PHASE TRANSITIONS IN THE ASEP AND STOCHASTIC SIX-VERTEX MODEL

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In this paper, we consider two models in the Kardar–Parisi–Zhang (KPZ) universality class, the asymmetric simple exclusion process (ASEP) and the stochastic six-vertex model. We introduce a new class of initial data (which we call *shape generalized step Bernoulli initial data*) for both of these models that generalizes the step Bernoulli initial data studied in a number of recent works on the ASEP. Under this class of initial data, we analyze the current fluctuations of both the ASEP and stochastic six-vertex model and establish the existence of a phase transition along a characteristic line, across which the fluctuation exponent changes from $1/2$ to $1/3$. On the characteristic line, the current fluctuations converge to the general (rank k) Baik–Ben–Arous–Péché distribution for the law of the largest eigenvalue of a critically spiked covariance matrix. For $k = 1$, this was established for the ASEP by Tracy and Widom; for $k > 1$ (and also $k = 1$, for the stochastic six-vertex model), the appearance of these distributions in both models is new.

CONTENTS

1. Introduction. In this paper, we study two interacting particle systems in the Kardar–Parisi–Zhang (KPZ) universality class, the asymmetric simple exclusion process (ASEP) and the stochastic six-vertex model. We begin in Section 1.1 by defining these two models and their associated observables. In Section 1.2, we provide some context for our results, which will be more carefully stated in Section 1.3.

1.1. *The ASEP and stochastic six-vertex model.*

1.1.1. *The asymmetric simple exclusion process.*   Introduced to the mathematics community by Spitzer [42] in 1970 (and also appearing two years earlier in the biology work of Macdonald, Gibbs and Pipkin [34]), the *asymmetric simple exclusion process* (ASEP) is a continuous time Markov process that can be described as follows. Particles are initially (at time 0) placed on $\mathbb{Z}$ in such a way that at most one particle occupies any site. Associated with each particle are two exponential clocks, one of rate L and one of rate R; we assume that $R > L \geq 0$ and that all clocks are mutually independent. When some particle's left clock rings, the particle attempts to jump one space to the left; similarly, when its right clock rings, it attempts to jump one space to the right. If the destination of the jump is unoccupied, the jump is performed; otherwise it is not. This is sometimes referred to as the *exclusion restriction*.

Associated with the ASEP is an observable called the *current*. For the purpose of this paper, the *current* [denoted $J_t(x)$] is the number of particles strictly to the right of x at time t, for $x \in \mathbb{R}$ and $t \in \mathbb{R}_{\geq 0}$. In all instances, we encounter in this paper, $J_t(x)$ will be almost surely finite since we only consider asymmetric simple exclusion processes in which no sites in $\mathbb{Z}_{>0}$ are initially occupied.

One of the purposes of this paper will be to understand the large-time current fluctuations of the ASEP under a certain type of initial data; we postpone further discussion on this to Section 1.2 and Section 1.3.

1.1.2. *The stochastic six-vertex model.*   The stochastic six-vertex model was first introduced to the mathematical physics community by Gwa and Spohn [27] in 1992 as a stochastic version of the more well-known six-vertex (ice) model studied by Lieb [33] and Baxter [10]; it was also studied more recently in [16, 21, 22]. This model can be defined in several equivalent ways, including as a ferromagnetic, asymmetric six-vertex model on a long rectangle with specific vertex weights (see [27] or Section 2.1 of [16]); as an interacting particle system with push dynamics and an exclusion restriction (see [27] or Section 2.2 of [16]); or as a probability measure on directed path ensembles (see Section 2 of [16] or Section 1 of [21]). In this section we will define the model through path ensembles.

A *six-vertex directed path ensemble* is a family of up-right directed paths in the nonnegative quadrant $\mathbb{Z}^2_{>0}$, such that each path emanates from either the x-axis or y-axis, and such that no two paths share an edge (although they may share vertices); see Figure 1. In particular, each vertex has six possible *arrow configurations*, which are listed in the top row of Figure 2. *Initial data*, or *boundary data*, for such an ensemble is prescribed by dictating which vertices on the positive x-axis and positive y-axis are entrance sites for a directed path. One example of initial data is *step initial data*, in which paths only enter through the y-axis, and all vertices on the y-axis are entrance sites for paths; see Figure 1.
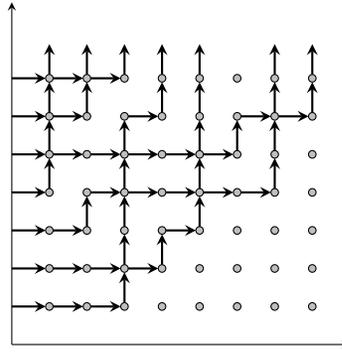
FIG. 1.   *A sample of the stochastic six-vertex model with step boundary data is depicted above.*

Now fix parameters $\delta_1, \delta_2 \in [0, 1]$ and some initial data. The *stochastic six-vertex model* $\mathcal{P} = \mathcal{P}(\delta_1, \delta_2)$ will be the infinite-volume limit of a family of probability measures $\mathcal{P}_n = \mathcal{P}_n(\delta_1, \delta_2)$ defined on the set of six-vertex directed path ensembles whose vertices are all contained in triangles of the form $\mathbb{T}_n = \{(x, y) \in \mathbb{Z}_{\geq 0}^2 : x + y \leq n\}$. The first such probability measure $\mathcal{P}_1$ is supported by the empty ensemble.

For each positive integer n, we define $\mathcal{P}_{n+1}$ from $\mathcal{P}_n$ through the following Markovian update rules. Use $\mathcal{P}_n$ to sample a directed path ensemble $\mathcal{E}_n$ on $\mathbb{T}_n$. This gives arrow configurations (of the type shown in Figure 2) to all vertices in the positive quadrant strictly below the diagonal $\mathbb{D}_n = \{(x, y) \in \mathbb{Z}_{>0}^2 : x + y = n\}$. Each vertex on $\mathbb{D}_n$ is also given "half" of an arrow configuration, in the sense that it is given the directions of all entering paths but no direction of any exiting path.

To extend $\mathcal{E}_n$ to a path ensemble on $\mathbb{T}_{n+1}$, we must "complete" the configurations (specify the exiting paths) at all vertices $(x, y) \in \mathbb{D}_n$. Any half-configuration can be completed in at most two ways; selecting between these completions is done randomly, according to the probabilities given in the second row of Figure 2, and independently among all vertices.

In this way, we obtain a random ensemble $\mathcal{E}_{n+1}$ on $\mathbb{T}_{n+1}$; the resulting probability measure on path ensembles with vertices in $\mathbb{T}_{n+1}$ is denoted $\mathcal{P}_{n+1}$. Now set $\mathcal{P} = \lim_{n \to \infty} \mathcal{P}_n$.

1	δ_1	$1 - \delta_1$	δ_2	$1 - \delta_2$	1

FIG. 2.   *The top row in the chart shows the six possible arrow configurations at vertices in the stochastic six-vertex model; the bottom row shows the corresponding probabilities.*

As in the ASEP, there exists an observable of interest for stochastic six-vertex model called the *current*. For our purposes, it will be defined as follows. Let $(X, Y) \in \mathbb{R}^2_{>0}$. The *current* (or *height function*) $\mathfrak{H}(X, Y)$ of the stochastic six-vertex model at (X, Y) is the number of paths that intersect the line $y = Y$ strictly to the right of (X, Y). Since in this paper we only consider stochastic six-vertex models in which no vertices on the x-axis are entrance sites for paths, the quantity $\mathfrak{H}(X, Y)$ is always finite.

We will again be interested in understanding the current fluctuations for the stochastic six-vertex model under certain types of initial data; we will discuss this further in the next two sections.

1.2. *Context and background.* The phenomenon that guides our results is commonly termed *KPZ universality*. About thirty years ago in their seminal paper [31], Kardar, Parisi and Zhang considered a family of random growth models that exhibit ostensibly unusual (although now known to be quite ubiquitous) scaling behavior. As part of this work, they predicted the scaling exponents for all one-dimensional models in this family; specifically, after running such a model (with certain deterministic initial data) for some large time T, they predicted fluctuations of order $T^{1/3}$ and nontrivial spacial correlation on scales $T^{2/3}$. This (rather vaguely defined) family of random growth models is now called the *Kardar–Parisi–Zhang (KPZ) universality class*, and consists of many more models (including the ASEP and stochastic six-vertex model) than those originally considered in [31]; we refer to the surveys [24] and [40] for more information.

In addition to predicting these exponents, Kardar, Parisi and Zhang proposed a stochastic differential equation that in a sense embodies all of the models in their class; this equation, now known as the *KPZ equation* is

$$(1.1) \qquad \partial_t h = \partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{\mathcal{W}},$$

where $\dot{\mathcal{W}}$ refers to space-time white noise.

Granting the well-posedness [12, 28, 29] of (1.1) [which by no means imme-diate, due to the nonlinearity $\frac{1}{2}(\partial_x h)^2$], it is widely believed that the long-time statistics of a stochastic model in the KPZ universality class should coincide with the long-time statistics of (1.1), whose initial data should be somehow chosen to "match" the initial data of the model, in a suitable way. In particular, distributions appearing as asymptotic marginals of the KPZ equation should also appear as the asymptotic one-point functions for models in the KPZ universality class.

For the ASEP and stochastic six-vertex model, these predictions have been ver-ified in only a few contexts. The first such result was established by Johansson [30] in 2000 for the *totally asymmetric simple exclusion process* (TASEP), which is the $L = 0$ case of the ASEP. Specifically, in the case of step initial data (all sites in $\mathbb{Z}_{\leq 0}$ are occupied and all other sites are unoccupied), he showed that, af-ter $T^{1/3}$ scaling, the current fluctuations converge to the Tracy–Widom distribu-tion for the limiting law of the largest eigenvalue of a large Gaussian Hermitian

random (GUE) matrix [43]. This is in agreement with the result later established by Amir–Corwin–Quastel [4], Calabrese–Le-Doussal–Rosso [23], Dotsenko [26] and Sasamoto–Spohn [41], who showed that the long-time height fluctuations of the KPZ equation under what is known as *narrow wedge initial data* also converge to the Tracy–Widom distribution, after $T^{1/3}$ scaling; see the survey [24] for a thorough discussion on the level of mathematical justification behind these developments.

The result of Johansson strongly relies on the *free-fermionicity* (complete determinantal structure) that underlies the TASEP. This property is not believed to hold for the more general ASEP, which had posed trouble for extending Johansson's result to the ASEP with nonzero L for several years. This was eventually overcome by Tracy and Widom in 2008 [44, 45]. Specifically, in the case of step initial data, they showed that the current fluctuations of the ASEP converge to the Tracy–Widom distribution after $T^{1/3}$ scaling [46], which again establishes the KPZ universality conjecture for the ASEP with step initial data.

Due to its two-dimensional nature and also its more complex Markov update rule, the stochastic six-vertex model has been less amenable to analysis. Only very recently has the analogous result been established for that model; this was done in the work [16] of Borodin, Corwin and Gorin. In that paper, the authors analyze the stochastic six-vertex model with step initial data and show that the fluctuations of the current are of order $T^{1/3}$ when (X, Y) lies in a certain cone (called a *rarefaction fan* or *liquid region*) of the positive quadrant. After rescaling by $T^{1/3}$, they show that the current fluctuations again converge to the Tracy–Widom distribution, thereby establishing the KPZ universality conjecture for the stochastic six-vertex model with step initial data.

One might ask whether this type of KPZ universality phenomenon can be proven for the ASEP and stochastic six-vertex model under other classes of initial data or, equally interesting, what other types of distributions can appear as the long-time current fluctuations for these two models. For the stochastic six-vertex model, we know of no results in this direction.

For the general ASEP with $L \neq 0$, we know of only one, which considers the case of *step Bernoulli initial data*, in which sites in $\mathbb{Z}_{\leq 0}$ are independently occupied with probability $b \in (0, 1]$ and no other sites are occupied (however, see the works [32, 36, 37] for partial results for the ASEP with flat and half-flat initial data; see also the works [7, 8] for partial results for the ASEP with stationary initial data, which are in fact completed to include the exact limiting statistics in the work [2], which further develops on some of the ideas presented in this paper). Under step Bernoulli initial data, Tracy and Widom [47] considered the fluctuations of the current $J_T(\eta T)$ as T tends to ∞, for fixed $\eta < 1$, and established the existence of a *phase transition* (already known in the case of the TASEP [5, 6, 11, 39]). Specifically, they showed the following:

1. If $\eta \in (1 - 2b, 1)$, then the fluctuations of $J_T(\eta T)$ are GUE Tracy–Widom and of order $T^{1/3}$.

2. If $\eta = 1 - 2b$, then the fluctuations of $J_T(\eta T)$ are F_1^2 and of order $T^{1/3}$, where F_1 is the limiting law of the largest eigenvalue for a large random Gaussian real symmetric matrix.

3. If $\eta < 1 - 2b$, then the fluctuations of $J_T(\eta T)$ are Gaussian and of order $T^{1/2}$.

In particular, across the line $x = (1 - 2b)T$, the fluctuation exponent of $J_T(x)$ changes from $1/2$ to $1/3$ and, on this line, the fluctuations are of order $T^{1/3}$ and are F_1^2 instead of GUE Tracy–Widom.

The corresponding initial data for the KPZ equation is half-Brownian initial data; this was analyzed by Corwin and Quastel in [25], where it was shown that the asymptotic height fluctuations of this KPZ equation are of order $T^{1/3}$ and scale to the same F_1^2 distribution, in agreement with the second part of Tracy–Widom's result.

Similar to the Tracy–Widom distribution, the phase transition described above was not first observed in the framework of interacting particle systems but rather in the context of random matrices [6]. Specifically, they originally arose as the limiting law of the largest eigenvalue λ_1 of large $N \times N$ *spiked covariance matrices*; these were studied at length by Baik, Ben Arous and Péché in the paper [6], which we refer to for more information. They also appeared in the work of Péché [38], as the limiting law of the largest eigenvalue of finite-rank perturbations of large GUE matrices.

We will not state Péché's result is precise detail, but informally it can be described as follows. Fix an integer $m \geq 1$ and consider a rank m perturbation of a $N \times N$ GUE matrix.

1. If the perturbation is sufficiently small, then the fluctuations λ_1 are of order $N^{-2/3}$ and Tracy–Widom.

2. If the perturbation is *critical* (in some appropriate sense), then the fluctuations of λ_1 are of order $N^{-2/3}$ and scale to the distribution $F_{\mathrm{BBP};0^m}$, which is a *rank m Baik–Ben–Arous–Péché distribution* (see Definition 1.3).

3. If the perturbation is sufficiently large, then the fluctuations of λ_1 are of order $N^{-1/2}$ and G_m, where G_m denotes the law of the largest eigenvalue of an $m \times m$ GUE matrix (see Definition 1.4).

The distribution G_1 is a Gaussian distribution, and the distribution $F_{\mathrm{BBP},0}$ is also known [6] to coincide with F_1^2. Thus, the results of Tracy–Widom [47] on the ASEP with step Bernoulli initial data strongly resemble the random matrix phase transitions described above in the case $m = 1$.

This stimulates two questions. The first is whether one can generalize the previously mentioned result of Corwin–Quastel [25] on the KPZ equation with half-Brownian initial data, that is, whether it is possible to observe any higher rank Baik–Ben–Arous–Péché distributions as marginals of the KPZ equation with some (explicit) initial data. The second is whether one can generalize Tracy–Widom's

[47] to produce some explicit class of initial data (generalizing the step Bernoulli initial data) for the ASEP (and stochastic six-vertex model) under which one observes a phase transition across a characteristic line and the higher rank Baik–Ben–Arous–Péché distributions along the characteristic line.

The first question was addressed by Borodin, Corwin and Ferrari in the paper [15]. In that work, the authors consider the KPZ equation with what they call *m-spiked initial data* and establish that its height fluctuations (after $T^{1/3}$ scaling) converge to rank m Baik–Ben–Arous–Péché distributions.

The purpose of the present paper is to affirmatively answer the second question, thereby establishing KPZ universality of the ASEP and stochastic six-vertex model under this type of m-spiked distribution. We state our results more precisely in the next section.

1.3. *Results.*   In this section we state our main results, which establish phase transitions for the ASEP and stochastic six-vertex model under what we call generalized step Bernoulli initial data; these are given by Theorem 1.6 and Theorem 1.7.

To state these results, we must define the relevant distributions, namely the GUE Tracy–Widom distribution, the Baik–Ben–Arous–Péché distributions, and the finite GUE distributions G_m. To define the former two distributions, we require the following kernels; each of these kernels will be normalized by a factor of $2\pi\mathbf{i}$ in order to simplify notation.

DEFINITION 1.1.   Denoting by $\mathrm{Ai}(x)$ the Airy function, the (normalized) *Airy kernel* $K_{\mathrm{Ai}}(x, y)$ is defined by

$$K_{\mathrm{Ai}}(x, y) = \frac{1}{2\pi\mathbf{i}} \oint \oint \exp\left(\frac{w^3}{3} - \frac{v^3}{3} - xv + yw\right) \frac{dw\,dv}{w - v},$$

where in the latter identity the contour for w is piecewise linear from $\infty e^{-\pi\mathbf{i}/3}$ to 0 to $\infty e^{\pi\mathbf{i}/3}$, and the contour for v is piecewise linear from $\infty e^{-2\pi\mathbf{i}/3}$ to -1 to $\infty e^{2\pi\mathbf{i}/3}$; see Figure 9 in Section 6 below.

DEFINITION 1.2.   Let m be a positive integer, and let $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{R}^m$ be a sequence of m real numbers. The (normalized) *Baik–Ben–Arous–Péché kernel* $K_{\mathrm{BBP};\mathbf{c}}(x, y)$ is defined by

$$K_{\mathrm{BBP};\mathbf{c}}(x, y) = \frac{1}{2\pi\mathbf{i}} \oint \oint \exp\left(\frac{w^3}{3} - \frac{v^3}{3} - xv + yw\right) \prod_{j=1}^{m} \frac{v + c_j}{w + c_j} \frac{dw\,dv}{w - v},$$

where the contour for w is piecewise linear from $\infty e^{-\pi\mathbf{i}/3}$ to $-E$ to $\infty e^{\pi\mathbf{i}/3}$, and the contour for v is piecewise linear from $\infty e^{-2\pi\mathbf{i}/3}$ to $-E - 1$ to $\infty e^{2\pi\mathbf{i}/3}$. Here, $E \in \mathbb{R}$ is chosen so that $E > \max_{1 \le i \le m} c_i$; see Figure 11 in Section 7 below.

Now we can define the GUE Tracy–Widom distribution F_{TW}; the Baik–Ben–Arous–Péché distributions $F_{BBP;\mathbf{c}}$; and the GUE distributions G_m. We refer to Appendix A of the Supplementary Material [3] for our conventions on Fredholm determinants (in particular, there will sometimes be a normalization of $2\pi\mathbf{i}$).

DEFINITION 1.3.   The *GUE Tracy–Widom distribution* $F_{TW}(s)$ is defined by

$$F_{TW}(s) = \det(\mathrm{Id} - K_{Ai})_{L^2(s,\infty)},$$

for each real number $s \in \mathbb{R}$.

Similarly, if m is a positive integer and $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{R}^m$ is a sequence of m real numbers, the Baik–Ben–Arous–Péché distribution $F_{BBP;\mathbf{c}}(s)$ is defined by

$$F_{BBP;\mathbf{c}}(s) = \det(\mathrm{Id} - K_{BBP;\mathbf{c}})_{L^2(s,\infty)}.$$

DEFINITION 1.4.   For any positive integer m, define the distribution $G_m(s)$ through

$$G_m(s) = Z_m^{-1} \int_{-\infty}^s \int_{-\infty}^s \cdots \int_{-\infty}^s \prod_{1 \le j < k \le m} |x_j - x_k|^2 \prod_{j=1}^m e^{-x_j^2/2} \, dx_j,$$

for each real number $s \in \mathbb{R}$; here, Z_m is a normalization constant chosen so that $G_m(\infty) = 1$.

Having defined the above distributions, we can now proceed to state our results. They concern the ASEP and stochastic six-vertex model with *generalized* $(b_1, b_2, \ldots, b_m)$-*Bernoulli initial data*. This initial data is not so quickly described; in particular, it relies on the *stochastic higher spin vertex models*, to be defined in Section 2. Thus, we reserve the definition of this initial data to Definition 3.3.

However, let us give a few properties of this initial data that might be of immediate interest:

1. In the case of the ASEP, all sites in $\mathbb{Z}_{>0}$ are initially unoccupied; in the case of the stochastic six-vertex model, no vertex on the x-axis is an entrance site for a path.

2. When $m = 1$ and $b_1 = b \in (0, 1)$, this degenerates to *step-Bernoulli initial data*. For the ASEP, this means that particles in $\mathbb{Z}_{\le 0}$ are initially occupied independently, with probability b. For the stochastic six-vertex model, this means that vertices on the positive y-axis are entrance sites for paths independently, with probability b.

3. Suppose $m > 1$ and $b_1 = b_2 = \cdots = b_m = b \in (0, 1)$. In the case of the ASEP, one expects (with high probability) the density of particles to the left of the origin to be approximately b (in the sense that one expects approximately $b|I|$ sites to initially be occupied in an interval $I \subset \mathbb{Z}_{<0}$). Similarly, in the case of the stochastic six-vertex model, one expects the density of entrance sites on the positive y-axis to be approximately b.

4. However, in the setting above (with $m > 1$ and all $b_j = b$), the events that different sites are occupied (for the ASEP) or different vertices are entrance sites (for the stochastic six-vertex model) are highly correlated. For example, suppose we restrict to the ASEP setting; the number of particles to the right of some negative site $-N$ at time 0, given by the current $J_0(-N)$, has law of large numbers bN but converges to the GUE G_m distribution after $N^{1/2}$-scaling as N tends to ∞. The fact that this is not a Gaussian distribution is due to the correlation between site occupations.

5. In the case of the ASEP, denote $q = L/R$, and in the case of the stochastic six-vertex model denote $q = \delta_1/\delta_2$ (q can be viewed as a measure of symmetry for these two models). Generalized Bernoulli initial data depends on this parameter q. If $q = 0$ (corresponding to the TASEP $L = 0$ setting), our initial data becomes the random particle configuration formed after running m steps of the geometric *Push TASEP* (with geometric rates $b_1, b_2, \ldots, b_m$) [19] on step initial data. This is known [20] to have a much more direct relationship with the spectral distribution of the spiked GUE matrices considered by Péché [38], which provides some explanation for the asymptotics stated in Theorem 1.6 below, when $q = 0$. In the general case $q \neq 0$, we know of no such explanation.

REMARK 1.5.   The ASEP is known to converge to the KPZ equation under a certain *weakly asymmetric scaling limit* [12] under which $q = L/R$ tends to 1. It seems plausible that, under this scaling limit, the generalized Bernoulli initial data will converge to the m-spiked boundary condition studied in [15].

The first property is directly stipulated as part of Definition 3.3, and the second property is explained in Remark 3.4. The third, fourth and fifth properties can also be derived from Definition 3.3, but establishing them carefully does not seem so relevant for the purposes of this paper. Thus, we omit their proofs.

Our results are as follows; Theorem 1.6 considers the ASEP with generalized step Bernoulli initial data and Theorem 1.7 considers the stochastic six-vertex model with this initial data.

THEOREM 1.6.   *Fix positive real numbers $R > L$, a real number $b \in (0, 1)$, an integer $m \geq 1$ and infinite families of real numbers $\{b_{1,T}\}_{T \in \mathbb{R}_{>0}}, \{b_{2,T}\}_{T \in \mathbb{R}_{>0}}, \ldots, \{b_{m,T}\}_{T \in \mathbb{R}_{>0}} \subset (0, 1)$. Assume that $R - L = 1$ and that there exist real numbers $d_1, d_2, \ldots, d_m \in \mathbb{R}$ such that $\lim_{T \to \infty} T^{1/3}(b_{i,T} - b) = d_i$, for each $i \in [1, m]$.*

Let T be a positive real number. Consider the ASEP, run for time T, with left jump rate L, right jump rate R and generalized $(b_{1,T}, b_{2,T}, \ldots, b_{m,T})$-Bernoulli initial data (given by Definition 3.3).

Then we have the following results, where in the below $\theta = 1 - 2b$ and $\chi = b(1 - b)$:

1. *Assume that $\eta \in (\theta, 1)$ is a real number. Set*

(1.2)
$$m_\eta = \left(\frac{1-\eta}{2}\right)^2; \qquad f_\eta = \left(\frac{1-\eta^2}{4}\right)^{2/3}.$$

Then, for any real number $s \in \mathbb{R}$, we have that

$$\lim_{T\to\infty} \mathbb{P}\left[\frac{m_\eta T - J_T(\eta T)}{f_\eta T^{1/3}} \le s\right] = F_{\mathrm{TW}}(s).$$

2. *Assume that $\{\eta_T\}_{T\ge 0}$ is a sequence of real numbers such that $\lim_{T\to\infty} T^{1/3} \times (\eta_T - \theta) = d$, for some real number d. Set m_η and f_η as in* (1.2), *and for each index $j \in [1, m]$, define*

(1.3)
$$c_j = -\frac{f_\eta(2d_j + d)}{2\chi}; \qquad \mathbf{c} = (c_1, c_2, \dots, c_m).$$

Then, for any real number $s \in \mathbb{R}$, we have that

$$\lim_{T\to\infty} \mathbb{P}\left[\frac{m_{\eta_T} T - J_T(\eta_T T)}{f_\theta T^{1/3}} \le s\right] = F_{\mathrm{BBP};\mathbf{c}}(s).$$

3. *Assume that $\eta \in (-b, \theta)$ is a real number, and that $b_j = b$ for all indices $j \in [1, m]$. Set*

(1.4)
$$m'_\eta = \chi - b\eta; \qquad f'_\eta = \chi^{1/2}(\theta - \eta)^{1/2}.$$

Then, for any real number $s \in \mathbb{R}$, we have that

$$\lim_{T\to\infty} \mathbb{P}\left[\frac{m'_\eta T - J_T(\eta T)}{f'_\eta T^{1/2}} \le s\right] = G_h(s).$$

THEOREM 1.7. *Fix positive real numbers $\delta_1 < \delta_2 < 1$, a real number $b \in (0, 1)$, an integer $m \ge 1$, and infinite families of real numbers $\{b_{1,T}\}_{T\in\mathbb{Z}_{>0}}$, $\{b_{2,T}\}_{T\in\mathbb{Z}_{>0}}, \dots, \{b_{m,T}\}_{T\in\mathbb{Z}_{>0}} \subset (0, 1)$. Assume that there exist real numbers $d_1, d_2, \dots, d_m \in \mathbb{R}$ such that $\lim_{T\to\infty} T^{1/3}(b_{i,T} - b) = d_i$, for each $i \in [1, m]$.*

Let $T > 0$ be a positive integer. Consider the stochastic six-vertex model, run for time T, with left jump probability δ_1, right jump probability δ_2, and generalized $(b_{1,T}, b_{2,T}, \dots, b_{m,T})$-Bernoulli initial data (given by Definition 3.3).

We have the following results, where in the below we set

(1.5)
$$\chi = b(1-b); \qquad \kappa = \frac{1-\delta_1}{1-\delta_2};$$
$$\Lambda = b + \kappa(1-b); \qquad \theta = \kappa^{-1}\Lambda^2.$$

1. *Assume that x and y are positive real numbers such that $x/y \in (\theta, \kappa)$. Set*

(1.6)
$$\mathcal{H}(x, y) = \frac{(\sqrt{(1-\delta_1)y} - \sqrt{(1-\delta_2)x})^2}{\delta_1 - \delta_2},$$

$$\mathcal{F}(x, y) = \frac{\kappa^{1/6}(\sqrt{\kappa x} - \sqrt{y})^{2/3}(\sqrt{\kappa y} - \sqrt{x})^{2/3}}{(\kappa - 1)x^{1/6}y^{1/6}}.$$

Then, for any real number $s \in \mathbb{R}$, we have that

$$\lim_{T \to \infty} \mathbb{P}\left[\frac{\mathcal{H}(x, y)T - \mathfrak{H}(xT, yT)}{\mathcal{F}(x, y)T^{1/3}} \le s\right] = F_{\mathrm{TW}}(s).$$

2. *Assume that $\{x_T\}_{T \in \mathbb{Z}_{>0}}$ and $\{y_T\}_{T \in \mathbb{Z}_{>0}}$ are sequences of positive real numbers such that $\eta_T = x_T/y_T$ satisfies $\lim_{T \to \infty} T^{1/3}(\eta_T - \theta) = d$, for some real number d. Set $\mathcal{H}(x, y)$ and $\mathcal{F}(x, y)$ as in (1.2), and for each index $j \in [1, m]$ define*

(1.7) $$c_j = -\frac{f_\eta d_j}{\chi} - \frac{\kappa f_\eta d}{2(\kappa - 1)\chi \Lambda}; \qquad \mathbf{c} = (c_1, c_2, \ldots, c_m).$$

Then, for any real number $s \in \mathbb{R}$, we have that

$$\lim_{T \to \infty} \mathbb{P}\left[\frac{\mathcal{H}(x, y)T - \mathfrak{H}(xT, yT)}{\mathcal{F}(x, y)T^{1/3}} \le s\right] = F_{\mathrm{BBP};\mathbf{c}}(s).$$

3. *Assume that x and y are positive real numbers such that $x \in (\Lambda^{-1}\theta y, \theta y)$, and that $b_j = b$ for all indices $j \in [1, m]$. Set*

(1.8) $$\mathcal{H}'(x, y) = by - \Lambda^{-1}bx; \qquad \mathcal{F}'(x, y) = \chi^{1/2}(y - \theta^{-1}x)^{1/2}.$$

Then, for any real number $s \in \mathbb{R}$, we have that

$$\lim_{T \to \infty} \mathbb{P}\left[\frac{\mathcal{H}'(x, y)T - \mathfrak{H}(xT, yT)}{\mathcal{F}'(x, y)T^{1/2}} \le s\right] = G_m(s).$$

REMARK 1.8.   We expect Theorem 1.6 to hold for all $\eta < \theta$ and Theorem 1.7 to hold for all $x/y \in (0, \theta)$ but will not pursue this potential improvement further. However, in Appendix B of the Supplementary Material [3], we outline a different method to establish asymptotics through a comparison with Schur measures [13, 35]. It seems likely that one can use this method to remove the limitation on η in the Gaussian regime.

REMARK 1.9.   The G_m fluctuations in the third parts of Theorem 1.6 and Theorem 1.7 can be traced back to the initial data, due to property 4 mentioned above.

What enables the proofs of Theorem 1.6 and Theorem 1.7 is the integrability of the *inhomogeneous stochastic higher spin vertex models* [21], which we will define

in Section 2. These are a family of quite general vertex models in the positive quadrant that exhibit remarkable integrability properties and degenerate to many known models in the KPZ universality class; in particular, they degenerate to the stochastic six-vertex model and the ASEP (both with generalized step Bernoulli initial data), facts which will be discussed further in Section 3 and Section 4.1.1.

These inhomogeneous stochastic higher spin vertex models were studied at length in the paper [21], in which multi-fold contour integral identities were obtained for certain observables (q-moments) of these models. In particular, it will be possible to directly degenerate these identities to the stochastic six-vertex model and ASEP with generalized step Bernoulli initial data; this will be done in Section 4.1.2. It is known [9, 14–16, 18] how to use such identities to produce Fredholm determinant identities that are directly amenable to asymptotic analysis, which we will do in Section 4.2. We especially emphasize here the work [9] of Barraquand, who used similar Fredholm determinant identities to derive Baik–Ben–Arous–Péché phase transitions (of the type given in Theorem 1.6 and Theorem 1.7) in the context of the q-TASEP with several slow particles.

In Section 5, we will begin the asymptotic analysis of these Fredholm determinant identities and explain how they can be used for the proofs of Theorem 1.6 and Theorem 1.7. In Section 6, Section 7 and Section 8, we will complete this analysis in the first, second and third regimes, respectively, of Theorem 1.6 and Theorem 1.7.

2. Stochastic higher spin vertex models.

Both the ASEP and stochastic six-vertex model are degenerations of a larger class of vertex models called the *inhomogeneous stochastic higher spin vertex models*, which were recently introduced by Borodin and Petrov in [21]. These models are in a sense the original source of integrability for the ASEP, the stochastic six-vertex model, and in fact most models proven to be in the Kardar–Parisi–Zhang (KPZ) universality class.

For that reason, we will begin our discussion by defining these vertex models. Similar to the stochastic six-vertex model, these models will take place on *directed path ensembles*. We will first carefully define what we mean by a directed path ensemble in Section 2.1, and then we will define the stochastic higher spin vertex model in Section 2.2. In Section 2.3, we will reinterpret these models as interacting particle systems, which will provide a useful framework for discussing observables in Section 4.

2.1. *Directed path ensembles.*   For the purpose of this paper, a *directed path* is a collection of *vertices*, which are lattice points in the nonnegative quadrant $\mathbb{Z}_{\geq 0}^2$, connected by *directed edges* (which we may also refer to as *arrows*). A directed edge can connect a vertex (i, j) to either $(i + 1, j)$ or $(i, j + 1)$, if $(i, j) \in \mathbb{Z}_{>0}^2$; we also allow directed edges to connect $(k, 0)$ to $(k, 1)$ or $(0, k)$ to $(1, k)$, for any positive integer k. Thus, directed edges connect adjacent vertices, always point either up or to the right, and do not lie on the x-axis or y-axis.

A *directed path ensemble* is a collection of paths with the following two properties:

- Each path must contain an edge connecting $(0, k)$ to $(1, k)$ or $(k, 0)$ to $(k, 1)$ for some $k > 0$; stated alternatively, every path "emanates" from either the x-axis or the y-axis.
- No two distinct paths can share a horizontal edge; however, in contrast with the six-vertex case, they may share vertical edges.

An example of a directed path ensemble was previously shown in Figure 1. See also Figure 5 in Section 3 below for more examples.

Associated with each $(x, y) \in \mathbb{Z}_{>0}^2$ in a path ensemble is an *arrow configuration*, which is a quadruple $(i_1, j_1; i_2, j_2) = (i_1, j_1; i_2, j_2)_{(x,y)}$ of nonnegative integers. Here, i_1 denotes the number of directed edges from $(x, y-1)$ to (x, y); equivalently, i_1 denotes the number of vertical incoming arrows at (x, y). Similarly, j_1 denotes the number of horizontal incoming arrows; i_2 denotes the number of vertical outgoing arrows (i.e., the number of directed edges from (x, y) to $(x, y+1)$); and j_2 denotes the number of horizontal outgoing arrows. Thus $j_1, j_2 \in \{0, 1\}$ at every vertex in a path ensemble, since no two paths share a horizontal edge. An example of an arrow configuration (which cannot be a vertex in a directed path ensemble, since $j_1, j_2 \notin \{0, 1\}$) is depicted in Figure 3.

Assigning values j_1 to vertices on the line $(1, y)$ and values i_1 to vertices on the line $(x, 1)$ can be viewed as imposing boundary conditions on the vertex model. If $j_1 = 1$ at $(1, k)$ and $i_1 = 0$ at $(k, 1)$ for each $k > 0$, then all paths enter through the y-axis, and every vertex on the positive y-axis is an entrance site for some path. We will refer to this particular assignment as *step initial data*; it was depicted in Figure 1, and it is also depicted on the left-hand side of Figure 5. In general, we will refer to any assignment of i_1 to $\mathbb{Z}_{>0} \times \{1\}$ and j_1 to $\{1\} \times \mathbb{Z}_{>0}$ as *initial data*, which can be deterministic (like step) or random.

Observe that, at any vertex in the positive quadrant, the total number of incoming arrows is equal to the total number of outgoing arrows; that is, $i_1 + j_1 = i_2 + j_2$. This is sometimes referred to as *arrow conservation* (or *spin conservation*). Any arrow configuration to all vertices of $\mathbb{Z}_{>0}^2$ that satisfies arrow conservation corresponds to a unique directed path ensemble, in which paths can share both vertical and horizontal edges.
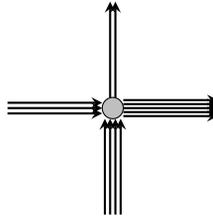

FIG. 3.   *Above is a vertex at which* $(i_1, j_1; i_2, j_2) = (4, 3; 2, 5)$.

2.2. *Probability measures on path ensembles.* The definition of the stochastic higher spin vertex models will closely resemble the definition of the stochastic six-vertex model given in Section 1.1.2. Specifically, we will first define probability measures $\mathbb{P}_n$ on the set of directed path ensembles whose vertices are all contained in triangles of the form $\mathbb{T}_n = \{(x, y) \in \mathbb{Z}_{\geq 0}^2 : x + y \leq n\}$, and then we will take a limit as n tends to infinity to obtain the vertex models in infinite volume. The first two measures $\mathbb{P}_0$ and $\mathbb{P}_1$ are both supported by the empty ensembles.

For each positive integer $n \geq 1$, we will define $\mathbb{P}_{n+1}$ from $\mathbb{P}_n$ through the following Markovian update rules. Use $\mathbb{P}_n$ to sample a directed path ensemble $\mathcal{E}_n$ on $\mathbb{T}_n$. This yields arrow configurations for all vertices in the triangle $\mathbb{T}_{n-1}$. To extend this to a path ensemble on $\mathbb{T}_{n+1}$, we must prescribe arrow configurations to all vertices (x, y) on the complement $\mathbb{T}_n \setminus \mathbb{T}_{n-1}$, which is the diagonal $\mathbb{D}_n = \{(x, y) \in \mathbb{Z}_{>0}^2 : x + y = n\}$. Since any incoming arrow to $\mathbb{D}_n$ is an outgoing arrow from $\mathbb{D}_{n-1}$, $\mathcal{E}_n$ and the initial data prescribe the first two coordinates, i_1 and j_1, of the arrow configuration to each $(x, y) \in \mathbb{D}_n$. Thus, it remains to explain how to assign the second two coordinates (i_2 and j_2) to any vertex on $\mathbb{D}_n$, given the first two coordinates.

This is done by producing $(i_2, j_2)_{(x,y)}$ from $(i_1, j_1)_{(x,y)}$ according to the transition probabilities

$$\mathbb{P}_n\big[(i_2, j_2) = (k, 0) \mid (i_1, j_1) = (k, 0)\big] = \frac{1 - q^k s_x \xi_x u_y}{1 - s_x \xi_x u_y},$$

$$\mathbb{P}_n\big[(i_2, j_2) = (k - 1, 1) \mid (i_1, j_1) = (k, 0)\big] = \frac{(q^k - 1) s_x \xi_x u_y}{1 - s_x \xi_x u_y},$$

(2.1)

$$\mathbb{P}_n\big[(i_2, j_2) = (k + 1, 0) \mid (i_1, j_1) = (k, 1)\big] = \frac{1 - q^k s_x^2}{1 - s_x \xi_x u_y},$$

$$\mathbb{P}_n\big[(i_2, j_2) = (k, 1) \mid (i_1, j_1) = (k, 1)\big] = \frac{q^k s_x^2 - s_x \xi_x u_y}{1 - s_x \xi_x u_y},$$

for any nonnegative integer k. We also set $\mathbb{P}_n[(i_2, j_2) \mid (i_1, j_1)] = 0$ for all $(i_1, j_1; i_2, j_2)$ not of the above form. In the above, $q \in \mathbb{C}$ is a complex number and $U = (u_1, u_2, \ldots) \subset \mathbb{C}$, $\Xi = (\xi_1, \xi_2, \ldots) \subset \mathbb{C}$, and $S = (s_1, s_2, \ldots) \subset \mathbb{C}$ are infinite sets of complex numbers, chosen to ensure that all of the above probabilities are nonnegative. This can be arranged for instance when $q \in (0, 1)$, $U \subset (-\infty, 0]$, and $S, \Xi \subset [0, 1]$, although there are also other suitable choices.

Choosing (i_2, j_2) according to the above transition probabilities yields a random directed path ensemble $\mathcal{E}_{n+1}$, now defined on $\mathbb{T}_{n+1}$; the probability distribution of $\mathcal{E}_{n+1}$ is then denoted by $\mathbb{P}_{n+1}$. We define $\mathbb{P} = \lim_{n \to \infty} \mathbb{P}_n$. Then, $\mathbb{P}$ is a probability measure on the set of directed path ensembles that is dependent on the complex parameters q, U, Ξ, and S. The variables $U = (u_1, u_2, \ldots)$ are occasionally referred to as *spectral parameters*, the variables $\Xi = (\xi_1, \xi_2, \ldots)$ as *spacial inhomogeneity parameters* and the variables $S = (s_1, s_2, \ldots)$ as *spin parameters*.

If there exists a positive integer I such that $s^2 q^I = 1$ for all $s \in S$, then $\mathbb{P}[(i_2, j_2) = (I+1, 0) \mid (i_1, j_1) = (I, 1)] = 0$; thus, the number of vertical incoming or outgoing arrows at any vertex remains less than $I + 1$. In this case, the vertex model is said to have *spin* $I/2$; if no such I exists, then the spin of the model is said to be infinite.

2.3. *Vertex models and interacting particle systems.* Let $\mathbb{P}^{(T)}$ denote the restriction of the random path ensemble with step initial data (given by the measure $\mathbb{P}$ from the previous section) to the strip $\mathbb{Z}_{>0} \times [0, T]$. Assume that all T paths in this restriction almost surely exit the strip $\mathbb{Z}_{>0} \times [0, T]$ through its top boundary; this will be the case, for instance, if $\mathbb{P}_n[(i_2, j_2) = (0, k) \mid (i_1, j_1) = (0, k)]$ is uniformly bounded below 1 over all n and k.

We will use the probability measure $\mathbb{P}^{(T)}$ to produce a discrete-time interacting particle system on $\mathbb{Z}_{>0}$, defined up to time $T - 1$, as follows. Sample a line ensemble $\mathcal{E}$ randomly under $\mathbb{P}^{(T)}$, and consider the arrow configuration it associates with some vertex $(p, t) \in \mathbb{Z}_{>0} \times [1, T]$. We will place k particles at position p and time $t - 1$ if and only if $i_1 = k$ at the vertex (p, t). Therefore, the paths in the path ensemble $\mathcal{E}$ correspond to space-time trajectories of the particles; see Figure 4.

Let us introduce notation for particle positions. A *nonnegative signature* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of *length* n is a nonincreasing sequence of n integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We denote the set of nonnegative signatures of length n by Sign_n^+, and the set of all nonnegative signatures by $\mathrm{Sign}^+ = \bigcup_{N=0}^{\infty} \mathrm{Sign}_n^+$. For any signature λ and integer j, let $m_j(\lambda)$ denote the number of indices i for which $\lambda_i = j$; that is, $m_j(\lambda)$ is the multiplicity of j in λ.

We can associate a configuration of n particles in $\mathbb{Z}_{\geq 0}$ with a signature of length n as follows. A signature $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is associated with the particle configuration that has $m_j(\lambda)$ particles at position j, for each nonnegative integer j. Stated alternatively, λ is the ordered set of positions in the particle configuration. If $\mathcal{E}$ is a directed line ensemble on $\mathbb{Z}_{\geq 0} \times [0, n]$, let $p_n(\mathcal{E}) \in \mathrm{Sign}_n^+$ denote the signature associated with the particle configuration produced from $\mathcal{E}$ at time n.

Then, $\mathbb{P}^{(n)}$ induces a probability measure on Sign_n^+, defined as follows.

DEFINITION 2.1. For any positive integer n, let $\mathbf{M}_n$ denote the measure on Sign_n^+ (dependent on the parameters U, S, and Ξ) defined by setting $\mathbf{M}_n(\lambda) = \mathbb{P}^{(n)}[p_n(\mathcal{E}) = \lambda]$, for each $\lambda \in \mathrm{Sign}_n^+$.

FIG. 4. *Shown above and to the left is a path ensemble; shown above and to the right is the associated interacting particle system.*

If a model has spin $I/2$, then it exhibits a type of *weak exclusion principle*, which states that at most I particles can occupy a given location [or equivalently that $m_j(p(\mathcal{E})) \leq I$ for each $j \in p(\mathcal{E})$]. We will be interested in the case when $I = 1$, which yields the stochastic six-vertex model.

3. The spin 1/2 vertex models. We will be particularly interested in the stochastic higher spin vertex model in the *spin* $1/2$ setting, when all of the s_k are equal to $s = q^{-1/2}$. Under this degeneration, the model becomes the stochastic six-vertex model. In Section 3.1, we will explain this degeneration in more detail. Then, in Section 3.2, we will discuss the ASEP and state how the stochastic six-vertex model degenerates to it.

3.1. *The stochastic six-vertex model.* The goal of this section is twofold. First, we explain how the stochastic six-vertex model can be recovered directly from a degeneration of the stochastic higher spin vertex model. Second, we formally define and give examples of initial data for the stochastic six-vertex model.

3.1.1. *The stochastic six-vertex model through higher spin vertex models.* Let us examine what happens to the probabilities (2.1) when $u_k = u$, $\xi_k = 1$, and $s_k = q^{-1/2}$ for all $k \geq 0$. Since $\mathbb{P}_n[(i_2, j_2) = (2, 0) \mid (i_1, j_1) = (1, 1)] = (1 - qs^2)/(1 - su) = 0$, we have that $i_1, i_2 \in \{0, 1\}$ at each vertex. Thus no two paths can share a vertical edge, meaning that $\mathbb{P}$ is supported on the set of six-vertex directed path ensembles defined in Section 1.1.2.

Now let us reparameterize. Let $\delta_1, \delta_2 \in (0, 1)$ satisfy $\delta_1 < \delta_2$, and define $\kappa = (1 - \delta_1)/(1 - \delta_2) > 1$. Denote

$$q = \frac{\delta_1}{\delta_2} < 1; \qquad u = \kappa s = \frac{1 - \delta_1}{1 - \delta_2}\sqrt{\frac{\delta_2}{\delta_1}} > 1.$$

Substituting this specialization into the other probabilities in (2.1), we find that

$$
\begin{aligned}
&\mathbb{P}_n[(i_2, j_2) = (0, 0) \mid (i_1, j_1) = (0, 0)] = 1 \\
&\qquad = \mathbb{P}_n[(i_2, j_2) = (1, 1) \mid (i_1, j_1) = (1, 1)], \\
&\mathbb{P}_n[(i_2, j_2) = (1, 0) \mid (i_1, j_1) = (1, 0)] = \delta_1; \\
&\mathbb{P}_n[(i_2, j_2) = (0, 1) \mid (i_1, j_1) = (1, 0)] = 1 - \delta_1, \\
&\mathbb{P}_n[(i_2, j_2) = (0, 1) \mid (i_1, j_1) = (0, 1)] = \delta_2; \\
&\mathbb{P}_n[(i_2, j_2) = (1, 0) \mid (i_1, j_1) = (0, 1)] = 1 - \delta_2.
\end{aligned}
$$

(3.1)

Since the probabilities (3.1) coincide with the probabilities depicted in Figure 2, it follows that the measure $\mathbb{P}$ of Section 2.2 (under the above specialization) is equal to the stochastic six-vertex measure $\mathcal{P}(\delta_1, \delta_2)$ defined in Section 1.1.2.

3.1.2. *Initial data for the stochastic six-vertex model.* We start with the following definition.

DEFINITION 3.1. Let $\varphi = (\varphi(1), \varphi(2), \ldots)$, where $\varphi(k) = (\varphi_k^{(x)}, \varphi_k^{(y)}) \in \mathbb{Z}_{\geq 0} \times \{0, 1\}$ for each k, be a stochastic process. The stochastic higher spin vertex model with *initial data* φ is the stochastic higher spin vertex model [as defined in Section 2.2 that evolves according to the transition probabilities given by (2.1)] whose boundary condition is defined by setting $j_1 = \varphi_k^{(y)}$ at $(1, k)$ and $i_1 = \varphi_k^{(x)}$ at $(k, 1)$, for each k.

EXAMPLE 3.2. *Step Bernoulli initial data* with *parameter b* arises when the $\varphi(i)$ are all mutually independent, $\varphi_i^{(x)} \equiv 0$, and the $\varphi_i^{(y)}$ are independent $0 - 1$ Bernoulli random variables with mean b.

Under this initial data, paths can only enter from the y-axis, and each vertex on the positive y-axis is an entrance point for a path with probability b. The case of step initial data is recovered when $b = 1$.

In the example above, the $\varphi(i)$ are mutually independent, but they need not be in general. For instance, this is the case for the following example, which introduces a class of initial data that generalizes step Bernoulli initial data.

DEFINITION 3.3. If m is a positive integer and $b_1, b_2, \ldots, b_m \in [0, 1]$ are positive real numbers, then *step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data* is defined as follows. First, we set $\varphi_i^{(x)} \equiv 0$, so that no paths enter through the x-axis.

Next, to define $\varphi_i^{(y)}$, consider the stochastic higher spin vertex model (with step initial data), restricted to the strip $[0, m + 1] \times [0, \infty)$, in which the transition probabilities given by (2.1) are replaced with the probabilities

$$\mathbb{P}_n[(i_2, j_2) = (k, 0) \mid (i_1, j_1) = (k, 0)] = 1 - (1 - q^k)b_x,$$

(3.2)
$$\mathbb{P}_n[(i_2, j_2) = (k - 1, 1) \mid (i_1, j_1) = (k, 0)] = (1 - q^k)b_x,$$

$$\mathbb{P}_n[(i_2, j_2) = (k + 1, 0) \mid (i_1, j_1) = (k, 1)] = 1 - b_x,$$

$$\mathbb{P}_n[(i_2, j_2) = (k, 1) \mid (i_1, j_1) = (k, 1)] = b_x,$$

at the vertex (x, y) for all integers $x \in [1, m]$ and $y \in [1, \infty)$. Then, define the random variable $\varphi_i^{(y)}$ to be the value of j_2 at the vertex (m, i), for each positive integer i.

REMARK 3.4. Observe that this coincides with step b_1-Bernoulli initial data when $m = 1$. Indeed, in that case, $j_1 = 1$ deterministically at each vertex on the positive y-axis due to the step initial data. Therefore, the top two probabilities in (3.2) have no effect, and the latter two probabilities imply that j_2 is a Bernoulli $0 - 1$ random variable with mean b_1 at each vertex on the positive y-axis.
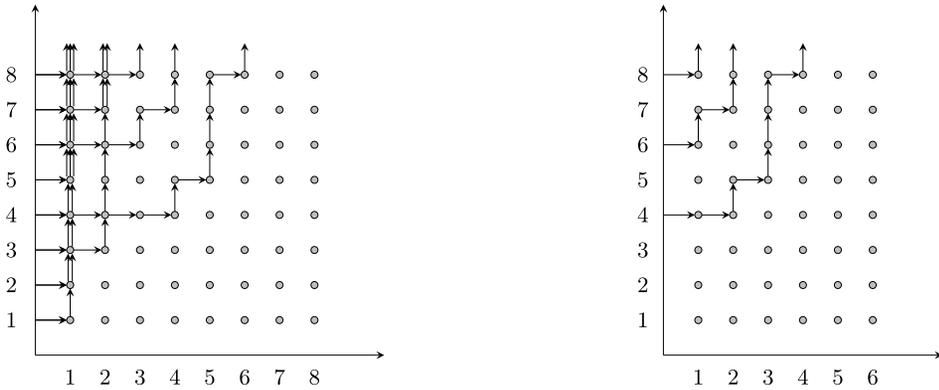
FIG. 5. *Shown above and to the left is the stochastic higher spin vertex model specialized as in Definition* 3.3, *with* $m = 2$. *Shown to the right is this model shifted to the left by* 2, *yielding a stochastic six-vertex model with generalized step Bernoulli initial data.*

REMARK 3.5. Running a stochastic six-vertex model with this initial data can be viewed as running a stochastic higher spin vertex model with step initial data, whose transition probabilities in the first m columns match those given by (3.2), and whose probabilities in all other columns match those of (3.1). This is depicted on the left-hand side of Figure 5 in the case $m = 2$.

On the right-hand side of Figure 5 is the same model "shifted" to the left by 2 with all arrows originally in the first two columns removed; this is how a stochastic six-vertex model with generalized step Bernoulli initial data might look.

3.2. *The ASEP.*

In this section we discuss the asymmetric simple exclusion process (ASEP). Our goals are again twofold. First, we formally define initial data for the ASEP; then, we explain a limit degeneration that maps the stochastic six-vertex model to the ASEP. The latter point will be particularly useful to us later, in Section 4.2, where it will allow us to produce identities for the ASEP from identities for the stochastic six-vertex model.

3.2.1. *Initial data for the ASEP.*

Recall from Section 1.1.1 that the ASEP is a continuous time Markov process $\{\eta_t(i)\}_{i\in\mathbb{Z},t\in\mathbb{R}_{\geq 0}} \subset \{0,1\}^{\mathbb{Z}} \times \mathbb{R}_{\geq 0}$, whose dynamics can be defined as follows.

For each pair of consecutive integers $(i, i + 1) \in \mathbb{Z}^2$, the variables $\eta_t(i)$ and $\eta_t(i + 1)$ are interchanged (meaning we set $\eta_t(i) = \eta_{t^-}(i + 1)$ and $\eta_t(i + 1) = \eta_{t^-}(i)$, but leave all other $\eta_t(j)$ fixed) at exponential rate $L \geq 0$ if $(\eta_{t^-}(i), \eta_{t^-}(i + 1)) = (0, 1)$; similarly, the variables $\eta_t(i)$ and $\eta_t(i + 1)$ are interchanged at exponential rate $R \geq 0$ if $(\eta_{t^-}(i), \eta_{t^-}(i + 1)) = (1, 0)$. The exponential clocks corresponding to each pair $(i, i + 1)$ are mutually independent. Here, t^- refers to the time infinitesimally before time t.

The random variables $\eta_t(i)$ can be viewed as indicators for the event that a particle is at position i at time t. Therefore, the dynamics at ASEP are equivalent to particles independently attempting to jump left at rate L and right at rate R, subject to the exclusion restriction. Specifically, if the destination of the jump is unoccupied, the jump is performed; otherwise, it is not. Throughout, we will assume that $R > L$, so that particles drift to the right on average.

Now we can define initial data for the ASEP.

DEFINITION 3.6.   Let $\varphi = (\varphi(1), \varphi(2), \ldots)$ be a stochastic process, where $\varphi(i) = (\varphi_i^{(x)}, \varphi_i^{(y)}) \in \{0, 1\} \times \{0, 1\}$. Define the *ASEP with initial data φ* to be the ASEP with $\eta_0(i) = \varphi_i^{(x)}$ if i is positive, and $\eta_0(i) = \varphi_{1-i}^{(y)}$ if i is nonpositive.

EXAMPLE 3.7.   If $b \in [0, 1]$, then *step b-Bernoulli initial data* arises when the $\varphi(i)$ are mutually independent, $\varphi_i^{(x)} \equiv 0$, and the $\varphi_i^{(y)}$ are independent $0 - 1$ Bernoulli random variables with mean b, for each i. That is, particles are initially placed at or to the left of 0 with probability b, and no particles are placed to the right of 0; all placements are independent.

3.2.2. *Degeneration of the stochastic six-vertex model to the ASEP.*   In this section we explain a certain limit degeneration under which the stochastic six-vertex model converges to the ASEP.

To briefly provide a heuristic as to why such a convergence should exist, consider the stochastic six-vertex model $\mathcal{P}(\delta_1, \delta_2)$ in which $\delta_1 = \delta_2 = 0$ (see Section 3.1.1 for definitions). In this case, all paths will almost surely alternate between turning one space up and one space right. In terms of the interacting particle system interpretation discussed in Section 2.3, each particle almost surely jumps one space to the right at each time step; thus, if we translate each particle t spaces to the left at time t (we refer to this as *offsetting by the diagonal*), then the particle system will not evolve over time.

Now assume that $\delta_1 = \varepsilon L$ and $\delta_2 = \varepsilon R$, for some positive real numbers $R, L > 0$ and some very small number $\varepsilon > 0$. Then, paths will "usually" turn up and right; thus, after offsetting by the diagonal, particles will usually not move. However, very rarely, an offset particle will jump to the left (with probability εL) and to the right (with probability εR), subject to the restriction that a particle cannot jump to an occupied location. After rescaling time by ε^{-1}, and taking the limit as ε tends to 0, this coincides with the dynamics of the ASEP.

Thus, after offsetting by the diagonal, one expects that the stochastic six-vertex model $\mathcal{P}(\varepsilon L, \varepsilon R)$ should converge to the ASEP. This heuristic was provided in Section 2.2 of [16] (see also Section 6.5 of [21]) and later established in Theorem 1 and Corollary 2 of [1].

The following proposition, which appears as Corollary 2 of [1], states this heuristic precisely in terms of convergence of currents. In what follows, we recall

the definitions of the currents J of the ASEP and $\mathfrak{H}$ of the stochastic six-vertex model from Section 1.1.1 and Section 1.1.2.

PROPOSITION 3.8 ([1], Corollary 2). *Fix real numbers $R, L \geq 0$ and a stochastic process $\varphi = (\varphi(1), \varphi(2), \ldots)$, with $\varphi(i) = (\varphi_i^{(x)}, \varphi_i^{(y)}) \in \{0, 1\} \times \{0, 1\}$. Let $\varepsilon > 0$ be a real number, and denote $\delta_1 = \delta_{1;\varepsilon} = \varepsilon L$ and $\delta_2 = \delta_{2;\varepsilon} = \varepsilon R$; assume that $\delta_1, \delta_2 \in (0, 1)$. Also fix $r \in \mathbb{R}$, $t \in \mathbb{R}_{>0}$ and $x \in \mathbb{Z}$.*

Let $p_\varepsilon(x; r) = \mathbb{P}_V[\mathfrak{H}(x + \lfloor \varepsilon^{-1} t \rfloor, \lfloor \varepsilon^{-1} t \rfloor) \geq r]$, where the probability $\mathbb{P}_V$ is under the stochastic six-vertex model $\mathcal{P}(\delta_1, \delta_2)$ with initial data φ. Furthermore, let $p(x; r) = \mathbb{P}_A[J_t(x) \geq r]$, where the probability $\mathbb{P}_A$ is under the ASEP with left jump rate L, right jump rate R, and initial data φ.

Then, $\lim_{\varepsilon \to 0} p_\varepsilon(x; r) = p(x; r)$.

4. Observables for models with generalized step Bernoulli initial data.

We now turn to the analysis of the ASEP and stochastic six-vertex model with generalized step Bernoulli initial data. Our results in this section are primarily algebraic; our goal is to establish Theorem 4.10 and Theorem 4.11, which are Fredholm determinant identities for the q-Laplace transform of the current of the stochastic six-vertex model and ASEP, respectively, with generalized step Bernoulli initial data. These identities will be amenable to asymptotic analysis and will therefore enable the proofs of the phase transition results Theorem 1.6 and Theorem 1.7 later in the paper.

The proofs of these Fredholm determinant identities rely on the integrability of the stochastic six-vertex model under generalized step Bernoulli initial data. In particular, we will first show how to degenerate the stochastic higher spin vertex model with step initial data to the stochastic six-vertex model with generalized step Bernoulli initial data. Using the results of [21], this then yields contour integral identities for q-moments of the current of this model, which will give rise to the Fredholm determinant identities Theorem 4.10 and Theorem 4.11.

4.1. *Contour integral identities for q-moments of the current.* Our goal in this section is to establish Proposition 4.4, which is a multi-fold contour integral identity for q-moments of the current of the stochastic six-vertex model with step Bernoulli initial data.

Obtaining such identities for the stochastic higher spin vertex models with step initial data has been investigated at length in [21]; see, for example, Theorem 9.8 of that article. We will show that, under a certain specialization, the stochastic higher spin vertex model with step initial data degenerates to the stochastic six-vertex model with different initial data, namely, generalized step Bernoulli initial data. Then it will be possible to degenerate the results of [21] to obtain the contour integral identities relevant for the proofs of Theorem 4.10 and Theorem 4.11.

4.1.1. *A specialization of the stochastic higher spin vertex model.* Consider the stochastic higher spin vertex model as defined in Section 2.2, with the following specialization of parameters. Let $m \geq 1$ be some positive integer and $b_1, b_2, \ldots, b_m \in (0, 1)$ be positive real numbers. Let $\delta_1 < \delta_2$ be positive real numbers in the interval $(0, 1)$, and let

$$q = \frac{\delta_1}{\delta_2} < 1; \qquad \kappa = \frac{1 - \delta_1}{1 - \delta_2} > 1;$$

(4.1)

$$s = q^{-1/2}; \qquad u = \kappa s; \qquad \beta_i = \frac{b_i}{1 - b_i},$$

for each $i \in [1, m]$, as in Section 3.1. Now set $u_j = u$ for each $j \geq 1$; $s_j = s$ and $\xi_j = 1$ for each $j \geq m + 1$; $s_i = a_i$ for each $i \in [1, m]$; and $\xi_i = -\beta_i / a_i u$ for each $i \in [1, m]$.

For $x > m$, the probabilities in (2.1) match with those of the stochastic six-vertex model given by (3.1). If $x \in [1, m]$, then the probabilities become

$$\mathbb{P}_n\big[(i_2, j_2) = (k, 0) \mid (i_1, j_1) = (k, 0)\big] = \frac{1 + q^k \beta_x}{1 + \beta_x},$$

$$\mathbb{P}_n\big[(i_2, j_2) = (k - 1, 1) \mid (i_1, j_1) = (k, 0)\big] = \frac{(1 - q^k)\beta_x}{1 + \beta_x},$$

(4.2)

$$\mathbb{P}_n\big[(i_2, j_2) = (k + 1, 0) \mid (i_1, j_1) = (k, 1)\big] = \frac{1 - q^k a_x^2}{1 + \beta_x},$$

$$\mathbb{P}_n\big[(i_2, j_2) = (k, 1) \mid (i_1, j_1) = (k, 1)\big] = \frac{q^k a_x^2 + \beta_x}{1 + \beta_x}.$$

Taking the limit as a_i tends to 0 for each $i \in [1, m]$ and recalling that $\beta_i = b_i/(1 - b_i)$ for each i yields the probabilities in (3.2).

In particular, let us define $\varphi_1^{(y)}, \varphi_2^{(y)}, \ldots$ by setting $\varphi_k^{(y)} = 1$ if $j_2 = 1$ at the vertex (k, m) and $\varphi_k^{(y)} = 0$ otherwise; equivalently, $\varphi_k^{(y)}$ is the indicator that there exists some path containing an arrow from (m, k) to $(m + 1, k)$. Then the initial data defined by $\varphi = (\varphi(1), \varphi(2), \ldots)$, where $\varphi(i) = (\varphi_i^{(x)}, \varphi_i^{(y)}) = (0, \varphi_i^{(y)})$, coincides with the step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data given by Definition 3.3.

Thus, if we translate the vertex model to the left by m (so that the line $y = m$ is shifted onto the y-axis) and then ignore all arrows to the left of the y-axis, we obtain the stochastic six-vertex model with step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data; see Remark 3.5 and Figure 5.

4.1.2. *Observables for the stochastic six-vertex model.* In this section we will establish contour integral identities for q-moments of the current of the stochastic six-vertex model with generalized step Bernoulli initial data. We will begin with Proposition 4.3, which gives an identity for q-moments of the height function of

the stochastic higher spin vertex model in a reasonably general setting. However, we first require some notation.

In what follows, n will be a positive integer; $q \in (0, 1)$ will be a positive real number; $U = (u_1, u_2, \ldots, u_n) \subset \mathbb{C}$ will be a finite set of nonzero complex numbers; and $\Xi = (\xi_1, \xi_2, \ldots) \subset \mathbb{C}$ and $S = (s_1, s_2, \ldots) \subset \mathbb{C}$ will be infinite sets of nonzero complex numbers.

The following definition is a restriction on our parameters so that the contours we consider exist.

DEFINITION 4.1.   We call the triple $(U; \Xi, S)$ *suitably spaced* if the following two conditions are satisfied:

- The elements of U are real and sufficiently close together so that they all have the same sign and $q \max_{1 \leq i \leq n} |u_i| < \min_{1 \leq i \leq n} |u_i|$.
- No number of the form $s_i \xi_i$ is contained in the interval $(\min_{1 \leq i \leq n} u_i^{-1}, \max_{1 \leq i \leq n} u_i^{-1})$.

The following defines the contours we will use in this section.

DEFINITION 4.2.   Suppose that $(U; \Xi, S)$ is suitably spaced. Let us define the set of k positively oriented contours $\gamma_1(U; \Xi, S), \gamma_2(U; \Xi, S), \ldots, \gamma_k(U; \Xi, S)$ as follows. Each contour $\gamma_i = \gamma_i(U; \Xi, S)$ will be the disjoint union of two positively oriented circles $c_i^{(1)}$ and $c_i^{(2)}$.

Here, $c^{(1)} = c_1^{(1)} = c_2^{(1)} = \cdots = c_k^{(1)}$ are all the same circle that contain each of the u_i^{-1} and leave outside $\xi_1 s_1, \xi_2 s_2, \ldots$. We also assume that $c^{(1)}$ is sufficiently small so that its interior is disjoint with the image of its interior under multiplication by q.

The circles $c_1^{(2)}, c_2^{(2)}, \ldots, c_k^{(2)}$ will be centered at 0 and sufficiently small such that the following two properties hold:

- The circles are "nested" in the sense that, for each integer $i \in [1, k-1]$, the circle $c_{i+1}^{(2)}$ strictly contains $q^{-1} c_i^{(2)}$ (defined to be the image of $c_i^{(2)}$ under multiplication by q^{-1}).
- All circles are sufficiently small so that the interior of $c_k^{(2)}$ is disjoint with the interior of $q c^{(1)}$ and so that the interior of $c_k^{(2)}$ does not contain $s_1 \xi_1, s_2 \xi_2, \ldots$.

The fact that $(U; \Xi, S)$ is suitably spaced ensures the existence of such contours. See Figure 6 for an example.

Now we can state Proposition 9.5 of [21]. In what follows, $\mathfrak{h}_\lambda(x)$ denotes the number of indices i for which $\lambda_i \geq x$, for any $\lambda \in \mathrm{Sign}^+$.

PROPOSITION 4.3 ([21], Proposition 9.5).   *Let k, x and t be positive integers; $q \in (0, 1)$ be a positive real number; $U = (u_1, u_2, \ldots, u_t)$ be a set of t positive*
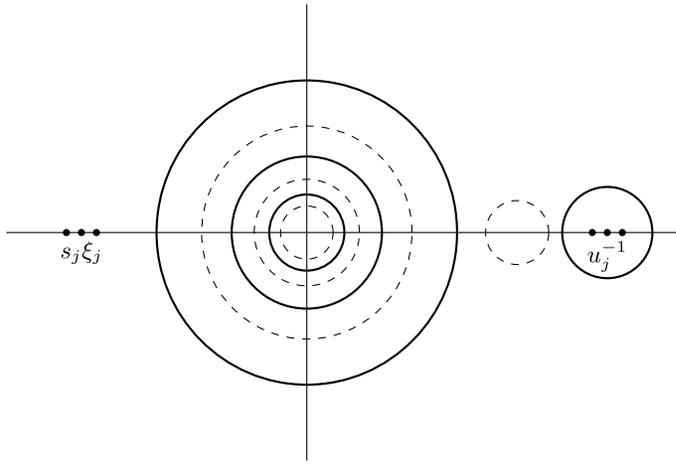
FIG. 6. *Shown above in solid is an example of possible contours $\gamma_1(U; \Xi, S)$, $\gamma_2(U; \Xi, S)$ and $\gamma_3(U; \Xi, S)$. The dashed curves are the images of the solid curves under multiplication by q.*

real numbers; and $S = (s_0, s_1, s_2, \ldots)$ and $\Xi = (\xi_0, \xi_1, \ldots) \in (0, \infty)$ be infinite sets of real numbers, such that $(U; \Xi, S)$ is suitably spaced and $|s_j \xi_j u_i - s_j^2| < |1 - s_j \xi_j u_i|$ for each i, j.

Under the measure $\mathbf{M} = \mathbf{M}_t$ given by Definition 2.1, with spectral parameters U, spin parameters $S = (s_1, s_2, \ldots)$ and spectral inhomogeneity parameters $\Xi = (\xi_1, \xi_2, \ldots)$, we have that

(4.3)
$$\mathbb{E}_{\mathbf{M}}\left[q^{k\mathfrak{h}_\lambda(x)}\right] = \frac{q^{\binom{k}{2}}}{(2\pi\mathbf{i})^k} \oint \cdots \oint \prod_{1 \leq i < j \leq k} \frac{w_i - w_j}{w_i - q w_j}$$
$$\times \prod_{i=1}^{k}\left(\prod_{j=1}^{x-1} \frac{s_j \xi_j - s_j^2 w_i}{s_j \xi_j - w_i} \prod_{j=1}^{t} \frac{1 - q u_j w_i}{1 - u_j w_i}\right)\frac{dw_i}{w_i},$$

where each w_i is integrated along the contour $\gamma_i = \gamma_i(U; \Xi, S)$ of Definition 4.2.

In fact, as stated above, Proposition 4.3 does not precisely coincide with Proposition 9.5 of [21]; the latter was originally stated with more stringent analytic constraints on the parameters U, S and Ξ. However, Lemma 9.1 of [21] states that the left-hand side of (4.3) is a rational function in all of these parameters. Thus, Proposition 4.3 above follows from Proposition 9.5 of [21] and a standard (see, e.g., Section 10 of [21]) analytic continuation argument, which we will not detail further.

Using Proposition 4.3, we can obtain q-moments of the current for the stochastic six-vertex model with generalized step Bernoulli initial data; recall Definition 3.3 for a description of this initial data. In what follows, for any $(x, t) \in \mathbb{Z}_{\geq 0}^2$,

$\mathfrak{h}_t(x)$ denotes the *height function* of a stochastic higher spin vertex model, that is, the number of integers $k \geq x$ such that $j_2 = 1$ at (k, t); stated alternatively, $\mathfrak{h}_t(x)$ denotes the number of paths that are strictly to the right of $x - 1$ at time t. Observe that $\mathfrak{h}_t(x) \leq t$ and is thus finite.

PROPOSITION 4.4. *Fix $m, t \in \mathbb{Z}_{>0}$; $\delta_1, \delta_2 \in (0, 1)$; and $b_1, b_2, \ldots, b_m \in (0, 1)$. Consider the stochastic six-vertex model with left jump probability δ_1, right jump probability δ_2, and step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data. Denoting q, u, κ, s, and β_i as in* (4.1), *we have that*

$$
(4.4) \quad
\begin{aligned}
\mathbb{E}[q^{k \mathfrak{h}_t(x)}] = {} & \frac{q^{\binom{k}{2}}}{(2\pi \mathbf{i})^k} \oint \cdots \oint \prod_{1 \leq i < j \leq k} \frac{y_i - y_j}{y_i - q y_j} \\
& \times \prod_{i=1}^{k} \left(\frac{1 + y_i}{1 + q^{-1} y_i} \right)^t \left(\frac{1 + q^{-1} \kappa^{-1} y_i}{1 + \kappa^{-1} y_i} \right)^{x-1} \\
& \times \prod_{i=1}^{k} \left(\prod_{j=1}^{m} \frac{1}{1 - q^{-1} \beta_j^{-1} y_i} \right) \frac{d y_i}{y_i},
\end{aligned}
$$

where each y_i is integrated along the contour $\gamma_i(\widehat{U}; \widehat{\Xi}, \widehat{S})$ of Definition 4.2. *Here, we have denoted $\widehat{U} = (-q^{-1}, -q^{-1}, \ldots)$, $\widehat{S} = (1, 1, \ldots)$ and $\widehat{\Xi} = (q\beta_1, q\beta_2, \ldots, q\beta_m, -\kappa, -\kappa, \ldots)$.*

PROOF. We will apply Proposition 4.3 to the stochastic higher spin vertex model, specialized as in Section 4.1.1, run for t discrete time steps. That is, set $u_i = u$ for each $i \in [1, t]$; $\xi_i = 1$ and $s_i = s$ if $i > m$; $s_i = a_i \in (-1, 0)$ to be some small negative real number if $i \in [1, m]$; and $\xi_i = -\beta_i / a_i u$ if $i \in [1, m]$. Also set $\xi_0 = 1$ and $s_0 = s$.

Now, applying Proposition 4.3, we obtain

$$
\begin{aligned}
\mathbb{E}_{\mathbf{M}}[q^{k \mathfrak{h}_\lambda(x+m)}] = {} & \frac{q^{\binom{k}{2}}}{(2\pi \mathbf{i})^k} \oint \cdots \oint \prod_{1 \leq i < j \leq k} \frac{w_i - w_j}{w_i - q w_j} \\
& \times \prod_{i=1}^{k} \left(\frac{1 - q u w_i}{1 - u w_i} \right)^t \left(\frac{s - s^2 w_i}{s - w_i} \right)^{x-1} \\
& \times \prod_{i=1}^{k} \left(\prod_{j=1}^{m} \frac{\beta_j + a_j^2 u w_i}{\beta_j + u w_i} \right) \frac{d w_i}{w_i},
\end{aligned}
$$

where each w_i is integrated along the contour $\gamma_i = \gamma(U; \Xi, S)$. Let us take the limit as each a_j tends to 0 (as in Section 4.1.1) and perform the change of variables

sending w_i to $-y_i/qu = -sy_i/\kappa$. We obtain

$$
\mathbb{E}_{\mathbf{M}}[q^{k\mathfrak{h}_\lambda(x+m)}] = \frac{q^{\binom{k}{2}}}{(2\pi\mathbf{i})^k} \oint \cdots \oint \prod_{1 \le i < j \le k} \frac{y_i - y_j}{y_i - qy_j}
$$

(4.5)
$$
\times \prod_{i=1}^{k} \left(\frac{1 + y_i}{1 + q^{-1}y_i}\right)^t \left(\frac{1 + q^{-1}\kappa^{-1}y_i}{1 + \kappa^{-1}y_i}\right)^{x-1}
$$

$$
\times \prod_{i=1}^{k} \left(\prod_{j=1}^{m} \frac{1}{1 - q^{-1}\beta_j^{-1}y_i}\right) \frac{dy_i}{y_i},
$$

where each y_i is integrated along the contour $\gamma_i(\widehat{U}; \widehat{\Xi}, \widehat{S})$

In the above, $\mathbf{M}$ refers to the measure on particle configurations induced by the stochastic higher spin vertex model with spectral parameters $(u, u, \ldots, u)$, in-homogeneity parameters $(-\beta_1/a_1 u, -\beta_2/a_2 u, \ldots, -\beta_m/a_m u, 1, 1, \ldots)$, and spin parameters $(a_1, a_2, \ldots, a_m, s, s, \ldots)$, where the first m spins $a_1, a_2, \ldots, a_m$ all tend to 0. Recall from the explanation in Section 4.1.1 that this stochastic higher spin vertex model in fact coincides with the stochastic six-vertex model with step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data, shifted to the right m spaces. Hence, $\mathbb{E}[q^{k\mathfrak{h}_t(x)}] = \mathbb{E}_{\mathbf{M}}[q^{k\mathfrak{h}_\lambda(x+m)}]$, where the former expectation is with respect to the stochastic six-vertex model with step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data. Thus, (4.4) follows from (4.5). □

REMARK 4.5. After having defined generalized Bernoulli initial data as in Definition 3.3, it might also be possible to establish Proposition 4.4 through the spectral theory introduced in [17]. However, we are not aware of an alternative way to guess the integrability of generalized Bernoulli initial data without using the stochastic higher spin vertex model as in Section 3.1.

4.2. *Fredholm determinants for models with generalized step Bernoulli initial data.* The goal of this section is to establish Fredholm determinant identities for the q-Laplace transform of the current of the stochastic six-vertex model (Theorem 4.10) and the ASEP (Theorem 4.11), both with generalized step Bernoulli initial data; we refer to Appendix A of [3] for the definition and properties of Fredholm determinants. Given Proposition 4.4, the derivation of these two theorems will largely follow Section 5 of [18].

4.2.1. *A general Fredholm determinant identity.* The goal of this section is to state Lemma 4.9, which produces Fredholm determinant identities from q-moment identities of the type given in Proposition 4.4; this lemma is similar to Proposition 3.6 of [18], and it can be proven analogously.

DEFINITION 4.6 ([18], Definition 5.1).   Let $a \in \mathbb{C}$ be a nonzero complex number and $q \in (0, 1)$ be a positive real number. Let f be a meromorphic function that has a pole at a, but no other poles in a neighborhood of the segment joining 0 and a. Define $\mathbf{m}_0 = 1$ and, for each positive integer k, define

$$\mathbf{m}_k = \mathbf{m}_{k,f} = \frac{q^{\binom{k}{2}}}{(2\pi \mathbf{i})^k} \oint \cdots \oint \prod_{1 \le i < j \le k} \frac{z_i - z_j}{z_i - q z_j} \prod_{i=1}^{k} f(z_i) z_i^{-1} \, dz_i,$$

where each z_i is integrated along a positively oriented contour C_i that satisfies the following two properties. First, each contour contains 0 and a, but does not contain any other poles of f. Second, for all integers $i < j \in [1, k]$, the interior of C_i does not intersect the contour $q C_j$.

REMARK 4.7.   The equality (4.4) gives an example for $\mathbf{m}_k$ if we set $\mathbf{m}_k = \mathbb{E}[q^{k\mathfrak{h}_t(x)}]$ and $a = -q$.

We also require the following types of contours; they appeared previously in [18] as Definition 3.5 (in the case $\delta = 1/2$).

DEFINITION 4.8.   Let $R, d, \delta > 0$ be positive real numbers with $d, \delta < 1$ and $R > 1$. Let $D_{R,d,\delta} \subset \mathbb{C}$ denote the contour in the complex plane, with nondecreasing imaginary part, formed by taking the union of intervals

$$(R - \mathbf{i}\infty, R - \mathbf{i}d) \cup [R - \mathbf{i}d, \delta - \mathbf{i}d]$$
$$\cup [\delta - \mathbf{i}d, \delta + \mathbf{i}d]$$
$$\cup [\delta + \mathbf{i}d, R + \mathbf{i}d]$$
$$\cup (R + \mathbf{i}d, R + \mathbf{i}\infty).$$

Furthermore, let $k > R$ be a positive integer. Let $z \in \mathbb{C}$ be the complex number satisfying $\Re z = R$, $\Im z > 0$ and $|z - \delta| = k$. Let I denote the minor arc of the circle in the complex half-plane, centered at δ, with radius k, connecting z to its conjugate $\overline{z}$. Then let $D_{R,d,\delta;k} \subset \mathbb{C}$ denote the negatively oriented contour in the complex plane, formed by the union

$$(\overline{z}, R - \mathbf{i}d) \cup [R - \mathbf{i}d, \delta - \mathbf{i}d]$$
$$\cup [\delta - \mathbf{i}d, \delta + \mathbf{i}d]$$
$$\cup [\delta + \mathbf{i}d, R + \mathbf{i}d] \cup (R + \mathbf{i}d, z) \cup I.$$

Observe that the contour $D_{R,d,\delta;k}$ approximates the contour $D_{R,d,\delta}$ as k tends to ∞. Examples of these contours are given in Figure 7.

The following proposition, which is similar to Proposition 3.6 of [18], states that one has a Fredholm determinant identity for a certain generating series of the sequence $\{\mathbf{m}_0, \mathbf{m}_1, \ldots\}$.
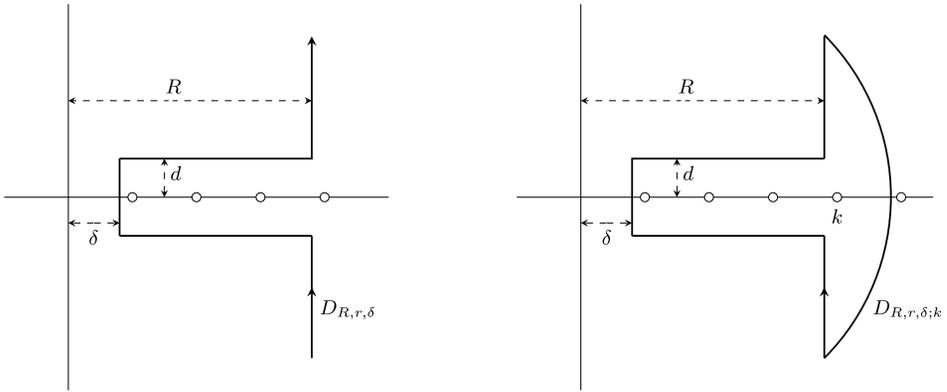
FIG. 7.   *An example of $D_{R,r,\delta}$ is shown to the left, and an example of $D_{R,r,\delta;k}$ is shown to the right;
the circles depict the locations of positive integers.*

LEMMA 4.9.   *Let $a \in \mathbb{C}$ be a complex number, and let f be a meromorphic
function satisfying the properties of Definition 4.6. Assume that f is of the form
$f(z) = g(z)/g(qz)$, for some function g. Let $C \subset \mathbb{C}$ be a compact contour con-
taining 0 and a in its interior, but no other pole of f; moreover, assume that the
interior of qC is contained inside C.*

*Suppose that there exists some constant $B > 0$ such that $f(q^n w) < B$ and
$|q^n w - w'| > B^{-1}$ for all $w, w' \in C$ and integers $n \geq 1$. Also assume that there
exist positive real numbers $d, \delta \in (0, 1)$ and $R > 1$ such that*

$$(4.6) \qquad \inf_{\substack{w, w' \in C \\ k \in (2R, \infty) \cap \mathbb{Z} \\ s \in D_{R,d,\delta;k}}} |q^s w - w'| > 0; \qquad \sup_{\substack{w \in C \\ k \in (2R, \infty) \cap \mathbb{Z} \\ s \in D_{R,d,\delta;k}}} \frac{g(w)}{g(q^s w)} < \infty.$$

*Then, for any complex number $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ satisfying $|\zeta| < B^{-1}(1 - q)^{-1}$, we
have that*

$$(4.7) \qquad \sum_{k=0}^{\infty} \frac{\mathbf{m}_k \zeta^k}{(q; q)_k} = \det(\mathrm{Id} + K_\zeta)_{L^2(C)},$$

where the kernel K is defined through

$$K_\zeta(w, w') = \frac{1}{2\mathbf{i}} \int_{D_{R,d,\delta}} \frac{g(w)}{g(q^r w)} \frac{(-\zeta)^r \, dr}{\sin(\pi r)(q^r w - w')}.$$

Furthermore, the right-hand side of (4.7) is analytic in $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

The proof of this lemma is very similar to that of Proposition 3.6 of [18] and is
thus omitted.

4.2.2. *Fredholm determinants for the stochastic six-vertex model and ASEP.*
In this section we will establish the Fredholm determinant identities Theorem 4.10
and Theorem 4.11. As stated previously, these two identities will follow from
Proposition 4.4 and Lemma 4.9.

THEOREM 4.10. *Let m be a nonnegative integer and t be a positive integer.
Fix $0 < \delta_1 < \delta_2 < 1$ and $b_1, b_2, \ldots, b_m \in (0, 1)$. Denoting q, u, κ, s and β_i as in
(4.1), let*

$$(4.8) \qquad g_V(z; x, t) = \frac{(\kappa^{-1}z + q)^{x-1}}{(z + q)^t} \prod_{j=1}^{m} \frac{1}{(q^{-1}\beta_j^{-1}z; q)_\infty}.$$

*Let $\mathcal{C}_V$ be a positively oriented circle in the complex plane, centered at some non-
positive real number, containing $-q$ and 0, but leaving outside -1 and $q\beta_j$ for
each integer $j \in [1, m]$.*

*Then, there exist a sufficiently large real number $R > 1$, a real number $\delta \in (0, 1)$
sufficiently close to 1 and a sufficiently small real number $d \in (0, 1)$ such that*

$$q^R < \min_{w, w' \in \mathcal{C}_V} |w^{-1}w'|; \qquad q^{1-\delta} > \max_{w \in \mathcal{C}_V} \{|\kappa^{-1}w|, |qw|\};$$

$$(4.9) \qquad |\Im q^{id}| < \max_{\substack{w, w' \in \mathcal{C}_V \\ \Re w^{-1}w' \geq 0 \\ |w^{-1}w'| \leq q^\delta}} |\Im w^{-1}w'|.$$

*Consider the stochastic six-vertex model with jump probabilities δ_1 and δ_2, and
with $(b_1, b_2, \ldots, b_m)$-step Bernoulli initial data (as defined in Section 3.1). For
any $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, we have that*

$$(4.10) \qquad \mathbb{E}\left[\frac{1}{(\zeta q^{\mathfrak{h}_t(x)}; q)_\infty}\right] = \det(\mathrm{Id} + V_\zeta)_{L^2(\mathcal{C}_V)},$$

where V_ζ is defined by

$$V_\zeta(w, w') = \frac{1}{2\mathbf{i}} \int_{D_{R,d,\delta}} \frac{g_V(w; x, t)}{g_V(q^r w; x, t)} \frac{(-\zeta)^r \, dr}{\sin(\pi r)(q^r w - w')}.$$

PROOF. Let us begin with the first statement. The existence of R follows from
the compactness of $\mathcal{C}_V$ and the fact that 0 does not lie on the contour $\mathcal{C}_V$. The
existence of δ follows from the compactness $\mathcal{C}_V$ and the fact that $\mathcal{C}_V$ and does
not intersect the circle centered at 0 with radius κ or radius q^{-1}. The existence of
d follows from the compactness of $\mathcal{C}_V$ and the fact that $\Im w^{-1}w' \neq 0$ whenever
$\Re w^{-1}w' \geq 0$ and $|w^{-1}w'| < q^\delta$. Thus, R, δ and d satisfying (4.9) exist.

To establish the Fredholm determinant identity (4.10), we will apply
Lemma 4.9, with $\mathbf{m}_k = \mathbb{E}[q^{k\mathfrak{h}_t(x)}]$ and $a = -q$; Proposition 4.4 shows that $\mathbf{m}_k$

has the form given by Definition 4.6, with

$$f(z) = \frac{g_V(z)}{g_V(qz)} = \left(\frac{1+z}{1+q^{-1}z}\right)^t \left(\frac{1+q^{-1}\kappa^{-1}z}{1+\kappa^{-1}z}\right)^{x-1} \prod_{j=1}^{m} \frac{1}{1-q^{-1}\beta_j^{-1}z}.$$

To apply Lemma 4.9, we must verify several conditions on the contours $\mathcal{C}_V$ and $D_{R,d,\delta}$.

First, we require that $\mathcal{C}_V$ contains 0, $-q$, and its image under multiplication by q, but no other pole of f; these statements all quickly follow from the definition of $\mathcal{C}_V$.

Second, we require the existence of some positive real number $B > 0$ such that $|q^n w - w'| > B^{-1}$ and $f(q^n w) < B$ for all integers $n \geq 1$ and all $w, w' \in \mathcal{C}_V$. By compactness of $\mathcal{C}_V$, the existence of such a B would follow if $|q^n w - w'| > 0$ and $f(q^n w) < \infty$ for all $w, w' \in \mathcal{C}$ and $n \geq 1$. The first inequality holds by convexity of $\mathcal{C}_V$. To establish the second inequality, recall that the only poles of f are at $-q$, κ and the $q\beta_j$. The $q\beta_j$ and κ are outside the contour $\mathcal{C}_V$, so no element of the form $q^n w$, with $w \in \mathcal{C}_V$, is equal to some $q\beta_j$. Furthermore, since no nonpositive power of q lies on $\mathcal{C}_V$, we have that $q^n w \neq q^{-1}$ for any $w \in \mathcal{C}_V$. Thus, the second inequality holds since $q^n w$ is not a pole of f for any $w \in \mathcal{C}_V$.

Third, we require that

$$(4.11) \qquad \inf_{\substack{w,w' \in \mathcal{C}_V \\ k \in (2R, \infty) \cap \mathbb{Z} \\ s \in D_{R,d,\delta;k}}} |q^s w - w'| > 0; \qquad \sup_{\substack{w \in \mathcal{C}_V \\ k \in (2R, \infty) \cap \mathbb{Z} \\ s \in D_{R,d,\delta;k}}} \frac{g_V(w)}{g_V(q^s w)} < \infty.$$

To see that the first inequality in (4.11) holds, observe that if $s \in D_{R,d,\delta;k}$ then either $\Re s \geq R$, $\Re s = \delta$ and $\Im s \in (-d, d)$, or $\Im s = d$ and $\Re s \in (\delta, R)$. The first inequality in (4.9) implies that $q^s \neq w^{-1}w'$ in the first case. Furthermore, the third inequality in (4.9) implies that $q^s \neq w^{-1}w'$ in either the second or third case. Therefore, $q^s \neq w^{-1}w'$, for any $w, w' \in \mathcal{C}_V$ and $s \in D_{R,d,\delta;k}$ with $k > 2R$; thus, the first inequality in (4.11) follows from compactness of $\mathcal{C}_V$.

Now let us establish the second inequality in (4.11). Again, since $\mathcal{C}_V$ is compact, it suffices to show that no pole of g_V is on $\mathcal{C}_V$, and that no element of the form $q^s w$ (with $w \in \mathcal{C}_V$ and $s \in D_{R,d,\delta;k}$) is a zero of g_V. The first statement holds by the definition of $\mathcal{C}_V$. To establish the second statement, observe that g_V can equal zero only at $-q\kappa$. If $q^s w = -q\kappa$, then we must have that $|w| = \kappa|q^{1-s}| \leq q^{1-\delta}\kappa$, which cannot hold due to the second inequality in (4.9). Thus, the second statement also holds.

Hence, the contours $\mathcal{C}_V$ and $D_{R,d,\delta}$ satisfy the conditions of Lemma 4.9. Applying this lemma to $\mathbf{m}_k = \mathbb{E}[q^{k\mathfrak{h}_t(x)}]$, we deduce that

$$(4.12) \qquad \sum_{k=0}^{\infty} \frac{\zeta^k \mathbb{E}[q^{k\mathfrak{h}_t(x)}]}{(q;q)_k} = \det(\mathrm{Id} + V_\zeta)_{L^2(\mathcal{C}_V)},$$

for any $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ satisfying $|\zeta| < (1-q)^{-1}B^{-1}$. Since $q \in (0,1)$, we have that $\mathbb{E}[q^{k\mathfrak{h}_t(x)}] < 1$. Hence, if $|\zeta| < 1$, then the left-hand side of (4.12) converges absolutely, meaning that we can commute the sum with the expectation. Therefore, if $|\zeta| < 1$, we have that

$$\sum_{k=0}^{\infty} \frac{\zeta^k \mathbb{E}[q^{k\mathfrak{h}_t(x)}]}{(q;q)_k} = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{(\zeta q^{\mathfrak{h}_t(x)})^k}{(q;q)_k}\right] = \mathbb{E}\left[\frac{1}{(\zeta q^{\mathfrak{h}_t(x)};q)_{\infty}}\right],$$

where we have used the q-binomial identity in the second equality. Substituting this into (4.12) yields

$$(4.13) \qquad \mathbb{E}\left[\frac{1}{(\zeta q^{\mathfrak{h}_t(x)};q)_{\infty}}\right] = \det(\mathrm{Id} + V_{\zeta})_{L^2(\mathcal{C}_V)},$$

for any $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ such that $|\zeta| < \min\{1, (1-q)^{-1}B^{-1}\}$.

Finally, for any real number $z \geq 0$, the function $(z\zeta;q)_{\infty}^{-1}$ is analytic in ζ in the domain $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Since the left-hand side of (4.13) is a locally uniformly convergent sum of such expressions, we deduce that the left-hand side of (4.13) is analytic in ζ in the domain $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Furthermore, Proposition 4.9 states that the right-hand side of (4.13) is also analytic in ζ in the domain $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Uniqueness of analytic continuation concludes the proof. $\square$

Degenerating the stochastic six-vertex model to the ASEP as explained in Section 3.2.2 yields the following Fredholm determinant identity for the ASEP with generalized step Bernoulli initial data.

THEOREM 4.11. *Let m be a nonnegative integer and t be a positive real number. Fix $0 < L < R$ and $b_1, b_2, \ldots, b_m \in (0,1)$. Denote $q = L/R < 1$ and $\beta_i = b_i/(1-b_i)$ for each integer $i \in [1, m]$. Define*

$$(4.14) \quad g_A(z;x,T) = (z+q)^{x-1} \exp\left(\frac{qT(R-L)}{z+q}\right) \prod_{j=1}^{m} \frac{1}{(q^{-1}\beta_j^{-1}z;q)_{\infty}}.$$

Let $\mathcal{C}_A$ be a positively oriented circle in the complex plane, centered at a nonpositive real number, containing $-q$ and 0, but leaving outside -1 and $q\beta_j$ for each integer $j \in [1, m]$. There exist a sufficiently large real number $R > 1$, a real number $\delta \in (0,1)$ sufficiently close to 1, and a sufficiently small real number $d \in (0,1)$ such that

$$q^R < \min_{w, w' \in \mathcal{C}_A} |w^{-1}w'|; \qquad q^{1-\delta} > \max_{w \in \mathcal{C}_A} |w|;$$

$$(4.15) \qquad |\Im q^{id}| < \max_{\substack{w, w' \in \mathcal{C}_A \\ \Re w^{-1}w' \geq 0 \\ |w^{-1}w'| \leq q^{\delta}}} |\Im w^{-1}w'|.$$

Consider the ASEP with left jump rate L and right jump rate R, with step
$(b_1, b_2, \ldots, b_m)$-*Bernoulli initial data* (*as defined in Section* 3.1.2). *For any* $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, *we have that*

$$(4.16) \qquad \mathbb{E}\left[\frac{1}{(\zeta q^{\mathfrak{h}_t(x)}; q)_\infty}\right] = \det(\mathrm{Id} + A_\zeta)_{L^2(\mathcal{C}_A)},$$

where $\mathfrak{h}_t(x) = J_t(x-1)$, *and* A_ζ *is defined by*

$$A_\zeta(w, w') = \frac{1}{2\mathbf{i}} \int_{D_{R,d,\delta}} \frac{g_A(w; x, T)}{g_A(q^r w; x, T)} \frac{(-\zeta)^r \, dr}{\sin(\pi r)(q^r w - w')}.$$

PROOF. The proof of the first statement is similar to that of the first statement in Theorem 4.10 (in fact, it is the $\kappa = 1$ degeneration of that statement), and is thus omitted.

For the proof of the second part of the theorem, we apply the degeneration of the stochastic six-vertex model to the ASEP explained in Section 3.2.2. Specifically, set $\delta_1 = \varepsilon L$, $\delta_2 = \varepsilon R$, $t = \varepsilon^{-1} T$, and let ε tend to 0. Then the distribution of $\mathfrak{h}_t(x+t) = \mathfrak{H}(x+t-1, t)$ under the stochastic six-vertex model with jump rates δ_1 and δ_2 (as defined in Section 3.1) converges to the distribution of $J_t(x)$ under the ASEP with left jump rate L and right jump rate R, due to Proposition 3.8.

Therefore, $\mathbb{E}_V[(\zeta q^{\mathfrak{h}_t(x+t)}; q)_\infty^{-1}]$ converges to $\mathbb{E}_A[(\zeta q^{\mathfrak{h}_t(x)}; q)_\infty^{-1}]$, since both $(\zeta q^{\mathfrak{h}_t(x+t)}; q)_\infty^{-1}$ and $(\zeta q^{\mathfrak{h}_t(x)}; q)_\infty^{-1}$ are uniformly bounded; here, the first expectation is with respect to the stochastic six-vertex model and the second expectation is with respect to the ASEP. Thus, the left-hand side of (4.10) (with x replaced by $x + t$) converges to the left-hand side of (4.16).

Moreover, it is quickly verified that

$$\lim_{\varepsilon \to 0} g_V(z; x+t, t) = e^{T(L-R)} g_A(z; x, T).$$

Hence, $V_\zeta(w, w')$ tends to $A_\zeta(w, w')$ as ε tends to 0, for all $w, w' \in \mathcal{C}_A$.

Now, as ε tends to 0, κ tends to 1. Therefore, if ε is sufficiently small (in comparison to R, δ and d), the estimates (4.9) are implied by the estimates (4.15). Hence, we may take the contour $\mathcal{C}_V$ from Theorem 4.10 to be the contour $\mathcal{C}_A$.

Since $\mathcal{C}_A$ is compact, the fact that $\lim_{\varepsilon \to 0} V_\zeta(w, w') = A_\zeta(w, w')$ implies that the right-hand side of (4.10) converges to the right-hand side of (4.16) as ε tends to 0 (for instance, by Corollary A.5 of [3]).

Therefore, (4.16) is a limit degeneration of (4.10), and the proof is complete. □

REMARK 4.12. Taking the weakly asymmetric limit [12] of Theorem 4.11 might provide an alternative proof of Theorem 1.10 of [15], which is a Fredholm

determinant identity for the Laplace transform of the free energy of the continuum directed random polymer with an m-spiked perturbation (see Remark 1.5).

5. Preliminary remarks for the asymptotic analysis. In this section we begin the analysis of the Fredholm determinant identities given by Theorem 4.10 and Theorem 4.11. Our goal is twofold. First, in Section 5.1, we will rewrite the kernels V_ζ and A_ζ (defined in Theorem 4.10 and Theorem 4.11, resp.) in a way that will be useful for asymptotic analysis later. Then, in Section 5.2, we will state the asymptotics (see Proposition 5.2 and Proposition 5.3) for the corresponding Fredholm determinants and also show how these asymptotics imply Theorem 1.7 and Theorem 1.6. The proofs of Proposition 5.2 and Proposition 5.3 will be the topics of later sections of the paper.

5.1. *Reformulations of Theorem 4.10 and Theorem 4.11.* Theorem 4.10 and Theorem 4.11 are of a similar form. Specifically, they both state that

$$(5.1) \qquad \mathbb{E}_K\left[\frac{1}{(\zeta q^{\mathfrak{h}_T(x)}; q)_\infty}\right] = \det(\mathrm{Id} + K_\zeta)_{L^2(\mathcal{C}_K)},$$

where $\mathfrak{h}_T(x) = \mathfrak{H}(x-1, T)$ in the case of the stochastic six-vertex model, and $\mathfrak{h}_T(x) = J_T(x-1)$ in the case of the ASEP. In (5.1), the notation K either stands for V (for the stochastic six-vertex model) or A (for the ASEP); the kernels V_ζ and A_ζ are defined in Theorem 4.10 and Theorem 4.11, respectively. If $K = V$, then the expectation on the left-hand side of (5.1) is with respect to the stochastic six-vertex model with step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data, run to some integer time T. If $K = A$, then the expectation on the left-hand side of (5.1) is with respect to the ASEP with step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data, run up to some real time T.

Recall from Theorem 4.10 and Theorem 4.11 that

$$K_\zeta(w, w') = \frac{1}{2\mathbf{i}} \int_{D_{R,d,\delta}} \frac{g_K(w; x, T)}{g_K(q^r w; x, t)} \frac{(-\zeta)^r \, dr}{\sin(\pi r)(q^r w - w')},$$

for any $w, w' \in \mathcal{C}_K$; the functions g_V and g_A are given by (4.8) and (4.14), respectively.

Let us set $\zeta = -q^p$, for some real number p, and change variables from r to $v = q^r w$. To understand how this affects the above identity for K_ζ, we must analyze the contour for v.

To that end, first observe that if d is sufficiently small (which we will always assume to be the case), the contour $D_{R,d,\delta}$ can be written as the union $D_{R,d,\delta} = I' \cup \bigcup_{k \in \mathbb{Z} \setminus \{-1\}} I_k$. Here, I_k is defined to be the interval

$$I_k = \left[R + \mathbf{i}(d + 2\pi |\log q|^{-1} k), R + \mathbf{i}(d + 2\pi |\log q|^{-1}(k+1))\right],$$

for each integer k, and I' is defined to be the piecewise linear curve

$$I' = \left[R + \mathbf{i}(d - 2\pi |\log q|^{-1}), R - \mathbf{i}d \right]$$
$$\cup \left[R - \mathbf{i}d, \delta - \mathbf{i}d \right] \cup \left[\delta - \mathbf{i}d, \delta + \mathbf{i}d \right] \cup \left[\delta + \mathbf{i}d, R + \mathbf{i}d \right].$$

Now, for each integer k, the map from r to $q^r w$ is a bijection from I_k to $-Q^{(1)}_{R,d,\delta;w}$, where the negative refers to reversal of orientation and the contour $Q^{(1)}_{R,d,\delta;w}$ is a positively oriented circle of radius q^R, centered at 0; here, the negative orientation is due to the fact that $\log q < 0$.

Furthermore, the map from r to $q^r w$ is a bijection from I' to the contour $-Q_{R,d,\delta;w}$, where $Q_{R,d,\delta;w}$ is defined to be the union of four curves $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3 \cup \mathcal{J}_4$, which are defined are as follows. The curve $\mathcal{J}_1$ is the major arc of the circle centered at 0, with radius $q^R |w|$, connecting $q^{R-id} w$ to $q^{R+id} w$. The curve $\mathcal{J}_2$ is the line segment in the complex plane connecting $q^{R+id} w$ to $q^{\delta+id} w$; the curve $\mathcal{J}_3$ is the minor arc of the circle centered at 0, with radius $q^\delta |w|$, connecting $q^{\delta+id} w$ to $q^{\delta-id} w$; and the curve $\mathcal{J}_4$ is the line segment in the complex plane connecting $q^{\delta-id} w$ to $q^{R-id} w$. We refer to Figure 8 for an example of the contour $Q_{R,d,\delta;w}$.

Thus, since $dv = v \log q \, dr$ and $(-\zeta)^r = v^p w^{-p}$, we obtain that

$$K_\zeta(w, w')$$

$$= -\frac{1}{2\mathbf{i} \log q}$$

$$\times \sum_{j \neq 0} \oint_{Q^{(1)}_{R,d,\delta;w}} \frac{g_K(w; x, T)}{g_K(v; x, t)} \frac{v^{p-1} w^{-p}}{\sin(\frac{\pi}{\log q}(\log v - \log w + 2\pi \mathbf{i}j))} \frac{dv}{v - w'}$$

$$- \frac{1}{2\mathbf{i} \log q} \oint_{Q_{R,d,\delta;w}} \frac{g_K(w; x, T)}{g_K(v; x, T)} \frac{v^{p-1} w^{-p}}{\sin(\frac{\pi}{\log q}(\log v - \log w))} \frac{dv}{v - w'},$$

where we have taken the principal branch of the logarithm.
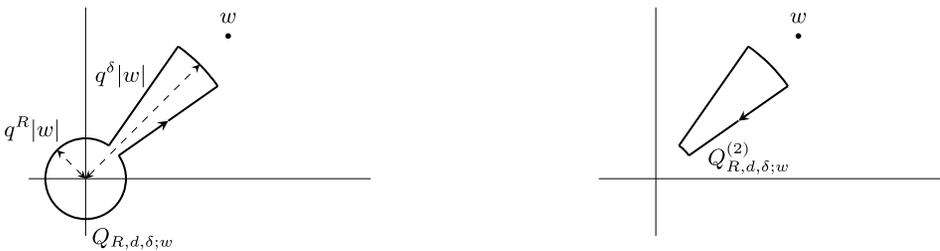
FIG. 8.   *Shown to the left is the contour $Q_{R,d,\delta;w}$. Shown to the right is the contour $Q^{(2)}_{R,d,\delta;w}$.*

We can rewrite $K_\zeta = K_\zeta(w, w')$ as

$$
\begin{aligned}
K_\zeta = &-\frac{1}{2\mathbf{i}\log q} \\
&\times \sum_{j=-\infty}^{\infty} \oint_{Q_{R,d,\delta;w}} \frac{g_K(w; x, T)}{g_K(v; x, t)} \frac{v^{p-1}w^{-p}}{\sin(\frac{\pi}{\log q}(\log v - \log w + 2\pi\mathbf{i}j))} \frac{dv}{v - w'} \\
&-\frac{1}{2\mathbf{i}\log q} \\
&\times \sum_{j\neq 0} \oint_{Q^{(2)}_{R,d,\delta;w}} \frac{g_K(w; x, T)}{g_K(v; x, T)} \frac{v^{p-1}w^{-p}}{\sin(\frac{\pi}{\log q}(\log v - \log w + 2\pi\mathbf{i}j))} \frac{dv}{v - w'},
\end{aligned}
$$

(5.2)

where the contour $Q^{(2)}_{R,d,\delta;w} = Q^{(1)}_{R,d,\delta;w} - Q_{R,d,\delta;w}$ is the union of four curves $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$, which are defined as follows. The curve $\mathcal{C}_1$ is the line segment in the complex plane connecting $q^{R-id}w$ to $q^{\delta-id}w$. The curve $\mathcal{C}_2$ is the minor arc of the circle centered at 0, with radius $q^\delta|w|$, connecting $q^{\delta-id}w$ to $q^{\delta+id}w$. The curve $\mathcal{C}_3$ is the line segment connecting $q^{\delta+id}w$ to $q^{R+id}w$. The curve $\mathcal{C}_4$ is the minor arc of the circle centered at 0, with radius $q^R|w|$, connecting $q^{R+id}w$ to $q^{R-id}w$. We refer to Figure 8 for an example of the contour $Q^{(2)}_{R,d,\delta;w}$.

We claim that each summand of the second sum on the right-hand side of (5.2) is equal to 0. To verify this, we analyze the poles of the corresponding integrands. When $K = V$, these poles are at $v \in \{0, w', -q\kappa\}$ and, when $K = A$, these poles are at $v \in \{0, w', -q\}$. Observe that there is no pole at $v = w$ (or at $v \in wq^{\mathbb{Z}}$) since j is nonzero in each summand.

The pole 0 is not contained inside $Q^{(2)}_{R,d,\delta;w}$. Furthermore, in the case $K = V$, the inequalities (4.9) imply that the poles at $v \in \{w', -q\kappa\}$ are not contained inside $Q^{(2)}_{R,d,\delta;w}$; similarly, in the case $K = A$, the inequalities (4.15) imply that the poles at $v \in \{w', -q\}$ are not contained inside $Q^{(2)}_{R,d,\delta;w}$. Thus, no poles of the integrand of the second sum on the right-hand side of (5.2) are contained in the interior of $Q^{(2)}_{R,d,\delta;w}$, from which we deduce that the corresponding integrals are equal to 0.

Next we would like to deform the contour $Q_{R,d,\delta;w}$ on the right-hand side of (5.2) to one (to be chosen later) that will be better suited for asymptotic analysis. To that end, recall that a contour $\gamma \subset \mathbb{C}$ is called *star-shaped* (with respect to the origin) if, for each real number $a \in \mathbb{R}$, there exists exactly one complex number $z_a \in \gamma$ such that $z_a/|z_a| = e^{\mathbf{i}a}$.

The following proposition is a reformulation of Theorem 4.10 and Theorem 4.11, specialized to the case $\zeta = -q^p$, but with more general, star-shaped contours. As above, the notation $K \in \{V, A\}$ will be used to discuss the stochastic six-vertex model and ASEP simultaneously.

PROPOSITION 5.1.   *Fix $\delta_1, \delta_2 \in (0, 1)$; $R > L > 0$; $m \in \mathbb{Z}_{\geq 1}$; $b_1, b_2, \ldots, b_m \in (0, 1)$; $x \in \mathbb{Z}$; and $p \in \mathbb{R}$. Denote $\beta_j = b_j / (1 - b_j)$ for each $j \in [1, m]$.*

Let $\Gamma_V \subset \mathbb{C}$ be a positively oriented, star-shaped contour in the complex plane containing 0, but leaving outside $-q\kappa$ and all $q\beta_j$. Let $\mathcal{C}_V \subset \mathbb{C}$ be a positively oriented, star-shaped contour contained inside $q^{-1}\Gamma_V$; that contains 0, $-q$, and Γ_V; but that leaves outside all $q\beta_j$. Let $\Gamma_A \subset \mathbb{C}$ be a positively oriented, star-shaped contour containing 0, but leaving outside $-q$ and all $q\beta_j$. Furthermore, let $\mathcal{C}_A \subset \mathbb{C}$ be a positively oriented, star-shaped contour contained inside $q^{-1}\Gamma_A$; that contains 0, $-q$, and Γ_A; but that leaves outside all $q\beta_j$.

Let $\mathbb{E}_V$ denote the expectation with respect to the stochastic six-vertex model with left jump probability δ_1, right jump probability δ_2 and step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data. Similarly, let $\mathbb{E}_A$ denote the expectation with respect to the ASEP with left jump rate L, right jump rate R and step $(b_1, b_2, \ldots, b_m)$-Bernoulli initial data.

Then, for each $K \in \{V, A\}$, we have that

$$(5.3) \qquad \mathbb{E}_K\left[\frac{1}{(-q^{\mathfrak{h}_T(x)+p}; q)_\infty}\right] = \det(\mathrm{Id} + K^{(p)})_{L^2(\mathcal{C}_K)},$$

where

$$K^{(p)}(w, w')$$

$$(5.4) \qquad = \frac{1}{2\mathbf{i}\log q}$$

$$\times \sum_{j=-\infty}^{\infty} \oint_{\Gamma_K} \frac{g_K(w; x, T)}{g_K(v; x, t)} \frac{v^{p-1} w^{-p}}{\sin(\frac{\pi}{\log q}(\log v - \log w + 2\pi\mathbf{i}j))} \frac{dv}{w' - v},$$

and g_V and g_A are defined in (4.8) *and* (4.14), *respectively.*

PROOF.   In view of (5.2) and the fact that the second sum on the right-hand side of this identity is equal to 0, there are two differences between (5.1) and (5.3). The first is that the contour along which the Fredholm determinants are taken is different; in the former it is $\mathcal{C}_K$ and in the latter it is C_K. The second is that the contour of integration for v in (5.2) is different from that in (5.4); in the former it is $Q_{R,d,\delta;w}$ and in the latter it is Γ_K. Thus, it suffices (see, e.g., Remark A.3 of [3]) to show that we can continuously and simultaneously deform $Q_{R,d,\delta;w}$ and $\mathcal{C}_K$ to Γ_K and C_K, respectively, without crossing any poles of any integrand on the right-hand side of (5.4).

We will only do this in the case $K = V$; the case $K = A$ is entirely analogous. To that end, observe that the poles for v of the integrands in the first sum on the right-hand side of (5.2) are at $v \in \{0, w', -q\kappa\} \cup \{q^j w\}_{j \in \mathbb{Z}}$; of these, the poles at $v \in \{0, qw, q^2w, \ldots\}$ are contained inside the contour $Q_{R,d,\delta;w}$ and the others are left outside. The poles for w of the same integrands are at

$w \in \{-q, 0, q\beta_1, q\beta_2, \ldots, q\beta_m\}$; of these, the poles at $w \in \{0, -q\}$ are contained inside the contour $\mathcal{C}_V$ and the others are left outside.

Now recall that $\mathcal{C}_V$ contains 0, $-q$, and Γ_V, but leaves outside all $q\beta_j$; similarly, Γ_V contains 0 and $q\mathcal{C}_V$, but leaves outside $-q\kappa$. Thus, $\mathcal{C}_V$ and $\mathcal{C}_V$ contain and leave outside the same poles of $K^{(p)}$; similarly, $Q_{R,d,\delta;w}$ and Γ_V contain and leave outside the same poles of the integrand on the right-hand side of (5.4). Therefore, since $\mathcal{C}_V$ and Γ_V are star-shaped, it follows by moving along rays through the origin that $Q_{R,d,\delta;w}$ and $\mathcal{C}_V$ can be continuously and simultaneously deformed to Γ_V and $\mathcal{C}_V$, respectively, without crossing any poles of any integrand on the right-hand side of (5.4). Thus, (5.3) follows from (5.1) and (5.2). $\quad\square$

Observe that $(-q^{\mathfrak{h}_T(x)+p}; q)_\infty^{-1}$ from (5.3) tends to 0 when p is much less than than $-\mathfrak{h}_T(x)$, and that it tends to 1 when p is much greater than than $-\mathfrak{h}_T(x)$. Therefore, one might expect the left-hand side of (5.3) to approximate the quantity $\mathbb{E}_{\mathbb{P}^K}[\mathbf{1}_{\mathfrak{h}_T(x) \geq -p}] = \mathbb{P}^K[\mathfrak{h}_T(x) \geq -p]$. Hence, in order to analyze this probability, one requires asymptotics of the determinant on the right-hand side of (5.3).

In the next section, we will clarify this heuristic and explain how to reduce Theorem 1.6 and Theorem 1.7 to an asymptotic analysis of Fredholm determinants of the type given on the right-hand side of (5.3).

5.2. *Applications of Proposition* 5.1. The following two results provide the large T asymptotics for the determinant on the right-hand side of (5.3). Proposition 5.2 addresses the case $K = A$ (ASEP), and Proposition 5.3 addresses the case $K = V$ (stochastic six-vertex model). In what follows, we recall that the integer x appears in the definition of the function g_K [of (4.8) and (4.14)] that is used to define the kernel $K^{(p)}$ [of (5.4)].

PROPOSITION 5.2. *Adopt the notation of Theorem* 1.6 *and Proposition* 5.1:

1. *Let* $\eta \in (\theta, 1)$. *For each positive real number* T, *let* $x = x(T) = \lfloor \eta T \rfloor + 1$ *and* $p_T = sf_\eta T^{1/3} - m_\eta T$. *We have that*

$$\lim_{T \to \infty} \det\bigl(\mathrm{Id} + A_\zeta^{(p_T)}\bigr)_{L^2(\mathcal{C}_A)} = F_{\mathrm{TW}}(s).$$

2. *Assume that* $\{\eta = \eta_T\}_{T \in \mathbb{Z}_{>0}}$ *is a sequence of real numbers such that* $\lim_{T \to \infty} T^{1/3}(\eta_T - \theta) = d$. *For each positive real number* T, *let* $x = x(T) = \lfloor \eta T \rfloor + 1$ *and* $p_T = sf_\eta T^{1/3} - m_\eta T$. *We have that*

$$\lim_{T \to \infty} \det\bigl(\mathrm{Id} + A_\zeta^{(p_T)}\bigr)_{L^2(\mathcal{C}_A)} = F_{\mathrm{BBP};\mathbf{c}}(s).$$

3. *Let* $\eta \in (-b, \theta)$, *and assume that* $b_j = b$ *for all indices* $j \in [1, m]$. *For each positive real number* T, *let* $x = x(T) = \lfloor \eta T \rfloor + 1$ *and* $p_T = sf_\eta' T^{1/3} - m_\eta' T$. *We have that*

$$\lim_{T \to \infty} \det\bigl(\mathrm{Id} + A_\zeta^{(p_T)}\bigr)_{L^2(\mathcal{C}_A)} = G_m(s).$$

PROPOSITION 5.3. *Recall the definitions of χ, κ, Λ, and θ given in* (1.5), *and also the definition of $\mathbf{c} = (c_1, c_2, \ldots, c_m)$ given in* (1.7). *Adopt the notation of Proposition* 5.1, *and further define*

$$(5.5) \qquad m_\eta = \frac{(\sqrt{\kappa} - \sqrt{\eta})^2}{\kappa - 1}; \qquad f_\eta = \frac{(\sqrt{\kappa\eta} - 1)^{2/3}(\kappa - \sqrt{\kappa\eta})^{2/3}}{(\kappa - 1)\kappa^{1/6}\eta^{1/6}}$$

and

$$(5.6) \qquad\qquad m'_\eta = b - \Lambda^{-1}b\eta; \qquad f'_\eta = \chi^{1/2}(1 - \theta^{-1}\eta)^{1/2}.$$

1. *Let $\eta \in (\theta, \kappa)$. For each positive integer T, let $x = x(T) = \lfloor \eta T \rfloor + 1$ and $p_T = sf_\eta T^{1/3} - m_\eta T$. We have that*

$$\lim_{T\to\infty} \det\bigl(\mathrm{Id} + V_\zeta^{(p_T)}\bigr)_{L^2(\mathcal{C}_V)} = F_{\mathrm{TW}}(s).$$

2. *Assume that $\{\eta = \eta_T\}_{T\in\mathbb{Z}_{>0}}$ is a sequence of real numbers such that $\lim_{T\to\infty} T^{1/3}(\eta_T - \theta) = d$. For each positive integer T, let $x = x(T) = \lfloor \eta T \rfloor + 1$ and $p_T = sf_\eta T^{1/3} - m_\eta T$. We have that*

$$\lim_{T\to\infty} \det\bigl(\mathrm{Id} + V_\zeta^{(p_T)}\bigr)_{L^2(\mathcal{C}_V)} = F_{\mathrm{BBP};\mathbf{c}}(s).$$

3. *Let $\eta \in (\Lambda^{-1}\theta, \theta)$, and assume that $b_j = b$ for all indices $j \in [1, m]$. For each positive integer T, let $x = x(T) = \lfloor \eta T \rfloor + 1$ and $p_T = sf'_\eta T^{1/3} - m'_\eta T$. We have that*

$$\lim_{T\to\infty} \det\bigl(\mathrm{Id} + V_\zeta^{(p_T)}\bigr)_{L^2(\mathcal{C}_V)} = G_m(s).$$

Now let us explain how to deduce Theorem 1.6 and Theorem 1.7 from Proposition 5.2 and Proposition 5.3, respectively. Both of these reductions use the following lemma, which is Lemma 4.1.39 of [14].

LEMMA 5.4 ([14], Lemma 4.1.39). *Let $g_1, g_2, \ldots : \mathbb{R} \to [0, 1]$ be decreasing functions, such that $\lim_{x\to-\infty} g_j(x) = 1$ and $\lim_{x\to\infty} g_j(x) = 0$ for each positive integer j. Assume moreover that, for each $\varepsilon > 0$, the $\{g_j(x)\}_{j\in\mathbb{Z}_{>0}}$ converge uniformly to $\mathbf{1}_{x\leq 0}$ on $x \in \mathbb{R} \setminus [-\varepsilon, \varepsilon]$, as j tends to ∞.*

Suppose that F is a continuous probability distribution and that $X_1, X_2, \ldots$ are random variables such that $\lim_{n\to\infty} \mathbb{E}[g_n(X_n - s)] = F(s)$ for each real number $s \in \mathbb{R}$. Then $\lim_{n\to\infty} \mathbb{P}[X_n \leq s] = F(s)$ for each $s \in \mathbb{R}$.

Using Lemma 5.4, we can establish Theorem 1.6 from Proposition 5.2.

PROOF OF THEOREM 1.6 ASSUMING PROPOSITION 5.2. Let $g_N(x) = (-q^{-N^{1/3}f_\eta x}; q)_\infty^{-1}$, for each $N > 0$, and suppose that $\eta \in (\theta, 1)$. From Proposition 5.1 and the first part of Proposition 5.2, it follows that

$$\lim_{T \to \infty} \mathbb{E}\big[g_T\big(T^{-1/3}f_\eta^{-1}(m_\eta T - \mathfrak{h}_t(\lfloor \eta T \rfloor + 1)) - s\big)\big]$$

(5.7)
$$= \lim_{T \to \infty} \det\big(\mathrm{Id} + A_\zeta^{(p_T)}\big)$$

$$= F_{\mathrm{TW}}(s).$$

Now observe that the sequence of functions $\{g_N\}_{N \geq 0}$ satisfies the conditions of Lemma 5.4. Applying this lemma with $X_T = T^{-1/3}f_\eta^{-1}(m_\eta T - \mathfrak{h}_t(\lfloor \eta T \rfloor + 1))$ and using (5.7), we find that

$$\lim_{T \to \infty} \mathbb{P}\big[m_\eta T - \mathfrak{h}_t(\lfloor \eta T \rfloor + 1) \leq s f_\eta T^{1/3}\big] = F_{\mathrm{TW}}(s).$$

From this, we deduce the first part of Theorem 1.6, since $\mathfrak{h}_T(\lfloor \eta T \rfloor + 1) = J_T(\lfloor \eta T \rfloor) = J_T(\eta T)$.

The second and third parts follow similarly. $\square$

We omit the verification of Theorem 1.7 from Proposition 5.3 since it follows in a very similar way, after setting $\eta = x/y$ and observing that the height function $\mathfrak{H}(xT, yT)$ from Theorem 1.7 satisfies $\mathfrak{H}(xT, yT) = \mathfrak{h}_{yT}(xT - 1)$ in Proposition 5.3.

In the remaining sections, we establish Proposition 5.2 and Proposition 5.3.

6. Tracy–Widom fluctuations.

In this section we establish the first parts of Proposition 5.2 and Proposition 5.3 through a saddle point analysis of the Fredholm determinant $\det(\mathrm{Id} + K_\zeta^{(p_T)})_{L^2(\mathcal{C}_K)}$, defined in Proposition 5.1. Similar analyses have been performed in several previous papers; in particular, we will largely follow the reasoning presented in Section 5 of [16], although we will be more detailed here.

To implement this analysis we will first select the contours $\mathcal{C}_K$ and Γ_K from Proposition 5.1 in a specific way, so as to ensure that the kernel $K_\zeta^{(p_T)}(w, w')$ decays exponentially (in T) for all $w, w' \in \mathcal{C}_K$ away from a certain point $\psi \in \mathcal{C}_K$; this choice of contours will be made in Section 6.1. Once the contours have been fixed, we will be able to proceed with the asymptotic analysis through the Laplace method; this will be done in Section 6.2.

6.1. *Choosing the contours $\mathcal{C}_K$ and Γ_K.* In this section we exhibit one choice of contours $\mathcal{C}_K$ and Γ_K that will later lead to proofs of the first parts of Proposition 5.2 and Proposition 5.3. The choice of these contours will depend on whether $K = A$ or $K = V$. In Section 6.1.2, we consider the case $K = V$, and in Section 6.1.3 we consider the case $K = A$. In both cases our definitions are based on the same observation, which we now describe.

6.1.1. *An exponential form for the kernel $K_\zeta^{(p_T)}$.* For notational convenience, we set $x(T) = \eta T + 1$, as opposed to $x(T) = \lfloor \eta T \rfloor + 1$ (which is what was stated in Proposition 5.2 and Proposition 5.3); this will not affect the asymptotics.

Now let us rewrite the kernel $K(w, w') = K^{(p_T)}(w, w')$ given in (5.4) through the identity

$$
\begin{aligned}
K(w, w') = {} & \frac{1}{2\mathbf{i}\log q} \\
& \times \sum_{j \in \mathbb{Z}} \oint_{\Gamma_V} \frac{\exp(T(G_K(w) - G_K(v)))}{\sin(\pi(\log q)^{-1}(2\pi \mathbf{i} j + \log v - \log w))} \\
& \times \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}v; q)_\infty}{(q^{-1}\beta_k^{-1}w; q)_\infty} \\
& \times \left(\frac{v}{w}\right)^{s f_{\eta;K} T^{1/3}} \frac{dv}{v(w' - v)},
\end{aligned}
$$
(6.1)

where $G_K(z)$ and $f_{\eta;K}$ depend on K. Explicitly, when $K = A$,

$$
\begin{aligned}
G_A(z) &= \frac{q}{z + q} + \eta \log(z + q) + m_{\eta;A} \log z; \\
f_{\eta;A} &= \left(\frac{1 - \eta^2}{4}\right)^{2/3},
\end{aligned}
$$
(6.2)

and when $K = V$,

$$
\begin{aligned}
G_V(z) &= \eta \log(\kappa^{-1}z + q) - \log(z + q) + m_{\eta;V} \log z; \\
f_{\eta;V} &= \frac{(\sqrt{\kappa\eta} - 1)^{2/3}(\kappa - \sqrt{\kappa\eta})^{2/3}}{(\kappa - 1)\kappa^{1/6}\eta^{1/6}}.
\end{aligned}
$$
(6.3)

Here,

$$
m_{\eta;A} = \left(\frac{1 - \eta}{2}\right)^2; \qquad m_{\eta;V} = \frac{(\sqrt{\kappa} - \sqrt{\eta})^2}{\kappa - 1}.
$$
(6.4)

Let us now locate the critical points of G_K; these will again depend on whether $K = A$ or $K = V$. We find that

$$
(6.5) \quad G_A'(z) = \left(\frac{\eta + 1}{2}\right)^2 \frac{(z - \psi_A)^2}{z(z + q)^2} \quad \text{with } \psi_A = \frac{q(1 - \eta)}{1 + \eta};
$$

$$
(6.6) \quad G_V'(z) = \frac{(\sqrt{\kappa\eta} - 1)^2}{\kappa - 1} \frac{(z - \psi_V)^2}{z(z + q\kappa)(z + q)} \quad \text{with } \psi_V = \frac{q(\kappa - \sqrt{\kappa\eta})}{\sqrt{\kappa\eta} - 1}.
$$

Thus, in both cases $K \in \{V, A\}$, ψ_K is a critical point of G_K. Furthermore, $G_K''(\psi_K) = 0$ and

$$\frac{G_A'''(\psi_K)}{2} = \frac{(\eta + 1)^5}{16q^3(1 - \eta)} = \left(\frac{f_{\eta;A}}{\psi_K}\right)^3;$$

$$\frac{G_V'''(\psi_K)}{2} = \frac{(\sqrt{\kappa\eta} - 1)^5}{q^3(\kappa - 1)^3(\kappa - \sqrt{\kappa\eta})\sqrt{\kappa\eta}} = \left(\frac{f_{\eta;V}}{\psi_K}\right)^3.$$

Thus, for both $K \in \{V, A\}$, we have that $G_K'''(\psi_K) = (\psi_K^{-1} f_{\eta;K})^3$, which implies from a Taylor expansion that

$$G_K(z) - G_K(\psi_K)$$

(6.7)
$$= \frac{1}{3}\left(\frac{f_{\eta;K}(z - \psi_K)}{\psi_K}\right)^3 + R_K\left(\frac{f_{\eta;K}(z - \psi_K)}{\psi_K}\right)$$

$$= \frac{1}{3}\left(\frac{f_{\eta;K}(z - \psi_K)}{\psi_K}\right)^3 + \mathcal{O}(|z - \psi_K|^4) \qquad \text{as } |z - \psi_K| \to 0.$$

where

(6.8)
$$R_K\left(\frac{f_{\eta;K}(z - \psi_K)}{\psi_K}\right)$$

$$= G_K(z) - G_K(\psi_K) - \frac{1}{3}\left(\frac{f_{\eta;K}(z - \psi_K)}{\psi_K}\right)^3.$$

We would now like to select contours $\mathcal{C}_K$ and Γ_K such that $\Re(G_K(w) - G_K(v)) < 0$, for all $w \in \mathcal{C}_K$ and $v \in \Gamma_K$ away from ψ_K, subject to the restrictions stated in Proposition 5.1. If we are able to do this, then the integrand on the right-hand side of (6.1) will decay exponentially away from ψ_K. This will allow us to localize the integral around the critical point ψ_K and use the estimate (6.7) to simplify the asymptotics.

6.1.2. *Choice of contours $\mathcal{C}_V$ and Γ_V.* In this section we show how to choose contours $\mathcal{C}_V$ and Γ_V satisfying the conditions of Proposition 5.1 such that $\Re(G_V(w) - G_V(v)) < 0$ for $w \in \mathcal{C}_V$ and $v \in \Gamma_V$ both away from ψ_V. In what follows, we omit the subscript V to simplify notation.

Our choice for $\mathcal{C}$ and Γ will be similar to the contours selected in Section 5 of [16], in which $\mathcal{C}$ and Γ are chosen to follow level lines of the equation $\Re G(z) = G(\psi)$.

That work required some properties of these level lines that are summarized in the following proposition; see Section 5.1 of [16] and Figure 9.

PROPOSITION 6.1 ([16], Section 5.1). *There exist three simple, closed curves, $\mathcal{L}_1 = \mathcal{L}_{1;V}$, $\mathcal{L}_2 = \mathcal{L}_{2;V}$, and $\mathcal{L}_3 = \mathcal{L}_{3;V}$, that all pass through ψ and satisfy the following properties*:
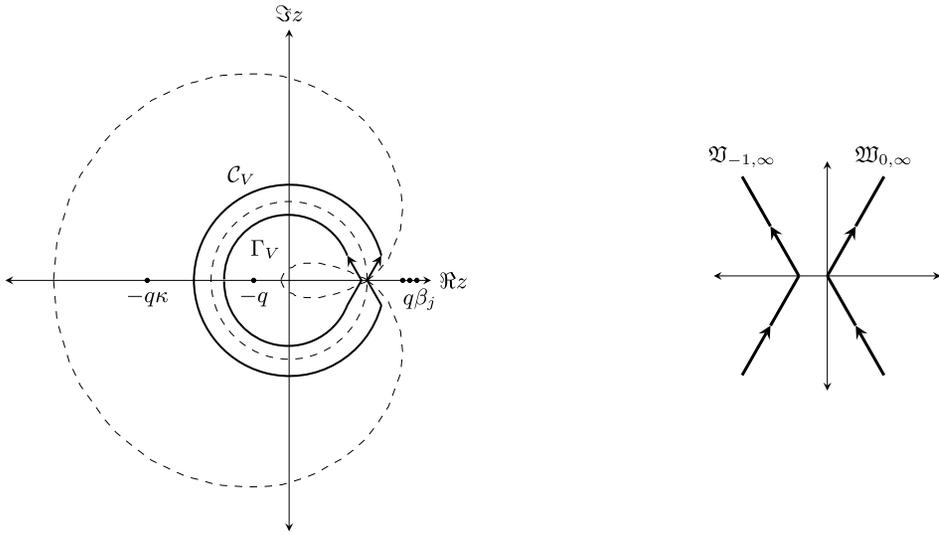
FIG. 9. *Above and to the left, the three level lines* $\mathcal{L}_1$, $\mathcal{L}_2$ *and* $\mathcal{L}_3$ *are depicted as dashed curves; the contours* Γ_V *and* $\mathcal{C}_V$ *are depicted as solid curves and are labeled. Above and to the right are the two contours* $\mathfrak{W}_{0,\infty}$ *and* $\mathfrak{V}_{-1,\infty}$.

1. *Any* $z \in \mathbb{C}$ *satisfying* $\Re G(z) = G(\psi)$ *lies on* $\mathcal{L}_j$ *for some* $j \in \{1, 2, 3\}$.
2. *Any complex number* $z \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ *satisfies* $\Re G(z) = G(\psi)$.
3. *The level lines* $\mathcal{L}_1$, $\mathcal{L}_2$ *and* $\mathcal{L}_3$ *are all star-shaped.*
4. *We have that* $\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}_2 \cap \mathcal{L}_3 = \mathcal{L}_1 \cap \mathcal{L}_3 = \{\psi\}$. *Furthermore,* $\mathcal{L}_1 \setminus \{\psi\}$ *is contained in the interior of* $\mathcal{L}_2$, *and* $\mathcal{L}_2 \setminus \{\psi\}$ *is contained in the interior of* $\mathcal{L}_3$.
5. *The interior of* $\mathcal{L}_1$ *contains* 0, *but not* $-q$; *the interior of* $\mathcal{L}_2$ *contains* 0 *and* $-q$, *but not* $-q\kappa$; *and the interior of* $\mathcal{L}_3$ *contains* 0, $-q$ *and* $-q\kappa$.
6. *The level line* $\mathcal{L}_1$ *meets the positive real axis (at* ψ) *at angles* $5\pi/6$ *and* $-5\pi/6$; *the level line* $\mathcal{L}_2$ *meets the positive real axis (at* ψ) *at angles* $\pi/2$ *and* $-\pi/2$; *and the level line* $\mathcal{L}_3$ *meets the positive real axis (at* ψ) *at angles* $\pi/6$ *and* $-\pi/6$.
7. *For all* z *in the interior of* $\mathcal{L}_2$ *but strictly outside of* $\mathcal{L}_1$, *we have that* $\Re(G(z) - G(\psi)) > 0$.
8. *For all* z *in the interior of* $\mathcal{L}_3$ *but strictly outside of* $\mathcal{L}_2$, *we have that* $\Re(G(z) - G(\psi)) < 0$.

Now let us explain how to select the contours $\mathcal{C}$ and Γ. They will each be the union of two contours, a "small piecewise linear part" near ψ, and a "large curved part" that closely follows the level line $\mathcal{L}_2$.

Let us be more specific. The contours given by the following definition will be the linear parts of $\mathcal{C}$ and Γ.

DEFINITION 6.2.   For a real number $r \in \mathbb{R}$ and a positive real number $\varepsilon > 0$ (possibly infinite), let $\mathfrak{W}_{r,\varepsilon}$ denote the piecewise linear curve in the complex plane that connects $r + \varepsilon e^{-\pi \mathbf{i}/3}$ to r to $r + \varepsilon e^{\pi \mathbf{i}/3}$. Similarly, let $\mathfrak{V}_{r,\varepsilon}$ denote the piecewise linear curve in the complex plane that connects $r + \varepsilon e^{-2\pi \mathbf{i}/3}$ to r to $r + \varepsilon^{2\pi \mathbf{i}/3}$.

Examples of these contours are given in Figure 9. Now let us make the following definitions; Definition 6.3 and Definition 6.4 define the piecewise linear and curved parts of the contours $\mathcal{C}$ and Γ, respectively. Then Definition 6.5 defines the contours $\mathcal{C}$ and Γ.

DEFINITION 6.3.   Let $\mathcal{C}^{(1)} = \mathfrak{W}_{\psi,\varepsilon}$ and $\Gamma^{(1)} = \mathfrak{V}_{\psi - \psi f_\eta^{-1} T^{-1/3}, \varepsilon}$, where ε is chosen to be sufficiently small (independent of T) so that the following four properties hold:

1. For all sufficiently large T, the two conjugate endpoints of $\mathcal{C}^{(1)}$ are strictly between the level lines $\mathcal{L}_2$ and $\mathcal{L}_3$, so that their distance from $\mathcal{L}_2$ and $\mathcal{L}_3$ is bounded away from 0, independent of T.
2. For all sufficiently large T, the two conjugate endpoints of $\Gamma^{(1)}$ are strictly between $\mathcal{L}_1$ and $\mathcal{L}_2$, so that their distance from $\mathcal{L}_1$ and $\mathcal{L}_2$ is bounded away from 0, independent of T.
3. We have that $|R(\psi^{-1} f_\eta (z - \psi))| < |f_\eta (z - \psi)/2\psi|^3$, for all $z \in \mathcal{C}^{(1)} \cup \Gamma^{(1)}$, where R was given in (6.8).
4. We have that $|v/w| \in (q^{1/2}, 1)$ for all $v \in \Gamma^{(1)}$ and $w \in \mathcal{C}^{(1)}$.

Such a positive real number ε is guaranteed to exist by part 6 of Proposition 6.1 and the estimate (6.7).

DEFINITION 6.4.   Let $\mathcal{C}^{(2)}$ denote a positively oriented contour from the top endpoint $\psi + \varepsilon e^{\pi \mathbf{i}/3}$ of $\mathcal{C}^{(1)}$ to the bottom endpoint $\psi + \varepsilon e^{-\pi \mathbf{i}/3}$ of $\mathcal{C}^{(1)}$, and let $\Gamma^{(2)}$ denote a positively oriented contour from the top endpoint $\psi - \psi f_\eta^{-1} T^{-1/3} + \varepsilon e^{2\pi \mathbf{i}/3}$ of $\Gamma^{(1)}$ to the bottom endpoint $\psi + \psi f_\eta^{-1} T^{-1/3} + \varepsilon e^{-2\pi \mathbf{i}/3}$ of $\Gamma^{(1)}$, satisfying the following five properties:

1. The contour $\mathcal{C}^{(2)}$ remains strictly between the level lines $\mathcal{L}_2$ and $\mathcal{L}_3$, so that the distance from $\mathcal{C}^{(2)}$ to $\mathcal{L}_2$ and $\mathcal{L}_3$ remains bounded away from 0, independent of T.
2. The contour $\Gamma^{(2)}$ remains strictly between the level lines $\mathcal{L}_1$ and $\mathcal{L}_2$, so that the distance from $\mathcal{C}^{(2)}$ to $\mathcal{L}_1$ and $\mathcal{L}_2$ remains bounded away from 0, independent of T.
3. The contour $\mathcal{C}^{(1)} \cup \mathcal{C}^{(2)}$ is star-shaped.
4. The contour $\Gamma^{(1)} \cup \Gamma^{(2)}$ is star-shaped and does not contain $-q\kappa$.
5. The contours $\mathcal{C}^{(2)}$ and $\Gamma^{(2)}$ are both sufficiently close to $\mathcal{L}_2$ so that the interior of $\Gamma^{(1)} \cup \Gamma^{(2)}$ contains the image of $\mathcal{C}^{(1)} \cup \mathcal{C}^{(2)}$ under multiplication by $q^{1/2}$.

Such contours $\mathcal{C}^{(2)}$ and $\Gamma^{(2)}$ are guaranteed to exist by part 3 and part 5 of Proposition 6.1.

DEFINITION 6.5.   Set $\mathcal{C} = \mathcal{C}^{(1)} \cup \mathcal{C}^{(2)}$ and $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)}$.

Examples of the contours $\mathcal{C}$ and Γ are depicted in Figure 9.

LEMMA 6.6.   *The contour Γ is positively oriented and star-shaped; it contains 0, but leaves outside $-q\kappa$ and $q\beta_j$ for each $j \in [1, m]$. Furthermore, $\mathcal{C}$ is a positively oriented, star-shaped contour that is contained inside $q^{-1}\Gamma$; that contains 0, $-q$ and Γ; but that leaves outside $q\beta_j$ for each $j \in [1, m]$.*

We omit the proof of this lemma, since it can be quickly deduced from the definitions of $\mathcal{C}$ and Γ and the fact that $q\beta_j > \psi$; the latter fact can be verified using the definition (6.6) of ψ_K and the fact that $\eta > \theta$ (recall that θ was defined by (1.5)).

In the next section, we choose the contours Γ_K and $\mathcal{C}_K$ in the case $K = A$.

6.1.3. *Choice of contours $\mathcal{C}_A$ and Γ_A.*   Similar to in Section 6.1.2, our goal in this section is to choose contours $\mathcal{C}_A$ and Γ_A satisfying the conditions of Proposition 5.1 such that $\Re(G_A(w) - G_A(v)) < 0$ for $w \in \mathcal{C}_A$ and $v \in \Gamma_A$ both away from ψ_A. In what follows, we omit the subscript A to simplify notation.

As in Section 6.1.2, we will take the contours Γ and $\mathcal{C}$ to follow level lines of the equation $\Re G(z) = G(\psi)$. Again, we require some properties of these level lines that are summarized in the following proposition. Unlike Proposition 6.1, this proposition does not seem to have appeared previously; we will establish it later, in Section 6.1.4.

PROPOSITION 6.7.   *There exist three simple, closed curves, $\mathcal{L}_1 = \mathcal{L}_{1;A}$, $\mathcal{L}_2 = \mathcal{L}_{2;A}$ and $\mathcal{L}_3 = \mathcal{L}_{3;A}$, that all pass through ψ and satisfy properties 1, 3, 4, 6, 7 and 8 of Proposition 6.1, with properties 2 and 5, respectively, replaced by the following*:

2. Any complex number $z \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$, not equal to q, satisfies $\Re G(z) = G(\psi)$.

5. The interior of $\mathcal{L}_1$ contains 0, but not $-q$; the interior of $\mathcal{L}_2$ contains 0; and the interior of $\mathcal{L}_3$ contains 0 and $-q$. Furthermore, $-q$ lies on the curve $\mathcal{L}_2$.

Thus, these contours are very similar to those depicted in Figure 9, except that now $-q$ lies on $\mathcal{L}_2$ instead of in its interior (and $-q\kappa$ does not exist).

Given Proposition 6.7, the selection of the contours $\mathcal{C}$ and Γ will proceed as in Section 6.1.2. Again, these contours will each be the union of two contours, a small piecewise linear part near ψ and a large curved part that closely follows the level line $\mathcal{L}_2$.

Specifically, define $\mathcal{C}^{(1)}$ and $\Gamma^{(1)}$ as in Definition 6.3, where now ψ_V is replaced with ψ_A, $f_{\eta;V}$ is replaced with $f_{\eta;A}$, R_V is replaced by R_A, and $\mathcal{L}_{j;V}$ is replaced with $\mathcal{L}_{j;A}$ for each $j \in \{1, 2, 3\}$. Similarly, define $\mathcal{C}^{(2)}$ and $\Gamma^{(2)}$, under the same replacements (and ignoring the requirement that $\Gamma^{(1)} \cup \Gamma^{(2)}$ must avoid $-q\kappa$, since the parameter κ does not exist for the ASEP). As in Section 6.1.2, the existence of these contours is guaranteed by Proposition 6.7 and the estimate (6.7).

Then, as in Definition 6.5, define $\mathcal{C} = \mathcal{C}^{(1)} \cup \mathcal{C}^{(2)}$ and $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)}$. The following lemma states that $\mathcal{C}$ and Γ satisfy their required conditions; we omit its proof.

LEMMA 6.8. *The contour Γ is positively oriented and star-shaped; it contains* 0, *but leaves outside $-q$ and $q\beta_j$ for each $j \in [1, m]$. Furthermore, $\mathcal{C}$ is a positively oriented, star-shaped contour that is contained inside $q^{-1}\Gamma$; that contains 0, $-q$ and Γ; but that leaves outside $q\beta_j$ for each $j \in [1, m]$.*

6.1.4. *Proof of Proposition 6.7.* The goal of this section is to establish Proposition 6.7, whose proof will be similar to that of Proposition 6.1 given in Section 5 of [16]. To do this, we require the following two lemmas, which provide some preliminary properties of the level lines of $\Re G_A(z) = G_A(\psi_A)$. As in Section 6.1.3, we assume that $K = A$ throughout, and thus omit the subscript A from notation.

LEMMA 6.9. *The set of $z \in \mathbb{C}$ for which $\Re G(z) = G(\psi)$ is bounded.*

PROOF. Observe that

$$
(6.9) \quad
\begin{aligned}
\Re G(z) &= \Re\left(\frac{q}{z+q}\right) + \eta \log|z+q| + m_\eta \log|z| \\
&= \left(\frac{1+\eta}{2}\right)^2 \log|z| + \mathcal{O}(|z|^{-1}),
\end{aligned}
$$

where we have used the definition of $m_\eta = m_{\eta;A}$ given in (6.4) to establish the estimate above. This implies that $|\Re G(z)|$ tends to ∞ as $|z|$ tends to ∞, so the set of z satisfying $\Re G(z) = G(\psi)$ is bounded. $\square$

LEMMA 6.10. *Let $\ell \subset \mathbb{C}$ be any line through 0. There exist at most 6 complex numbers $z \in \ell$ such that $\Re G(z) = G(\psi)$. Moreover, if ℓ is the real axis, then there exist only three such z; one is $z = \psi$, one is in the interval $(-q, 0)$, and one is in the interval $(-\infty, -q)$.*

PROOF. Let us establish the second statement first, so suppose that ℓ is the real axis. Then, from (6.5), we deduce that $G'(z) > 0$ for $z > 0$, and that $G'(z) < 0$ for $z < 0$. Due to the singularity of $G(z)$ at $z = -q$, this implies that there are at most three real numbers z for which $\Re G(z) = G(\psi)$, and that at most one of them is positive; the one positive value of z is $z = \psi$.

Furthermore, observe that $G(z)$ tends to ∞ when z tends to $-\infty$ (see (6.9)) or when z tends to $-q$ from the right. Moreover, $G(z)$ tends to $-\infty$ when z tends to 0 or when z tends to $-q$ from the left. By the continuity of G on $(-\infty, -q) \cup (-q, 0)$, it follows that there exists one value of $z = z_1 \in (-q, 0)$ and one value of $z = z_2 \in (-\infty, -q)$ such that $\Re G(z_1) = G(\psi) = \Re G(z_2)$. This establishes the lemma in the case when $\ell = \mathbb{R}$.

Now assume that $\ell \neq \mathbb{R}$, and let $z_0 \in \ell$ be a nonzero complex number on ℓ. It suffices to show that there exist at most six real numbers $\omega \in \mathbb{R}$ such that $\Re G(\omega z_0) = G(\psi)$. To that end, we differentiate $\Re G(\omega z_0)$ with respect to ω to find that

$$
\frac{\partial}{\partial \omega}(\Re G(\omega z_0) - G(\psi))
$$

(6.10)
$$
= \Re\left(\frac{\partial}{\partial \omega}\left(\frac{q}{\omega z_0 + q} + \eta \log(\omega z_0 + q) + m_\eta \log(\omega z_0)\right)\right)
$$

$$
= \left(\frac{(\eta + 1)^2}{4\omega|\omega + q z_0^{-1}|^4}\right)\Re(\omega - z_0^{-1}\psi)^2(\omega + q\overline{z_0^{-1}})^2.
$$

Fixing $z_0 \in \ell$, the right-hand side of (6.10) is a rational function in ω whose numerator is a polynomial of degree four; in particular, it can be zero for at most four distinct values of ω.

Furthermore, (6.10) shows that the only real value of ω for which the derivative of $\Re(G(\omega z_0) - G(\psi))$ is singular is $\omega = 0$; this is due to the fact that $\omega + q z_0^{-1} \neq 0$, which holds since $\ell \neq \mathbb{R}$. Thus, there are at most five real numbers ω, at which the derivative $\partial(\Re(G(\omega z_0) - G(\psi)))/\partial\omega$ can change sign. It follows that there exist at most six real numbers ω for which $\Re G(\omega z_0) = G(\psi)$. $\square$

Now we can establish Proposition 6.7.

PROOF OF PROPOSITION 6.7. Let $S = \mathbb{C} \setminus \{\psi, -q\}$. Then $\Re G(z)$ and $\Re(G'(z))$ are smooth on $S \setminus \{0\}$. Furthermore, from the explicit form (6.5) of $G'(z)$, we deduce that $G'(z)$ is nonzero on $S \setminus \{0\}$, which implies that $\Re G(z)$ has no critical points on $S \setminus \{0\}$. Thus, the implicit function theorem shows that the set of $z \in S$ satisfying $\Re G(z) = G(\psi)$ is a one-dimensional submanifold of S. In particular, it is a disjoint union of connected components that are each either homeomorphic to a line or a circle.

By Lemma 6.9, all of these connected components are bounded. Therefore, the closure $\overline{\mathcal{M}}$ (in $\mathbb{C}$) of any such component $\mathcal{M}$ is either homeomorphic to a circle or has endpoints $-q$ and ψ and is homeomorphic to a compact interval. If $\overline{\mathcal{M}}$ is homeomorphic to a circle, then the maximum principle for harmonic functions shows that one of the singularities 0 or $-q$ of $\Re G(z)$ must lie on or in the interior of $\overline{\mathcal{M}}$ [since $\Re G(z)$ is harmonic for $z \in \mathbb{C} \setminus \{-q, 0\}$].
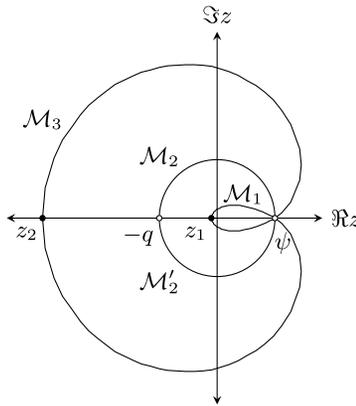
FIG. 10.   *Shown above are the curves $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_2'$ and $\mathcal{M}_3$ from the proof of Proposition 6.7.*

Now, due to (6.7), six (not necessarily distinct) connected components exit from ψ, at angles $\pi/6$, $\pi/2$, $5\pi/6$, $-5\pi/6$, $-\pi/2$ and $-\pi/6$. Furthermore, due to the explicit form (6.2) of G, we deduce that two components also exit from $-q$ at angles $\pi/2$ and $-\pi/2$. Lemma 6.10 also shows that there exist $z_1 \in (-q, 0)$ and $z_2 \in (-\infty, -q)$ such that $\Re G(z_1) = G(\psi) = \Re G(z_2)$; this implies that one of the components passes through z_1 and one of the components passes through z_2. By the second part of Lemma 6.10, we also find that there are no other real numbers $z \notin \{z_1, z_2, \psi\}$ such that $\Re G(z) = G(\psi)$.

These facts imply that there are four disjoint connected components $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_2'$, $\mathcal{M}_3 \subset S$ (see Figure 10) such that the set of $z \in S$ satisfying $\Re G(z) = G(\psi)$ is equal to the union $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_2' \cup \mathcal{M}_3$; these components can be described as follows. First, we have that $\psi \in \overline{\mathcal{M}_1}, \overline{\mathcal{M}_2}, \overline{\mathcal{M}_2'}, \overline{\mathcal{M}_3}$. Furthermore, $z_1 \in \mathcal{M}_1$, and $\overline{\mathcal{M}_1}$ is homeomorphic to a circle; similarly, $z_2 \in \mathcal{M}_3$, and $\overline{\mathcal{M}_3}$ is homeomorphic to a circle.

Moreover, $\overline{\mathcal{M}_2}$ and $\overline{\mathcal{M}_2'}$ are both homeomorphic to compact intervals; we also have that $-q \in \overline{\mathcal{M}_2} \cap \overline{\mathcal{M}_2'}$, that $\mathcal{M}_2$ is in the upper half plane, and that $\mathcal{M}_2'$ is in the lower half plane. We can describe the intersections $\overline{\mathcal{M}_1} \cap \mathbb{R} = \{z_1, \psi\}$, $\overline{\mathcal{M}_2} \cap \mathbb{R} = \{-q, \psi\} = \overline{\mathcal{M}_2'} \cap \mathbb{R}$ and $\overline{\mathcal{M}_3} \cap \mathbb{R} = \{z_2, \psi\}$.

Now set $\mathcal{L}_1 = \overline{\mathcal{M}_1}$, $\mathcal{L}_2 = \overline{\mathcal{M}_2} \cup \overline{\mathcal{M}_2'}$ and $\mathcal{L}_3 = \overline{\mathcal{M}_3}$. Then, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are all simple, closed curves passing through ψ. Parts 1 and 2 of Proposition 6.7 are satisfied by these curves since $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_2' \cup \mathcal{M}_3 \cup \{\psi\} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$.

Furthermore, only one positive real number and only one negative real number lie on each of the closures $\overline{\mathcal{M}_1}$, $\overline{\mathcal{M}_2}$, $\overline{\mathcal{M}_2'}$ and $\overline{\mathcal{M}_3'}$. This implies that 0 is in the interior of $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.

Now property 3 holds from Lemma 6.10; indeed, this lemma implies that any line $\ell \subset \mathbb{C}$ can only intersect $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ in at most six places. Since each $\mathcal{L}_j$ is a closed curve containing 0 in its interior, ℓ intersects each $\mathcal{L}_j$ at least, and thus

only, twice; furthermore, 0 is between these two intersection points. This implies that for each angle $a \in \mathbb{R}$ and each index $j \in \{1, 2, 3\}$, there is only one complex number $z_a \in \mathcal{L}_j$ such that $z_a/|z_a| = e^{\mathbf{i}a}$. This verifies property 3.

To see property 4, observe that $\mathcal{L}_i$ and $\mathcal{L}_j$ can only intersect at critical points of $G(z)$, for any distinct $i, j \in \{1, 2, 3\}$. These critical points are ψ and $-q$. Furthermore, only $\mathcal{L}_2$ contains $-q$; this implies that $\mathcal{L}_i \cap \mathcal{L}_j = \{\psi\}$, for any $i, j \in \{1, 2, 3\}$. Now the containment follows from the fact that $\mathcal{L}_1$ intersects the negative real axis only once at z_1, that $\mathcal{L}_2$ intersects the negative real axis only once at $-q$, that $\mathcal{L}_3$ intersects the negative real axis only once at z_2, and that $z_2 < -q < z_1$.

Property 5 follows from the fact that $z_2 < -q < z_1$, and property 6 follows from the containment property 4 and the estimate (6.7).

Furthermore, observe that $\Re G(z)$ can only change sign across some $\mathcal{L}_j$; thus, property 7 and property 8 follow from the facts that $\lim_{\varepsilon \to 0^+} \Re G(\varepsilon - q) = \infty$ and $\lim_{\varepsilon \to 0^-} \Re G(\varepsilon - q) = -\infty$, respectively. $\square$

6.2. *Proof of part* 1 *of Proposition* 5.2 *and Proposition* 5.3.

The goal of this section is to establish the first parts of Proposition 5.2 and Proposition 5.3. Since both proofs are similar, we will do them simultaneously. Thus, recall the notation $K \in \{V, A\}$ that corresponds to the stochastic six-vertex model when $K = V$ and the ASEP when $K = A$.

If accordance with the statements of Proposition 5.2 and Proposition 5.3, we assume that $\eta \in (\theta, 1)$ when $K = A$ and that $\eta \in (\theta, \kappa)$ when $K = V$. In either case, set $x = x(T) = \lfloor \eta T \rfloor + 1$, and let $p_T = f_\eta s T^{1/3} - m_\eta T$, where we recall that $m_\eta = m_{\eta;K}$ is defined through (6.4) and $f_\eta = f_{\eta;K}$ are defined through (6.2) and (6.3) in the cases $K = A$ and $K = V$, respectively.

As in Section 6.1.1, we replace $x(T)$ with $\eta T + 1$ for notational convenience; this will not affect the asymptotics. We also simplify notation by omitting the subscript K.

Now, in order to establish Proposition 5.2 and Proposition 5.3, we must asymptotically analyze the kernel $K^{(p_T)}$ given by (6.1). To that end, we will follow a Laplace method similar to what was outlined in Section 5 of [16]. Specifically, in Section 6.2.1, we will analyze the contribution to the integral on the right-hand side of (6.1) when v is integrated in a small neighborhood of ψ; this will yield a kernel resembling the Airy kernel, which will lead to the F_{TW} asymptotics. Then, in Section 6.2.2, we will show that the contribution to the integral decays exponentially when v is integrated outside a neighborhood of ψ; we will then be able to ignore this contribution, which will lead to the proofs of the first parts of Proposition 5.2 and Proposition 5.3.

6.2.1. *Contribution near* ψ.

In this section we analyze the contribution to the right-hand side of (6.1) when $w \in \mathcal{C}^{(1)}$ and $v \in \Gamma^{(1)}$, that is, when both w and v

are near ψ. To that end, define

$$\tilde{K}(w, w') = \frac{1}{2\mathbf{i}\log q} \sum_{j=-\infty}^{\infty} \int_{\Gamma^{(1)}} \frac{\exp(T(G(w) - G(v)))}{\sin(\frac{\pi}{\log q}(2\pi\mathbf{i}j + \log v - \log w))}$$

(6.11)
$$\times \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}v; q)_\infty}{(q^{-1}\beta_k^{-1}w; q)_\infty}$$

$$\times \left(\frac{v}{w}\right)^{sf_\eta T^{1/3}} \frac{dv}{v(w' - v)},$$

for any $w, w' \in \mathcal{C}$. Observe that the integral in (6.11) is similar to the integral in (6.1); the difference is that the former integral is along $\Gamma^{(1)}$, and the latter integral is along Γ.

Now let us change variables, a procedure that will in effect "zoom into" the region around ψ. Denote $\sigma = \psi f_\eta^{-1} T^{-1/3}$, and set

(6.12)
$$w = \psi + \sigma\widehat{w}; \qquad w' = \psi + \sigma\widehat{w'};$$
$$v = \psi + \sigma\widehat{v}; \qquad \widehat{K}(\widehat{w}, \widehat{w'}) = \sigma\tilde{K}(w, w').$$

Also, for any contour $\mathcal{D}$, set $\widehat{\mathcal{D}} = \sigma^{-1}(\mathcal{D} - \psi)$, where $\sigma^{-1}(\mathcal{D} - \psi)$ denotes all numbers of the form $\sigma^{-1}(z - \psi)$ with $z \in \mathcal{D}$. In particular, from Definition 6.2 and Definition 6.3, we find that $\widehat{\mathcal{C}^{(1)}} = \mathfrak{W}_{0,\varepsilon/\sigma}$ and $\widehat{\Gamma^{(1)}} = \mathfrak{V}_{-1,\varepsilon/\sigma}$.

Our goal in this section is to establish the following lemma.

LEMMA 6.11. *There exist constants $c, C > 0$ such that*

(6.13)
$$|\widehat{K}(\widehat{w}, \widehat{w'})| \leq \frac{C}{1 + |\widehat{w'}|} \exp(-c|\widehat{w}|^3),$$

for each $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ and $\widehat{w'} \in \widehat{\mathcal{C}}$.

Furthermore, if we define the kernel $L_s(\widehat{w}, \widehat{w'})$ by

(6.14)
$$L_s(\widehat{w}, \widehat{w'}) = \frac{1}{2\pi\mathbf{i}} \int_{\mathfrak{V}_{-1,\infty}} \frac{1}{(\widehat{v} - \widehat{w})(\widehat{w'} - \widehat{v})}$$

$$\times \exp\left(\frac{\widehat{w}^3}{3} - \frac{\widehat{v}^3}{3} + s(\widehat{v} - \widehat{w})\right) d\widehat{v},$$

then we have that

(6.15)
$$\lim_{T\to\infty} \widehat{K}(\widehat{w}, \widehat{w'}) = L_s(\widehat{w}, \widehat{w'}),$$

for each fixed $\widehat{w}, \widehat{w'} \in \mathfrak{W}_{0,\infty}$.

To establish this lemma, we first rewrite the kernel $\widehat{K}$. By (6.12) and the fact that $\sigma = \psi f_\eta^{-1} T^{-1/3}$, we deduce that

$$(6.16) \qquad \widehat{K}(\widehat{w}, \widehat{w'}) = \frac{1}{2\pi \mathbf{i}} \int_{\widehat{\Gamma}^{(1)}} I(\widehat{w}, \widehat{w'}; \widehat{v}) \, d\widehat{v},$$

where

$$
\begin{aligned}
I(\widehat{w}, \widehat{w'}; \widehat{v}) = & \frac{1}{(1 + \psi^{-1}\sigma\widehat{v})(\widehat{w'} - \widehat{v})} \\
& \times \exp\left(\frac{\widehat{w}^3 - \widehat{v}^3}{3} + T\left(R(T^{-1/3}\widehat{w}) - R(T^{-1/3}\widehat{v})\right) \right) \\
(6.17) \qquad & \times \left(\frac{1 + \psi^{-1}\sigma\widehat{v}}{1 + \psi^{-1}\sigma\widehat{w}} \right)^{\psi\sigma^{-1}s} \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}(\psi + \sigma\widehat{v}); q)_\infty}{(q^{-1}\beta_k^{-1}(\psi + \sigma\widehat{w}); q)_\infty} \\
& \times \frac{\pi\psi^{-1}\sigma}{\log q} \\
& \times \sum_{j=-\infty}^{\infty} \frac{1}{\sin(\frac{\pi}{\log q}(2\pi\mathbf{i}j + \log(1 + \psi^{-1}\sigma\widehat{v}) - \log(1 + \psi^{-1}\sigma\widehat{w})))}.
\end{aligned}
$$

Lemma 6.11 will follow from the large T asymptotics of (and a uniform estimate on) I, given by the following lemma.

LEMMA 6.12.   *There exist constants $c, C > 0$ such that*

$$(6.18) \qquad \left| I(\widehat{w}, \widehat{w'}; \widehat{v}) \right| \le \frac{C}{1 + |\widehat{w'}|} \exp\left(-c(|\widehat{w}|^3 + |\widehat{v}|^3)\right),$$

for all $\widehat{w} \in \mathfrak{W}_{0,\varepsilon/\sigma}$, $\widehat{w'} \in \widehat{\mathcal{C}}$, and $\widehat{v} \in \mathfrak{V}_{-1,\varepsilon/\sigma}$.
    Furthermore,

$$(6.19) \quad \lim_{T \to \infty} I(\widehat{w}, \widehat{w'}; \widehat{v}) = \frac{1}{(\widehat{v} - \widehat{w})(\widehat{w'} - \widehat{v})} \exp\left(\frac{\widehat{w}^3}{3} - \frac{\widehat{v}^3}{3} + s(\widehat{w} - \widehat{v}) \right),$$

for each fixed $\widehat{w}, \widehat{w'}, \widehat{v} \in \mathbb{C}$.

PROOF.   Let us first establish the uniform estimate (6.18). To that end, observe that there exists a constant $C_1 > 0$ such that the six inequalities

$$
\left| \frac{1}{1 + \psi^{-1}\sigma\widehat{v}} \right| < C_1; \qquad \frac{1}{\widehat{v} - \widehat{w'}} < \frac{C_1}{1 + |\widehat{w'}|};
$$

$$(6.20) \qquad \left| \frac{1 + \psi^{-1}\sigma\widehat{v}}{1 + \psi^{-1}\sigma\widehat{w}} \right|^{\psi\sigma^{-1}s} \le C_1 \exp\left(C_1(|\widehat{w}| + |\widehat{v}|)\right);$$

$$\frac{\pi \psi^{-1}\sigma}{|\log q|} \sum_{j\neq 0}\left|\frac{1}{\sin(\frac{\pi}{\log q}(2\pi \mathbf{i} j + \log(1+\psi^{-1}\sigma\widehat{v}) - \log(1+\psi^{-1}\sigma\widehat{w})))}\right| \leq C_1 T^{-1/3};$$

$$\left|\left(\frac{\pi\psi^{-1}\sigma}{\log q}\right)\frac{1}{\sin(\frac{\pi}{\log q}(\log(1+\psi^{-1}\sigma\widehat{v}) - \log(1+\psi^{-1}\sigma\widehat{w})))}\right| < C_1;$$

$$\left|\prod_{k=1}^{m}\frac{(q^{-1}\beta_k^{-1}(\psi+\sigma\widehat{v});q)_\infty}{(q^{-1}\beta_k^{-1}(\psi+\sigma\widehat{w});q)_\infty}\right| < C_1\exp(C_1(|\widehat{w}|+|\widehat{v}|)),$$

all hold for each $\widehat{w} \in \mathfrak{W}_{0,\varepsilon/\sigma}$, $\widehat{w'} \in \widehat{\mathcal{C}}$, and $\widehat{v} \in \mathfrak{V}_{-1,\varepsilon/\sigma}$.

Indeed, the first inequality holds since $v = \psi(1 + \psi^{-1}\sigma\widehat{v})$ is bounded away from 0 for $v \in \Gamma$. The second inequality holds since $|\widehat{w'} - \widehat{v}| \geq 1$ for $\widehat{w'} \in \widehat{\mathcal{C}}$ and $\widehat{v} \in \mathfrak{V}_{-1,\varepsilon/\sigma}$. The third inequality holds since $1 + \psi^{-1}\sigma\widehat{w}$ is bounded away from 0 for $\widehat{w} \in \widehat{\mathcal{C}}$.

The fourth inequality holds since $\sigma = \mathcal{O}(T^{-1/3})$, since $\sin(\frac{\pi}{\log q}(2\pi \mathbf{i} j + \log(1 + \psi^{-1}\sigma\widehat{v}) - \log(1 + \psi^{-1}\sigma\widehat{w})))$ increases exponentially in $|j|$, and since that term is also bounded away from 0. The latter statement is true since j is nonzero.

The fifth inequality follows from a Taylor expansion, the fact that v/w is always bounded away from any integral power of q, and the fact that $|\widehat{v} - \widehat{w}|^{-1} < C_1$ for sufficiently large $C_1 > 0$. The sixth inequality is true since its left-hand side grows polynomially in $|\widehat{w}|$ and $|\widehat{v}|$, while its right-hand side grows exponentially in these two quantities.

The estimates (6.20) address all terms on the right-hand side of (6.17), except for the exponential term. To analyze this term, first recall $|R(z)| < |z|^3/8$, for all $z \in \widehat{\mathcal{C}}^{(1)} \cup \widehat{\Gamma}^{(1)}$; this was stipulated as the third part of Definition 6.3. Thus, increasing C_1 if necessary, we find that

$$(6.21) \quad \begin{aligned} &\left|\exp\left(\frac{\widehat{w}^3}{3} - \frac{\widehat{v}^3}{3} + T\left(R(T^{-1/3}\widehat{w}) - R(T^{-1/3}\widehat{v})\right)\right)\right| \\ &= \exp\left(\frac{\widehat{w}^3}{3} - \frac{\widehat{v}^3}{3} + \frac{|\widehat{v}|^3}{8} + \frac{|\widehat{w}|^3}{8}\right) \\ &< C_1\exp\left(-\frac{1}{5}(|\widehat{w}|^3 + |\widehat{v}|^3)\right). \end{aligned}$$

In (6.21), the last estimate follows from the fact that $\widehat{w}^3 - \widehat{v}^3 < 0$, for sufficiently large $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ and $\widehat{v} \in \widehat{\Gamma}^{(1)}$, and that it decreases cubically in $|\widehat{w}|$ and $|\widehat{v}|$ as they tend to ∞.

Now the estimate (6.18) follows from the definition (6.17) of I, the six estimates (6.20), and the exponential estimate (6.21).

In order to establish the limiting statement (6.19), we observe that

$$\lim_{T \to \infty} \frac{1}{1 + \psi^{-1} \sigma \widehat{v}} = 1;$$

$$\lim_{T \to \infty} \left(\frac{1 + \psi^{-1} \sigma \widehat{v}}{1 + \psi^{-1} \sigma \widehat{w}} \right)^{\psi \sigma^{-1} s} = \exp(s(\widehat{v} - \widehat{w}));$$

$$\lim_{T \to \infty} \exp\left(\frac{\widehat{w}^3 - \widehat{v}^3}{3} + T\left(R(T^{-1/3} \widehat{w}) - R(T^{-1/3} \widehat{v}) \right) \right)$$

(6.22)
$$= \exp\left(\frac{\widehat{w}^3}{3} - \frac{\widehat{v}^3}{3} \right);$$

$$\lim_{T \to \infty} \frac{\pi \psi^{-1} \sigma}{\log q} \sum_{j \neq 0} \frac{1}{\sin(\frac{\pi}{\log q}(2\pi \mathbf{i} j + \log(1 + \psi^{-1} \sigma \widehat{v}) - \log(1 + \psi^{-1} \sigma \widehat{w})))} = 0;$$

$$\lim_{T \to \infty} \left(\frac{\pi \psi^{-1} \sigma}{\log q} \right) \frac{1}{\sin(\frac{\pi}{\log q}(\log(1 + \psi^{-1} \sigma \widehat{v}) - \log(1 + \psi^{-1} \sigma \widehat{w})))} = \frac{1}{\widehat{v} - \widehat{w}};$$

and

(6.23)
$$\lim_{T \to \infty} \prod_{k=1}^{m} \frac{(q^{-1} \beta_k^{-1}(\psi + \sigma \widehat{v}); q)_\infty}{(q^{-1} \beta_k^{-1}(\psi + \sigma \widehat{w}); q)_\infty} = 1,$$

for each $\widehat{w}, \widehat{w'}, \widehat{v} \in \mathbb{C}$. Indeed, the first limit in (6.22) and the limit (6.23) follow from the fact that σ tends to 0 as T tends to ∞ and, in the latter limit, the fact that $q^{-1} \beta_k^{-1} \psi$ is uniformly bounded away from any nonpositive power of q. The second and fifth limits in (6.22) follow from a Taylor expansion and the fact that σ tends to 0 as T tends to ∞. The fourth limit in (6.22) follows from the fourth estimate in (6.20). Furthermore, the third limit in (6.22) follows from the fact that $|R(z)| = \mathcal{O}(|z|^4)$; see (6.7). Thus, (6.19) follows from multiplying the results of (6.22) and (6.23). $\quad \square$

Now we can establish Lemma 6.11.

PROOF OF LEMMA 6.11. The uniform estimate (6.13) follows from integrating the estimate (6.18) over $\widehat{v} \in \mathfrak{V}_{-1,\varepsilon/\sigma}$. By (6.16), (6.18), (6.19) the dominated convergence theorem, and the fact that $\widehat{\Gamma}^{(1)} = \mathfrak{V}_{-1,\varepsilon/\sigma}$, we deduce that

$$\lim_{T \to \infty} \widehat{K}(\widehat{w}, \widehat{w'})$$

(6.24)
$$= \lim_{T \to \infty} \frac{1}{2\pi \mathbf{i}} \int_{\mathfrak{V}_{-1,\varepsilon/\sigma}} \frac{1}{(\widehat{v} - \widehat{w})(\widehat{w'} - \widehat{v})} \exp\left(\frac{\widehat{w}^3}{3} - \frac{\widehat{v}^3}{3} + s(\widehat{v} - \widehat{w}) \right) d\widehat{v},$$

for each fixed $\widehat{w} \in \mathfrak{W}_{0,\infty}$ and $\widehat{w'} \in \mathfrak{W}_{0,\infty}$.

Hence, the limit (6.15) follows from (6.24), the fact that σ tends to 0 as T tends to ∞, and the exponential decay in $|\widehat{v}|^3$ of the integrand defining the kernel L_s (see (6.14)). $\square$

6.2.2. *Exponential decay of K on $\mathcal{C}^{(2)}$ and $\Gamma^{(2)}$.* In this section we analyze the integral (6.1) defining $K(w, w')$ when either w or v is not close to ψ, that is, when either $w \in \mathcal{C}^{(2)}$ or $v \in \Gamma^{(2)}$. In this case, will show that the integral decays exponentially in T, which will allow us to ignore this contribution and apply the results of the previous section.

We begin with the following lemma, which estimates the exponential term in the integrand on the right-hand side of (6.1).

LEMMA 6.13. *There exists some positive real number $c_1 > 0$, independent of T, such that*

$$\max\Big\{ \sup_{\substack{w \in \mathcal{C} \\ v \in \Gamma^{(2)}}} \Re\big(G(w) - G(v)\big), \ \sup_{\substack{w \in \mathcal{C}^{(2)} \\ v \in \Gamma}} \Re\big(G(w) - G(v)\big) \Big\} < -c_1.$$

PROOF. From part 7 and part 8 of Proposition 6.1, we find that $\Re(G(w) - G(v)) < 0$ for all $w \in \mathcal{C}$ and $v \in \Gamma^{(2)}$, and that $\Re(G(w) - G(v)) < 0$ for all $w \in \mathcal{C}^{(2)}$ and $v \in \Gamma$ (uniformly in T). Thus, the existence of the claimed constant c_1 follows from the compactness of the contours $\mathcal{C}$ and Γ. $\square$

Now recall the definitions of $\sigma = \psi f_\eta^{-1} T^{-1/3}$; $\widehat{w}$, $\widehat{w'}$, and $\widehat{v}$ from (6.12); and I from (6.17). Define

$$(6.25) \qquad \overline{K}(\widehat{w}, \widehat{w'}) = \sigma K(w, w') = \frac{1}{2\pi \mathbf{i}} \int_{\widehat{\Gamma}} I(\widehat{w}, \widehat{w'}; \widehat{v}) \, d\widehat{v},$$

for each $\widehat{w}, \widehat{w'} \in \widehat{\mathcal{C}}$. From the change of variables (6.12), we have that

$$(6.26) \qquad \det(\mathrm{Id} + K)_{L^2(\mathcal{C})} = \det(\mathrm{Id} + \overline{K})_{L^2(\widehat{\mathcal{C}})}.$$

Lemma 6.13 implies the following result.

COROLLARY 6.14. *There exist constants $c, C > 0$ such that*

$$(6.27) \qquad \big|\overline{K}(\widehat{w}, \widehat{w'}) - \widehat{K}(\widehat{w}, \widehat{w'})\big| < C \exp\big(-c\big(T + |\widehat{w}|^3\big)\big),$$

for all $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ and $\widehat{w'} \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{0,\infty}$, and such that

$$(6.28) \qquad \big|\overline{K}(\widehat{w}, \widehat{w'})\big| < C \exp\big(-c\big(T + |\widehat{w}|^3\big)\big),$$

for all $\widehat{w} \in \widehat{\mathcal{C}}^{(2)}$ and $\widehat{w'} \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{0,\infty}$.

PROOF. Let us begin with the proof of (6.27). Defining the constant $c_1 > 0$ as in Lemma 6.13, we have that

$$
\begin{aligned}
&\bigl|\overline{K}(\widehat{w}, \widehat{w'}) - \widehat{K}(\widehat{w}, \widehat{w'})\bigr| \\
&\quad \leq \frac{\sigma}{2} \int_{\Gamma^{(2)}} \sum_{j=-\infty}^{\infty} \left| \frac{\exp(T(G(w) - G(v)))}{v(w' - v) \sin(\frac{\pi}{\log q}(2\pi \mathbf{i} j + \log v - \log w))} \left(\frac{v}{w}\right)^{s f_\eta T^{1/3}} \right. \\
&\quad\quad \times \prod_{k=1}^{m} \left. \frac{(q^{-1}\beta_k^{-1}v; q)_\infty}{(q^{-1}\beta_k^{-1}w; q)_\infty} \right| dv \\
&\quad = \mathcal{O}(e^{-c_1 T}).
\end{aligned}
$$

(6.29)

The first estimate in (6.29) follows from the facts that $\widehat{K} = \sigma \widetilde{K}$, that $\overline{K} = \sigma K$ and that the definition for K given in (6.1) and the definition for $\widetilde{K}$ given in (6.11) coincide except that the former integral is over Γ, and the latter integral is over $\Gamma^{(1)}$.

The second estimate in (6.29) follows from the following four facts. First, by Lemma 6.13 $\Re(G(w) - G(v)) < c_1$, for $w \in \mathcal{C}$ and $v \in \Gamma^{(2)}$; this gives rise to the $\mathcal{O}(e^{-c_1 T})$ in (6.29). Second, the denominator of the integrand is bounded away from 0 (independently of T) if $\widehat{w} \in \widehat{\mathcal{C}}$, $\widehat{w'} \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{0,\infty}$, and $\widehat{v} \in \widehat{\Gamma}^{(2)}$. Third, the denominator of the integrand increases exponentially in $|j|$; this verifies convergence of the sum in the integrand. Fourth, the contour Γ is compact; this verifies that the integral is bounded by a constant multiple of the supremum of its integrand.

Thus, the estimate (6.27) follows from (6.29) and the fact that $|\widehat{w}| = \mathcal{O}(T^{1/3})$ for $\widehat{w} \in \widehat{\mathcal{C}}$.

Now let us establish the estimate (6.28). We have that

$$
\begin{aligned}
\bigl|\overline{K}(\widehat{w}, \widehat{w'})\bigr| &\leq \frac{\sigma}{2} \sum_{j=-\infty}^{\infty} \oint_{\Gamma} \left| \frac{\exp(T(G(w) - G(v)))}{v(w' - v) \sin(\frac{\pi}{\log q}(2\pi \mathbf{i} j + \log v - \log w))} \right. \\
&\quad \times \left(\frac{v}{w}\right)^{s f_\eta T^{1/3}} \prod_{k=1}^{m} \left. \frac{(q^{-1}\beta_k^{-1}v; q)_\infty}{(q^{-1}\beta_k^{-1}w; q)_\infty} \right| dv \\
&= \mathcal{O}(e^{-c_1 T}),
\end{aligned}
$$

(6.30)

for all $\widehat{w} \in \widehat{\mathcal{C}}^{(2)}$. The first estimate is due to the definitions (6.1) and $\overline{K} = \sigma K$. The second estimate in (6.30) follows from the estimate on $\Re(G(w) - G(v))$, for $w \in \mathcal{C}^{(2)}$ and $v \in \Gamma$, given by Lemma 6.13; the compactness of the contour Γ; and the fact that the denominator of the integrand is bounded below by σ if $w' \in \mathcal{C}$ and $v \in \Gamma$ (since the factor $|w' - v|$ is bounded below by σ), and also increases exponentially in $|j|$.

Thus, the estimate (6.28) follows from (6.30) and the fact that $|\widehat{w}| = \mathcal{O}(T^{1/3})$ for $\widehat{w} \in \widehat{\mathcal{C}}$. $\square$

Now we can use Corollary A.5 of [3], Lemma 6.11, and Corollary 6.14 to establish the first parts of Proposition 5.2 and Proposition 5.3.

PROOF OF PART 1 OF PROPOSITION 5.2 AND PROPOSITION 5.3. The proof will follow from an application of Corollary A.5 of [3] to the sequence of kernels $\{\overline{K}(\widehat{w}, \widehat{w'})\}$, the sequence of contours $\{\mathfrak{W}_{0,\infty} \cup \widehat{\mathcal{C}}^{(2)}\}$ (recall that these kernels and contours depend on, and are hence indexed by, the time T), and the kernel $L_s(\widehat{w}, \widehat{w'})$ [given by (6.14)].

Let us verify that the conditions of that corollary hold. First, we require the existence of a dominating function $\mathbf{K}$ satisfying the three conditions of this corollary. In fact, we may take $\mathbf{K}(z)$ to be of the form $C \exp(-c|z|^3)$, for some sufficiently small constant $c > 0$ and some sufficiently large constant $C > 0$; such a $\mathbf{K}$ satisfies the second and third conditions of Corollary A.5 of [3]. That such a $\mathbf{K}$ satisfies the first condition (for sufficiently small c and sufficiently large C) follows from (6.13) and (6.27) in the case $z \in \widehat{\mathcal{C}}^{(1)}$, and from (6.28) in the case $z \in \widehat{\mathcal{C}}^{(2)}$.

Second, we require the limit condition, which states that for any $\widehat{w}, \widehat{w'} \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{0,\infty}$ we have that $\lim_{N\to\infty} \mathbf{1}_{\widehat{w},\widehat{w'}\in\widehat{\mathcal{C}}} \overline{K}(\widehat{w}, \widehat{w'}) = \mathbf{1}_{\widehat{w},\widehat{w'}\in\mathfrak{W}_{0,\infty}} L_s(\widehat{w}, \widehat{w'})$. When $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ and $\widehat{w'} \in \mathfrak{W}_{0,\infty}$, this follows from (6.15) and (6.27). When $\widehat{w'} \in \widehat{\mathcal{C}}^{(2)} = \widehat{\mathcal{C}}_T^{(2)}$, this follows from the fact that both $\overline{K}(\widehat{w}, \widehat{w'})$ and $L_s(\widehat{w}, \widehat{w'})$ tend to 0 as T tends to ∞; this can be deduced from (6.13) and (6.14), respectively. When $\widehat{w} \in \widehat{\mathcal{C}}^{(2)}$, this follows from (6.28) and the fact that $L_s(\widehat{w}, \widehat{w'})$ decays exponentially in $|\widehat{w}|^3$. Furthermore, when $\widehat{w} \in \mathfrak{W}_{0,\infty} \setminus \widehat{\mathcal{C}}^{(1)} = \mathfrak{W}_{0,\infty} \setminus \mathfrak{W}_{0,\varepsilon/\sigma}$, this again follows from the fact that $L_s(\widehat{w}, \widehat{w'})$ exhibits exponential decay in $|\widehat{w}|^3$. This verifies the limit condition.

Thus, Corollary A.5 of [3] applies, and we obtain that

$$
(6.31) \qquad \lim_{T\to\infty} \det(\mathrm{Id} + K)_{L^2(\mathcal{C})} = \lim_{T\to\infty} \det(\mathrm{Id} + \overline{K})_{L^2(\widehat{\mathcal{C}})} = \det(\mathrm{Id} + L_s)_{L^2(\mathfrak{W}_{0,\infty})},
$$

where we have used (6.26) to deduce the first identity.

Now it is known (see, e.g., Lemma 8.6 of [15]) that $\det(\mathrm{Id} + L_s)_{\mathfrak{W}_{0,\infty}} = F_{\mathrm{TW}}(s)$. Thus, we deduce from (6.31) that

$$
(6.32) \qquad \lim_{T\to\infty} \det(\mathrm{Id} + K)_{L^2(\mathcal{C})} = F_{\mathrm{TW}}(s).
$$

Applying (6.32) in the case $K = A$ yields the first part of Proposition 5.2, and applying (6.32) in the case $K = V$ yields the first part of Proposition 5.3. □

7. Baik–Ben–Arous–Péché phase transitions.

Our goal in this section is to establish the second parts of Proposition 5.2 and Proposition 5.3, which examine the case when η is "near" the phase transition θ. This will be similar to what was explained in Section 6.2, but there will be two main differences. Both are based on the fact that the critical point ψ_K (defined in (6.5) and (6.6) when $K = A$ and $K = V$, resp.) is now close to $q\beta$ when η is close to θ.

The first difference is that we cannot take exactly the same contours as in Section 6.1. Indeed, these contours will intersect the positive real line near $\psi = q\beta$, meaning that they might pass through some pole $q\beta_j$ of the integrand in (6.1); this is not permitted by the statement of Proposition 5.1. The second difference is that the approximation (6.23), which was based on the assumption that $1 - q^{-1}\beta^{-1}\psi$ is bounded away from 0 (independently of T), no longer holds.

To remedy the first issue, we perturb the contours by translating them to the left by some sufficiently large multiple of $T^{-1/3}$, so that they avoid the poles $q\beta_j$. To remedy the second issue, we Taylor expand the left-hand side of (6.23), taking into account the new location of ψ. Both of these will be done in Section 7.1. Then, in Section 7.2, we will establish the second parts of Proposition 5.2 and Proposition 5.3; this will be similar to what was done in Section 6.2.

7.1. *Modifying the approximation and contours.* The goal of this section is twofold. First, we modify the contours $\mathcal{C}_K$ and Γ_K to ensure that they still satisfy the requirements of Proposition 5.1. Second, we modify the approximation (6.23) so that it holds when ψ_K is close to $q\beta$. Let us begin with the contours; in what follows, we recall the notation of Theorem 1.6 and Theorem 1.7, and that $\sigma = \sigma_K = \psi_K f_{\eta;K}^{-1} T^{-1/3}$.

We would like to shift the contours $\mathcal{C}_K$ and Γ_K to the left so that they both avoid each of the $q\beta_j$. To that end, recall that $|b_{j,T} - b| = \mathcal{O}(T^{-1/3})$ for each $j \in [1, m]$, and $|\eta - \theta_K| = \mathcal{O}(T^{-1/3})$. The first estimate implies that $|q\beta_{T,j} - q\beta| = \mathcal{O}(T^{-1/3})$, for each j [a more precise approximation for the error is given by (7.1)], and the second estimate implies that $|\psi_K - q\beta| = \mathcal{O}(T^{-1/3})$ [again, a more precise approximation for the error is given by (7.2) and (7.3), in the cases $K = A$ and $K = V$, resp.].

Thus, there exists some finite positive real number $E = E_K$ (independent of T) so that $\psi - \sigma E < \min_{1 \le j \le m} q\beta_j$; in fact, one can verify using (7.1), (7.2) and (7.3) that it suffices to take $E > \max_{j \in [1,k]} c_j$, where $c_1, c_2, \ldots, c_m$ are given by (1.3) and (1.7) in the cases $K = A$ and $K = V$, respectively. The quantity E_K will signify the translation of the contours $\mathcal{C}_K$ and Γ_K. Specifically, we make the following definitions, which are analogous to Definition 6.3, Definition 6.4 and Definition 6.5. In what follows, we recall the level lines $\mathcal{L}_{1;K}$, $\mathcal{L}_{2;K}$ and $\mathcal{L}_{3;K}$ defined in Section 6.1.

DEFINITION 7.1. Let $\mathcal{C}_K^{(1)} = \mathfrak{W}_{\psi - \sigma E, \varepsilon}$ and $\Gamma_K^{(1)} = \mathfrak{V}_{\psi - \sigma E - \sigma, \varepsilon}$, where ε is chosen to be sufficiently small (independent of T) such that the four properties listed in Definition 6.3 all hold.

Such a positive real number ε is guaranteed to exist by part 6 of Proposition 6.1 (in the case $K = V$), part 6 of Proposition 6.7 (in the case $K = A$), and the estimate (6.7).

See Figure 11 for (rescaled) examples of these contours.
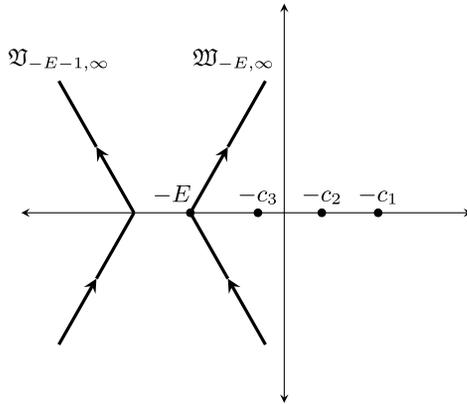
FIG. 11. *Shown above are the contours $\mathfrak{W}_{-E,\infty}$ and $\mathfrak{V}_{-E-1,\infty}$.*

DEFINITION 7.2. Let $\mathcal{C}_K^{(2)}$ denote a positively oriented contour from the top endpoint $\psi - \sigma E + \varepsilon e^{\pi \mathbf{i}/3}$ of $\mathcal{C}_K^{(1)}$ to the bottom endpoint $\psi - \sigma E + \varepsilon e^{-\pi \mathbf{i}/3}$ of $\mathcal{C}^{(1)}$, and let $\Gamma_K^{(2)}$ denote a positively oriented contour from the top endpoint $\psi - \sigma E - \sigma + \varepsilon e^{2\pi \mathbf{i}/3}$ of $\Gamma_K^{(1)}$ to the bottom endpoint $\psi - \sigma E - \sigma + \varepsilon e^{-2\pi \mathbf{i}/3}$ of $\Gamma^{(1)}$, satisfying the five properties listed in Definition 6.4 (if $K = A$, then ignore that part of the fourth condition stating that $\Gamma_K^{(1)} \cup \Gamma_K^{(2)}$ does not contain $-q\kappa$).

Such contours $\mathcal{C}_K^{(2)}$ and $\Gamma_K^{(2)}$ are guaranteed to exist by part 3 and part 5 of Proposition 6.1 (when $K = V$) and by part 3 and part 5 of Proposition 6.7 (in the case $K = A$).

DEFINITION 7.3. Set $\mathcal{C}_K = \mathcal{C}_K^{(1)} \cup \mathcal{C}_K^{(2)}$ and $\Gamma_K = \Gamma_K^{(1)} \cup \Gamma_K^{(2)}$.

The following lemma states that $\mathcal{C}_K$ and Γ_K satisfy their required properties, for sufficiently large T. We omit its proof, but let us mention that the reason as to why $\mathcal{C}_K$ and Γ_K avoid the poles $\{q\beta_j\}_{1 \le j \le m}$ is that they both intersect the positive real axis to the left of any $q\beta_j$ (since $\psi - \sigma E < \min_{1 \le j \le m} q\beta_j$).

LEMMA 7.4. *If $K = A$ and T is sufficiently large, then Γ_K is a positively oriented, star-shaped curve that contains 0, but leaves outside $-q$ and $q\beta_j$ for each $j \in [1, m]$. Furthermore, $\mathcal{C}_K$ is a positively oriented, star-shaped contour that is contained inside $q^{-1}\Gamma_K$; that contains 0, $-q$ and Γ_K; but that leaves outside $q\beta_j$ for each $j \in [1, m]$.*

If $K = V$ and T is sufficiently large, then Γ_K is a positively oriented, star-shaped curve that contains 0, but leaves outside $-q\kappa$ and $q\beta_j$ for each $j \in [1, m]$. Furthermore, $\mathcal{C}_K$ is a positively oriented, star-shaped contour that is contained inside $q^{-1}\Gamma_K$; that contains 0, $-q$ and Γ_K; but that leaves outside $q\beta_j$ for each $j \in [1, m]$.

Having defined the contours $\mathcal{C}_K$ and Γ_K, we now turn to the approximation (6.23).

Observe that the left-hand side of (6.23) is a product of terms involving $q\beta_j$ and ψ. When η is close to θ and all of the b_j are close to b, the $q\beta_j$, ψ_A and ψ_V are all close to $q\beta$; in particular, this implies that the numerator and denominator of (6.23) both tend to 0 as T tends to ∞. Thus, to analyze (6.23) further, we must obtain a more refined estimate on the error in approximating these quantities by $q\beta$.

Let us begin with $q\beta_j$. To that end, we recall from the second part of Theorem 1.6 (or Theorem 1.7) that the $\{b_{j,T}\}_{T\in\mathbb{Z}_{>0}}$ satisfy $\lim_{T\to\infty} T^{1/3}(b_{j,T} - b) = d_j$. Thus, since $\beta_{j,T} = b_{j,T}/(1 - b_{j,T})$, Taylor expanding yields

$$q\beta_{T,j} = \frac{qb_{T,j}}{1 - b_{T,j}}$$

(7.1)
$$= q\left(\frac{b + d_j T^{-1/3} + o(T^{-1/3})}{1 - b - d_j T^{-1/3} + o(T^{-1/3})}\right)$$

$$= q\beta\left(1 + \frac{d_j T^{-1/3}}{b(1 - b)}\right) + o(T^{-1/3}).$$

Next we analyze the error in approximating ψ_K. This depends on whether $K = A$ or $K = V$.

First, assume that $K = A$. Recall that ψ_A is defined in (6.5) and that $\{\eta_T\}_{T\in\mathbb{Z}_{>0}}$ satisfies $\lim_{T\to\infty} T^{1/3}(\eta_T - \theta_A) = d$. Thus, Taylor expanding ψ_A and using the fact that $\theta_A = 1 - 2b$, we find that

(7.2)
$$\psi_A = q\left(\frac{1 - \theta_A - dT^{-1/3} + o(T^{-1/3})}{1 + \theta_A + dT^{-1/3} + o(T^{-1/3})}\right)$$

$$= q\beta\left(1 - \frac{dT^{-1/3}}{2b(1 - b)}\right) + o(T^{-1/3}).$$

Next, assume that $K = V$. Recalling that ψ_V is defined in (6.6), Taylor expanding ψ_V and recalling the facts that $\Lambda = b + \kappa(1 - b)$ and $\theta_V = \kappa^{-1}\Lambda^2$ (see (1.5)), we find that

$$\psi_V = \frac{q(\kappa - \sqrt{\kappa\eta})}{\sqrt{\kappa\eta} - 1}$$

(7.3)
$$= \frac{q(\kappa - (1 + \frac{dT^{-1/3}}{\theta_V} + o(T^{-1/3}))^{1/2}\sqrt{\kappa\theta_V})}{(1 + \frac{dT^{-1/3}}{\theta_V} + o(T^{-1/3}))^{1/2}\sqrt{\kappa\theta_V} - 1}$$

$$= q\beta\left(1 - \frac{d(b + \kappa(1 - b))T^{-1/3}}{2\theta_V(\kappa - 1)b(1 - b)} + o(T^{-1/3})\right)$$

$$= q\beta\left(1 - \frac{d\kappa T^{-1/3}}{2(\kappa - 1)b(1 - b)\Lambda}\right) + o(T^{-1/3}).$$

Using (7.1), (7.2) and (7.3), we can establish the following lemma.

LEMMA 7.5. *Recall that the $c_j = c_{j;K}$ were defined in the second part of Theorem* 1.6 *(in the case $K = A$) and in the second part of Theorem* 1.7 *(in the case $K = V$). For any fixed $\widehat{w}, \widehat{v} \in \mathbb{C}$ (with $\widehat{w}$ not equal to any $-c_j$), we have that*

$$\lim_{T \to \infty} \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}(\psi_K + \sigma \widehat{v}); q)_\infty}{(q^{-1}\beta_k^{-1}(\psi_K + \sigma \widehat{w}); q)_\infty} = \prod_{j=1}^{m} \frac{\widehat{v} + c_j}{\widehat{w} + c_j}.$$

PROOF. Observe that

(7.4)
$$\prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}(\psi + \sigma \widehat{v}); q)_\infty}{(q^{-1}\beta_k^{-1}(\psi + \sigma \widehat{w}); q)_\infty}$$
$$= \prod_{k=1}^{m} \frac{1 - q^{-1}\beta_k^{-1}(\psi + \sigma \widehat{v})}{1 - q^{-1}\beta_k^{-1}(\psi + \sigma \widehat{w})} \prod_{k=1}^{m} \frac{(\beta_k^{-1}(\psi + \sigma \widehat{v}); q)_\infty}{(\beta_k^{-1}(\psi + \sigma \widehat{w}); q)_\infty}.$$

Using the fact that $\lim_{T \to \infty} \beta_k = \beta$ and $\lim_{T \to \infty}(\psi + \sigma \widehat{v}) = q\beta = \lim_{T \to \infty}(\psi + \sigma \widehat{w})$, we deduce that $\lim_{T \to \infty}(\beta_k^{-1}(\psi + \sigma \widehat{v}); q)_\infty = (q; q)_\infty = \lim_{T \to \infty}(\beta_k^{-1} \times (\psi + \sigma \widehat{w}); q)_\infty$.

Thus, (7.4) implies that

(7.5)
$$\lim_{T \to \infty} \prod_{k=1}^{m} \frac{(q^{-1}\beta_k^{-1}(\psi + \sigma \widehat{v}); q)_\infty}{(q^{-1}\beta_k^{-1}(\psi + \sigma \widehat{w}); q)_\infty}$$
$$= \lim_{T \to \infty} \prod_{k=1}^{m} \frac{1 - q^{-1}\beta_k^{-1}(\psi + \sigma \widehat{v})}{1 - q^{-1}\beta_k^{-1}(\psi + \sigma \widehat{v})}$$
$$= \lim_{T \to \infty} \prod_{k=1}^{m} \frac{\psi q^{-1}\beta_k^{-1}\widehat{v} + T^{1/3} f_{\eta;K}(q^{-1}\beta_k^{-1}\psi - 1)}{\psi q^{-1}\beta_k^{-1}\widehat{w} + T^{1/3} f_{\eta;K}(q^{-1}\beta_k^{-1}\psi - 1)},$$

where in the second identity we used the fact that $\sigma = \psi f_\eta^{-1} T^{-1/3}$. Thus, it suffices to analyze the limit on the right-hand side of (7.5); this will depend on whether $K = A$ or $K = V$.

In the case $K = A$, we apply (7.1) and (7.2) to find that

(7.6)
$$\lim_{T \to \infty} \prod_{k=1}^{m} \frac{\psi q^{-1}\beta_k^{-1}\widehat{v} + T^{1/3} f_{\eta;A}(q^{-1}\beta_k^{-1}\psi - 1)}{\psi q^{-1}\beta_k^{-1}\widehat{w} + T^{1/3} f_{\eta;A}(q^{-1}\beta_k^{-1}\psi - 1)}$$
$$= \prod_{j=1}^{m} \frac{\widehat{v} - \frac{f_{\eta;A}d_j}{b(1-b)} - \frac{f_{\eta;A}d}{2b(1-b)}}{\widehat{w} - \frac{f_{\eta;A}d_j}{b(1-b)} - \frac{f_{\eta;A}d}{2b(1-b)}}.$$

In the case $K = V$, we apply (7.1) and (7.3) to find that

$$\lim_{T \to \infty} \prod_{k=1}^{m} \frac{\psi q^{-1} \beta_k^{-1} \widehat{v} + T^{1/3} f_{\eta;V}(q^{-1}\beta_k^{-1}\psi - 1)}{\psi q^{-1} \beta_k^{-1} \widehat{w} + T^{1/3} f_{\eta;V}(q^{-1}\beta_k^{-1}\psi - 1)}$$

(7.7)

$$= \prod_{j=1}^{m} \frac{\widehat{v} - \frac{f_{\eta;V} d_j}{b(1-b)} - \frac{f_{\eta;V} d\kappa}{2(\kappa-1)b(1-b)\Lambda}}{\widehat{w} - \frac{f_{\eta;V} d_j}{b(1-b)} - \frac{f_{\eta;V} d\kappa}{2(\kappa-1)b(1-b)\Lambda}}.$$

Now, in the case $K = A$, the lemma follows from (7.5), (7.6) and the definition of $c_{j;A}$ (given in the statement of Theorem 1.6). Similarly, in the case $K = V$, the lemma follows from (7.5), (7.7) and the definition of $c_{j;V}$ (given in the statement of Theorem 1.7). $\square$

7.2. *Recovery of the Baik–Ben–Arous–Péché asymptotics.* In what follows, we omit the subscript K from our notation.

The goal of this section is to establish the second parts of Theorem 1.6 and Theorem 1.7. This will be similar to what was done in Section 6.2. In particular, we first analyze the kernel K (defined in (6.1)) when $w \in \mathcal{C}^{(1)}$ and $v \in \Gamma^{(1)}$, that is, when both w and v are near ψ. Then we analyze the kernel when either $w \in \mathcal{C}^{(2)}$ or $v \in \mathcal{C}^{(2)}$.

Let us perform the former task first. To that end, recall the change of variables (6.12), which also defined the kernel $\widehat{K}$. Furthermore, recall that, for any contour $\mathcal{D} \subset \mathbb{C}$, we set $\widehat{\mathcal{D}} = \sigma^{-1}(\mathcal{D} - \psi)$. In particular, from Definition 7.1, we find that $\mathcal{C}^{(1)} = \mathfrak{W}_{-E,\varepsilon/\sigma}$ and $\Gamma^{(1)} = \mathfrak{V}_{-E-1,\varepsilon/\sigma}$; see Figure 11 for an example of these contours in the case $m = 3$.

Now we would like to establish the following lemma about the asymptotics of the kernel $\widehat{K}$.

LEMMA 7.6. *There exist constants $c, C > 0$ such that*

(7.8)
$$|\widehat{K}(\widehat{w}, \widehat{w'})| \leq \frac{C}{1 + |\widehat{w'}|} \exp(c\widehat{w}^3),$$

for each $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ and $\widehat{w'} \in \widehat{\mathcal{C}}$.

Furthermore, if we define the kernel $L_{s;\mathbf{c}}(\widehat{w}, \widehat{w'})$ by

$$L_{s;\mathbf{c}}(\widehat{w}, \widehat{w'}) = \frac{1}{2\pi \mathbf{i}} \int_{\mathfrak{V}_{-1,\infty}} \frac{1}{(\widehat{v} - \widehat{w})(\widehat{w'} - \widehat{v})}$$

(7.9)
$$\times \exp\left(\frac{\widehat{w}^3}{3} - \frac{\widehat{v}^3}{3} + s(\widehat{w} - \widehat{v})\right)$$

$$\times \prod_{j=1}^{m} \frac{\widehat{v} + c_j}{\widehat{w} + c_j} d\widehat{v},$$

then we have that

$$\lim_{T \to \infty} \widehat{K}(\widehat{w}, \widehat{w'}) = L_s(\widehat{w}, \widehat{w'}), \tag{7.10}$$

for each fixed $\widehat{w} \in \mathfrak{W}_{-E,\infty}$ and $\widehat{w'} \in \mathfrak{W}_{-E,\infty}$.

To establish this lemma, we recall from (6.16) that $\widehat{K}$ can be written as an integral of the function I defined by (6.17). Lemma 7.6 will follow from the large T asymptotics of (and uniform estimate on) I given by the following lemma.

LEMMA 7.7. *There exist constants $c, C > 0$ such that*

$$\left| I(\widehat{w}, \widehat{w'}; \widehat{v}) \right| \le \frac{C}{1 + |\widehat{w'}|} \exp\left(c(\widehat{w}^3 - \widehat{v}^3)\right) \tag{7.11}$$

for all $\widehat{w} \in \mathfrak{W}_{-E,\varepsilon/\sigma}$, $\widehat{w'} \in \widehat{\mathcal{C}}$, and $\widehat{v} \in \mathfrak{V}_{-E-1,\varepsilon/\sigma}$.
   Furthermore,

$$\lim_{T \to \infty} I(\widehat{w}, \widehat{w'}; \widehat{v})$$
$$= \frac{1}{(\widehat{v} - \widehat{w})(\widehat{w'} - \widehat{v})} \exp\left(\frac{\widehat{w}^3}{3} - \frac{\widehat{v}^3}{3} + s(\widehat{w} - \widehat{v})\right) \prod_{j=1}^{m} \frac{\widehat{v} + c_j}{\widehat{w} + c_j}, \tag{7.12}$$

for each fixed $\widehat{w}, \widehat{w'}, \widehat{v} \in \mathbb{C}$.

PROOF.   The derivation of (7.11) is very similar to that of (6.18) and is thus omitted.
   Now let us establish the limiting statement (7.12); this will also be similar to the proof of (6.19). Indeed, the five limit identities listed in (6.22) still hold for all $\widehat{w}, \widehat{v} \in \mathbb{C}$. However, (6.23) no longer holds and must be replaced by Lemma 7.5. Multiplying the five limits in (6.22) and the statement of Lemma 7.5 yields (7.12). $\square$

Now we can establish Lemma 7.6.

PROOF OF LEMMA 7.6.   The uniform estimate (7.8) follows from integrating the estimate (7.11) over $\widehat{v} \in \mathfrak{V}_{-E-1,\varepsilon/\sigma}$. Then, the limit (7.10) follows from (7.12), (7.11), the fact that $\widehat{\Gamma}^{(1)}$ is contained in and converges to $\mathfrak{V}_{-E-1,\infty}$ as T tends to ∞, the exponential decay of the kernel $L_{s;\mathbf{c}}(\widehat{w}, \widehat{w'})$ in $|\widehat{w}|^3$, and the dominated convergence theorem. $\square$

Next we analyze the integral (6.1) defining $K(w, w')$ when either w or v is not close to ψ, that is, when either $w \in \mathcal{C}^{(2)}$ or $v \in \Gamma^{(2)}$.
   Recalling the rescaled kernel $\overline{K}$ defined in (6.25), we obtain the following result, which is the analog of Corollary 6.14.

COROLLARY 7.8. *There exist constants* $c, C > 0$ *such that*

$$(7.13) \qquad |\overline{K}(\widehat{w}, \widehat{w'}) - \widehat{K}(\widehat{w}, \widehat{w'})| < C \exp(c|\widehat{w}|^3 - cT),$$

for all $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ *and* $\widehat{w'} \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{-E,\infty}$, *and such that*

$$(7.14) \qquad |\overline{K}(\widehat{w}, \widehat{w'})| < C \exp(c|\widehat{w}|^3 - cT),$$

for all $\widehat{w} \in \widehat{\mathcal{C}}^{(2)}$ *and* $\widehat{w'} \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{-E,\infty}$.

The proof of this corollary is very similar to that of Corollary 6.14 and is thus omitted.

Now we can use Corollary A.5 of [3], Lemma 7.6, and Corollary 7.8 to establish the second parts of Proposition 5.2 and Proposition 5.3.

PROOF OF PART 2 OF PROPOSITION 5.2 AND PROPOSITION 5.3. This will be similar to the proof of the first parts of Proposition 5.2 and Proposition 5.3, given in Section 6.2.2. In particular, it will follow from an application of Corollary A.5 of [3] to the sequence of kernels $\{\overline{K}(\widehat{w}, \widehat{w'})\}$, the sequence of contours $\{\mathfrak{W}_{-E,\infty} \cup \widehat{\mathcal{C}}^{(2)}\}$ and the kernel $L_{s;\mathbf{c}}(\widehat{w}, \widehat{w'})$ [given by (6.14)].

The verification of the conditions of that corollary in this setting is similar to the verification provided in the proof of the first parts of Proposition 5.2 and Proposition 5.3 in Section 6.2.2. Indeed, the existence of the dominating function $\mathbf{K}$ (satisfying the three conditions of Corollary A.5 of [3]) follows from the uniform estimates (7.8), (7.13) and (7.14).

Furthermore, the convergence $\lim_{N \to \infty} \mathbf{1}_{\widehat{w}, \widehat{w'} \in \widehat{\mathcal{C}}} \overline{K}(\widehat{w}, \widehat{w'}) = \mathbf{1}_{\widehat{w}, \widehat{w'} \in \mathfrak{W}_{-E,\infty}} \times L_{s;\mathbf{c}}(\widehat{w}, \widehat{w'})$, for each $\widehat{w}, \widehat{w'} \in \widehat{\mathcal{C}} \cup \mathfrak{W}_{-E,\infty}$ follows from (6.15) and (6.27) (when $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$), and from (6.28) and the exponential decay of $L_{s;\mathbf{c}}(\widehat{w}, \widehat{w'})$ in $|\widehat{w}|^3$ (when $\widehat{w} \in \widehat{\mathcal{C}}^{(2)}$ or $\widehat{w} \in \mathfrak{W}_{-E,\infty} \setminus \widehat{\mathcal{C}}^{(1)}$).

Now applying Corollary A.5 of [3] yields that

$$(7.15) \qquad \begin{aligned} \lim_{T \to \infty} \det(\mathrm{Id} + K)_{L^2(\mathcal{C})} &= \lim_{T \to \infty} \det(\mathrm{Id} + \overline{K})_{L^2(\widehat{\mathcal{C}})} \\ &= \det(\mathrm{Id} + L_{s;\mathbf{c}})_{L^2(\mathfrak{W}_{-E,\infty})}, \end{aligned}$$

where we have used (6.26) to deduce the first identity.

Now it is known (see, e.g., Lemma 8.7 of [15]) that $\det(\mathrm{Id} + L_{s;\mathbf{c}})_{\mathfrak{W}_{-E,\infty}} = F_{\mathrm{BBP};\mathbf{c}}(s)$. Thus, we deduce from (7.15) that

$$(7.16) \qquad \lim_{T \to \infty} \det(\mathrm{Id} + K)_{L^2(\mathcal{C})} = F_{\mathrm{BBP};\mathbf{c}}(s).$$

Applying (7.16) in the case $K = A$ yields the second part of Proposition 5.2 and applying (7.16) in the case $K = V$ yields the second part of Proposition 5.3. $\quad\square$

8. Gaussian fluctuations. We now turn to the third parts of Proposition 5.2 and Proposition 5.3, whose proofs will be partly similar to those of the first two parts discussed previously. In Section 8.1, we select contours $\mathcal{C}_K$ and Γ_K that will be suitable for saddle point analysis. In Section 8.2, we use these contours to implement this analysis, which will conclude the proofs of Proposition 5.2 and Proposition 5.3.

8.1. *The contours $\mathcal{C}_K$ and Γ_K for $T^{1/2}$ fluctuations.* Observe that $m_{\eta;K}$ and $f_{\eta;K}$ from the first and second parts of Proposition 5.2 and Proposition 5.3 have now changed to $m'_{\eta;K}$ and $f'_{\eta;K}$, respectively. Thus, the function $G_K(z)$ (defined by (6.2) and (6.3) in the cases $K = A$ and $K = V$, resp.) changes, so we will change the contours $\mathcal{C}_K$ and Γ_K as well.

The goal of this section is to explain how to do this, which we will discuss further in Section 8.1.2. However, before describing these contours, let us see how the identities in Section 6.1.1 change with our new $m'_{\eta;K}$ and $f'_{\eta;K}$.

8.1.1. *A new exponential form for the kernel $K_\zeta^{(PT)}$.* As in Section 6.1, we rewrite the kernel $K(w, w') = K^{(PT)}(w, w')$ given in (5.4) through the identity

$$
\begin{aligned}
K(w, w') = {}& \frac{1}{2\mathbf{i}\log q} \\
& \times \sum_{j\in\mathbb{Z}} \oint_{\Gamma_V} \frac{\exp(T(G_K(w) - G_K(v)))}{\sin(\pi(\log q)^{-1}(2\pi\mathbf{i}j + \log v - \log w))} \\
& \times \frac{(q^{-1}\beta^{-1}v; q)_\infty^m}{(q^{-1}\beta^{-1}w; q)_\infty^m} \\
& \times \left(\frac{v}{w}\right)^{sf'_{\eta;K}T^{1/2}} \frac{dv}{v(w' - v)},
\end{aligned}
$$
(8.1)

where we have used the fact that all b_j are equal to b in the third parts of Proposition 5.2 and Proposition 5.3. In (8.1), $G_K(z)$ and $f'_{\eta;K}$ depend on K. Explicitly,

$$
\begin{aligned}
G_A(z) &= \frac{q}{z + q} + \eta\log(z + q) + m'_{\eta;A}\log z; \\
f'_{\eta;A} &= \chi^{1/2}(\theta - \eta)^{1/2};
\end{aligned}
$$
(8.2)

$$
\begin{aligned}
G_V(z) &= \eta\log(\kappa^{-1}z + q) - \log(z + q) + m'_{\eta;V}\log z; \\
f'_{\eta;V} &= \chi^{1/2}(1 - \theta^{-1}\eta)^{1/2}.
\end{aligned}
$$
(8.3)

In (8.2) and (8.3),

$$m'_{\eta;A} = b(1-b) - b\eta = \chi - b\eta;$$

(8.4)

$$m'_{\eta;V} = b - \frac{b\eta}{b + \kappa(1-b)} = b - b\Lambda^{-1}\eta,$$

where we recall the definition of Λ from (1.5). As in Section 6.1.1, we require the critical points of G_K. We find that

(8.5) $$G'_A(z) = \left(\frac{(1-b)(b+\eta)}{z(z+q)^2}\right)(z-\psi)(z-\vartheta_A);$$

(8.6) $$G'_V(z) = \frac{1-b}{z(z+q)(z+q\kappa)}(\Lambda^{-1}\kappa\eta - 1)(z-\psi)(z-\vartheta_V),$$

where $\psi = \psi_A = \psi_V = q\beta$ and

(8.7) $$\vartheta_A = q\left(\frac{1-b-\eta}{b+\eta}\right); \qquad \vartheta_V = q\kappa\left(\frac{\Lambda-\eta}{\kappa\eta-\Lambda}\right).$$

In both cases, $\psi = q\beta$ is a critical point of G_K. Furthermore, observe that

$$G''_A(\psi) = \frac{(1-b)^3(2b-1+\eta)}{q^2 b} = -\frac{f'^2_{\eta;A}}{\psi^2};$$

$$G''_V(\psi) = \frac{(1-b)^3}{q^2 b\Lambda^2}(\kappa\eta - \Lambda^2) = -\frac{f'^2_{\eta;V}}{\psi^2}.$$

Thus, $G''_K(\psi) = -\psi^{-2} f^2_{\eta;K}$ for $K \in \{V, A\}$, which implies by a Taylor expansion that

$$G_K(z) - G_K(\psi)$$

(8.8) $$= -\frac{1}{2}\left(\frac{f_{\eta;K}(z-\psi)}{\psi}\right)^2 + R_K(z)$$

$$= -\frac{1}{2}\left(\frac{f_{\eta;K}(z-\psi)}{\psi}\right)^2 + \mathcal{O}(|z-\psi|^3) \qquad \text{as } |z-\psi| \to 0.$$

Here $R_K(z) = G_K(z) - G_K(\psi) + (\psi^{-1} f_{\eta;K}(z-\psi))^2/2$.

8.1.2. *New choice of contours $\mathcal{C}_K$ and Γ_K.* The goal of this section is to exhibit contours $\mathcal{C}_K$ and Γ_K satisfying the conditions of Proposition 5.1 and such that $\Re(G_K(w) - G_K(v)) < 0$ for $w \in \mathcal{C}_K$ and $v \in \Gamma_K$ both away from ψ. When $K = A$, we will only be able to do this when $\eta \geq -b$; when $K = V$, we will only be able to do this when $\eta \geq \kappa^{-1}\Lambda = \Lambda^{-1}\theta$.

Throughout, we denote

(8.9) $$\alpha_A = -b; \qquad \alpha_V = \kappa^{-1}b + 1 - b = \kappa^{-1}\Lambda,$$
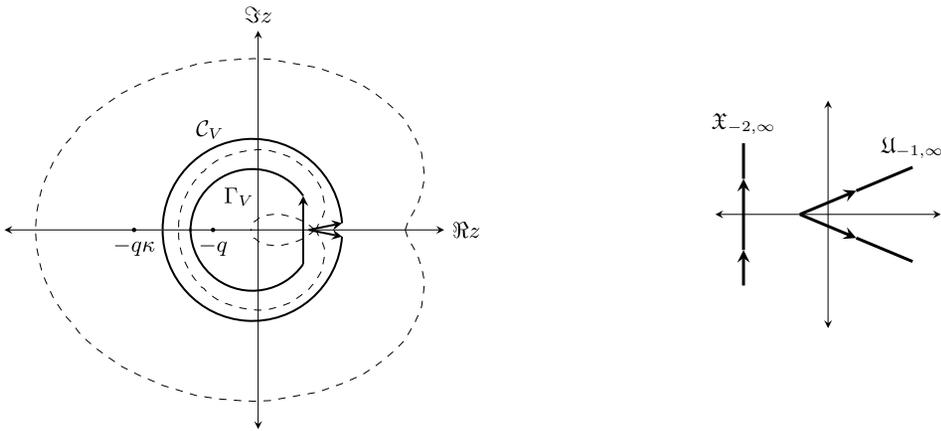
FIG. 12. *Above and to the left, the three level lines $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are depicted as dashed curves; the contours $\mathcal{C}$ and Γ are depicted as solid curves and are labeled. Shown above and to the right are the contours $\mathfrak{U}_{-1,\infty}$ and $\mathfrak{X}_{-2,\infty}$.*

as the values of η at which ϑ_A and ϑ_V are infinite. According to our restriction on η, we have that $\eta > \alpha_K$; this implies that $\vartheta_K > q\beta$, for each $K \in \{V, A\}$. In what follows, we will simplify notation by omitting the subscript K.

As in Section 6.1, we will take $\mathcal{C}$ and Γ to follow level lines of the equation $\Re G(z) = G(\psi)$. However, since the function G is different from the one used in Section 6.1, we must perform a new analysis of these level lines; see Figure 12.

Specifically, we require the following proposition, which we will establish in Section 8.1.3.

PROPOSITION 8.1. *Assume that $\eta > \alpha$. Then there exist three simple, closed curves, $\mathcal{L}_1 = \mathcal{L}_{1;K}$, $\mathcal{L}_2 = \mathcal{L}_{2;K}$ and $\mathcal{L}_3 = \mathcal{L}_{3;K}$ satisfying properties 1, 2, 3, 5, 7 and 8 of Proposition 6.1 (in the case $K = V$) and Proposition 6.7 (in the case $K = A$), with properties 4 and 6, respectively, replaced by the following:*

4. *We have that $\mathcal{L}_1 \cap \mathcal{L}_2 = \{\psi\}$, that $\mathcal{L}_2$ and $\mathcal{L}_3$ are disjoint and that $\mathcal{L}_1$ and $\mathcal{L}_3$ are disjoint. Furthermore, $\mathcal{L}_1 \setminus \{\psi\}$ is contained in the interior of $\mathcal{L}_2$, which is contained in the interior of $\mathcal{L}_3$.*

6. *The level line $\mathcal{L}_1$ meets the positive real axis (at ψ) at angles $3\pi/4$ and $-3\pi/4$, and the level line $\mathcal{L}_2$ meets the positive real axis (at ψ) at angles $\pi/4$ and $-\pi/4$.*

REMARK 8.2. When $\eta = \alpha$ or η is slightly smaller than α, the level line $\mathcal{L}_3$ from Proposition 8.1 does not exist; this alone does not pose an issue for asymptotic analysis, since we will chose the contours $\mathcal{C}$ and Γ to be very close to $\mathcal{L}_2$, instead of $\mathcal{L}_3$ (see Figure 12). More troublesome for us is that the contour $\mathcal{L}_2$ does not remain star-shaped, and also that it can contain $-q\kappa$ (in the case $K = V$), for

η sufficiently small. Then we cannot choose $\mathcal{C}$ and Γ to closely follow $\mathcal{L}_2$ without violating the restrictions imposed by Proposition 5.1.

Assuming this proposition, let us define the contours $\mathcal{C}$ and Γ. As in Section 6.1, they will each be the union of two contours, a small piecewise linear part near ψ and a large curved part that closely follows the level line $\mathcal{L}_2$.

Let us be more specific. In view of the fact that the level lines $\mathcal{L}_j$ now pass through $q\beta$ at angles $\pi/4$, $3\pi/4$, $-3\pi/4$ and $-\pi/4$, we must select different linear contours from those given by Definition 6.2. Instead, we use the following contours; see Figure 12.

DEFINITION 8.3.   For a real number $r \in \mathbb{R}$ and a positive real number $\varepsilon > 0$, let $\mathfrak{U}_{r,\varepsilon}$ denote the piecewise linear curve in the complex plane that connects $r + \varepsilon e^{-\pi \mathbf{i}/8}$ to r to $r + \varepsilon e^{\pi \mathbf{i}/8}$. Similarly, let $\mathfrak{X}_{r,\varepsilon}$ denote the linear curve in the complex plane that connects $r - i\varepsilon$ to $r + i\varepsilon$.

Now let us make the following definitions: Definition 6.3 and Definition 6.4 define the piecewise linear and curved parts of the contours $\mathcal{C}$ and Γ, respectively. Then Definition 6.5 defines the contours $\mathcal{C}$ and Γ.

DEFINITION 8.4.   Let $\mathcal{C}^{(1)} = \mathfrak{U}_{\psi - \psi f_\eta'^{-1} T^{-1/2}, \varepsilon}$ and $\Gamma^{(1)} = \mathfrak{X}_{\psi - 2\psi f_\eta'^{-1} T^{-1/2}, \varepsilon}$, where ε is chosen to be sufficiently small (independent of T) so that the first, second and fourth properties listed in Definition 6.3 all hold for sufficiently large T. The third is replaced by the following:

- We have that $|R(\psi^{-1} f_\eta(z - \psi))| < |f_\eta(z - \psi)/3\psi|^2$, for all $z \in \mathcal{C}^{(1)} \cup \Gamma^{(1)}$, where we recall the definition of R from (8.8).

Such a positive real number ε is guaranteed to exist by part 6 of Proposition 8.1, and also by the estimate (8.8).

DEFINITION 8.5.   Let $\mathcal{C}^{(2)}$ denote a positively oriented contour starting from the top endpoint $\psi - \psi f_\eta'^{-1} T^{-1/2} + \varepsilon e^{\pi \mathbf{i}/8}$ of $\mathcal{C}^{(1)}$ to the bottom endpoint $\psi - \psi f_\eta'^{-1} T^{-1/2} + \varepsilon e^{-\pi \mathbf{i}/8}$ of $\mathcal{C}^{(1)}$, and let $\Gamma^{(2)}$ denote a positively oriented contour starting from the top endpoint $\psi - 2\psi f_\eta'^{-1} T^{-1/2} + i\varepsilon$ of $\Gamma^{(1)}$ to the bottom endpoint $\psi - 2\psi f_\eta'^{-1} T^{-1/2} - i\varepsilon$ of $\Gamma^{(1)}$, satisfying the five properties listed in Definition 6.4.

Such contours $\mathcal{C}^{(2)}$ and $\Gamma^{(2)}$ are guaranteed to exist by part 3 and part 5 of Proposition 8.1.

DEFINITION 8.6.   Set $\mathcal{C} = \mathcal{C}^{(1)} \cup \mathcal{C}^{(2)}$ and $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)}$.

Examples of the contours $\mathcal{C}$ and Γ are depicted in Figure 12. It can be quickly verified that Lemma 7.4 again holds with the contours $\mathcal{C}$ and Γ defined above, meaning that we can use them for the proof of Proposition 5.2 and Proposition 5.3 in Section 8.2.

8.1.3. *Proof of Proposition* 8.1. The goal of this section is to establish Proposition 8.1. The proof of this proposition will be similar to that of Proposition 6.7.

To that end, we have the following two lemmas, which are the analogs of Lemma 6.9 and Lemma 6.10. In what follows, we omit the subscript K from notation.

LEMMA 8.7. *For all* $\eta \in (\alpha, \theta)$, *we have that* $\lim_{z \to \infty} |\Re G(z)| = \infty$. *In particular, the set of* $z \in \mathbb{C}$ *for which* $\Re G(z) = G(\psi)$ *is bounded.*

PROOF. To establish this lemma, one Taylor expands $\Re G_K(z)$ in each case $K = V$ and $K = A$. Here, we assume that $K = V$, for the case $K = A$ is entirely analogous. Then we obtain

$$\Re G(z) = \eta \log |z + q\kappa| - \log |z + q| + m'_\eta \log |z|$$
$$= (\eta + m'_\eta - 1) \log |z| + \mathcal{O}(|z|^{-1}).$$

Thus, $|\Re G(z)|$ tends to ∞ as $|z|$ tends to ∞; this implies the first, and hence also the second, statement of the lemma. $\square$

LEMMA 8.8. *Let* $\ell \subset \mathbb{C}$ *be any line through* 0. *There exist at most* 6 *complex numbers* $z \in \ell$ *such that* $\Re G(z) = G(\psi)$.

Moreover, if $K = A$ *and* ℓ *is the real axis, then there exist only four such* z; *one is* $z = \psi$, *one is in* (ϑ, ∞), *one is in* $(-q, 0)$ *and one is in* $(-\infty, -q)$.

Similarly, if $K = V$ *and* ℓ *is the real axis, then there exist only five such* z; *one is* $z = \psi$, *one is in* (ϑ, ∞), *one is in* $(-q, 0)$, *one is in* $(-q\kappa, -q)$ *and one is in* $(-\infty, -q\kappa)$.

The proof of this lemma is very similar to that of Lemma 8.8 and is thus omitted. Now we can establish Proposition 8.1.

PROOF OF PROPOSITION 8.1. We only consider the case $K = V$, since the proof is entirely analogous in the case $K = A$ (see also the proof of Proposition 6.7 in Section 6.1.4 for the types of modifications necessary for the case $K = A$). In what follows, we omit the subscript V to simplify notation.

Let $S = \mathbb{C} \setminus \{\psi\}$. Then $\Re G(z)$ and $\Re(G'(z))$ are smooth on $S \setminus \{-q\kappa, -q, 0, \vartheta\}$. Furthermore, from (8.5), we deduce that $G'(z)$ is nonzero on $S \setminus \{-q\kappa, -q, 0, \vartheta\}$, which implies that $\Re G(z)$ has no critical points on $S \setminus \{-q\kappa, -q, 0, \vartheta\}$. Thus, the implicit function theorem shows that the set of $z \in S$ satisfying $\Re G(z) = G(\psi)$ is

a one-dimensional sub-manifold of S; here, we used the fact that $\Re G(z) \neq \Re G(\psi)$ for any $z \in \{-q\kappa, -q, 0, \vartheta\}$ [where the fact that $G(\vartheta) \neq G(\psi)$ follows from the fact that $G'(z) < 0$ for all $z \in (\psi, \vartheta)$]. In particular, the set of such z is a union of connected components, each of which is homeomorphic to a line or circle.

By Lemma 8.7, all of these connected components are bounded. Therefore, the closure $\overline{\mathcal{M}}$ (in $\mathbb{C}$) of any such component $\mathcal{M}$ is homeomorphic to a circle. In particular, by the maximum principle for harmonic functions, one of the singularities $-q\kappa$, $-q$, or 0 of $\Re G(z)$ must lie on or in the interior of any closure $\overline{\mathcal{M}}$ [since $\Re G(z)$ harmonic for $z \in \mathbb{C} \setminus \{-q\kappa, -q, 0\}$].

Now, due to (8.8), four (not necessarily distinct) components exit from ψ, at angles $\pi/4$, $3\pi/4$, $-3\pi/4$ and $-\pi/4$. Lemma 8.8 also shows that there exist $z_1 \in (\vartheta, \infty)$, $z_2 \in (-q, 0)$, $z_3 \in (-q\kappa, -q)$, and $z_4 \in (-\infty, -q\kappa)$ such that $\Re G(z_1) = \Re G(z_2) = \Re G(z_3) = \Re G(z_4) = G(\psi)$; this implies that one of the components passes through each of the points z_1, z_2, z_3 and z_4. The second part of Lemma 8.8 states that there are no other real numbers $z \notin \{z_1, z_2, z_3, z_4, \psi\}$ such that $\Re G(z) = G(\psi)$.

These facts imply that there are three one-dimensional components $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3 \subset S$ such that the set of $z \in S$ satisfying $\Re G(z) = G(\psi)$ is equal to the union $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$; these components can be described as follows. First, we have that $\psi \in \overline{\mathcal{M}_1}, \overline{\mathcal{M}_2}$ and $z_1 \in \mathcal{M}_3$. Furthermore, $z_2 \in \mathcal{M}_1$, $z_3 \in \mathcal{M}_2$ and $z_4 \in \mathcal{M}_3$. We can describe the intersections $\overline{\mathcal{M}_1} \cap \mathbb{R} = \{z_2, \psi\}$, $\overline{\mathcal{M}_2} \cap \mathbb{R} = \{z_3, \psi\}$, and $\mathcal{M}_3 \cap \mathbb{R} = \{z_1, z_4\}$.

Now set $\mathcal{L}_1 = \overline{\mathcal{M}_1}$, $\mathcal{L}_2 = \overline{\mathcal{M}_2}$, and $\mathcal{L}_3 = \mathcal{M}_3$. Then $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are all simple, closed curves. Parts 1 and 2 of Proposition 8.1 are satisfied by these curves since $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \{\psi\} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$.

Furthermore, only one positive real number and only one negative real number lie on each of the curves $\overline{\mathcal{M}_1}$, $\overline{\mathcal{M}_2}$ and $\mathcal{M}_3$. This implies that 0 is in the interior of $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.

Now property 3 holds from Lemma 8.8; indeed, this lemma implies that any line $\ell \subset \mathbb{C}$ can only intersect $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ in at most six places. Since each $\mathcal{L}_j$ is a closed curve containing 0 in its interior, ℓ intersects each $\mathcal{L}_j$ at least, and thus only, twice; furthermore, 0 is between these two intersection points. This implies that for each angle $a \in \mathbb{R}$ and each index $j \in \{1, 2, 3\}$, there is only one complex number $z_a \in \mathcal{L}_j$ such that $z_a/|z_a| = e^{\mathbf{i}a}$. This verifies property 3.

To see property 4, observe that $\mathcal{L}_i$ and $\mathcal{L}_j$ can only intersect at critical points of $G(z)$, for any distinct $i, j \in \{1, 2, 3\}$. These critical points are ψ, ϑ, 0, $-q$ and $-q\kappa$, but none of the $\mathcal{L}_j$ pass through ϑ, 0, $-q$, or $-q\kappa$. Now the containment follows from the fact that $\mathcal{L}_1$ intersects the negative real axis only once at z_2, that $\mathcal{L}_2$ intersects the negative real axis only once at z_3, that $\mathcal{L}_3$ intersects the negative real axis only once at z_4, and that $z_4 < z_3 < z_2$.

Property 5 follows from the fact that $z_4 < -\kappa q < z_3 < -q < z_2$, and property 6 follows from the containment property 4 and the estimate (8.8).

Furthermore, observe that $\Re G(z)$ can only change sign across some $\mathcal{L}_j$; thus, property 7 and property 8 follow since $\lim_{z \to -q} \Re G(z) = \infty$ and $\lim_{z \to 0} \Re G(z) = -\infty$. $\square$

8.2. *Proof of part* 3 *of Proposition* 5.2 *and Proposition* 5.3. The goal of this section is to establish the third parts of Proposition 5.2 and Proposition 5.3, which exhibit Gaussian-type asymptotic fluctuations in the case $\eta \in (\alpha, \theta)$. As in Section 6.2 and Section 7.2, the proofs of both statements are similar, so we will do them simultaneously. Thus, recall the notation $K \in \{V, A\}$ that corresponds to the stochastic six-vertex model when $K = V$, and to the ASEP when $K = A$. In what follows, we simplify notation by omitting the subscript K, and we always assume that $\eta \in (\alpha, \theta)$.

In either case $K \in \{V, A\}$, let $x = \eta T + 1$; as in Section 6.2, the replacement of $\lfloor \eta T \rfloor + 1$ (which was in the statement of Proposition 5.2 and Proposition 5.3) by $\eta T + 1$ will not affect the asymptotics. Furthermore, let $p_T = f'_\eta s T^{1/2} - m'_\eta T$, where we recall that m'_η is defined through (8.4) and f'_η are defined through (8.2) and (8.3) in the cases $K = A$ and $K = V$, respectively.

The proofs of the third parts of Proposition 5.2 and Proposition 5.3 will be very similar the proofs of the first parts given in Section 6.2 in the case $\eta > \theta$. Specifically, we will analyze the contribution to the integral on the right-hand side of (8.1) when v is integrated in a small neighborhood of ψ; this will yield a Gaussian-type kernel. Then, in Section 8.2.2, we will show that the contribution to the integral decays exponentially when v is integrated outside a neighborhood of ψ; we will then be able to ignore this contribution, which will lead to the proofs of the third parts of Proposition 5.2 and Proposition 5.3.

8.2.1. *Contribution near* ψ. In this section we analyze the contribution to the kernel $K(w, w')$, when $w \in \mathcal{C}^{(1)}$ and when v is integrated along $\Gamma^{(1)}$. This will be similar to what was done in Section 6.2.1.

Similar to (6.11), define

$$
\begin{aligned}
\widetilde{K}(w, w') = {} & \frac{1}{2\mathbf{i} \log q} \\[1ex]
& \times \sum_{j=-\infty}^{\infty} \int_{\Gamma^{(1)}} \frac{\exp(T(G(w) - G(v)))}{\sin(\frac{\pi}{\log q}(2\pi \mathbf{i} j + \log v - \log w))} \\[1ex]
& \times \frac{(q^{-1}\beta^{-1}v; q)_\infty^m}{(q^{-1}\beta^{-1}w; q)_\infty^m} \\[1ex]
& \times \left(\frac{v}{w}\right)^{s f'_\eta T^{1/2}} \frac{dv}{v(w' - v)},
\end{aligned}
$$

(8.10)

for any $w, w' \in \mathcal{C}$.

Now let us change variables. Denote $\sigma = \psi f_\eta'^{-1} T^{-1/2}$, and set

(8.11)
$$w = \psi + \sigma \widehat{w}; \qquad w' = \psi + \sigma \widehat{w'};$$
$$v = \psi + \sigma \widehat{v}; \qquad \widehat{K}(\widehat{w}, \widehat{w'}) = \sigma \widetilde{K}(w, w').$$

Also, for any contour $\mathcal{D}$, set $\widehat{\mathcal{D}} = \sigma^{-1}(\mathcal{D} - \psi)$.

The following lemma provides the asymptotics for the kernel $\widehat{K}$.

LEMMA 8.9. *There exist constants $c, C > 0$ such that*

(8.12)
$$\left| \widehat{K}(\widehat{w}, \widehat{w'}) \right| \leq \frac{C}{1 + |\widehat{w'}|} \exp(-c|\widehat{w}|^2),$$

for each $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ and $\widehat{w'} \in \widehat{\mathcal{C}}$.

Furthermore, if we define the kernel $L_{s;m}(\widehat{w}, \widehat{w'})$ by

(8.13)
$$L_{s;m}(\widehat{w}, \widehat{w'}) = \frac{1}{2\pi \mathbf{i}} \int_{\mathfrak{X}_{-2,\infty}} \frac{1}{(\widehat{v} - \widehat{w})(\widehat{w'} - \widehat{v})}$$
$$\times \exp\left(\frac{\widehat{v}^2}{2} - \frac{\widehat{w}^2}{2} + s(\widehat{v} - \widehat{w}) \right) \left(\frac{\widehat{v}}{\widehat{w}} \right)^m d\widehat{v},$$

then we have that

(8.14)
$$\lim_{T \to \infty} \widehat{K}(\widehat{w}, \widehat{w'}) = L_{s;m}(\widehat{w}, \widehat{w'}),$$

for each fixed $\widehat{w}, \widehat{w'} \in \mathfrak{U}_{-1,\infty}$.

To establish this lemma, we first rewrite the kernel $\widehat{K}$. Analogously to (6.16) and (6.17), we have that

(8.15)
$$\widehat{K}(\widehat{w}, \widehat{w'}) = \frac{1}{2\pi \mathbf{i}} \int_{\widehat{\Gamma}^{(1)}} I(\widehat{w}, \widehat{w'}; \widehat{v}) \, d\widehat{v},$$

where

$$I(\widehat{w}, \widehat{w'}; \widehat{v})$$

$$= \frac{1}{(1 + \psi^{-1} \sigma \widehat{v})(\widehat{w'} - \widehat{v})}$$

$$\times \exp\left(\frac{\widehat{v}^2 - \widehat{w}^2}{2} + T\left(R(T^{-1/2}\widehat{w}) - R(T^{-1/2}\widehat{v}) \right) \right)$$

(8.16)
$$\times \left(\frac{1 + \psi^{-1} \sigma \widehat{v}}{1 + \psi^{-1} \sigma \widehat{w}} \right)^{\psi \sigma^{-1} s} \frac{(q^{-1}\beta^{-1}(\psi + \sigma \widehat{v}); q)_\infty^m}{(q^{-1}\beta^{-1}(\psi + \sigma \widehat{w}); q)_\infty^m} \frac{\pi \psi^{-1} \sigma}{\log q}$$

$$\times \sum_{j=-\infty}^{\infty} \frac{1}{\sin\left(\frac{\pi}{\log q} (2\pi \mathbf{i} j + \log(1 + \psi^{-1}\sigma \widehat{v}) - \log(1 + \psi^{-1}\sigma \widehat{w})) \right)}.$$

Lemma 8.9 will follow from the large T asymptotics of (and uniform estimate on) I, given by the following lemma.

LEMMA 8.10.   *There exist constants $c, C > 0$ such that*

$$(8.17) \qquad |I(\widehat{w}, \widehat{w'}; \widehat{v})| \leq \frac{C}{1 + |\widehat{w'}|} \exp(-c(|\widehat{v}|^2 + |\widehat{w}|^2))$$

for all $\widehat{w} \in \mathfrak{U}_{-1,\varepsilon/\sigma}$, $\widehat{w'} \in \widehat{\mathcal{C}} \cup \mathfrak{U}_{-1,\infty}$, and $\widehat{v} \in \mathfrak{X}_{-2,\varepsilon/\sigma}$.
Furthermore,

$$(8.18) \qquad \begin{aligned} &\lim_{T \to \infty} I(\widehat{w}, \widehat{w'}; \widehat{v}) \\ &= \frac{1}{(\widehat{v} - \widehat{w})(\widehat{w'} - \widehat{v})} \exp\left(\frac{\widehat{v}^2}{2} - \frac{\widehat{w}^2}{2} + s(\widehat{w} - \widehat{v})\right)\left(\frac{\widehat{v}}{\widehat{w}}\right)^m, \end{aligned}$$

for each fixed $\widehat{w}, \widehat{w'}, \widehat{v} \in \mathbb{C}$.

PROOF.   The proof of this lemma will be similar to that of Lemma 6.12.

Let us first establish the uniform estimate (6.18). To that end, observe that there exists a constant $C_1 > 0$ such that the inequalities

$$\left|\frac{1}{1 + \psi^{-1}\sigma\widehat{v}}\right| < C_1; \qquad \frac{1}{\widehat{v} - \widehat{w'}} < \frac{C_1}{1 + |\widehat{w'}|};$$

$$\left|\frac{1 + \psi^{-1}\sigma\widehat{v}}{1 + \psi^{-1}\sigma\widehat{w}}\right|^{\psi\sigma^{-1}s} \leq C_1 \exp(C_1(|\widehat{w}| + |\widehat{v}|));$$

$$\frac{\pi\psi^{-1}\sigma}{|\log q|} \sum_{j \neq 0} \left|\frac{1}{\sin(\frac{\pi}{\log q}(2\pi \mathbf{i} j + \log(1 + \psi^{-1}\sigma\widehat{v}) - \log(1 + \psi^{-1}\sigma\widehat{w})))}\right|$$

$$(8.19) \qquad \leq C_1 T^{-1/2};$$

$$\left|\left(\frac{\pi\psi^{-1}\sigma}{\log q}\right)\frac{1}{\sin(\frac{\pi}{\log q}(\log(1 + \psi^{-1}\sigma\widehat{v}) - \log(1 + \psi^{-1}\sigma\widehat{w})))}\right|$$

$$< C_1 \exp(C_1(|\widehat{w}| + |\widehat{v}|));$$

$$\left|\frac{(q^{-1}\beta^{-1}(\psi + \sigma\widehat{v}); q)_\infty^m}{(q^{-1}\beta^{-1}(\psi + \sigma\widehat{w}); q)_\infty^m}\right|$$

$$< C_1 \exp(C_1(|\widehat{w}| + |\widehat{v}|)),$$

all hold for each $\widehat{w} \in \mathfrak{U}_{-1,\varepsilon/\sigma}$, $\widehat{w'} \in \widehat{\mathcal{C}}$, and $\widehat{v} \in \mathfrak{X}_{-2,\varepsilon/\sigma}$. The verification of these estimates is entirely analogous to the verification of (6.20) and is thus omitted.

The estimates (8.19) address all terms on the right-hand side of (8.16), except for the exponential term. To analyze this term, first recall $|R(z)| < |z|^2/9$, for all

$z \in \widehat{\mathcal{C}}^{(1)} \cup \widehat{\Gamma}^{(1)}$; this was stipulated in Definition 8.4. Thus, increasing C_1 if necessary, we obtain that

$$
\left| \exp\left(\frac{\widehat{v}^2}{2} - \frac{\widehat{w}^2}{2} + T \big(R(T^{-1/2}\widehat{w}) - R(T^{-1/2}\widehat{v}) \big) \right) \right|
$$

(8.20)
$$
= \exp\left(\frac{\widehat{v}^2}{2} - \frac{\widehat{w}^2}{2} + \frac{|\widehat{w}|^2}{9} + \frac{|\widehat{v}|^2}{9} \right)
$$

$$
< C_1 \exp\left(\frac{|\widehat{v}|^2}{9} - \frac{|\widehat{w}|^2}{9} \right).
$$

In (8.20), the last estimate follows from the facts that there exist constants C_2 and C_3 such that $\widehat{v}^2 < C_2 - |\widehat{v}|^2/4$ and $\widehat{w}^2 > C_2 + |\widehat{w}|^2/4$, for all $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ and $\widehat{v} \in \widehat{\Gamma}^{(1)}$.

Now the estimate (8.17) follows from the definition (8.16) of I, the six estimates (8.19) and the exponential estimate (8.20).

In order to establish the limiting statement (8.18), we observe that

$$
\lim_{T \to \infty} \frac{1}{1 + \psi^{-1}\sigma\widehat{v}} = 1;
$$

$$
\lim_{T \to \infty} \left(\frac{1 + \psi^{-1}\sigma\widehat{v}}{1 + \psi^{-1}\sigma\widehat{w}} \right)^{\psi\sigma^{-1}s} = \exp(s(\widehat{v} - \widehat{w}));
$$

$$
\lim_{T \to \infty} \exp\left(\frac{\widehat{v}^2 - \widehat{w}^2}{2} + T \big(R(T^{-1/2}\widehat{w}) - R(T^{-1/2}\widehat{v}) \big) \right)
$$

(8.21)
$$
= \exp\left(\frac{\widehat{v}^2}{2} - \frac{\widehat{w}^2}{2} \right);
$$

$$
\lim_{T \to \infty} \frac{\pi\psi^{-1}\sigma}{\log q}
$$

$$
\times \sum_{j \neq 0} \frac{1}{\sin\big(\frac{\pi}{\log q}(2\pi\mathbf{i}j + \log(1 + \psi^{-1}\sigma\widehat{v}) - \log(1 + \psi^{-1}\sigma\widehat{w})) \big)} = 0;
$$

$$
\lim_{T \to \infty} \left(\frac{\pi\psi^{-1}\sigma}{\log q} \right) \frac{1}{\sin\big(\frac{\pi}{\log q}(\log(1 + \psi^{-1}\sigma\widehat{v}) - \log(1 + \psi^{-1}\sigma\widehat{w})) \big)} = \frac{1}{\widehat{v} - \widehat{w}};
$$

and

(8.22)
$$
\lim_{T \to \infty} \frac{(q^{-1}\beta^{-1}(\psi + \sigma\widehat{v}); q)_\infty^m}{(q^{-1}\beta^{-1}(\psi + \sigma\widehat{w}); q)_\infty^m} = \left(\frac{\widehat{v}}{\widehat{w}} \right)^m,
$$

for each $\widehat{w}, \widehat{w'}, \widehat{v} \in \mathbb{C}$. The verification of the five identities (8.21) is very similar to that of (6.22) and is thus omitted. The limit (8.22) follows from the fact that $\psi = q\beta$ and that σ tends to 0 as T tends to ∞.

Now (8.18) follows from multiplying the results of (8.21) and (8.22). □

We can now establish Lemma 8.9.

PROOF OF LEMMA 8.9.   The uniform estimate (8.12) follows from integrating the estimate (8.17) over $\widehat{v} \in \mathfrak{X}_{-2,\varepsilon/\sigma}$. Then the limit (8.14) follows from (8.18), (8.17), the fact that $\widehat{\Gamma}^{(1)}$ is contained in and converges to $\mathfrak{X}_{-2,\infty}$ as T tends to ∞, the exponential decay of the kernel $L_{s;m}$ in $|\widehat{v}|^2$ and the dominated convergence theorem. □

8.2.2. *Exponential decay of K on $\mathcal{C}^{(2)}$ and $\Gamma^{(2)}$.*   In this section we analyze the integral (8.1) defining $K(w, w')$ when either $w \in \mathcal{C}^{(2)}$ or $v \in \Gamma^{(2)}$. In this case, we show that the integral decays exponentially in T, which will allow us to ignore this contribution and apply the results of Section 8.2.1.

We begin with the following lemma, which is the analog of Lemma 6.13 and estimates the exponential term in the integrand on the right-hand side of (8.1).

LEMMA 8.11.   *There exists some positive real number $c_1 > 0$, independent of T, such that*

$$\max\Big\{ \sup_{\substack{w \in \mathcal{C} \\ v \in \Gamma^{(2)}}} \Re\big(G(w) - G(v)\big), \ \sup_{\substack{w \in \mathcal{C}^{(2)} \\ v \in \Gamma}} \Re\big(G(w) - G(v)\big) \Big\} < -c_1.$$

PROOF.   From Proposition 8.1, we find that $\Re(G(w) - G(v)) < 0$ for all $w \in \mathcal{C}$ and $v \in \Gamma^{(2)}$ and that $\Re(G(w) - G(v)) < 0$ for all $w \in \mathcal{C}^{(2)}$ and $v \in \Gamma$, where both estimates are uniform in T. Thus, the existence of the claimed constant c_1 follows from the compactness of the contours $\mathcal{C}$ and Γ. □

Now recall the definitions of $\sigma = \psi f_\eta'^{-1} T^{-1/2}$; $\widehat{w}, \widehat{w'}$ and $\widehat{v}$ from (8.11); and I from (8.16). Define

$$(8.23) \qquad \overline{K}(\widehat{w}, \widehat{w'}) = \sigma K(w, w') = \frac{1}{2\pi\mathbf{i}} \int_{\widehat{\Gamma}} I(\widehat{w}, \widehat{w'}; \widehat{v}) \, d\widehat{v},$$

for each $\widehat{w}, \widehat{w'} \in \mathcal{C}$. From the change of variables (8.11), we have that

$$(8.24) \qquad \det(\mathrm{Id} + K)_{L^2(\mathcal{C})} = \det(\mathrm{Id} + \overline{K})_{L^2(\widehat{\mathcal{C}})}.$$

Given Lemma 8.11, the proof of the following result is entirely analogous to that of Corollary 6.14 and Corollary 7.8 and is thus omitted.

COROLLARY 8.12.   *There exist constants $c, C > 0$ such that*

$$(8.25) \qquad \big|\overline{K}(\widehat{w}, \widehat{w'}) - \widehat{K}(\widehat{w}, \widehat{w'})\big| < C \exp\big(-c(T + |\widehat{w}|^2)\big),$$

for all $\widehat{w} \in \widehat{\mathcal{C}}^{(1)}$ and $\widehat{w'} \in \widehat{\mathcal{C}}$, and such that

$$(8.26) \qquad |\overline{K}(\widehat{w}, \widehat{w'})| < C \exp(-c(T + |\widehat{w}|^2)),$$

for all $\widehat{w} \in \widehat{\mathcal{C}}^{(2)}$ and $\widehat{w'} \in \widehat{\mathcal{C}}$.

Now we can use Corollary A.5 of [3], Lemma 8.9 and Corollary 8.12 to establish the third parts of Proposition 5.2 and Proposition 5.3.

PROOF OF PART 3 OF PROPOSITION 5.2 AND PROPOSITION 5.3. This will be very similar to the proof of the first (and second) parts of Proposition 5.2 and Proposition 5.3, given in Section 6.2.2 (and Section 7.2). In particular, it will follow from an application of Corollary A.5 of [3] to the sequence of kernels $\{\overline{K}(\widehat{w}, \widehat{w'})\}$, the sequence of contours $\{\mathfrak{U}_{-1,\infty} \cup \widehat{\mathcal{C}}^{(2)}\}$ and the kernel $L_{s;m}(\widehat{w}, \widehat{w'})$ [given by (8.13)].

The verification of the conditions of that corollary in this setting is very similar to the verification provided in the proofs of the first (in Section 6.2.2) and second (in Section 7.2) parts of Proposition 5.2 and Proposition 5.3, and is thus omitted. Applying Corollary A.5 of [3], we obtain that

$$(8.27) \qquad \begin{aligned} \lim_{T \to \infty} \det(\mathrm{Id} + K)_{L^2(\mathcal{C})} &= \lim_{T \to \infty} \det(\mathrm{Id} + \overline{K})_{L^2(\widehat{\mathcal{C}})} \\ &= \det(\mathrm{Id} + L_{s;m})_{L^2(\mathfrak{U}_{-1,\infty})}, \end{aligned}$$

where we have used (8.24) to deduce the first identity.

Now it is known (see, e.g., Proposition 5 of [9]) that $\det(\mathrm{Id} + L_{s;m})_{\mathfrak{U}_{-1,\infty}} = G_m(s)$. Thus, we deduce from (6.31) that

$$(8.28) \qquad \lim_{T \to \infty} \det(\mathrm{Id} + K)_{L^2(\mathcal{C})} = G_m(s).$$

Applying (6.32) in the case $K = A$ yields the third part of Proposition 5.2 and applying (6.32) in the case $K = V$ yields the third part of Proposition 5.3. □

SUPPLEMENTARY MATERIAL

Supplement to "Phase transitions in the ASEP and stochastic six-vertex model." (DOI: 10.1214/17-AOP1253SUPP; .pdf). This supplement serves as the Appendix for the present paper. In Appendix A, we provide some results about Fredholm determinants that are used in the asymptotic analysis above, and in Appendix B we outline an alternative way to establish Theorem 1.7 through a comparison with Schur measures.

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