INTERLACEMENTS AND THE WIRED UNIFORM SPANNING FOREST

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We extend the Aldous–Broder algorithm to generate the wired uniform spanning forests (WUSFs) of infinite, transient graphs. We do this by replacing the simple random walk in the classical algorithm with Sznitman's random interlacement process. We then apply this algorithm to study the WUSF, showing that every component of the WUSF is one-ended almost surely in any graph satisfying a certain weak anchored isoperimetric condition, that the number of 'excessive ends' in the WUSF is nonrandom in any graph, and also that every component of the WUSF is one-ended almost surely in any transient unimodular random rooted graph. The first two of these results answer positively two questions of Lyons, Morris and Schramm [*Electron. J. Probab.* **13** (2008) 1702–1725], while the third extends a recent result of the author.

Finally, we construct a counterexample showing that almost sure one-endedness of WUSF components is not preserved by rough isometries of the underlying graph, answering negatively a further question of Lyons, Morris and Schramm.

1. Introduction. The *uniform spanning forests* (USFs) of an infinite, locally finite, connected graph G are defined as weak limits of uniform spanning trees (USTs) of large finite subgraphs of G. These weak limits can be taken with either free or wired boundary conditions (see Section 3.1), yielding the *free uniform spanning forest* (FUSF) and *wired uniform spanning forest* (WUSF), respectively. The USFs are closely related to several other topics in probability theory, including loop-erased random walks [24, 44], potential theory [5, 8], conformally invariant scaling limits [25, 36], domino tiling [22] and the Abelian sandpile model [11, 19]. In this paper, we develop a new connection between the wired uniform spanning forest and Sznitman's *interlacement process* [39, 41].

A key theoretical tool in the study of the UST and USFs is *Wilson's algorithm* [44], which allows us to sample the UST of a finite graph by joining together loop-erasures of random walk paths. In their seminal work [5], Benjamini, Lyons, Peres and Schramm (henceforth referred to as BLPS) extended Wilson's algorithm to infinite transient graphs and used this extension to establish several fundamental properties of the WUSF. For example, they proved that the WUSF of an infinite,

locally finite, connected graph is connected almost surely (a.s.) if and only if the sets of vertices visited by two independent random walks on the graph have infinite intersection a.s. This recovered the earlier, pioneering work of Pemantle [32], who proved that the FUSF and WUSF of $\mathbb{Z}^d$ coincide for all d and are a.s. connected if and only if $d \leq 4$. Wilson's algorithm has also been instrumental in the study of scaling limits of uniform spanning trees and forests [3, 25, 34, 36, 37].

Prior to the introduction of Wilson's algorithm, the best known algorithm for sampling the UST of a finite graph was the *Aldous–Broder algorithm* [2, 7], which generates a uniform spanning tree of a finite connected graph G as the collection of first-entry edges of a random walk on G. We now describe this algorithm in detail. Let ρ be a fixed vertex of G, and let $\langle X_n \rangle_{n \geq 0}$ be a simple random walk on G started at ρ. For each vertex v of G, let $e(v)$ be the edge of G incident to v that is traversed by the random walk X_n as it enters v for the first time, and let $T = \{e(v) : v \in V \setminus \{\rho\}\}$ be set of first-entry edges. Aldous [2] and Broder [7] proved independently that the resulting random spanning tree T is distributed uniformly on the set of spanning trees of G (see also [27], Section 4.4). If we orient the edge in the direction opposite to that in which it was traversed by the random walk, then the spanning tree is oriented towards ρ, meaning that every vertex of G other than ρ has exactly one oriented edge emanating from it in the tree.

While the algorithm extends without modification to generate USTs of recurrent infinite graphs, the collection of first entry edges of a random walk on a transient graph might not span the graph. Thus, naively running the Aldous–Broder on a transient graph will not necessarily produce a spanning forest of the graph. Moreover, unlike in Wilson's algorithm, we cannot simply continue the algorithm by starting another random walk from a new location. As such, it has hitherto been unclear how to extend the Aldous–Broder algorithm to infinite transient graphs and, as a result, the Aldous–Broder algorithm has been of limited theoretical use in the study of USFs of infinite graphs.

In this paper, we extend the Aldous–Broder algorithm to infinite, transient graphs by replacing the random walk with the random interlacement process. The interlacement process was originally introduced by Sznitman [39] to study the disconnection of cylinders and tori by a random walk trajectory, and was generalised to arbitrary transient graphs by Teixeira [41]. The interlacement process $\mathscr{I}$ on a transient graph G is a point process on the space $\mathcal{W}^* \times \mathbb{R}$, where $\mathcal{W}^*$ is the space of doubly-infinite paths in G modulo time-shift (see Section 3.3 for precise definitions), and should be thought of as a collection of random walk excursions from infinity. Let us note that the interlacement process as we define it is stationary under shifts in the second coordinate. This stationarity is important for us, and is why we consider the interlacement process to be a point process on $\mathcal{W}^* \times \mathbb{R}$ rather than $\mathcal{W}^* \times [0, \infty)$ as is more common in the literature. We refer the reader to the monographs [12] and [9] for an introduction to the extensive literature on the random interlacement process.

We state our results in the natural generality of networks. Recall that a *network* (G, c) is a connected, locally finite graph $G = (V, E)$, possibly containing self-loops and multiple edges, together with a function $c : E \to (0, \infty)$ assigning a positive *conductance* $c(e)$ to each edge e of G. The conductance $c(v)$ of a vertex v is defined to be the sum of the conductances of the edges emanating from v. Graphs without specified conductances are considered as networks by setting $c(e) \equiv 1$. We will usually suppress the notation of conductances, and write simply G for a network. See Section 3.1 for detailed definitions of the USFs on general networks.

Oriented edges e are oriented from their tail e^- to their head e^+. The reversal of an oriented edge e is denoted $-e$.

THEOREM 1.1 (Interlacement Aldous–Broder). *Let G be a transient, connected, locally finite network, let $\mathscr{I}$ be the interlacement process on G and let $t \in \mathbb{R}$. For each vertex v of G, let $\tau_t(v)$ be the smallest time greater than t such that there exists a trajectory $(W_{\tau_t(v)}, \tau_t(v)) \in \mathscr{I}$ passing through v, and let $e_t(v)$ be the oriented edge of G that is traversed by the trajectory $W_{\tau_t(v)}$ as it enters v for the first time. Then*

$$\mathsf{AB}_t(\mathscr{I}) := \big\{ -e_t(v) : v \in V \big\}$$

has the law of the oriented wired uniform spanning forest of G.

A useful feature of the interlacement Aldous–Broder algorithm is that it allows us to consider the wired uniform spanning forest of an infinite transient graph as the stationary measure of the ergodic Markov process $\langle \mathsf{AB}_t(\mathscr{I}) \rangle_{t \in \mathbb{R}}$. For example, a key step in proving that the number of excessive ends of the WUSF is nonrandom is to show that the number of *indestructible* excessive ends is a.s. monotone in the time evolution of the process $\langle \mathsf{AB}_t(\mathscr{I}) \rangle_{t \in \mathbb{R}}$.

2. Applications.

2.1. *Ends.*   Other than connectivity, the most basic topological property of a forest is the number of ends its components have. Here, an infinite, connected graph $G = (V, E)$ is said to be *k-ended* if, over all finite subsets W of V, the subgraph of G induced by $V \setminus W$ has a supremum of k infinite connected components. In particular, an infinite tree is k-ended if and only if there exist exactly k distinct infinite simple paths starting at each vertex of the tree. Components of the WUSF are known to be one-ended a.s. in several large classes of graphs. The first result of this kind is due to Pemantle [32], who proved that the WUSF of $\mathbb{Z}^d$ is one-ended a.s. for $2 \leq d \leq 4$, and that every component of the WUSF of $\mathbb{Z}^d$ has at most two ends a.s. for every $d \geq 5$ (the WUSF of $\mathbb{Z}$ is the whole of $\mathbb{Z}$ and is therefore two-ended). BLPS [5] later completed this work, showing in particular that every component of the WUSF is one-ended a.s. in any transient Cayley graph.

We note that one-endedness of WUSF components has important consequences for the Abelian sandpile model [19–21].

Taking a different approach, Lyons, Morris and Schramm [26] gave an isoperimetric criterion for one-endedness of WUSF components, from which they deduced that the every component of the WUSF is one-ended in every transitive graph not rough isometric to $\mathbb{Z}$, and also every nonamenable graph. Unlike the earlier results of BLPS, the results of Lyons, Morris and Schramm are robust in the sense that their assumptions depend only upon the coarse geometry of the graph and do not require any kind of homogeneity. They asked [26], Question 7.9, whether the isoperimetric assumption in their theorem could be replaced by the *anchored* version of the same condition, and in particular whether every WUSF component is one-ended a.s. in any graph with anchored expansion (defined below). Unlike classical isoperimetric conditions, anchored isoperimetric conditions are often preserved under random perturbations such as supercritical Bernoulli percolation [10, 35].

Given a network G and a set K of vertices of G, we write $\partial_E K$ for the set of edges of G with exactly one endpoint in K, and write $|K|$ for the sum of the conductances of the vertices in K. Similarly, if W is a set of edges in G, we write $|W|$ for the sum of the conductances of the edges in W. Given an increasing function $f : (0, \infty) \to (0, \infty)$, we say that G satisfies an *anchored $f(t)$-isoperimetric inequality* if

$$\inf\left\{\frac{|\partial_E K|}{f(|K|)} : K \subset V \text{ connected}, v \in K, |K| < \infty\right\} > 0$$

for every vertex v of G. (In contrast, the graph is said to satisfy a (nonanchored) $f(t)$-isoperimetric inequality if the infimum $\inf |\partial_E K|/f(|K|)$ is positive when taken over *all* sets of vertices K with $|K| < \infty$.) In particular, G is said to have *anchored expansion* if and only if it satisfies an anchored t-isoperimetric inequality, and is said to satisfy a *d-dimensional anchored isoperimetric inequality* if it satisfies an anchored $t^{(d-1)/d}$-isoperimetric inequality. Such anchored isoperimetric inequalities are known to hold on, for example, supercritical percolation clusters on $\mathbb{Z}^d$ and related graphs, such as half-spaces and wedges [35].

THEOREM 2.1. *Let G be a network with $c_0 := \inf_e c(e) > 0$, and suppose that G satisfies an anchored $f(t)$-isoperimetric inequality for some increasing function $f : (0, \infty) \to (0, \infty)$ for which there exists a constant α such that $f(t) \leq t$ and $f(2t) \leq \alpha f(t)$ for all $t \in (0, \infty)$. Suppose that f also satisfies each of the following conditions*:

1.

$$\int_{c_0}^{\infty} \frac{1}{f(t)^2}\, dt < \infty$$

and

2.

$$\int_{c_0}^{\infty} \exp\left(-\varepsilon\left(\int_s^{\infty} \frac{1}{f(t)^2}\, \mathrm{d}t\right)^{-1}\right) \mathrm{d}s < \infty$$

for every $\varepsilon > 0$.

Then every component of the wired uniform spanning forest of G is one-ended almost surely.

In particular, Theorem 2.1 applies both to every graph with anchored expansion and to every graph satisfying a d-dimensional anchored isoperimetric inequality with $d > 2$. The graph formed by joining two copies of $\mathbb{Z}^2$ together with a single edge between their origins satisfies a 2-dimensional isoperimetric inequality but has a two-ended WUSF. The theorem can fail if edge conductances are not bounded away from zero, as can be seen by attaching an infinite path with exponentially decaying edge conductances to the root of a 3-regular tree.

Theorem 2.1 comes very close to giving a complete answer to [26], Question 7.9. The isoperimetric condition of [26] is essentially that G satisfies an $f(t)$-isoperimetric inequality for some f satisfying all conditions of Theorem 2.1 with the possible exception of (2); the precise condition required is slightly weaker than this but also more technical. Our formulation of Theorem 2.1 is adapted from the presentation of the results of [26] given in [27], Theorem 10.43. The difference in requirements on the function $f(t)$ between Theorem 2.1 and [27], Theorem 10.43, can be seen by considering $f(t)$ of the form $t^{1/2}\log^{\alpha}(1 + t)$: In particular, we observe that [27], Theorem 10.43, applies to graphs satisfying a $t^{1/2}\log^{\alpha}(1 + t)$-isoperimetric inequality for some $\alpha > 1/2$, while our theorem applies to graphs satisfying an *anchored* $t^{1/2}\log^{\alpha}(1 + t)$-isoperimetric inequality only if $\alpha > 1$.

In Section 6, we give an example of two bounded degree, rough-isometric graphs G and G' such that every component of the WUSF of G is one-ended, while the WUSF of G' a.s. contains a component with uncountably many ends. This answers negatively Question 7.6 of [26], and shows that the behaviour of the WUSF of a graph cannot always be determined from the coarse geometric properties of the graph alone.

2.2. *Excessive ends.* One example of a transient graph in which the WUSF has multiply-ended components is the subgraph of $\mathbb{Z}^6$ spanned by the vertex set

$$\left(\mathbb{Z}^5 \times \{0\}\right) \cup \left(\{(0, 0, 0, 0, 0), (2, 0, 0, 0, 0)\} \times \mathbb{N}\right),$$

which is obtained from $\mathbb{Z}^5$ by attaching an infinite path to each of the vertices $u = (0, 0, 0, 0, 0)$ and $v = (2, 0, 0, 0, 0)$. The WUSF of this graph, which we denote $\mathfrak{F}$, is equal in distribution to the union of the WUSF of $\mathbb{Z}^5$ with each of the two added paths. If u and v are in the same component of $\mathfrak{F}$, then there is a single component

of $\mathfrak{F}$ with three ends and all other components are one-ended. Otherwise, u and v are in different components of $\mathfrak{F}$, so that there are exactly two components of $\mathfrak{F}$ that are two-ended and all other components are one-ended. Each of these events has positive probability, so that the event that there exists a two-ended component of the WUSF has probability strictly between 0 and 1. Nevertheless, the *number of excessive ends* of $\mathfrak{F}$, that is, the sum over all components of $\mathfrak{F}$ of the number of ends minus 1, is equal to two a.s.

In light of this example, Lyons, Morris and Schramm [26], Question 7.8, asked whether the number of excessive ends of the WUSF is nonrandom (i.e., equal to some constant a.s.) for any graph. Our next application of the interlacement Aldous–Broder algorithm is to answer this question positively.

THEOREM 2.2.   *Let G be a network. Then the number of excessive ends of the wired uniform spanning forest of G is nonrandom.*

When combined with the spatial Markov property of the wired uniform spanning forest, Theorem 2.2 has the following immediate corollary, which states that a natural weakening of [26], Question 7.6, has a positive answer. (As mentioned above, we show the original question to have a negative answer in Section 6.)

COROLLARY 2.3.   *Let G be a network, and suppose that G' is a network obtained from G by adding and deleting finitely many edges. Then the wired uniform spanning forests of G and G' have the same number of excessive ends almost surely. In particular, if every tree of the wired uniform spanning forest of G is one-ended a.s., then the same is true of G'.*

2.3. *Ends in unimodular random rooted graphs.*   Another generalisation of the one-endedness theorem of BLPS [5] concerns transient *unimodular random rooted graphs*. A *rooted graph* is a connected, locally finite graph G together with a distinguished vertex ρ, the *root*. An isomorphism of graphs is an isomorphism of rooted graphs if it preserves the root. The *local topology* on the space $\mathcal{G}_\bullet$ of isomorphism classes of rooted graphs is defined so that two rooted graphs are close if they have large graph distance balls around their respective roots that are isomorphic as rooted graphs. Similarly, a doubly rooted graph is a graph together with an ordered pair of distinguished vertices, and the local topology on the space $\mathcal{G}_{\bullet\bullet}$ of isomorphism classes of doubly rooted graphs is defined similarly to the local topology on $\mathcal{G}_\bullet$. A random rooted graph (G, ρ) is said to be *unimodular* if it satisfies the *Mass-Transport Principle*, which states that for every Borel function $f : \mathcal{G}_{\bullet\bullet} \to [0, \infty]$ (which we call a *mass transport*),

$$\mathbb{E}\left[\sum_{v \in V} f(G, \rho, v)\right] = \mathbb{E}\left[\sum_{v \in V} f(G, v, \rho)\right].$$

In other words, the expected mass sent by the root is equal to the expected mass received by the root. Unimodular random rooted networks are defined similarly by allowing the mass-transport to depend on the edge conductances. We refer the reader to Aldous and Lyons [1] for a systematic development and overview of the theory of unimodular random rooted graphs and networks, as well as several examples.

Aldous and Lyons [1] proved that every component of the WUSF is one-ended a.s. in any bounded degree unimodular random rooted graph, and the author of this article [17] later extended this to all transient unimodular random rooted graphs with finite expected degree at the root, deducing that every component of the WUSF is one-ended a.s. in any supercritical Galton–Watson tree. (The assumption of finite expected degree was implicit in [17] since there we considered *reversible* random rooted graphs, which correspond to unimodular random rooted graphs with finite expected degree.) Our final application of the interlacement Aldous–Broder algorithm is to extend the main result of [17] by removing the condition that the expected degree of the root is finite.

THEOREM 2.4.   *Let (G, ρ) be a transient unimodular random rooted network. Then every component of the wired uniform spanning forest of G is one-ended almost surely.*

3. Background and definitions.

3.1. *Uniform spanning forests.*   For each finite graph $G = (V, E)$, let UST_G denote the uniform measure on the set of spanning trees of G (i.e., connected, cycle-free subgraphs of G containing every vertex), which are considered for measure-theoretic purposes to be functions $E \to \{0, 1\}$. More generally, for each finite network G, let UST_G denote the probability measure on the set of spanning trees of G such that the probability of a tree is proportional to the product of the conductances of its edges.

Let G be an infinite network, and let $\langle V_n \rangle_{n \geq 0}$ be an exhaustion of V by finite connected subsets, that is, an increasing sequence of finite connected subsets of V such that $\bigcup V_n = V$. For each n, let G_n denote the subnetwork of G induced by V_n, and let G_n^* denote the finite network obtained by identifying (wiring) every vertex of G in $V \setminus V_n$ into a single vertex ∂_n, and deleting the infinitely many self-loops from ∂_n to itself. The *wired uniform spanning forest* measure is defined as the weak limit of the uniform spanning tree measures on G_n^*. That is, for every finite set $S \subset E$,

$$\mathsf{WUSF}_G(S \subseteq \mathfrak{F}) := \lim_{n \to \infty} \mathsf{UST}_{G_n^*}(S \subseteq T).$$

In contrast, the *free uniform spanning forest* measure is defined as the weak limit of the uniform spanning tree measures on the finite induced subnetworks G_n. It

is easily seen that both the free and wired measures are supported on the set of *essential spanning forests* of G, that is, the set of cycle-free subgraphs of G that contain every vertex and do not have any finite connected components.

It will also be useful to consider oriented trees and forests. Given an infinite network G with exhaustion $\langle V_n \rangle_{n \geq 0}$, let $\mathsf{OUST}_{G_n^*}$ denote the law of a uniform spanning tree of G_n^* that has been oriented towards the boundary vertex ∂_n, meaning that every vertex of G_n^* other than ∂_n has exactly one oriented edge emanating from it in the tree. BLPS [5] proved that if G is transient, then the measures $\mathsf{OUST}_{G_n^*}$ converge weakly to a measure OWUSF, the *oriented wired uniform spanning forest* (OWUSF) measure. This measure is supported on the set of essential spanning forests of G that are oriented so that every vertex of G has exactly one oriented edge emanating from it in the forest. The WUSF of a transient graph can be obtained from the OWUSF of the graph by forgetting the orientations of the edges.

3.2. *The space of trajectories.* Let G be a graph. For each $-\infty \leq n \leq m \leq \infty$, let $L(n, m)$ be the graph with vertex set $\{i \in \mathbb{Z} : n \leq i \leq m\}$ and with edge set $\{(i, i + 1) : n \leq i \leq m - 1\}$. We define $\mathcal{W}(n, m)$ to be the set of multigraph homomorphisms from $L(n, m)$ to G such that the preimage of each vertex in G is finite, and define $\mathcal{W}$ to be the union

$$\mathcal{W} := \bigcup \{\mathcal{W}(n, m) : -\infty \leq n \leq m \leq \infty\}.$$

For each set $K \subseteq V$, we let $\mathcal{W}_K(n, m)$ denote the set of $w \in \mathcal{W}(n, m)$ that visit K [i.e., for which there exists $n \leq i \leq m$ such that $w(i) \in K$], and let $\mathcal{W}_K$ be the union $\mathcal{W}_K = \bigcup \{\mathcal{W}_K(n, m) : -\infty \leq n \leq m \leq \infty\}$.

Given $w \in \mathcal{W}(n, m)$ and $a \leq b \in \mathbb{Z}$, we define $w|_{[a,b]} \in \mathcal{W}(n \vee a, m \wedge b)$ to be the restriction of w to the subgraph $L(n \vee a, m \wedge b)$ of $L(n, m)$. Given $w \in \mathcal{W}_K(n, m)$, let $H_K^-(w) = \inf\{n \leq i \leq m : w(i) \in K\}$, let $H_K^+(w) = \sup\{n \leq i \leq m : w(i) \in K\}$, and let

$$w_K = w|_{[H_k^-(w), H_K^+(w)]}$$

be the restriction of w to between the first time it visits K and last time it visits K. We equip $\mathcal{W}$ with the topology generated by open sets of the form

$$\{w \in \mathcal{W} : w \text{ visits K and } w_K = w'_K\},$$

where $K \subset V$ is finite and $w' \in \mathcal{W}_K$. [Note that this topology is not the weakest topology making the evaluation maps $w \mapsto w(i)$ and $w \mapsto w(i, i + 1)$ continuous. First and last hitting times of finite sets are not continuous with respect to that topology, but are continuous with respect to ours. The Borel σ-algebras generated by the two topologies are the same.] We also equip $\mathcal{W}$ with the Borel σ-algebra generated by this topology.

The *time shift* $\theta_k : \mathcal{W} \to \mathcal{W}$ is defined by $\theta_k : \mathcal{W}(n, m) \longrightarrow \mathcal{W}(n - k, m - k)$,

$$\theta_k(w)(i) = w(i + k), \qquad \theta_k(w)(i, i + 1) = w(i + k, i + k + 1).$$

The space $\mathcal{W}^*$ is defined to be the quotient

$$\mathcal{W}^* = \mathcal{W}/\sim \qquad \text{where} \quad w_1 \sim w_2 \quad \text{if and only if} \quad w_1 = \theta_k(w_2) \qquad \text{for some } k.$$

Let $\pi : \mathcal{W} \to \mathcal{W}^*$ denote the quotient map. $\mathcal{W}^*$ is equipped with the quotient topology and associated quotient σ-algebra. An element of $\mathcal{W}^*$ is called a *trajectory*.

3.3. *The interlacement process.* Given a network $G = (G, c)$, the conductance $c(v)$ of a vertex v is defined to be the sum of the conductances of the edges emanating from v, and, for each pair of vertices (u, v), the conductance $c(u, v)$ is defined to be the sum of the conductances of the edges connecting u to v. Recall that the *random walk* X on the network G is the Markov chain on V with transition probabilities $p(u, v) = c(u, v)/c(u)$. In case G has multiple edges, we will also keep track of the edges crossed by X, considering X to be a random element of $\mathcal{W}(0, \infty)$. We write either P_u or P_u^G for the law of the random walk started at u on the network G, depending on whether the network under consideration is clear from context. When X is a random walk on a network G and K is a set of vertices in G, we let τ_K denote the first time that X hits K and let τ_K^+ denote the first positive time that X hits K.

Let $G = (V, E)$ be a transient network. Given $w \in \mathcal{W}(n, m)$, let $w^{\leftarrow} \in \mathcal{W}(-n, -m)$ be the reversal of w, defined by setting $w^{\leftarrow}(i) = w(-i)$ for all $-m \leq i \leq -n$, and $w^{\leftarrow}(i, i + 1) = w(-i - 1, -i)$ for all $-m \leq i \leq -n - 1$. For each subset $\mathscr{A} \subseteq \mathcal{W}$, let $\mathscr{A}^{\leftarrow}$ denote the set

$$\mathscr{A}^{\leftarrow} := \{w \in \mathcal{W} : w^{\leftarrow} \in \mathscr{A}\}.$$

For each finite set $K \subset V$, define a measure Q_K on $\mathcal{W}_K$ by setting

$$Q_K(\{w \in \mathcal{W} : w(0) \notin K\}) = 0$$

and, for each $u \in K$ and each two Borel subsets $\mathscr{A}, \mathscr{B} \subseteq \mathcal{W}$,

$$Q_K(\{w \in \mathcal{W} : w|_{(-\infty, 0]} \in \mathscr{A}, w(0) = u \text{ and } w|_{[0, \infty)} \in \mathscr{B}\})$$

$$= c(u) P_u(\langle X_k \rangle_{k \geq 0} \in \mathscr{A}^{\leftarrow} \text{ and } \tau_K^+ = \infty) P_u(\langle X_k \rangle_{k \geq 0} \in \mathscr{B}).$$

Let $\mathcal{W}_K^* = \pi(\mathcal{W}_K)$ be the set of trajectories that visit K.

THEOREM 3.1 (Sznitman [39] and Teixeira [41]: Existence and uniqueness of the interlacement intensity measure). *Let G be a transient network. There exists a unique σ-finite measure Q^* on $\mathcal{W}^*$ such that for every Borel set $\mathscr{A} \subseteq \mathcal{W}^*$ and every finite $K \subset V$,*

$$(3.1) \qquad\qquad Q^*(\mathscr{A} \cap \mathcal{W}_K^*) = Q_K(\pi^{-1}(\mathscr{A})).$$

The measure Q^* is referred to as the *interlacement intensity measure*.

DEFINITION 3.2.   Let Λ denote the Lebesgue measure on $\mathbb{R}$. The *interlacement process* $\mathscr{I}$ on G is defined to be a Poisson point process on $\mathcal{W}^* \times \mathbb{R}$ with intensity measure $Q^* \otimes \Lambda$. For each $t \in \mathbb{R}$, we denote by $\mathscr{I}_t$ the set of $w \in \mathcal{W}^*$ such that $(w, t) \in \mathscr{I}$. We also write $\mathscr{I}_{[a,b]}$ for the intersection of $\mathscr{I}$ with $\mathcal{W}^* \times [a, b]$.

Let $\langle V_n \rangle_{n \geq 0}$ be an exhaustion of an infinite transient network G. The interlacement process on G can be constructed as a limit of Poisson processes on random walk excursions from the boundary vertices ∂_n to itself in the networks G_n^*.

Let N be a Poisson point process on $\mathbb{R}$ with intensity measure $c(\partial_n) \Lambda$. Conditional on N, for every $t \in N$, let W_t be a random walk started at ∂_n and stopped when it first returns to ∂_n, where we consider each W_t to be an element of $\mathcal{W}^*$. We define $\mathscr{I}^n$ to be the point process

$$\mathscr{I}^n := \big\{ (W_t, t) : t \in N \big\}.$$

PROPOSITION 3.3.   *Let G be an infinite transient network and let $\langle V_n \rangle_{n \geq 0}$ be an exhaustion of G. Then the Poisson point processes $\mathscr{I}^n$ converge in distribution to the interlacement process $\mathscr{I}$ as $n \to \infty$.*

A similar construction of the random interlacement process is sketched in [40], Section 4.5.

PROOF.   Let $K \subset V$ be finite, and let n be sufficiently large that K is contained in V_n. Define a measure Q_K^n on $\mathcal{W}$ by setting

$$Q_K^n \big(\{ w \in \mathcal{W} : w(0) \notin K \} \big) = 0$$

and, for each $u \in K$, each $r, m \geq 0$ and each two Borel subsets $\mathscr{A}, \mathscr{B} \in \mathcal{W}$,

$$Q_K^n \big(\{ w \in \mathcal{W}(-r, m) : w|_{[-r,0]} \in \mathscr{A}, \, w(0) = u \text{ and } w|_{[0,m]} \in \mathscr{B} \} \big)$$

(3.2)
$$= c(u) P_u^{G_n^*} \big(\langle X_k \rangle_{k=0}^r \in \mathscr{A}^{\leftarrow} \text{ and } \tau_K^+ > \tau_{\partial_n} = r \big)$$

$$\cdot P_u^{G_n^*} \big(\langle X_k \rangle_{k=0}^m \in \mathscr{B} \text{ and } \tau_{\partial_n} = m \big).$$

By reversibility of the random walk, the right-hand side of (3.2) is equal to

(3.3)
$$c(\partial_n) P_{\partial_n}^{G_n^*} \big(\langle X_k \rangle_{k=0}^n \in \mathscr{A} \text{ and } X_{\tau_K} = u \text{ and } \tau_K = r < \tau_{\partial_n}^+ \big)$$

$$\cdot P_u^{G_n^*} \big(\langle X_k \rangle_{k=0}^m \in \mathscr{B} \text{ and } \tau_{V \setminus V_n} = m \big).$$

It follows that $Q_K^n(\mathcal{W}) = c(\partial_n) P_{\partial_n}^{G_n^*}(\tau_K < \tau_{\partial_n}^+)$ and that the normalized measure $Q_K^n / Q_K^n(\mathcal{W})$ coincides with the law of a random walk excursion from ∂_n to itself in G_n^* that has been conditioned to hit K and reparameterised so that it first hits

K at time 0. Thus, by the splitting property of Poisson processes, $\mathscr{I}^n$ is a Poisson point process on $\mathcal{W}^* \times \mathbb{R}$ with intensity measure $Q^{n*} \otimes \Lambda$, where Q^{n*} satisfies

$$Q^{n*}(\mathscr{A} \cap \mathcal{W}_K^*) = Q_K^n(\pi^{-1}(\mathscr{A})).$$

We conclude the proof by noting that Q_K^n converges weakly to Q_K as $n \to \infty$.
□

3.4. *Hitting probabilities.*   Recall that the *capacity* (a.k.a. the *conductance to infinity*) of a finite set of vertices K in a network G is defined to be

$$\operatorname{Cap}(K) = \sum_{v \in K} c(v) P_v(\tau_K^+ = \infty),$$

and observe that $Q_K(\mathcal{W}) = \operatorname{Cap}(K)$ for every finite set of vertices K. If K is infinite, we define $\operatorname{Cap}(K) = \lim_{n \to \infty} \operatorname{Cap}(K_n)$, where K_n is any increasing sequence of finite sets of vertices with $\bigcup K_n = K$. We say that a set K of vertices is *hit* by $\mathscr{I}_{[a,b]}$ if there exists $(W,t) \in \mathscr{I}_{[a,b]}$ such that W hits K. By the definition of $\mathscr{I}$, we have that

$$\mathbb{P}(K \text{ hit by } \mathscr{I}_{[a,b]}) = 1 - \exp(-(b-a)Q_K(\mathcal{W})) = 1 - \exp(-(b-a)\operatorname{Cap}(K))$$

for each finite set K. This formula extends to infinite sets by taking limits over exhaustions: If $K \subseteq V$ is infinite, let K_n be an exhaustion of K by finite sets. Then

$$\begin{aligned}
\mathbb{P}(K \text{ hit by } \mathscr{I}_{[a,b]}) &= \lim_{n \to \infty} \mathbb{P}(K_n \text{ hit by } \mathscr{I}_{[a,b]}) \\
&= 1 - \lim_{n \to \infty} \exp(-(b-a)\operatorname{Cap}(K_n)) \\
&= 1 - \exp(-(b-a)\operatorname{Cap}(K)).
\end{aligned}$$

Similarly, the expected number of trajectories in $\mathscr{I}_{[a,b]}$ that hit K is equal to $(b-a)\operatorname{Cap}(K)$. We apply the above formulas to deduce the following simple 0–1 law.

LEMMA 3.4.   *Let G be a transient network, let $\mathscr{I}$ be the interlacement process on G. Then for all $a < b \in \mathbb{R}$ and every set of vertices $K \subseteq V$, we have*

$$\mathbb{P}(K \text{ is hit by infinitely many trajectories in } \mathscr{I}_{[a,b]}) = \mathbb{1}(\operatorname{Cap}(K) = \infty).$$

PROOF.   If $\operatorname{Cap}(K)$ is finite, then the expected number of trajectories in $\mathscr{I}_{[a,b]}$ that hit K is finite, so that the number of trajectories in $\mathscr{I}_{[a,b]}$ that hit K is finite a.s. Conversely, if $\operatorname{Cap}(K)$ is infinite, then there is a trajectory in $\mathscr{I}_{[b-2^{-n}, b-2^{-n-1}]}$ that hits K a.s. for every $n \geq 1$ a.s. Since $b - 2^{-n} \geq a$ for all but finitely many n, it follows that $\mathscr{I}_{[a,b]}$ hits K infinitely often a.s.   □

We next prove that any set that has a positive probability to be hit infinitely often by any single trajectory will in fact be hit by infinitely many trajectories. Recall

the *method of random paths* [33], Theorem 10.1: If G is an infinite network, A is a finite subset of G and Γ is a random infinite simple path in G starting at A, then

$$\mathrm{Cap}(A)^{-1} \leq \sum_{e \in E} \mathbb{P}(e \in \Gamma)^2.$$

In particular, if the sum on the right-hand side is finite for some random infinite simple path Γ starting at A, then the capacity of A is positive and G is therefore transient. Moreover, for every finite set A in a transient network, there exists a random infinite simple path Γ starting in A such that

$$\mathrm{Cap}(A)^{-1} = \sum_{e \in E} \mathbb{P}(e \in \Gamma)^2.$$

The following lemma is presumably well known, but we were unable to find a reference.

LEMMA 3.5.   *Let G be a transient network and let $K \subseteq V$. If the random walk on G hits K infinitely often with positive probability, then $\mathrm{Cap}(K) = \infty$.*

PROOF.   Let c be the conductance function of G. First, suppose that K is hit infinitely often with probability one. Let $X = \langle X_i \rangle_{i \geq 0}$ be a random walk on G started at a vertex of K, and let N_i be the ith time that X visits K. Define

$$c_K(u, v) = c(u) P_u(X_{N_1} = v)$$

for all $u, v \in K$, so that X_{N_i} is the random walk on the (nonlocally finite) network $H := ((K, K^2), c_K)$. [It follows by time-reversal that $c_K(u, v) = c_K(v, u)$ for every pair $u, v \in K$.] Note that if A is a finite subset of K, then the capacity of A considered as a set of vertices in H is the same as the capacity of A considered as a set of vertices in G. That is,

$$\mathrm{Cap}(A) = \sum_{v \in A} c(v) P_v\big(\langle X_i \rangle_{i \geq 0} \text{ does not return to } A\big)$$

$$= \sum_{v \in A} c(v) P_v\big(\langle X_{N_i} \rangle_{i \geq 0} \text{ does not return to } A\big).$$

Let $\langle K_n \rangle_{n \geq 1}$ be an increasing sequence of finite sets with $\bigcup_{n \geq 1} K_n = K$, and let Γ be a random infinite simple path in H starting at K_1 such that

$$\frac{1}{2} \sum_{u, v \in K} \mathbb{P}(\{u, v\} \in \Gamma)^2 = \mathrm{Cap}(K_1)^{-1} < \infty.$$

For each $n \geq 2$, let Γ_n be the subpath of Γ beginning at the last time Γ visits K_n. Then, by the monotone convergence theorem,

$$\mathrm{Cap}(K) = \lim_{n \to \infty} \mathrm{Cap}(K_n) \geq \lim_{n \to \infty} \left(\frac{1}{2} \sum_{u, v \in K} \mathbb{P}(\{u, v\} \in \Gamma_n)^2 \right)^{-1} = \infty.$$

This concludes the proof in this case.

Now suppose that K is hit infinitely often by the random walk on G with positive probability, and, for each vertex u of G, let $h(u)$ be the probability that a random walk on G started at u hits K infinitely often. We have that, for each two vertices u and v of G,

$$P_u(X_1 = v \mid X \text{ hits } K \text{ infinitely often})$$

$$= \frac{P_u(X_1 = v) P_v(X \text{ hits } K \text{ infinitely often})}{P_u(X \text{ hits } K \text{ infinitely often})}$$

$$= \frac{h(v)c(u, v)}{h(u)c(u)}.$$

It follows by an elementary calculation that the random walk on G conditioned to hit K infinitely often is reversible, and is equal to the random walk on the network $(G, \hat{c})$, where the conductances $\hat{c}$ are defined by

$$\hat{c}(u, v) = c(u, v)h(u)h(v), \qquad \hat{c}(u) = c(u)h(u)^2.$$

This is an example of Doob's h-transform. Since $h \leq 1$, Rayleigh monotonicity implies that the capacity of K with respect to the h-transformed conductances $\hat{c}$ is at most the capacity of K with respect to the original conductances c. Thus, we may conclude the proof by applying the argument of the previous paragraph to the h-transformed network. $\square$

The following lemma is an immediate consequence of Lemmas 3.4 and 3.5.

LEMMA 3.6. *Let G be a transient network, let $\mathcal{I}$ be the interlacement process on G. Then for all $a < b \in \mathbb{R}$ and every set of vertices $K \subseteq V$, we have*

$$\mathbb{P}(\text{infinitely many vertices of } K \text{ are hit by } \mathcal{I}_{[a,b]})$$

$$= \mathbb{P}(K \text{ is hit by infinitely many trajectories in } \mathcal{I}_{[a,b]})$$

$$= \mathbb{1}(\text{Cap}(K) = \infty).$$

PROOF. If $\text{Cap}(K) = \infty$, then K is hit by infinitely many trajectories in $\mathcal{I}_{[a,b]}$ a.s. by Lemma 3.4, and since each vertex of G is hit by only finitely many trajectories of $\mathcal{I}_{[a,b]}$ a.s., we deduce that infinitely many vertices of K are hit by $\mathcal{I}_{[a,b]}$ a.s. On the other hand, if $\text{Cap}(K) < \infty$, then K is hit by only finitely many trajectories in $\mathcal{I}_{[a,b]}$ a.s. by Lemma 3.4. Conditional on the number of these trajectories and the locations that they first hit K, the rest of each trajectory after it first hits K is a simple random walk. Since $\text{Cap}(K) < \infty$, it follows from Lemma 3.5 that each of these trajectories visits K only finitely often a.s., and hence that only finitely many vertices of K are hit by $\mathcal{I}_{[a,b]}$ a.s. $\square$

4. Interlacement Aldous–Broder. In this section, we describe the interlacement Aldous–Broder algorithm and investigate its basic properties. Let G be an infinite transient network. For each set $A \subseteq \mathcal{W}^* \times \mathbb{R}$, and each vertex $v \in V$, define

$$\tau_t(A, v) := \inf\{s \geq t : \exists(W, s) \in A \text{ such that } W \text{ hits } v\}.$$

Let $\mathscr{I}$ be the interlacement process on G and write $\tau_t(v) = \tau_t(\mathscr{I}, v)$. Let $\mathcal{A}$ be the set of subsets A of $\mathcal{W}^* \times \mathbb{R}$ that satisfy the property that for every for every vertex $v \in V$ and every $t \in \mathbb{R}$ there exists a unique trajectory $W_{\tau_t(A,v)}$ such that $(W_{\tau_t(A,v)}, \tau_t(A, v)) \in A$ and $W_{\tau_t(A,v)}$ hits v. It is clear that $\mathscr{I} \in \mathcal{A}$ a.s. Define $e_t(v) = e_t(\mathscr{I}, v)$ to be the oriented edge pointing into v that is traversed by the trajectory $W_{\tau_t(v)}$ as it enters v for the first time. For each $t \in \mathbb{R}$ and $T \in (t, \infty]$, we define the set $\mathsf{AB}_t^T(\mathscr{I}) \subseteq E$ by

$$(4.1) \qquad \mathsf{AB}_t^T(\mathscr{I}) := \{-e_t(v) : v \in V, \tau_t(v) \leq T\}.$$

We write $\mathsf{AB}_t(\mathscr{I}) = \mathsf{AB}_t^\infty(\mathscr{I})$. We define $\mathsf{AB}_t^T(A)$ similarly for all $A \in \mathcal{A}$. Let $\langle V_n \rangle_{n \geq 0}$ be an exhaustion of G, and for each $n \geq 0$ let $\mathscr{I}^n$ be defined as in Section 3.3. Since the process $\mathscr{I}^n$ is just a decomposition of a single random walk trajectory into excursions, Theorem 1.1 follows immediately from the following lemma together with the correctness of the classical Aldous–Broder algorithm.

LEMMA 4.1. *Let G be a transient network with exhaustion $\langle V_n \rangle_{n \geq 0}$ and let $\mathscr{I}$ be the interlacement process on G. Then $\mathsf{AB}_t^T(\mathscr{I}^n)$ converges weakly to $\mathsf{AB}_t^T(\mathscr{I})$ for each $t \in \mathbb{R}$ and $T \in [t, \infty]$.*

PROOF. Let $E^\rightarrow$ be the set of oriented edges of G, let S be a finite subset of $E^\rightarrow$, and consider the set

$$C_t^T(S) := \{A \subseteq \mathcal{W}^* \times \mathbb{R} : S \subseteq \mathsf{AB}_t^T(A)\}.$$

Let $\partial C_t^T(S)$ denote the topological boundary of $C_t^T(S)$. Observe that

$$\partial C_t^T(S) \subseteq \{A \subseteq \mathcal{A} : \tau_t(e^-) \in \{t, T\} \text{ for some } e \in S\} \cup (\mathcal{W}^* \times \mathbb{R} \setminus \mathcal{A}).$$

It follows that $\mathbb{P}(\mathscr{I} \in \partial C_t^T(S)) = 0$. The Portmanteau theorem [23], Theorem 13.16, therefore implies that

$$\mathbb{P}(S \subseteq \mathsf{AB}_t^T(\mathscr{I})) = \mathbb{P}(\mathscr{I} \in C_t^T(S))$$
$$= \lim_{n \to \infty} \mathbb{P}(\mathscr{I}^n \in C_t^T(S)) = \lim_{n \to \infty} \mathbb{P}(S \subseteq \mathsf{AB}_t^T(\mathscr{I}^n))$$

for every finite set $S \subseteq E^\rightarrow$. □

We next establish the basic properties of the process $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}} = \langle \mathsf{AB}_t^\infty(\mathscr{I}) \rangle_{t \in \mathbb{R}}$. We first recall that a process $\langle X_t \rangle_{t \in \mathbb{R}}$ is said to be *ergodic* if $\mathbb{P}(\langle X_t \rangle_{t \in \mathbb{R}} \in \mathscr{A}) \in$

$\{0, 1\}$ whenever $\mathscr{A}$ is an event that is shift invariant in the sense that $\langle X_t \rangle_{t \in \mathbb{R}} \in \mathscr{A}$ implies that $\langle X_{t+s} \rangle_{t \in \mathbb{R}} \in \mathscr{A}$ for every $s \in \mathbb{R}$. The process $\langle X_t \rangle_{t \in \mathbb{R}}$ is said to be *mixing* if for every two events $\mathscr{A}$ and $\mathscr{B}$,

$$\mathbb{P}(\langle X_t \rangle_{t \in \mathbb{R}} \in \mathscr{A} \text{ and } \langle X_{t+s} \rangle_{t \in \mathbb{R}} \in \mathscr{B}) \xrightarrow[s \to \infty]{} \mathbb{P}(\langle X_t \rangle_{t \in \mathbb{R}} \in \mathscr{A})\mathbb{P}(\langle X_t \rangle_{t \in \mathbb{R}} \in \mathscr{B}).$$

Every mixing process is clearly ergodic, but the converse need not hold in general [23], Section 20.5.

PROPOSITION 4.2. *Let G be a transient network and let $\mathscr{I}$ be the interlacement process on G. Then $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}} = \langle \mathsf{AB}_t^\infty(\mathscr{I}) \rangle_{t \in \mathbb{R}}$ is an ergodic, mixing, Markov process.*

PROOF. The fact that $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}}$ is a Markov process follows from the following identity, which immediately implies that $\langle \mathfrak{F}_s \rangle_{s \leq t}$ and $\langle \mathfrak{F}_s \rangle_{s \geq t}$ are conditionally independent given $\mathfrak{F}_t$ for each $t \in \mathbb{R}$: whenever $t \in \mathbb{R}$ and $s \geq 0$,

$$(4.2) \qquad \mathsf{AB}_{t-s}(\mathscr{I}) = \mathsf{AB}_{t-s}^t(\mathscr{I}) \cup \{e \in \mathsf{AB}_t(\mathscr{I}) : e^- \text{ is not hit by } \mathscr{I}_{[t-s,t)}\}.$$

We now prove that $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}}$ is mixing. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$ be two increasing sequences of real numbers, and let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_m$ be finite subsets of $E^\rightarrow$. Let K denote the set of endpoints of edges in the union $\bigcup_{i=1}^n A_i$. Let s be sufficiently large that $b_1 + s \geq a_n$. Let $\mathscr{A} = \{A_i \subseteq \mathsf{AB}_{a_i}^\infty(\mathscr{I}) \text{ for all } 1 \leq i \leq n\}$, $\mathscr{B}_s = \{B_i \subseteq \mathsf{AB}_{b_i+s}^\infty(\mathscr{I}) \text{ for all } 1 \leq i \leq m\}$, and $\mathscr{A}'_s = \{A_i \subseteq \mathsf{AB}_{a_i}^{b_1+s}(\mathscr{I}) \text{ for all } 1 \leq i \leq n\} \subseteq \mathscr{A}$. The events $\mathscr{A}'_s$ and $\mathscr{B}_s$ are independent and

$$\mathbb{P}(\mathscr{A} \setminus \mathscr{A}'_s) \leq \mathbb{P}(\mathscr{I}_{[a_n, b_1+s]} \text{ does not hit some vertex in } K)$$

$$\leq \sum_{v \in K} \exp(-(b_1 + s - a_n)\mathrm{Cap}(v)).$$

We deduce that

$$\left| \mathbb{P}(\mathscr{A} \cap \mathscr{B}_s) - \mathbb{P}(\mathscr{A})\mathbb{P}(\mathscr{B}_s) \right|$$
$$\leq \left| \mathbb{P}(\mathscr{A} \cap \mathscr{B}_s) - \mathbb{P}(\mathscr{A}'_s \cap \mathscr{B}_s) \right| + \left| \mathbb{P}(\mathscr{A}'_s \cap \mathscr{B}_s) - \mathbb{P}(\mathscr{A})\mathbb{P}(\mathscr{B}_s) \right|$$
$$= \left| \mathbb{P}(\mathscr{A} \cap \mathscr{B}_s) - \mathbb{P}(\mathscr{A}'_s \cap \mathscr{B}_s) \right| + \left| \mathbb{P}(\mathscr{A}'_s)\mathbb{P}(\mathscr{B}_s) - \mathbb{P}(\mathscr{A})\mathbb{P}(\mathscr{B}_s) \right|$$
$$\leq 2 \sum_{v \in K} \exp(-(b_1 + s - a_n)\mathrm{Cap}(v)) \xrightarrow[s \to \infty]{} 0.$$

Since events of the form $\{A_i \subseteq \mathsf{AB}_{a_i}^\infty(\mathscr{I}) \text{ for all } 1 \leq i \leq n\}$ generate the Borel σ-algebra on the space of $(E^\rightarrow)^{\{0,1\}}$-valued processes, it follows that $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}}$ is mixing and, therefore, ergodic. □

5. Proofs of Theorems 2.1, 2.2, 2.4.   Recall that an *end* of a tree is an equivalence class of infinite simple paths in the tree, where two infinite simple paths are equivalent if their traces have finite symmetric difference. Similarly, if $\mathfrak{F}$ is a spanning forest of a network G, we define an end of $\mathfrak{F}$ to be an equivalence class of infinite simple paths in G that eventually only use edges of $\mathfrak{F}$, where, again, two infinite simple paths are equivalent if their traces have finite symmetric difference. If an infinite tree T is oriented so that every vertex of the tree has exactly one oriented edge emanating from it, then there is exactly one end ξ of T for which the paths representing ξ eventually follow the orientation of T. We call this end the *primary end* of T. We call ends of T that are not the primary end *excessive ends* of T. Note that if ξ is an excessive end of T and γ is a simple path representing ξ, then all but finitely many of the edges traversed by the path γ are traversed in the opposite direction to their orientation in T.

Given an oriented forest $\mathfrak{F}$ and a vertex v in $\mathfrak{F}$, we define the *past of v in $\mathfrak{F}$*, denoted $\mathrm{past}_{\mathfrak{F}}(v)$, to be the subgraph of $\mathfrak{F}$ spanned by the set of vertices u of $\mathfrak{F}$ such that there exists an oriented path from u to v in $\mathfrak{F}$. Observe that if $\mathfrak{F}$ is oriented so that every vertex has exactly one oriented edge emanating from it, then every component of $\mathfrak{F}$ is one-ended if and only if the past of every vertex is finite.

The proofs of Theorems 2.1 and 2.4 both rely on the following criterion. See Figure 1 for an illustration of the proof. If $\mathscr{I}$ is the interlacement process on a transient network G, we write $\mathcal{I}_{[a,b]}$ for the set of vertices of G hit by $\mathscr{I}_{[a,b]}$

LEMMA 5.1.   *Let G be a transient network, let $\mathscr{I}$ be the interlacement process on G, and let $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}} = \langle \mathsf{AB}_t(\mathscr{I}) \rangle_{t \in \mathbb{R}}$. If the connected component containing v of $\mathrm{past}_{\mathfrak{F}_0}(v) \setminus \mathcal{I}_{[-\varepsilon,0]}$ is finite a.s. for every vertex v of G and every $\varepsilon > 0$, then every component of $\mathfrak{F}_0$ is one-ended a.s.*

PROOF.   If u is in the past of v in $\mathfrak{F}_{-\varepsilon}$, then $\tau_{-\varepsilon}(u) \geq \tau_{-\varepsilon}(v)$. Thus, on the event that v is not hit by $\mathscr{I}_{[-\varepsilon,0]}$, the past of v in $\mathfrak{F}_{-\varepsilon}$ is equal to the component containing v in the subgraph of $\mathrm{past}_{\mathfrak{F}_0}(v)$ induced by the complement of $\mathcal{I}_{[-\varepsilon,0]}$
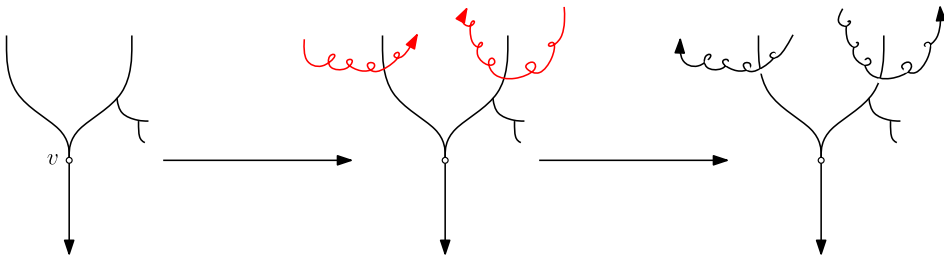

FIG. 1.   *A schematic illustration of the proof of Lemma 5.1. Left: a tree in $\mathfrak{F}_0$ with two excessive ends in the past of the vertex v. Centre: each of the excessive ends is hit by a trajectory of $\mathscr{I}_{[-\varepsilon,0]}$ (red), but v is not hit by such a trajectory. Right: the resulting forest $\mathfrak{F}_{-\varepsilon}$.*

(see Figure 1). By assumption, the connected component containing v in this subgraph is finite a.s., and so, by stationarity,

$$\mathbb{P}\big(\text{past}_{\mathfrak{F}_0}(v) \text{ is infinite}\big) = \mathbb{P}\big(\text{past}_{\mathfrak{F}_{-\varepsilon}}(v) \text{ is infinite}\big)$$

$$\leq \mathbb{P}\big(v \in \mathcal{I}_{[-\varepsilon,0]}\big) = 1 - e^{-\varepsilon \operatorname{Cap}(v)} \xrightarrow[\varepsilon \to 0]{} 0.$$

Since v was arbitrary, we deduce that every component of $\mathfrak{F}_0$ is one-ended a.s. □

The proof of Theorem 2.1 requires both of the following theorems.

THEOREM 5.2 (Lyons, Morris and Schramm [26]; Lyons and Peres [27], Theorem 6.41). *Let G be a network satisfying an anchored $f(t)$-isoperimetric inequality, where f is an increasing function such that $f(t) \leq t$, $f(2t) \leq \alpha f(t)$ for some constant α, and $\int_1^\infty f(t)^{-2} dt < \infty$. Then for every vertex v of G there exists a positive constant c_v such that, for every connected set K containing v,*

$$\operatorname{Cap}(K) \geq \frac{c_v^2}{4\alpha^2} \left(\int_{|K|}^\infty \frac{1}{f(t)^2} \right)^{-1}.$$

In particular, G is transient.

THEOREM 5.3 (Morris [31], Theorem 9: WUSF components are recurrent). *Let G be an infinite network with edge conductances bounded above. Then every component of the wired uniform spanning forest of G is recurrent a.s.*

An equivalent statement of Theorem 5.3 is the following.

LEMMA 5.4. *Let G be an infinite network with $\inf_e c(e) > 0$. Then every component of the wired uniform spanning forest of G is recurrent when given unit conductances a.s.*

PROOF. Form a network $\hat{G}$ by replacing each edge e of G with $\lceil c(e) \rceil$ parallel edges each with conductance $c(e)/\lceil c(e) \rceil$. It follows immediately from the definition of the UST of a network that the WUSF $\hat{\mathfrak{F}}$ of $\hat{G}$ may be coupled with the WUSF $\mathfrak{F}$ of G so that an edge e of G is contained in $\mathfrak{F}$ if and only if one of the edges corresponding to e in $\hat{G}$ is contained in $\hat{\mathfrak{F}}$. Since $\hat{G}$ has edge conductances bounded above, Theorem 5.3 implies that every component of $\hat{\mathfrak{F}}$ is recurrent a.s. Since the edge conductances of $\hat{G}$ are bounded away from zero, it follows by Rayleigh monotonicity that every component of $\hat{\mathfrak{F}}$ is recurrent a.s. when given unit conductances, and consequently that the same is true of $\mathfrak{F}$. □

PROOF OF THEOREM 2.1. By Theorem 5.2, G is transient. Let $\mathscr{I}$ be the interlacement process on G. Let v be a fixed vertex of G. Then for each vertex u of

G contained in the same component of $\mathfrak{F} = \mathsf{AB}_0(\mathscr{I})$ as v, the conditional probability given $\mathfrak{F}$ that u is connected to v in $\mathfrak{F} \setminus \mathcal{I}_{[-\varepsilon,0]}$ is equal to $\exp(-\varepsilon \operatorname{Cap}(\gamma_{u,v}))$, where $\gamma_{u,v}$ is the trace of the path connecting u to v in $\mathfrak{F}$. Write $d_{\mathfrak{F}}(u,v)$ for the graph distance in $\mathfrak{F}$ between two vertices u and v in G. Since the conductances of G are bounded below by some positive constant δ, we have that $|\gamma_{u,v}| \geq \delta d_{\mathfrak{F}}(u,v)$ and so, by Theorem 5.2,

$$\mathbb{P}(u \text{ connected to } v \text{ in } \mathfrak{F} \setminus \mathcal{I}_{[-\varepsilon,0]} \mid \mathfrak{F})$$

(5.1)
$$\leq \mathbb{1}(u \text{ connected to } v \text{ in } \mathfrak{F}) \exp\left(-\varepsilon \frac{c_v^2}{4\alpha^2} \left(\int_{\delta d_{\mathfrak{F}}(u,v)}^{\infty} \frac{1}{f(t)^2}\right)^{-1}\right).$$

LEMMA 5.5. *Let T be an infinite tree, let v be a vertex of T and let ω be a random subgraph of T. For each vertex u of G, let $\|u\|$ denote the distance between u and v in T, and suppose that there exists a function $p : \mathbb{N} \to [0,1]$ such that*

$$\mathbb{P}(u \text{ is connected to } v \text{ in } \omega) \leq p(\|u\|)$$

for every vertex u in T and

$$\sum_{n \geq 1} p(n) < \infty.$$

Then

$$\mathbb{P}(\text{The component of } v \text{ in } \omega \text{ is infinite}) \leq \operatorname{Cap}(v) \sum_{n \geq 1} p(n).$$

In particular, if T is recurrent then the component containing v in ω is finite a.s.

PROOF. Suppose that the component containing v in ω is infinite with positive probability; the inequality holds trivially otherwise. Denote this event $\mathscr{A}$. Fix a drawing of T in the plane rooted at v. On the event $\mathscr{A}$, let $\Gamma = \Gamma(\omega)$ be the leftmost simple path from v to infinity in ω. Observe that

$$\mathbb{P}(u \in \Gamma \mid \mathscr{A}) \leq \frac{\mathbb{P}(u \text{ is connected to } v \text{ in } \omega)}{\mathbb{P}(\mathscr{A})} \leq \frac{p(n)}{\mathbb{P}(\mathscr{A})}$$

and $\sum_{\|u\|=n} \mathbb{P}(u \in \Gamma \mid \mathscr{A}) = 1$, so that

$$\sum_{\|u\|=n} \mathbb{P}(u \in \Gamma \mid \mathscr{A})^2 \leq \frac{p(n)}{\mathbb{P}(\mathscr{A})}$$

and hence

$$\sum_{e \in E} \mathbb{P}(e \in \Gamma \mid \mathscr{A})^2 = \sum_{u \in V \setminus \{v\}} \mathbb{P}(u \in \Gamma \mid \mathscr{A})^2$$

$$= \sum_{n \geq 1} \sum_{\|u\|=n} \mathbb{P}(u \in \Gamma \mid \mathscr{A})^2 \leq \frac{1}{\mathbb{P}(\mathscr{A})} \sum_{n \geq 1} p(n).$$

Applying the method of random paths (taking our measure on random paths to be the conditional distribution of Γ given $\mathscr{A}$), we deduce that

$$\mathrm{Cap}(v) \geq \left(\frac{1}{\mathbb{P}(\mathscr{A})} \sum_{n \geq 1} p(n) \right)^{-1},$$

which rearranges to give the desired inequality. $\quad\square$

Comparing sums with integrals, hypothesis (2) of Theorem 2.1 implies that

$$\sum_{n \geq 1} \exp\left(-\varepsilon \frac{c_v^2}{4\alpha^2} \left(\int_{\delta n}^{\infty} \frac{1}{f(t)^2} \right)^{-1} \right) < \infty$$

for every $\varepsilon > 0$, and we deduce from 5.1, Lemma 5.5 and Theorem 5.4 that the component containing v in $\mathfrak{F} \setminus \mathcal{I}_{[-\varepsilon, 0]}$ is finite a.s. Since v was arbitrary, every component of $\mathfrak{F} \setminus \mathcal{I}_{[-\varepsilon, 0]}$ is finite a.s. We conclude by applying Lemma 5.1. $\quad\square$

5.1. *Unimodular random rooted graphs.*

PROOF OF THEOREM 2.4.   Let (G, ρ) be a transient unimodular random rooted network, let $\mathscr{I}$ be the interlacement process on G and let $\mathfrak{F} = \mathsf{AB}_0(\mathscr{I})$. It is known [1], Theorem 6.2, Proposition 7.1, that every component of $\mathfrak{F}$ has at most two ends a.s. Suppose for contradiction that $\mathfrak{F}$ contains a two-ended component with positive probability. The *trunk* of a two-ended component of $\mathfrak{F}$ is defined to be the unique doubly infinite simple path that is contained in the component. Define trunk($\mathfrak{F}$) to be the set of vertices of G that are contained in the trunk of some two-ended component of $\mathfrak{F}$. For each vertex v of G, let $e(v)$ be the unique oriented edge of G emanating from v that is contained in $\mathfrak{F}$. For each vertex $v \in \mathrm{trunk}(\mathfrak{F})$, let $s(v)$ be the unique vertex in trunk($\mathfrak{F}$) that has $e(s(v))^+ = v$, and let $s^n(v)$ be defined recursively for $n \geq 0$ by $s^0(v) = v$, $s^{n+1}(v) = s(s^n(v))$.

Let $\varepsilon > 0$. We claim that $s^n(\rho) \in \mathcal{I}_{[-\varepsilon, 0]}$ for infinitely many n a.s. on the event that $\rho \in \mathrm{trunk}(\mathfrak{F})$. Let $k \geq 1$ and define the mass transport

$$f_k(G, u, v, \mathfrak{F}, \mathcal{I}_{[-\varepsilon, 0]}) = \mathbb{1}\left(\begin{array}{l} u \text{ is in the trunk of its component in } \mathfrak{F}, \\ v = s^k(u) \text{ and } \mathcal{I}_{[-\varepsilon, 0]} \cap \{ s^n(v) : n \geq 0 \} \neq \varnothing \end{array} \right).$$

Applying the mass-transport principle to f_k, we deduce that

$$\mathbb{P}\big(\mathcal{I}_{[-\varepsilon, 0]} \cap \{ s^n(\rho) : n \geq 0 \} \neq \varnothing \mid \rho \in \mathrm{trunk} \big)$$
$$= \mathbb{P}\big(\mathcal{I}_{[-\varepsilon, 0]} \cap \{ s^n(\rho) : n \geq k \} \neq \varnothing \mid \rho \in \mathrm{trunk} \big)$$

for all $k \geq 0$. (Here, we are using the fact that $(G, \rho, \mathfrak{F}, \mathcal{I}_{[-\varepsilon, 0]})$ is a unimodular rooted marked graph; see [1] for appropriate definitions.) By taking the limit as $k \to \infty$, we deduce that

$$\mathbb{P}\big(s^n(\rho) \in \mathcal{I}_{[-\varepsilon, 0]} \text{ for infinitely many } n \mid \rho \in \mathrm{trunk} \big)$$
$$= \mathbb{P}\big(\mathcal{I}_{[-\varepsilon, 0]} \cap \{ s^n(\rho) : n \geq 0 \} \neq \varnothing \mid \rho \in \mathrm{trunk} \big).$$

It follows that $s^n(\rho) \in \mathcal{I}_{[-\varepsilon,0]}$ for infinitely many n almost surely on the event that $\rho \in \text{trunk}(\mathfrak{F}) \cap \mathcal{I}_{[-\varepsilon,0]}$. Since $\rho \in \text{trunk}(\mathfrak{F}) \cap \mathcal{I}_{[-\varepsilon,0]}$ with positive probability conditional on $\mathfrak{F}$ and the event that $\rho \in \text{trunk}(\mathfrak{F})$, it follows from Lemma 3.6 that infinitely many vertices of $\{s^n(\rho) : n \geq 1\}$ are hit by $\mathcal{I}_{[-\varepsilon,0]}$ a.s. on the event that $\rho \in \text{trunk}(\mathfrak{F})$.

It follows from [1], Lemma 2.3, that for every vertex $v \in \text{trunk}(\mathfrak{F})$, $s^n(v) \in \mathcal{I}_{[-\varepsilon,0]}$ for infinitely many n a.s., and consequently that the component containing v in $\text{past}_{\mathfrak{F}}(v) \setminus \mathcal{I}_{[-\varepsilon,0]}$ is finite for every vertex v of G a.s. We conclude by applying Lemma 5.1. $\square$

5.2. *Excessive ends.*

PROOF OF THEOREM 2.2.   We may assume that G is transient: if not, the WUSF of G is connected, the number of excessive ends of G is tail measurable, and the claim follows by tail-triviality of the WUSF [5]. Let $\mathscr{I}$ be the interlacement process on G and let $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}} = \langle \mathsf{AB}_t(\mathscr{I}) \rangle_{t \in \mathbb{R}}$. The event that $\mathfrak{F}_0$ has uncountably many ends is tail measurable, and hence has probability either 0 or 1, again by tail-triviality of the WUSF. If the number of ends of $\mathfrak{F}_0$ is uncountable a.s., then $\mathfrak{F}_0$ must also have uncountably many excessive ends a.s., since the number of components of $\mathfrak{F}_0$ is countable. Thus, it suffices to consider the case that $\mathfrak{F}_0$ has countably many ends a.s.

For each $t \in \mathbb{R}$, we call an excessive end ξ of $\mathfrak{F}_t$ *indestructible* if $\text{Cap}(\{\gamma_i : i \geq 0\})$ is finite for some (and hence every) simple path $\langle \gamma_i \rangle_{i \geq 0}$ in G representing ξ, and *destructible* otherwise. Given a simple path $\gamma = \langle \gamma_i \rangle_{i \geq 0}$, write $\gamma_{i,i+1}$ for the oriented edge that is traversed by γ as it moves from γ_i to γ_{i+1}, and let $\gamma_{i,i-1} = -\gamma_{i-1,i}$. Observe, as we did at the beginning of Section 5, that a simple path $\langle \gamma_i \rangle_{i \geq 0}$ in G represents an excessive end of $\mathfrak{F}_t$ if and only if $e_t(\gamma_i, \mathscr{I}) = \gamma_{i,i-1}$ for all sufficiently large values of i (equivalently, if and only if the reversed oriented edges $-\gamma_{i,i+1}$ are contained in $\mathfrak{F}_t$ for all sufficiently large values of i). Since $\mathfrak{F}_0$ has countably many ends a.s. by assumption, it follows from Lemma 3.4 that for every destructible end ξ of $\mathfrak{F}_0$ and every infinite simple path $\langle \gamma_i \rangle_{i \geq 0}$ in G representing ξ, the trace $\{\gamma_i : i \geq 0\}$ of γ is hit by $\mathscr{I}_{[-\varepsilon,0]}$ infinitely often a.s. for every $\varepsilon > 0$. Recall from the proof of Lemma 5.1 that, on the event that $v \notin \mathcal{I}_{[-\varepsilon,0]}$, the past of v in $\mathfrak{F}_{-\varepsilon}$ is contained in subgraph of $\text{past}_{\mathfrak{F}_0}(v)$ induced by the complement of $\mathcal{I}_{[-\varepsilon,0]}$. It follows that v a.s. does not have any destructible ends in its past in $\mathfrak{F}_{-\varepsilon}$ a.s. on the event that $v \notin \mathcal{I}_{[-\varepsilon,0]}$, and so, by stationarity,

$$\mathbb{P}(v \text{ has a destructible end in its past in } \mathfrak{F}_0) \leq \mathbb{P}(v \in \mathcal{I}_{[-\varepsilon,0]}) \xrightarrow[\varepsilon \to 0]{} 0.$$

Since the vertex v was arbitrary, we deduce that $\mathfrak{F}_0$ does not contain any destructible excessive ends a.s.

Since every excessive end of $\mathfrak{F}_0$ is indestructible a.s., it follows from Lemma 3.6 that for every excessive end ξ of $\mathfrak{F}_0$ and every path $\langle v_i \rangle$ representing ξ, only finitely

many of the vertices v_i are hit by $\mathscr{I}_{[t,0]}$ a.s. for every $t \leq 0$. Since $\mathfrak{F}_0$ has at most countably many excessive ends a.s., we deduce that every path $\langle v_i \rangle_{i \geq 0}$ that represents an excessive end of $\mathfrak{F}_0$ also represents an excessive end of $\mathfrak{F}_t$ for every $t \leq 0$. In particular, the cardinality of the set of excessive ends of $\mathfrak{F}_t$ is at least the cardinality of the set of excessive ends of $\mathfrak{F}_0$ a.s. for every $t \leq 0$. Since, by Proposition 4.2, $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}}$ is stationary and ergodic, we deduce that the cardinality of the set of excessive ends of $\mathfrak{F}_0$ is a.s. equal to some constant. $\quad \square$

PROOF OF COROLLARY 2.3. Let G'' be the network that has all the edges of both G and G'. By symmetry, it suffices to show that the wired uniform spanning forests of G and G'' have the same number of excessive ends a.s. Let $\mathfrak{F}$ and $\mathfrak{F}''$ be samples of the WUSFs of G and G'', respectively, and let A be the set of edges of G'' that are not edges of G. Since A is finite and G is connected, the event $\mathscr{A} = \{A \cap \mathfrak{F}'' = \varnothing\}$ has positive probability (this implication is easily proven in several ways, for example, using either Wilson's algorithm, the Aldous–Broder algorithm or the transfer current theorem [8]). The spatial Markov property of the WUSF implies that the conditional distribution of $\mathfrak{F}''$ given $\mathscr{A}$ is equal to the distribution of $\mathfrak{F}$, and in particular the conditional distribution of the number of excessive ends of $\mathfrak{F}''$ given $\mathscr{A}$ has the same distribution as the number of excessive ends of $\mathfrak{F}$. The claim now follows from Theorem 2.2. $\quad \square$

6. Ends and rough isometries. Recall that a *rough isometry* from a graph $G = (V, E)$ to a graph $G' = (V', E')$ is a function $\phi : V \to V'$ such that, letting d_G and $d_{G'}$ denote the graph distances on V and V', there exist positive constants α and β such that the following conditions are satisfied:

1. (*ϕ roughly preserves distances*.) For every pair of vertices $u, v \in V$,

$$\alpha^{-1} d_G(u, v) - \beta \leq d_{G'}(\phi(u), \phi(v)) \leq \alpha d_G(u, v) + \beta.$$

2. (*ϕ is almost surjective*.) For every vertex $v' \in V'$, there exists a vertex $v \in V$ such that $d_G(\phi(v), v') \leq \beta$.

For background on rough isometries, see [27], Section 2.6. The final result of this paper answers negatively Question 7.6 of Lyons, Morris and Schramm [26], which asked whether the property of having one-ended WUSF components is preserved under rough isometry of graphs.

THEOREM 6.1. *There exist two rough-isometric, bounded degree graphs G and G' such that every component of the wired uniform spanning forest of G has one-end a.s., but the wired uniform spanning forest of G' contains a component with uncountably many ends a.s.*

The proof of Theorem 6.1 uses Wilson's algorithm rooted at infinity. We refer the reader to [27], Proposition 10.1, for an exposition of this algorithm. The description as a branching process of the past of the WUSF of a regular trees with

height-dependent exponential edge stretching is adapted from [5], Section 11, and first appeared in the work of Häggström [14].

PROOF OF THEOREM 6.1.   Let $T = (V, E)$ be a 3-regular tree with root ρ. We write $\|u\|$ for the distance between $u \in V$ and ρ. For each positive integer k, let $T_k = (V_k, E_k)$ denote the tree obtained from T by replacing every edge connecting a vertex u of T to its parent by a path of length $k^{\|u\|}$. We identify the degree 3 vertices of T_k with the vertices of T. For each vertex $u \in V$, let $S(u)$ be a binary tree with root ρ_u and let $S_k(u)$ be the tree obtained from $S(u)$ by replacing every edge with a path of length $k^{\|u\|+1}$. Finally, for each pair of positive integers (k, m), let G_k^m be the graph obtained from T_k by, for each vertex $u \in V$, adding a path of length $k^{\|u\|+1}$ connecting u to ρ_u and then replacing every edge in each of these added paths and every edge in each of the trees $S_k(u)$ by m parallel edges. The vertex degrees of G_k^m are bounded by $3 + m$, and the identity map is an isometry (and hence a rough isometry) between G_k^m and $G_k^{m'}$ whenever k, m and m' are positive integers. See Figure 2 for an illustration.

Let k and m be positive integers. Observe that for every vertex v of T and every child u of v in T, the probability that simple random walk on G_k^m started at u ever hits v does not depend on the choice of v or u. Denote this probability $p(m, k)$. We can bound $p(m, k)$ as follows:

$$(6.1) \qquad \frac{k}{k+2+m} \leq p(m, k) \leq \frac{k+2}{k+2+2m}.$$

The lower bound of $k/(k+2+m)$ is exactly the probability that the random walk started at u visits v before visiting any other vertex of T or visiting ρ_u. The upper
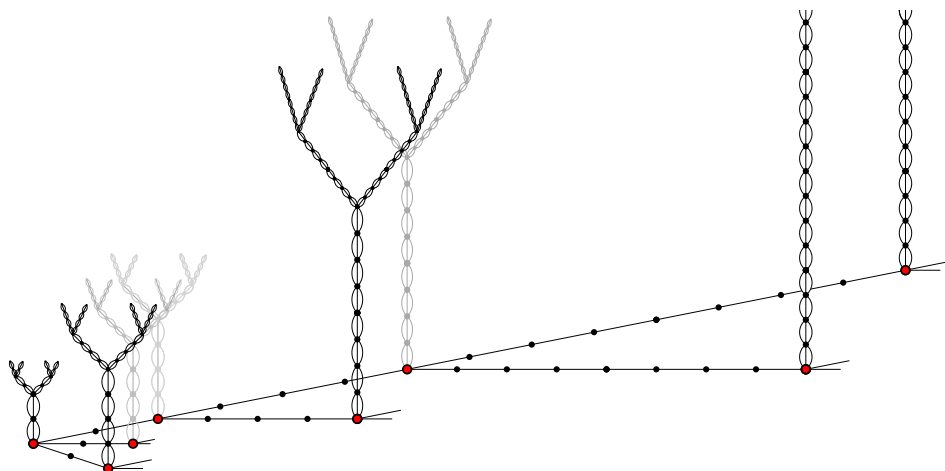

FIG. 2.   *An illustration of the graph G_2^3. Red vertices correspond to vertices of the 3-regular tree T. Only three generations of each of the trees $S(v)$ are pictured.*

bound of $(k + 2)/(k + 2 + 2m)$ is exactly the probability that the random walk started at u ever visits a neighbour of u in T. This can be computed by a straightforward network reduction (see [27] for background): The conductance to infinity from the root of a binary tree is 1, so that, by the series and parallel laws, the effective conductance to infinity from u in the subgraph of G_k^m spanned by the vertices of $S_k(u)$ and the path connecting u to $S_k(u)$ is $2mk^{-\|u\|-1}$. On the other hand, the effective conductance between u and its parent v is $k^{-\|u\|}$, while the effective conductance between u and each of its children is $k^{-\|u\|-1}$. It follows that the probability that a random walk started at u ever visits a neighbour of u in T is exactly

$$\frac{k^{-\|u\|} + 2k^{-\|u\|-1}}{k^{-\|u\|} + 2k^{-\|u\|-1} + 2mk^{-\|u\|-1}} = \frac{k+2}{k+2+2m}$$

as claimed.

Let $\mathfrak{F}_k^m$ be a sample of $\mathsf{WUSF}_{G_k^m}$ generated using Wilson's algorithm on G_k^m, starting with the root ρ of T. Let ξ be the loop-erased random walk in G_k^m beginning at ρ that is used to start our forest. The path ξ includes either one or none of the neighbours of ρ in T and so, in either case, there are at least two neighbours v_1 and v_2 of ρ in T that are not contained in this path. Continuing to run Wilson's algorithm from v_1 and v_2, we see that, conditional on ξ, the events $A_1 = \{v_1$ is in the past of ρ in $\mathfrak{F}_k^m\}$ and $A_2 = \{v_2$ is in the past of ρ in $\mathfrak{F}_k^m\}$ are independent and each have probability $p(m, k)$. Furthermore, on the event A_i, we add only the path connecting v_i and ρ in G_k^m to the forest during the corresponding step of Wilson's algorithm. Recursively, we see that the restriction to T of the past of ρ in $\mathfrak{F}$ contains a Galton–Watson branching process with Binomial offspring distribution $(2, p(m, k))$. If $k \geq m + 3$, this branching process is supercritical, so that $\mathfrak{F}_k^m$ contains a component with uncountably many ends with positive probability. By tail triviality of the WUSF [27], Theorem 10.18, $\mathfrak{F}_k^m$ contains a component with uncountably many ends a.s. when $k \geq m + 3$.

On the other hand, a similar analysis shows that the restriction to T of past of ρ in $\mathfrak{F}_k^m$ is stochastically dominated by a binomial $(3, p(m, k))$ branching process. (The 3 here is to account for the possibility that every child of ρ in T is in its past). If $m \geq k + 2$, this branching process is either critical or subcritical, and we conclude that the restriction to T of the past of ρ in $\mathfrak{F}_k^m$ is finite a.s. Condition on this restriction. Similarly, again to the above, the restriction to $S_k(v)$ of the past of v in $\mathfrak{F}_m^k$ is stochastically dominated by a critical binomial $(2, 1/2)$ branching process for each vertex v of T, and is therefore finite a.s. We conclude that the past of ρ in $\mathfrak{F}_k^m$ is finite a.s. whenever $m \geq k + 2$. A similar analysis shows that the past in $\mathfrak{F}_k^m$ of every vertex of G_k^m is finite a.s., and consequently that every component of $\mathfrak{F}_k^m$ is one-ended a.s. whenever $m \geq k + 2$.

Since $4 \geq 1 + 3$ and $6 \geq 4 + 2$, the wired uniform spanning forest $\mathfrak{F}_4^1$ of G_4^1 contains an infinitely-ended component a.s., and every component of the wired uniform spanning forest $\mathfrak{F}_4^6$ of G_4^6 is one-ended a.s. $\square$

We note that an example without multiple edges can be obtained by replacing all edges in both G and G' with paths of length two.

7. Closing discussion and open problems.

7.1. *The FMSF of the interlacement ordering.*   One way to think about the interlacement Aldous–Broder algorithm is as follows. Given the interlacement process $\mathscr{I}$ on a transient network G, we can define a total ordering of the edges of G according to the order in which they are traversed by the trajectories of $\mathscr{I}_{[0,\infty)}$. That is, we define a strict total ordering $\prec$ of E by setting $e_1 \prec e_2$ if and only if either e_1 is first traversed by a trajectory of $\mathscr{I}_{[0,\infty)}$ at a smaller time than e_2 is first traversed by a trajectory of $\mathscr{I}_{[0,\infty)}$, or if e_1 and e_2 are both traversed for the first time by the same trajectory of $\mathscr{I}_{[0,\infty)}$, and this trajectory traverses e_1 before it traverses e_2. We call $\prec$ the *interlacement ordering* of the edge set E associated to $\mathscr{I}_{[0,\infty)}$.

It is easily verified that $\mathsf{AB}_0(\mathscr{I})$ is the *wired minimal spanning forest* of G with respect to the interlacement ordering. That is, an edge $e \in E$ is included in $\mathsf{AB}_0(\mathscr{I})$ if and only if there does not exist either a finite cycle or a bi-infinite path in G containing e for which e is the $\prec$-maximal element. See [27] for background on minimal spanning forests. In light of this, it is natural to wonder what might be said about the *free* minimal spanning forest of the interlacement ordering, that is, the spanning forest of G that includes an edge $e \in E$ if and only if there does not exist a finite cycle in G containing e for which e is the $\prec$-maximal element. Indeed, if this forest were the FUSF of G, it would be possible to use this construction to solve the monotone coupling problem [27], Question 10.6 (see also [6, 29, 30]) and perhaps also the almost-connectivity problem [27], Question 10.12 (e.g., by adapting the methods of [28], Theorem 1.3, but using the set of edges visited by an independent interlacement process over a short time window $[0, \varepsilon]$ in the place of the Bernoulli ε percolation used there).

Unfortunately, there is little reason for this to be the case other than wishful thinking. Indeed, let $\mathscr{I}$ be the interlacement process on a transient network G, and define

$$t_c = \inf\{t \in (0, \infty) : \mathcal{I}_{[0,t]} \text{ is connected a.s.}\}.$$

Teixeira and Tykesson [42] proved that if G is transitive, then t_c is positive if and only if G is nonamenable. (The amenable case of their result generalises the corresponding result for $\mathbb{Z}^d$, due to Sznitman [39].) We can apply this result to prove that the free minimal spanning forest of the interlacement ordering is distinct from the WUSF on any nonamenable transitive graph: This is similar to how the usual FMSF and WMSF (where the edge weights are i.i.d. uniform random variables) are distinct if and only there is a nonempty nonuniqueness phase for Bernoulli bond percolation [28]. Since there are many nonamenable transitive graphs where the WUSF and FUSF coincide (e.g., the product of a 3-regular tree with $\mathbb{Z}$; see [27],

Chapter 10), we deduce that there are transitive graphs (indeed, Cayley graphs) for which the FUSF does not coincide with the free minimal spanning forest of the interlacement ordering.

We now give a quick sketch of this argument. Suppose that G is a transitive nonamenable graph, and let ρ be a fixed root vertex of G. Observe that for every $t < t_c$, there must exist a connected component K of $\mathcal{I}_{[0,t)}$ and a vertex u of G such that a random walk started at u has a positive probability not to hit the component: If not, we would have that $\mathcal{I}_{[0,s]}$ was connected for every $s > t$, contradicting the assumption that $t < t_c$. Moreover, by finding a path from u to K and considering the last vertex of the path before we reach K, the vertex u can be taken to be adjacent to the component K. By translation invariance, for each $t < t_c$ there is a positive probability that there exists a connected component K of $\mathcal{I}_{[0,t]}$ adjacent to ρ such that a random walk started at ρ has a positive probability to never visit K.

Let $\tau = \tau_0(\rho)$ be the first time after time zero that ρ is hit by a trajectory of $\mathcal{I}$. Denote this trajectory by W. No other trajectories of $\mathcal{I}$ appear at time τ a.s. Let $(\tilde{W}_1, \tilde{\tau}_1)$ and $(\tilde{W}_2, \tilde{\tau}_2)$ be independent of each other and of $\mathcal{I}$, and have law given by the conditional distribution of (W, τ) given that $\tau < t_c$. It is clear that the law of $\mathcal{I}' = \mathcal{I} \cup \{(\tilde{W}_1, \tilde{\tau}_1), (\tilde{W}_2, \tilde{\tau}_2)\}$ is absolutely continuous with respect to the law of $\mathcal{I}$. Conditional on $\tilde{\tau}_1$, $\tilde{\tau}_2$ and the event that $\tilde{\tau}_1 < \tilde{\tau}_2$, we have that the following events all occur together with positive probability:

- There exists a connected component K of $\mathcal{I}_{[0,\tilde{\tau}_2]}$ adjacent to ρ such that a random walk started at ρ never visits K with positive probability.
- The trajectory $\tilde{W}_1$ does not visit K.
- The trajectory $\tilde{W}_2$ does not visit K before hitting ρ, but hits K immediately after hitting ρ for the first time.

On this event, it is easily verified that the edge traversed by $\tilde{W}_2$ as it enters K for the first time is included in the free minimal spanning forest of the interlacement ordering associated to $\mathcal{I}'_{[0,\infty)}$. On the other hand, this edge is not included in $\mathsf{AB}_0(\mathcal{I}')$, since it is not the first entry edge of any vertex. Thus, the two forests do not coincide with positive probability. Since the law of $\mathcal{I}'$ is absolutely continuous with respect to the law of $\mathcal{I}$, this concludes the proof.

7.2. *Exceptional times.*

A natural question raised by the Interlacement Aldous–Broder algorithm concerns the existence or nonexistence of *exceptional times* for the process $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}} = \langle \mathsf{AB}_t(\mathcal{I}) \rangle_{t \in \mathbb{R}}$, that is, times at which $\mathfrak{F}_t$ has properties markedly different from the a.s. properties of $\mathfrak{F}_0$. For example, we might ask whether, considering the process $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}}$ on $\mathbb{Z}^d$ ($d \geq 3$), there are exceptional times when the forest has multiply ended components, is disconnected (if $d = 3, 4$), or is connected (if $d \geq 5$). (Note that the proof of Theorem 2.2 implies that there do not exist exceptional times at which $\mathfrak{F}_t$ contains *indestructible* excessive ends.)

The answers to the first of these questions turn out to rather simple. Given a trajectory W in a graph G and a vertex u of G visited by the path, we define $e(W, u)$ to be the oriented edge pointing into u that is traversed by W as it enters u for the first time, and define

$$\mathsf{AB}(W) = \{-e(W, u) : u \text{ is visited by } W\}.$$

Note that if the trace of W is infinite then $\mathsf{AB}(W)$ is an infinite oriented tree. Observe also that if $(W, t) \in \mathscr{I}$ then the tree $\mathsf{AB}(W)$ is contained in the forest $\mathsf{AB}_t(\mathscr{I})$ on the almost sure event that no two trajectories of $\mathscr{I}$ arrive at the same time. We define the tree of first entry edges $\mathsf{AB}(X)$ similarly when $X = \langle X_n \rangle_{n \geq 0}$ is a path in G.

PROPOSITION 7.1 (Exceptional times for excessive ends). *Let G be a transient network, let $\mathscr{I}$ be the interlacement process, let $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}} = \langle \mathsf{AB}_t(\mathscr{I}) \rangle_{t \in \mathbb{R}}$ and suppose that every component of $\mathfrak{F}_0$ is one-ended almost surely. Let $\mathscr{E}$ be the set of times $t \in \mathbb{R}$ such that $\mathfrak{F}_t$ has a multiply ended component, and let $\mathscr{E}'$ be the set of times $t \in \mathbb{R}$ for which there exists a trajectory W_t in $\mathscr{I}_t$ such that $\mathsf{AB}(W_t)$ is multiply ended. Then the following hold almost surely:*

1. *$\mathscr{E} = \mathscr{E}'$.*
2. *$\mathscr{E} = \varnothing$ if and only if $\mathfrak{F}_0$ is connected almost surely.*
3. *For every $t \in \mathscr{E}$, there is exactly one two-ended component T_t of $\mathfrak{F}_t$, and all other components are one-ended. The unique two-ended component T_t contains the tree $\mathsf{AB}(W_t)$, and every connected component of $T_t \setminus \mathsf{AB}(W_t)$ is finite.*

Since Proposition 7.1 is tangential to the rest of paper, we leave out some details from the proof.

PROOF.   We first prove that $\mathscr{E} = \mathscr{E}'$ almost surely. Let Ω be the almost sure event that every component of $\mathfrak{F}_t$ is one-ended for every rational t, and that no two trajectories of $\mathscr{I}$ have the same arrival time. We claim that $\mathscr{E} = \mathscr{E}'$ pointwise on the event Ω. The containment $\mathscr{E}' \subseteq \mathscr{E}$ is immediate since if $(W, t) \in \mathscr{I}$ then the tree $\mathsf{AB}(W)$ is contained in the forest $\mathsf{AB}_t(\mathscr{I})$ on the event that no two trajectories of $\mathscr{I}$ arrive at the same time.

Now suppose that Ω holds and that $t \in \mathscr{E}$, so that there exists a sequence of vertices $\langle v_i \rangle_{i \geq 0}$ such that $v_i = e_t(v_{i+1})^-$ for each $i \geq 0$. In particular, the arrival times $\tau_t(v_i)$ are increasing. We claim that we must have $\tau_t(v_i) = t$ for all $i \geq 0$. Indeed, if $\tau_t(v_i) \geq t + \varepsilon$ for some $\varepsilon > 0$ and all i larger than some i_0, then we would have that $e_{t+\delta}(v_i) = e_t(v_i)$ for all $0 < \delta \leq \varepsilon$ and all $i \geq i_0$. In this situation, we would therefore have that $\mathfrak{F}_{t+\delta}$ contained a multiply-ended component for every $0 < \delta \leq \varepsilon$, contradicting the assumption that the event Ω occurred. Thus, we must have that there exists a trajectory $W_t \in \mathscr{I}_t$ (which is unique by definition of Ω), and the sequence v_i gives an excessive end in the tree $\mathsf{AB}(W_t)$. Since the sequence

v_i represented an arbitrary excessive end of $\mathfrak{F}_t$, it follows that every excessive end of $\mathfrak{F}_t$ arises from the tree $\mathsf{AB}(W_t)$ on the event Ω. This establishes that $\mathscr{E} = \mathscr{E}'$ on the event Ω as claimed.

If $\mathfrak{F}_0$ is not connected a.s., then there is a vertex v of G such that two independent random walks from v do not intersect with positive probability, and it follows that there a.s. exists a trajectory W_t in $\mathscr{I}$ such that $\mathsf{AB}(W_t)$ has at least two ends. In particular, $\mathscr{E}' \neq \varnothing$ a.s., and hence $\mathscr{E} \neq \varnothing$ a.s. as claimed. It remains to prove that the trees $\mathsf{AB}(W_t)$ have at most two ends for every trajectory W_t in $\mathscr{I}_t$ a.s., and are all one-ended a.s. if $\mathfrak{F}_0$ is a.s. connected. Since there are only countably many trajectories in $\mathscr{I}$, it suffices to show that if W is a bi-infinite random walk on G, the tree $\mathsf{AB}(W)$ has at most two ends a.s. when $\mathfrak{F}_0$ is disconnected a.s., and is one-ended a.s. when $\mathfrak{F}_0$ is connected a.s.

To prove this, it is convenient to introduce a variant of the interlacement Aldous–Broder algorithm in which we first run a simple random walk started from a fixed vertex (considered to arrive at time zero), then run the interlacement process $\mathscr{I}_{[0,\infty)}$, and then form a forest from the first entry edges. It is not difficult to see, by a slight modification of the proof Theorem 1.1, that the forest produced this way is the wired uniform spanning forest: In the exhaustion by finite graphs, this corresponds to first running a random walk from v until hitting the distinguished boundary vertex, and then decomposing the rest of the walk into excursions from the boundary vertex. Using this algorithm, it follows that $\mathsf{AB}(X)$ is one-ended a.s. whenever X is a random walk on a transient graph G for which the wired uniform spanning forest is one-ended.

Now suppose that $W = \langle W_n \rangle_{n \in \mathbb{Z}}$ is a bi-infinite random walk. If $\langle v_i \rangle_{i \geq 0}$ is a sequence of vertices in G corresponding to an excessive end of $\mathsf{AB}(W)$, then we must have that $v_i = e(v_{i+1})^-$ for all i sufficiently large, and it follows that this excessive end must be an end of the tree $\mathsf{AB}(\langle W_i \rangle_{i \geq 0})$, completing the proof that $\mathsf{AB}(W)$ has at most two ends. On the other hand, we note that the unique path to infinity from W_0 in $\mathsf{AB}(\langle W_i \rangle_{i \geq 0})$ is exactly the loop-erasure of $\langle W_i \rangle_{i \geq 0}$, and if $\mathfrak{F}_0$ is connected a.s. then this path is hit infinitely often a.s. by $\langle W_i \rangle_{i < 0}$. We deduce that in this case this end is not present in the tree $\mathsf{AB}(W)$, completing the proof. □

We do not know if there exist exceptional times for (dis)connectivity.

QUESTION 7.2.   Let $d \geq 3$, let $\mathscr{I}$ be the interlacement process on $\mathbb{Z}^d$, and let $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}} = \langle \mathsf{AB}_t(\mathscr{I}) \rangle_{t \in \mathbb{R}}$. If $d = 3, 4$, do there exist times at which $\mathfrak{F}_t$ is disconnected? If $d \geq 5$, do there exist times at which $\mathfrak{F}_t$ is connected?

We expect that exceptional times do *not* exist for $d \geq 5$. For $d = 3, 4$, we are more circumspect. If the answer to Question 7.2 is positive, it would be interesting to further understand the structure of the set of exceptional times and the geometry of the forest $\mathfrak{F}_t$ at a typical exceptional time. It is easy to see that, unlike for

excessive ends, the arrival times of trajectories are *not* exceptional times for connectivity, so that if such exceptional times do exist then they are likely to have a more interesting structure. We note that there is a rich theory of exceptional times for other models such as dynamical percolation, addressing many analogous questions; see, for example, [13, 15, 16, 38].

A related (and possibly quite easy) question concerns the decorrelation of connectivity events under the dynamics.

QUESTION 7.3.   Let $d \geq 5$, let $\mathscr{I}$ be the interlacement process on $\mathbb{Z}^d$ and let $\langle \mathfrak{F}_t \rangle_{t \in \mathbb{R}} = \langle \mathsf{AB}_t(\mathscr{I}) \rangle_{t \in \mathbb{R}}$. How does

$$\mathbb{P}(x \text{ is connected to } y \text{ in both } \mathfrak{F}_0 \text{ and } \mathfrak{F}_t)$$

behave as a function of the vertices $x, y \in \mathbb{Z}^d$ and the number $t > 0$? Is there a way of scaling t as a function of $\|x - y\|$ so that interesting behaviour is observed? $t = \alpha \|x - y\|^{-2} \log \|x - y\|$ for some constant $\alpha > 0$ is a likely candidate.

Recall that for the USF of $\mathbb{Z}^d$, $d \geq 5$, the probability that two vertices x and y are in the same component of the USF decays like $\|x - y\|^{-(d-4)}$ as $\|x - y\| \to \infty$ [4]. A successful approach to Questions 7.2 and 7.3 might need to draw more deeply on the interlacement literature than we have needed to in this paper.

7.3. *Excessive ends via update tolerance.*   A key tool in the study of the USFs carried out in [17, 18, 43] is the *update-tolerance* of the USFs (referred to as *weak insertion tolerance* by Timár [43]). Given a sample $\mathfrak{F}$ of either the WUSF and the FUSF of a network G and an oriented edge e of G not in $\mathfrak{F}$, update-tolerance states that there exists a forest $U(\mathfrak{F}, e)$, obtained from $\mathfrak{F}$ by adding e and deleting some other appropriately chosen edge d, such that the law of $U(\mathfrak{F}, e)$ is absolutely continuous with respect to that of $\mathfrak{F}$. The forest $U(\mathfrak{F}, e)$ is called the update of $\mathfrak{F}$ at e; see [17, 18, 43] for further details.

Since the number of excessive ends does not change when we perform an update, a positive solution of the following conjecture would yield an alternative proof of Theorem 2.2. We say that a Borel set $\mathscr{A} \subseteq \{0, 1\}^E$ is *update-stable* if for every oriented edge e of G, the updated forest $U(\mathfrak{F}, e)$ is in $\mathscr{A}$ if and only if $\mathfrak{F}$ is in $\mathscr{A}$ almost surely. The conjecture would also imply a positive solution to [5], Question 15.7.

CONJECTURE 7.4.   *Let G be an infinite network, and let $\mathfrak{F}$ be either the wired or free spanning forest of G. Then for every update-stable Borel set $\mathscr{A} \subseteq \{0, 1\}^E$, the probability that $\mathfrak{F}$ is in $\mathscr{A}$ is either zero or one.*

7.4. *Ends in uniformly transient networks*.   The following natural question remains open. If true, it would strengthen the results of Lyons, Morris and Schramm [26]. A network is said to be *uniformly transient* if the capacities of the vertices of the network are bounded below by a positive constant.

QUESTION 7.5.   Let G be a uniformly transient network with $\inf_e c(e) > 0$. Does it follow that every component of the wired uniform spanning forest of G is one-ended almost surely?

The argument used in the proof of Theorem 2.2 can be adapted to show that, under the hypotheses of Question 7.5, every component of the WUSF is either one-ended or has uncountably many ends, with no isolated excessive ends. To answer Question 7.5 positively, it remains to rule this second case out.

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