson structure. Poisson levels have been used extensively since [e.g., Buhr (2002), Donnelly et al. (2000), Greven, Limic and Winter (2005), Véber and Wakolbinger (2015)]. In Kurtz and Rodrigues (2011), levels are again conditionally Poisson, but now they are allowed to evolve continuously with time, a device which we shall also exploit in this work. The main novelty in the examples presented here is that we are

able to (flexibly) incorporate "event-based" updating mechanisms in the lookdown construction.

Our approach in this article will be to define population models in which individuals are assigned levels, to average out over those levels in order to identify the unlabeled population model, and to pass to an infinite population limit. Justification of this approach to constructing the unlabeled population model is based upon filtering arguments, that is, the "averaging out" corresponds to conditioning on all information about the past of the process *except* the levels of the particles. Ensuring the validity of this conditioning argument requires that the assignment of individuals to levels be done in such a way that past observations of the distribution of spatial positions and genetic types does not give any information about the current levels of individuals in the population. It is important to realize that such assignments are far from unique. For example, in Section 3.1 we provide three possible ways for levels to evolve in a simple pure death process and in Section 4.1, we give two different particle constructions of the spatial Λ-Fleming–Viot process.

In the models we consider, new individuals have a single parent. This assumption is common in both genetic and ecological models. To specify a model, one must specify the rules by which a parent is selected, the rules by which the number and types of offspring are determined, the rules that determine the time of death of an individual, and the rules by which types change through movement, mutation or other process. One can also include such processes as immigration. The primary goal of the paper is to outline how one can obtain a lookdown construction for any such model, and hence determine the genealogy of a sample of individuals from the population. In Section 3, we consider each of the pieces separately. One then constructs a model by selecting "one of these" and "one of those" and "one of something else." Since we are considering Markov models, each piece corresponds to a generator, and the final model is essentially obtained by adding the generators. Since each piece has a lookdown representation built in, the lookdown representation of the final model is obtained. This description of the construction is formal and additional work must be done to ensure that the generator obtained uniquely determines a process. One useful approach to proving uniqueness is to show that the martingale problem is equivalent to a system of stochastic equations (cf., Theorem A.6) and then prove uniqueness for the system of equations. For example, see Lemma 4.3, which gives a new proof of uniqueness for the spatial Λ-Fleming–Viot process under conditions given in Véber and Wakolbinger (2015).

1.3. *Structure of paper.* The rest of the paper is laid out as follows. In Section 2, we lay out the notation that we need for our discrete and continuous population models and for the "averaging" operations that we apply when we use the Markov mapping theorem. In order to construct our general models, we exploit the fact that sums of generators are typically generators [see, e.g., Problem 32 in Section 4.11 of Ethier and Kurtz (1986)], and so we can break our models apart into

component pieces. In Section 3, we examine each of these components in turn. In Section 4, we draw these together into a collection of familiar, and not so familiar, examples. For convenience, some useful identities for Poisson random measures are gathered together in Appendix A.1, and the Markov mapping theorem is stated in Appendix A.2. We refer to Appendix A.2 of Kurtz and Rodrigues (2011) for necessary results on conditionally Poisson systems.

We need to emphasize that although Section 3 is the main focus of the paper, it contains calculations, not proofs. These calculations give the first step in the application of the Markov mapping theorem, Theorem A.2, which ensures that the lookdown constructions actually represent the desired processes, but additional details must be checked for particular applications. We spell this out in the simplest example of a pure death process in Section 3.1 and in the novel setting of the spatial Λ-Fleming–Viot process and its extensions in Section 4.1 and Section 4.2. In addition, the discrete particle models, indexed by $\lambda > 0$, should converge to measure-valued models as $\lambda \to \infty$. For many of the models, convergence of the lookdown constructions is obvious while in other cases, convergence follows easily by standard generator/martingale problem arguments. It is then useful to know that convergence of the lookdown constructions implies convergence of the corresponding measure-valued processes. Appendix A.3 of Kurtz and Rodrigues (2011) provides the results needed to verify this convergence.

The results given in Section 4 are intended to be rigorous unless otherwise indicated.

2. Notation. We consider continuous-time, time-homogeneous, Markov models specified by their generators. Each individual will have a *type* chosen from a complete separable metric space (E, d). We emphasize that here we are using "type" as shorthand for both spatial location and genetic type. The distribution of types over E may be discrete, that is, given by a counting measure that "counts" the number of individuals in each subset of E, or continuous, that is, the distribution of types is given by a measure on E as in the classical examples of Dawson–Watanabe and Fleming–Viot. In addition, each individual will be assigned a "level" which in the discrete case will be sampled from an interval $[0, \lambda]$ and in the continuous case from $[0, \infty)$. No two individuals will have the same level, and in the continuous case, the types along with their levels give a countable collection of particles that determines the measure.

A state of one of our discrete population models will be of the form $\eta = \sum \delta_{(x,u)}$, where $(x, u) \in E \times [0, \lambda]$. We shall abuse notation and treat η both as a set and a counting measure, with the understanding that multiple points are treated as distinct individuals. In other words,

$$\sum_{(x,u)\in\eta} g(x, u) = \int g(x, u)\eta(dx, du)$$

and

$$\prod_{(x,u)\in\eta} g(x,u) = \exp\left\{\int \log g(x,u)\eta(dx,du)\right\}.$$

The projection of η on E will be denoted $\overline{\eta} = \sum_{(x,u)\in\eta} \delta_x$ and η will have the property that conditional on $\overline{\eta}$, the levels of the individuals in the population are independent uniform random variables on $[0,\lambda]$. It will be crucial that this conditioning property be preserved by the transformations of η induced by the components in our generator. Notice that allocating levels as independent uniform random variables is the natural continuous analogue of the way in which we allocated discrete levels through a uniform random sample from all possible permutations. We shall write $\alpha(\overline{\eta}, \cdot)$ for the joint distribution of independent uniform $[0,\lambda]$ random variables U_x indexed by the points $x \in \overline{\eta}$. If f is a function of the U_x, then αf will denote the corresponding expectation.

When there is a need to be precise about the state space $\mathcal{N}_0$ for the counting measures η, we will assume that $\mathcal{N}_0$ satisfies the following condition.

CONDITION 2.1. There exist $c_k \in C(E \times [0,\infty))$ (or $c_k \in C(E \times [0,\lambda])$ if $\lambda < \infty$), $k = 1, 2, \ldots$, $c_k \geq 0$, $\sum_{k=1}^{\infty} c_k(x,u) > 0$, $(x,u) \in E \times [0,\infty)$ such that $\eta \in \mathcal{N}_0$ if and only if $\int_{E\times[0,\infty)} c_k(x,u)\eta(dx,du) < \infty$ for each k, and for $\eta_n, \eta \in \mathcal{N}_0$, $\eta_n \to \eta$ if and only if $\int_{E\times[0,\infty)} f\,d\eta_n \to \int_{E\times[0,\infty)} f\,d\eta$ for each $f \in C(E \times [0,\infty))$ such that $|f| \leq a_f c_k$ for some k and some $a_f \in (0,\infty)$.

We note that $\mathcal{N}_0$ defined in this way will be a Polish space, and if all the c_k have compact support, convergence is just vague convergence.

Under appropriate conditions (which we make explicit for the examples in Section 4) we can pass from the discrete population models to an infinite density limit. The resulting continuous population models arise as limits of states η_λ under assumptions that imply $\lambda^{-1}\eta_\lambda(\cdot, [0,\lambda])$ converges (at least in distribution) to a (possibly random) measure Ξ on E. This is the analogue of convergence of the empirical distribution in the simple case of a fixed number of discrete levels described in Section 1.2. Since we require that the levels in η_λ be conditionally independent uniform random variables given $\overline{\eta}_\lambda$, it follows that η_∞, the limit of the η_λ, will be a counting measure on $E \times [0,\infty)$ that is conditionally Poisson with Cox measure $\Xi \times \ell$, ℓ being Lebesgue measure. That is, for example,

$$E\left[e^{-\int_{E\times[0,\infty)} f(x,u)\eta(dx,du)}\big|\Xi\right] = e^{-\int_E \int_0^\infty (1-e^{-f(x,u)})\,du\,\Xi(dx)}.$$

To mirror our notation in the discrete setting, in the continuous case, $\alpha(\Xi, \cdot)$ will denote the distribution of a conditionally Poisson random measure η on $E \times [0,\infty)$ with mean measure $\Xi(dx) \times \ell$. See Appendix A.1 and Appendices A.1, A.2 and A.3 of Kurtz and Rodrigues (2011).

To describe the generators of our population models, we take the domain to consist of functions of the form

$$(2.1) \qquad f(\eta) = \prod_{(x,u)\in\eta} g(x,u) = \exp\left\{\int \log g(x,u)\eta(dx,du)\right\},$$

where g is continuous in (x, u), differentiable in u and $0 \le g \le 1$. In order for the generator to be defined in specific examples, g may, for example, be required to satisfy additional regularity conditions, but the key point is that the collection of g employed will be large enough to ensure that the domain is separating. In what follows, we will frequently write expressions in which $f(\eta)$ is multiplied by one or more factors of the form $1/g(x, u)$. It should be understood that if $g(x, u) = 0$, it simply cancels the corresponding factor in $f(\eta)$. Since linear combinations of martingales are martingales, we could, of course, extend the domain to include finite linear combinations of functions of the form (2.1).

In the discrete case, if a transformation moves the level of an individual above λ, then the individual dies. We therefore impose the condition $g(x, u) = 1$ if $u \ge \lambda$. In this case $\alpha f(\overline{\eta}) = \prod_{x\in\overline{\eta}} \overline{g}(x)$, where $\overline{g}(x) = \lambda^{-1} \int_0^\lambda g(x, u)\,du$.

In the continuous case, we assume that there exists some u_g such that $g(x, u) = 1$ for $u \ge u_g$. Consequently, $h(x) = \int_0^\infty (1 - g(x, u))\,du$ is finite, and we have

$$\alpha f(\Xi) = e^{-\int_E \int_0^\infty (1-g(x,u))\,du\,\Xi(dx)} = e^{-\int_E h(x)\Xi(dx)}.$$

3. Components of our generators.
Having established our notation, we now turn to the building blocks of our population models. By combining these components, we will be able to consider models which incorporate a wide range of reproduction mechanisms.

3.1. *Pure death process.*
In this subsection, we introduce a component which, when we average over levels, corresponds to each individual in the population, independently, dying at an instantaneous rate $d_0(x) \ge 0$ which may depend on its type, x. We reiterate that x encodes both spatial position and genetic type. In particular, we do not require the population to be selectively neutral.

We assume that the level of an individual of type x evolves according to the differential equation $\dot{u} = d_0(x)u$. The individual will be killed when its level first reaches λ. Note that since the initial level $u(0)$ of an individual must be uniformly distributed on $[0, \lambda]$, if nothing else affects the level, the lifetime of the individual (that is the time τ until the level hits λ) is exponentially distributed,

$$P\{\tau > t\} = P\{u(0)e^{d_0(x)t} < \lambda\} = P\{u(0) < \lambda e^{-d_0(x)t}\} = e^{-d_0(x)t},$$

and conditional on $\{\tau > t\} = \{u(0)e^{d_0(x)t} < \lambda\}$, $u(0)e^{d_0(x)t}$ is uniformly distributed on $[0, \lambda]$.

The generator of this process is

$$A_{pd} f(\eta) = \int_{E \times [0,\lambda]} f(\eta) d_0(x) u \frac{\partial_u g(x,u)}{g(x,u)} \eta(dx, du).$$

Note that $g(x,u)$ in the denominator cancels the corresponding factor in $f(\eta)$. Consequently,

$$\alpha A_{pd} f(\overline{\eta}) = \alpha f(\overline{\eta}) \int_E \frac{1}{\overline{g}(x)} \lambda^{-1} \int_0^\lambda d_0(x) u \partial_u g(x,u) \, du \overline{\eta}(dx).$$

Observing that

$$\lambda^{-1} \int_0^\lambda u \partial_u g(x,u) \, du = \lambda^{-1} u(g(x,u) - 1)|_0^\lambda - \lambda^{-1} \int_0^\lambda (g(x,u) - 1) \, du$$

$$= 1 - \overline{g}(x),$$

we see that

$$(3.1) \qquad \alpha A_{pd} f(\overline{\eta}) = \alpha f(\overline{\eta}) \int_E d_0(x) \left(\frac{1}{\overline{g}(x)} - 1 \right) \overline{\eta}(dx),$$

so that in this case, the projected population model is indeed just a pure death process in which the death rates may depend on the types of the individuals.

The calculation above was purely formal. It is instructive to illustrate the work required to apply Theorem A.2 in the context of this simple example. The key is that we must be able to check (A.7); that is, we restrict the domain of A_{pd} and exhibit a function $\psi \geq 1$ for which, for each f in this smaller domain, we can find a constant c_f such that $|A_{pd} f(\eta)| \leq c_f \psi(\eta)$. To this end, suppose $K_1 \subset K_2 \subset \cdots$ are subsets of E such that $E = \bigcup_k K_k$ [e.g., if $E = \mathbb{R}^d$, we might take $K_k = B_k(0)$]. Then let $\mathcal{D}(A)$ be the collection of f of the form $f(\eta) = \prod_{(x,u)} g(x,u)$ for $g(x,u) \in C_b(E \times [0,\lambda])$ satisfying $\partial_u g(x,u) \in C_b(E \times [0,\lambda])$ and $g(x,u) = 1$ for $(x,u) \notin K_k \times [0, u_g]$ for some $k \in \mathbb{N}$ and $0 \leq u_g \leq \lambda$. We can take ψ in Theorem A.2 to be of the form

$$\psi(\eta) = \int_{E \times [0,\lambda]} \sum_k d_0(x) \delta_k \mathbf{1}_{K_k}(x) \eta(dx, du) + 1$$

for some $\{\delta_k\}$ satisfying $\sum_k \delta_k \sup_{x \in K_k} d_0(x) < \infty$. (The "+1" is just to guarantee that $\psi \geq 1$.) Then for $g(x,u) = 1$ outside $K_k \times [0, u_g]$, we can take $c_f = u_g \|\partial_u g\| \delta_k^{-1}$ in (A.7), where $\| \cdot \|$ denotes the sup norm. The function $\widetilde{\psi}$ of Theorem A.2, which is just $\alpha \psi(\overline{\eta})$, takes the form

$$\widetilde{\psi}(\overline{\eta}) = \int_E \sum_k d_0(x) \delta_k \mathbf{1}_{K_k}(x) \overline{\eta}(dx) + 1.$$

Then, by Theorem A.2, any solution of the martingale problem for αA_{pd} satisfying $E[\int_0^t \widetilde{\psi}(\overline{\eta}_s) \, ds] < \infty$ (which will hold provided $E[\widetilde{\psi}(\overline{\eta}_0)] < \infty$) can be obtained from a solution of the martingale problem for A_{pd}.

Other choices of the dynamics of the process with levels would have projected onto the same population model on averaging out the levels. For example, we could equally have obtained (3.1) by starting with

$$\tilde{A}_{pd} f(\eta) = \int_{E \times [0,\lambda]} f(\eta) d_0(x) \left(\frac{1}{g(x,u)} - 1 \right) \eta(dx, du)$$

(the levels do not move; the particles just disappear) or

$$\widehat{A}_{pd} f(\eta) = \int_{E \times [0,\lambda]} f(\eta) d_0(x) \left(\frac{\lambda^2}{2} - \frac{u^2}{2} \right) \frac{\partial_u^2 g(x,u)}{g(x,u)} \eta(dx, du)$$

for g such that $\partial_u g(x,u)|_{u=0} = 0$ (the levels diffuse and absorption at λ corresponds to death of the particle). Checking that $\alpha \widehat{A}_{pd} f = \alpha A_{pd}$ given in (3.1) is an exercise in integration by parts.

For the continuous population limit, conditionally Poisson as described in Section 1.2, it is immediate that

$$A_{pd} f(\eta) = \int_{E \times [0,\infty)} f(\eta) d_0(x) u \frac{\partial_u g(x,u)}{g(x,u)} \eta(dx, du).$$

Recall that $g(x,u) = 1$ for u above some u_g. Defining

$$h(x) = \int_0^\infty \left(1 - g(x,u) \right) du,$$

and using the identities of Lemma A.1,

$$\begin{aligned}
\text{(3.2)} \qquad \alpha A_{pd} f(\Xi) &= \alpha f(\Xi) \int_E \int_0^\infty d_0(x) u \partial_u g(x,u) \, du \, \Xi(dx) \\
&= \alpha f(\Xi) \int_E d_0(x) h(x) \Xi(dx),
\end{aligned}$$

where $\alpha f(\Xi) = e^{-\int_E h(x) \Xi(dx)}$. Define

$$\text{(3.3)} \qquad \Xi_t(dx) = e^{-d_0(x)t} \Xi_0(dx),$$

and note that

$$\frac{d}{dt} \alpha f(\Xi_t) = \alpha f(\Xi_t) \int_E d_0(x) h(x) \Xi_t(dx),$$

so αA_{pd} is the generator corresponding to the evolution of Ξ given by (3.3).

3.2. *Multiple deaths.* Whereas in the pure death process of the previous subsection, individuals are removed from the population one at a time, we now turn to a model that allows for multiple simultaneous deaths. Moreover, in place of individual based death rates, deaths in the population will be driven by a series of "events" at which a specified number of deaths occur. Since in the discrete setting,

death occurs when the level of an individual crosses level λ, in order to have multiple simultaneous deaths, the levels must evolve through a series of jumps (cf. the thinning transformation in Section 3.6).

We parametrize the multiple death events by points from some abstract space $\mathbb{U}_d$. Corresponding to each $z \in \mathbb{U}_d$ is a pair $(k(z), d_1(\cdot, z))$, where $k(z)$ is an integer and $d_1(\cdot, z)$ is a nonnegative function on E, which allows us to weight each individual's relative probability of death during an event according to its type and spatial position. We shall focus on the case in which events happen with intensity determined by a measure μ_d on $\mathbb{U}_d$, but exactly the same approach applies if we demand that the events occur at discrete times.

For a given pair $(k, d_1(\cdot))$, let

$$\tau(k, d_1, \eta) = \inf\{v : \eta\{(x, u) : e^{vd_1(x)}u \geq \lambda\} \geq k\},$$

where the infimum of an empty set is infinite. After the death event, the configuration becomes

$$\theta_{k,d_1}\eta \equiv \{(x, e^{\tau(k,d_1,\eta)d_1(x)}u) : (x, u) \in \eta \text{ and } e^{\tau(k,d_1,\eta)d_1(x)}u < \lambda\}.$$

Note that k individuals will die if $\eta(\{(x, u) : d_1(x) > 0\}) \geq k$. Otherwise, all individuals in $\{(x, u) : d_1(x) > 0\}$ are killed.

Now assuming that k and d_1 depend on $z \in \mathbb{U}_d$, the generator for the model in which discrete death events occur with intensity $\mu_d(dz)$ then takes the form

$$A_{md}f(\eta) = \int_{\mathbb{U}_d}\left(\prod_{(x,u)\in\eta} g\big(x, ue^{\tau(k(z),d_1(\cdot,z),\eta)d_1(x,z)}\big) - f(\eta)\right)\mu_d(dz).$$

Since, conditional on $\overline{\eta}$, the levels of individuals in the population are independent uniformly distributed random variables on $[0, \lambda]$, $\tau_{x,z}$ given by $U_x e^{d_1(x,z)\tau_{x,z}} = \lambda$ is exponential with parameter $d_1(x, z)$. The lack of memory property of the exponential distribution guarantees that the levels of individuals in the population after the event are still uniformly distributed on $[0, \lambda]$. Moreover, since the $\tau_{x,z}$ are independent,

$$\alpha A_{md}f(\overline{\eta}) = \int_{\mathbb{U}_d} \mathbf{1}_{\{k(z)\leq|\overline{\eta}|\}} \sum_{S\subset\overline{\eta}, |S|=k(z)} d(S, z)\alpha f(\overline{\eta})\left(\frac{1}{\prod_{x\in S}\overline{g}(x)} - 1\right)\mu_d(dz)$$

$$+ \int_{\mathbb{U}_d} \mathbf{1}_{\{k(z)>|\overline{\eta}|\}}(1 - \alpha f(\overline{\eta}))\mu_d(dz),$$

where

$$d(S, z) = P\Big\{\max_{x\in S}\tau_{x,z} < \min_{x\in\overline{\eta}\setminus S}\tau_{x,z}\Big\}.$$

Note that while all the points in η will be distinct (no two points can have the same level), we have not ruled out the possibility that multiple points may have the

same type. Consequently, the same value of x may appear multiple times in S, that is, we allow $\overline{\eta}$ and S to be multisets.

As particular examples, if $k(z) = 1$ and $S = \{x\}$, then

$$d(S, z) = \frac{d_1(x, z)}{\int_E d_1(y, z)\overline{\eta}(dy)},$$

and if $d_1(x, z) = \zeta_z \mathbf{1}_{C_z}(x)$ and $S \subset C_z$, $|S| = k(z)$ [so that $k(z)$ individuals will be chosen at random from the region C_z to die], then

$$d(S, z) = \binom{\overline{\eta}(C_z)}{k(z)}^{-1}.$$

Many interesting high density limits require a balance between birth and death events. However, we close this subsection with a high density limit for the discrete death process above when there are no balancing births. Suppose that $\lambda^{-1}\overline{\eta}_\lambda \to \Xi$ as $\lambda \to \infty$. At an event of type $z \in \mathbb{U}_d$,

$$P\{\tau(k(z), d_1(\cdot, z), \eta_\lambda) > \lambda^{-1}c\} = P\{\eta_\lambda\{(x, u) : e^{cd_1(x,z)/\lambda}u \geq \lambda\} < k(z)\}.$$

Now since, conditional on $\overline{\eta}_\lambda$, the levels u are independent uniform random variables on $[0, \lambda]$, for a single $(x, u) \in \eta_\lambda$, the probability that $u \geq \lambda e^{-cd_1(x,z)/\lambda}$ is $1 - e^{-cd_1(x,z)/\lambda}$ and the events $\{u \geq \lambda e^{-cd_1(x,z)/\lambda}\}$ are independent. Consequently, a Poisson approximation argument implies that $P\{\tau(k(z), d_1(\cdot, z), \eta_\lambda) > \lambda^{-1}c\}$ converges to $P\{Z_c < k(z)\}$ where, conditional on Ξ, Z_c is Poisson distributed with parameter $\int cd_1(x, z)\Xi(dx)$.

Consider the motion of a single level. The jumps of size

$$(e^{\tau(k(z), d_1(\cdot, z), \eta_\lambda)d_1(x, z)} - 1)u$$

that it experiences whenever a death event falls are independent (by lack of memory of the exponential distribution) and so if we speed up time by λ and apply the law of large numbers, observing that $\lambda E[\tau(k(z), d_1(\cdot, z), \eta_\lambda)] = \int_0^\infty P[Z_c < k(z)] \, dc = k(z)/\int d_1(x, z)\Xi(dx)$, we see that, in the limit as $\lambda \to \infty$, the motion of a single level converges to

$$\dot{u} = \int_{\mathbb{U}_d} \frac{k(z)}{\theta(z, \Xi(t))} d_1(x, z)\mu_d(dz)u,$$

where $\theta(z, \Xi) = \int d_1(x, z)\Xi(dx)$. The limit of λA is

$$A_{md}^\infty f(\eta) = \int_{\mathbb{U}_d} f(\eta) \int \frac{k(z)}{\theta(z, \Xi)} d_1(x, z)u \frac{\partial_u g(x, u)}{g(x, u)} \eta(dx, du)\mu_d(dz).$$

Integrating the limiting form of the generator by parts, exactly as we did to obtain (3.2), yields

$$\alpha A_{md}^\infty f(\Xi) = \alpha f(\Xi) \int_{\mathbb{U}_d} \int_E \frac{k(z)d_1(x, z)h(x)}{\theta(z, \Xi)} \Xi(dx)\mu_d(z).$$

Note that there is a time change relative to the generator (3.2) even in the case when $k(z) \equiv 1$ and $d_1(x, z) \equiv d_1(x)$, since deaths are driven by "events" and not linked to individuals.

3.3. *Discrete birth events.* We shall consider two different approaches to birth events. Just as in the case of deaths, a fundamental distinction will be that in the approach outlined in this subsection, births will be based on events and particle levels will evolve in a series of jumps, whereas in the next subsection, births will be individual based and levels will evolve continuously, according to the solution of a differential equation. To emphasize this point, we shall refer to discrete and continuous birth events.

A discrete birth event involves the selection of a parent, the determination of the number of offspring, and the placement of the offspring. Selection of the parent is controlled by a function r with $r(x) \geq 0$ [the larger $r(x)$, the more likely an individual of type x is to be the parent]; the number of offspring is specified by an integer k; and the placement of the offspring is determined by a transition function $q(x, dy)$ from E to E^k. In this discrete model, we can either assume that the parent is eliminated from the population or that it is identified with the offspring at level v^* defined below [in which case it jumps according to $q(x, dy)$ as a result of the event].

For a birth event to occur for (r, k, q), we must have $\int r(x)\overline{\eta}(dx) > 0$, otherwise no individual is available to be the parent. If there is a parent available, then k points, $v_1, \ldots, v_k$, are chosen independently and uniformly on $[0, \lambda]$. These will be the levels of the offspring of the event. Let v^* denote the minimum of the k new levels. For old points $(x, u) \in \eta$ with $u > v^*$ and $r(x) > 0$, let τ_x be defined by $e^{-r(x)\tau_x} = \frac{\lambda - u}{\lambda - v^*}$ and for $(x, u) \in \eta$ satisfying $u < v^*$ and $r(x) > 0$, let τ_x be determined by $e^{-r(x)\tau_x} = \frac{u}{v^*}$. Note that conditioned on $u > v^*$, $\frac{\lambda - u}{\lambda - v^*}$ is uniformly distributed on $[0, 1]$ and similarly, conditioned on $u < v^*$, $\frac{u}{v^*}$ is uniformly distributed on $[0, 1]$, so in both cases, τ_x is exponentially distributed with parameter $r(x)$. Take (x^*, u^*) to be the point in η with $\tau_{x^*} = \min_{(x,u)\in\eta} \tau_x$. This point will be the parent. We have

$$P\{x^* = x'\} = \frac{r(x')}{\int r(x)\overline{\eta}(dx)}, \qquad x' \in \overline{\eta}.$$

After the event, the configuration $\gamma_{k,r,q}\eta$ of levels and types in the population is obtained by assigning types $(y_1, \ldots, y_k)$ with joint distribution $q(x^*, dy)$ uniformly at random to the k new levels and transforming the old levels so that

$$\gamma_{k,r,q}\eta = \{(x, \lambda - (\lambda - u)e^{r(x)\tau_{x^*}}) : (x, u) \in \eta, \tau_x > \tau_{x^*}, u > v^*\}$$
$$\cup \{(x, ue^{r(x)\tau_{x^*}}) : (x, u) \in \eta, \tau_x > \tau_x^*, u < v^*\}$$
$$\cup \{(y_i, v_i), i = 1, \ldots, k\}.$$

Notice that the parent has been removed from the population and that if $r(x) = 0$, the point (x, u) is unchanged.

Since x^* and τ_{x^*} are deterministic functions of η and $v^* \equiv \bigwedge_{j=1}^{k} v_k$, for $(x, u) \in \eta$, $(x, u) \neq (x^*, u^*)$, that is an "old" individual which is not the parent, we can

write the new level as $\mathcal{J}_r^{\lambda}(x, u, \eta, v^*)$. Then

$$f(\gamma_{k,r,q}\eta) = \prod_{(x,u)\in\eta, u\neq u^*} g(x, \mathcal{J}_r^{\lambda}(x, u, \eta, v^*)) \prod g(y_i, v_i).$$

The crucial feature of this construction is captured by the following lemma.

LEMMA 3.1. *Conditional on $\{(y_i, v_i)\}$ and $\overline{\eta}$, $\{\mathcal{J}_r^{\lambda}(x, u, \eta, v^*) : (x, u) \in \eta, u \neq u^*\}$ are independent and uniformly distributed on $[0, \lambda]$.*

PROOF. Conditioned on $\overline{\eta}$ and the vector $v = (v_1, \ldots, v_k)$, the levels u are independent and uniformly distributed on $[0, \lambda]$. Conditioned further on $u < v^*$, u is uniform on $[0, v^*]$, whereas conditioned on $u > v^*$, u is uniform on $[v^*, \lambda]$.

Now if $u < v^*$ and $u \neq u^*$, then, by definition, $ue^{r(x)\tau_{x^*}} < v^*$, that is, $u < v^*e^{-r(x)\tau_{x^*}}$ and, conditional on τ_{x^*} and this event, u is uniform on $[0, v^*e^{-r(x)\tau_{x^*}}]$.

Similarly, conditioning on $u > v^*$ and $u \neq u^*$, knowing τ_{x^*}, u is uniform on $[\lambda - (\lambda - v^*)e^{-r(x)\tau_{x^*}}, \lambda]$.

Consequently, for $(x, u) \in \eta$, we compute

$$E[g(\mathcal{J}_r^{\lambda}(x, u, \eta, v^*))|\overline{\eta}, u \neq u^*, v^*]$$

$$= E[g(\lambda - (\lambda - u)e^{r(x)\tau_{x^*}}), u > v^*|\overline{\eta}, u \neq u^*, v^*]$$

$$\quad + E[g(ue^{r(x)\tau_{x^*}}), u < v^*|\overline{\eta}, u \neq u^*, v^*]$$

$$= E[g(\lambda - (\lambda - u)e^{r(x)\tau_{x^*}})|\overline{\eta}, \lambda - (\lambda - u)e^{r(x)\tau_{x^*}} > v^*, u > v^*, v^*]\frac{\lambda - v^*}{\lambda}$$

$$\quad + E[g(ue^{r(x)\tau_{x^*}})|\overline{\eta}, ue^{r(x)\tau_{x^*}} < v^*, u < v^*, v^*]\frac{v^*}{\lambda}$$

$$= \frac{1}{\lambda - v^*}\int_{v^*}^{\lambda} g(z)\,dz\,\frac{\lambda - v^*}{\lambda} + \frac{1}{v^*}\int_0^{v^*} g(z)\,dz\,\frac{v^*}{\lambda}$$

$$= \frac{1}{\lambda}\int_0^{\lambda} g(z)\,dz,$$

where the conditioning $\lambda - (\lambda - u)e^{r(x)\tau_{x^*}} > v^*$ in the first line of the second equality captures $u \neq u^*$ and to pass from the third line to the fourth we partition over τ_{x^*} (and perform a change of variable in the integral). $\square$

By Lemma 3.1,

$$E[f(\gamma_{k,r,q}\eta)|\overline{\eta}] = \sum_{x'\in\overline{\eta}} \frac{r(x')}{\int r(x)\overline{\eta}(dx)}\alpha f(\overline{\eta})\frac{1}{\overline{g}(x')}\int \prod_{i=1}^k \overline{g}(y_i)q(x', dy).$$

Of course, if $\int r(x)\overline{\eta}(dx) = 0$, there is no parent and $\gamma_{k,r,q}\eta = \eta$.

This tells us how a configuration will be transformed by a single discrete birth event. Now, just as in the previous subsection, we suppose that the events are parametrized by some abstract space, this time denoted by $\mathbb{U}_b$, equipped with a measure μ_{db} that determines the intensity of events. The discrete birth generator will then be of the form

$$A_{db}f(\eta) = \int_{\mathbb{U}_b} \mathbf{1}_{\{\int r(x,z)\overline{\eta}(dx)>0\}}\big(H_{k(z),r(\cdot,z),q(\cdot,z,\cdot)}(g,\eta) - f(\eta)\big)\mu_{db}(dz),$$

where

$$H_{k(z),r(\cdot,z),q(\cdot,z,\cdot)}(g,\eta) = \lambda^{-k(z)}\int_{[0,\lambda]^{k(z)}} \prod_{(x,u)\in\eta,u\neq u^*(\eta,v^*)} g\big(x, \mathcal{J}_{r(\cdot,z)}^{\lambda}(x,u,\eta,v^*)\big)$$

$$\times \int_{E^{k(z)}} \prod_{i=1}^{k(z)} g(y_i,v_i)q\big(x^*(\eta,v^*),z,dy\big)\,dv_1\cdots dv_{k(z)}.$$

Integrating out the levels gives

$$\alpha A_{db}f(\overline{\eta}) = \int_{\mathbb{U}_{db}} \mathbf{1}_{\{\int r(x,z)\overline{\eta}(dx)>0\}}$$

$$\times\left(\sum_{x'\in\overline{\eta}} \frac{r(x',z)}{\int r(x,z)\overline{\eta}(dx)} \frac{\alpha f(\overline{\eta})}{\overline{g}(x')}\left(\int_{E^{k(z)}} \prod_{i=1}^{k(z)} \overline{g}(y_i)q\big(x',z,dy\big)\right)\right.$$

$$\left.- \alpha f(\overline{\eta})\right)\mu_{db}(dz).$$

If we wish to pass to a high density limit, we must control the size and frequency of the jumps in the level of an individual, so that the level process converges as we increase λ. To investigate the restriction that this will impose on the discrete birth events, we examine $\mathcal{J}_r^{\lambda}(x,u,\eta,v^*)$ more closely. Recall that for $u > v^*$, τ_x is defined by $e^{-r(x)\tau_x} = (\lambda - u)/(\lambda - v^*)$. Evidently, we are only interested in the case when $\lambda^{-1}\overline{\eta}_\lambda$ converges to a nontrivial limit, and in changes in those levels that we actually "see" in our limiting model, that is, to levels that are of order one. For such changes, v^* will also be order one, and then it is easy to see that for λ sufficiently large, we will have $u^* > v^*$ and, since $(\lambda - u^*)/(\lambda - v^*) \to 1$, $\tau_{x^*} \to 0$ as $\lambda \to \infty$. Now

(3.4)
$$\mathcal{J}_r^{\lambda}(x,u,\eta,v^*)$$
$$= \mathbf{1}_{\{u>v^*\}}\big(\lambda - (\lambda - u)e^{r(x)\tau_{x^*}}\big) + \mathbf{1}_{\{u<v^*\}}ue^{r(x)\tau_{x^*}}$$
$$= \mathbf{1}_{\{u>v^*\}}\big(ue^{r(x)\tau_{x^*}} - \lambda(e^{r(x)\tau_{x^*}} - 1)\big) + \mathbf{1}_{\{u<v^*\}}ue^{r(x)\tau_{x^*}},$$

and so it follows that for $u < v^*$, $\mathcal{J}_r^\lambda(x, u, \eta, v^*) = ue^{r(x)\tau_{x^*}} \to u$. However, for $(x, u) \in \eta$ with $u > v^*$, $u \neq u^*$, and $r(x) > 0$,

$$
\begin{aligned}
\mathcal{J}_r^\lambda(x, u, \eta, v^*) &= ue^{r(x)\tau_{x^*}} - \lambda\big(e^{r(x)\tau_{x^*}} - 1\big) \\
&= ue^{r(x)\tau_{x^*}} - \lambda\frac{(e^{r(x)\tau_{x^*}} - 1)}{(e^{r(x^*)\tau_{x^*}} - 1)}\big(e^{r(x^*)\tau_{x^*}} - 1\big) \\
(3.5)\qquad &= ue^{r(x)\tau_{x^*}} - \lambda\frac{(e^{r(x)\tau_{x^*}} - 1)}{(e^{r(x^*)\tau_{x^*}} - 1)}\Big(\frac{\lambda - v^*}{\lambda - u^*} - 1\Big) \\
&= ue^{r(x)\tau_{x^*}} - \frac{\lambda}{\lambda - u^*}(u^* - v^*)\frac{(e^{r(x)\tau_{x^*}} - 1)}{(e^{r(x^*)\tau_{x^*}} - 1)} \\
&\to u - (u^* - v^*)\frac{r(x)}{r(x^*)}.
\end{aligned}
$$

Thus if a level jumps, then that jump will be order one. It is clear that, regardless of balancing death events, to have stable behavior of the levels as $\lambda \to \infty$, we must have $v^* < u$ and $r(x) > 0$ only finitely often per unit time. Since for a given k, the probability that v^* will be less than u is $1 - (\frac{\lambda-u}{\lambda})^k$, we need

$$
\begin{aligned}
(3.6)\qquad &\lim_{\lambda \to \infty}\int_{\mathbb{U}_b}\Big(1 - \Big(\frac{\lambda - u}{\lambda}\Big)^{k(z)}\Big)\mathbf{1}_{\{r(x,z)>0\}}\mu_b^\lambda(dz) \\
&= u\lim_{\lambda \to \infty}\frac{1}{\lambda}\int_{\mathbb{U}_b}k(z)\mathbf{1}_{\{r(x,z)>0\}}\mu_b^\lambda(dz) < \infty
\end{aligned}
$$

for each x. If the limit were infinite for some x, then each individual of that type would instantaneously become a parent and be removed from the population.

3.4. *Continuous birth.* As an alternative to the birth process described above, in which levels move by discrete jumps, in this subsection we consider a birth process in which births are based on individuals and levels move continuously. Our aim is to obtain a construction of a pure birth process in which an individual of type x gives birth to k offspring at a rate $r(x)$. For simplicity, we assume that offspring adopt the type of their parent. In the model with levels, an individual $(x, u) \in \eta$ gives birth to k offspring at rate $r(x, u) = (k + 1)(\lambda - u)^k \lambda^{-k} r(x)$. The parent remains in the population and the offspring are assigned levels independently and uniformly distributed above the level of the parent. Evidently this will result in an increase in the proportion of individuals with higher levels and so to preserve the conditionally uniform distribution of levels, we make them move downwards. We shall do this by making them evolve according to a differential equation $\dot{u} = r(x)G_k^\lambda(u)$, for an appropriate choice of the function G_k^λ.

At first sight, there is something arbitrary about the choice of the dependence of branching rate on level. It is essential that $\lambda^{-1}\int_0^\lambda r(x, u)\,du = r(x)$, so that

when we average out over its level, the expected branching rate of an individual of type x is indeed $r(x)$. However, in principle, other choices of $r(x, u)$ with this property would work, provided we change the differential equation driving the levels. This particular choice has the advantage that it makes calculation of the averaged generator, and hence identification of G_k^λ, very straightforward.

The generator of the process with levels is of the form

$$A_{cb,k} f(\eta) = f(\eta) \sum_{(x,u) \in \eta} r(x) \left[\frac{(k+1)}{\lambda^k} \right.$$

$$(3.7) \qquad \times \int_u^\lambda \cdots \int_u^\lambda \left(\prod_{i=1}^k g(x, v_i) - 1 \right) dv_1 \cdots dv_k$$

$$\left. + G_k^\lambda(u) \frac{\partial_u g(x, u)}{g(x, u)} \right].$$

For brevity, for the rest of this subsection, we drop the subscript k in the generator. In order to calculate αA_{cb}, for each $x \in \overline{\eta}$, write $\overline{\eta}_x$ for $\overline{\eta} \setminus x$. Then

$$\alpha A_{cb} f(\eta) = \sum_{x \in \overline{\eta}} r(x) f(\overline{\eta}_x) \left[\frac{1}{\lambda} \int_0^\lambda g(x, u) \frac{(k+1)}{\lambda^k} \right.$$

$$\times \int_u^\lambda \cdots \int_u^\lambda \left(\prod_{i=1}^k g(x, v_i) - 1 \right) dv_1 \cdots dv_k \, du$$

$$\left. + \frac{1}{\lambda} \int_0^\lambda G_k^\lambda(u) \partial_u g(x, u) \, du \right].$$

Now observe that

$$\frac{k+1}{\lambda^{k+1}} \int_0^\lambda g(x, u) \int_u^\lambda \cdots \int_u^\lambda \prod_{i=1}^k g(x, v_i) \, dv_1 \cdots dv_k \, du = \left(\frac{1}{\lambda} \int_0^\lambda g(x, u) \, du \right)^{k+1}.$$

To see this, notice that on the right-hand side we have the result of averaging over $k + 1$ independent uniform levels, while on the left we have $(k + 1)$ times the result of averaging over those levels if we specify that the first level is the smallest, and by symmetry any of the $k + 1$ uniform variables is equally likely to be the smallest. This deals with the first term of the averaged generator. All that remains of the expression in square brackets is

$$(3.8) \qquad \frac{1}{\lambda} \int_0^\lambda \left(G_k^\lambda(u) \partial_u g(x, u) - \frac{(k+1)(\lambda - u)^k}{\lambda^k} g(x, u) \right) du.$$

Now we make a judicious choice of G_k^λ. Suppose that

$$(3.9) \qquad G_k^\lambda(u) = \lambda^{-k} (\lambda - u)^{k+1} - (\lambda - u).$$

Then, noting that $G_k^\lambda(0) = G_k^\lambda(\lambda) = 0$, and integrating by parts, we see that (3.8) reduces to

$$-\frac{1}{\lambda} \int_0^\lambda g(x, u)\, du$$

and so we obtain

$$\alpha A_{cb} f(\overline{\eta}) = \alpha f(\overline{\eta}) \sum_{x \in \overline{\eta}} r(x) [\overline{g}(x)^k - 1],$$

which is the generator of a branching process, as required.

Now consider what happens as $\lambda \to \infty$. Since $g(x, u) \equiv 1$ for $u > u_g$,

$$\frac{1}{\lambda} \int_u^\lambda g(x, v)\, dv \to 1 \qquad \text{as } \lambda \to \infty,$$

and so the first term on the right-hand side of (3.7) vanishes, and observing that

$$G_k^\lambda(u) = u \frac{(1 - \frac{u}{\lambda})^{k+1} - (1 - \frac{u}{\lambda})}{\frac{u}{\lambda}} \to -ku,$$

we obtain

$$A_{cb}^\infty f(\eta) = -f(\eta) \sum_{(x,u) \in \eta} r(x) ku \frac{\partial_u g(x, u)}{g(x, u)}.$$

Assuming $\lambda^{-1} \eta_\lambda(\cdot \times [0, \lambda]) \to \Xi$, we have $\alpha f(\Xi) = e^{-\int_E h(x) \Xi(dx)}$ and, using (A.3),

$$\alpha A_{cb}^\infty f(\Xi) = -e^{-\int_E h(x) \Xi(dx)} \int_E r(x) k \int_0^\infty u \partial_u g(x, u)\, du\, \Xi(dx)$$

$$= e^{-\int_E h(x) \Xi(dx)} \int_E r(x) k \int_0^\infty (g(x, u) - 1)\, du\, \Xi(dx)$$

$$= -e^{-\int_E h(x) \Xi(dx)} \int_E r(x) k h(x) \Xi(dx),$$

[where to perform the integration by parts we have used that $\partial_u g(x, u) = \partial_u(g(x, u) - 1)$] which corresponds to the evolution of Ξ given by

$$\Xi_t(dx) = e^{r(x)kt} \Xi_0(dx).$$

3.5. *One for one replacement.* So far, we have considered separately the births and deaths of individuals. In some models, it is natural to think of offspring as replacing individuals in the population and, thereby, maintaining constant population size. In this section, we consider three different models of one-for-one replacement. For $\lambda < \infty$, we shall suppose that the population size is finite. In the first model, we specify a number $k < |\eta|$ of individuals to be replaced. Those indi-

viduals are then sampled uniformly at random from the population. In the second model, the probability, $r(x)$, that an individual of type x is replaced is specified. Either of these models can be modified in such a way that events affect only a subset $C \subset E$ and this allows us to replace the requirement that η be finite by a local condition [e.g., $\eta(C) < \infty$]. Both are special cases of a third model in which there is a probability distribution $p(S)$ over subsets $S \subset \eta$ that determines the subset to be replaced. We require $p(S)$ to depend only on the types of the members of S and not on their levels. By focusing on the case in which events happen with intensity determined by a measure $\mu_{dr,3}$ on $\mathbb{U}_{dr,3}$ we can replace $p(S)$, a probability, by $r(S)$, a rate, giving the intensity for a replacement event involving the individuals in S.

In all three cases, we can take the levels to be fixed. The parent (x^*, u^*) is taken to be the individual chosen to be replaced that has the lowest level. We assume that the types of the new individuals are chosen independently with distribution given by a transition function $q(x^*, dy)$, but we could allow dependence provided the new individuals are assigned to the chosen levels uniformly at random.

For the first model, it is natural to take a generator of the form

$$
A_{dr,1} f(\eta)
$$
$$
= \int_{\mathbb{U}_{dr,1}} \binom{|\eta|}{k(z)}^{-1} \sum_{S \subset \eta, |S| = k(z)} f(\eta) \left(\prod_{(x,u) \in S} \frac{\int g(y,u) q(x^*(S), z, dy)}{g(x,u)} - 1 \right)
$$
$$
\times \mu_{dr,1}(dz),
$$

where $x^*(S) = x'$ if $(x', u') \in S$ and $u' = \min\{u : (x, u) \in S\}$. As usual, $\mathbb{U}_{dr,1}$ parametrizes the events and they occur with intensity $\mu_{dr,1}$. The levels are fixed and the individuals chosen to be replaced "look down," just as in the simple example of Section 1.2, to identify their parental type. Averaging over levels yields

$$
\alpha A_{dr,1} f(\overline{\eta})
$$
$$
= \int_{\mathbb{U}_{dr,1}} \binom{|\overline{\eta}|}{k(z)}^{-1} \sum_{S \subset \overline{\eta}, |S| = k(z)} \alpha f(\overline{\eta}) \frac{1}{k(z)} \sum_{x' \in S} \left(\prod_{x \in S} \frac{\int \overline{g}(y) q(x', z, dy)}{\overline{g}(x)} - 1 \right)
$$
$$
\times \mu_{dr,1}(dz).
$$

In the second case, let $\xi_{x,u}$ be independent random variables with $P\{\xi_{x,u} = 1\} = 1 - P\{\xi_{x,u} = 0\} = r(x)$. Then $(x^*, u^*) \in \eta$ is the parent if $u^* = \min\{u : \xi_{x,u} = 1\}$. Let

(3.10)
$$
\widehat{g}(x, z, u) = \int_E g(y, u) q(x, z, dy).
$$

Once again we can fix the levels, in which case the generator will take the form

$$A_{dr,2}f(\eta)$$

(3.11)
$$= \int_{\mathbb{U}_{dr,2}} \left(E\left[\prod_{(x,u)\in\eta} (\xi_{x,u}\widehat{g}(x^*,z,u) + (1 - \xi_{x,u})g(x,u)) \right] - f(\eta) \right)$$

$$\times \mu_{dr,2}(dz),$$

where the expectation is with respect to the $\xi_{x,u}$ and x^* is a function of η and the $\{\xi_{x,u}\}$. (More precisely, x^* is a function of η and the subset S of the individuals for which $\xi_{x,u} = 1$.) Recall our assumption that there is a u_g such that $g(x,u) = 1$ for all $u > u_g$. This property is inherited by $\widehat{g}$. Thus the factor in the product in (3.11) is 1 if $u \geq u_g$, and so the expectation in the integral can be written as

$$H(g, \widehat{g}, \eta, z)$$

(3.12)
$$\equiv E\left[\prod_{(x,u)\in\eta, u\leq u_g} (\xi_{x,u}\widehat{g}(x^*,z,u) + (1 - \xi_{x,u})g(x,u)) \right]$$

$$= \sum_{S\subset\eta|_{E\times[0,u_g]}} E\left[\prod_{(x,u)\in S} (\xi_{x,u}\widehat{g}(x^*(S),z,u)) \right.$$

$$\times \prod_{(x,u)\notin S, u\leq u_g} (1 - \xi_{x,u})g(x,u) \right]$$

$$= \sum_{S\subset\eta|_{E\times[0,u_g]}} \prod_{(x,u)\in S} (r(x,z)\widehat{g}(x^*(S),z,u))$$

$$\times \prod_{(x,u)\notin S, u\leq u_g} (1 - r(x,z))g(x,u).$$

Partitioning on the lowest level particle, we see that this expression can also be written as

(3.13)
$$\sum_{(x^*,u^*)\in\eta} r(x^*,z)\widehat{g}(x^*,z,u^*) \prod_{(x,u)\in\eta, u<u^*} (1 - r(x,z))g(x,u)$$

$$\times \prod_{(x,u)\in\eta, u>u^*} (r(x,z)\widehat{g}(x^*,z,u) + (1 - r(x,z))g(x,u)).$$

It will be useful to write $A_{dr,2}$ as a sum of two terms,

$$A_{dr,2}f(\eta)$$

$$= f(\eta) \int_{\mathbb{U}_{dr,2}} \left[\sum_{(x^*,u^*)\in\eta|_{E\times[0,u_g]}} \frac{r(x^*,z)(\widehat{g}(x^*,z,u^*) - g(x^*,u^*))}{g(x^*,u^*)} \right.$$

$$(3.14) \qquad \times \prod_{(x,u)\in\eta_{|E\times[0,u_g]},u\neq u^*} \left(1 - r(x,z)\right)$$

$$+ \sum_{S\subset\eta_{|E\times[0,u_g]},|S|\geq 2} \left(\frac{\prod_{(x,u)\in S}\widehat{g}(x^*(S),z,u)}{\prod_{(x,u)\in S}g(x,u)} - 1\right)\prod_{(x,u)\in S} r(x,z)$$

$$\times \prod_{(x,u)\in\eta_{|E\times[0,u_g]}-S}\left(1 - r(x,z)\right)\Bigg]\mu_{dr,2}(dz).$$

We separate the first term, in which only one individual is replaced in the event, because it looks like the generator for simple, *almost* independent evolution of the particle types. We exploit this observation in Section 4.1.

Since

$$\lambda^{-k}\int_0^\lambda g(u')\left(\int_{u'}^\lambda g(u)\,du\right)^{k-1}du' = \frac{1}{k}\left(\lambda^{-1}\int_0^\lambda g(u)\,du\right)^k$$

(cf. the calculations in Section 3.4) it follows from (3.12) that

$$\alpha A_{dr,2}f(\overline{\eta}) = \int_{\mathbb{U}_{dr,2}}\Bigg[\sum_{\overline{S}\subset\overline{\eta}}\left(\prod_{x\in\overline{S}}r(x,z)\right)\left(\frac{1}{|\overline{S}|}\sum_{y\in\overline{S}}\overline{g}(y,z)^{|S|}\right)$$

$$\times\left(\prod_{x\in\overline{\eta}-\overline{S}}(1 - r(x,z))\overline{g}(x)\right) - \alpha f(\overline{\eta})\Bigg]\mu_{dr,2}(dz),$$

where $\overline{g}(y,z) = \lambda^{-1}\int_0^\lambda\widehat{g}(y,z,u)\,du$.

In the third case, in which we specify the rate at which subsets of individuals are replaced, we can write

$$A_{dr,3}f(\eta)$$

$$(3.15)\qquad = \int_{\mathbb{U}_{dr,3}}\sum_{S\subset\eta}r(S,z)f(\eta)\left(\prod_{(x,u)\in S}\frac{\int g(y,u)q(x^*(S),z,dy)}{g(x,u)} - 1\right)$$

$$\times\mu_{dr,3}(dz)$$

and

$$\alpha A_{dr,3}f(\overline{\eta})$$

$$= \int_{\mathbb{U}_{dr,3}}\sum_{\overline{S}\subset\overline{\eta}}r(\overline{S},z)\alpha f(\overline{\eta})\frac{1}{|\overline{S}|}\sum_{x'\in\overline{S}}\left(\prod_{x\in\overline{S}}\frac{\int\overline{g}(y)q(x',z,dy)}{\overline{g}(x)} - 1\right)$$

$$\times\mu_{dr,3}(dz).$$

So far we have dealt with finite population models with one-for-one replacement. We now turn our attention to infinite population limits. For the first model, $A_{dr,1}$, there are two natural ways to pass to an infinite population limit. In one,

the rate at which birth events occur remains the same, but the size of the event (by which we mean the number of individuals replaced) grows with λ, that is,

$$\lambda^{-1} k_\lambda(z) \to \kappa(z) < |\Xi| = \lim_{\lambda \to \infty} \lambda^{-1} |\eta_\lambda|.$$

Asymptotically, this model behaves in the same way as $A_{dr,2}$ in the special case in which $r(x, z) \equiv \kappa(z)/|\Xi|$ and so we do not consider it here.

The other possibility is for $k(z)$ to remain fixed, but for $\mu_{dr,1}$ to increase with λ, that is, to have replacement events occur at an increasingly rapid rate (e.g., this is the approach when we pass from a Moran model to a Fleming–Viot process).

First, we identify the appropriate scaling. Assume that $\lambda^{-1} \eta_\lambda(t, \cdot) \Rightarrow \Xi(t, dx)$, where $\Xi(t, E) < \infty$. [Of course, unless other factors are acting, $\Xi(t, E)$ is constant in time, but recall that we are thinking of our components as "building blocks" of population models.] If a discrete birth event z occurs at time t, then conditional on $\eta_\lambda(t, \cdot \times [0, \lambda])$ and z, the number of individuals selected with levels below a, where $0 < a < \lambda$, is binomial with parameters $k(z)$ and $\frac{\eta_\lambda(t, E \times [0,a])}{\eta_\lambda(t, E \times [0,\lambda])} = O(\lambda^{-1})$. Since the probability of selecting two levels below a is $O(\lambda^{-2})$, if we are to see any interaction between levels in the limiting model, we need to scale $\mu_{dr,1}$ by λ^2. On the other hand, if we scale $\mu_{dr,1}$ by λ^2, the rate at which the individual at a fixed level is selected is of order λ. When this happens, unless it is one of the (finite rate) events in which more than one level below a is selected, the individual at the selected level will necessarily be the parent of the event and so will jump to a new position determined by the transition density q. If the limiting model is to make sense, we must therefore rescale q in such a way that in the limit, the motion of a fixed level will be well defined.

To make this more precise, suppose that an event of type z occurs at time t. If an individual has level u, the probability that they are the parent of the event is

$$\frac{\binom{\eta_\lambda(t, E \times (u, \lambda])}{k(z) - 1}}{\binom{\eta_\lambda(t, E \times [0, \lambda])}{k(z)}} \approx \frac{k(z)}{\eta_\lambda(t, E \times [0, \lambda])}.$$

Assume that q depends on λ. Then the motion of a particle at level u due to its being chosen as a parent is essentially (since we ignore the asymptotically negligible number of times when a particle with level below u is also chosen) Markov with generator

$$\widetilde{B}_\lambda g(x) = \frac{\lambda^2}{\eta_\lambda(t, E \times [0, \lambda])} \int_{\mathbb{U}_{dr,1}} k(z) \big(g(y) - g(x)\big) q_\lambda(x, z, dy) \mu_{dr,1}(dz).$$

We assume that the Markov process with generator

$$B_\lambda g(x) = \lambda \int_{\mathbb{U}_{dr,1}} k(z) \big(g(y) - g(x)\big) q_\lambda(x, z, dy) \mu_{dr,1}(dz)$$

converges in distribution to a Markov process with generator B. Then [up to a time change which, in the limit, will be $1/\Xi(E)$] this Markov process will describe the

motion of a particle at a fixed level that results from it being selected as parent of a replacement event. Note that this convergence implies that for each $\epsilon > 0$,

$$(3.16) \qquad \int k(z)\mathbf{1}_{\{d(y,x)>\epsilon\}}q_\lambda(x,z,dy)\mu_{dr,1}(dz) = O(\lambda^{-1}).$$

Similarly, we identify the interaction between distinct levels in the limiting process. If there are individuals at levels $u_1 < u_2$ and an event of type z occurs at time t, then the probability that u_1, u_2 are the lowest two levels selected is

$$\frac{\binom{\eta_\lambda(t,E\times(u_2,\lambda])}{k(z)-2}}{\binom{\eta_\lambda(t,E\times[0,\lambda])}{k(z)}} \approx \frac{k(z)(k(z)-1)}{\eta_\lambda(t,E\times[0,\lambda])^2} = O(\lambda^{-2}).$$

We chose our rescaling in such a way that events involving two levels below a fixed level a will occur at a rate $O(1)$, and by (3.16), after the event, asymptotically, both the parent and the offspring will have the type of the parent immediately before the event. In this limit, we will never see events involving three or more levels below a fixed level a.

If the replacement process is the only process affecting the population, then

$$|\Xi| = \Xi(t,E) = \lim_{\lambda\to\infty} \frac{\eta_\lambda(t,E\times[0,\lambda])}{\lambda}$$

is constant in time and [recalling that $g(x,u) = 1$ for $u > u_g$] the limiting model will have generator

$$A_{dr,1}^\infty f(\eta)$$

$$= \int_{E\times[0,\infty)} \frac{1}{|\Xi|} f(\eta)\frac{Bg(x,u)}{g(x,u)}\eta(dx,du) + \int_{\mathbb{U}_{dr,1}} \frac{k(z)(k(z)-1)}{|\Xi|^2}$$

$$\times \sum_{(x_1,u_1),(x_2,u_2)\in\eta, u_1<u_2} f(\eta)\left(\frac{g(x_1,u_2)}{g(x_2,u_2)} - 1\right)\mu_{dr,1}(dz)$$

$$= \int_{E\times[0,\infty)} \frac{1}{|\Xi|} f(\eta)\frac{Bg(x,u)}{g(x,u)}\eta(dx,du)$$

$$+ \int_{\mathbb{U}_{dr,1}} \frac{k(z)(k(z)-1)}{|\Xi|^2} \sum_{(x_1,u_1),(x_2,u_2)\in\eta}\left(\frac{f(\eta)}{g(x_1,u_1)g(x_2,u_2)}\right.$$

$$\left.\times\left[\mathbf{1}_{\{u_1<u_2\}}\big(g(x_1,u_2)g(x_1,u_1) - g(x_2,u_2)g(x_1,u_1)\big)\right]\right)\mu_{dr,1}(dz).$$

Applying (A.3) and (A.5), the averaged generator becomes

$$\alpha A_{dr,1}^\infty f(\Xi)$$

$$= e^{-\int_E h(x)\Xi(dx)}\left[-\int_E \frac{1}{|\Xi|}\int_E Bh(x)\Xi(dx)\right.$$

$$+ \int_{\mathbb{U}_{dr,1}} \frac{k(z)(k(z)-1)}{2|\Xi|^2} \int_{E \times E} \big(h(x_1)^2 - h(x_1)h(x_2) \big) \Xi(dx_1) \Xi(dx_2)$$

$$\times \mu_{dr,1}(dz) \bigg].$$

The dependence of the first term on $\mu_{dr,1}$ is absorbed into our definition of B. If $|\Xi| \equiv 1$ and $k(z) = 2$ for all z, then we recognize the generator of a Fleming–Viot diffusion [see, e.g., Section 1.11 of Etheridge (2000)].

It is elementary to identify the limit of our second model as λ tends to infinity. Since $g(x, u) = 1$ for $u > u_g$, the only changes that we "see" are those that affect $\eta^{u_g} = \sum_{(x,u) \in \eta, u \le u_g} \delta_{(x,u)}$ and these are determined by the generator when $\lambda = u_g$, so that

$$(3.17) \qquad A^\infty_{dr,2} f(\eta) = \int_{\mathbb{U}_{dr,2}} \big(H(g, \widehat{g}, \eta, z) - f(\eta) \big) \mu_{dr,2}(dz),$$

with H given by (3.12). If η is conditionally Poisson with Cox measure $\Xi(dx)\,du$, $\{\xi_{x,u,z}\}$ are independent with $P\{\xi_{x,u,z} = 1\} = 1 - P\{\xi_{x,u,z} = 0\} = r(x, z)$, and

$$\eta_1 = \sum_{(x,u) \in \eta} \xi_{x,u,z} \delta_{(x,u)}, \qquad \eta_2 = \sum_{(x,u) \in \eta} (1 - \xi_{x,u,z}) \delta_{(x,u)},$$

then η_1 and η_2 are conditionally independent given Ξ, η_1 and η_2 are conditionally Poisson with Cox measures $r(x, z)\Xi(dx)\,du$ and $(1 - r(x, z))\Xi(dx)\,du$ respectively and the cumulative distribution function of the level of the lowest particle to be replaced is $1 - e^{-u \int r(x,z)\Xi(dx)}$. We now recall that the x coordinates of the points in η_1, ordered according to the u coordinates, are exchangeable with de Finetti measure

$$\frac{r(x, z)\Xi(dx)}{\int_E r(y, z)\Xi(dy)},$$

and partition on the lowest level particle as in (3.13). Using (A.3), this yields

$$E\big[H(g, \widehat{g}, \eta, z) | \Xi \big]$$

$$= e^{-\int_E h(x)(1-r(x,z))\Xi(dx)} \int_E \int_0^\infty r(x^*, z) \widehat{g}(x^*, z, u) e^{-u \int_E r(x,z)\Xi(dx)}$$

$$\times e^{-\int_u^\infty (1 - \widehat{g}(x^*,z,v))\,dv \int_E r(x,z)\Xi(dx)} \Xi(dx^*)\,du$$

$$= e^{-\int_E h(x)(1-r(x,z))\Xi(dx)} \int_E \int_0^\infty r(x^*, z) \widehat{g}(x^*, z, u)$$

$$\times e^{-\int_0^u \widehat{g}(x^*,z,v)\,dv \int_E r(x,z)\Xi(dx)} e^{-\widehat{h}(x^*,z) \int_E r(x,z)\Xi(dx)} \Xi(dx^*)\,du$$

$$= e^{-\int_E h(x)(1-r(x,z))\Xi(dx)} \int_E \frac{r(x^*, z)}{\int_E r(x, z)\Xi(dx)} e^{-\widehat{h}(x^*,z) \int_E r(x,z)\Xi(dx)} \Xi(dx^*)$$

$$\equiv \mathbb{H}(h, \widehat{h}, \Xi, z),$$

where the factor $\int r(x, z) \Xi(dx)$ in the density function of the lowest level has canceled with the denominator in the de Finetti measure of η_1 on the right-hand side in the first line and to get from the second line to the third we integrated with respect to u and used that $\widehat{g} = 1$ for $u > u_g$. Thus

$$\alpha A_{dr,2}^\infty = \int_{\mathbb{U}_{dr,2}} \left(\mathbb{H}(h, \widehat{h}, \Xi, z) - \alpha f(\Xi)\right)\mu_{dr,2}(dz).$$

Evidently, since $A_{dr,1}$ and $A_{dr,2}$ are special cases of $A_{dr,3}$ and their continuous density limits are quite different, we cannot expect a general result for the continuous density limit of $A_{dr,3}$, but a large class of limits should retain the discrete model form

$$A_{dr,3}^\infty f(\eta)$$

$$= \sum_{S \subset \eta} \int_{\mathbb{U}_{dr,3}} r(S, z) f(\eta) \left(\prod_{(x,u) \in S} \frac{\int g(y, u) q(x^*(S), z, dy)}{g(x, u)} - 1 \right)$$

$$\times \mu_{dr,3}(dz),$$

provided there is a sufficiently large class of functions g satisfying

$$(3.18) \quad \begin{aligned} &\sum_{S \subset \eta} \int_{\mathbb{U}_{dr,3}} r(S, z) \\ &\times \sum_{(x,u) \in S} \left| \int (g(y, u) - g(x, u)) q(x^*(S), z, dy) \right| \mu_{dr,3}(dz) < \infty \end{aligned}$$

with $0 \le g \le 1$ and $g(x, u) \equiv 1$ for $u > u_g$. In Section 4.1, we consider an example in which we can center $g(y, u) - g(x, u)$ in order to weaken the condition in (3.18). The form of the averaged generator is problem dependent, but convex combinations of $\alpha A_{dr,1}^\infty$ and $\alpha A_{dr,2}^\infty$ can arise.

3.6. *Independent thinning.* Independent thinning will work in essentially the same way as the pure death process. However, whereas in the pure death process the levels grew continuously, here we scale them up by a (type-dependent) factor at discrete times. Levels which are above level λ after this multiplication are removed. The generator with finite λ is then of the form

$$A_{th} f(\eta) = \int_{\mathbb{U}_{th}} \left(\prod_{(x,u) \in \eta} g(x, u\rho(x, z)) - f(\eta) \right) \mu_{th}(dz),$$

for some $\rho(x, z) \ge 1$. Setting $\rho(x, z) = \frac{1}{1-p(x,z)}$, we see that the probability that $\rho(x, z)U_x > \lambda$, for U_x uniformly distributed on $[0, \lambda]$, is $P\{U_x > \lambda/\rho(x, z)\} = p(x, z)$. Recalling that $g(x, u) = 1$ for $u \ge \lambda$ and integrating out the levels gives

$$\alpha A_{th} f(\overline{\eta}) = \int_{\mathbb{U}_{th}} \left(\prod_{x \in \overline{\eta}} ((1 - p(x, z))\overline{g}(x) + p(x, z)) - \alpha f(\overline{\eta}) \right) \mu_{th}(dz),$$

which says that when a thinning event of type z occurs, individuals are independently eliminated with (type-dependent) probability $p(x, z)$.

In the continuous population limit, the form of A_{th} remains unchanged, and the projected operator becomes

$$\alpha A_{th} f(\Xi) = \int_{\mathbb{U}_{th}} \left(e^{-\int_E \frac{1}{\rho(x,z)} h(x) \Xi(dx)} - \alpha f(\Xi) \right) \mu_{th}(dz),$$

where as usual $h(x) = \int_0^\infty (1 - g(x, u)) \, du$ and $\alpha f(\Xi) = e^{-\int_E h(x) \Xi(dx)}$.

3.7. *Event based models.* Motivated by the model considered in Berestycki, Etheridge and Hutzenthaler (2009), we combine independent thinning and discrete birth so that both transformations take place at the same time. Event times and types (t, z) are determined by a Poisson random measure with mean measure $dt \, \mu_{th,db}(dz)$. The value of z determines the number of offspring $k(z)$, the relative chance $r(x, z)$ that an individual of type x will be the parent, and the parameter $\rho(x, z)$ that determines the probability

$$p(x, z) = \frac{\rho(x, z) - 1}{\rho(x, z)}$$

that an individual of type x is killed. Let

$$\overline{\eta}(r, z) = \int r(x, z) \overline{\eta}(dx),$$

and note that for there to be a parent, we must have $\overline{\eta}(r, z) > 0$. We will assume that the parent is killed, although alternatively, we could interpret the model as saying the parent jumps to the location of the particle at level v^*.

The form of the generator will be

$$A_{th,db}^\lambda f(\eta) = \int_{\mathbb{U}} \mathbf{1}_{\{\overline{\eta}(r,z)>0\}} (H_z^\lambda(g, \eta) - f(\eta)) \mu_{th,db}(dz),$$

where, for $\mathcal{J}_r^\lambda$ given by (3.4), if $\overline{\eta}(r, z) > 0$,

$$H_z^\lambda(g, \eta) = \lambda^{-k(z)} \int_{[0,\lambda]^{k(z)}} \prod_{(x,u) \in \eta, u \neq u^*(\eta, v^*)} g\big(x, \rho(x, z) \mathcal{J}_{r(\cdot, z)}^\lambda(x, u, \eta, v^*)\big)$$

$$\times \prod_{i=1}^{k(z)} \int_E g(y_i, v_i) q\big(x^*(\eta, v^*), z, dy_i\big) \, dv_1 \cdots dv_{k(z)}.$$

The first product in the integral accounts for the thinning of the existing population (after the removal of the parent), and the second product accounts for the births. Note that (x^*, u^*) is a function of η and v^*, and if an event z occurs at time t and

$\overline{\eta}_{t-}(r, z) > 0$, then

$$\eta_t = \sum_{(x,u)\in\eta_{t-},u\neq u^*} \mathbf{1}_{\{\rho(x,z)\mathcal{J}_{r(\cdot,z)}^\lambda(x,u,\eta_{t-},v^*)<\lambda\}} \delta_{(x,\rho(x,z)\mathcal{J}_{r(\cdot,z)}^\lambda(x,u,\eta_{t-},v^*))}$$

$$+ \sum_{i=1}^{k(z)} \delta_{(y_i,v_i)}.$$

Averaging gives

$$\alpha A_{th,db}^\lambda f(\overline{\eta}) = \int_{\mathbb{U}} \mathbf{1}_{\{\overline{\eta}(r,z)>0\}} \sum_{x^*\in\overline{\eta}} \frac{r(x^*,z)}{\int r(x,z)\overline{\eta}(dx)} (\overline{H}_z^\lambda(g,\overline{\eta},x^*) - \alpha f(\overline{\eta}))$$

$$\times \mu_{th,db}(dz),$$

where, recalling that $p(x,z) = \frac{\rho(x,z)-1}{\rho(x,z)}$ and $\overline{\eta}_{x^*} = \overline{\eta} - \delta_{x^*}$,

$$\overline{H}_z^\lambda(g,\overline{\eta},x^*) = \prod_{x\in\overline{\eta}_{x^*}} ((1-p(x,z))\overline{g}(x) + p(x,z))$$

$$\times \prod_{i=1}^{k(z)} \int_E \overline{g}(y_i)q(x^*,z,dy_i).$$

Note that if $\frac{k(z)}{\lambda} \to \zeta$ as $\lambda \to \infty$, then calculating as in Section 3.3,

$$H_z^\lambda(g,\eta)$$

$$= \lambda^{-k(z)} \int_{[0,\lambda]^{k(z)}} \left[\prod_{(x,u)\in\eta,u\neq u^*(\eta,v^*)} g(x,\rho(x,z)\mathcal{J}_{r(\cdot,z)}^\lambda(x,u,\eta,v^*)) \right.$$

$$\times \prod_{i=1}^{k(z)} \int_E g(y_i,v_i)q(x^*,z,dy_i) \Bigg] dv_1 \cdots dv_{k(z)}$$

$$\to \int_0^\infty \left[\zeta e^{-\zeta v^*} \right.$$

$$\times \prod_{(x,u)\in\eta,u\neq u^*(\eta,v^*)} g\left(x,\rho(x,z)\left(u - \mathbf{1}_{\{u>u^*\}}(u^*-v^*)\frac{r(x,z)}{r(x^*,z)}\right)\right)$$

$$\times \int_E g(y,v^*)q(x^*,z,dy)$$

$$\times \exp\left\{-\zeta \int_E \int_{v^*}^\infty (1-g(y,v))q(x^*,z,dy)\,dv\right\} \Bigg] dv^*$$

$$\equiv H_\zeta^\infty(g,\eta).$$

Consequently, at least in the simple setting when $\mu_{th,db}^\lambda(\mathbb{U}) < \infty$ and the various parameters are continuous, if we assume that as $\lambda \to \infty$, for each $\varphi \in C_b(\mathbb{R} \times \mathbb{U})$,

$$\int_{\mathbb{U}} \varphi\left(\frac{k(z)}{\lambda}, z\right) \mu_{th,db}^\lambda(dz) \to \int_{\mathbb{U}} \int_0^\infty \varphi(\zeta, z) \mu_\zeta(d\zeta, z) \mu_{th,db}^\infty(dz),$$

where $\mu_\zeta(d\zeta, z)$ is a probability distribution on $[0, \infty)$, then $A_{th,db} f(\eta)$ converges to

$$A_{th,db}^\infty f(\eta) = \int_{\mathbb{U}} \int_0^\infty \mathbf{1}_{\{\overline{\eta}(r,z)>0\}} \big(H_\zeta^\infty(g, \eta) - f(\eta)\big) \mu_\zeta(d\zeta, z) \mu_{th,db}^\infty(dz).$$

If $\int r(x, z) \Xi(dx) > 0$, define

$$\beta(x^*, \Xi) = \frac{r(x^*, z)}{\int_E r(x, z) \Xi(dx)}$$

and

$$\mathcal{H}_z(g, \Xi) = \int_0^\infty \int_0^\infty \left[\exp\left\{ -\int_E \frac{1}{\rho(x, z)} h(x) \Xi(dx) \right\} \right.$$
$$\left. \times \int_E \beta(x^*, \Xi) \exp\left\{ -\zeta \int_E h(y) q(x^*, z, dy) dv \right\} \Xi(dx^*) \right] \mu_\zeta(d\zeta, z).$$

The projected generator then becomes

$$\alpha A_{th,db}^\infty f(\Xi) = \int_{\mathbb{U}} \mathbf{1}_{\{\Xi(r,z)>0\}} \big(\mathcal{H}_z(g, \Xi) - \alpha f(\Xi)\big) \mu_{th,db}^\infty(dz).$$

3.8. *Immigration.* Immigration can be modeled by simply assigning each new immigrant a randomly chosen level. This approach gives a generator of the form

$$A_{im} f(\eta) = \int_{\mathbb{U}_{im}} f(\eta) \left(\lambda^{-1} \int_0^\lambda g(x(z), v) dv - 1 \right) \mu_{im}(dz)$$
$$= \int_{\mathbb{U}_{im}} f(\eta) \big(\overline{g}(x(z)) - 1 \big) \mu_{im}(dz),$$

which gives

$$\alpha A_{im} f(\overline{\eta}) = \int_{\mathbb{U}_{im}} \alpha f(\overline{\eta}) \big(\overline{g}(x(z)) - 1 \big) \mu_{im}(dz).$$

Again setting $h(x) = \int_0^\infty (1 - g(x, u)) \, du$, replacing μ_{im} by $\lambda \mu_{im}$, and passing to the limit as $\lambda \to \infty$ gives

$$A_{im} f(\eta) = -\int_{\mathbb{U}_{im}} f(\eta) h(x(z)) \mu_{im}(dz),$$

and integrating out the levels

$$\alpha A_{im} f(\overline{\eta}) = -\int_{\mathbb{U}_{im}} \alpha f(\overline{\eta}) h(x(z)) \mu_{im}(dz)$$

which implies

$$\frac{d}{dt} \int_E h(x) \Xi_t(dx) = \int_{\mathbb{U}_{im}} h(x(z)) \mu_{im}(dz),$$

as we would expect.

3.9. *Independent and exchangeable motion.* Typically, population models assume independent motion or mutation causing individual types to change between birth/death events. Some models allow common stochastic effects to influence type changes so that particle types evolve in an exchangeable fashion. In either case, we assume the existence of a collection of process generators $\{B_n\}$, where B_n determines a process with state space E^n, B_n is exchangeable in the sense that if $(X_1, \ldots, X_n)$ is a solution of the martingale problem for B_n, then any permutation of the indices $(X_{\sigma_1}, \ldots, X_{\sigma_n})$ also gives a solution of the martingale problem for B_n, and the B_n are consistent in the sense that if $(X_1, \ldots, X_{n+1})$ is a solution of the martingale problem for B_{n+1}, then $(X_1, \ldots, X_n)$ is a solution of the martingale problems for B_n. Of course, if B_n is the generator for n independent particles, each with generator B_1, then the collection $\{B_n\}$ has the desired properties.

To combine motion with the other possible elements of a model described above, we need a sufficiently rich class of function $g(x, u)$ such that for each n, and fixed $u_1, \ldots, u_n$, $\prod_{i=1}^n g(x_i, u_i)$ gives a function in the domain of B_n. In the independent case, this requirement simply means that $g(x, u)$ is in the domain of $B \equiv B_1$, and

$$B_{|\eta|} f(\eta) = f(\eta) \sum_{(x,u) \in \eta} \frac{Bg(x, u)}{g(x, u)}.$$

For finite λ, if $\overline{\eta}(E) < \infty$, then the motion generator is just given by

$$\widehat{B} f(\eta) = B_{|\eta|} \prod_{(x,u) \in \eta} g(x, u).$$

For $\lambda = \infty$, since we assume that $g(x, u) \equiv 1$ for $u \geq u_g$, the same formula works provided $\eta(E \times [0, u_g]) < \infty$.

For models with infinitely many particles with levels below a fixed level, we can require the existence of a sequence $K_k \subset E$ such that $\bigcup_k K_k = E$ and $\eta(K_k \times [0, u_0]) < \infty$ for each k and u_0. Requiring $g(x, u) = 1$ and $B_1 g(x, u) = 0$ for $(x, u) \notin K_{k_g} \times [0, u_g]$ for some k_g would give

(3.19) $$\widehat{B} f(\eta) = B_{\eta(K_{k_g} \times [0, u_g])} \prod_{(x,u) \in \eta_{|K_{k_g} \times [0, u_g]}} g(x, u).$$

Note that this condition simply places restrictions on the size or direction of jumps by the motion process.

For finite λ and $\overline{\eta}(E) < \infty$,

$$\alpha \widehat{B} f(\overline{\eta}) = B_{|\eta|} \prod_{x \in \overline{\eta}} \overline{g}(x),$$

and similarly for (3.19). For $\lambda = \infty$, a general derivation for exchangeable but not independent motion is not clear, but for independent motion, observing that $Bg = B(g - 1)$ we have

$$\alpha \widehat{B} f(\Xi) = -e^{-\int_E h(x) \Xi(dx)} \int Bh(x) \Xi(dx).$$

3.10. *Selecting a random sample.* The various recipes described above allow one to construct population models in a way that parent–offspring relationships can be identified knowing the evolution of the state in the model. In particular, one can select a random "sample" from an appropriately finite region of the type space (even in the $\lambda = \infty$ case) and trace its genealogy. For example, let $C \subset E$ satisfy $\overline{\eta}(t, C) < \infty$ in the $\lambda < \infty$ case and $\Xi(t, C) < \infty$ in the $\lambda = \infty$ case. Then the set of particles with types in C at the n lowest levels is a uniform random sample of size n drawn from the subpopulation of particles with types in C and the genealogies of these n particles can be traced by following the evolution of the levels back in time.

If the levels are constant in time, then as noted in Remark 4.6 and Section 5 of Donnelly and Kurtz (1999), one can define a family of counting processes and a system of stochastic equations driven by these counting processes whose solution gives the desired genealogy. Tracing the genealogy for a model with moving levels is much less elegant; however, complete genealogical information is present in the levels and the stochastic inputs of the birth events.

4. Examples. So far, we have largely performed formal calculations, not proofs. In this section, we illustrate our results in some specific examples and here, unless otherwise stated, our results are mathematically rigorous. In Section 4.1, we present two different approaches to the process known as the spatial Λ-Fleming–Viot process (which we shall also define). The first, based on one-for-one replacement, yields, in the high intensity limit, the process with levels of Véber and Wakolbinger (2015) (under somewhat weaker conditions). The second, based on discrete births of Poisson numbers of offspring and death by independent thinning, corresponds in the prelimit to the particle system studied in Berestycki, Etheridge and Hutzenthaler (2009). In Section 4.2, we extend this second approach to discrete birth mechanisms in which the number of offspring is no longer required to be Poisson. This yields a new class of population models, in which the replacement mechanism mirrors that of the spatial Λ-Fleming–Viot process, but the population intensity can vary with spatial position. In particular, these models provide one approach to combining ecology and genetics as described in the Introduction. In Section 4.3, we revisit branching processes and the Dawson–Watanabe

superprocess. In Section 4.4, we use one-for-one replacement, in the special case in which just two individuals are involved in each event, to recover, in particular, the lookdown construction of Greven, Limic and Winter (2005) for a spatially interacting Moran model. In Section 4.5, we use the lookdown construction to derive a stochastic partial differential equation as the limit of rescaled spatially interacting Moran models of the type discussed in Section 4.4. Finally, in Section 4.6, we give a lookdown construction for a class of voter models and use the construction to give a heuristic argument for a result of Müller and Tribe (1995) showing that the rescaled voter model converges to a solution of the stochastic partial differential equation obtained in Section 4.5.

4.1. *Spatial Λ-Fleming–Viot process.* The spatial Λ-Fleming–Viot process was introduced in Etheridge (2008) and rapidly developed by a number of authors [Barton, Etheridge and Véber (2010), Berestycki, Etheridge and Hutzenthaler (2009), Véber and Wakolbinger (2015)]. The primary motivation is to model a spatially distributed population in such a way that the distribution of the population is stable in space and one can recover the genealogical trees relating individuals in a sample from the population in an analytically tractable way. A survey can be found in Barton, Etheridge and Véber (2013). The process is driven by spatially distributed birth/death events in which a significant fraction of the local population is replaced. The location, spatial extent and "impact" of these events (by which we mean the proportion of the local population replaced in an event) is determined by a Poisson random measure, and stability of the population is maintained by ensuring that the numbers of births and deaths balance.

We now explicitly distinguish between the location of a particle $x \in \mathbb{R}^d$ and its type $\kappa \in \mathbb{K}$. Let $E = \mathbb{R}^d \times \mathbb{K}$, $\mathbb{U} = \mathbb{R}^d \times [0, 1] \times [0, \infty)$, and $\mu = \ell^d \times \nu^1(w, d\zeta) \times \nu^2(dw)$ where ℓ^d is Lebesgue measure on $\mathbb{R}^d$, ν^2 is a σ-finite measure on $[0, \infty)$ and ν^1 is a transition function from $[0, \infty)$ to $[0, 1]$.

If $C \subset \mathbb{R}^d$ is Borel measurable, then $|C| = \ell^d(C)$. If C is a finite or countable set, then $|C|$ will denote the number of elements in C; which interpretation applies should be clear in context.

Each point in $\mathbb{U}$ specifies a point $y \in \mathbb{R}^d$, $w \in [0, \infty)$ and $\zeta \in [0, 1]$. The corresponding reproduction event will affect the population in the ball $D_{y,w} \subseteq \mathbb{R}^d$ centered at y with radius w, and ζ will determine the impact within the ball. The model is driven by a space-time Poisson random measure on $\mathbb{U} \times [0, \infty)$ with mean measure $\mu \times \ell$. If a birth/death event occurs at time t corresponding to $(y, \zeta, w) \in \mathbb{U}$, an individual located in $D_{y,w}$ is selected at random to be the "parent," a fraction ζ of the individuals in $D_{y,w}$ are killed and replaced by individuals of the same type as the parent, with the locations of the new individuals uniformly distributed over $D_{y,w}$.

We will give two constructions of processes following this recipe which differ substantially for finite λ but, under conditions for which both constructions are

valid, yield the same measure-valued model in the limit. The first construction follows ideas of Véber and Wakolbinger (2015).

In order to rigorously define the generators of our processes, we will need to restrict the domains. In both cases, the domains will be subsets of

$$
\mathcal{D}_\lambda = \left\{ f(\eta) = \prod_{(x,\kappa,u)\in\eta} g(x,\kappa,u) : 0 \le g \le 1, \right.
$$

$$
\exists \text{ compact } K_g \subset \mathbb{R}^d, 0 < u_g \le \lambda,
$$

(4.1)

$$
\left. g(x,\kappa,u) = 1 \text{ for } (x,u) \notin K_g \times [0, u_g] \right\}
$$

$$
\mathcal{D}_\infty = \bigcup_{\lambda>0} \mathcal{D}_\lambda.
$$

Without loss of generality, we can assume that $K_g = D_{0,\rho_g}$ for $0 < \rho_g < \infty$.

Consider $A_{dr,2}$ defined in (3.11). Recall that with this mechanism, for each replacement event, we specify the probability $r(x)$ that an individual of type x is replaced and the parent is taken to be the individual chosen to be replaced that has the lowest level. For an event corresponding to $z = (y, \zeta, w)$, let $r(x, z)$ be $\zeta \mathbf{1}_{D_{y,w}}(x)$ and for $(x, \kappa) \in E$, the transition function q of Section 3.5 becomes

$$
q(x,\kappa,z,dx' \times d\kappa') = \upsilon_{y,w}(dx')\delta_\kappa(d\kappa'),
$$

where $\upsilon_{y,w}$ is the uniform distribution over the ball $D_{y,w}$, that is, the offspring have the same type as the parent and are independently and uniformly distributed over the ball. Consequently, $\widehat{g}$ in (3.10) becomes

$$
\widehat{g}_{y,w}(\kappa, u) \equiv \int g(x', \kappa, u)\upsilon_{y,w}(dx').
$$

In addition, recalling that $\overline{g}(x, \kappa) = \lambda^{-1} \int_0^\lambda g(x, \kappa, u) \, du$, we define

$$
\overline{g}_{y,w}(\kappa) \equiv \int \overline{g}(x', \kappa)\upsilon_{y,w}(dx')
$$

(4.2)

$$
= \frac{1}{\lambda} \int_0^\lambda \widehat{g}_{y,w}(\kappa, u) \, du
$$

$$
= \int \frac{1}{\lambda} \int_0^\lambda g(x', \kappa, u) \, du \upsilon_{y,w}(dx').
$$

We postpone giving precise conditions on ν^1 and ν^2 until we have formally derived the generators.

We define

(4.3) $\eta_{y,w} = \displaystyle\sum_{(x,\kappa,u)\in\eta:x\in D_{y,w}} \delta_{(x,\kappa,u)}$ and $\eta^g_{y,w} = \displaystyle\sum_{(x,\kappa,u)\in\eta:x\in D_{y,w}, u\le u_g} \delta_{(x,\kappa,u)}.$

That is, $\eta^g_{y,w} = \eta(\cdot \cap D_{y,w} \times \mathbb{K} \times [0, u_g))$ is the restriction of η to $D_{y,w} \times \mathbb{K} \times [0, u_g)$. From (3.17) and (3.12), $A^\infty_{dr,2}$ is given by

$$
(4.4) \quad A^\infty_{dr,2} f(\eta) = f(\eta) \int_{\mathbb{R}^d \times [0,1] \times [0,\infty)} \left(\frac{\sum_{S \subset \eta^g_{y,w}} H(g, \widehat{g}, S, y, \zeta, w)}{\prod_{(x,\kappa,u) \in \eta^g_{y,w}} g(x, \kappa, u)} - 1 \right) dy
$$
$$
\times \nu^1(w, d\zeta)\nu^2(dw),
$$

where

$$
H(g, \widehat{g}, S, y, \zeta, w)
$$
$$
= \prod_{(x,\kappa,u) \in S} (\widehat{g}_{y,w}(\kappa^*(S), u)\zeta) \prod_{(x,\kappa,u) \in \eta^g_{y,w}, (x,\kappa,u) \notin S} ((1 - \zeta)g(x, \kappa, u)),
$$

$\kappa^*(S)$ being the type of the lowest level particle in S. $A^\infty_{dr,2}$ is the generator for the lookdown construction of Véber and Wakolbinger (2015). Again, for an event corresponding to (y, ζ, w), a particle in $D_{y,w}$ is involved in the event with probability ζ.

The relationship between the martingale problems for finite and infinite λ is particularly simple in this setting. For finite λ, $A^\lambda_{dr,2} f(\eta) = A^\infty_{dr,2} f(\eta)$ provided $u_g \leq \lambda$. Consequently, any solution of the martingale problem for $A^\infty_{dr,2}$ restricted to levels in $[0, \lambda]$ gives a solution of the martingale problem for $A^\lambda_{dr,2}$. In particular, existence and uniqueness for $A^\lambda_{dr,2}$ for all $\lambda > 0$ implies existence and uniqueness for $A^\infty_{dr,2}$.

Setting $\overline{\eta}_{y,w} = \overline{\eta}(\cdot \cap D_{y,w} \times \mathbb{K})$, for finite λ,

$$
(4.5) \quad \alpha A^\lambda_{dr,2} f(\overline{\eta}) = \alpha f(\overline{\eta}) \int_{\mathbb{R}^d \times [0,1] \times [0,\infty)} \left(\frac{\sum_{S \subset \overline{\eta}_{y,w}} \overline{H}(\overline{g}, \widehat{g}, S, y, \zeta, w)}{\prod_{(x,\kappa) \in \overline{\eta}_{y,w}} \overline{g}(x, \kappa)} - 1 \right) dy
$$
$$
\times \nu^1(w, d\zeta)\nu^2(dw),
$$

where, recalling the notation defined in (4.2),

$$
\overline{H}(\overline{g}, \widehat{g}, S, y, \zeta, w)
$$
$$
(4.6) \quad = \frac{1}{|S|} \sum_{(x,\kappa) \in S} (\overline{g}_{y,w}(\kappa)\zeta)^{|S|} \prod_{(x,\kappa) \in \overline{\eta}_{y,w}, (x,\kappa) \notin S} ((1 - \zeta)\overline{g}(x, \kappa)).
$$

Finally, setting $h^*_{y,w}(\kappa) = \int_0^\infty (1 - \widehat{g}_{y,w}(\kappa, u)) \, du$ [recall $h(x, \kappa) = \int_0^\infty (1 - g(x, \kappa, u)) \, du$] and

$$
\mathbb{H}_1(h^*_{y,w}, \Xi, y, \zeta, w) = \frac{1}{\Xi(D_{y,w} \times \mathbb{K})} \int_{D_{y,w} \times \mathbb{K}} e^{-\zeta h^*_{y,w}(\kappa) \Xi(D_{y,w} \times \mathbb{K})} \Xi(dx \times d\kappa),
$$

we have

$$\alpha A_{dr,2}^{\infty} f(\Xi)$$

$$= e^{-\int h(x,\kappa)\Xi(dx,d\kappa)}$$

(4.7)

$$\times \int_{\mathbb{R}^d \times [0,1] \times [0,\infty)} \left(\mathbb{H}_1\left(h_{y,w}^*, \Xi, y, \zeta, w\right) e^{\zeta \int_{D_{y,w} \times \mathbb{K}} h(x,\kappa)\Xi(dx,d\kappa)} \right.$$

$$\left. - 1 \right) dy \nu^1(w, d\zeta) \nu^2(dw).$$

Note that if Ξ is a solution of the martingale problem for $\alpha A_{dr,2}^{\infty}$ and $\Xi(0, dx \times \mathbb{K})$ is Lebesgue measure, then $\Xi(t, dx \times \mathbb{K})$ is Lebesgue measure for all $t \geq 0$. (Consider the generator with h not depending on κ.)

Before establishing conditions under which the construction above is valid, let us describe an alternative lookdown construction of the spatial Λ-Fleming–Viot process employing discrete births (Section 3.3) and independent thinning (Section 3.6) as in Section 3.7. With $z = (y, \zeta, w)$ as above, the thinning parameter is

(4.8)
$$\rho(x, \kappa, z) = 1 + \frac{\zeta}{1 - \zeta} \mathbf{1}_{D_{y,w}}(x).$$

As we saw in Section 3.6, this assumption ensures that the probability that an existing individual (other than the parent) dies is zero outside the ball $D_{y,w}$ and ζ within it.

If there is at least one individual in $D_{y,w}$ (to serve as parent), the discrete birth event corresponding to z produces a Poisson number of offspring with parameter $\lambda \alpha_z$ conditioned to be positive, where $\alpha_z = \zeta |D_{y,w}|$, $r(x, z) = \mathbf{1}_{D_{y,w}}(x)$, and

$$q(x, \kappa, z, dx', d\kappa') = \upsilon_{y,w}(dx') \delta_\kappa(d\kappa').$$

The finite intensity model is then essentially that considered in Berestycki, Etheridge and Hutzenthaler (2009), differing only in the assumptions that the parent is selected before the thinning and the offspring distribution is conditioned to be positive. Note that the definition of r in this construction is different from the definition in the previous construction. There, r determined the chance of being involved in the event; here we use it to weight the chance of being a parent. This distinction becomes important in modelling different forms of natural selection when we would choose r to depend on type.

As in Section 3.7, but with a slight change of notation, let $\eta(y, w) = \eta(D_{y,w} \times \mathbb{K} \times [0, \lambda])$ and

(4.9)
$$A_{th,db}^{\lambda} f(\eta) = \int_{\mathbb{U}} \mathbf{1}_{\{\eta(y,w)>0\}} \left(H_z^{\lambda}(g, \eta) - f(\eta) \right)$$

$$\times \left(1 - e^{-\alpha_z \lambda} \right) dy \nu^1(w, d\zeta) \nu^2(dw).$$

We introduce the factor $1 - e^{-\alpha_z \lambda}$ in the event measure, and then condition on there being at least one offspring. If $\eta(D_{y,w} \times \mathbb{K} \times [0, \lambda]) = \eta(y, w) \neq 0$, we obtain an expression for $H_z^{\lambda}(g, \eta)$ by partitioning on the lowest level selected for

the offspring. Since the levels $\{v_i\}$ selected for the offspring are the jump times in $[0, \lambda]$ of a Poisson process with intensity α_z, this yields

$$H_z^\lambda(g, \eta)$$

$$= \prod_{(x, \kappa, u) \in \eta, x \notin D_{y,w}} g(x, \kappa, u)$$

(4.10)

$$\times \frac{1}{1 - e^{-\alpha_z \lambda}} \int_0^\lambda \left[\alpha_z e^{-\alpha_z v^*} \widehat{g}_{y,w}(\kappa^*, v^*) e^{-\alpha_z \int_{v^*}^\lambda (1 - \widehat{g}_{y,w}(\kappa^*, v)) \, dv} \right.$$

$$\left. \times \prod_{(x, \kappa, u) \in \eta, x \in D_{y,w}, u \neq u^*} g\left(x, \kappa, \frac{1}{1 - \zeta} \mathcal{J}_{y,w}^\lambda(x, u, \eta, v^*) \right) \right] dv^*,$$

where (x^*, κ^*, u^*) is the point in η satisfying $x^* \in D_{y,w}$ and

$$u^* = \operatorname{argmax}\left\{ \frac{\lambda - u}{\lambda - v^*} : (x, \kappa, u) \in \eta, x \in D_{y,w}, u \geq v^* \right\}$$

$$\cup \left\{ \frac{u}{v^*} : (x, \kappa, u) \in \eta, x \in D_{y,w}, u \leq v^* \right\},$$

and $\mathcal{J}_{y,w}^\lambda(x, u, \eta, v^*)$ is obtained as in (3.4) with $r = \mathbf{1}_{D_{y,w}}$. Recall that we thin the existing population after we select the parent, and the thinning is accomplished by multiplying $\mathcal{J}_{y,w}^\lambda$ by ρ defined in (4.8).

Let $\overline{\eta}_{|D_{y,w}}$ denote $\overline{\eta}$ restricted to $D_{y,w} \times \mathbb{K}$. Since conditional on $\overline{\eta}$ and v^*, (x^*, κ^*) is selected uniformly at random from $\overline{\eta}_{|D_{y,w}}$ and, for $u \neq u^*$ (see Lemma 3.1), the $\mathcal{J}_{y,w}^\lambda(x, u, \eta, v^*)$ are independent and uniform over $[0, \lambda]$, partitioning on the level of the lowest offspring, define

$$\mathcal{H}_z^\lambda(\overline{g}, \overline{\eta})$$

$$= \frac{1}{|\overline{\eta}_{|D_{y,w}}|} \sum_{(x^*, \kappa^*) \in \overline{\eta}_{|D_{y,w}}} \prod_{(x, \kappa) \in \overline{\eta}_{|D_{y,w}}, (x, \kappa) \neq (x^*, \kappa^*)} ((1 - \zeta) \overline{g}(x, \kappa) + \zeta)$$

$$\times \frac{1}{1 - e^{-\lambda \alpha_z}} \int_0^\lambda \alpha_z e^{-\alpha_z v^*} \widehat{g}_{y,w}(\kappa^*, v^*) e^{-\alpha_z \int_{v^*}^\lambda (1 - \widehat{g}_{y,w}(\kappa^*, v)) \, dv} \, dv^*$$

$$= \frac{1}{|\overline{\eta}_{|D_{y,w}}|} \sum_{(x^*, \kappa^*) \in \overline{\eta}_{|D_{y,w}}} \prod_{(x, \kappa) \in \overline{\eta}_{|D_{y,w}}, (x, \kappa) \neq (x^*, \kappa^*)} ((1 - \zeta) \overline{g}(x, \kappa) + \zeta)$$

$$\times \frac{1}{1 - e^{-\lambda \alpha_z}} \left(e^{-\alpha_z \int_0^\lambda (1 - \widehat{g}_{y,w}(\kappa^*, v)) \, dv} - e^{-\lambda \alpha_z} \right)$$

$$= \frac{1}{|\overline{\eta}_{|D_{y,w}}|} \sum_{(x^*, \kappa^*) \in \overline{\eta}_{|D_{y,w}}} \prod_{(x, \kappa) \in \overline{\eta}_{|D_{y,w}}, (x, \kappa) \neq (x^*, \kappa^*)} ((1 - \zeta) \overline{g}(x, \kappa) + \zeta)$$

$$\times \frac{1}{1 - e^{-\lambda \alpha_z}} \left(e^{-\zeta |D_{y,w}| h_{y,w}^*(\kappa^*)} - e^{-\alpha_z \lambda} \right),$$

where, as before, $h^*_{y,w}(\kappa) = \int_0^\infty (1 - \widehat{g}_{y,w}(\kappa, u))\, du$. To understand this quantity, recall first that in our discrete births model, the parent is eliminated from the population. Next, for points within $D_{y,w}$, they survive with probability $(1 - \zeta)$, otherwise they are removed (giving the product on the right-hand side of the first line). The final term corresponds to the offspring [recalling the notation $\overline{g}_{y,w}(\kappa)$ from (4.2) and that we have conditioned on there being at least one offspring]. Then

$$\alpha A^\lambda_{th,db} f(\overline{\eta}) = \alpha f(\overline{\eta}) \int_{\mathbb{U}} \mathbf{1}_{\{\overline{\eta}(y,w)>0\}} \left(\frac{\mathcal{H}^\lambda_z(\overline{g}, \overline{\eta})}{\prod_{(x,\kappa)\in\overline{\eta}_{|D_{y,w}}} \overline{g}(x,\kappa)} - 1 \right)$$

$$\times (1 - e^{-\alpha_z})\, dy\, \nu^1(w, d\zeta) \nu^2(dw).$$

Note that $\alpha A^\lambda_{th,db}$ constructed here is not the same as $\alpha A^\lambda_{dr,2}$ given in (4.5). Here, at each birth/death event, existing particles are randomly killed and an independent number of new particles are created while in the previous construction, the number of births equaled the number of deaths. However, taking $\lambda \to \infty$, by (3.5),

$$(4.11) \qquad A^\infty_{th,db} f(\eta) = \int_{\mathbb{U}} \mathbf{1}_{\{\eta(y,w)>0\}} (H_z(g, \eta) - f(\eta))\, dy\, \nu^1(w, d\zeta) \nu^2(dw),$$

with

$$H_z(g, \eta) = \prod_{(x,\kappa,u)\in\eta, x\notin D_{y,w}} g(x, \kappa, u)$$

$$\times \int_0^\infty \left[\alpha_z e^{-\alpha_z v^*} \widehat{g}_{y,w}(\kappa^*, v^*) e^{-\alpha_z \int_{v^*}^\infty (1-\widehat{g}_{y,w}(\kappa^*, v))\, dv} \right.$$

$$\times \prod_{(x,\kappa,u)\in\eta, x\in D_{y,w}, u>u^*} g\left(x, \kappa, \frac{1}{1-\zeta}(u - u^* + v^*)\right)$$

$$\times \left. \prod_{(x,\kappa,u)\in\eta, x\in D_{y,w}, u<u^*} g\left(x, \kappa, \frac{1}{1-\zeta}u\right) \right] dv^*.$$

Just as in Lemma 3.1, [and using (3.5)], it is easy to see that η^* satisfying

$$\int g\, d\eta^* = \sum_{(x,\kappa,u)\in\eta, x\in D_{y,w}, u>u^*} g\left(x, \kappa, \frac{1}{1-\zeta}(u - u^* + v^*)\right)$$

$$+ \sum_{(x,\kappa,u)\in\eta, x\in D_{y,w}, u<u^*} g\left(x, \kappa, \frac{1}{1-\zeta}u\right)$$

is conditionally Poisson with Cox measure $(1 - \zeta)\mathbf{1}_{D_{y,w}}(x)\Xi(dx, d\kappa)$ and recalling the definition of $h^*_{y,w}(\kappa)$ from just below equation (4.6) an integration by parts gives

$$\int_0^\infty \alpha_z e^{-\alpha_z v^*} \widehat{g}_{y,w}(\kappa^*, v^*) e^{-\alpha_z \int_{v^*}^\infty (1-\widehat{g}_{y,w}(\kappa^*, v))\, dv}\, dv^* = e^{-\alpha_z h^*_{y,w}(\kappa^*)}.$$

Averaging (4.11) gives

$$\alpha A_{th,db}^{\infty} f(\Xi) = e^{-\int_{\mathbb{R}^d \times \mathbb{K}} h(x,\kappa)\Xi(dx,d\kappa)}$$
$$\times \int_{\mathbb{U}} \big(\mathbb{H}_2(h_{y,w}^*, \Xi, z) e^{\zeta \int_{D_{y,w} \times \mathbb{K}} h(x,\kappa)\Xi(dx,d\kappa)} - 1\big) \, dy$$
$$\times \nu^1(w, d\zeta)\nu^2(dw),$$

where

$$\mathbb{H}_2(h_{y,w}^*, \Xi, z) = \frac{1}{\Xi(D_{y,w} \times \mathbb{K})} \int_{D_{y,w} \times \mathbb{K}} e^{-\zeta|D_{yw}|h_{y,w}^*(\kappa)} \Xi(dx \times d\kappa),$$

[defined to be 1 if $\Xi(D_{y,w} \times \mathbb{K}) = 0$] which, in general, differs from $\mathbb{H}_1$. However, if Ξ is a solution of the martingale problem for $\alpha A_{th,db}^{\infty}$ with $\Xi(0, dx \times \mathbb{K})$ Lebesgue measure, then $\Xi(t, dx \times \mathbb{K})$ is Lebesgue measure for all $t \geq 0$ and $\mathbb{H}_2(h_{y,w}^*, \Xi, z) = \mathbb{H}_1(h_{y,w}^*, \Xi, z)$. Consequently, in this case, Ξ is also a solution of the martingale problem for $\alpha A_{dr,2}^{\infty}$ in the previous construction.

Our calculations so far in this subsection have been entirely formal. We now turn to actually constructing the processes that correspond to the generators described above.

4.1.1. *First construction of spatial Λ-Fleming–Viot with levels.* The process corresponding to $A_{dr,2}^{\infty}$ appears already in Véber and Wakolbinger (2015), but the strategy of our construction, based on writing down stochastic equations for the type of the particle at the ith level for each i, is somewhat different, and we obtain our process under somewhat weaker conditions. In particular, for existence of our construction, we require

$$(4.12) \qquad \int_{[0,1]\times(1,\infty)} \zeta w^d \nu^1(w, d\zeta)\nu^2(dw) < \infty$$

and

$$(4.13) \qquad \begin{cases} \int_{[0,1]\times[0,1]} \zeta |w|^2 \nu^1(w, d\zeta)\nu^2(dw) < \infty & \text{if } d = 1, \\ \int_{[0,1]\times[0,1]} \zeta |w|^{2+d} \nu^1(w, d\zeta)\nu^2(dw) < \infty & \text{if } d \geq 2, \end{cases}$$

while Véber and Wakolbinger (2015) assume

$$(4.14) \qquad \int_{[0,1]\times(0,\infty)} \zeta w^d \nu^1(w, d\zeta)\nu^2(dw) < \infty.$$

We should point out, however, that up to now, we do not have a proof of uniqueness for the system of stochastic equations under the weaker conditions, except in the case $d = 1$ where uniqueness is proved in Zheng and Xiong (2017). The solution is unique under (4.14).

To rigorously cover the more general conditions, we need to be more careful in the description of the generators, which for simplicity we will call A^λ and A^∞. In particular, we appeal to the construction in Appendix A.3. With reference to (4.1), we restrict the domain to

$$\mathcal{D}(A^\infty) = \left\{ f(\eta) = \prod_{(x,\kappa,u)\in\eta} g(x,\kappa,u) \in \mathcal{D}_\infty : g(\cdot,\kappa,u) \in C^2(\mathbb{R}^d) \right\}.$$

To avoid additional complication of notation, we will also assume that for each $k = 1, 2, \ldots,$

$$(4.15) \qquad\qquad v^2(2^{-k}, 2^k) < \infty.$$

With the results of Appendix A.3 in mind, define $\Gamma_0 = \varnothing$ and for $k = 1, 2, \ldots,$

$$(4.16) \qquad\qquad \Gamma_k = D_{0,k} \times [0,1] \times [2^{-k}, 2^k].$$

Set

$$B_k f(\eta) = \int_{\Gamma_k - \Gamma_{k-1}} f(\eta) \left(\frac{\sum_{S \subset \eta^g_{y,w}} H(g, \widehat{g}, S, y, \zeta, w)}{\prod_{(x,\kappa,u)\in\eta^g_{y,w}} g(x,\kappa,u)} - 1 \right) dy$$
$$\times v^1(w, d\zeta)v^2(dw),$$

where as before

$$H(g, \widehat{g}, S, y, \zeta, w)$$
$$= \prod_{(x,\kappa,u)\in S} \big(\widehat{g}_{y,w}(\kappa^*(S), u)\zeta\big) \prod_{(x,\kappa,u)\in\eta^g_{y,w}, (x,\kappa,u)\notin S} \big((1-\zeta)g(x,\kappa,u)\big).$$

Note that, writing v_d for the volume of the unit ball, λ_k in (A.13) is

$$(4.17) \qquad \begin{aligned} \lambda_k &= v_d k^d \int_{2^{-k}}^{2^k} v^1(w, [0,1])v^2(dw) - v_d(k-1)^d \\ &\quad \times \int_{2^{-(k-1)}}^{2^{k-1}} v^1(w, [0,1])v^2(dw). \end{aligned}$$

The definition of H_k is somewhat more complicated than the form used in Appendix A.3, but arguments used there carry over immediately. Let $\mathbb{U}_k = (\Gamma_k - \Gamma_{k-1}) \times ([0,1] \times D_{0,1})^\eta$, and

$$v_k(dy, d\zeta, dw, \ldots, dz_u, dv_u, \ldots)$$
$$= \frac{1}{\lambda_k} dy v^1(w, d\zeta)v^2(dw) \prod_{(x,\kappa,u)\in\eta} dz_u v_{0,1}(dv_u),$$

that is, for each k, we associate a pair of random variables $(Z_{k,u}, V_{k,u})$ with each element of η, where $Z_{k,u}$ is uniformly distributed on $[0,1]$ and $V_{k,u}$ is uniformly

distributed on $D_{0,1}$. We can index these random variables by u since in our model the levels u will be distinct. Then

$$H_k\big(\eta, y, \zeta, w, (z, v)^\eta\big)$$

(4.18)
$$= \sum_{(x,\kappa,u)\in\eta} \big((1 - \mathbf{1}_{D_{y,w}}(x)\mathbf{1}_{[0,\zeta]}(z))\delta_{(x,\kappa,u)}$$

$$+ \mathbf{1}_{D_{y,w}}(x)\mathbf{1}_{[0,\zeta]}(z)\delta_{(y+wv,\kappa_{u^*},u)}\big),$$

where $u^* = \min\{u : (x,\kappa,u)\in\eta, x\in D_{y,w}, z_u \le \zeta\}$. Note that if V is uniformly distributed on $D_{0,1}$, then $y + wV$ is uniformly distributed on $D_{y,w}$.

To verify Condition A.5, we split Γ_k, setting

$$\Gamma_k = \Gamma_k^1 \cup \Gamma_k^2 \equiv D_{0,k} \times [0,1] \times [2^{-k}, 1] \cup D_{0,k} \times [0,1] \times (1, 2^k],$$

$$\Gamma_\infty = \Gamma_\infty^1 \cup \Gamma_\infty^2 \equiv \mathbb{R}^d \times [0,1] \times (0,1] \cup \mathbb{R}^d \times [0,1] \times (1,\infty),$$

and for $i = 1, 2$, define

$$B_k^i f(\eta) = \int_{\Gamma_k^i - \Gamma_{k-1}^i} f(\eta)\bigg(\frac{\sum_{S\subset\eta_{y,w}^g} H(g, \widehat{g}, S, y, \zeta, w)}{\prod_{(x,\kappa,u)\in\eta_{y,w}^g} g(x,\kappa,u)} - 1\bigg)\,dy\,v^1(w, d\zeta)v^2(dw).$$

Recall the definition of $\eta_{y,w}^g$ from (4.3). For $i = 2$, as in (3.11), let $\{\xi_{x,\kappa,u}^\zeta\}$ be independent with $P\{\xi_{x,\kappa,u}^\zeta = 1\} = 1 - P\{\xi_{x,\kappa,u}^\zeta = 0\} = \zeta$. Then

$$\bigg|\sum_{k=m+1}^\infty B_k^2 f(\eta)\bigg|$$

$$= \bigg|\int_{\Gamma_\infty^2 - \Gamma_m^2} \prod_{(x,\kappa,u)\in\eta-\eta_{y,w}^g} g(x,\kappa,u)$$

$$\times \bigg(E\bigg[\prod_{(x,\kappa,u)\in\eta_{y,w}^g} (\xi_{x,\kappa,u}^\zeta \widehat{g}_{y,w}(\kappa^*, u) + (1 - \xi_{x,\kappa,u}^\zeta)g(x,\kappa,u))\bigg]$$

$$- f\big(\eta_{y,w}^g\big)\bigg)\,dy\,v^1(w, d\zeta)v^2(dw)\bigg|$$

$$\le \int_{\Gamma_\infty^2 - \Gamma_m^2} \sum_{(x,\kappa,u)\in\eta_{y,w}^g} E\big[\xi_{x,\kappa,u}^\zeta|\widehat{g}_{y,w}(\kappa^*, u) - 1|\big]\,dy\,v^1(w, d\zeta)v^2(dw)$$

$$+ \int_{\Gamma_\infty^2 - \Gamma_m^2} \sum_{(x,\kappa,u)\in\eta_{y,w}^g\cap D_{0,\rho_g}\times\mathbb{K}\times[0,u_g)} E\big[\xi_{x,\kappa,u}^\zeta|1 - g(x,\kappa,u)|\big]\,dy$$

$$\times v^1(w, d\zeta)v^2(dw)$$

(4.19)
$$\le \int_{\Gamma_\infty^2} |\eta_{y,w}^g|\frac{|D_{0,\rho_g}\cap D_{y,w}|}{|D_{y,w}|}\zeta\,dy\,v^1(w, d\zeta)v^2(dw)$$

$$+ \int_{\Gamma_\infty^2} \eta\big(D_{0,\rho_g} \times \mathbb{K} \times [0, u_g)\big)$$

$$\times \mathbf{1}_{\{D_{y,w} \cap D_{0,\rho_g} \neq \varnothing\}} \zeta \, dy \nu^1(w, d\zeta) \nu^2(dw)$$

$$\leq |D_{0,\rho_g}| \int_{[0,1] \times (1,\infty)} \frac{|\eta_{y,w}^g|}{|D_{y,w}|}$$

$$\times \mathbf{1}_{\{D_{y,w} \cap D_{0,\rho_g} \neq \varnothing\}} \zeta \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ \eta\big(D_{0,\rho_g} \times \mathbb{K} \times [0, u_g)\big) \int_{[0,1] \times (1,\infty)} v_d(\rho_g + w)^d$$

$$\times \zeta \nu^1(w, d\zeta) \nu^2(dw),$$

where to obtain the first inequality we have used the identity

$$\prod_{k=1}^{m} a_k - \prod_{k=1}^{m} b_k = \sum_{k=1}^{m} \left(\prod_{1 \leq l < k} a_l \right) (a_k - b_k) \prod_{k < l \leq m} b_l,$$

observing that, in our case, all factors are less than or equal to one and so we can estimate the right-hand side by $\sum_{k=1}^{m} |a_k - b_k|$, and the differences are

$$\xi_{x,\kappa,u}^\zeta \widehat{g}_{y,w}(\kappa^*, u) + (1 - \xi_{x,\kappa,u}^\zeta) g(x, \kappa, u) - g(x, \kappa, u)$$

$$= \xi_{x,\kappa,u}\big(\widehat{g}_{y,w}(\kappa^*, u) - g(x, \kappa, u)\big)$$

$$= \xi_{x,\kappa,u}^\zeta\big(\widehat{g}_{y,w}(\kappa^*, u) - 1\big)$$

$$+ \xi_{x,\kappa,u}^\zeta\big(1 - g(x, \kappa, u)\big).$$

Recalling that g vanishes outside D_{0,ρ_g}, in the second inequality we have then used that for $u < u_g$,

$$(4.20) \qquad |1 - \widehat{g}_{y,w}(\kappa, u)| \leq \frac{|D_{0,\rho_g} \cap D_{y,w}|}{|D_{y,w}|}.$$

The sums in the two integrals are over the (x, κ, u) for which the term is nonzero. If there exists $0 < c < \infty$ such that $E[\eta(D_{y,w}) \times \mathbb{K} \times [0, r])] \leq cr|D_{y,w}|$ for all y, w, r, as would be the case if $\eta(dx \times \mathbb{K} \times du)$ were a Poisson random measure with Lebesgue mean measure, then the expectation of the right side of (4.19) is bounded by

$$c|D_{0,\rho_g}|(1 + u_g) \int_{[0,1] \times (1,\infty)} v_d(\rho_g + w)^d \zeta \, dy \nu^1(w, d\zeta) \nu^2(dw),$$

which is finite under (4.12).

If $m > \rho_g + 1$, then

$$\sum_{k=1}^{m} B_k^1 f(\eta)$$

$$= \int_{\Gamma_m^1} f(\eta) \left(\frac{\sum_{S \subset \eta_{y,w}^g} H(g, \widehat{g}, S, y, \zeta, w)}{\prod_{(x,\kappa,u) \in \eta_{y,w}^g} g(x, \kappa, u)} - 1 \right) dy \nu^1(w, d\zeta) \nu^2(dw)$$

$$= \int_{\Gamma_m^1} f(\eta) \sum_{S \subset \eta_{y,w}^g} \int_{D_{y,w}^{|S|}} \left(\frac{\prod_{(x,\kappa,u) \in S} g(x_u, \kappa^*(S), u)}{\prod_{(x,\kappa,u) \in S} g(x, \kappa, u)} - 1 \right) \prod \upsilon_{y,w}(dx_u)$$

$$\times \zeta^{|S|} (1 - \zeta)^{|\eta_{y,w}^g| - |S|} dy \nu^1(w, d\zeta) \nu^2(dw)$$

$$= \int_{\Gamma_m^1} f(\eta) \left(\sum_{(x^*,\kappa^*,u^*) \in \eta_{y,w}^g} \int_{D_{y,w}} \left[\frac{g(x_{u^*}, \kappa^*, u^*)}{g(x^*, \kappa^*, u^*)} - 1 \right. \right.$$

$$\left. - \frac{(x_{u^*} - x^*) \cdot \nabla g(x^*, \kappa^*, u^*)}{g(x^*, \kappa^*, u^*)} \right] \upsilon_{y,w}(dx_{u^*}) \Bigg)$$

$$\times \zeta (1 - \zeta)^{|\eta_{y,w}^g| - 1} dy \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ \int_{\Gamma_m^1} f(\eta) \sum_{S \subset \eta_{y,w}^g, |S| \geq 2} \int_{D_{y,w}^{|S|}} \left(\frac{\prod_{(x,\kappa,u) \in S} g(x_u, \kappa^*(S), u)}{\prod_{(x,\kappa,u) \in S} g(x, \kappa, u)} - 1 \right)$$

$$\times \prod \upsilon_{y,w}(dx_u) \zeta^{|S|} (1 - \zeta)^{|\eta_{y,w}^g| - |S|} dy \nu^1(w, d\zeta) \nu^2(dw),$$

where in the first term on the right, we are summing over $S \subset \eta_{y,w}^g$ with $|S| = 1$ and in the second term, we are summing over $S \subset \eta_{y,w}^g$ with $|S| \geq 2$. If we assume that $m > \rho_g + 1$, then since in the integral over Γ_m^1 we have $w < 1$, for each x^* for which $\nabla g(x^*, \kappa^*, u^*)$ is nontrivial we have

$$(4.21) \qquad \int_{D_{0,m} \times [2^{-m}, 1]} \mathbf{1}_{D_{y,w}}(x^*) \int_{D_{y,w}} (x' - x^*) \upsilon_{y,w}(dx') \, dy \nu^2(dw) = 0,$$

and so including the gradient term has no effect. Also, observe that the ∇g term plays the same role here as it does in the generator of a Lévy process (in fact, the location of the particle at a fixed level u is a Lévy process).

Define

$$(4.22) \qquad \begin{aligned} C_{y,w} g(x, \kappa, u) = \int_{D_{y,w}} & (g(x', \kappa, u) - g(x, \kappa, u) \\ & - (x' - x) \cdot \nabla g(s, \kappa, u)) \upsilon_{y,w}(dx'). \end{aligned}$$

Then, for $m > \rho_g + 1$,

$$\left| \sum_{k=m+1}^{\infty} B_k^1 f(\eta) \right|$$

$$\leq \int_{\Gamma_\infty^1 - \Gamma_m^1} \sum_{S \subset \eta_{y,w}^g, |S| \geq 2} \left| \prod_{(x,\kappa,u) \in S} \widehat{g}_{y,w}\big(\kappa^*(S), u\big) - \prod_{(x,\kappa,u) \in S} g(x, \kappa, u) \right|$$

$$\times \, \zeta^{|S|} (1 - \zeta)^{|\eta_{y,w}^g| - |S|} \, dy \, \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ \int_{\Gamma_\infty^1 - \Gamma_m^1} \sum_{(x^*,\kappa^*,u^*) \in \eta_{y,w}^g} \left| C_{y,w} g\big(x^*, \kappa^*, u^*\big) \right|$$

$$\times \, \zeta (1 - \zeta)^{|\eta_{y,w}^g| - 1} \, dy \, \nu^1(w, d\zeta) \nu^2(dw)$$

$$\leq \int_{\Gamma_\infty^1 - \Gamma_m^1} \big(1 - (1 - \zeta)^{|\eta_{y,w}^g|} - |\eta_{y,w}^g| \zeta (1 - \zeta)^{|\eta_{y,w}^g| - 1}\big)$$

$$\times \, \mathbf{1}_{\{D_{0,\rho_g} \cap D_{y,w} \neq \varnothing\}} \, dy \, \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ \int_{\Gamma_\infty^1 - \Gamma_m^1} \|\partial^2 g\| |\eta_{y,w}^g| w^2 \zeta \mathbf{1}_{\{D_{0,\rho_g} \cap D_{y,w} \neq \varnothing\}} \, dy \, \nu^1(w, d\zeta) \nu^2(dw)$$

$$\leq \int_{\Gamma_\infty^1} |\eta_{y,w}^g| \big(|\eta_{y,w}^g| - 1\big) \mathbf{1}_{\{D_{0,\rho_g} \cap D_{y,w} \neq \varnothing\}} \zeta^2 \, dy \, \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ \int_{\Gamma_\infty^1} \|\partial^2 g\| |\eta_{y,w}^g| w^2 \zeta \mathbf{1}_{\{D_{0,\rho_g} \cap D_{y,w} \neq \varnothing\}} \, dy \, \nu^1(w, d\zeta) \nu^2(dw).$$

We are primarily interested in solutions $\{\eta_t\}$ of the martingale problem for A^∞ such that at each time t, $\eta_t(\cdot \times \mathbb{K} \times \cdot)$ is a Poisson point process on $\mathbb{R}^d \times [0, \infty)$ with mean measure ℓ^{d+1}. Consequently, if, as in the discussion of $\sum B_k^2$, we require that

$$(4.23) \qquad E\big[\eta(D_{y,w} \times \mathbb{K} \times [0, r])\big] \leq cr|D_{y,w}| \qquad \text{for all } y \in \mathbb{R}^d, \, w > 0,$$

and in addition require

$$(4.24) \qquad \begin{aligned} &E\big[\eta(D_{y,w} \times \mathbb{K} \times [0, r])\big(\eta(D_{y,w} \times \mathbb{K} \times [0, r]) - 1\big)\big] \\ &\qquad \leq cr^2 |D_{y,w}|^2 \qquad \text{for all } 0 < w \leq 1, \end{aligned}$$

the solution of primary interest will meet these requirements. Under these assumptions,

$$E\left[\left| \sum_{k=m+1}^{\infty} B_k^1 f(\eta) \right| \right]$$

$$\leq c u_g^2 v_d (\rho_g + 1)^d \int_{[0,1] \times [0,1]} v_d^2 \zeta^2 w^{2d} \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ \|\partial^2 g\| u_g (\rho_g + 1)^d \int_{[0,1]\times[0,1]} v_d w^{d+2}$$

$$\times \zeta \nu^1(w, d\zeta) \nu^2(dw).$$

Note that by (4.12) and (4.13), the right-hand side is finite. Comparing the two terms, for $d = 1$, the first term dominates, while for $d \geq 2$, the second term dominates [explaining the need for alternative conditions in (4.13)].

From this point on, our approach is reminiscent of that in Section 3.1. For $l = 1, 2, \ldots$, define $\eta^l_{y,w} = \eta(\cdot \cap D_{y,w} \times \mathbb{K} \times [0,l))$. Set

$$\psi_l(\eta) = \eta(D_{0,l} \times \mathbb{K} \times [0,l)) \int_{[0,1]\times(1,\infty)} v_d(l+w)^d \zeta \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ v_d l^d \int_{\mathbb{R}^d \times [0,1] \times (1,\infty)} \frac{|\eta^l_{y,w}|}{v_d w^d} \mathbf{1}_{\{D_{0,l} \cap D_{y,w} \neq \varnothing\}} \zeta \, dy \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ \int_{\mathbb{R}^d \times [0,1] \times [0,1]} |\eta^l_{y,w}| (|\eta^l_{y,w}| - 1) \mathbf{1}_{\{D_{0,l} \cap D_{y,w} \neq \varnothing\}} \, dy \nu^1(w, d\zeta) \nu^2(dw)$$

$$+ l \int_{\mathbb{R}^d \times [0,1] \times [0,1]} |\eta^l_{y,w}| w^2 \mathbf{1}_{\{D_{0,l} \cap D_{y,w} \neq \varnothing\}} \zeta \, dy \nu^1(w, d\zeta) \nu^2(dw).$$

Select $\delta_l > 0$ so that if (4.23) and (4.24) are satisfied, then $\sum_l \delta_l E[\psi_l(\eta_t)] < \infty$, and define

$$\psi(\eta) = 1 + \sum_l \delta_l \psi_l(\eta).$$

Then for each g such that $f(\eta) = \prod_{(x,\kappa,u)\in\eta} g(x, \kappa, u) \in \mathcal{D}(A^\infty)$, there exists l such that $\rho_g \leq l$, $u_g \leq l$ and $\|\partial^2 g\| \leq l$, and hence for $m \geq 0$,

$$\left| \sum_{k=m+1}^{\infty} B_k \right| \leq \frac{1}{\delta_l} \psi(\eta).$$

Consequently, we can take c_f in Theorem A.2 and Condition A.5 to be δ_l^{-1} and m_f in Condition A.5 to be $[\rho_g + 2]$. We have the following.

THEOREM 4.1. *Assume that* (4.12) *and* (4.13) *hold and that η is a solution of the martingale problem for $A^\infty_{dr,2}$ given by* (4.4) *satisfying* (4.23) *and* (4.24). *Then with the H_k given by* (4.18) *and λ_k given by* (4.17), *the conclusion of Theorem A.6 holds.*

Let Ξ be a solution of the martingale problem for $\alpha A^\infty_{dr,2}$ given by (4.7) *satisfying*

$$E\left[\int_0^t E[\psi(\eta_s)|\Xi(s)] \, ds \right] < \infty,$$

for all $t > 0$, where for each s, η_s is a conditionally Poisson process with Cox measure $\Xi(s) \times \ell$. Then Ξ can be obtained from a solution of the martingale problem for $A^\infty_{dr,2}$. In particular, the conclusion holds for any solution with $\Xi(0, dx \times \mathbb{K})$ equal to Lebesgue measure.

REMARK 4.2. With the above formulation of the generator, for finite λ,

$$
\alpha A^\lambda_{dr,2} f(\overline{\eta}) = \alpha f(\overline{\eta}) \int_{\mathbb{R}^d \times [0,1] \times [0,\infty)} \sum_{(x,\kappa) \in \overline{\eta}_{y,w}} \frac{C_{y,w} \overline{g}(x,\kappa)}{\overline{g}(x,\kappa)}
$$

$$
\times \zeta (1-\zeta)^{|\overline{\eta}_{y,w}|-1} \, dy \nu^1(w, d\zeta) \nu^2(dw) + \alpha f(\overline{\eta})
$$

$$
\times \int_{\mathbb{R}^d \times [0,1] \times [0,\infty)} \left(\frac{\sum_{S \subset \overline{\eta}_{y,w}, |S| \geq 2} \overline{H}(\overline{g}, \widehat{g}, S, y, \zeta, w)}{\prod_{(x,\kappa) \in \overline{\eta}_{y,w}} \overline{g}(x,\kappa)} - 1 \right) dy
$$

$$
\times \nu^1(w, d\zeta) \nu^2(dw),
$$

where $\overline{H}$ is as in (4.6) for $|S| \geq 2$.

For $\lambda = \infty$, $\alpha A^\infty_{dr,2}$ is as in (4.7), that is,

$$
\alpha A^\infty_{dr,2} f(\Xi)
$$

$$
= e^{-\int h(x,\kappa) \Xi(dx,d\kappa)}
$$

$$
\times \int_{\mathbb{R}^d \times [0,1] \times [0,\infty)} \left(\mathbb{H}_1 \left(h^*_{y,w}, \Xi, y, \zeta, w \right) e^{\zeta \int_{D_{y,w} \times \mathbb{K}} h(x,\kappa) \Xi(dx,d\kappa)} - 1 \right) dy
$$

$$
\times \nu^1(w, d\zeta) \nu^2(dw),
$$

where $h^*_{y,w}(\kappa) = \int_0^\infty (1 - \widehat{g}_{y,w}(\kappa, u)) \, du$.

4.1.2. *Stochastic equations for locations and types.* Recall

$$
H_k(\eta, y, \zeta, w, (z,v)^\eta)
$$

$$
= \sum_{(x,\kappa,u) \in \eta} \left((1 - \mathbf{1}_{D_{y,w}}(x) \mathbf{1}_{[0,\zeta]}(z)) \delta_{(x,\kappa,u)} \right.
$$

$$
\left. + \mathbf{1}_{D_{y,w}}(x) \mathbf{1}_{[0,\zeta]}(z) \delta_{(y+wv,\kappa_{u*},u)} \right).
$$

Write

$$
\eta(t) = \sum \delta_{(X_u(t),\kappa_u(t),u)}.
$$

Under the conditions of Theorem 4.1, setting

$$
H_k(s-) = H_k\left(\eta(s-), Y_k(s-), \zeta_k(s-), W_k(s-), \{(Z_{k,u}(s-), V_{k,u}(s-))\} \right)
$$

we have

$$
f(\eta(t)) = f(\eta(0)) + \lim_{m \to \infty} \sum_{k=1}^m \int_0^t \left(f(H_k(s-)) - f(\eta(s-)) \right) dN_k(s),
$$

where, for each k, N_k is a Poisson process with parameter λ_k (as defined in Theorem 4.1) and at each jump τ of N_k, if $X_u(\tau-) \in D_{Y_k(\tau-), W_k(\tau-)}$, then $(Z_{k,u}(\tau-), V_{k,u}(\tau-))$ is replaced by an independent pair of random variables $(Z_{k,u}(\tau), V_{k,u}(\tau)) \in [0,1] \times D_{0,1}$ with distribution $dz \times \upsilon_{0,1}(dv)$. Consequently, the location of the particle with level u will satisfy

$$
X_u(t) = X_u(0)
$$

$$
(4.25) \qquad + \lim_{m \to \infty} \sum_{k=1}^{m} \int_0^t \mathbf{1}_{D_{Y_k(s-), W_k(s-)}}(X_u(s-)) \mathbf{1}_{[0, \zeta_k(s-)]}(Z_{k,u}(s-))
$$

$$
\times \big(Y_k(s-) + W_k(s-) V_{k,u}(s-) - X_u(s-) \big) \, dN_k(s).
$$

Note that ξ_0 defined by

$$
\int_{[0,t] \times \mathbb{R}^d \times [0,1] \times [0,\infty)} f(y, \zeta, w) \xi_0(ds, dy, d\zeta, dw)
$$

$$
= \sum_{k=1}^{\infty} \int_0^t f\big(Y_k(s-), \zeta_k(s-), W_k(s-) \big) \, dN_k(s)
$$

is a Poisson random measure with mean measure $ds \, dy \, \nu^1(w, d\zeta) \nu^2(dw)$, and ξ_u defined by

$$
\int_{[0,t] \times \{0,1\} \times D_{0,1} \times \mathbb{R}^d \times [0,1] \times [0,\infty)} f(\theta, v, y, \zeta, w) \xi_u(ds, d\theta, dv, dy, d\zeta, dw)
$$

$$
= \sum_{k=1}^{\infty} \int_0^t f\big(\mathbf{1}_{[0, \zeta_k(s-)]}(Z_{k,u}(s-)), V_{k,u}(s-),
$$

$$
Y_k(s-), \zeta_k(s-), W_k(s-) \big) \, dN_k(s)
$$

is a Poisson random measure with mean measure

$$
ds \big((1 - \zeta) \delta_0(\theta) + \zeta \delta_1(\theta) \big) \upsilon_{0,1}(dv) \, dy \, \nu^1(w, d\zeta) \nu^2(dw).
$$

Then letting $z = (\theta, v, y, \zeta, w)$, so $\xi_u(ds, d\theta, dv, dy, d\zeta, dw) = \xi_u(ds, dz)$, (4.25) becomes

$$
X_u(t) = X_u(0)
$$

$$
+ \lim_{k \to \infty} \int_{[0,t] \times \{0,1\} \times D_{0,1} \times \Gamma_k} \mathbf{1}_{D_{y,w}}(X_u(s-))
$$

$$
(4.26) \qquad \times \theta \big(y + wv - X_u(s-) \big) \xi_u(ds, dz)
$$

$$
= X_u(0) + \int_{[0,t] \times \{0,1\} \times D_{0,1} \times \mathbb{R}^d \times [0,1] \times [0,\infty)} \mathbf{1}_{D_{y,w}}(X_u(s-))
$$

$$
\times \theta \big(y + wv - X_u(s-) \big) \widetilde{\xi}_u(ds, dz),
$$

where $\widetilde{\xi}_u$ is ξ_u centered by its mean measure. The centering has no effect on the right-hand side of the first equality once k is large enough that $X_u(s) \in D_{0,k-1}$ for $0 \le s \le t$. In particular, if $X_u(s) \in D_{0,k-1}$,

$$
\begin{aligned}
\int_{\{0,1\} \times D_{0,1} \times D_{0,k} \times [0,1] \times [2^{-k},1]} &\mathbf{1}_{D_{y,w}}(X_u(s))\theta(y + wv - X_u(s)) \\
&\times ((1-\zeta)\delta_0(\theta) + \zeta\delta_1(\theta))\upsilon_{0,1}(dv)\, dy \nu^1(w, d\zeta)\nu^2(dw) \\
&= \int_{D_{0,k} \times [0,1] \times [0,1]} \mathbf{1}_{D_{0,w}}(X_u(s) - y)\zeta(y - X_u(s))\, dy \nu^1(w, d\zeta)\nu^2(dw) \\
&= \int_{\mathbb{R}^d \times [0,1] \times [0,1]} \mathbf{1}_{D_{0,w}}(y)\zeta y\, dy \nu^1(w, d\zeta)\nu^2(dw) \\
&= 0,
\end{aligned}
$$

where in the first equality we have used that v is uniformly distributed on $D_{0,1}$ and so has mean zero. Furthermore, (4.12) and (4.13) imply the existence of the stochastic integrals in the limiting equation.

Set $R_{y,w,v}(x) = (y + wv - x)(y + wv - x)^T$. Assuming existence of a solution, the centered integral in (4.26) is a square integrable martingale M_u with covariation matrix

$$
\begin{aligned}
[M_u]_t = \int_{[0,t] \times \{0,1\} \times D_{0,1} \times \mathbb{R}^d \times [0,1] \times [0,1]} &\mathbf{1}_{D_{y,w}}(X_u(s-))\theta R_{y,w,v}(X_u(s-)) \\
&\times \xi_u(ds, d\theta, dv, dy, d\zeta, dw)
\end{aligned}
$$

and, by translation invariance,

$$
\begin{aligned}
E[[M_u]_t] &= t \int_0^1 \int_0^1 \int_{D_{0,w}} \int_{D_{0,1}} \zeta(y + wv)(y + wv)^T \\
&\qquad\qquad \times \upsilon_{0,1}(dv)\, dy \nu^1(w, d\zeta)\nu^2(dw) \\
&= t \int_0^1 \int_0^1 \int_{D_{0,w}} \zeta(yy^T + |w|^2 c_d I)\, dy \nu^1(w, d\zeta)\nu^2(dw) \\
&= t C_d \int_0^1 \int_0^1 \zeta|w|^{2+d} \nu^1(w, d\zeta)\nu^2(dw) I,
\end{aligned}
$$

for appropriate choices of c_d and C_d, which is finite by (4.13).

LEMMA 4.3. *Assume* (4.12) *and* (4.13). *Then weak* (*distributional*) *existence holds for the system* (4.26). *If, in addition,* (4.14) *holds, then strong uniqueness* (*and hence strong existence*) *holds.*

REMARK 4.4. Weak uniqueness (uniqueness in distribution) for a single X_u follows by uniqueness of the corresponding martingale problem (X_u is a Lévy

process). Unfortunately, weak uniqueness for a single X_u does not imply weak uniqueness for the system. If we consider the joint distribution of X_u and $X_{u'}$, weak uniqueness only implies uniqueness of the marginal distributions. Strong existence means that X_u can be written as a function of the stochastic inputs, and strong uniqueness implies there is only one such function (up to modification on events of probability zero). Strong uniqueness for a single X_u would give strong uniqueness for the system. In the case of $d = 1$, strong uniqueness is proved in Zheng and Xiong (2017) under the more general conditions (4.12) and (4.13)

PROOF. It is enough to consider an arbitrary but finite subsystem $\{X_{u_i}, 1 \leq i \leq m\}$. With reference to (4.22), let $f(x) = \prod_{i=1}^{m} g(x_i)$, $g \in C^2(\mathbb{R}^d)$, $0 \leq g \leq 1$, $g(x) = 1$ for x outside D_{0,ρ_g}, and for $S \subset \{i : x_i \in D_{y,w}, 1 \leq i \leq m\}$, let $f_S(x) = \prod_{i \in S} g(x_i)$ and

$$
\begin{aligned}
(4.27) \quad B_{y,w}^{|S|} f_S(x) = \int_{D_{y,w}^{|S|}} &\left(\prod_{i \in S} g(x_i') - \prod_{i \in S} g(x_i) \right. \\
&\left. - \mathbf{1}_{\{w \leq 1\}} f_S(x) \sum_{i \in S} \frac{(x_i' - x_i) \cdot \nabla g(x_i)}{g(x_i)} \right) \prod_{i \in S} \upsilon_{y,w}(dx_i').
\end{aligned}
$$

Then setting $S_{y,w}(x) = \{i : x_i \in D_{y,w}\}$, the generator for the subsystem becomes

$$
\begin{aligned}
(4.28) \quad A^m f(x) = f(x) \int_{\mathbb{R}^d \times [0,1] \times [0,\infty)} &\sum_{S \subset S_{y,w}(x)} \frac{B_{y,w}^{|S|} f_S(x)}{\prod_{i \in S} g(x_i))} \\
&\times \zeta^{|S|} (1 - \zeta)^{|S_{y,w}(x)| - |S|} \, dy \nu^1(w, d\zeta) \nu^2(dw).
\end{aligned}
$$

Note that $A^m f(x)$ is a continuous function of x.

Existence of solutions of the martingale problem for (4.28) follows by approximation. To obtain an approximation $X^\epsilon = (X_1^\epsilon, \ldots, X_m^\epsilon)$, consider the system obtained by replacing ν^2 by ν_ϵ^2 given by $\nu_\epsilon^2(C) = \nu^2(C \cap [\epsilon, \infty))$. The generator $A^{m,\epsilon}$ is then a bounded operator (the gradient term integrates to zero), and existence and uniqueness for the martingale problem is immediate. For each i, X_i^ϵ is a Lévy process with Lévy measure

$$
\begin{aligned}
\nu_\epsilon(C) &= \int_{\mathbb{R}^d \times [0,1] \times [\epsilon,\infty)} \int_{D_{y,w}} \mathbf{1}_C(x' - x) \mathbf{1}_{D_{y,w}}(x) \upsilon_{y,w}(dx') \\
&\quad \times \zeta \, dy \nu^1(w, d\zeta) \nu^2(dw) \\
&= \int_{\mathbb{R}^d \times [0,1] \times [\epsilon,\infty)} \int_{D_{0,w}} \mathbf{1}_C(x' + y - x) \mathbf{1}_{D_{0,w}}(x - y) \upsilon_{0,w}(dx') \\
&\quad \times \zeta \, dy \nu^1(w, d\zeta) \nu^2(dw) \\
&= \int_{\mathbb{R}^d \times [0,1] \times [\epsilon,\infty)} \int_{D_{0,w}} \mathbf{1}_C(x' - z) \mathbf{1}_{D_{0,w}}(z) \upsilon_{0,w}(dx') \\
&\quad \times \zeta \, dz \nu^1(w, d\zeta) \nu^2(dw),
\end{aligned}
$$

and convergence in distribution of $\{X_i^\epsilon\}$ follows from convergence of the Lévy measures. Convergence for each component implies relative compactness of $\{X^\epsilon\}$ at least in $D_{\mathbb{R}^d}[0, \infty) \times \cdots \times D_{\mathbb{R}^d}[0, \infty)$ [Ethier and Kurtz (1986), Proposition 3.2.4], if not in $D_{(\mathbb{R}^d)^m}[0, \infty)$. For a convergent subsequence, convergence in the product topology still implies convergence of the integrals

$$\int_0^t A^{m,\epsilon} f(X^\epsilon(s)) \, ds \quad \Rightarrow \quad \int_0^t A^m f(X(s)) \, ds,$$

which in turn ensures that the limit is a solution of the martingale problem for A^m. The fact that the limit is a weak solution of the stochastic differential equation follows by Theorem 2.3 of Kurtz (2011).

If (4.14) holds, then a solution of (4.26) jumps only finitely often in a finite time interval, that is, $\{(s, z) \in \xi_u : s \le t, X_u(s-) \in D_{y,w}, \theta = 1\}$ is finite for each $t > 0$. Consequently, the equation is uniquely solved by moving from one such (s, z) to the next, and this solution depends only on the stochastic inputs, that is, it is a strong solution. □

We still need to consider the evolution of the type of each particle. Note that the particle with index u changes type only if it is involved in a birth/death event with a particle having a lower level. The number of times that particle u_1 and particle u_2 are involved in the same birth/death event up to time t can be written as

$$
\begin{aligned}
N_{u_1 u_2}(t) \\
= \int_{[0,t] \times \mathbb{R}^d \times [0,1] \times [0,\infty)} &\mathbf{1}_{D_{y,w}}(X_{u_2}(s-)) \mathbf{1}_{D_{y,w}}(X_{u_1}(s-)) \theta_{u_1}(s) \theta_{u_2}(s) \\
&\times \xi_0(ds, dy, d\zeta, dw)
\end{aligned}
$$

and since θ_{u_1} and θ_{u_2} are conditionally independent given ξ_0,

$$
\begin{aligned}
E[N_{u_1 u_2}(t)] = \int_{[0,t] \times \mathbb{R}^d \times [0,1] \times [0,\infty)} &E[\mathbf{1}_{D_{y,w}}(X_{u_2}(s)) \mathbf{1}_{D_{y,w}}(X_{u_1}(s))] \\
&\times \zeta^2 \, ds \, dy \, v^1(w, d\zeta) v^2(dw).
\end{aligned}
$$

Let $C \subset \mathbb{R}^d$ be bounded and $u > 0$, and let $N_{C,u}(t)$ be the number of times by time t that two particles with levels below u and locations in C are involved in the same birth/death event. Then, assuming (4.23) and (4.24),

$$
\begin{aligned}
E[N_{C,u}(t)] \\
= \int_{[0,t] \times \mathbb{R}^d \times [0,1] \times [0,\infty)} &E\left[\sum_{u_1 < u_2 \le u} \mathbf{1}_C(X_{u_1}(s)) \mathbf{1}_C(X_{u_2}(s)) \right. \\
&\left. \times \mathbf{1}_{D_{y,w}}(X_{u_2}(s)) \mathbf{1}_{D_{y,w}}(X_{u_1}(s)) \right]
\end{aligned}
$$

$$\times \zeta^2 \, ds \, dy \, v^1(w, d\zeta) v^2(dw)$$

$$= \int_{[0,t] \times \mathbb{R}^d \times [0,1] \times [0,\infty)} E\left[\binom{\eta(s, D_{y,w} \cap C \times [0, u])}{2} \right]$$

$$\times \zeta^2 \, ds \, dy \, v^1(w, d\zeta) v^2(dw)$$

$$\leq \int_{[0,t] \times \mathbb{R}^d \times [0,1] \times [0,\infty)} u^2 |D_{y,w} \cap C|^2 \zeta^2 \, ds \, dy \, v^1(w, d\zeta) v^2(dw)$$

$$\leq u^2 \int_{[0,t] \times \mathbb{R}^d \times [0,1] \times [0,\infty)} (v_d^2 w^{2d} \wedge |C|^2) \zeta^2 \, ds \, dy \, v^1(w, d\zeta) v^2(dw)$$

$$< \infty.$$

It follows that no single particle will change type more than finitely often by time t.

It is now straightforward to write down an equation for the way in which individuals' types change with time. For $u_1 < u_2$, define

$$L_{u_1 u_2}(t) = \#\{s \leq t : N_{u_1 u_2}(s) - N_{u_1 u_2}(s-) = 1,$$
$$N_{u_3 u_2}(s) - N_{u_3 u_2}(s-) = 0, \forall u_3 < u_1\}.$$

Then, writing κ_u for the type of the individual with level u,

$$(4.29) \qquad \kappa_{u_2}(t) = \kappa_{u_2}(0) + \sum_{u_1 < u_2} \int_0^t (\kappa_{u_1}(s-) - \kappa_{u_2}(s-)) \, dL_{u_1 u_2}(s).$$

As in Section 5 of Donnelly and Kurtz (1999), the genealogy of the particles alive at time t is determined by the $L_{u_1 u_2}$. In particular, the index of the ancestor at time $r < t$ of the particle at level u_2 at time t satisfies

$$(4.30) \qquad J_{u_2}(t, r) = u_2 - \sum_{u_3 < u_1 \leq u_2} \int_r^t (u_1 \mathbf{1}_{\{J_{u_2}(t,s) = u_1\}} - u_3) \, dL_{u_3 u_1}(s).$$

Since the lookdown construction for the discrete population model is simply the restriction of the lookdown construction of the infinite density population model, the genealogies of the discrete model converge to those of the infinite density model. To be precise, we have the following.

THEOREM 4.5. *For any solution of the infinite system* (4.26) (*regardless of the uniqueness question*), *the counting processes $N_{u_1 u_2}$ and $L_{u_1 u_2}$, the type processes κ_u, and the ancestral index J_u are uniquely determined, and the genealogies of the $\lambda < \infty$ model converge to those of the $\lambda = \infty$ model.*

REMARK 4.6. As noted in Section 3.10, one can model the selection of a random sample of size n from a region C satisfying $0 < \Xi(t, C \times \mathbb{K}) < \infty$ simply by selecting the particles located in C with the n lowest levels. The genealogy of the sample can then be obtained using equation (4.30).

4.1.3. *Second construction of spatial Λ-Fleming–Viot with levels.* The particle dynamics for $A_{dr,2}^\lambda$ given by (4.26) and (4.29) are very different from the particle dynamics that are natural for $A_{th,db}^\lambda$ defined in (4.9). The event measures $\mu(dz) \equiv dy\nu^1(w, d\zeta)\nu^2(dw)$ are of the same form, but what happens at each event $z = (y, \zeta, w)$ is very different. In particular, for a birth/death event in the ball $D_{y,w}$, the total population size in $D_{y,w}$ does not change for $A_{dr,2}^\lambda$, but typically it will change for $A_{th,db}^\lambda$. As will become apparent when we analyze the behavior of the levels, we will need to assume stronger conditions on the event measures than were used in the previous construction.

With $\mathcal{D}_\lambda$ defined in (4.1), we take the domain of $A^\lambda \equiv A_{th,db}^\lambda$ to be

$$\left\{ f(\eta) = \prod_{(x,\kappa,u)\in\eta} g(x, \kappa, u) \in \mathcal{D}_\lambda : g(\cdot, \kappa, \cdot) \in C^2, \right.$$

$$\left. \|\partial_u g\| \equiv \sup_{x,\kappa,u} |\partial_u g(x, \kappa, u)| < \infty \right\},$$

and $\mathcal{D}(A^\infty) = \bigcup_\lambda \mathcal{D}(A^\lambda)$. In a birth/death event determined by $z = (y, \zeta, w)$, the parent is killed and, with probability 1, for $\lambda < \infty$ all other particles in the event region $D_{y,w}$ will change levels and for $\lambda = \infty$, all particles with levels above that of the parent will change levels.

For finite λ, v^* has density $(1 - e^{-\lambda\alpha_z})^{-1}\alpha_z e^{-\alpha_z v}$ on $[0, \lambda]$, where $\alpha_z = \zeta|D_{y,w}|$. Let $u_1^* = \min\{u : u > v^*, (x, \kappa, u) \in \eta_{|D_{y,w}}\}$ and $u_2^* = \max\{u : u < v^*, (x, \kappa, u) \in \eta_{|D_{y,w}}\}$ and define

$$\tau_1^* = \log\frac{\lambda - v^*}{\lambda - u_1^*}, \qquad \tau_2^* = \log\frac{\lambda}{u_2^*} \quad \text{and} \quad \tau^* = \tau_1^* \wedge \tau_2^*.$$

Setting

$$u^* = u_1^* \mathbf{1}_{\{\tau_1^* < \tau_2^*\}} + u_2^* \mathbf{1}_{\{\tau_1^* > \tau_2^*\}},$$

for $(x, \kappa, u) \in \eta_{|D_{y,w}}$, we have

(4.31) $$\mathcal{J}_{y,w}^\lambda(x, u, \eta, v^*) = \mathbf{1}_{\{u>v^*\}}(ue^{\tau^*} - \lambda(e^{\tau^*} - 1)) + \mathbf{1}_{\{u<v^*\}}ue^{\tau^*},$$

and for $\lambda = \infty$,

(4.32) $$\mathcal{J}_{y,w}^\infty(x, u, \eta, v^*) = u - \mathbf{1}_{[v^*,\infty)}(u)(u^* - v^*).$$

If $(x, \kappa, u) \in \eta_{|D_{y,w}}$ is not the parent, that is, $u \neq u^*$, then (x, κ, u) jumps to $(x, \kappa, \frac{\mathcal{J}_{y,w}^\lambda(x,u,\eta,v^*)}{1-\zeta})$.

For reasons that will become clear below, we also require the stronger condition (4.14), that is,

(4.33) $$\int_{(0,\infty)\times[0,1]} \zeta w^d \nu^1(w, d\zeta)\nu^2(dw) < \infty.$$

Recall (4.20), and note that at an event $z = (y, \zeta, w)$, the expected number of new particles with level below u_g is bounded by

$$\frac{1}{1 - e^{-\lambda \alpha_z}} \zeta |D_{y,w}| u_g.$$

Setting $\mathbb{U} = \mathbb{R}^d \times [0, 1] \times [0, \infty)$ and assuming (4.15), define Γ_k as in (4.16). With reference to Appendix A.3, define

$$B_k^\lambda f(\eta) = \int_{\Gamma_k - \Gamma_{k-1}} \left(H_z^\lambda(g, \eta) - f(\eta) \right)(1 - e^{-\lambda \alpha_z}) \, dy \nu^1(w, d\zeta) \nu^2(dw),$$

where $H_z^\lambda(g, \eta)$ is defined in (4.10). Note that u^* and κ^* are determined by v^* and $\eta_{y,w}$. Then

$$
\begin{aligned}
&\left| B_k^\lambda f(\eta) \right| \\
&\quad \leq \int_{\Gamma_k - \Gamma_{k-1}} \left| H_z^\lambda(g, \eta) - f(\eta) \right|(1 - e^{-\alpha_z \lambda}) \, dy \nu^1(w, d\zeta) \nu^2(dw) \\
&\quad \leq \int_{\Gamma_k - \Gamma_{k-1}} \int_0^\lambda \alpha_z \\
&\qquad \times e^{-\alpha_z v^*} \left(1 - \widehat{g}_{y,w}(\kappa^*, v^*) e^{-\alpha_z \int_{v^*}^\lambda (1 - \widehat{g}_{y,w}(\kappa^*, v)) \, dv} \right) dv^* \mu(dz) \\
&\quad\quad (4.34) \\
&\qquad + \int_{\Gamma_k - \Gamma_{k-1}} \int_0^\lambda \alpha_z \\
&\qquad \times e^{-\alpha_z v^*} \left| \prod_{(x, \kappa, u) \in \eta, x \in D_{y,w}, u \neq u^*} g\left(x, \kappa, \frac{1}{1 - \zeta} \mathcal{J}_{y,w}^\lambda(x, u, \eta, v^*) \right) \right. \\
&\qquad \left. - \prod_{(x, \kappa, u) \in \eta, x \in D_{y,w}} g(x, \kappa, u) \right| dv^* \mu(dz).
\end{aligned}
$$

Note that the integrand in the first term on the right is zero if $v^* \geq u_g$ and

$$\widehat{g}_{y,w}(\kappa^*, v) \geq 1 - \frac{|D_{y,w} \cap D_{0,\rho_g}|}{|D_{y,w}|} \equiv \underline{\widehat{g}}_{y,w}.$$

Then, bounding the two terms on the right of (4.34),

$$
\begin{aligned}
\left| B_k^\lambda f(\eta) \right| &\leq \int_{\Gamma_k - \Gamma_{k-1}} \int_0^{u_g} \alpha_z e^{-\alpha_z v^*} \left(1 - \underline{\widehat{g}}_{y,w} e^{-\alpha_z (u_g - v^*)(1 - \widehat{g}_{y,w})} \right) dv^* \mu(dz) \\
&\quad + \int_{\Gamma_k - \Gamma_{k-1}} (1 - e^{-\alpha_z \lambda}) \overline{\eta}(D_{y,w} \cap D_{0,\rho_g} \times \mathbb{K}) \mu(dz) \\
&\leq \int_{\Gamma_k - \Gamma_{k-1}} \left((1 - \underline{\widehat{g}}_{y,w})(1 - e^{-\alpha_z u_g}) \right.
\end{aligned}
$$

$$+ \int_0^{u_g} \alpha_z e^{-\alpha_z v^*} \widehat{\underline{g}}_{y,w} \big(1 - e^{-\alpha_z (u_g - v^*)(1 - \widehat{\underline{g}}_{y,w})}\big) \, dv^* \bigg) \mu(dz)$$

$$+ \int_{\Gamma_k - \Gamma_{k-1}} (1 - e^{-\alpha_z \lambda}) \overline{\eta}(D_{y,w} \cap D_{0,\rho_g} \times \mathbb{K}) \mu(dz)$$

$$\leq \int_{\Gamma_k - \Gamma_{k-1}} 2\alpha_z u_g (1 - \widehat{\underline{g}}_{y,w}) \mu(dz)$$

$$+ \int_{\Gamma_k - \Gamma_{k-1}} (1 - e^{-\alpha_z \lambda}) \overline{\eta}(D_{y,w} \cap D_{0,\rho_g} \times \mathbb{K}) \mu(dz)$$

$$\leq \int_{\Gamma_k - \Gamma_{k-1}} 2u_g \zeta |D_{y,w}| \frac{|D_{y,w} \cap D_{0,\rho_g}|}{|D_{y,w}|} \mu(dz)$$

$$+ \int_{\Gamma_k - \Gamma_{k-1}} \zeta |D_{y,w}| \lambda \overline{\eta}(D_{y,w} \cap D_{0,\rho_g} \times \mathbb{K}) \mu(dz).$$

The construction of the ψ needed to apply Theorem A.2 and Theorem A.6 is similar to the construction in the previous section. Bounding the parameters in the estimates above that depend on g by a positive integer l, we have

$$(4.35) \quad \begin{aligned} \sum_{k=1}^{\infty} |B_k^{\lambda} f(\eta)| &\leq \int_{[0,1] \times [0,\infty)} 2v_d \zeta l (w \wedge l)^d \nu^1(w, d\zeta) \nu^2(dw) \\ &\quad + \int_{\mathbb{U}} \zeta |D_{y,w}| \lambda \overline{\eta}(D_{y,w} \cap D_{0,l} \times \mathbb{K}) \mu(dz) \\ &\equiv \psi_l(\eta), \end{aligned}$$

provided u_g and ρ_g are less than l.

We are primarily interested in solutions of the martingale problem for which $\eta(\cdot \times \mathbb{K} \times \cdot)$ will be dominated by a Poisson random measure on $\mathbb{R}^d \times [0, \lambda]$ with Lebesgue mean measure, so restricting our attention to solutions of the martingale problem satisfying

$$(4.36) \qquad\qquad E\big[\overline{\eta}(D_{y,w} \times \mathbb{K})\big] \leq c|D_{y,w}| = cv_d w^d,$$

we have

$$E[\psi_l(\eta)] \leq \int_{[0,1] \times [0,\infty)} \big(v_d \zeta l (w \wedge l)^d + c\lambda v_d^2 \zeta w^d (w \wedge l)^d\big) \nu^1(w, d\zeta) \nu^2(dw),$$

which is finite under (4.33). Then, as before, we set $\psi(\eta) = 1 + \sum_{l=1}^{\infty} \delta_l \psi_l(\eta)$, where we select $\delta_l > 0$ satisfying

$$\sum_{l=1}^{\infty} \delta_l \int_{[0,1] \times [0,\infty)} \big(v_d \zeta l (w \wedge l)^d + c\lambda v_d^2 \zeta w^d (w \wedge l)^d\big) \nu^1(w, d\zeta) \nu^2(dw) < \infty.$$

The $\lambda = \infty$ case takes a little more care. Note that we exploit the fact that if $u < v^*$ and $x \in D_{y,w}$, then

$$\mathcal{J}_{y,w}^{\infty}(x, u, \eta, v^*) = u,$$

so after a birth-death event, the new level is $\frac{1}{1-\zeta}u$. Let

$$\mathbb{U}_1 = \left\{(y, \zeta, w) \in \mathbb{U} : \zeta \le \frac{1}{2}\right\} \quad \text{and} \quad \mathbb{U}_2 = \left\{(y, \zeta, w) \in \mathbb{U} : \zeta > \frac{1}{2}\right\}.$$

$$|B_k f(\eta)|$$

$$\le \int_{\Gamma_k - \Gamma_{k-1}} |H_z(g, \eta) - f(\eta)| \, dy \nu^1(w, d\zeta) \nu^2(dw)$$

$$\le \int_{\Gamma_k - \Gamma_{k-1}} \int_0^\infty \alpha_z$$

$$\times e^{-\alpha_z v^*}\left(1 - \widehat{g}_{y,w}(\kappa^*, v^*)e^{-\alpha_z \int_{v^*}^\infty (1 - \widehat{g}_{y,w}(\kappa^*, v)) \, dv}\right) dv^* \mu(dz)$$

$$+ \int_{\Gamma_k - \Gamma_{k-1}} \int_0^\infty \alpha_z$$

$$\times e^{-\alpha_z v^*}\Bigg| \prod_{(x, \kappa, u) \in \eta, x \in D_{y,w}, u \ne u^*} g\left(x, \kappa, \frac{1}{1-\zeta}\mathcal{J}_{y,w}^\infty(x, u, \eta, v^*)\right)$$

$$(4.37) \qquad - \prod_{(x, \kappa, u) \in \eta, x \in D_{y,w}} g(x, \kappa, u)\Bigg| dv^* \mu(dz)$$

$$\le \int_{\Gamma_k - \Gamma_{k-1}} 2u_g \zeta |D_{y,w}| \frac{|D_{y,w} \cap D_{0, \rho_g}|}{|D_{y,w}|} \mu(dz)$$

$$+ \int_{(\Gamma_k - \Gamma_{k-1})} \int_0^{u_g} \alpha_z$$

$$\times e^{-\alpha_z v^*} \eta\left(D_{y,w} \cap D_{0, \rho_g} \times \mathbb{K} \times [0, u_g + u_1^* - v^*)\right) dv^* \mu(dz)$$

$$+ \int_{(\Gamma_k - \Gamma_{k-1}) \cap \mathbb{U}_1} e^{-\alpha_z u_g}$$

$$\times \eta\left(D_{y,w} \cap D_{0, \rho_g} \times \mathbb{K} \times [0, u_g]\right) \|\partial_u g\| \frac{\zeta}{1-\zeta} u_g \mu(dz)$$

$$+ \int_{(\Gamma_k - \Gamma_{k-1}) \cap \mathbb{U}_2} e^{-\alpha_z u_g}$$

$$\times \eta\left(D_{y,w} \cap D_{0, \rho_g} \times \mathbb{K} \times [0, u_g]\right) \mu(dz).$$

The first term corresponds to offspring of the event, the second accounts for the change in levels of individuals already present in the population in the case $v^* < u_g$ and the final two terms to the corresponding changes when $v^* > u_g$. As in Section 4.1.1, we are bounding the difference of two products in which all the factors are less than or equal to one, by a sum of differences of factors.

As before, for u_g, $\|\partial_u g\|$, and ρ_g less than l,

$$
\sum_{k=1}^{\infty} |B_k f(\eta)| \leq \int_{\mathbb{U}} 2l\zeta |D_{y,w}| \frac{|D_{y,w} \cap D_{0,l}|}{|D_{y,w}|} \mu(dz)
$$
$$
+ \int_{\mathbb{U}} \int_0^l \alpha_z e^{-\alpha_z v^*}
$$
$$
\times \eta(D_{y,w} \cap D_{0,l} \times \mathbb{K} \times [0, l + u_1^* - v^*)) \, dv^* \mu(dz)
$$
$$
+ \int_{\mathbb{U}_1} e^{-\alpha_z l} \eta(D_{y,w} \cap D_{0,l} \times \mathbb{K} \times [0, l]) \frac{\zeta}{1-\zeta} l^2 \mu(dz)
$$
$$
+ \int_{\mathbb{U}_2} e^{-\alpha_z l} \eta(D_{y,w} \cap D_{0,l} \times \mathbb{K} \times [0, l]) \mu(dz)
$$
$$
\equiv \psi_l(\eta).
$$

Note that in the second term on the right, $v^* \leq l$, and $u_1^* - v^* < u_1^* \leq 2l$, if $\eta(D_{y,w} \times \mathbb{K} \times (l, 2l]) > 0$. In general, we have

$$
\eta(D_{y,w} \cap D_{0,l} \times \mathbb{K} \times [0, l + u_1^* - v^*))
$$
$$
\leq \eta(D_{y,w} \cap D_{0,l} \times \mathbb{K} \times [0, 3l))
$$
$$
+ \sum_{k=2}^{\infty} \mathbf{1}_{\{\eta(D_{y,w} \times \mathbb{K} \times (l, kl]) = 0, \eta(D_{y,w} \times \mathbb{K} \times (kl, (k+1)l]) > 0\}}
$$
$$
\times \eta(D_{y,w} \cap D_{0,l} \times \mathbb{K} \times ((k+1)l, (k+2)l)),
$$

and assuming η is conditionally Poisson with Cox measure $\Xi(dx, d\kappa) \, du$, the conditional independence of η on disjoint sets gives

$$
E[\eta(D_{y,w} \cap D_{0,l} \times \mathbb{K} \times [0, l + u_1^* - v^*))]
$$
$$
\leq E[\Xi(D_{y,w} \cap D_{0,l} \times \mathbb{K})]3l
$$
$$
+ \sum_{k=2}^{\infty} E[e^{-\Xi(D_{y,w} \times \mathbb{K})(k-1)l}(1 - e^{-\Xi(D_{y,w} \times \mathbb{K})l}) \Xi(D_{y,w} \cap D_{0,l} \times \mathbb{K})]2l
$$
$$
\leq 5l E[\Xi(D_{y,w} \cap D_{0,l} \times \mathbb{K})].
$$

Consequently, if there exists $c > 0$ such that

$$
(4.38) \qquad\qquad E[\Xi(D_{y,w} \times \mathbb{K})] \leq c|D_{y,w}|,
$$

then $E[\psi_l(\eta)] < \infty$, and the conclusions of Theorem A.2 and Theorem A.6 hold.

For solutions of the martingale problem for $A_{th,db}^{\lambda}$ or $A_{th,db}^{\infty}$, the initial level of each particle will be distinct, and we will index particles by their initial level. Each particle has birth time b_u, which we will take to be 0 for the particles in the

population at time 0, and an initial location $x_u = X_u(b_u)$ and a type κ_u which do not change with time.

Let $\mathcal{N} = \mathcal{N}(\mathbb{R}^d \times [0, \infty))$ be the space of counting measures on $\mathbb{R}^d \times [0, \infty)$. The evolution of the process is determined by a Poisson random measure ξ on $[0, \infty) \times \mathcal{N} \times \mathbb{R}^d \times [0, 1] \times [0, \infty)$ with mean measure

$$ds \, \nu^3(y, \zeta, w, d\gamma) \, dy \, \nu^1(w, d\zeta) \nu^2(dw),$$

where $\nu^3(y, \zeta, w, d\gamma)$ is the distribution of the Poisson random measure on $\mathbb{R}^d \times [0, \infty)$ with mean measure

$$\zeta \mathbf{1}_{D_{y,w}}(x) \, dx \, dv = \zeta |D_{y,w}| \upsilon_{y,w}(dx) \, dv.$$

Note that a "point" in ξ is of the form $\beta = (s, \{(x_k, v_k), k \geq 1\}, y, \zeta, w)$, where we will assume that the $\{(x_k, v_k)\}$ are indexed in increasing order of the v_k. Then, the birth times, locations and levels of "new" particles are given by

$$\mathcal{B} = \bigcup_{\beta \in \xi} \{(s, x_k, v_k), k \geq 1, v_k < \lambda\}.$$

Then $v^* \equiv v(\beta) = v_1$ and $x(\beta) = x_1$, and

$$\left(x^*, \kappa^*, u^*\right) \equiv \left(x^*(\beta, \eta), \kappa^*(\beta, \eta), u^*(\beta, \eta)\right)$$

is the point in η satisfying $x^* \in D_{y,w}$ and

$$u^* = \operatorname{argmax} \left\{ \frac{\lambda - u}{\lambda - v^*} : (x, \kappa, u) \in \eta, x \in D_{y,w}, u \geq v^* \right\}$$

$$\cup \left\{ \frac{u}{v^*} : (x, \kappa, u) \in \eta, x \in D_{y,w}, u \leq v^* \right\}.$$

Set $G = \mathcal{N} \times \mathbb{R}^d \times [0, 1] \times [0, \infty)$. By Theorem A.6, we have the following.

THEOREM 4.7. *For $0 < \lambda \leq \infty$, any solution of the martingale problem for $A^\lambda = A^\lambda_{th,db}$ given in* (4.9) *that satisfies*

$$E\left[\int_0^t \psi(\eta(s)) \, ds\right] < \infty \qquad \text{for all } t \geq 0,$$

can be obtained as a solution of the stochastic equation

$$f(\eta(t)) = f(\eta(0))$$

$$+ \int_{[0,t] \times G} \left[\frac{f(\eta(s-))}{f(\eta_{y,w}(s-))} \prod_{(x,u) \in \gamma(\beta)} g(x, \kappa^*(\beta, \eta(s-)), u) \right.$$

$$\times \prod_{(x,\kappa,u) \in \eta_{y,w}(s-), u \neq u^*(\beta, \eta(s-))} g\left(x, \kappa, \frac{\mathcal{J}^\lambda_{y,w}(x, u, \eta(s-), v(\beta))}{1 - \zeta}\right)$$

$$\left. - f(\eta(s-)) \right] \mathbf{1}_{\{v(\beta) < \lambda\}} \xi(ds, d\beta).$$

To construct a more useful system of equations, if $(x, \kappa, u) \equiv (x_u, \kappa_u, u) \in \eta(0)$, the level evolves by

$$U_u(t) = u + \int_{(0,t] \times G} \mathbf{1}_{D_{y,w}}(x_u)$$

(4.39)
$$\times \left(\frac{\mathcal{J}_{y,w}^{\lambda}(x_u, U_u(s-), \eta(s-), v(\beta))}{1 - \zeta} - U_u(s-) \right)$$

$$\times \mathbf{1}_{\{v(\beta) < \lambda\}} \xi(ds, d\beta),$$

and the particle dies at time

(4.40) $\qquad d_u = \inf\{t > 0 : U_u(t) > \lambda \text{ or } U_u(t-) = u^*(\beta, \eta(t-)), (t, \beta) \in \xi\}.$

If there is a birth/death event at time s,

$$(s, \beta) = (s, \{(x_k, v_k), k \geq 1\}, y, \zeta, w) \in \xi,$$

then for $u = v_k$, we set $x_u = x_k$ and $b_u = s$. The levels for the new particles satisfy

$$U_u(t) = u + \int_{(b_u, t] \times G} \mathbf{1}_{D_{y,w}}(x_u)$$

(4.41)
$$\times \left(\frac{\mathcal{J}_{y,w}^{\lambda}(x_u, U_u(s-), \eta(s-), v(\beta))}{1 - \zeta} - U_u(s-) \right)$$

$$\times \mathbf{1}_{\{v(\beta) < \lambda\}} \xi(ds, d\beta),$$

for $t \geq b_u$, and the type is given by $\kappa_u = \kappa^*(\beta, \eta(s-))$. Again, the particle dies at time d_u given by (4.40), so

$$\eta(t) = \sum \mathbf{1}_{[b_u, d_u)}(t) \delta_{(x_u, \kappa_u, U_u(t))}.$$

With reference to (3.5), passing to the limit as $\lambda \to \infty$, the equations become

$$U_u(t)$$
$$= u + \int_{(b_u, t] \times G} \mathbf{1}_{D_{y,w}}(x_u) \left[\frac{U_u(s-) - \mathbf{1}_{\{U_u(s-) \geq v(\beta)\}}(u^*(\beta, \eta(s-)) - v(\beta))}{1 - \zeta} \right.$$

$$\left. - U_u(s-) \right] \xi(ds, d\beta),$$

for $t \geq b_u$, and defining

(4.42) $\qquad \tau_u = \lim_{k \to \infty} \inf\{t : U_u(t) > k\},$

the particle dies at time

(4.43) $\qquad d_u = \tau_u \wedge \inf\{t > 0 : U_u(t-) = u^*(\beta, \eta(t-)), (t, \beta) \in \xi\}.$

Since the downward jumps in U_u, when they occur, will typically be $O(1)$, we can only allow finitely many per unit time. Conditional on U_u, the intensity of downward jumps is

$$\int_0^\infty \int_0^1 v_d w^d \big(1 - e^{-\zeta v_d w^d U_u(t)}\big) \nu^1(w, d\zeta) \nu^2(dw),$$

which is finite by (4.33). (Recall that v_d is the volume of the unit ball.) The cumulative effect of the upward jumps on $\log U_u$ is bounded by

$$-\int_{G_t} \mathbf{1}_{D_{y,w}}(x_u) \log(1 - \zeta) \xi(ds, d\beta),$$

which has expectation

$$-\int_0^t \int_0^\infty \int_0^1 v_d w^d \log(1 - \zeta) \nu^1(w, d\zeta) \nu^2(dw),$$

which is again finite by (4.33), that is, assuming (4.33), τ_u defined in (4.42) is infinite.

We are going to prove existence by a tightness and weak convergence argument, so we need to view ξ as a random variable in an appropriate metric space. Let $\varphi \in C_b([0, \infty) \times \mathbb{R}^d)$ be strictly positive and satisfy $\int_{\mathbb{R}^d} \int_0^\infty \varphi(s, y) \, ds \, dy < \infty$. Let $\mathbb{M}$ be the space of measures on $\mathbb{S} = [0, \infty) \times \mathcal{N} \times \mathbb{R}^d \times [0, 1] \times [0, \infty)$ and define convergence in $\mathbb{M}$ by the requirement that $\mu_n \to \mu$ if and only if

$$\int_{\mathbb{S}} \varphi(s, y) f(s, \gamma, y, \zeta, w) \mu_n(ds, d\gamma, dy, d\zeta, dw)$$

$$\to \int_{\mathbb{S}} \varphi(s, y) f(s, \gamma, y, \zeta, w) \mu(ds, d\gamma, dy, d\zeta, dw),$$

for all $f \in C_b(\mathbb{S})$. Then $\mathbb{M}$ is metrizable and complete.

THEOREM 4.8. *For $\lambda < \infty$, assume that with probability one, $\overline{\eta}^\lambda(0, K \times \mathbb{K}) < \infty$ for every compact $K \subset \mathbb{R}^d$ and that conditioned on $\overline{\eta}^\lambda(0)$, the levels in $\eta^\lambda(0)$ are independent and uniform on $[0, \lambda]$. Then existence holds for the solution of the system of stochastic equations (4.39) and (4.41), and hence for the corresponding martingale problem.*

For $\lambda = \infty$, assume that $\eta(0)$ is conditionally Poisson with Cox measure $\Xi(0) \times \ell$ on $\mathbb{R}^d$ and $\sup_{y \in \mathbb{R}^d} E[\Xi(0, D_{y,1} \times \mathbb{K})] < \infty$. For $\lambda < \infty$, let U^λ be a solution of the system (4.39) and (4.41) with $\eta^\lambda(0)$ the restriction of $\eta(0)$ to $u \in [0, \lambda]$. Then $\{(U^\lambda, \xi)\}$ is relatively compact in $D_{\mathbb{R}}[0, \infty)^\infty \times \mathbb{M}$ and any limit point is a solution for the system with $\lambda = \infty$. Consequently, existence holds for the $\lambda = \infty$ system of stochastic equations, and hence for the corresponding martingale problem, and along the convergent subsequence, the genealogies corresponding to U^λ converge to the genealogies of the limit.

PROOF.   Assume $\lambda < \infty$. There are only countably many particles that ever live, and the levels must satisfy the countable system of equations

$$U_u(t) = u + \int_{(b_u,t]\times G} \mathbf{1}_{D_{y,w}}(x_u)\left[\frac{\mathcal{J}^\lambda_{y,w}(x_u, U_u(s-), \eta(s-), v(\beta))}{1-\zeta} - U_u(s-)\right]$$
$$\times \mathbf{1}_{\{v(\beta)<\lambda\}}\xi(ds, d\beta),$$

including the initial particles with $b_u = 0$.

Let U_u^ε satisfy

$$U_u^\varepsilon(t)$$
$$= u + \int_{(b_u,t]\times G} \mathbf{1}_{D_{y,w}}(x_u)\left[\frac{\mathcal{J}^\lambda_{y,w}(x_u, U_u^\varepsilon(s-), \eta^\varepsilon([s/\varepsilon]\varepsilon), v(\beta))}{1-\zeta} - U_u^\varepsilon(s-)\right]$$
$$\times \mathbf{1}_{\{v(\beta)<\lambda\}}\xi(ds, d\beta).$$

With probability one, no jump in ξ occurs at times of the form $[s/\varepsilon]\varepsilon$, and it follows that U^ε is uniquely determined. On any bounded time interval, each particle is involved in only finitely many events, that is, U_u^ε jumps only finitely often, and the jumps are bounded. Consequently, $\{(U^\varepsilon, \xi)\}$ is relatively compact in $D_\mathbb{R}[0,\infty)^\infty \times \mathbb{M}$ in the sense of convergence in distribution. Selecting a convergent subsequence with limit (U, ξ), the only issue is the continuity of $\mathcal{J}^\lambda_{y,w}$. Suppose $(\beta, t) \in \xi$. Then since $\mathcal{J}^\lambda_{y,w}$ only depends on finitely many of the U_u, and, with probability one, no particle locations are on the boundary of $D_{y,w}$, the necessary continuity will be satisfied if $U_{u_1}(t-) \neq U_{u_2}(t-)$ for all u_1 and u_2 with $x_{u_1}, x_{u_2} \in D_{y,w}$ and there are no ties in the determination of $u^*(\beta, \eta)$. But the first requirement holds since $U_{u_1}(t-)$ and $U_{u_2}(t-)$ will be independent and uniform and the second holds since $v(\beta)$ will be independent of $U(t-)$.

Essentially the same argument works for the relative compactness of $\{(U^\lambda, \xi)\}$ and taking a convergent subsequence, we obtain existence for $\lambda = \infty$ and convergence of the genealogies.   □

REMARK 4.9.   At this point, we do not have a uniqueness result for the martingale problem or the stochastic equations. This question will be pursued elsewhere.

4.2. *Spatial Λ-Fleming–Viot process with general offspring distribution.*   In the discrete birth/independent thinning model described in the previous section, the offspring distribution was Poisson and the model was constructed so that for $\lambda = \infty$, the locations and levels of the particles form a spatial Poisson process that is stationary in time. We now drop the Poisson assumption and allow an offspring distribution restricted only by the requirement that the expected number of offspring for an event $z = (y, \zeta, w)$ in the ball $D_{y,w}$ with thinning probability ζ is

$$\sum_{k=0}^\infty kp(k, z) = \lambda\zeta|D_{y,w}|.$$

To avoid the uniqueness problem mentioned in Remark 4.9, we replace $\mathbb{R}^d$ by a torus $\mathbb{T}$. Taking $\mathbb{U} = \mathbb{T} \times [0,1] \times [0,\infty)$ and setting $\mu(dy, d\zeta, dw) = dy\,v^1(w, d\zeta)v^2(dw)$, we assume $\mu(\mathbb{U}) < \infty$ and define

$$A^\lambda f(\eta) = \int_{\mathbb{U}} \sum_{k=1}^\infty p(k,z)\big(H_{k,z}^\lambda(g,\eta) - f(\eta)\big)\,dy\,v^1(w, d\zeta)v^2(dw),$$

where as before, if $\eta(D_{y,w} \times \mathbb{K}) = 0$, $H_{k,z}^\lambda(g,\eta) = f(\eta)$, and if $\eta(D_{y,w} \times \mathbb{K}) \neq 0$,

$$H_{k,z}^\lambda(g,\eta)$$

$$= \prod_{(x,\kappa,u)\in\eta,\,x\notin D_{y,w}} g(x,\kappa,u)$$

$$\times \int_0^\lambda \left[\frac{k}{\lambda}\left(1 - \frac{v^*}{\lambda}\right)^{k-1} \widehat{g}_{y,w}(\kappa^*, v^*)\left(\frac{1}{\lambda - v^*}\int_{v^*}^\lambda \widehat{g}_{y,w}(\kappa^*, v)\,dv\right)^{k-1} \right.$$

$$\times \left. \prod_{(x,\kappa,u)\in\eta,\,x\in D_{y,w},\,u\neq u^*} g\left(x,\kappa, \frac{1}{1-\zeta}\mathcal{J}_{y,w}^\lambda(x,u,\eta,v^*)\right) \right] dv^*,$$

where as before $\widehat{g}_{y,w}(\kappa,u) \equiv \int g(x',\kappa,u)v_{y,w}(dx')$.

Again, (x^*, κ^*, u^*) is the point in η satisfying $x^* \in D_{y,w}$ and

$$u^* = \operatorname{argmax}\left\{ \frac{\lambda - u}{\lambda - v^*} : (x,\kappa,u) \in \eta, x \in D_{y,w}, u \geq v^* \right\}$$

$$\cup \left\{ \frac{u}{v^*} : (x,\kappa,u) \in \eta, x \in D_{y,w}, u \geq v^* \right\},$$

and $\mathcal{J}_{y,w}^\lambda(x,u,\eta,v^*)$ is obtained as in (4.31).

Recalling that $\overline{g}_{y,w}(\kappa) = \lambda^{-1}\int_0^\lambda \int g(x,\kappa,u)v_{y,w}(dx)\,du$ and averaging, we define

$$\mathcal{H}_{k,z}^\lambda(\overline{g}, \overline{\eta}) = \frac{1}{|\overline{\eta}_{|D_{y,w}}|} \sum_{(x^*,\kappa^*)\in\overline{\eta}_{|D_{y,w}}} \overline{g}_{y,w}(\kappa^*)^k \frac{1}{\overline{g}(x^*,\kappa^*)}$$

$$\times \prod_{(x,\kappa)\in\overline{\eta}_{|D_{y,w}},\,(x,\kappa)\neq(x^*,\kappa^*)} \left((1-\zeta) + \zeta\frac{1}{\overline{g}(x,\kappa)}\right)$$

and obtain

$$\alpha A^\lambda f(\overline{\eta}) = \alpha f(\overline{\eta})\int_{\mathbb{U}} \sum_{k=1}^\infty p(k,z)\big(\mathcal{H}_{k,z}^\lambda(\overline{g}, \overline{\eta}) - 1\big)\,dy\,v^1(w, d\zeta)v^2(dw).$$

To obtain a limit as $\lambda \to \infty$, for each z, let $\mu(dq, z)$ be a probability distribution on $[0,\infty)$ satisfying

$$\int_0^\infty q\mu(dq, z) = \alpha_z \equiv \zeta|D_{y,w}|,$$

and assume that as $\lambda \to \infty$, for each $\varphi \in C_b(\mathbb{R})$,

$$\sum_k \varphi\left(\frac{k}{\lambda}\right) p^\lambda(k, z) \to \int_0^\infty \varphi(q)\mu(dq, z).$$

These conditions imply

$$\sum_k p^\lambda(k, z) \int_0^\lambda \frac{k}{\lambda}\left(1 - \frac{v^*}{\lambda}\right)^{\lambda\frac{k-1}{\lambda}} f(v^*)\, dv^*$$

$$\to \int_0^\infty \int_0^\infty q e^{-qv^*} f(v^*)\, dv^* \mu(dq, z).$$

Observing that $\frac{k}{\lambda} \to q$ implies

$$\left(\frac{1}{\lambda - v^*} \int_{v^*}^\lambda \widehat{g}_{y,w}\left(\kappa^*, \frac{1}{1-\zeta}v\right) dv\right)^{k-1} \to \exp\left\{-q \int_{v^*}^\infty (1 - \widehat{g}_{y,w}(\kappa^*, v)\, dv)\right\}$$

$$= \exp\{-q(\widehat{h}_{y,w}(\kappa^*, v^*)\},$$

where $\widehat{h}_{y,w}(\kappa, u) = \int_u^\infty (1 - \widehat{g}_{y,w}(\kappa, v))\, dv$, it follows that $\sum_k p^\lambda(k, z) H^\lambda_{k,z}(g, \eta)$ converges to

$$H_z(g, \eta) = \prod_{(x,\kappa,u)\in\eta, x\notin D_{y,w}} g(x, \kappa, u)$$

$$\times \int_0^\infty \int_0^\infty \left[q e^{-qv^*} \widehat{g}_{y,w}(\kappa^*, v^*) \exp\{-q(1-\zeta)\widehat{h}_{y,w}(\kappa^*, v^*)\} \right.$$

$$\times \prod_{(x,\kappa,u)\in\eta, x\in D_{y,w}, u>u^*} g\left(x, \kappa, \frac{1}{1-\zeta}(u - (u^* - v^*))\right)$$

$$\times \left. \prod_{(x,\kappa,u)\in\eta, x\in D_{y,w}, u<u^*} g\left(x, \kappa, \frac{1}{1-\zeta}u\right)\right] dv^* \mu(dq, z)$$

and

(4.44) $$A^\infty f(\eta) = \int_{\mathbb{U}} (H_z(g, \eta) - f(\eta))\, dy\nu^1(w, d\zeta)\nu^2(dw).$$

As before, setting $h^*_{y,w}(\kappa) = \int_0^\infty (1 - \widehat{g}_{y,w}(\kappa, u))\, du$ and

$$\mathbb{H}_3(h^*_{y,w}, q, \Xi, y, \zeta, w) = \frac{1}{\Xi(D_{y,w} \times \mathbb{K})} \int_{D_{y,w} \times \mathbb{K}} e^{-q(1-\zeta)h^*_{y,w}(\kappa)} \Xi(dx \times d\kappa),$$

for $f(\Xi) = e^{-\int h(x,\kappa)\Xi(dx,d\kappa)}$, we have

$$\alpha A^\infty f(\Xi) = e^{-\int h(x,\kappa)\Xi(dx,d\kappa)}$$

$$\times \int_{[0,\infty)\times\mathbb{R}^d\times[0,1]\times[0,\infty)} \left[\mathbb{H}_3(h^*_{y,w}, q, \Xi, y, \zeta, w)\right.$$

$$\times e^{\zeta \int_{D_{y,w} \times \mathbb{K}} h(x,\kappa) \Xi(dx,d\kappa)} - 1]$$

$$\times \mu(dq,z) \, dy \nu^1(w,d\zeta) \nu^2(dw).$$

Since we are assuming that $\mu(\mathbb{U}) < \infty$, the martingale problems for the A^λ and A^∞ are well-posed, and we have the following.

THEOREM 4.10. *If η^λ, $0 < \lambda < \infty$, is a solution of the martingale problems for A^λ, and $\eta^\lambda(0) \Rightarrow \eta(0)$, then η^λ converges in distribution to the unique solution of the martingale problem for A^∞ with initial distribution the distribution of $\eta(0)$.*

If $\mu(dq,z)$ is degenerate for every z, that is, $\mu(dq,z) = \delta_{\alpha_z}$, then (4.44) is the same as (4.11). Of course, if $\{p^\lambda(k,z)\}$ is the Poisson distribution with mean $\lambda \zeta |D_{y,w}|$, then degeneracy holds. However, we can also construct nondegenerate examples, for example, by choosing a geometric offspring distribution, in which case $\mu(dq,z)$ is exponential.

For $\lambda < \infty$, let $\mathcal{N}^\lambda$ be the collection of counting measures on $\mathbb{T} \times [0,\infty)$ and let ξ^λ be a Poisson random measure on $[0,\infty) \times \mathcal{N}^\lambda \times \mathbb{T} \times [0,1] \times [0,\infty)$ with mean measure

$$ds\nu^3(\{p(k,z)\}, y, w, d\gamma) \, dy \nu^1(w,d\zeta) \nu^2(dw),$$

where $\nu^3(\{p(k,z)\}, y, w, d\gamma)$ is the probability distribution on $\mathcal{N}^\lambda$ of the point process

$$\sum_{i=1}^{K} \delta_{(X_i, V_i)},$$

where K is integer-valued with distribution $\{p(k,z)\}$ and the (X_i, V_i) are independent and uniformly distributed over $D_{y,w} \times [0,\lambda]$.

For $\lambda = \infty$, let ξ be a Poisson random measure on $[0,\infty) \times \mathcal{N} \times [0,\infty) \times \mathbb{T} \times [0,1] \times [0,\infty)$ with mean measure

$$ds\nu^3(q, y, w, d\gamma)\mu(dq,z) \, dy \nu^1(w,d\zeta) \nu^2(dw),$$

where $\nu^3(q, y, w, d\gamma)$ is the probability distribution of the Poisson random measure on $\mathbb{T} \times [0,\infty)$ with mean measure

$$q\mathbf{1}_{D_{y,w}}(x) \, dx \, dv = q|D_{y,w}|\upsilon_{y,w}(dx) \, dv.$$

Under our boundedness assumption, we can take $\psi \equiv 1$ in Theorem A.2 and in Theorem A.6. The form of the stochastic equation is the same as in the previous section.

Set $G^\lambda = \mathcal{N}^\lambda \times \mathbb{T} \times [0,1] \times [0,\infty)$.

LEMMA 4.11.   *Any solution of the martingale problem for A^λ can be obtained as a solution of the stochastic equation*

$$
f\big(\eta(t)\big) = f\big(\eta(0)\big)
$$
$$
+ \int_{[0,t]\times G^\lambda} \Big(\frac{f(\eta(s-))}{f(\eta_{y,w}(s-))} \prod_{(x,u)\in\gamma(\beta)} g\Big(x, \kappa^*(\beta, \eta(s-), u)
$$
$$
\times \prod_{(x,\kappa,u)\in\eta(s-),\, u\neq u^*(\eta(s-),v^*(\beta))} g\Big(x, \kappa, \frac{\mathcal{J}^\lambda_{y,w}(x,u,\eta(s-),v^*)}{1-\zeta}\Big)
$$
$$
- f\big(\eta(s-)\big)\Big)\xi^\lambda(ds, d\beta).
$$

For $\lambda = \infty$, the equation is the same with G^λ replaced by G, ξ^λ replaced by ξ and $\mathcal{J}^\lambda_{y,w}$ replaced by $\mathcal{J}^\infty_{y,w}$.

As before, the level processes satisfy

$$
U_u(t) = u + \int_{(b_u,t]\times G} \mathbf{1}_{D_{y,w}}(x_u)\Big(\frac{\mathcal{J}^\lambda_{y,w}(x_u, U_u(s-), \eta(s-), v(\beta))}{1-\zeta}
$$
$$
\tag{4.45}
$$
$$
- U_u(s-)\Big)\xi^\lambda(ds, d\beta),
$$

where $b_u = 0$ if $(x,\kappa,u) \in \eta^\lambda(0)$, and the death time of a particle satisfies

$$
\tag{4.46} \quad d_u = \inf\{t > 0 : U_u(t) > \lambda \text{ or } U_u(t-) = u^*(\beta, \eta(t-)), (t, \beta) \in \xi\}.
$$

Passing to the $\lambda = \infty$ limit, we can derive the equation for the population distribution. Let $\Xi(t, dx, d\kappa)\, du$ be the Cox measure for $\eta(t)$. Define

$$
P(t, C) = \Xi(t, C \times \mathbb{K}), \qquad C \in \mathcal{B}(\mathbb{T}).
$$

If $P(0, dx) = P(0, x)\, dx$, that is, $P(0, \cdot)$ is absolutely continuous with respect to Lebesgue measure, then since locations of new points are uniformly distributed over disks, $P(t, dx) = P(t, x)\, dx$ for all $t \geq 0$. Since $\mathbb{T} \times \mathbb{K}$ is a complete, separable metric space, we can write

$$
\Xi(t, dx, d\kappa) = P(t, x)\Xi_x(t, d\kappa)\, dx,
$$

where $\Xi_x(t, \cdot) \in \mathcal{P}(\mathbb{K})$.

THEOREM 4.12.   *For $\lambda = \infty$, if $\eta(0) = \sum_{(x,\kappa,u)\in\eta(0)} \delta_{(x,\kappa,u)}$ is conditionally Poisson with Cox measure $\Xi(0, dx, d\kappa)\, du = P(0, x)\Xi_x(0, d\kappa)\, dx\, du$, then $\eta(t)$ is conditionally Poisson with Cox measure $\Xi(t, dx, d\kappa)\, du$, $\Xi(t, dx, d\kappa) =$*

$P(t, x) \Xi_x(t, d\kappa) \, dx$, where $\Xi_x(t, \mathbb{K}) \equiv 1$. Then (dropping the γ coordinate from ξ),

$$P(t, x) = P(0, x) + \int_{[0,t] \times [0,\infty) \times \mathbb{T} \times [0,1] \times [0,\infty)} \left(\frac{q}{|D_{y,w}|} - \zeta P(s-, x) \right)$$

$$\times \mathbf{1}_{D_{y,w}}(x) \xi(ds, dq, dy, d\zeta, dw).$$

REMARK 4.13. Note that in the degenerate case, $q \equiv \alpha_z = \zeta |D_{y,w}|$, and $P(t, x) \equiv 1$ is a solution of this equation.

To write an equation including Ξ_x, we need to enrich ξ so that each point includes a coordinate that is independent and uniformly distributed over $[0, 1]$, that is, for $\widehat{G} = [0, 1] \times [0, \infty) \times \mathbb{T} \times [0, 1] \times [0, \infty)$, we let ξ be the Poisson random measure on $[0, \infty) \times \widehat{G}$ with mean measure

$$ds \, dr \, \mu(dq, z) \, dy \nu^1(w, d\zeta) \nu^2(dw).$$

Let $K : [0, 1] \times \mathcal{P}(\mathbb{K}) \to \mathbb{K}$ be a measurable function such that if R is uniformly distributed over $[0, 1]$ and $\rho \in \mathcal{P}(\mathbb{K})$, then $K(R, \rho)$ has distribution ρ. Note that if an event $z = (y, \zeta, w)$ occurs at time t, then the distribution of the type of the parent will be

$$\int_{D_{y,w}} \Xi_{x'}(t-, \cdot) \upsilon_{y,w}(dx').$$

THEOREM 4.14. For $\varphi \in C_c(\mathbb{T} \times \mathbb{K})$,

$$\langle \Xi(t), \varphi \rangle = \langle \Xi(0), \varphi \rangle$$

$$+ \int_{[0,t] \times \widehat{G}} \left[q \int_{D_{y,w}} \varphi \left(x, K \left(r, \int_{D_{y,w}} \Xi_{x'}(s-, \cdot) \upsilon_{y,w}(dx') \right) \right) \upsilon_{y,w}(dx) \right.$$

$$\left. - \zeta \langle \Xi(s-), \mathbf{1}_{D_{y,w}} \varphi \rangle \right] \xi(ds, dr, dq, dy, d\zeta, dw).$$

REMARK 4.15. The above construction is more than complicated enough at least for a first reading, but still keep in mind that the parameters of the this model, as well as other kinds of population models, could be taken to be functions of $\overline{\eta}$ for $\lambda < \infty$ or Ξ for $\lambda = \infty$. For example, $\mu(dq, z)$ could be replaced by $\mu(dq, z, \Xi(t))$, or in a genealogical construction of the model introduced by Bolker and Pacala (1999), the death rate would be $d_0(x, \overline{\eta}) = \int d(x - y) \overline{\eta}(dy)$. Equally, we could consider frequency dependent selection, in which the strength of selection in favour of a particular genetic type at a specific location depends on the current frequency of types there. For example, Forien and Penington (2017) consider the spatial Λ-Fleming–Viot model for a haploid population with general frequency dependent selection. Variations like this lead to a rich class of models in which we can combine the forces of ecology and genetics.

4.3. *Branching processes.*   Next, we recover a lookdown construction for the Dawson–Watanabe superprocess. Let $A_{cb,k}$ be given by (3.7), and let $A_{pd,k}$ be the pure death generator with $d_0(x) = r(x)k$. Let $\mathcal{D}_\lambda$ be defined as in (4.1) with $\mathbb{R}^d$ replaced by E, and let $\mathcal{D}(A^\lambda) = \{f \in \mathcal{D}_\lambda : \partial_u g$ is continuous$\}$. Then, recalling the definition of $G_k^\lambda(u)$ from (3.9),

$$
\begin{aligned}
&A^\lambda f(\eta) \\
&\quad = \lambda\big(A_{cb,k} f(\eta) + A_{pd,k} f(\eta)\big) \\
&\quad = f(\eta) \sum_{(x,u)\in\eta} \lambda r(x) \left[\frac{(k+1)}{\lambda^k} \int_u^\lambda \cdots \int_u^\lambda \left(\prod_{i=1}^k g(x, v_i) - 1 \right) dv_1 \cdots dv_k \right. \\
&\qquad\qquad\qquad\qquad \left. + (G_k^\lambda(u) + ku) \frac{\partial_u g(x, u)}{g(x, u)} \right] \\
&\quad \to f(\eta) \sum_{(x,u)\in\eta} r(x)(k+1)k \left(\int_u^\infty (g(x, v) - 1)\, dv + \frac{1}{2} u^2 \frac{\partial_u g(x, u)}{g(x, u)} \right) \\
&\quad = A^\infty f(\eta)
\end{aligned}
$$

and

$$
\alpha A^\infty f(\Xi) = e^{-\int_E h(x)\Xi(dx)} \int_E r(x) \frac{k(k+1)}{2} h^2(x)\,\Xi(dx),
$$

which is the generator of a Dawson–Watanabe process without any spatial motion [see Section 1.5 of Etheridge (2000) or Section 3.4 of Kurtz and Rodrigues (2011)]. Note that for finite λ, each birth event produces k offspring.

For more general offspring distribution, one can take

$$
\begin{aligned}
A^\lambda f(\eta) &= \lambda \int_{\mathbb{U}} \big(A_{cb,k(z)} f(\eta) + A_{pd,k(z)} f(\eta)\big) \mu(dz) \\
&= f(\eta) \int_{\mathbb{U}} \sum_{(x,u)\in\eta} \lambda r(x, z) \left[\frac{(k(z) + 1)}{\lambda^{k(z)}} \right. \\
&\qquad\qquad\qquad\quad \times \int_u^\lambda \cdots \int_u^\lambda \left(\prod_{i=1}^{k(z)} g(x, v_i) - 1 \right) dv_1 \cdots dv_{k(z)} \\
&\qquad\qquad\qquad\quad \left. + (G_{k(z)}^\lambda(u) + k(z)u) \frac{\partial_u g(x, u)}{g(x, u)} \right] \mu(dz) \\
&\to f(\eta) \int_{\mathbb{U}} \sum_{(x,u)\in\eta} r(x, z)\big(k(z) + 1\big)k(z) \\
&\qquad\qquad\qquad \times \left(\int_u^\infty (g(x, v) - 1)\, dv + \frac{1}{2} u^2 \frac{\partial_u g(x, u)}{g(x, u)} \right) \mu(dz) \\
&= A^\infty f(\eta),
\end{aligned}
$$

assuming $\sup_{x \in E} \int_{\mathbb{U}} r(x, z)(k(z) + 1)k(z)\mu(dz) < \infty$. We can take ψ in Theorem A.2 to be of the form $\sum_l \delta_l \eta(K_l \times [0, l])$ for appropriately selected δ_l.

This construction is a special case of the results in Kurtz and Rodrigues (2011) which considers more general offspring distributions (e.g., offspring distributions without second moments), and other variants of branching processes including random environments and processes conditioned on extinction and nonextinction.

4.4. *Spatially interacting Moran model.* Consider $A_{dr,3}$, as defined in Section 3.5, in the special case in which the sum is over all subsets with $|S| = 2$. In other words, each replacement event involves just two individuals. Specifically, we take $r(S, z) = r(x, x')$ for $S = \{x, x'\}$. We include independent motion with generator $B \subset C_b(E) \times C_b(E)$, set $q(x, z, dy) = \delta_x(dy)$, and assume $r(x, x') = r(x', x)$. (Note that this symmetry is needed for $\alpha A f$ to be a generator applied to αf.) The generator becomes

$$
(4.47) \quad
\begin{aligned}
Af(\eta) = {}& f(\eta) \sum_{(x,u) \in \eta} \frac{Bg(x, u)}{g(x, u)} \\
& + f(\eta) \sum_{(x,u) \neq (x',u') \in \eta} r(x, x') \mathbf{1}_{\{u' < u\}} \left(\frac{g(x', u)}{g(x, u)} - 1 \right)
\end{aligned}
$$

for

$$
f \in \mathcal{D}(A) = \{ f \in \mathcal{D}_\lambda : g \in \mathcal{D}(B) \}
$$

and

$$
(4.48) \quad
\begin{aligned}
\alpha A f(\overline{\eta}) = {}& \alpha f(\overline{\eta}) \sum_{x \in \overline{\eta}} \frac{B\overline{g}(x)}{\overline{g}(x)} \\
& + \alpha f(\overline{\eta}) \sum_{\{x,x'\} \subset \overline{\eta}} r(x, x') \left(\frac{1}{2} \frac{\overline{g}(x')}{\overline{g}(x)} + \frac{1}{2} \frac{\overline{g}(x)}{\overline{g}(x')} - 1 \right),
\end{aligned}
$$

that is, at rate $r(x, x')$ one of the pair is killed and replaced by a copy of the other.

Since either particles move or a particle of one type is replaced by a particle of another type, if the initial number of particles is finite then, as in the classical Moran model, the total number of particles is preserved. Consequently, if $r(x, x')$ is bounded, we can apply Theorem A.2 with $\psi(\eta) = 1 + |\eta|^2$. If the number of particles is infinite, the following condition is useful.

CONDITION 4.16. Let $\mathcal{K} = \{K_1, K_2, \ldots\}$, $K_k \subset E$. For each $f \in \mathcal{D}(A)$, $f(\eta) = \prod_{(x,u) \in \eta} g(x, u)$, there exists $K_g \in \mathcal{K}$ such that $g(x, u) = 1$ and $Bg(x, u) = 0$ for all $x \notin K_g$ and $r(x, x') = 0$ for $x \notin K_g$ and x' in the support of $1 - g$.

LEMMA 4.17.   *Assume Condition 4.16. Then for each $f \in \mathcal{D}(A)$, there exists c_f such that*

$$\left| Af(\eta) \right| \leq c_f \left(\overline{\eta}(K_g) + \int_{K_g \times K_g} r(x, x') \overline{\eta}(dx) \overline{\eta}(dx') \right).$$

Then for $\delta_k > 0$, $k = 1, 2, \ldots$, ψ of the form

$$\psi(\eta) = \sum_k \delta_k \left(\overline{\eta}(K_k) + \int_{K_k \times K_k} r(x, x') \overline{\eta}(dx) \overline{\eta}(dx') \right)$$

satisfies (A.7).

REMARK 4.18.   Of course, to apply Theorem A.2 one must verify that

(4.49) $$\int_0^t E\big[\widetilde{\psi}(\overline{\eta}(s)) \big] \, ds < \infty, \qquad t \geq 0$$

for the solution of interest. For example, in the spatially interacting Moran model in Greven, Limic and Winter (2005), particles have a location and type $[(x, \kappa) \in E = G \times \mathbb{K}$ rather than $x]$ for a countable set G,

$$r\big((x, \kappa), (x', \kappa')\big) = \gamma \mathbf{1}_{\{x = x'\}},$$

the locations evolve independently according to a Markov chain with transition intensities $q(x, y)$, that is,

$$Bg(x, \kappa) = \sum_{y \in G} q(x, y)\big(g(y, \kappa) - g(x, \kappa)\big) + Cg(x, \kappa),$$

where C is a mutation operator that acts only on the type. The location Markov chain is assumed to satisfy estimates that imply $E[\eta(t, \{x\} \times \mathbb{K})^2] < \infty$ provided $\eta(0)$ satisfies specified conditions. Consequently, if we take $K_k = G_k \times \mathbb{K}$ for finite subsets G_k, we can select δ_k so that (4.49) is satisfied.

Note that λ does not appear in the formula for the generator (4.47). Consequently, the same formula gives the limiting generator as $\lambda \to \infty$, and with reference to (A.5),

$$\alpha A^{\infty} f(\Xi)$$

$$= e^{-\int_E h(x) \Xi(dx)} \left[-\int_E Bh(x) \Xi(dx) \right.$$

$$\left. + \int_{E \times E} r(x, x') \left(\frac{1}{2} h^2(x) + \frac{1}{2} h^2(x') - h(x')h(x) \right) \Xi(dx) \Xi(dx') \right].$$

For $\lambda = \infty$, if the number of particles below any level is finite, we can take $\psi(\eta) = \sum_{l=1}^{\infty} \delta_l (1 + \eta(E \times [0, l])^2)$. If the number of particles below a level is

infinite, then ψ of the form

$$\psi(\eta) = \sum_{k,l} \delta_{k,l}\bigg(\eta\big(K_k \times [0,l]\big)$$

$$+ \int_{K_k \times [0,l] \times K_k \times [0,l]} r(x,x')\eta(dx,du)\eta(dx',du')\bigg)$$

meets the requirements of Theorem A.2.

For the limiting process, one can also see that mass is preserved directly from the limiting generator. Suppose Ξ is a solution of the martingale problem with $\Xi(0,E) < \infty$. Take $h(x) \equiv c > 0$, and observe that $e^{-c\Xi(t,E)}$ is a martingale. But, in general, if M and M^2 are both martingales, then M must be constant, so consider $e^{-c\Xi(t,E)}$ and $e^{-2c\Xi(t,E)}$.

If $r(x,x') \equiv \gamma$ and $\Xi(0,E) = 1$, then Ξ is a neutral Fleming–Viot process. Since the set of levels is fixed, in this case, the lookdown construction is equivalent to the construction given in Donnelly and Kurtz (1996). If as above, $r((x,\kappa),(x',\kappa')) = \gamma \mathbf{1}_{\{x=x'\}}$, then the lookdown construction for $\lambda = \infty$ is just the lookdown construction for the interacting Fisher–Wright diffusions discussed in Greven, Limic and Winter (2005).

4.5. *A stochastic partial differential equation.* Consider a spatially interacting Moran model with both location $x \in \lambda^{-1}\mathbb{Z}$ and type $\kappa \in \mathbb{K}$. Assume that the particle locations follow a simple symmetric random walk, and for simplicity, assume that the types of the particles do not change. Killing and replacement of the previous section now takes place locally at each site. The generator then becomes

$$Af(\eta) = f(\eta) \sum_{(x,\kappa,u)\in\eta} \lambda^2 \frac{g(x+\lambda^{-1},\kappa,u) + g(x-\lambda^{-1},\kappa,u) - 2g(x,\kappa,u)}{2g(x,\kappa,u)}$$

$$+ f(\eta) \sum_{(x,\kappa,u)\neq(x',\kappa',u')\in\eta} \lambda \mathbf{1}_{\{x=x'\}}\mathbf{1}_{\{u'<u\}}\bigg(\frac{g(x,\kappa',u)}{g(x,\kappa,u)} - 1\bigg).$$

Note that particles move independently, so that the number of particles at a site will fluctuate; however, if the initial site occupancies are i.i.d. Poisson, then they will remain i.i.d. Poisson. The averaged generator becomes

$$\alpha Af(\eta) = \alpha f(\overline{\eta}) \sum_{(x,\kappa)\in\overline{\eta}} \lambda^2 \frac{\overline{g}(x+\lambda^{-1},\kappa) + \overline{g}(x-\lambda^{-1},\kappa) - 2\overline{g}(x,\kappa)}{2\overline{g}(x,\kappa)}$$

$$+ \alpha f(\overline{\eta}) \sum_{(x,\kappa)\neq(x',\kappa')\in\overline{\eta}} \frac{\lambda}{2}\mathbf{1}_{\{x=x'\}}\bigg(\frac{\overline{g}(x,\kappa')}{\overline{g}(x,\kappa)} - 1\bigg)$$

[cf. (4.47) and (4.48)].

Let $(X_u^\lambda(t), \kappa_u(t))$ denote the position and type of a particle at level u, Assume that $\{(X_u^\lambda(0), \kappa_u(0), u)\}$ determines a conditionally Poisson random measure with Cox measure $\lambda^{-1} \times \ell^\lambda(dx) \times \nu_0(x, d\kappa) \times du$ on $(\lambda^{-1}\mathbb{Z} \times \mathbb{K} \times [0, \lambda])$, where ℓ^λ is counting measure on $\lambda^{-1}\mathbb{Z}$ and ν_0 is a random mapping $\nu_0 : x \in \mathbb{R} \to \nu_0(x, \cdot) \in \mathcal{P}(\mathbb{K})$. Note that as $\lambda \to \infty$, the $\{X_u^\lambda - X_u^\lambda(0)\}$ converge to independent standard Brownian motions $\{W_u\}$.

For $u' < u$, let $L_{u'u}^\lambda(t)$ be the number of times by time t that there has been a "lookdown" from u to u'. Then $L_{u'u}^\lambda$ is a counting process with integrated intensity

$$\Lambda_{u'u}^\lambda(t) = \lambda \int_0^t \mathbf{1}_{\{X_u^\lambda(s) = X_{u'}^\lambda(s)\}} \, ds,$$

and we can write

$$L_{u'u}(t) = Y_{u'u}\big(\Lambda_{u'u}^\lambda(t)\big),$$

where the $Y_{u'u}$ are independent unit Poisson processes and are independent of $X_{u'u}^\lambda(t) \equiv X_{u'}^\lambda(t) - X_u^\lambda(t)$. To identify the limit of $\Lambda_{u'u}^\lambda$ as $\lambda \to \infty$, define

$$N_{u'u}^\lambda(t) = \#\{s \le t : X_{u'u}^\lambda(s-) = 0, X_{u'u}^\lambda(s) \ne 0\}.$$

Then $N_{u'u}^\lambda$ is a counting process with intensity $\lambda^2 \mathbf{1}_{\{X_{u'u}^\lambda(t) = 0\}}$. Define

$$\widetilde{N}_{u'u}^\lambda(t) = N_{u'u}^\lambda(t) - \int_0^t \lambda^2 \mathbf{1}_{\{X_{u'u}^\lambda(s) = 0\}} \, ds.$$

Then

$$\big|X_{u'u}^\lambda(t)\big| = \big|X_{u'u}^\lambda(0)\big| + \int_0^t \mathrm{sign}\big(X_{u'u}^\lambda(s-)\big) \, dX_{u'u}^\lambda(s) + \frac{1}{\lambda} \widetilde{N}_{u'u}^\lambda(t)$$

$$+ \lambda \int_0^t \mathbf{1}_{\{X_{u'u}^\lambda(s) = 0\}} \, ds.$$

Since $X_{u'u}^\lambda \Rightarrow X_{u'u} = X_{u'} - X_u$ and $\lambda^{-1} \widetilde{N}_{u'u}^\lambda \Rightarrow 0$, it follows that $X_{u'u}^\lambda$ and $\Lambda_{u'u}^\lambda = \lambda \int_0^t \mathbf{1}_{\{X_{u'u}^\lambda(s) = 0\}} \, ds$ converge to $X_{u'u}$ and $\Lambda_{u'u}$, respectively, satisfying Tanaka's formula

(4.50) $$\big|X_{u'u}(t)\big| = \big|X_{u'u}(0)\big| + \int_0^t \mathrm{sign}\big(X_{u'u}(s-)\big) \, dX_{u'u}(s) + \Lambda_{u'u}(t).$$

An application of Itô's formula gives

(4.51) $$\Lambda_{u'u}(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{(-\varepsilon, \varepsilon)}\big(X_{u'}(s) - X_u(s)\big) \, ds.$$

To summarize, $\{(X_u(0), \kappa_u(0), u)\}$ determines a conditionally Poisson random measure with Cox measure $dx \times \nu_0(x, d\kappa) \times du$ and $X_u(t) = X_u(0) + W_u(t)$, where the W_u are independent, standard Brownian motions. $L_{u'u}$ is determined by (4.50) and

$$L_{u'u}(t) = Y_{u'u}\big(\Lambda_{u'u}(t)\big),$$

where the $Y_{u'u}$ are independent unit Poisson processes that are independent of $\{(X_u(0), \kappa_u, u)\}$ and $\{W_u\}$. The particle types satisfy

$$\kappa_u(t) = \kappa_u(0) + \sum_{u' < u} \int_0^t \left(\kappa_{u'}(s-) - \kappa_u(s-) \right) dL_{u'u}(s).$$

Then $\{(X_u(t), \kappa_u(t), u)\}$ determines a conditionally Poisson random measure with Cox measure

$$\Xi_t(dx, d\kappa) \times du = dx \times \nu_t(x, d\kappa) \times du.$$

For details and related results, see Buhr (2002). In particular, for $\varphi(x, \kappa)$ bounded, C^2 in x, and having compact support in x,

$$M_\varphi(t) = \langle \Xi_t, \varphi \rangle - \int_0^t \left\langle \Xi_s, \frac{1}{2} \partial_x^2 \varphi \right\rangle ds$$

is a $\{\mathcal{F}_t^\Xi\}$-martingale with quadratic variation

$$[M_\varphi]_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{K} \times \mathbb{K}} \left(\varphi(x, \kappa') - \varphi(x, \kappa) \right)^2 \nu_s(x, d\kappa') \nu_s(x, d\kappa) \, dx \, ds,$$

identifying Ξ as a solution of a martingale problem.

Suppose $\mathbb{K} = \{0, 1\}$ and $\nu_s(x) \equiv \nu_s(x, \{1\})$. Then taking $\varphi(x, \kappa) = \kappa \psi(x)$,

$$M_\psi(t) = \int_{\mathbb{R}} \psi(x) \nu_t(x) \, dx - \int_0^t \int_{\mathbb{R}} \frac{1}{2} \psi''(x) \nu_s(x) \, dx \, ds$$

is a martingale with quadratic variation

$$[M_\psi]_t = \int_0^t \int_{\mathbb{R}} 2\psi^2(x) \nu_s(x) \left(1 - \nu_s(x) \right) dx \, ds,$$

which implies ν_t is a weak solution of the stochastic partial differential equation

$$
\begin{aligned}
(4.52) \quad \int_{\mathbb{R}} \psi(x) \nu_t(x) \, dx &= \int_{\mathbb{R}} \psi(x) \nu_0(x) \, dx + \int_0^t \int_{\mathbb{R}} \frac{1}{2} \psi''(x) \nu_s(x) \, dx \, ds \\
&\quad + \int_{[0,t] \times \mathbb{R}} \psi(x) \sqrt{2\nu_s(x)(1 - \nu_s(x))} \, W(ds, dx),
\end{aligned}
$$

where W is Gaussian white noise on $[0, \infty) \times \mathbb{R}$ with $E[W(A)W(B)] = \ell(A \cap B)$ for Lebesgue measure ℓ on $[0, \infty) \times \mathbb{R}$.

4.6. *Voter model.* The stochastic partial differential equation (4.52) is a special case of the equation that arises as the limit of rescaled voter models in the work of Müller and Tribe (1995). To see the relationship of their work to our current approach, we give a construction of a class of voter models.

Let $E = \mathbb{Z} \times \mathbb{K}$, where $\mathbb{Z}$ is the space of locations and $\mathbb{K}$ the space of types. We assume that there is one particle at each location, and consider

$$
\begin{aligned}
A_{dr,3} f(\eta) = f(\eta) \sum_{i \neq j} & r(|x_i - x_j|) \mathbf{1}_{\{u_i < u_j\}} \\
& \times \left(\frac{g(x_i, \kappa_i, u_i) g(x_j, \kappa_i, u_j) + g(x_i, \kappa_i, u_j) g(x_j, \kappa_i, u_i)}{2 g(x_i, \kappa_i, u_i) g(x_j, \kappa_j, u_j)} - 1 \right),
\end{aligned}
$$

where

$$
\sigma^2 \equiv \frac{1}{2} \sum_l l^2 r(l) < \infty.
$$

Then

$$
\alpha A_{dr,3} f(\overline{\eta}) = \alpha f(\overline{\eta}) \sum_{i < j} r(|x_i - x_j|) \left(\frac{1}{2} \frac{\overline{g}(x_j, \kappa_i)}{\overline{g}(x_j, \kappa_j)} + \frac{1}{2} \frac{\overline{g}(x_i, \kappa_j)}{\overline{g}(x_i, \kappa_i)} - 1 \right)
$$

which is the generator for a voter model. Particle motion involves two particles exchanging places, so in this model, the occupancy at each site is preserved.

Note that the collection of levels does not change, and the location of the particle associated with level u will satisfy a stochastic equation of the form

$$
\begin{aligned}
X_u(t) = X_u(0) \\
+ \sum_{k < l} \int_{[0,t] \times \{0,1\}} \theta \big(\mathbf{1}_{\{X_u(s-)=l\}}(k - l) + \mathbf{1}_{\{X_u(s-)=k\}}(l - k) \big) \xi_{kl}(ds, d\theta),
\end{aligned}
$$

where the ξ_{kl} are independent Poisson random measures with mean measures

$$
r(|k - l|) \left(\frac{1}{2} \delta_1(d\theta) + \frac{1}{2} \delta_0(d\theta) \right) ds.
$$

For $k > l$, assume $\xi_{kl} \equiv \xi_{lk}$. Let $U_l(t)$ and $\widehat{K}_l(t)$ denote the level and type of the particle with location l. Then the type for the particle with level u satisfies

$$
\begin{aligned}
K_u(t) = K_u(0) \\
+ \sum_{l \neq k} \int_{[0,t] \times \{0,1\}} \mathbf{1}_{\{U_l(s-)<u\}} \mathbf{1}_{\{X_u(s-)=k\}} \big(\widehat{K}_l(s-) - K_u(s-) \big) \xi_{kl}(ds, d\theta).
\end{aligned}
$$

Now, as $\lambda \to \infty$, assume that $\{(\lambda^{-1} X_u(0), K_u(0), u)\}$ converges to a conditionally Poisson point process on $\mathbb{R} \times \mathbb{K} \times [0, \infty)$ with Cox measure $dx \times \nu_0(x, d\kappa) \times du$. Set $X_u^\lambda(t) = \frac{1}{\lambda} X_u(\lambda^2 t)$ and $K_u^\lambda(t) = K_u(\lambda^2 t)$. Then X_u^λ is a martingale with quadratic variation

$$
[X_u^\lambda]_t = \sum_{k < l} \frac{1}{\lambda^2} \int_{[0,\lambda^2 t] \times \{0,1\}} \theta \big(\mathbf{1}_{\{X_u(s-)=l\}}(k-l)^2 + \mathbf{1}_{\{X_u(s-)=k\}}(l-k)^2 \big) \xi_{kl}(ds, d\theta)
$$

and

$$[X_u^\lambda]_t \to \frac{1}{2}\sum_{k<l}(k-l)^2 r(|k-l|)t = \sigma^2 t.$$

In addition, for $u \neq u'$,

$$[X_u^\lambda, X_{u'}^\lambda]_t \to 0,$$

so the X_u^λ converge to a collection of independent Brownian motions X_u.

For $u' < u$, let

$$N_{u',u}^\lambda(t) = \sum_{l \neq k}\int_{[0,\lambda^2 t]\times\{0,1\}}\mathbf{1}_{\{X_{u'}(s-)=l\}}\mathbf{1}_{\{X_u(s-)=k\}}\xi_{kl}(ds,d\theta).$$

Then $N_{u',u}^\lambda$ is a counting process with integrated intensity

$$\int_0^t \lambda^2 r\big(\lambda|X_{u'}^\lambda(s) - X_u^\lambda(s)|\big)\,ds.$$

Under appropriate time-scaling conditions, this integral should converge to a constant times the intersection local time given in (4.51). Then, up to changes in parameters, the limit of the lookdown construction would be the same as in Section 4.5.

APPENDIX

A.1. Poisson identities.

LEMMA A.1. *If ξ is a Poisson random measure on S with σ-finite mean measure v and $f \in L^1(v)$, then*

$$(A.1) \qquad E\big[e^{\int_S f(z)\xi(dz)}\big] = e^{\int_S(e^f-1)\,dv},$$

$$(A.2) \quad E\bigg[\int_S f(z)\xi(dz)\bigg] = \int_S f\,dv, \qquad \mathrm{Var}\bigg(\int_S f(z)\xi(dz)\bigg) = \int_S f^2\,dv,$$

allowing $\infty = \infty$.

Letting $\xi = \sum_i \delta_{Z_i}$, for $g \geq 0$ with $\log g \in L^1(v)$,

$$E\bigg[\prod_i g(Z_i)\bigg] = e^{\int_S(g-1)\,dv}.$$

Similarly, if $hg, g - 1 \in L^1(v)$, then

$$(A.3) \qquad E\bigg[\sum_j h(Z_j)\prod_i g(Z_i)\bigg] = \int_S hg\,dv\,e^{\int_S(g-1)\,dv},$$

$$(A.4) \qquad E\bigg[\sum_{i\neq j}h(Z_i)h(Z_j)\prod_k g(Z_k)\bigg] = \bigg(\int_S hg\,dv\bigg)^2 e^{\int_S(g-1)\,dv},$$

and more generally, if ν has no atoms and $r \in M(S \times S)$, $r \geq 0$,

$$(A.5) \quad E\left[\sum_{i \neq j} r(Z_i, Z_j) \prod_{k \neq i,j} g(Z_k)\right] = \int_{S \times S} r(x, y)\nu(dx)\nu(dy)e^{\int (g-1)\,d\nu},$$

allowing $\infty = \infty$.

PROOF. The independence properties of ξ imply (A.1) and (A.2) for simple functions. The general case follows by approximation.

To prove (A.5), it is enough to consider a finite measure ν and bounded continuous r and g and extend by approximation. Let $\{B_k^n\}$ be a partition of S with $\text{diam}(B_k^n) \leq n^{-1}$, and let $x_k^n \in B_k^n$. Define

$$\xi_n = \sum_k \delta_{x_k^n}\mathbf{1}_{\{\xi(B_k^n)>0\}}.$$

Then $\xi_n \to \xi$ in the sense that $\int f\,d\xi_n \to \int f\,d\xi$ for every bounded continuous f, and

$$\sum_{i \neq j} r(x_i^n, x_j^n)\mathbf{1}_{\{\xi(B_i^n)>0\}}\mathbf{1}_{\{\xi(B_j^n)>0\}} \prod_{k \neq i,j} \left(g(x_k^n)\mathbf{1}_{\{\xi(B_k^n)>0\}} + \mathbf{1}_{\{\xi(B_k^n)=0\}}\right)$$

$$\to \sum_{i \neq j} r(Z_i, Z_j) \prod_{k \neq i,j} g(Z_k).$$

By independence, the expectation of the left-hand side is

$$\sum_{i \neq j} r(x_i^n, x_j^n)(1 - e^{-\nu(B_i^n)})(1 - e^{-\nu(B_j^n)}) \prod_{k \neq i,j} \left(g(x_k^n)(1 - e^{-\nu(B_k^n)}) + e^{-\nu(B_k^n)}\right)$$

$$\approx \sum_{i \neq j} r(x_i^n, x_j^n)\nu(B_i^n)\nu(B_j^n) \exp\left\{\sum_{k \neq i,j} (g(x_k^n) - 1)\nu(B_k^n)\right\}$$

$$\to \int_{S \times S} r(x, y)\nu(dx)\nu(dy)e^{\int (g-1)\,d\nu},$$

where the convergence follows from the assumed continuity of r and g and the fact that $\sum_i \nu(B_i^n)^2 \to 0$.

The other identities follow in a similar manner. Note that the integrability of the random variables in the expectations above can be verified by replacing g by $(g \vee (-a)) \wedge a\mathbf{1}_A + \mathbf{1}_{A^c}$ and h by $(h \vee (-a)) \wedge a\mathbf{1}_A$ for $0 < a < \infty$ and $\nu(A) < \infty$ and passing to the limit as $a \to \infty$ and $A \nearrow E$. □

A.2. Markov mapping theorem. The following theorem [extending Corollary 3.5 from Kurtz (1998)] plays an essential role in justifying the particle representations and can also be used to prove uniqueness for the corresponding measure-valued processes. Let (S, d) and (S_0, d_0) be complete, separable metric spaces, $B(S) \subset M(S)$ be the Banach space of bounded measurable functions on S, with $\|f\| = \sup_{x \in S} |f(x)|$, and $C_b(S) \subset B(S)$ be the subspace of

bounded continuous functions. An operator $A \subset B(S) \times B(S)$ is *dissipative* if $\|f_1 - f_2 - \epsilon(g_1 - g_2)\| \geq \|f_1 - f_2\|$ for all $(f_1, g_1), (f_2, g_2) \in A$ and $\epsilon > 0$; A is a *pre-generator* if A is dissipative and there are sequences of functions $\mu_n : S \to \mathcal{P}(S)$ and $\lambda_n : S \to [0, \infty)$ such that for each $(f, g) \in A$

$$(A.6) \qquad g(x) = \lim_{n \to \infty} \lambda_n(x) \int_S (f(y) - f(x)) \mu_n(x, dy)$$

for each $x \in S$. A is *countably determined* if there exists a countable subset $\{g_k\} \subset \mathcal{D}(A) \cap \overline{C}(S)$ such that every solution of the martingale problem for $\{(g_k, Ag_k)\}$ is a solution of the martingale problem for A [e.g., A is countably determined if it is *graph separable* in the sense that there exists $\{(g_k, h_k)\} \subset A \cap \overline{C}(S) \times B(S)$ such that A is contained in the bounded pointwise closure of $\{(g_k, h_k)\}$]. These conditions are satisfied by essentially all operators A that might reasonably be thought to be generators of Markov processes. Note that A is graph separable if $A \subset L \times L$, where $L \subset B(S)$ is separable in the sup norm topology, for example, if S is locally compact and L is the space of continuous functions vanishing at infinity.

A collection of functions $D \subset \overline{C}(S)$ is *separating* if $\nu, \mu \in \mathcal{P}(S)$ and $\int_S f \, d\nu = \int_S f \, d\mu$ for all $f \in D$ imply $\mu = \nu$.

For a S_0-valued, measurable process Y, $\widehat{\mathcal{F}}_t^Y$ will denote the completion of the σ-algebra $\sigma(Y(0), \int_0^r h(Y(s)) \, ds, r \leq t, h \in B(S_0))$. For almost every t, $Y(t)$ will be $\widehat{\mathcal{F}}_t^Y$-measurable, but in general, $\widehat{\mathcal{F}}_t^Y$ does not contain $\mathcal{F}_t^Y = \sigma(Y(s) : s \leq t)$. Let $\mathbf{T}^Y = \{t : Y(t) \text{ is } \widehat{\mathcal{F}}_t^Y \text{ measurable}\}$. If Y is càdlàg and has no fixed points of discontinuity [i.e., for every t, $Y(t) = Y(t-)$ a.s.], then $\mathbf{T}^Y = [0, \infty)$. Let $D_S[0, \infty)$ denote the space of càdlàg, S-valued functions with the Skorohod topology, and $M_S[0, \infty)$ denotes the space of Borel measurable functions, $x : [0, \infty) \to S$, topologized by convergence in Lebesgue measure.

THEOREM A.2. *Let (S, d) and (S_0, d_0) be complete, separable metric spaces. Let $A \subset \overline{C}(S) \times C(S)$ and $\psi \in C(S)$, $\psi \geq 1$. Suppose that for each $f \in \mathcal{D}(A)$ there exists $c_f > 0$ such that*

$$(A.7) \qquad |Af(x)| \leq c_f \psi(x), \qquad x \in A,$$

and define $A_0 f(x) = Af(x)/\psi(x)$.

Suppose that A_0 is a countably determined pre-generator, and suppose that $\mathcal{D}(A) = \mathcal{D}(A_0)$ is closed under multiplication and is separating. Let $\gamma : S \to S_0$ be Borel measurable, and let α be a transition function from S_0 into S [$y \in S_0 \to \alpha(y, \cdot) \in \mathcal{P}(S)$ is Borel measurable] satisfying $\int h \circ \gamma(z) \alpha(y, dz) = h(y)$, $y \in S_0$, $h \in B(S_0)$, that is, $\alpha(y, \gamma^{-1}(y)) = 1$. Assume that $\widetilde{\psi}(y) \equiv \int_S \psi(z) \alpha(y, dz) < \infty$ for each $y \in S_0$ and define

$$C = \left\{ \left(\int_S f(z) \alpha(\cdot, dz), \int_S Af(z) \alpha(\cdot, dz) \right) : f \in \mathcal{D}(A) \right\}.$$

Let $\mu_0 \in \mathcal{P}(S_0)$, and define $\nu_0 = \int \alpha(y, \cdot) \mu_0(dy)$.

(a) *If $\widetilde{Y}$ satisfies $\int_0^t E[\widetilde{\psi}(\widetilde{Y}(s))] \, ds < \infty$ for all $t \geq 0$ and $\widetilde{Y}$ is a solution of the martingale problem for (C, μ_0), then there exists a solution X of the martingale problem for (A, v_0) such that $\widetilde{Y}$ has the same distribution on $M_{S_0}[0, \infty)$ as $Y = \gamma \circ X$. If Y and $\widetilde{Y}$ are càdlàg, then Y and $\widetilde{Y}$ have the same distribution on $D_{S_0}[0, \infty)$.*

(b) *For $t \in \mathbf{T}^Y$,*

$$(A.8) \qquad P\{X(t) \in \Gamma | \widehat{\mathcal{F}}_t^Y\} = \alpha(Y(t), \Gamma), \qquad \Gamma \in \mathcal{B}(S).$$

(c) *If, in addition, uniqueness holds for the martingale problem for (A, v_0), then uniqueness holds for the $M_{S_0}[0, \infty)$-martingale problem for (C, μ_0). If $\widetilde{Y}$ has sample paths in $D_{S_0}[0, \infty)$, then uniqueness holds for the $D_{S_0}[0, \infty)$-martingale problem for (C, μ_0).*

(d) *If uniqueness holds for the martingale problem for (A, v_0), then Y restricted to $\mathbf{T}^Y$ is a Markov process.*

REMARK A.3.   Theorem A.2 can be extended to cover a large class of generators whose range contains discontinuous functions [see Kurtz (1998), Corollary 3.5 and Theorem 2.7]. In particular, suppose $A_1, \ldots, A_m$ satisfy the conditions of Theorem A.2 for a common domain $\mathcal{D} = \mathcal{D}(A_1) = \cdots = \mathcal{D}(A_m)$ and $\beta_1, \ldots, \beta_m$ are nonnegative functions in $B(S)$. Then the conclusions of Theorem A.2 hold for

$$Af = \beta_1 A_1 f + \cdots + \beta_m A_m f.$$

By (A.8), X and Y are "intertwined" in the sense of Rogers and Pitman (1981).

PROOF.   Theorem 3.2 of Kurtz (1998) can be extended to operators satisfying (A.7) by applying Corollary 1.12 of Kurtz and Stockbridge (2001) (with the operator B in that corollary set equal zero) in place of Theorem 2.6 of Kurtz (1998). Alternatively, see Corollary 3.2 of Kurtz and Nappo (2011).   □

A.3. Stochastic equations for processes built from bounded generators. We are primarily interested in generators of the form

$$(A.9) \qquad Af(x) = \int_{\mathbb{U}} (P_z f(x) - f(x)) \mu(dz),$$

where for each $z \in \mathbb{U}$, P_z is a transition operator on a complete, separable metric space E, appropriately measurable as a function of $z \in \mathbb{U}$ and μ is a σ-finite measure on $\mathbb{U}$. To illustrate the type of stochastic equation we have in mind, let

$$A_0 f(x) = \lambda_0 \int_E (f(y) - f(x)) \eta(x, dy),$$

where $0 < \lambda_0 < \infty$ and η is a transition function on E. We can always find a probability measure v_0 on a measurable space $\mathbb{U}_0$ and a measurable function

$H_0(x, u) : E \times \mathbb{U}_0 \to E$ satisfying $\eta(x, C) = \int_{\mathbb{U}_0} \mathbf{1}_C(H_0(x, u)) \nu(du)$, $C \in \mathcal{B}(E)$, so that

$$\lambda_0 \int_E (f(y) - f(x)) \eta(x, dy) = \lambda_0 \int_{\mathbb{U}_0} (f(H_0(x, u)) - f(x)) \nu_0(du).$$

See, for example, the construction in Blackwell and Dubins (1983).

If N is a Poisson process with parameter λ_0, $U_0, U_1, \dots$ are independent $\mathbb{U}_0$-valued random variables with distribution ν_0, and $X(0)$ is an E-valued random variable, N, $\{U_i\}$, and $X(0)$ independent, then there is a unique, E-valued process X satisfying

$$(\text{A.10}) \quad f(X(t)) = f(X(0)) + \int_0^t (f(H_0(X(s-), U_{N(s-)})) - f(X(s-)) \, dN(s),$$

for all $f \in B(E)$, and X will be a solution of the martingale problem for A_0. Since in this case, A_0 is a bounded operator and the martingale problem is well-posed, it follows that the martingale problem and the stochastic equation are equivalent in the sense that every solution of the stochastic equation is a solution of the martingale problem and every solution of the martingale problem is a weak solution of the stochastic equation.

In general, we are interested in situations where uniqueness is not necessarily known for either the martingale problem or the stochastic equation, but we still want to know that the two are equivalent. We will obtain our result by application of the Markov mapping theorem using arguments similar to those used in Kurtz (2011). Let us illustrate these arguments by proving what we already know regarding the martingale problem for A_0 and (A.10).

Let $\widehat{B}_0$ be the generator for a process in $S = E \times \mathbb{U}_0 \times \{-1, 1\}$ given by

$$\widehat{B}_0 \widehat{f}(x, u, \theta) = \lambda_0 \int_{\mathbb{U}_0} (\widehat{f}(H_0(x, u), u', -\theta) - \widehat{f}(x, u, \theta)) \nu_0(du'), \qquad \widehat{f} \in B(S),$$

and setting

$$f(x) = \frac{1}{2} \int_{\mathbb{U}_0} \widehat{f}_0(x, u, 1) \nu_0(du) + \frac{1}{2} \int_{\mathbb{U}_0} \widehat{f}_0(x, u, -1) \nu_0(du),$$

observe that

$$A_0 f(x) = \frac{1}{2} \int_{\mathbb{U}_0} \widehat{B}_0 \widehat{f}(x, u, 1) \nu_0(du) + \frac{1}{2} \int_{\mathbb{U}_0} \widehat{B}_0 \widehat{f}(x, u, -1) \nu_0(du).$$

The Markov mapping theorem implies that if $\widehat{X}$ is a solution of the martingale problem for A_0, there exists a solution $Z = (X, U, \Theta)$ of the martingale problem for $\widehat{B}_0$ such that X has the same distribution as $\widehat{X}$.

Let $N(t)$ be the counting process satisfying $\Theta(t) = \Theta(0)(-1)^{N(t)}$. Note that setting $\widehat{f}(x, u, \theta) = \theta$,

$$M_\theta(t) = \Theta(t) - \int_0^t \widehat{B}_0 f(Z(s)) \, ds = \Theta(t) + 2 \int_0^t \lambda_0 \Theta(s) \, ds$$

is a martingale and

$$N(t) = -\frac{1}{2} \int_0^t \Theta(s-) \, d\Theta(s) = -\frac{1}{2} \int_0^t \Theta(s-) \, dM_\theta(s) + \lambda_0 t.$$

Consequently, $N(t) - \lambda_0 t$ is a martingale, and hence N is a Poisson process with intensity λ_0.

LEMMA A.4. *For any bounded function f on E,*

$$\text{(A.11)} \quad f(X(t)) = f(X(0)) + \int_0^t (f(H_0(X(s-), U(s-))) - f(X(s-))) \, dN(s).$$

PROOF. To see that this identity holds, let

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t \widehat{B}_0 f(X(s), U(s), \Theta(s)) \, ds.$$

We have the following Meyer processes [see Lemma 5.1 of Kurtz (2011)]:

$$\langle M_f \rangle_t = \int_0^t (\lambda_0 (f^2(H_0(X(s), U(s))) - f^2(X(s))$$
$$- 2f(X(s))\lambda_0 (f(H_0(X(s), U(s))) - f(X(s))) \, ds$$
$$= \int_0^t \lambda_0 (f(H_0(X(s), U(s))) - f(X(s)))^2 \, ds,$$

$$\langle M_f, M_\theta \rangle_t = \int_0^t [\lambda_0 (f(H_0(X(s), U(s)))(-1)\Theta(s) - f(X(s))\Theta(s))$$
$$+ 2\lambda_0 f(X(s))\Theta(s) - \Theta(s)\lambda_0 (f(H_0(X(s), U(s)) - f(X(s)))] \, ds$$
$$= -\int_0^t 2\Theta(s)\lambda_0 (f(H_0(X(s), U(s)) - f(X(s))) \, ds,$$

$$\langle M_\theta \rangle_t = \int_0^t 2\Theta(s) 2\lambda_0(s)\Theta(s) = 4\lambda_0 t.$$

Then

$$M(t) = f(X(t)) - f(X(0))$$
$$- \int_0^t (f(H_0(X(s-), U(s-))) - f(X(s-))) \, dN(s)$$
$$= M_f(t) + \frac{1}{2} \int_0^t (f(H_0(X(s-), U(s-))) - f(X(s-)))\Theta(s-) \, dM_\theta(s)$$

is a martingale and

$$\langle M \rangle = \langle M_f \rangle + \int_0^t \left(f\left(H_0\big(X(s-), U(s-)\big)\right) - f\big(X(s-)\big)\right)\Theta(s-)\,d\langle M_f, M_\theta\rangle$$

$$+ \frac{1}{4}\int_0^t \left(f\big(H_0(X(s), U(s))\big) - f\big(X(s)\big)\right)^2 d\langle M_\theta\rangle_s$$

$$= 0,$$

so $M = 0$ and (A.11) holds. $\quad\square$

We now assume that μ is in (A.9) is infinite, but σ-finite. Writing $\mathbb{U} = \bigcup_{k=1}^\infty \mathbb{U}_k$ as a disjoint union of sets of finite measure, we can write

(A.12) $$Af(x) = \sum_{k=1}^\infty \int_{\mathbb{U}_k} \left(P_z f(x) - f(x)\right)\mu(dz) \equiv \sum_{k=1}^\infty B_k f(x),$$

where each B_k is a bounded generator, and hence can be written as

$$B_k f(x) = \lambda_k \int_E \left(f(y) - f(x)\right)\eta_k(x, dy)$$

(A.13)

$$= \lambda_k \int_{\mathbb{U}_k} \left(f\big(H_k(x, u)\big) - f(x)\right)\nu_k(du),$$

for $\lambda_k = \mu(\mathbb{U}_k)$, and some $H_k : E \times \mathbb{U}_k \to E$ and $\nu_k \in \mathcal{P}(\mathbb{U}_k)$. We are implicitly assuming that $\mathbb{U}_k$ is rich enough to support a measure ν_k for which the desired H_k will exist. One can always replace $\mathbb{U}$ by $\mathbb{U} \times [0, 1]$ and μ by $\mu \times \ell$.

To be specific, we will simply assume that B_k is given by the right-hand side of (A.13). To make the definition of A as the sum of the B_k precise, let $\mathcal{D} \subset C_b(E)$, and assume the following conditions.

CONDITION A.5. (a) $\mathcal{D}$ is closed under multiplication and separates points in E.

(b) For each $f \in \mathcal{D}$,

$$Af(x) \equiv \lim_{m \to \infty} \sum_{k=1}^m B_k f(x)$$

exists pointwise in E.

(c) There exists $\psi \in M(E)$ such that $\psi \geq 1$ and for each $f \in \mathcal{D}$, there exists c_f and m_f such that for $m \geq m_f$,

$$\left| \sum_{k=m+1}^\infty B_k f(x) \right| \equiv \left| Af(x) - \sum_{k=1}^m B_k f(x) \right| \leq c_f \psi(x), \qquad x \in E.$$

Let $\mathbb{E}_m = E \times \mathbb{U}_1 \times \cdots \times \mathbb{U}_m \times \{-1, 1\}^m$,

$$\mathcal{D}(\widehat{A}_m) = \left\{ \widehat{f}(x, u, \theta) = f(x) \prod_{k=1}^m g_k(u_k, \theta_k) : \right.$$

$$\left. f \in \mathcal{D}, g_k \in C_b(\mathbb{U}_k \times \{-1, 1\}), 1 \le k \le m \right\},$$

and define a generator $\widehat{A}_m$ for a process in $\mathbb{E}_m$ by

$$\widehat{A}_m \widehat{f}(x, u_1, \ldots, u_m, \theta_1, \ldots, \theta_m)$$

$$= \sum_{k=1}^m \lambda_k \int_{\mathbb{U}_k} (\widehat{f}(H_k(x, u_k), \eta_k(u|u_k'), \eta_k(\theta| - \theta_k)) - \widehat{f}(x, u, \theta)) \nu_k(du_k')$$

$$+ \prod_{k=1}^m g(u_k, \theta_k) \sum_{l=m+1}^\infty B_l f(x),$$

where for an arbitrary set S, for $z \in S^\infty$ and $z_k' \in S$, $\eta_k(z|z_k')$ is the element of S^∞ obtained from z by replacing z_k by z_k'. If $\widehat{X}$ is a solution of the martingale problem for A satisfying

$$E\left[\int_0^t \psi(\widehat{X}(s)) \, ds\right] < \infty, \qquad t \ge 0,$$

the Markov mapping theorem implies that for each m, there exists a solution $(X^{(m)}, U^{(m)}, \Theta^{(m)})$ of the martingale problem for $\widehat{A}_m$ such that $X^{(m)}$ and $\widehat{X}$ have the same distribution. By induction, the sequence of processes can be constructed so that the restriction of $(X^{(m+1)}, U^{(m+1)}, \Theta^{(m+1)})$ to $\mathbb{E}_m$ has the same distribution as $(X^{(m)}, U^{(m)}, \Theta^{(m)})$, and it follows that there exists a process $(X, \mathbb{U}, \Theta)$ in $\mathbb{E} = E \times \mathbb{U}_1 \times \mathbb{U}_2 \times \cdots \times \{-1, 1\}^\infty$ so that the restriction of (X, U, Θ) to $\mathbb{E}_m$ has the same distribution as $(X^{(m)}, U^{(m)}, \Theta^{(m)})$.

Consequently,

$$\widehat{M}_f^m(t) = f(X(t)) - f(X(0))$$

$$- \sum_{k=1}^m \int_0^t (f(H_k(X(s-), U_k(s-))) - f(X(s-))) \, dN_k(s)$$

$$- \int_0^t \sum_{k \ge m+1} B_k f(X(s)) \, ds$$

$$= f(X(t)) - f(X(0))$$

$$- \sum_{k=1}^m \int_0^t \lambda_k (f(H_k(X(s-), U_k(s-))) - f(X(s-))) \, ds$$

$$- \int_0^t \sum_{k \geq m+1} B_k f(X(s)) \, ds$$

$$- \sum_{k=1}^m \int_0^t \left(f(H_k(X(s-), U_k(s-))) - f(X(s-)) \right) d\widetilde{N}_k(s)$$

$$= M_f^m(t)$$

$$+ \sum_{k=1}^m \frac{1}{2} \int_0^t \left(f(H_k(X(s-), U_k(s-))) - f(X(s-)) \right) \Theta_k(s-) \, dM_{\theta_k}(s)$$

is a $\{\mathcal{F}_t^m\}$-martingale for $\mathcal{F}_t^m = \sigma(\widehat{X}(s), U_1(s), \ldots, U_m(s), \Theta_1(s), \ldots, \Theta_m(s) : s \leq t)$.

Note that

$$\langle M_f^m \rangle = \sum_{k=1}^m \int_0^t \lambda_k \left(f^2(H_k(X(s), U_k(s))) - f^2(X(s)) \right.$$

$$\left. - 2f(X(s)) \lambda_k \left(f(H_k(X(s-), U_k(s-))) - f(X(s-)) \right) \right) ds$$

$$+ \int_0^t \sum_{k \geq m+1} \left(B_k f^2(X(s)) - 2f(X(s)) B_k f(X(s)) \right) ds,$$

$$\langle M_{\theta_k} \rangle_t = 4\lambda_k t$$

and

$$\langle M_f^m, M_{\theta_k} \rangle = \int_0^t \left(\sum_{1 \leq l \neq k \leq m} \Theta_k(s) \lambda_l \left(f(H_l(X(s), U_l(s))) - f(X(s)) \right) \right.$$

$$- \lambda_k \Theta_k(s) \left(f(H_k(X(s), U(s))) + f(X(s)) \right)$$

$$+ \Theta_k(s) \sum_{l \geq m+1} B_l f(X(s)) - \Theta_k(s) \sum_{l \geq m+1} B_l f(X(s))$$

$$- \Theta_k(s) \sum_{l=1}^m \lambda_l \left(f(H_l(X(s), U_l(s))) - f(X(s)) \right)$$

$$\left. + 2\lambda_k \Theta_k(s) f(X(s)) \right) ds$$

$$= - \int_0^t 2\lambda_k \Theta_k(s) \left(f(H_k(X(s), U_k(s))) - f(X(s)) \right) ds.$$

Consequently,

$$\langle \widehat{M}_f^m \rangle_t = \langle M_f^m \rangle_t + \sum_{k=1}^m \int_0^t \left(f(H_k(X(s), U_k(s))) \right.$$

$$\left. - f(X(s)) \right) \Theta_k(s) \, d\langle M_f^m, M_{\theta_k} \rangle_s$$

$$+ \sum_{k=1}^{m} \int_{0}^{t} \frac{1}{4} \big(f\big(H_k(X(s), U_k(s)) \big) - f(X(s)) \big)^2 \, d\langle M_{\theta_k} \rangle_s$$

$$= \int_{0}^{t} \Bigg(\sum_{k=1}^{m} \lambda_k \big(f^2\big(H_k(X(s), U_k(s)) \big) - f^2(X(s)) \big)$$

$$- 2f(X(s)) \sum_{k=1}^{m} \lambda_k \big(f\big(H_k(X(s-), U_k(s-)) \big) - f(X(s-)) \big) \Bigg) \, ds$$

$$+ \sum_{k \geq m+1} \big(B_k f^2(X(s)) - 2f(X(s)) B_k f(X(s)) \big)$$

$$- 2 \sum_{k=1}^{m} \lambda_k \big(f\big(H_k(X(s), U_k(s)) \big) - f(X(s)) \big)^2$$

$$+ \sum_{k=1}^{m} \lambda_k \big(f\big(H_k(X(s), U_k(s)) \big) - f(X(s)) \big)^2 \big) \, ds$$

$$= \int_{0}^{t} \sum_{k \geq m+1} \big(B_k f^2(X(s)) - 2f(X(s)) B_k f(X(s)) \big).$$

THEOREM A.6. *Let $\{B_k\}$ be a sequence of bounded generators of the form (A.13), and assume that Condition A.5 holds. Suppose that $\widehat{X}$ is a solution of the martingale problem for A satisfying*

$$E\left[\int_{0}^{t} \psi(\widehat{X}(s)) \, ds \right] < \infty, \qquad t \geq 0.$$

Then, for each $f \in \mathcal{D}$,

$$f(X(t)) = f(X(0))$$

$$+ \sum_{k=1}^{\infty} \int_{0}^{t} \big(f\big(H_k(X(s-), U_k(s-)) \big) - f(X(s-)) \big) \, dN_k(s),$$

in the sense that, for each $T \geq 0$,

$$\lim_{m \to \infty} \sup_{t \leq T} \Bigg| f(X(t)) - f(X(0))$$

$$- \sum_{k=1}^{m} \int_{0}^{t} \big(f\big(H_k(X(s-), U_k(s-)) \big) - f(X(s-)) \big) \, dN_k(s) \Bigg| = 0$$

in probability.

PROOF. Since $\langle \widehat{M}_f^m \rangle_t \to 0$, it follows that $\sup_{t \le T} |\widehat{M}_f^m(t)| \to 0$, and since

$$\widehat{M}_f^m(t) = f(X(t)) - f(X(0))$$
$$- \sum_{k=1}^{m} \int_0^t \big(f(H_k(X(s-), U_k(s-))) - f(X(s-)) \big) \, dN_k(s)$$
$$- \int_0^t \sum_{k \ge m+1} B_k f(X(s)) \, ds$$

and the last term goes to zero, the lemma follows. $\square$

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