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STOCHASTIC HEAT EQUATION WITH ROUGH DEPENDENCE IN SPACE

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This paper studies the nonlinear one-dimensional stochastic heat equation driven by a Gaussian noise which is white in time and which has the covariance of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in the space variable. The existence and uniqueness of the solution u are proved assuming the nonlinear coefficient $\sigma(u)$ is differentiable with a Lipschitz derivative and $\sigma(0) = 0$.

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1. Introduction. In this paper, we are interested in the one-dimensional stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W}, \qquad t \geq 0, x \in \mathbb{R}, \tag{1.1}$$

where W is a centered Gaussian process with covariance given by

$$\mathbf{E}\big[W(s,x)W(t,y)\big] = \frac{1}{2}\big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big)(s \wedge t), \tag{1.2}$$

with $\frac{1}{4} < H < \frac{1}{2}$. That is, W is a standard Brownian motion in time and a fractional Brownian motion with Hurst parameter H in the space variable and $\dot{W} = \frac{\partial^2 W}{\partial t\, \partial x}$. For this stochastic heat equation with a rough noise in space, understood in the Itô sense, our aim is to obtain the existence and uniqueness of the solution for a differentiable coefficient σ with a Lipschitz derivative and satisfying $\sigma(0) = 0$. We now detail the main points.

Since the pioneering work by Peszat–Zabczyk [13] and Dalang (see [4]), there has been a lot of interest in stochastic partial differential equations driven by a Brownian motion in time with spatial homogeneous covariance. After more than a decade of investigations, the standard assumptions on W under which existence and uniqueness hold take the following form:

(i) $\mathbf{E}[\dot{W}(s,x)\dot{W}(t,y)] = \Lambda(x-y)\delta_0(s-t)$, where Λ is a positive distribution of positive type.

(ii) The Fourier transform of the spatial covariance Λ is a tempered measure μ that satisfies the integrability condition $\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$.

In case of the covariance (1.2) under consideration, one can easily compute the measure μ, whose explicit expression is $\mu(d\xi) = c_{1,H}|\xi|^{1-2H}\, d\xi$, where $c_{1,H}$ is a constant depending on H [see expression (2.2) below]. In addition, it is readily checked that μ fulfills the condition $\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$ for all $H \in (0,1)$. However, the corresponding covariance Λ is a distribution which fails to be positive when $H < \frac{1}{2}$, and the covariance of two stochastic integrals with respect to $\dot{W}$ is expressed in terms of fractional derivatives. For this reason, the standard methodology used in the classical references [4, 6, 13] to handle homogeneous spatial covariances does not apply to our case of interest.

In a recent paper, Balan, Jolis and Quer-Sardanyons [2] proved the existence of a unique mild solution for equation (1.1) in the case $\sigma(u) = au + b$, using techniques of Fourier analysis. The method used in [2] cannot be extended to general nonlinear coefficients. Indeed, the isometry property of stochastic integrals with respect to W involves the seminorm

$$\mathcal{N}_{\frac{1}{2}-H,2}u(t,x) = \left(\int_{\mathbb{R}} \mathbf{E}\big|u(t,x+h) - u(t,x)\big|^2 |h|^{2H-2}\, dh\right)^{\frac{1}{2}},$$

where $\mathcal{N}_{\beta,p}$ is defined in (3.2). Then, if u and v are two solutions, $\mathcal{N}_{\frac{1}{2}-H,2}(\sigma(u)-\sigma(v))$ cannot be bounded in terms of $\mathcal{N}_{\frac{1}{2}-H,2}(u-v)$, due to the presence of a double increment of the form $\sigma(u(s,z+h))-\sigma(v(s,z+h))-\sigma(u(s,z))+\sigma(v(s,z))$. To overcome this difficulty, we shall use a truncation argument to show the uniqueness of mild solutions, inspired by the work of Gyöngy and Nualart in [11] on the stochastic Burgers equation on the whole real line driven by a space–time white noise. The main ingredient is a uniform estimate of the $L^p(\Omega)$-norm of a stochastic convolution (see Lemma 4.9). Due to this argument, the uniqueness is obtained in the space $\mathcal{Z}_T^p$ [see (4.1) for the definition of the norm in $\mathcal{Z}_T^p$], which requires an integrability condition in the space variable.

The existence of a solution is much more involved. The methodology, inspired by the work of Gyöngy in [9] on semilinear stochastic partial differential equations, consists in taking approximations obtained by regularizing the noise and using a compactness argument on a suitable space of trajectories, together with the strong uniqueness result.

Once existence and uniqueness are obtained, we establish the Hölder continuity of the solution u in both space and time variables. We also derive upper bounds for the moments of the solution using a sharp Burkholder's inequality, as well as the matching lower bounds for the second moment by means of a Sobolev embedding argument. Summarizing, we get a complete basic picture of the solution to equation (1.1) in the case $\frac{1}{4}<H<\frac{1}{2}$. The critical parameter $H=\frac{1}{4}$ is worthwhile noting, since it is also the threshold under which rough differential equations driven by a fractional Brownian motion are ill-defined.

The paper is organized as follows. Section 2 contains some preliminaries on stochastic integration with respect to the noise W. Section 3 deals with basic moment estimates and Hölder continuity properties of stochastic convolutions. We establish the uniqueness of the solution in Section 4. To do this, first we derive moment estimates for the supremum norm in space and time for the stochastic convolution. In order to show the existence, we need to introduce several spaces of functions studied in the Appendix and derive compactness criteria.

2. Preliminaries. In this section, we introduce the noise structure and the corresponding stochastic integration.

Our noise W can be seen as a Brownian motion with values in an infinite dimensional Hilbert space. One might thus think that the stochastic integration theory with respect to W can be handled by classical theories (see, e.g., [3, 4, 7]). However, the spatial covariance function of W, which is formally equal to $H(2H-1)|x-y|^{2H-2}$, is not locally integrable when $H<1/2$ (in other words, the Fourier transform of $|\xi|^{1-2H}$ is not a function), and W thus lies outside the scope of application of these classical references. Due to this fact, we provide some details about the construction of a stochastic integral with respect to our noise.

Let us start by introducing our basic notation on Fourier transforms of functions. The space of Schwartz functions is denoted by $\mathcal{S}$. Its dual, the space of tempered distributions, is $\mathcal{S}'$. The Fourier transform of a function $u \in \mathcal{S}$ is defined with the normalization

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} e^{-i\xi x} u(x)\, dx,$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1}u(\xi) = (2\pi)^{-1}\mathcal{F}u(-\xi)$.

Let $\mathcal{D}((0,\infty)\times\mathbb{R})$ denote the space of real-valued infinitely differentiable functions with compact support on $(0,\infty)\times\mathbb{R}$. Taking into account the spectral representation of the covariance function of the fractional Brownian motion in the case $H < \frac{1}{2}$ proved in [14], Theorem 3.1, we represent our noise W by a zero-mean Gaussian family $\{W(\varphi), \varphi \in \mathcal{D}((0,\infty)\times\mathbb{R})\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, whose covariance structure is given by

$$\mathbf{E}\big[W(\varphi)W(\psi)\big] = c_{1,H} \int_{\mathbb{R}_+\times\mathbb{R}} \mathcal{F}\varphi(s,\xi)\overline{\mathcal{F}\psi(s,\xi)}|\xi|^{1-2H}\, ds\, d\xi, \tag{2.1}$$

where the Fourier transforms $\mathcal{F}\varphi, \mathcal{F}\psi$ are understood as Fourier transforms in space only and

$$c_{1,H} = \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H). \tag{2.2}$$

The inner product appearing in (2.1) can be expressed in terms of fractional derivatives. Let β be in $(0,1)$. The Marchaud fractional derivative D_-^β of order β with respect to the space variable is defined, for a function $\varphi : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, as follows:

$$D_-^\beta\varphi(s,x) = \lim_{\varepsilon\to 0} D_{-,\varepsilon}^\beta\varphi(s,x), \tag{2.3}$$

where

$$D_{-,\varepsilon}^\beta\varphi(s,x) = \frac{\beta}{\Gamma(1-\beta)}\int_\varepsilon^\infty \frac{\varphi(s,x)-\varphi(s,x+y)}{y^{1+\beta}}\, dy.$$

We also define the Riemann–Liouville fractional integral of order β of a function $\psi : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ by

$$I_-^\beta\psi(s,x) = \frac{1}{\Gamma(\beta)}\int_x^\infty \psi(s,u)(u-x)^{\beta-1}\, du.$$

Note again that here the fractional differentiation and integration are only with respect to space variables. Observe that if $\varphi = I_-^\beta\psi$ for some $\psi \in L^2(\mathbb{R}_+\times\mathbb{R})$, then by Theorem 6.1 in [15] we have

$$D_-^\beta\varphi = D_-^\beta\big(I_-^\beta\psi\big) = \psi$$

and hence,

$$\int_{\mathbb{R}_+\times\mathbb{R}} \big[D_-^\beta\varphi(s,x)\big]^2\, ds\, dx = \int_{\mathbb{R}_+\times\mathbb{R}} \psi^2(s,x)\, ds\, dx < \infty.$$

The previous notions can be related to our noise in the following way: it is known (cf. [14] for further details) that

$$\mathbf{E}\big[W(\varphi)W(\psi)\big] = c_{2,H}\int_{\mathbb{R}_+\times\mathbb{R}} D_-^{\frac{1}{2}-H}\varphi(s,x)D_-^{\frac{1}{2}-H}\psi(s,x)\,ds\,dx, \tag{2.4}$$

where

$$c_{2,H} = \left[\Gamma\left(H+\frac{1}{2}\right)\right]^2\left(\int_0^\infty ((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}})^2\,ds + \frac{1}{2H}\right)^{-1} \tag{2.5}$$

for any $\varphi, \psi \in \mathcal{D}((0,\infty)\times\mathbb{R})$.

Based on the previous observation and relation (2.4), we introduce a new set of function spaces. Indeed, let $\mathfrak{H}$ be the class of functions $\varphi : \mathbb{R}_+\times\mathbb{R} \to \mathbb{R}$ such that there exists $\psi \in L^2(\mathbb{R}_+\times\mathbb{R})$ satisfying $\varphi(s,x) = I_-^{\frac{1}{2}-H}\psi(s,x)$. The relation between $\mathfrak{H}$ and our noise W is given in the following proposition.

PROPOSITION 2.1. *The class of functions $\mathfrak{H}$ is a Hilbert space equipped with the inner product*

$$\langle\varphi,\psi\rangle_{\mathfrak{H}} := c_{2,H}\int_{\mathbb{R}_+\times\mathbb{R}} D_-^{\frac{1}{2}-H}\varphi(s,x)D_-^{\frac{1}{2}-H}\psi(s,x)\,ds\,dx, \tag{2.6}$$

and $\mathcal{D}((0,\infty)\times\mathbb{R})$ is dense in $\mathfrak{H}$. Moreover if $\mathfrak{H}_0$ denotes the class of functions $\varphi \in L^2(\mathbb{R}_+\times\mathbb{R})$ such that $\int_{\mathbb{R}_+\times\mathbb{R}} |\mathcal{F}\varphi(s,\xi)|^2|\xi|^{1-2H}\,d\xi\,ds < \infty$, then $\mathfrak{H}_0$ is not complete and the inclusion $\mathfrak{H}_0 \subset \mathfrak{H}$ is strict. Also for any $\varphi,\psi \in \mathfrak{H}_0$,

$$\langle\varphi,\psi\rangle_{\mathfrak{H}} = c_{1,H}\int_{\mathbb{R}_+\times\mathbb{R}} \mathcal{F}\varphi(s,\xi)\overline{\mathcal{F}\psi(s,\xi)}|\xi|^{1-2H}\,d\xi\,ds. \tag{2.7}$$

We refer to [14] for the proof of this proposition. Note that in [14], the functions considered there are from $\mathbb{R}$ to $\mathbb{R}$, but by scrutinizing the proofs we see that the results of this paper can be easily extended to our case, that is, for functions from $\mathbb{R}_+\times\mathbb{R}$ to $\mathbb{R}$. We omit the details.

Let us now identify our space $\mathfrak{H}$ with another classical space in harmonic analysis. Indeed, according to Proposition 1.37 in [1], for any $\beta \in (0,\frac{1}{2})$ the homogeneous Sobolev space $\dot{H}^\beta$ is defined as the completion of the space of infinitely differentiable functions with compact support with respect to the norm

$$\begin{aligned}\|f\|_{\dot{H}^\beta}^2 &= \int_{\mathbb{R}} \left|D_-^\beta f(x)\right|^2 dx\\ &= c_{3,\beta}^2\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(x+y)-f(x)\right|^2|y|^{-1-2\beta}\,dx\,dy,\end{aligned} \tag{2.8}$$

where $c_{3,\beta}^2 = (1/2-\beta)\beta c_{2,\frac{1}{2}-\beta}^{-1}$ and $c_{2,\frac{1}{2}-\beta}$ is defined by (2.5). As a consequence, our Hilbert space $\mathfrak{H}$ can be identified with the homogenous Sobolev space of order

$\beta = \frac{1}{2} - H$ of functions with values in $L^2(\mathbb{R}_+)$. Namely, $\mathfrak{H} = \dot{H}^{\frac{1}{2}-H}(L^2(\mathbb{R}_+))$, and for any $f \in \mathfrak{H}$ the quantity $\|f\|_{\mathfrak{H}}$ can be represented as

$$\|f\|_{\mathfrak{H}}^2 = c_{3,\frac{1}{2}-H}^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}} \left|f(s, x+y) - f(s,x)\right|^2 |y|^{2H-2}\, dx\, dy\, ds.$$

From Proposition 2.1, we see that the Gaussian family W can be extended as an isonormal Gaussian process $W = \{W(\phi), \phi \in \mathfrak{H}\}$ indexed by the Hilbert space $\mathfrak{H}$.

Let us now turn to the stochastic integration with respect to W. Since we are handling a Brownian motion in time, one can start by integrating elementary processes.

DEFINITION 2.2. For any $t \geq 0$, let $\mathcal{F}_t$ be the σ-algebra generated by W up to time t. An elementary process g is a process given by

$$g(s,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \mathbf{1}_{(a_i,b_i]}(s) \mathbf{1}_{(h_j,l_j]}(x),$$

where n and m are finite positive integers, $-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \dots, n$. The integral of such a process with respect to W is defined as

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x) W(ds, dx) \\ &\quad = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} W(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(h_j,l_j]}) \\ &\quad = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \left[W(b_i, l_j) - W(a_i, l_j) - W(b_i, h_j) + W(a_i, h_j)\right]. \end{aligned} \tag{2.9}$$

We can now extend the notion of integral with respect to W to a broad class of adapted processes.

PROPOSITION 2.3. *Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathfrak{H}$ and $\mathbf{E}[\|g\|_{\mathfrak{H}}^2] < \infty$. Then we have*:

(i) *The space of elementary processes defined in Definition 2.2 is dense in Λ_H.*

(ii) *For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x) W(ds, dx)$ is defined as the $L^2(\Omega)$-limit of Riemann sums along elementary processes approximating g, and we have*

$$\mathbf{E}\left[\left(\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x) W(ds, dx)\right)^2\right] = \mathbf{E}\left[\|g\|_{\mathfrak{H}}^2\right]. \tag{2.10}$$

PROOF. Let us prove item (i). To this aim, consider $g \in \Lambda_H$ and set $\varphi(t,x) = D_-^{\frac{1}{2}-H} g(t,x)$. According to the definition of Λ_H, we have $\mathbf{E}[\int_{\mathbb{R}_+}\int_{\mathbb{R}} |\varphi(s,x)|^2 \, dx \, ds] < \infty$. Then we will show that $g(t,x)$ can be approximated by elementary processes in $L^2(\Omega; \mathfrak{H})$ in three steps.

Step 1. Recall that $\dot{H}^{\frac{1}{2}-H}$ denotes the class of functions f such that there exists $h \in L^2(\mathbb{R})$ satisfying $f = I_-^{1/2-H} h$. We show that the process g can be approximated in $L^2(\Omega; \mathfrak{H})$ by functions of the form

$$\psi(s,x;\omega) = \sum_{i=1}^{N} \mathbf{1}_{(a_i,b_i]}(s)\phi_i(x;\omega), \tag{2.11}$$

where for each i, $\phi_i(x;\omega)$ is an $\mathcal{F}_{a_i}$-measurable $L^2(\Omega; \dot{H}^{\frac{1}{2}-H})$-valued random field. To see this, we just set

$$\psi_m(s,x;\omega) = \sum_{k=1}^{m2^m} \mathbf{1}_{((k-1)2^{-m},k2^{-m}]}(s) 2^m \int_{(k-1)2^{-m}}^{k2^{-m}} g(r,x;\omega) \, dr,$$

and we easily get that $D_-^{\frac{1}{2}-H}\psi_m(s,x;\omega) \to D_-^{\frac{1}{2}-H} g(s,x;\omega)$ in $L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ as m tends to infinity. In this way we get the desired approximation.

Step 2. We show that each $\psi_m(s,x;\omega)$ of the form (2.11) can be approximated in $L^2(\Omega; \mathfrak{H})$ by a linear combination of elements of the form $X\mathbf{1}_{(a,b]}(s)h(x)$. Indeed, for each $\phi_i(x)$, we notice that since

$$\mathbf{E}\int_{\mathbb{R}} \left| D_-^{\frac{1}{2}-H}\phi_i(x)\right|^2 dx < \infty,$$

the random function $D_-^{\frac{1}{2}-H}\phi_i(x;\omega)$ can be approximated in $L^2(\Omega; L^2(\mathbb{R}))$ by functions of the form $\sum_{j=1}^{N} X_j h_j(x)$, where each X_j is an $\mathcal{F}_{a_i}$-measurable random variable and each h_j is an element in $L^2(\mathbb{R})$. Thus, it is easily seen that $\phi_i(x;\omega)$ can be approximated by a sequence of functions of the form

$$\sum_{j=1}^{N} X_j I_-^{\frac{1}{2}-H} h_j(x).$$

So we conclude that $\psi_m(s,x;\omega)$ can be approximated in $L^2(\Omega; \mathfrak{H})$ by

$$\sum_{i=1}^{m} \mathbf{1}_{(a_i,b_i]}(s) \sum_{j=1}^{N} X_{i,j} I_-^{\frac{1}{2}-H} h_{i,j}(x),$$

where for each (i,j), $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables and $h_{i,j} \in L^2(\mathbb{R})$.

Step 3. Owing to Theorem 3.3 in [14] we know that

$$\mathrm{Span}\{D_-^{\frac{1}{2}-H}\mathbf{1}_{(h,l]}, h < l\}$$

is dense in $\Lambda_0 := \{D_-^{\frac{1}{2}-H} f : f \in \dot{H}^\beta\}$, in $L^2(\mathbb{R})$ norm. This observation and the results in Step 2 immediately show that $\psi_m(s, x; \omega)$ can be approximated by elementary processes in $L^2(\Omega; \mathfrak{H})$.

For item (ii), it is easy to see that (2.10) holds for processes of the form (2.11). For a general $g \in \Lambda_H$, we obtain the result by a limiting argument. This completes the proof. $\square$

With this stochastic integral defined, we are ready to state the definition of the solution to equation (1.1).

DEFINITION 2.4. Let $u = \{u(t, x), 0 \leq t \leq T, x \in \mathbb{R}\}$ be a real-valued predictable stochastic process such that for all $t \in [0, T]$ and $x \in \mathbb{R}$ the process $\{p_{t-s}(x - y)\sigma(u(s, y))\mathbf{1}_{[0,t]}(s), 0 \leq s \leq t, y \in \mathbb{R}\}$ is an element of Λ_H, where $p_t(x)$ is the heat kernel on the real line related to $\frac{\kappa}{2}\Delta$. We say that u is a mild solution of (1.1) if for all $t \in [0, T]$ and $x \in \mathbb{R}$ we have

$$u(t, x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x - y)\sigma\big(u(s, y)\big) W(ds, dy) \qquad \text{a.s.,} \tag{2.12}$$

where the stochastic integral is understood in the sense of Proposition 2.3.

Along the paper, we denote by C a generic constant that may vary from line to line.

3. Moment estimates and Hölder continuity of stochastic convolutions. This section is devoted to a thorough study of the stochastic convolution related to our noise $\dot{W}$, including moment bounds and Hölder continuity estimates.

3.1. *Moment bound of the solution.* First, we introduce some notation, which makes some of our formulae easier to read, and which will prevail until the end of the article. Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$, and let $\beta \in (0, 1)$ be a fixed number. For every function $f : \mathbb{R} \to B$, we introduce the function $\mathcal{N}_\beta^B f : \mathbb{R} \to [0, \infty]$ defined by

$$\mathcal{N}_\beta^B f(x) = \left(\int_{\mathbb{R}} \|f(x + h) - f(x)\|^2 |h|^{-1-2\beta} \, dh\right)^{\frac{1}{2}}. \tag{3.1}$$

When $B = \mathbb{R}$, we abbreviate the notation $\mathcal{N}_\beta^{\mathbb{R}} f$ into $\mathcal{N}_\beta f$. With this notation, the norm of the homogeneous Sobolev space $\dot{H}^\beta$ can be written as $c_{3,\beta}\|\mathcal{N}_\beta f\|_{L^2(\mathbb{R})}$. The following technical lemma will be used along the paper.

LEMMA 3.1. *For any $\beta \in (0, 1)$,*

$$\int_{\mathbb{R}} [\mathcal{N}_\beta p_s(x)]^2 \, dx \leq C_\beta (\kappa s)^{-\frac{1}{2}-\beta}.$$

PROOF. The kernel p_s is an element of $\dot{H}^\beta$, where this space has been introduced in Section 2. Thus, from (2.7) and (2.8), we can write

$$\int_{\mathbb{R}} [\mathcal{N}_\beta p_s(x)]^2 \, dx = c_{3,\beta}^{-2} \|p_s\|_{\dot{H}^\beta}^2 = c_{3,\beta}^{-2} c_{1,\frac{1}{2}-\beta} \int_{\mathbb{R}} |\mathcal{F} p_s(\xi)|^2 |\xi|^{2\beta} \, d\xi$$
$$= C_{1,\beta} \int_{\mathbb{R}} e^{-\kappa s \xi^2} |\xi|^{2\beta} \, d\xi.$$

Setting now $\eta = (\kappa s)^{1/2} \xi$ in the integral in ξ, we get

$$\int_{\mathbb{R}} [\mathcal{N}_\beta p_s(x)]^2 \, dx \le C_\beta (\kappa s)^{-\frac{1}{2}-\beta},$$

where $C_\beta = C_{1,\beta} \int_{\mathbb{R}} e^{-\eta^2} |\eta|^{2\beta} \, d\eta$. $\square$

The transformation $\mathcal{N}_\beta^B$ can also be defined for functions f defined on $\mathbb{R}_+ \times \mathbb{R}$ acting on the spatial variable, and in this case, $\mathcal{N}_\beta^B f : \mathbb{R}_+ \times \mathbb{R} \to [0, \infty]$. Now fix $p \ge 2$, and suppose that $f = \{f(t,x), t \ge 0, x \in \mathbb{R}\}$ is a random field such that $\mathbf{E}|f(t,x)|^p < \infty$ for all (t,x). Then we can consider f as an $L^p(\Omega)$-valued function and we will denote by $\mathcal{N}_{\beta,p} f$ the transformation introduced in (3.1) for $B = L^p(\Omega)$, that is,

$$\mathcal{N}_{\beta,p} f(t,x) = \left(\int_{\mathbb{R}} \|f(t,x+h) - f(t,x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} \, dh \right)^{\frac{1}{2}}. \tag{3.2}$$

With the above notation in mind, the following proposition is essential in our approach.

PROPOSITION 3.2. *Let W be the Gaussian noise defined by the covariance* (2.1), *and consider a predictable random field $f \in \Lambda_H$. Then, for any $p \ge 2$ we have*

$$\begin{aligned} &\left\| \int_0^t \int_{\mathbb{R}} f(s,y) W(ds,dy) \right\|_{L^p(\Omega)} \\ &\qquad \le \sqrt{4p}\, c_{3,\frac{1}{2}-H} \left(\int_0^t \int_{\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H,p} f(s,y)]^2 \, dy \, ds \right)^{\frac{1}{2}}, \end{aligned} \tag{3.3}$$

where $c_{3,\beta}$ is defined by relation (2.8).

PROOF. Applying Burkholder's inequality, we have

$$\left\| \int_0^t \int_{\mathbb{R}} f(s,y) W(ds,dy) \right\|_{L^p(\Omega)} \le \sqrt{4p} \left\| \int_0^t \|f(s,\cdot)\|_{\dot{H}^{\frac{1}{2}-H}}^2 \, ds \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{1}{2}}. \tag{3.4}$$

Moreover, using (2.8) we can write

$$\|f(s,\cdot)\|_{\dot{H}^{\frac{1}{2}-H}}^2 = c_{3,\frac{1}{2}-H}^2 \int_{\mathbb{R}^2} |f(s,y+h) - f(s,y)|^2 |h|^{2H-2} \, dh \, dy. \tag{3.5}$$

We now invoke Minkowski's inequality, under the form

$$\left\| \int_S U(\xi)\mu(d\xi) \right\|_{L^q(\Omega)} \leq \int_S \|U(\xi)\|_{L^q(\Omega)}\mu(d\xi),$$

for a measure μ on the state space S. Together with (3.5), this yields

$$\begin{aligned}
&\left\| \int_0^t \|f(s,\cdot)\|^2_{\dot{H}^{\frac{1}{2}-H}} \, ds \right\|_{L^{\frac{p}{2}}(\Omega)} \\
&\quad \leq c^2_{3,\frac{1}{2}-H} \int_0^t \int_{\mathbb{R}^2} \left\| \left(f(s,y+h)-f(s,y)\right)^2 \right\|_{L^{\frac{p}{2}}(\Omega)} |h|^{2H-2} \, dh\, dy\, ds \\
&\quad = c^2_{3,\frac{1}{2}-H} \int_0^t \int_{\mathbb{R}^2} \|f(s,y+h)-f(s,y)\|^2_{L^p(\Omega)} |h|^{2H-2} \, dh\, dy\, ds,
\end{aligned}$$

from which identity (3.3) is easily deduced. □

From now on, we fix a finite time horizon T. We introduce the following function space which plays an important role throughout the paper.

DEFINITION 3.3. Let $\mathfrak{X}_T^\beta(B)$ be the space of all continuous functions $f : [0,T]\times\mathbb{R} \to B$ such that

$$\|f\|_{\mathfrak{X}_T^\beta(B)} := \sup_{t\in[0,T],x\in\mathbb{R}} \|f(t,x)\| + \sup_{t\in[0,T],x\in\mathbb{R}} \mathcal{N}_\beta^B f(t,x) < \infty,$$

where we recall that $\mathcal{N}_\beta^B$ is defined by (3.1).

We equip $\mathfrak{X}_T^\beta(B)$ with the norm $\|\cdot\|_{\mathfrak{X}_T^\beta(B)}$ defined above. Then $\mathfrak{X}_T^\beta(B)$ is a normed vector space. Moreover, it can be shown that $\mathfrak{X}_T^\beta(B)$ is a Banach space (see Proposition A.2 in the Appendix).

When $B = L^p(\Omega)$ with $p \in [1,\infty)$, we use the notation $\mathfrak{X}_T^{\beta,p} = \mathfrak{X}_T^\beta(L^p(\Omega))$. A function f in $\mathfrak{X}_T^{\beta,p}$ can be considered as a stochastic process indexed by (t,x) in $[0,T]\times\mathbb{R}$ such that

$$\begin{aligned}
&\sup_{t\in[0,T],x\in\mathbb{R}} \|f(t,x)\|_{L^p(\Omega)} \\
&\quad + \sup_{t\in[0,T],x\in\mathbb{R}} \left(\int_{\mathbb{R}} \|f(t,x+y)-f(t,x)\|^2_{L^p(\Omega)} |y|^{-2\beta-1} \, dy \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Next, for $\theta > 0$, $\varepsilon > 0$ and $\beta \in (0,1)$, we consider the following norm on $\mathfrak{X}_T^{\beta,p}$:

$$\begin{aligned}
\|f\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\beta,p}} := &\sup_{t\in[0,T],x\in\mathbb{R}} e^{-\theta t} \|f(t,x)\|_{L^p(\Omega)} \\
&+ \varepsilon \sup_{t\in[0,T],x\in\mathbb{R}} e^{-\theta t} \mathcal{N}_{\beta,p} f(t,x),
\end{aligned} \tag{3.6}$$

where we recall that $\mathcal{N}_{\beta,p}$ is defined by (3.2). In the case $\varepsilon = 1$, we simply write $\|\cdot\|_{\mathfrak{X}^{\beta,p}_{T,\theta}}$. Because T is finite, the norm $\|\cdot\|_{\mathfrak{X}^{\beta,p}_{T,\theta,\varepsilon}}$ defined as above is equivalent to the norm $\|\cdot\|_{\mathfrak{X}^{\beta,p}_{T}}$.

REMARK 3.4. (i) In (3.6), β is a parameter for regularity (see Proposition A.1), θ is a weight index for the interval $[0,T]$, which is introduced so that a certain map becomes a contraction without changing the value of T (see Proposition 3.5 and Remark 3.6) and ε is a parameter of dimension which is introduced so that correct upper bounds for the moments of the solution can be achieved (see Theorem 4.7).

(ii) The second term in the norm in (3.6) is not invariant by scaling while the first term is. Indeed, denote $f_\lambda(t,x) = f(t,\lambda x)$, then

$$\sup_{x\in\mathbb{R}}\left(\int_{\mathbb{R}}\|f_\lambda(t,x+h)-f_\lambda(t,x)\|^2_{L^p(\Omega)}|h|^{-1-2\beta}\,dh\right)^{\frac{1}{2}}$$

$$=\lambda^\beta\sup_{x\in\mathbb{R}}\left(\int_{\mathbb{R}}\|f(t,x+h)-f(t,x)\|^2_{L^p(\Omega)}|h|^{-1-2\beta}\,dh\right)^{\frac{1}{2}}.$$

This is the very reason why various orders of $(t-s)$ appear in the proof of Proposition 3.5 below. We bypass this technical difficulty by the introduction of an additional scaling factor ε in (3.6).

(iii) Another way to see the role of ε is via dimensional analysis. Suppose that the amplitude of f has unit L, the spatial variable x has unit S, while the randomness ω is dimensionless. Then the first term in (3.6) has unit L while the second term has unit L/S^β. Hence, in order for the two terms to have the same dimension, we multiply the second term with a constant ε having unit of S^β.

The next proposition gives a convenient bound on the stochastic convolution in term of the spaces $\mathfrak{X}^{\beta,p}_T$.

PROPOSITION 3.5. *Consider a predictable random field $f\in\mathfrak{X}^{\frac{1}{2}-H,p}_T$ and define a process $\{\Phi(t,x), t\ge 0, x\in\mathbb{R}\}$ by*

$$\Phi(t,x)=\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)f(s,y)W(ds,dy). \tag{3.7}$$

Then, for any $\theta>0$, $\varepsilon>0$, $\beta<H$ and $p\ge 2$, the following inequality holds:

$$\begin{aligned}&\|\Phi\|_{\mathfrak{X}^{\beta,p}_{T,\theta,\varepsilon}}\\ &\quad\le C_0\sqrt{p}\|f\|_{\mathfrak{X}^{\frac{1}{2}-H,p}_{T,\theta,\varepsilon}}\\ &\qquad\times\left(\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}}+\kappa^{-\frac{1}{4}-\frac{\beta}{2}}\theta^{\frac{\beta}{2}-\frac{1}{4}}+\varepsilon^{-1}\kappa^{-\frac{1}{4}}\theta^{-\frac{1}{4}}+\varepsilon\kappa^{\frac{H}{2}-\frac{\beta}{2}-\frac{1}{2}}\theta^{\frac{\beta}{2}-\frac{H}{2}}\right),\end{aligned}\tag{3.8}$$

where C_0 is a constant depending only on H and β.

REMARK 3.6. According to relation (3.8), the stochastic convolution induces some stability properties in the spaces $\mathfrak{X}_{\theta,\varepsilon}^{\beta,p}$ whenever $\frac{1}{2}-H\leq\beta<H$. This imposes the restriction $H>\frac{1}{4}$ already at this stage.

PROOF OF PROPOSITION 3.5. We begin by noting that since f is predictable and belongs to $\mathfrak{X}_T^{\frac{1}{2}-H,p}$, we have that $p_{t-\cdot}(x-\cdot)f(\cdot,\cdot)$ is in Λ_H. According to our definition (3.6), we get $\|\Phi\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\beta,p}}=\mathcal{A}_1+\varepsilon\mathcal{A}_2$, with

$$\mathcal{A}_1=\sup_{t\in[0,T],x\in\mathbb{R}}e^{-\theta t}\|\Phi(t,x)\|_{L^p(\Omega)}\quad\text{and}\quad\mathcal{A}_2=\sup_{t\in[0,T],x\in\mathbb{R}}e^{-\theta t}\mathcal{N}_{\beta,p}\Phi(t,x).$$

We now estimate those terms separately. Along the proof C will denote a generic constant depending only on H and β.

Step 1: *Upper bound for* $\mathcal{A}_1$. The term $\Phi(t,x)$ is of the form

$$\int_0^t\int_{\mathbb{R}}g_{t,x}(s,y)W(ds,dy)\qquad\text{with } g_{t,x}(s,y)=p_{t-s}(x-y)f(s,y).$$

Applying inequality (3.3), we thus have

$$\begin{aligned}&\|\Phi(t,x)\|_{L^p(\Omega)}\\&\quad\leq C\sqrt{p}\left(\int_0^t\int_{\mathbb{R}^2}\|g_{t,x}(s,y+h)-g_{t,x}(s,y)\|_{L^p(\Omega)}^2|h|^{2H-2}\,dh\,dy\,ds\right)^{\frac{1}{2}}.\end{aligned}$$

A simple decomposition of the increment $g_{t,x}(s,y+h)-g_{t,x}(s,y)$ then yields

$$\|\Phi(t,x)\|_{L^p(\Omega)}\leq C\sqrt{p}\left[\left(\int_0^t J_1(s)\,ds\right)^{\frac{1}{2}}+\left(\int_0^t J_2(s)\,ds\right)^{\frac{1}{2}}\right],$$

where

$$J_1(s)=\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x-y-h)-p_{t-s}(x-y)|^2\|f(s,y+h)\|_{L^p(\Omega)}^2|h|^{2H-2}\,dy\,dh$$

and

$$J_2(s)=\int_{\mathbb{R}}\int_{\mathbb{R}}p_{t-s}^2(x-y)\|f(s,y+h)-f(s,y)\|_{L^p(\Omega)}^2|h|^{2H-2}\,dy\,dh.$$

To estimate $J_1(s)$, we write

$$J_1(s)\leq\sup_{x\in\mathbb{R}}\|f(s,x)\|_{L^p(\Omega)}^2\int_{\mathbb{R}}\left[\mathcal{N}_{\frac{1}{2}-H}p_{t-s}(y)\right]^2dy.$$

Applying Lemma 3.1 with $\beta=\frac{1}{2}-H$, we obtain

$$J_1(s)\leq C\sup_{x\in\mathbb{R}}\|f(s,x)\|_{L^p(\Omega)}^2[\kappa(t-s)]^{H-1}.$$

Let us now turn to estimate $J_2(s)$. Recalling our notation (3.2), we have

$$
\begin{aligned}
J_2(s) &= \int_{\mathbb{R}} p_{t-s}^2(x-y)\big[\mathcal{N}_{\frac{1}{2}-H,p} f(s,y)\big]^2 \, dy \\
&\le \sup_{x\in\mathbb{R}} \big[\mathcal{N}_{\frac{1}{2}-H,p} f(s,x)\big]^2 \int_{\mathbb{R}} p_{t-s}^2(x-y)\, dy \qquad (3.9)\\
&\le \big[2\pi\kappa(t-s)\big]^{-\frac{1}{2}} \sup_{x\in\mathbb{R}} \big[\mathcal{N}_{\frac{1}{2}-H,p} f(s,x)\big]^2.
\end{aligned}
$$

Hence, putting together our bounds on J_1 and J_2, we get

$$
\begin{aligned}
& e^{-\theta t} \sup_{x\in\mathbb{R}} \|\Phi(t,x)\|_{L^p(\Omega)} \\
&\quad \le C\sqrt{p} \sup_{\substack{0\le s\le T\\ x\in\mathbb{R}}} e^{-\theta s} \|f(s,x)\|_{L^p(\Omega)} \left(\int_0^t e^{-2\theta(t-s)} \big[\kappa(t-s)\big]^{H-1} \, ds\right)^{\frac{1}{2}} \\
&\qquad + C\sqrt{p}\varepsilon \sup_{\substack{0\le s\le T\\ x\in\mathbb{R}}} e^{-\theta s} \sup_{x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H,p} f(s,x) \frac{(\int_0^t e^{-2\theta(t-s)} [\kappa(t-s)]^{-\frac{1}{2}} \, ds)^{\frac{1}{2}}}{\varepsilon},
\end{aligned}
$$

and some elementary computations for the integrals above yield

$$
\begin{aligned}
\mathcal{A}_1 &= \sup_{t\in[0,T], x\in\mathbb{R}} e^{-\theta t} \|\Phi(t,x)\|_{L^p(\Omega)} \\
&\le C\sqrt{p} \|f\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\frac{1}{2}-H,p}} \big(\kappa^{\frac{H}{2}-\frac{1}{2}} \theta^{-\frac{H}{2}} + \varepsilon^{-1} \kappa^{-\frac{1}{4}} \theta^{-\frac{1}{4}}\big).
\end{aligned}
$$

Step 2: *Upper bound for* $\mathcal{A}_2$. According to the definition of $\mathcal{A}_2$, we have to bound $\mathcal{N}_{\beta,p}\Phi(t,x)$, where we recall that

$$
\mathcal{N}_{\beta,p}\Phi(t,x) = \left(\int_{\mathbb{R}} \|\Phi(t,x+h)-\Phi(t,x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} \, dh\right)^{\frac{1}{2}}. \qquad (3.10)
$$

Furthermore, arguing as in Step 1 above, it is easily seen that

$$
\begin{aligned}
& \|\Phi(t,x+h)-\Phi(t,x)\|_{L^p(\Omega)} \qquad (3.11)\\
&\quad \le C\sqrt{p}\left[\left(\int_0^t J_1'(s,h)\, ds\right)^{1/2} + \left(\int_0^t J_2'(s,h)\, ds\right)^{1/2}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
J_1'(s,h) = \int_{\mathbb{R}}\int_{\mathbb{R}} & \big| p_{t-s}(x+h-y-z) - p_{t-s}(x-y-z) \\
& - p_{t-s}(x+h-y) + p_{t-s}(x-y)\big|^2 \\
& \times \|f(s,y+z)\|_{L^p(\Omega)}^2 |z|^{2H-2} \, dy\, dz
\end{aligned}
$$

and

$$J_2'(s,h) = \int_{\mathbb{R}}\int_{\mathbb{R}} |p_{t-s}(x+h-y) - p_{t-s}(x-y)|^2 \\ \times \|f(s,y+z) - f(s,y)\|_{L^p(\Omega)}^2 |z|^{2H-2}\,dy\,dz.$$

Plugging (3.11) into (3.10), we end up with

$$\mathcal{N}_{\beta,p}\Phi(t,x) \\ \le C\sqrt{p}\left[\int_0^t\int_{\mathbb{R}} J_1'(s,h)|h|^{-1-2\beta}\,dh\,ds + \int_0^t\int_{\mathbb{R}} J_2'(s,h)|h|^{-1-2\beta}\,dh\,ds\right].$$

In addition, arguing again as in the proof of Lemma 3.1, we can show that

$$\int_{\mathbb{R}} J_1'(s,h)|h|^{-1-2\beta}\,dh \le C\big[\kappa(t-s)\big]^{H-\beta-1}\sup_{x\in\mathbb{R}}\|f(s,x)\|_{L^p(\Omega)}^2.$$

On the other hand, applying Lemma 3.1 leads to

$$\int_{\mathbb{R}} J_2'(s,h)|h|^{-1-2\beta}\,dh \le C\big[\kappa(t-s)\big]^{-\frac{1}{2}-\beta}\sup_{x\in\mathbb{R}}\big[\mathcal{N}_{\frac{1}{2}-H,p}f(s,x)\big]^2.$$

Combining these estimates for J_1', J_2' and resorting to (3.11), similarly as the estimate for $e^{-\theta t}\|\Phi(t,x)\|_{L^p(\Omega)}$, we obtain

$$\mathcal{A}_2 \le C\sqrt{p}\|f\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\frac{1}{2}-H,p}}\big(\kappa^{\frac{H}{2}-\frac{\beta}{2}-\frac{1}{2}}\theta^{\frac{\beta}{2}-\frac{H}{2}} + \varepsilon^{-1}\kappa^{-\frac{1}{4}-\frac{\beta}{2}}\theta^{\frac{\beta}{2}-\frac{1}{4}}\big).$$

Putting together Step 1 and Step 2, our claim (3.8) is now easily checked. □

We conclude this section by a simple remark which is labeled for further use. In the particular case $\beta = \frac{1}{2} - H$, and using the simplified notation $\|\cdot\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{\frac{1}{2}-H,p}} = \|\cdot\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{p}}$, the estimate (3.8) can be written as

$$\|\Phi\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{p}} \le C_0\sqrt{p}\|f\|_{\mathfrak{X}_{T,\theta,\varepsilon}^{p}}\big(\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}} + \varepsilon^{-1}\kappa^{-\frac{1}{4}}\theta^{-\frac{1}{4}} + \varepsilon\kappa^{H-\frac{3}{4}}\theta^{\frac{1}{4}-H}\big). \tag{3.12}$$

3.2. *Hölder continuity estimates.* A natural question arising from the definition (3.7) of the process Φ is the derivation of Hölder type exponents in both time and space. Some estimates in this direction are provided in the next proposition. We set $\mathfrak{X}_T^p = \mathfrak{X}_T^{\frac{1}{2}-H,p}$, and the norm $\|\cdot\|_{\mathfrak{X}_{T,\theta}^p}$ is given by (3.6) with $\varepsilon = 1$ and $\beta = \frac{1}{2} - H$.

PROPOSITION 3.7. *Recall that the covariance of the noise W is given by* (2.1). *Consider $p \ge 2$ and a predictable random field $f \in \mathfrak{X}_T^p$, where T is a fixed finite* *time horizon. Let θ_0 be any positive number. We define the random field Φ as in* (3.7). *Then for every $x, h \in \mathbb{R}$, $t_1, t_2 \in [0, T]$ and every $\gamma \in [0, H]$ we have*

$$\begin{aligned}\tag{3.13} &\big\|\Phi\big([t_1,t_2],x+h\big)-\Phi\big([t_1,t_2],x\big)\big\|_{L^p(\Omega)} \\ &\quad\le C\sqrt{p}e^{\theta_0 T}\|f\|_{\mathfrak{X}^p_{T,\theta_0}}|t_2-t_1|^{\frac{H-\gamma}{2}}|h|^{\gamma}.\end{aligned}$$

In the above, the constant C depends on T and does not depend on p, and we are using the notation

$$\Phi\big([t_1,t_2],x\big)=\Phi(t_2,x)-\Phi(t_1,x).$$

In particular, if we let $t_1 = 0$, we get the Hölder estimate of the space variable. For the Hölder estimate of the time variable, we have

$$\tag{3.14} \big\|\Phi(t_2,x)-\Phi(t_1,x)\big\|_{L^p(\Omega)}\le C\sqrt{p}e^{\theta_0 T}\|f\|_{\mathfrak{X}^p_{T,\theta_0}}|t_2-t_1|^{\frac{H}{2}}.$$

PROOF. First, we prove (3.13). Without loss of generality, we assume $t_1 < t_2$ and denote $\Delta t = t_2 - t_1$. We also set

$$\begin{aligned}\tag{3.15} V_1(f)&=\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\big\|f(t,x)\big\|_{L^p(\Omega)}, \\ V_2(f)&=\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\mathcal{N}_{\frac12-H,p}f(t,x),\end{aligned}$$

and $V(f) = V_1(f) + V_2(f)$. Observe that according to (3.6), we have $V(f) \le \exp(\theta_0 T)\|f\|_{\mathfrak{X}^p_{T,\theta_0}}$.

As in the proof of Proposition 3.5, we first write $\Phi([t_1,t_2],x+h) - \Phi([t_1,t_2],x) = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1=\int_0^{t_1}\int_{\mathbb{R}}\big[p_{[t_1-s,t_2-s]}(x+h-y)-p_{[t_1-s,t_2-s]}(x-y)\big]f(s,y)W(ds,dy)$$

and

$$\mathcal{A}_2=\int_{t_1}^{t_2}\int_{\mathbb{R}}\big[p_{t_2-s}(x+h-y)-p_{t_2-s}(x-y)\big]f(s,y)W(ds,dy).$$

We now treat those two terms separately. To alleviate notation, we will include $\sqrt{p}$ into the constant C below.

Step 1: *Upper bound for $\mathcal{A}_1$*. The computations are carried out analogously to the proof of Proposition 3.5, and we have

$$\|\mathcal{A}_1\|^2_{L^p(\Omega)}\le C\int_0^{t_1}\big(A_{11}(s)+A_{12}(s)\big)\,ds,$$

where A_{11} and A_{12} are analogous to J_1, J_2 in the proof of Proposition 3.5, and are respectively defined by

$$\begin{aligned}A_{11}(s) = \int_{\mathbb{R}}\int_{\mathbb{R}} \big| & p_{[t_1-s,t_2-s]}(x+h-y-z) - p_{[t_1-s,t_2-s]}(x-y-z) \\ & - p_{[t_1-s,t_2-s]}(x+h-y) + p_{[t_1-s,t_2-s]}(x-y)\big|^2 \\ & \times \|f(s,y+z)\|_{L^p(\Omega)}^2 |z|^{2H-2}\,dy\,dz\end{aligned}$$

and

$$\begin{aligned}A_{12}(s) = \int_{\mathbb{R}}\int_{\mathbb{R}} \big| & p_{[t_1-s,t_2-s]}(x+h-y) - p_{[t_1-s,t_2-s]}(x-y)\big|^2 \\ & \times \|f(s,y+z) - f(s,y)\|_{L^p(\Omega)}^2 |z|^{2H-2}\,dy\,dz.\end{aligned}$$

Let us now bound A_{11}. Invoking Plancherel's identity with respect to y and the explicit formula for $\mathcal{F}p_t$, we have

$$\begin{aligned}A_{11}(s) &\le C V_1^2(f) \int_{\mathbb{R}}\int_{\mathbb{R}} \big| p_{[t_1-s,t_2-s]}(h+y-z) - p_{[t_1-s,t_2-s]}(y-z) \\ &\qquad - p_{[t_1-s,t_2-s]}(h+y) + p_{[t_1-s,t_2-s]}(y)\big|^2 |z|^{2H-2}\,dy\,dz \\ &\le C V_1^2(f) \int_{\mathbb{R}}\int_{\mathbb{R}} \big| e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\big|^2 \\ &\qquad \times |e^{-i\xi z} - 1|^2 |e^{i\xi h} - 1|^2 |z|^{2H-2}\,d\xi\,dz \\ &\le C V_1^2(f) \int_{\mathbb{R}} \big| e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\big|^2 |e^{i\xi h} - 1|^2 |\xi|^{1-2H}\,d\xi.\end{aligned}$$

Moreover, owing to the inequality

$$\int_0^{t_1} \big| e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\big|^2\,ds \le \frac{|e^{-\frac{\Delta t\kappa}{2}|\xi|^2} - 1|^2}{\kappa|\xi|^2}, \tag{3.16}$$

we obtain

$$\begin{aligned}\int_0^{t_1} A_{11}(s)\,ds &\le C\kappa^{-1} V_1^2(f) \int_{\mathbb{R}} \big| e^{-\frac{\Delta t\kappa}{2}|\xi|^2} - 1\big|^2 |e^{i\xi h} - 1|^2 |\xi|^{-1-2H}\,d\xi \\ &\le C\kappa^{-1} V_1^2(f) I,\end{aligned} \tag{3.17}$$

where

$$I := \int_{\mathbb{R}} \big|1 - e^{-\frac{\Delta t\kappa}{2}|\xi|^2}\big|^2 \sin^2(\xi h/2) |\xi|^{-1-2H}\,d\xi. \tag{3.18}$$

Our next step is to bound I in two elementary and different ways:

(i) The change of variable $h\xi := \xi$ yields

$$I = |h|^{2H} \int_{\mathbb{R}} \big(1 - e^{-\frac{\kappa\Delta t}{2|h|^2}|\xi|^2}\big)^2 \sin^2(\xi/2) |\xi|^{-1-2H}\,d\xi,$$

and we then bound $1-e^{-\frac{\kappa\Delta t}{2h^2}}$ by 1 to obtain $I \le C|h|^{2H}$.

(ii) On the other hand, the change of variable $(\kappa\Delta t)^{1/2}\xi := \xi$ in (3.18) leads to

$$I = (\kappa\Delta t)^H \int_{\mathbb{R}} (1-e^{-\xi^2/2})^2 \sin^2\left(\frac{h\xi}{2(\kappa\Delta t)^{1/2}}\right)|\xi|^{-1-2H}\, d\xi,$$

and we bound the trigonometric function $\sin^2$ by 1 to obtain $I \le C(\kappa\Delta t)^H$.

Interpolating the two estimates, we have obtained for I, with a coefficient $\delta = \frac{\gamma}{2H} \in [0,1]$, we see that

$$I \le C|h|^{2H\delta}(\kappa\Delta t)^{H(1-\delta)} \le C(\kappa\Delta t)^{\frac{2H-\gamma}{2}}|h|^{\gamma}. \tag{3.19}$$

Plugging this identity back into (3.17), we have shown

$$\int_0^{t_1} A_{11}(s)\, ds \le C\kappa^{-1}(\kappa\Delta t)^{\frac{2H-\gamma}{2}}|h|^{\gamma} V_1^2(f),$$

for all $\gamma \in [0, 2H]$. Let us now turn to the estimate for A_{12}. Similar to what has been done for A_{11}, we get

$$\begin{aligned}\int_0^{t_1} A_{12}(s)\, ds &\le C V_2^2(f)\int_0^{t_1}\int_{\mathbb{R}} \left|p_{[t_1-s,t_2-s]}(h+y) - p_{[t_1-s,t_2-s]}(y)\right|^2 dy\, ds \\ &\le C V_2^2(f)\int_{\mathbb{R}}\int_0^{t_1} \left|e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2}\right|^2 ds \left|e^{i\xi h}-1\right|^2 d\xi.\end{aligned}$$

Thanks to (3.16), we thus end up with

$$\int_0^{t_1} A_{12}(s)\, ds \le C\kappa^{-1}V_2^2(f)\int_{\mathbb{R}} \left|1-e^{-\frac{\Delta t\kappa}{2}|\xi|^2}\right|^2 \sin^2(h\xi/2)|\xi|^{-2}\, d\xi.$$

In addition, the integral on the right-hand side can be estimated as I above, and we get

$$\int_0^{t_1} A_{12}(s)\, ds \le C V_2^2(f)(\kappa\Delta t)^{\frac{1-\gamma'}{2}}|h|^{\gamma'}$$

for all $\gamma' \in [0,1]$. Since $1 > 2H$, we may choose $\gamma' = \gamma$ to obtain

$$\int_0^{t_1} A_{12}(s)\, ds \le C\kappa^{-1}(\kappa\Delta t)^{\frac{2H-\gamma}{2}}|h|^{\gamma} V_2^2(f),$$

for all $\gamma \in [0, 2H]$. Hence, the bounds on A_{11} and A_{12} yield

$$\|\mathcal{A}_1\|_{L^p(\Omega)}^2 \le C V^2(f)(\Delta t)^{\frac{2H-\gamma}{2}}|h|^{\gamma},$$

for all $\gamma \in [0, 2H]$.

Step 2: *Upper bound for* $\mathcal{A}_2$. The term $\|\mathcal{A}_2\|_{L^p(\Omega)}^2$ can be estimated analogously to $\mathcal{A}_1$. Indeed, the reader can check that, owing to inequality (3.3) and Plancherel's identity, we have

$$\|\mathcal{A}_2\|_{L^p(\Omega)}^2 \le C V_1^2(f)\int_0^{\Delta t}\int_{\mathbb{R}} e^{-s\kappa|\xi|^2}\sin^2(h\xi/2)\left(|\xi|^{1-2H}+1\right) d\xi\, ds,$$

where we recall that V_1 is defined by (3.15). Taking integration in ds first, we see that

$$\|\mathcal{A}_2\|_{L^p(\Omega)}^2 \le C\kappa^{-1} V_1^2(f) \int_{\mathbb{R}} (1 - e^{-\Delta t \kappa |\xi|^2}) \sin^2(h\xi/2) (|\xi|^{-1-2H} + |\xi|^{-2})\, d\xi.$$

These two integrals can be estimated as the term I in (3.19), and we get

$$\|\mathcal{A}_2\|_{L^p(\Omega)}^2 \le C V_1^2(f) (\Delta t)^{\frac{2H-\gamma}{2}} |h|^\gamma,$$

for all $\gamma \in [0, 2H]$. Let us remark that the constants in all previous estimates depend only on T, p and κ^{-1}. In addition, as functions of (p, κ^{-1}), these constants have at most polynomial growth. Hence, gathering the estimates for $\|\mathcal{A}_1\|_{L^p(\Omega)}^2$ and $\|\mathcal{A}_2\|_{L^p(\Omega)}^2$ the proof of our claim (3.13) is complete.

Step 3: *Proof of* (3.14). Again, we assume that $t_1 < t_2$, and we proceed as in the previous steps and the proof of Proposition 3.5. Indeed, we begin by writing

$$\left\|\Phi(t_2, x) - \Phi(t_1, x)\right\|_{L^p(\Omega)} \le B_1 + B_2,$$

where

$$B_1 = \left\| \int_0^{t_1} \int_{\mathbb{R}} p_{[t_1-s, t_2-s]}(x-y) f(s,y) W(ds, dy) \right\|_{L^p(\Omega)}$$

and

$$B_2 = \left\| \int_{t_1}^{t_2} \int_{\mathbb{R}} p_{t_2-s}(x-y) f(s,y) W(ds, dy) \right\|_{L^p(\Omega)}.$$

Once again we handle those two terms separately.

For the term B_1, we resort to inequality (3.3) in our usual way. We get

$$\begin{aligned} B_1 \le{}& C\biggl(\int_0^{t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} p_{[t_1-s,t_2-s]}^2(x-y) \|f(s,y) - f(s,y+z)\|_{L^p(\Omega)}^2 \\ &\times |z|^{2H-2}\, dz\, dy\, ds \biggr)^{\frac{1}{2}} \\ &+ C\biggl(\int_0^{t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \bigl| p_{[t_1-s,t_2-s]}(x-y) - p_{[t_1-s,t_2-s]}(x-y-z) \bigr|^2 \\ &\times \|f(s,y+z)\|_{L^p(\Omega)}^2 |z|^{2H-2}\, dz\, dy\, ds \biggr)^{\frac{1}{2}}. \end{aligned}$$

With the definition (3.15) in mind, it is now readily checked that

$$B_1 \le C\bigl(B_{11} V_2(f) + B_{12} V_1(f)\bigr), \tag{3.20}$$

with

$$B_{11} = \left(\int_0^{t_1} \int_{\mathbb{R}} \bigl| p_{[t_1-s,t_2-s]}(x-y) \bigr|^2\, dy\, ds \right)^{\frac{1}{2}}$$

and

$$B_{12} = \left(\int_0^{t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| p_{[t_1-s,t_2-s]}(x-y) \right. \right.$$
$$\left. \left. - p_{[t_1-s,t_2-s]}(x-y-z) \right|^2 |z|^{2H-2} \, dz \, dy \, ds \right)^{\frac{1}{2}}.$$

We now appeal to Plancherel's identity to get

$$B_{11} = C \left(\int_0^{t_1} \int_{\mathbb{R}} \left| e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2} \right|^2 d\xi \, ds \right)^{\frac{1}{2}} = C(t_2-t_1)^{\frac{1}{4}}$$

and

$$B_{12} = C \left(\int_0^{t_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2} \right|^2 \left| e^{-i\xi z} - 1 \right|^2 |z|^{2H-2} \, dz \, d\xi \, ds \right)^{\frac{1}{2}}$$
$$= C \left(\int_0^{t_1} \int_{\mathbb{R}} \left| e^{-\frac{t_2-s}{2}\kappa|\xi|^2} - e^{-\frac{t_1-s}{2}\kappa|\xi|^2} \right|^2 |\xi|^{1-2H} \, d\xi \, ds \right)^{\frac{1}{2}} = C(t_2-t_1)^{\frac{H}{2}}.$$

Reporting these estimates in (3.20) and observing that $H < \frac{1}{2}$, we end up with

$$B_1 \le C(t_2-t_1)^{\frac{H}{2}} \left[V_1(f) + V_2(f) \right] \le C(t_2-t_1)^{\frac{H}{2}} \|f\|_{\mathfrak{X}^p_{T,\theta_0}} e^{\theta_0 T}.$$

The patient reader might check that the same kind of upper bound is valid for B_2, and gathering the estimates for B_1 and B_2 yields inequality (3.14). $\square$

4. Existence and uniqueness of the solution. In this section we will first establish a result regarding the uniqueness of the solution. Then based on the analysis of the structure of some new spaces of stochastic processes, as described in the Appendix, we will show the existence of the solution. Finally, we derive moment estimates. For the reader's convenience, in the first subsection we will introduce the function spaces needed in the sequel. In the second subsection, we summarize the main results of the current section. Details and proofs are provided in the following subsections.

4.1. *Function spaces.* In this subsection, we list the functions spaces that will be used along the paper.

(a) *The spaces* $\mathfrak{X}^{\beta}_T(B)$ *and* $\mathfrak{X}^{\beta,p}_T$: We recall that $\mathfrak{X}^{\beta}_T(B)$, introduced in Definition 3.3, is the space of all continuous functions f from $[0,T] \times \mathbb{R}$ to a Banach space B quipped with the norm $\|\cdot\|$ such that

$$\|f\|_{\mathfrak{X}^{\beta}_T(B)} := \sup_{t\in[0,T],x\in\mathbb{R}} \|f(t,x)\|$$
$$+ \sup_{t\in[0,T],x\in\mathbb{R}} \left(\int_{\mathbb{R}} \|f(t,x+h) - f(t,x)\|^2 |h|^{-1-2\beta} \, dh \right)^{1/2} < \infty.$$

When $B = L^p(\Omega)$ with $p \in [1, \infty)$, we use the notation $\mathfrak{X}_T^{\beta,p} = \mathfrak{X}_T^{\beta}(L^p(\Omega))$.

(b) *The space* $\mathcal{Z}_T^p$: We first introduce a norm $\|\cdot\|_{\mathcal{Z}_T^p}$ for a random field $v(t,x)$ as follows:

$$\|v\|_{\mathcal{Z}_T^p} = \sup_{t\in[0,T]} \|v(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,p} v(t), \tag{4.1}$$

where $p \geq 2$ and

$$\mathcal{N}^*_{\frac{1}{2}-H,p} v(t) := \left(\int_{\mathbb{R}} \|v(t,\cdot) - v(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2}\, dh \right)^{\frac{1}{2}}. \tag{4.2}$$

Then the space $\mathcal{Z}_T^p$ will consist all the random fields $v = v(t,x)$ such that $\|v\|_{Z_T^p}$ is finite. Observe that according to definition (3.1), we have $\mathcal{N}^*_{\frac{1}{2}-H,p} v(t) = \mathcal{N}^{L^p(\Omega\times\mathbb{R})}_{\frac{1}{2}-H,p} v(t)$.

(c) *The norm* $\mathcal{N}_\beta^{B,(\delta)}$: We denote by $C_{\mathrm{uc}}([0,T]\times\mathbb{R})$ the space of all real-valued continuous functions on $[0,T]\times\mathbb{R}$ equipped with the topology of uniform convergence over compact sets. Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$. Let $\beta \in (0,1)$ be a fixed number. For every $\delta \in (0,\infty]$ and every function $f : \mathbb{R} \to B$, we introduce the function $\mathcal{N}_\beta^{B,(\delta)} f : \mathbb{R} \to [0,\infty]$ defined by

$$\mathcal{N}_\beta^{B,(\delta)} f(x) = \left(\int_{|h|\leq\delta} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta}\, dh \right)^{\frac{1}{2}}. \tag{4.3}$$

Notice that for $\delta = \infty$, the quantity (4.3) coincides with the function $\mathcal{N}_\beta^{B,(\infty)} f = \mathcal{N}_\beta^B f$ introduced in (3.1). In the same way as for the quantities $\mathcal{N}_\beta^B f$, the function $\mathcal{N}_\beta^{B,(\delta)} f$ can be defined for functions defined on $\mathbb{R}_+ \times \mathbb{R}$, acting only on the spatial variable. In this case, we have $\mathcal{N}_\beta^{B,(\delta)} f : \mathbb{R}_+ \times \mathbb{R} \to [0,\infty]$. As our usual practice, when $B = \mathbb{R}$ we omit the dependence of $\mathbb{R}$ in $\mathcal{N}_\beta^{\mathbb{R},(\delta)}$ and simply write $\mathcal{N}_\beta^{(\delta)}$.

As we will see later in the Appendix of the paper, $\mathcal{N}_\beta^{B,(\delta)} f$ plays a role analogous to the modulus of continuity of f near x. It follows from the triangular inequality, that $\mathcal{N}$ satisfies

$$\left|\mathcal{N}_\beta^{B,(\delta)} f(x) - \mathcal{N}_\beta^{B,(\delta)} g(x)\right| \leq \mathcal{N}_\beta^{B,(\delta)} (f-g)(x) \tag{4.4}$$

for all $\delta \in (0,\infty]$, functions f, g and x in $\mathbb{R}$. Thus, $\mathcal{N}$ is a seminorm.

REMARK 4.1. Whenever σ is an affine function [i.e. $\sigma(u) = au + b$ for some constants a, b], the spaces $\mathfrak{X}_T^{\beta,p}$ are sufficient to show existence and uniqueness for equation (1.1). On the other hand, the case of general Lipschitz function σ leads to the consideration of additional spaces, which we are going to introduce now.

(d) *The spaces* X_T^β: The spaces X_T^β are the underlying spaces for our treatment of the existence result. Since these spaces do not belong to standard classes of function spaces, we will analyze them in detail in the Appendix. For every $h \in \mathbb{R}$, let τ_h be the translation map in the spatial variable, that is $\tau_h f(t,x) = f(t,x-h)$.

DEFINITION 4.2. Let X_T^β be the space of all real-valued continuous functions f on $[0,T] \times \mathbb{R}$ such that:

(i) $(t,x) \mapsto \mathcal{N}_\beta^{(1)} f(t,x)$ is finite and continuous on $[0,T] \times \mathbb{R}$.
(ii) $\lim_{h \downarrow 0} \sup_{t \in [0,T], x \in [-R,R]} \mathcal{N}_\beta^{(1)}(\tau_h f - f)(t,x) = 0$ for every $R > 0$.

We equip X_T^β with the following topology. A sequence $\{f_n\}$ in X_T^β converges to f in X_T^β if for all $R > 0$, the sequences $\{f_n\}$ and $\{\mathcal{N}_\beta^{(1)}(f_n - f)\}$ converge uniformly on $[0,T] \times [-R,R]$ to f and 0 respectively. We define a metric on X_T^β as follows:

$$d_\beta(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f-g\|_{n,\beta}}{1 + \|f-g\|_{n,\beta}}, \tag{4.5}$$

where $\|\cdot\|_{n,\beta}$ is the seminorm

$$\|f\|_{n,\beta} := \sup_{t \in [0,T], x \in [-n,n]} |f(t,x)| + \sup_{t \in [0,T], x \in [-n,n]} \mathcal{N}_\beta^{(1)} f(t,x).$$

Since functions in X_T^β are locally bounded, the topology of X_T^β is not altered if in the previous definition $\mathcal{N}_\beta^{(1)} f$ is replaced by $\mathcal{N}_\beta^{(\delta)} f$ for some *finite* and positive δ. We emphasize that replacing δ by ∞ would create a strictly smaller space. It is shown in the Appendix, Corollary A.6, that X_T^β is a complete and separable space.

4.2. *Main results*. The first result is the uniqueness of the solution to (1.1) assuming it has enough regularity.

THEOREM 4.3. *Assume the following conditions hold true*:

(1) *For $p > \frac{6}{4H-1}$, the initial condition u_0 is in $L^p(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|_{L^p(\mathbb{R})}^2 |h|^{2H-2}\, dh < \infty. \tag{4.6}$$

(2) *σ is differentiable, its derivative is Lipschitz and $\sigma(0) = 0$.*
(3) *u and v are two solutions of (1.1) and $u, v \in \mathcal{Z}_T^p$.*

Then for every $t \in [0,T]$ and $x \in \mathbb{R}$, $u(t,x) = v(t,x)$, a.s.

REMARK 4.4. This is the first occurrence of the hypothesis $\sigma(0) = 0$, and one might wonder about the necessity of this assumption. To this respect, let us mention that if we define Φ as in (3.7) for $f \equiv \mathbf{1}$, then Φ does not belong to $\mathcal{Z}_T^p$.

The main ingredient of the proof of Theorem 4.3 is a localization argument based on uniform estimates (in space and time) of stochastic convolutions, which is provided by Lemma 4.9.

The next theorem is the existence result.

THEOREM 4.5. *Assume that for equation* (1.1) *the following conditions hold*:

(1) *For some* $\beta_0 > \frac{1}{2} - H$ *and some* $p > \max(\frac{6}{4H-1}, \frac{1}{\beta_0+H-1/2})$, *the initial condition* u_0 *is in* $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and*

$$\sup_{x\in\mathbb{R}} \mathcal{N}_{\beta_0} u_0(x) + \left(\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|_{L^p(\mathbb{R})}^2 |h|^{2H-2}\, dh\right)^{\frac{1}{2}} < \infty. \tag{4.7}$$

(2) σ *is differentiable*, *the derivative of* σ *is Lipschitz and* $\sigma(0) = 0$.

Then there exists a solution u *to* (1.1) *in* $\mathcal{Z}_T^p \cap \mathfrak{X}_T^{\frac{1}{2}-H,p}$. *In addition*, *the solution has sample paths in the space* $X_T^{\frac{1}{2}-H}$ *a.s.*

REMARK 4.6. (i) We can add a drift $b(u(t,x))$ in equation (1.1), and if the function b is Lipschitz continuous with $b(0) = 0$, the results we have obtained on the existence and uniqueness of a solution can be extended to equations with drift.

(ii) If we only assume that the initial condition u_0 is bounded and

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |u_0(x) - u_0(x+h)|^2 |h|^{2H-2}\, dh < \infty, \tag{4.8}$$

and we only assume that σ is Lipschitz, then from the proof of Theorem 4.5 we can show that we have the weak existence of a solution to equation (1.1). By this we mean the existence of a couple of predictable random fields (X, W) parameterized by $[0, T] \times \mathbb{R}$, defined in some filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, such that W is a centered Gaussian process with covariance (1.2) and X is a mild solution to (1.1) in the sense of Definition 2.4, belonging to the space $\mathfrak{X}_T^{\frac{1}{2}-H,p}$ with trajectories almost surely in $X_T^{\frac{1}{2}-H}$. The assumption (1) in Theorem 4.5 and the condition that the derivative of σ is Lipschitz and $\sigma(0) = 0$ are only used to show the uniqueness.

Finally, the techniques we have designed to get existence and uniqueness for equation (1.1) also allow us to obtain the following moment bound for the solution.

THEOREM 4.7. *Assuming the conditions in Theorem 4.5, then a solution of (1.1) satisfies following moment bounds*:

$$\sup_{x\in\mathbb{R}}\|u(t,x)\|_{L^p(\Omega)} \leq 2\|u_0\|_{\varepsilon_0} \exp\{\theta_0 p^{\frac{1}{H}} t\} \tag{4.9}$$

and

$$\sup_{x\in\mathbb{R}} \mathcal{N}_{1/2-H,p} u(t,x) \leq 2\|u_0\|_{\varepsilon_0} \varepsilon_0^{-1} \exp\{\theta_0 p^{\frac{1}{H}} t\},$$

where we recall that $\mathcal{N}_{1/2-H,p}$ *is defined by (3.2), and where for any* $\varepsilon > 0$ *we have used the notation*

$$\|u_0\|_\varepsilon := \sup_{x\in\mathbb{R}}|u_0(x)| + \varepsilon \sup_{x\in\mathbb{R}}\left(\int_{\mathbb{R}} |u_0(x+h) - u_0(x)|^2 |h|^{2H-2}\, dh\right)^{\frac{1}{2}}.$$

In the formulae above, $\theta_0 = (6C_0)^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} \|\sigma\|_{\mathrm{Lip}}^{\frac{2}{H}}$ *and*

$$\varepsilon_0 = (6C_0)^{1-\frac{1}{2H}} \kappa^{\frac{1}{4H}-\frac{1}{2}} p^{\frac{1}{2}-\frac{1}{4H}} \|\sigma\|_{\mathrm{Lip}}^{1-\frac{1}{2H}},$$

where C_0 *is defined in (3.12). In addition, from Proposition A.1, we see that the initial condition* u_0 *is Hölder continuous with order* β_0, *then by Proposition 3.7 we have*

$$\|u(t,x) - u(s,y)\|_{L^p(\Omega)} \leq C\big(|t-s|^{\frac{H}{2}\wedge\frac{\beta_0}{2}} + |x-y|^{H\wedge\beta_0}\big) \tag{4.10}$$

for all $s, t \in [0,T]$ *and* $x, y \in \mathbb{R}$.

We also have the following matching lower bound in terms of κ and t for the second moment.

PROPOSITION 4.8. *Under the conditions of Theorem 4.5, let* u *be a solution to equation (1.1). Suppose that* u_0 *is a bounded nontrivial function and there is a positive constant* σ_* *such that* $|\sigma(z)| \geq \sigma_*|z|$ *for all* $z \in \mathbb{R}$. *Then there exist some universal constants* C *and* L *such that*

$$\mathbf{E}|u(t,x)|^2 \geq C\frac{|p_t * u_0(x)|^3}{\|u_0\|_{L^\infty}} \exp\{L\sigma_*^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} t\}. \tag{4.11}$$

4.3. *Proof of the uniqueness.* In this subsection, we prove Theorem 4.3. First, we need the following result.

LEMMA 4.9. *Suppose that* $p > \frac{6}{4H-1}$. *Let* v *be a process in the space* $\mathcal{Z}_T^p$. *As in (3.7), define*

$$\Phi(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) v(s,y) W(ds,dy). \tag{4.12}$$

Then there exists a constant C depending on T, p and H, such that

$$\Big\| \sup_{t\in[0,T],x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)\Big\|_{L^p(\Omega)} \le C\|v\|_{\mathcal{Z}_T^p}, \tag{4.13}$$

where recalling our definition (3.1), *we have* $\mathcal{N}_{1/2-H}\Phi(t,x) = \mathcal{N}^{\mathbb{R}}_{1/2-H}\Phi(t,x)$.

REMARK 4.10. Let us stress the following facts:

(i) In relation (4.13), the operator $\mathcal{N}_{\frac{1}{2}-H}$ [defined in (3.2)] acts on the trajectories of the random field $\Phi(t,x)$. As a consequence, $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$ is a random variable.

(ii) With respect to Proposition 3.5, inequality (4.13) involves a supremum in the variable $x \in \mathbb{R}$ before taking $L^p(\Omega)$ norms. We thus get a stronger result with a different kind of assumptions (namely $v \in \mathcal{Z}_T^p$ instead of $v \in \mathfrak{X}_T^{\frac{1}{2}-H,p}$).

PROOF OF LEMMA 4.9. We shall apply the factorization method to handle the stochastic convolution (see, for instance, [7]). Namely, an application of a stochastic version of Fubini's theorem enables to write

$$\Phi(t,x) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}p_{t-r}(x-z)Y(r,z)\,dz\,dr,$$

with

$$Y(r,z) = \int_0^r\int_{\mathbb{R}}(r-s)^{-\alpha}p_{r-s}(z-y)v(s,y)W(ds,dy),$$

and where $\alpha \in (0,1)$ is a parameter whose value will be chosen later. The proof will be done in two steps.

Step 1: *Uniform estimate of* $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$. In order to estimate $\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$, we bound the difference $\Phi(t,x) - \Phi(t,x+h)$ as follows:

$$\begin{aligned}
&\big|\Phi(t,x) - \Phi(t,x+h)\big| \\
&\quad = \frac{\sin(\alpha\pi)}{\pi}\bigg|\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\big(p_{t-r}(x-z) \\
&\qquad - p_{t-r}(x+h-z)\big)Y(r,z)\,dz\,dr\bigg| \\
&\quad \le \frac{\sin(\alpha\pi)}{\pi}\int_0^t(t-r)^{\alpha-1}\|p_{t-r}(\cdot) - p_{t-r}(\cdot+h)\|_{L^q(\mathbb{R})}\|Y(r,\cdot)\|_{L^p(\mathbb{R})}\,dr,
\end{aligned}$$

where q satisfies $p^{-1}+q^{-1}=1$. So using Minkowski's integral inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}}\big|\Phi(t,x) - \Phi(t,x+h)\big|^2|h|^{2H-2}\,dh \\
&\quad \le C\int_{\mathbb{R}}\bigg(\int_0^t(t-r)^{\alpha-1}\|p_{t-r}(x-\cdot) - p_{t-r}(x+h-\cdot)\|_{L^q(\mathbb{R})}
\end{aligned} \tag{4.14}$$

$$
\times \|Y(r,\cdot)\|_{L^p(\mathbb{R})}\,dr\Big)^2 |h|^{2H-2}\,dh
$$

$$
\leq C\bigg(\int_0^t (t-r)^{\alpha-1}\|Y(r,\cdot)\|_{L^p(\mathbb{R})}\big[K_t(r)\big]^{1/2}\,dr\bigg)^2,
$$

where we have set

$$
K_t(r) := \int_{\mathbb{R}} \|p_{t-r}(x-z) - p_{t-r}(x+h-z)\|^2_{L^q(\mathbb{R},dz)}|h|^{2H-2}\,dh.
$$

Now the kernel K_t can be bounded by elementary methods: with the change of variable $z \to \sqrt{t-r}z$ and $h \to \sqrt{t-r}h$, we obtain

$$
\begin{aligned}
K_t(r) &= \int_{\mathbb{R}} \|p_{t-r}(x-z) - p_{t-r}(x+h-z)\|^2_{L^q(\mathbb{R},dz)}|h|^{2H-2}\,dh \\
&= C(t-r)^{-\frac{3}{2}+\frac{1}{q}+H}\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}\big|e^{-\frac{z^2}{2\kappa}} - e^{-\frac{(z+h)^2}{2\kappa}}\big|^q\,dz\bigg)^{\frac{2}{q}}|h|^{2H-2}\,dh \\
&= C(t-r)^{-\frac{1}{2}-\frac{1}{p}+H},
\end{aligned}
$$

where we have used the fact that $q^{-1} = 1 - p^{-1}$, and the constant C in the above equation and below in this proof may depend on κ. Going back to (4.14), the following holds true:

$$
\begin{aligned}
&\int_{\mathbb{R}} \big|\Phi(t,x) - \Phi(t,x+h)\big|^2|h|^{2H-2}\,dh \\
&\quad\leq C\bigg(\int_0^t (t-r)^{\alpha-1+\frac{1}{2}(H-\frac{1}{p}-\frac{1}{2})}\|Y(r,\cdot)\|_{L^p(\mathbb{R})}\,dr\bigg)^2 \\
&\quad\leq C\bigg(\int_0^t (t-r)^{q[\alpha-1+\frac{1}{2}(H-\frac{1}{2}-\frac{1}{p})]}\,dr\bigg)^{\frac{2}{q}}\bigg(\int_0^t \|Y(r,\cdot)\|^p_{L^p(\mathbb{R})}\,dr\bigg)^{\frac{2}{p}}.
\end{aligned}
$$

We can now start to tune our parameters. It is easily checked that the first integral in the right-hand side above is finite (uniformly in $0 < t \leq T$) if and only if

$$
\alpha > \frac{3}{2p} + \frac{1}{4} - \frac{H}{2}. \tag{4.15}
$$

With this choice of α, we get

$$
\int_{\mathbb{R}} \big|\Phi(t,x) - \Phi(t,x+h)\big|^2|h|^{2H-2}\,dh \leq C\bigg(\int_0^t \|Y(r,\cdot)\|^p_{L^p(\mathbb{R})}\,dr\bigg)^{\frac{2}{p}},
$$

and since this bound is uniform in x, this yields

$$
\sup_{t\in[0,T],x\in\mathbb{R}} \big[\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)\big]^2 \leq C\bigg(\int_0^T \|Y(r,\cdot)\|^p_{L^p(\mathbb{R})}\,dr\bigg)^{\frac{2}{p}}. \tag{4.16}
$$

Then, to prove (4.13) it suffices to show that

$$\mathbf{E}\int_{\mathbb{R}} \big|Y(r,z)\big|^p \, dz \le C\|v\|_{\mathcal{Z}_T^p}^p. \tag{4.17}$$

Step 2: *Proof of* (4.17). Set $g_{r,z}(s,y) = (r-s)^{-\alpha} p_{r-s}(z-y) v(s,y)$, so that

$$Y(r,z) = \int_0^r \int_{\mathbb{R}} g_{r,z}(s,y) W(ds,dy).$$

Then applying the Burkholder-type inequality (3.3), plus an elementary decomposition of the increments of $g_{r,z}$, we obtain

$$\mathbf{E}\int_{\mathbb{R}} \big|Y(r,z)\big|^p \, dz \le C\big[D_1(r) + D_2(r)\big],$$

where

$$D_1(r) = \int_{\mathbb{R}} \bigg(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \big| p_{r-s}(y) - p_{r-s}(y+h)\big|^2 \\ \times \big\| v(s, y+z+h)\big\|_{L^p(\Omega)}^2 |h|^{2H-2} \, dh \, dy \, ds \bigg)^{\frac{p}{2}} dz$$

and

$$D_2(r) = \int_{\mathbb{R}} \bigg(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \big| p_{r-s}(y)\big|^2 \\ \times \big\| v(s, y+z+h) - v(s, y+z)\big\|_{L^p(\Omega)}^2 |h|^{2H-2} \, dh \, dy \, ds \bigg)^{\frac{p}{2}} dz.$$

Let us now bound the term D_1. Invoking Minkowski's integral inequality, it is easily seen that

$$D_1(r) \le \bigg(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \big| p_{r-s}(y) - p_{r-s}(y+h)\big|^2 \\ \times \big\| v(s, \cdot)\big\|_{L^p(\Omega\times\mathbb{R})}^2 |h|^{2H-2} \, dh \, dy \, ds \bigg)^{\frac{p}{2}}.$$

Integrating this identity in h and y, we end up with

$$D_1(r) \le C\bigg(\int_0^r (r-s)^{-2\alpha+H-1} \big\| v(s,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})}^2 \, ds \bigg)^{\frac{p}{2}}.$$

Similarly, we get the following estimate for $D_2(r)$:

$$D_2(r) \le C\bigg(\int_0^r \int_{\mathbb{R}} (r-s)^{-2\alpha-\frac{1}{2}} \big\| v(s,\cdot+h) - v(s,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})}^2 |h|^{2H-2} \, dh \, ds \bigg)^{\frac{p}{2}} \\ = C\bigg(\int_0^r (r-s)^{-2\alpha-\frac{1}{2}} \big[\mathcal{N}^*_{\frac{1}{2}-H,p} v(s)\big]^2 \, ds \bigg)^{\frac{p}{2}}.$$

Combining the estimates for $D_1(r)$ and $D_2(r)$, we obtain

$$\begin{aligned}\mathbf{E}\int_{\mathbb{R}}\big|Y(r,z)\big|^p\,dz \le C\biggl(\int_0^r\bigl[(r-s)^{-2\alpha+H-1}\|v(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}\\ +(r-s)^{-2\alpha-\frac{1}{2}}\bigl[\mathcal{N}^*_{\frac{1}{2}-H,p}v(s)\bigr]^2\bigr]\,ds\biggr)^{\frac{p}{2}}.\end{aligned}\tag{4.18}$$

Let us go back now to the values of our parameters α, p. One can check that the two singularities in the integrals on the right-hand side above are nondivergent whenever $\alpha < \frac{H}{2}$. Combining this condition with the restriction (4.15), we end up with the relation

$$\frac{3}{2p}+\frac{1}{4}-\frac{H}{2}<\alpha<\frac{H}{2}.\tag{4.19}$$

Those two conditions can be jointly met if and only if $H > \frac{1}{4}$ and $p > \frac{6}{4H-1}$. This completes the proof of the lemma. □

REMARK 4.11. Notice that the previous lemma implies that for any process $v \in \mathcal{Z}_T^p$, the random variable $\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\mathcal{N}_{\frac{1}{2}-H}\Phi(t,x)$ is finite almost surely, if Φ is given by (4.12).

We are ready to prove Theorem 4.3.

PROOF OF THEOREM 4.3. Assume that u solves (1.1) and $u \in \mathcal{Z}_T^p$. From the mild formulation of the solution, we have

$$u(t,x)=p_t*u_0(x)+\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\sigma\bigl(u(s,y)\bigr)W(ds,dy).\tag{4.20}$$

We claim that

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\mathcal{N}_{\frac{1}{2}-H}u(t,x)<\infty,\qquad \text{a.s.}\tag{4.21}$$

This follows from the decomposition (4.20). Indeed, on one hand, (4.6) implies that, if $g(t,x)=p_t*u_0(x)$, then

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\mathcal{N}_{\frac{1}{2}-H}g(t,x)<\infty.$$

On the other hand, from the properties of σ, it follows that if $u \in \mathcal{Z}_T^p$, then $\sigma(u)$ also belongs to $\mathcal{Z}_T^p$ [notice that to estimate the first term of (4.1) for $\sigma(u)$, we need to assume $\sigma(0)=0$]. Hence, Remark 4.11 entails

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\mathcal{N}_{\frac{1}{2}-H}\sigma(u)(t,x)<\infty,\qquad \text{a.s.}$$

If v is another solution of equation (1.1) belonging also to $\mathcal{Z}_T^p$, then (4.21) also holds for v. In this way, we can define the stopping times

$$T_k = \inf\Big\{0 \le t \le T : \sup_{0\le s\le t, x\in\mathbb{R}} \int_{\mathbb{R}} |u(s,x) - u(s,x+h)|^2 |h|^{2H-2}\, dh \ge k \\ \text{or} \sup_{0\le s\le t, x\in\mathbb{R}} \int_{\mathbb{R}} |v(s,x) - v(s,x+h)|^2 |h|^{2H-2}\, dh \ge k\Big\},$$

and $T_k \uparrow T$, almost surely, as k tends to infinity. Our strategy will be to control the two following quantities:

$$I_1(t,x) = \mathbf{E}\big[\mathbf{1}_{\{t<T_k\}} |u(t,x) - v(t,x)|^2\big]$$

and

$$I_2(t,x) = \mathbf{E}\Big[\int_{\mathbb{R}} \mathbf{1}_{\{t<T_k\}} |u(t,x) - v(t,x) - u(t,x+h) + v(t,x+h)|^2 |h|^{2H-2}\, dh\Big].$$

We also set $\mathcal{I}_j(t) = \sup_{x\in\mathbb{R}} I_j(t,x)$ for $j = 1, 2$.

In order to bound I_1, let us first use elementary properties of Itô's integral, which yield

$$\begin{aligned}
&\mathbf{1}_{\{t<T_k\}}\big(u(t,x) - v(t,x)\big) \\
&\quad = \mathbf{1}_{\{t<T_k\}} \int_0^{t\wedge T_k} \int_{\mathbb{R}} p_{t-s}(x-y)\big[\sigma\big(u(s,y)\big) - \sigma\big(v(s,y)\big)\big] W(ds,dy) \\
&\quad = \mathbf{1}_{\{t<T_k\}} \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\mathbf{1}_{\{s<T_k\}}\big[\sigma\big(u(s,y)\big) - \sigma\big(v(s,y)\big)\big] W(ds,dy).
\end{aligned}$$

We thus get $I_1(t,x) \le C(I_{11}(t,x) + I_{12}(t,x))$, where

$$\begin{aligned}
I_{11}(t,x) = \mathbf{E} \int_0^t \int_{\mathbb{R}^2} &|p_{t-s}(x-y) - p_{t-s}(x-y-h)|^2 \\
&\times \mathbf{1}_{\{s<T_k\}} |\sigma\big(u(s,y+h)\big) - \sigma\big(v(s,y+h)\big)|^2 |h|^{2H-2}\, dh\, dy\, ds
\end{aligned}$$

and

$$\begin{aligned}
I_{12}(t,x) = \mathbf{E} \int_0^t \int_{\mathbb{R}^2} &p_{t-s}^2(x-y)\mathbf{1}_{\{s<T_k\}} |\sigma\big(u(s,y)\big) - \sigma\big(v(s,y)\big) \\
&- \sigma\big(u(s,y+h)\big) + \sigma\big(u(s,y+h)\big)|^2 |h|^{2H-2}\, dh\, dy\, ds.
\end{aligned}$$

Next, we bound the term $I_{11}(t,x)$ as follows:

$$\begin{aligned}
I_{11}(t,x) &\le C\mathbf{E} \int_0^t \int_{\mathbb{R}^2} |p_{t-s}(x-y) - p_{t-s}(x-y-h)|^2 \\
&\qquad \times \mathbf{1}_{\{s<T_k\}} |u(s,y+h) - v(s,y+h)|^2 |h|^{2H-2}\, dh\, dy\, ds \\
&\le C \int_0^t (t-s)^{H-1} \mathcal{I}_1(s)\, ds,
\end{aligned}$$

where we recall that $\mathcal{I}_1(t) = \sup_{x\in\mathbb{R}} I_1(t,x)$, and the constant C in the above inequality and below in this proof may depend on κ. Let us now invoke the following elementary bound on the rectangular increments of σ, valid whenever σ' is Lipschitz:

$$\big|\sigma(a)-\sigma(b)-\sigma(c)+\sigma(d)\big| \le C|a-b-c+d| + C|a-b|\big(|a-c|+|b-d|\big).$$

With this additional ingredient, and along the same lines as for $I_{11}(t,x)$, we get

$$I_{12}(t,x) \le Ck\int_0^t (t-s)^{-\frac{1}{2}}\big[\mathcal{I}_1(s)+\mathcal{I}_2(s)\big]\,ds.$$

Finally, gathering our estimates on I_{11} and I_{12} we end up with

$$\mathcal{I}_1(t) \le Ck\int_0^t (t-s)^{H-1}\big[\mathcal{I}_1(s)+\mathcal{I}_2(s)\big]\,ds.$$

The term $I_2(t,x)$ above is dealt with exactly the same way, and we leave to the reader the task of showing that

$$\mathcal{I}_2(t) \le Ck\int_0^t (t-s)^{2H-\frac{3}{2}}\big[\mathcal{I}_1(s)+\mathcal{I}_2(s)\big]\,ds.$$

As a consequence,

$$\mathcal{I}_1(t)+\mathcal{I}_2(t) \le Ck\int_0^t (t-s)^{2H-\frac{3}{2}}\big[\mathcal{I}_1(s)+\mathcal{I}_2(s)\big]\,ds,$$

which implies $\mathcal{I}_1(t)+\mathcal{I}_2(t)=0$ for all $t\in[0,T]$. In particular,

$$\mathbf{E}\big[\mathbf{1}_{\{t<T_k\}}\big|u(t,x)-v(t,x)\big|^2\big]=0,$$

which implies $u(t,x)=v(t,x)$ a.s. on $\{t<T_k\}$ for all $k\ge 1$ and $t\in[0,T]$. Therefore, taking into account that $T_k\uparrow\infty$ a.s. as k tends to infinity, we conclude that $u(t,x)=v(t,x)$ a.s. for all $(t,x)\in[0,T]\times\mathbb{R}$. This proves the uniqueness. □

4.4. *Proof of the existence*. In this subsection, we prove Theorem 4.5. The methodology, inspired by the work of Gyöngy [9] on semilinear stochastic partial differential equations, consists in proving tightness of a sequence of solutions obtained by regularizing the noise, and then using the uniqueness result. The space $\mathcal{Z}_T^p$, where we proved our uniqueness result, consists of $L^p(\mathbb{R})$-valued processes, and it is not clear how to characterize compactness of probability laws on the space of trajectories of these processes. For this reason, we prove the existence of a solution with paths in the space $X_T^{\frac{1}{2}-H}$ introduced in Definition 4.2, equipped with the metric (4.5).

PROOF OF THEOREM 4.5. As mentioned above, we follow the methodology developed in [9] and we consider a regularization of the noise in space. Indeed, for each $\varepsilon > 0$ and $\varphi \in \mathfrak{H}$, we define

$$(4.22)\quad \begin{aligned} W_\varepsilon(\varphi) &= \int_0^t \int_{\mathbb{R}} [\rho_\varepsilon * \varphi](s,y) W(ds,dy) \\ &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(s,x)\rho_\varepsilon(x-y) W(ds,dy)\,dx. \end{aligned}$$

The noise W_ε induces an approximation to equation (2.12), namely

$$(4.23)\quad u_\varepsilon(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma\big(u_\varepsilon(s,y)\big) W_\varepsilon(ds,dy),$$

where the integral is understood in the Itô sense. Applying Lemma 4.12 below, we see that for each $\varepsilon > 0$, equation (4.23) has a unique solution u_ε satisfying

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathfrak{X}_T^{\beta,p}} < \infty,$$

for all $\beta \le \beta_0$ and $\frac{1}{2} - H \le \beta < H$. In particular, because $\frac{1}{2} - H < \beta_0 - \frac{1}{p}$, we can choose β such that $\frac{1}{2} - H < \beta - \frac{1}{p}$. In addition, we can show that u_ε satisfies Condition (2) in Proposition A.17. With these properties, we can check that the three conditions in Proposition A.17 are satisfied. Hence, the laws of the processes u_ε, considered as probability measures on the space $X_T^{\frac{1}{2}-H}$ are tight, and hence weakly relatively compact.

We now base our final considerations on the forthcoming Lemmas 4.13–4.16. Fix a sequence ε_n converging to zero and set $u_n = u_{\varepsilon_n}$. We shall hinge on Lemma 4.14 in order to prove that the sequence u_n actually converges in probability. To apply this lemma, we consider now two sequences $u_{m(n)}$ and $u_{l(n)}$, where $\{m(n), n \ge 1\}$ and $\{l(n), n \ge 1\}$ are strictly increasing sequences of positive integers. For each $n \ge 1$, the triplet $(u_{m(n)}, u_{l(n)}, W)$ defines probability measure on the space

$$\mathcal{B} := X_T^{\frac{1}{2}-H} \times X_T^{\frac{1}{2}-H} \times C_{\mathrm{uc}}\big([0,T]\times\mathbb{R}\big).$$

Since the family $\{u_\varepsilon, \varepsilon > 0\}$ is weakly relatively compact, there exists a subsequence of the form $\{(u_{m(n_k)}, u_{l(n_k)}, W), k \ge 1\}$ which converges in distribution as k tends to infinity. Thus, by Skorokhod embedding theorem, there is a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$ and a sequence of random elements $z_k = (u'_{m(n_k)}, u'_{l(n_k)}, W')$ with values on $\mathcal{B}$ such that z_k has the same distribution as $(u_{m(n_k)}, u_{l(n_k)}, W)$ and z_k converges almost surely (in the topology of $\mathcal{B}$) to (u', v', W'). By Lemma 4.16, we see that both u' and v' are solutions to equation (2.12), with W replaced by W'. Then by Lemma 4.15 and the uniqueness result Theorem 4.3 we thus get that $u' = v'$ in $X_T^{\frac{1}{2}-H}$. We can now apply Lemma 4.14 in order to assert that u_n converges to some random field u in $X_T^{\frac{1}{2}-H}$, in probability. Moreover, taking a subsequence if necessary, we see that u_n converges to u in $X_T^{\frac{1}{2}-H}$ a.s. Hence, thanks to another application of Lemma 4.16 we see that u satisfies equation (2.12). This proves the existence of the solution. $\square$

We now state the lemmas on which the proof of Theorem 4.5 relies.

LEMMA 4.12. *Let the initial condition u_0 and β_0 be as in Theorem 4.5. Then for each $\varepsilon > 0$, equation (4.23) has a unique solution u_ε such that $u_\varepsilon \in \mathfrak{X}_T^{\beta,q}$ for every $\beta \in (0, \beta_0]$ and $q \geq 2$. Furthermore, for all $q \geq 2$, $\beta \leq \beta_0$ and $\frac{1}{2} - H \leq \beta < H$,*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathfrak{X}_T^{\beta,q}} < \infty. \tag{4.24}$$

PROOF. Fix $q \geq 2$. Since $|\xi|^{1-2H} e^{-\varepsilon|\xi|^2}$ is in $L^1(\mathbb{R})$, $|f_\varepsilon|$ is bounded. Thus, using Picard iteration, it is easy to see that (4.23) has a unique random field solution, and by estimating the qth moment of $|u_\varepsilon(t,x) - u_\varepsilon(t,x')|$, we see that each solution u_ε belongs to $\mathfrak{X}_T^{\frac{1}{2}-H,q}$. We remark that $\|u_\varepsilon\|_{\mathfrak{X}_T^{\frac{1}{2}-H,q}}$ may not be bounded uniformly in ε as seen from this procedure. Using the definition of W_ε and stochastic Fubini theorem, for any predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that $\mathbf{E}[\|g\|_{\mathfrak{H}}^2] < \infty$, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} g(s,y) W_\varepsilon(ds,dy) &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} g(s,y) \rho_\varepsilon(y-z) W(ds,dz)\, dy \\ &= \int_0^t \int_{\mathbb{R}} g(s,\cdot) * \rho_\varepsilon(z) W(ds,dz). \end{aligned}$$

Applying Proposition 3.2, we obtain

$$\left\| \int_0^t \int_{\mathbb{R}} g(s,y) W_\varepsilon(ds,dy) \right\|_{L^q(\Omega)}^2 \leq C_q \int_0^t \int_{\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H,q}\, g(s,\cdot) * \rho_\varepsilon(y)]^2\, dy\, ds.$$

In addition, from (4.13) and Minkowski inequality, we see that

$$\begin{aligned} &\int_{\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H,q}\, g(s,\cdot) * \rho_\varepsilon(y)]^2\, dy \\ &\quad \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \|g(s,y+h-z) - g(s,y-z)\|_{L^q(\Omega)}^2 \rho_\varepsilon(z)\, dz |h|^{2H-2}\, dh\, dy \\ &\quad = \int_{\mathbb{R}} [\mathcal{N}_{\frac{1}{2}-H,q}\, g(s,y)]^2\, dy. \end{aligned}$$

So, if we proceed as in the proof of Proposition 3.5, and take $\varepsilon = 1$ in (3.8), we are able to get

$$\|u_\varepsilon\|_{\mathfrak{X}_{T,\theta}^{\beta,q}} \leq C_0 + C_q \|u_\varepsilon\|_{\mathfrak{X}_{T,\theta}^{\frac{1}{2}-H,q}} \big(\theta^{-\frac{H}{2}} + \theta^{\frac{\beta}{2}-\frac{1}{4}} + \theta^{-\frac{1}{4}} + \theta^{\frac{\beta}{2}-\frac{H}{2}}\big) \tag{4.25}$$

for any $\beta \le \beta_0$ and $\beta < H$. By taking $\beta = \frac{1}{2} - H$ and θ large enough such that

$$C_q\big(\theta^{-\frac{H}{2}} + \theta^{\frac{\beta}{2}-\frac{1}{4}} + \theta^{-\frac{1}{4}} + \theta^{\frac{\beta}{2}-\frac{H}{2}}\big) \le \frac{1}{2},$$

noting that $\|u_\varepsilon\|_{\mathfrak{X}_{T,\theta}^{\frac{1}{2}-H,q}}$ is finite, (4.25) implies that $\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathfrak{X}_{T,\theta}^{\frac{1}{2}-H,q}}$ is at most $2C_0$. Plugging this back into (4.25) yields (4.24). $\square$

The next lemma is a version of Gronwall's lemma, borrowed from [4], Lemma 15, and the correction [5] to that paper.

LEMMA 4.13. *Let $g \in L^1([0,T];\mathbb{R}_+)$ and consider a sequence of functions $\{f_n; n \ge 0\}$ with $f_n : [0,T] \to \mathbb{R}_+$, such that f_0 is bounded and for all $n \ge 1$*

$$f_n(t) \le c_1 + c_2 \int_0^t g(t-s) f_{n-1}(s)\,ds, \tag{4.26}$$

for two positive constants c_1, c_2. Then $\sup_{n\ge1} f_n$ is bounded. If we assume moreover that $c_1 = 0$ in inequality (4.26), we obtain that $\sum_{n\ge0} f_n^{1/p}$ converges uniformly in $[0,T]$, for all $1 \le p < \infty$.

The third lemma is a general result on convergence of random variables borrowed from [9, 10].

LEMMA 4.14. *Let $\mathbb{E}$ be a Polish space equipped with the Borel σ-algebra. A sequence of $\mathbb{E}$-valued random elements z_n converges in probability if and only if for every pair of subsequences $z_{l(n)}$, $z_{m(n)}$ there exists a subsequence $w_k := (z_{l(n_k)}, z_{m(n_k)})$ converging weakly to a random element w supported on the diagonal $\{(x,y) \in \mathbb{E}\times\mathbb{E} : x = y\}$.*

The next result asserts that the approximate solution to the stochastic heat equation is uniformly bounded in the space $\mathcal{Z}_T^p$ defined by (4.2).

LEMMA 4.15. *The approximate solutions u_ε satisfy the condition*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathcal{Z}_T^p} < \infty. \tag{4.27}$$

Furthermore, if $u_\varepsilon \to u$ in $X_T^{\frac{1}{2}-H}$ a.s., as ε tends to zero, then u is also in $\mathcal{Z}_T^p$.

PROOF. We will use Picard's iteration to show that for each ε, $u_\varepsilon \in \mathcal{Z}_T^p$. Then we will use Gronwall's lemma to show that the processes u_ε are uniformly (in ε) bounded in $\mathcal{Z}_T^p$. To this end, we first define

$$u_\varepsilon^0(t,x) = p_t * u_0(x),$$

and recursively

$$u_\varepsilon^{n+1}(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma\big(u_\varepsilon^n(s,y)\big) W_\varepsilon(ds,dy).$$

We wish to bound $\|u_\varepsilon^n\|_{\mathcal{Z}_T^p}$ uniformly in n. First, recall that

$$\|u_\varepsilon^n\|_{\mathcal{Z}_T^p} = \sup_{t\in[0,T]} \|u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(t),$$

where $\mathcal{N}^*_{\frac{1}{2}-H,p}$ is defined in (4.2). Let us now bound the terms $\|u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ and $\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon^n(t)$.

Step 1. We shall bound $\|u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ uniformly in n by considering the differences of Picard's iterations. Indeed, by Burkholder's inequality we have

$$\begin{aligned}
&\mathbf{E}\big|u_\varepsilon^{n+1}(t,x) - u_\varepsilon^n(t,x)\big|^p \\
&\quad = \mathbf{E}\bigg|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\big[\sigma\big(u_\varepsilon^n(s,y)\big) - \sigma\big(u_\varepsilon^{n-1}(s,y)\big)\big] W_\varepsilon(ds,dy)\bigg|^p \\
&\quad \le C_p \mathbf{E}\bigg|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) p_{t-s}(x-z)\big[\sigma\big(u_\varepsilon^n(s,y)\big) - \sigma\big(u_\varepsilon^{n-1}(s,y)\big)\big] \\
&\qquad \times \big[\sigma\big(u_\varepsilon^n(s,z)\big) - \sigma\big(u_\varepsilon^{n-1}(s,z)\big)\big] f_\varepsilon(y-z)\,dy\,dz\,ds\bigg|^{\frac{p}{2}}.
\end{aligned}$$

Thus, since $\|f_\varepsilon\|_\infty \le C_\varepsilon$ and owing to the fact that σ is a Lipschitz function, we have

$$\begin{aligned}
&\mathbf{E}\big|u_\varepsilon^{n+1}(t,x) - u_\varepsilon^n(t,x)\big|^p \\
&\quad \le C_\varepsilon \mathbf{E}\bigg|\int_0^t \bigg(\int_{\mathbb{R}} p_{t-s}(y)|u_\varepsilon^n(s,x+y) - u_\varepsilon^{n-1}(s,x+y)|\,dy\bigg)^2 ds\bigg|^{\frac{p}{2}},
\end{aligned}$$

where C_ε denotes a generic constant depending on ε and p. We now integrate with respect to the space variable and invoke Minkowski's inequality. In this way, we obtain

$$\begin{aligned}
&\mathbf{E}\big\|u_\varepsilon^{n+1}(t,\cdot) - u_\varepsilon^n(t,\cdot)\big\|_{L^p(\mathbb{R})}^p \\
&\quad \le C_\varepsilon \mathbf{E}\bigg\|\int_0^t \bigg(\int_{\mathbb{R}} p_{t-s}(y)\big|u_\varepsilon^n(s,y+\cdot) - u_\varepsilon^{n-1}(s,y+\cdot)\big|\,dy\bigg)^2 ds\bigg\|_{L^{\frac{p}{2}}(\mathbb{R})}^{\frac{p}{2}} \\
&\quad \le C_\varepsilon \mathbf{E}\bigg(\int_0^t \bigg(\int_{\mathbb{R}} p_{t-s}(y)\big\|u_\varepsilon^n(s,\cdot) - u_\varepsilon^{n-1}(s,\cdot)\big\|_{L^p(\mathbb{R})}\,dy\bigg)^2 ds\bigg)^{\frac{p}{2}} \\
&\quad \le C_\varepsilon \bigg(\int_0^t \big\|u_\varepsilon^n(s,\cdot) - u_\varepsilon^{n-1}(s,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})}^2\,ds\bigg)^{\frac{p}{2}}.
\end{aligned}$$

This relation easily entails

$$\|u_\varepsilon^{n+1}(t,\cdot)-u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2 \le C_\varepsilon\int_0^t \|u_\varepsilon^n(s,\cdot)-u_\varepsilon^{n-1}(s,\cdot)\|_{L^p(\Omega\times\mathbb{R})}^2\,ds,$$

and a direct application of Gronwall's lemma as stated in Lemma 4.13 yields that the quantity $\sup_n\sup_{t\in[0,T]}\|u_\varepsilon^n(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ is finite for each fixed $\varepsilon>0$. This implies that $\sup_{t\in[0,T]}\|u_\varepsilon(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}<\infty$ for each fixed $\varepsilon>0$.

Step 2. Next, we estimate $\mathcal{N}^*_{\frac12-H,p}u_\varepsilon(t)$, and observe that we are able to handle this term directly (namely without invoking Picard's iterations). We can write

$$\begin{aligned}
&\int_{\mathbb{R}}\mathbf{E}\big|u_\varepsilon(t,x)-u_\varepsilon(t,x+h)\big|^p\,dx\\
&\quad\le C\int_{\mathbb{R}}\big|p_t*u_0(x)-p_t*u_0(x+h)\big|^p\,dx\\
&\qquad + C_\varepsilon\int_{\mathbb{R}}\mathbf{E}\bigg|\int_0^t\bigg(\int_{\mathbb{R}}\big|p_{t-s}(y)-p_{t-s}(y+h)\big|\big|u_\varepsilon(s,y+x)\big|\,dy\bigg)^2ds\bigg|^{\frac p2}dx\\
&\quad\le C\int_{\mathbb{R}}\big|p_t*u_0(x)-p_t*u_0(x+h)\big|^p\,dx\\
&\qquad + C_\varepsilon\bigg(\int_0^t\bigg(\int_{\mathbb{R}}\big|p_{t-s}(y)-p_{t-s}(y+h)\big|\,dy\bigg)^2\big\|u_\varepsilon(s,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})}^2\,ds\bigg)^{\frac p2}.
\end{aligned}$$

We thus end up with

$$\begin{aligned}
&\mathcal{N}^*_{\frac12-H,p}u_\varepsilon(t)\\
&\quad=\int_{\mathbb{R}}\big\|u_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot+h)\big\|_{L^p(\Omega\times\mathbb{R})}^2|h|^{2H-2}\,dh\\
&\quad\le C\int_{\mathbb{R}}\big\|p_t*u_0(\cdot)-p_t*u_0(\cdot+h)\big\|_{L^p(\mathbb{R})}^2|h|^{2H-2}\,dh\\
&\qquad + C_\varepsilon\sup_{s\in[0,T]}\big\|u_\varepsilon(s,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})}^2\\
&\qquad\times\int_0^t\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}\big|p_{t-s}(y)-p_{t-s}(y+h)\big|\,dy\bigg)^2|h|^{2H-2}\,dh\,ds,
\end{aligned}$$

and the right-hand side in the above inequality is easily seen to be finite. Putting together the last two steps, we can conclude that for each fixed ε, $u_\varepsilon\in\mathcal{Z}_T^p$.

Step 3: *Uniform bounds in* ε. To prove the norms of u_ε in $\mathcal{Z}_T^p$ are uniformly bounded in ε, we note that u_ε satisfies the equation

$$u_\varepsilon(t,x)=p_t*u_0(x)+\int_0^t\int_{\mathbb{R}}\big[\big(p_{t-s}(x-\cdot)\sigma\big(u_\varepsilon(s,\cdot)\big)\big)*\rho_\varepsilon\big](y)W(ds,dy).$$

Hence, we have

$$
\begin{aligned}
&\mathbf{E}\big|u_\varepsilon(t,x)\big|^p \\
&\quad\le C\big|p_t * u_0(x)\big|^p \\
&\qquad + C\mathbf{E}\Big(\int_0^t\int_{\mathbb{R}}\big|\mathcal{F}\big(p_{t-s}(x-\cdot)\sigma\big(u_\varepsilon(s,\cdot)\big)\big)(\xi)\big|^2 e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}\,d\xi\,ds\Big)^{\frac{p}{2}} \qquad (4.28)\\
&\quad\le C\big|p_t * u_0(x)\big|^p \\
&\qquad + C\mathbf{E}\Big(\int_0^t\int_{\mathbb{R}}\big|\mathcal{F}\big(p_{t-s}(x-\cdot)\sigma\big(u_\varepsilon(s,\cdot)\big)\big)(\xi)\big|^2|\xi|^{1-2H}\,d\xi\,ds\Big)^{\frac{p}{2}}.
\end{aligned}
$$

Going back from Fourier to direct coordinates, one can check that

$$
\mathbf{E}\big|u_\varepsilon(t,x)\big|^p \le C\big|p_t * u_0(x)\big|^p + \mathcal{D}_1(t) + \mathcal{D}_2(t),
$$

with

$$
\begin{aligned}
\mathcal{D}_1(t) = \Big(\int_0^t\int_{\mathbb{R}^2}&\big|p_{t-s}(y) - p_{t-s}(y+h)\big|^2 \\
&\times \big\|u_\varepsilon(s, y+x+h)\big\|_{L^p(\Omega)}^2|h|^{2H-2}\,dh\,dy\,ds\Big)^{\frac{p}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}_2(t) = \Big(\int_0^t\int_{\mathbb{R}^2}&\big|p_{t-s}(y)\big|^2\big\|u_\varepsilon(s, y+x+h) - u_\varepsilon(s, y+x)\big\|_{L^p(\Omega)}^2 \\
&\times |h|^{2H-2}\,dh\,dy\,ds\Big)^{\frac{p}{2}}.
\end{aligned}
$$

These terms are treated exactly as the terms D_1, D_2 in the proof of Lemma 4.9, except for the fact that $\alpha = 0$ in the current situation. We obtain

$$
\begin{aligned}
&\big\|u_\varepsilon(t,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})}^2 \\
&\quad\le C\|u_0\|_{L^p(\mathbb{R})}^2 + C\int_0^t (t-s)^{H-1}\big\|u_\varepsilon(s,\cdot)\big\|_{L^p(\Omega\times\mathbb{R})}^2\,ds \qquad (4.29)\\
&\qquad + C\int_0^t (t-s)^{-\frac{1}{2}}\int_{\mathbb{R}}\big\|u_\varepsilon(s,\cdot) - u_\varepsilon(s,\cdot+h)\big\|_{L^p(\Omega\times\mathbb{R})}^2|h|^{2H-2}\,dh\,ds.
\end{aligned}
$$

Similarly, we get (see also the bounds for the terms $\mathcal{I}_1, \mathcal{I}_2$ in the proof of Theorem 4.3)

$$
\begin{aligned}
&\big[\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon(t)\big]^2 \\
&\quad\le C\int_{\mathbb{R}}\big\|u_0(\cdot) - u_0(\cdot+h)\big\|_{L^p(\mathbb{R})}^2|h|^{2H-2}\,dh \qquad (4.30)
\end{aligned}
$$

$$+ C \int_0^t (t-s)^{2H-\frac{3}{2}} \|u_\varepsilon(s,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}\, ds$$
$$+ C \int_0^t \int_{\mathbb{R}} (t-s)^{H-1} \|u_\varepsilon(s,\cdot) - u_\varepsilon(s,\cdot+l)\|^2_{L^p(\Omega\times\mathbb{R})} |l|^{2H-2}\, dl\, ds.$$

Set

$$\Psi(t) = \|u_\varepsilon(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} + [\mathcal{N}^*_{\frac{1}{2}-H,p} u_\varepsilon(t)]^2.$$

Thus combining the estimates (4.29) and (4.30) yields

$$\Psi(t) \le C\|u_0\|^2_{L^p(\mathbb{R})} + C\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot+h)\|^2_{L^p(\mathbb{R})} |h|^{2H-2}\, dh$$
$$+ C\int_0^t (t-s)^{2H-\frac{3}{2}} \Psi(s)\, ds.$$

Since we have shown that for each fixed ε, $\|u_\varepsilon\|_{\mathcal{Z}_T^p} < \infty$, we can apply the Gronwall-type Lemma 4.13 to the above inequality to show that

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathcal{Z}_T^p} < \infty.$$

Step 4: *u is an element of $\mathcal{Z}_T^p$.* Recall once again that we have decomposed $\|u\|_{\mathcal{Z}_T^p}$ according to relation (4.1). We now bound $\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ and $\mathcal{N}^*_{\frac{1}{2}-H,p} u(t)$ in this decomposition.

Since u_ε converges to u in $X_T^{\frac{1}{2}-H}$ a.s., we have $u_\varepsilon(t,x) \to u(t,x)$ a.s. for each $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Thus, by Fatou's lemma,

$$\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} = \left(\mathbf{E}\int_{\mathbb{R}} \lim_{\varepsilon\to 0} |u_\varepsilon(t,x)|^p\, dx\right)^{\frac{1}{p}}$$
$$\le \varliminf_{\varepsilon\to 0} \left(\mathbf{E}\int_{\mathbb{R}} |u_\varepsilon(t,x)|^p\, dx\right)^{\frac{1}{p}} \le C.$$

Therefore, we conclude that $\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$ is finite. On the other hand, for each x and h we have $|u_\varepsilon(t,x+h) - u_\varepsilon(t,x)|^2 \to |u(t,x+h) - u(t,x)|^2$ a.s., so by Fatou's lemma again we obtain

$$\int_{|h|\le 1} \|u(t,\cdot+h) - u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2}\, dh$$
$$\le \int_{|h|\le 1} \varliminf_{\varepsilon\to 0} \|u_\varepsilon(t,\cdot+h) - u_\varepsilon(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2}\, dh$$
$$\le \varliminf_{\varepsilon\to 0} \int_{|h|\le 1} \|u_\varepsilon(t,\cdot+h) - u_\varepsilon(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2}\, dh.$$

The desired bound on $\mathcal{N}^*_{\frac{1}{2}-H,p}u(t)$ is obtained from the inequality above, by handling the integral on the domains $|h| \le 1$ and $|h| > 1$. In the latter case, we simply bound $\|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}$ by $2\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$. By doing so, we conclude that

$$\sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,p}u(t) = \sup_{t\in[0,T]} \int_{\mathbb{R}} \|u(t,\cdot+h)-u(t,\cdot)\|^2_{L^p(\Omega\times\mathbb{R})}|h|^{2H-2}\,dh < \infty.$$

Together with the previous estimate on $\|u(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}$, we conclude that $u \in \mathcal{Z}^p_T$. $\square$

We now state a convergence result for stochastic integrals, with respect to the approximating noise W_ε.

LEMMA 4.16. *Let $u_n(t,x)$ be a solution to the equation*

$$u_n(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma\big(u_n(s,y)\big)W_n(ds,dy),$$

where we have set $W_n = W_{\varepsilon_n}$ [recall that W_ε is defined by (4.22)] for a sequence $\{\varepsilon_n, n \ge 1\}$ satisfying $\lim_{n\to\infty}\varepsilon_n = 0$. We assume the following conditions:

(i) *with probability one, u_n converges to u in $X_T^{\frac{1}{2}-H}$,*
(ii) $\sup_n \|u_n\|_{\mathfrak{X}_T^{\beta,p}} < \infty$, *with $\frac{1}{2}-H < \beta < H$ and $p > \frac{2}{H}$.*

Then the process u belongs to $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. Furthermore, for any fixed $t \le T$ and $x \in \mathbb{R}$, the random variable $\Phi^n(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))W_n(ds,dy)$ converges a.s. to $\Phi(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)$, as $n\to\infty$.

PROOF. We focus on the convergence part and decompose the difference $\Phi(t,x)-\Phi^n(t,x)$ into $(\Phi(t,x)-\Phi^{n,1}(t,x)) + (\Phi^{n,1}(t,x)-\Phi^n(t,x))$, where

$$\Phi^{n,1}(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma\big(u(s,y)\big)W^n(ds,dy).$$

Now we note that $\Phi(t,x)-\Phi^{n,1}(t,x)$ can be expressed as

$$\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma\big(u(s,y)\big)W(ds,dy) \\ - \int_0^t \int_{\mathbb{R}} \big[\big(p_{t-s}(x-\cdot)\sigma\big(u(s,\cdot)\big)\big) * \rho_{\varepsilon_n}\big](y)W(ds,dy),$$

and thus

$$\mathbf{E}\big|\Phi(t,x)-\Phi^{n,1}(t,x)\big|^2 \\ = C\mathbf{E}\int_0^t \int_{\mathbb{R}} \big|e^{-\frac{\varepsilon_n|\xi|^2}{2}}-1\big|^2 \big|\mathcal{F}\big(p_{t-s}(x-\cdot)\sigma\big(u(s,\cdot)\big)\big)(\xi)\big|^2 |\xi|^{1-2H}\,d\xi\,ds.$$

The latter quantity obviously converges to 0 as ε_n goes to 0 because of the finiteness of

$$\mathbf{E}\int_0^t \int_{\mathbb{R}} \left|\mathcal{F}\big(p_{t-s}(x-\cdot)\sigma\big(u(s,\cdot)\big)\big)(\xi)\right|^2 |\xi|^{1-2H}\, d\xi\, ds,$$

which can be seen by an application of Fatou's lemma (as in Step 4 of the proof of Lemma 4.15).

It remains to show that $\lim_{n\to\infty}\mathbf{E}|\Phi^{n,1}(t,x)-\Phi^n(t,x)|^2=0$. However, similar to (4.28), we have

$$\mathbf{E}\big[\left|\Phi^{n,1}(t,x)-\Phi^n(t,x)\right|^2\big] \le \mathbf{E}\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f_n(s,y) W(ds,dy)\right|^2,$$

where we have set $f_n=\sigma(u_n)-\sigma(u)$. Furthermore, appealing to Proposition A.14, we see that f_n converges to 0 in $X_T^{\frac{1}{2}-H}$. We will verify that f_n satisfies the conditions (C1)–(C3) of Lemma 4.17 below. Indeed, (C1) is verified by assumption (i). (C2) is verified by assumption (ii) and the estimate (3.14). (C3) is readily assumption (ii). Then an application of Lemma 4.17 completes the proof. □

LEMMA 4.17. *Suppose that* $\{f_n, n\ge 1\}$ *is a sequence of stochastic processes belonging to* $\mathfrak{X}_T^{\beta,p}$ *with* $\frac{1}{2}-H<\beta<H$ *and* $p>\frac{2}{H}$. *Assume that the following conditions hold*:

(C1) *With probability one*, f_n *converges uniformly to* 0 *over compact sets of* $[0,T]\times\mathbb{R}$.

(C2) *For every* $R>0$, $\sup_n \sup_{s,t\in[0,T],|x|\le R}\mathbf{E}|f_n(t,x)-f_n(s,x)|^p \le C|t-s|^{p\frac{H}{2}}$.

(C3) $\sup_n \|f_n\|_{\mathfrak{X}_T^{\beta,p}}\le M$, *where* M *is a finite number.*

Then for every $t\le T$ *and* $x\in\mathbb{R}$ *the random variable* $Y_n(t,x)$ *defined by*

$$Y_n(t,x)=\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) f_n(s,y) W(ds,dy)$$

converges to 0 *in* $L^2(\Omega)$.

PROOF. We first observe that Proposition A.9 asserts that f_n belongs to $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. Next, we show that $\{f_n, n\ge 1\}$ is relatively compact and converges to 0 in $\mathfrak{X}_T^{\frac{1}{2}-H,2}$. For this purpose, we verify the three conditions (1)–(3) of Proposition A.13. Condition (2) in Proposition A.13 is evident from (C2). Condition (3) in Proposition A.13 follows from the following inequality, where

$\delta \le 1$:

$$\int_{|y|\le\delta} \|f(t,x+y)-f(t,x)\|^2_{L^2(\Omega)}|y|^{2H-2}\,dy$$
$$\le \Big(\sup_{|y|\le 1}\|f(t,x+y)-f(t,x)\|^2_{L^2(\Omega)}|y|^{-2\beta}\Big)\int_{|y|\le\delta}|y|^{2\beta+2H-2}\,dy.$$

In fact, the first factor on the right-hand side of the above inequality is uniformly bounded in $(t,x)\in[0,T]\times\mathbb{R}$ because of inequality (A.1) and the fact that f_n is bounded in $\mathfrak{X}_T^{\beta,2}$ by condition (C3). Taking into account that $\beta > 1/2-H$, the second factor converges to zero as δ tends to zero. To verify condition (1) in Proposition A.13, we fix t,x and note that (C1) implies that $f_n(t,x)$ converges almost surely to 0. On the other hand, $\mathbf{E}|f_n(t,x)|^p$ is uniformly bounded, where $p>2$. These two facts imply that $\{f_n(t,x)\}$ converges to 0 in $L^2(\Omega)$, thus condition (1) in Proposition A.13 is verified. Furthermore, condition (C1) ensures that 0 is the only possible limit point of $\{f_n\}$ in $\mathfrak{X}_T^{1/2-H,2}$. We conclude that f_n converges to 0 in $\mathfrak{X}_T^{1/2-H,2}$.

Let us now prove that $Y_n(t,x)$ converges to 0 in $L^2(\Omega)$. Applying (3.3), we get $\mathbf{E}|Y_n(t,x)|^2 \le C(J_1(t)+J_2(t))$ with

$$J_1(t)=\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x-y-z)-p_{t-s}(x-y)|^2\mathbf{E}f_n^2(s,y+z)|z|^{2H-2}\,dy\,dz\,ds$$

and

$$J_2(t)=\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x-y)|^2\mathbf{E}|f_n(s,y+z)-f_n(s,y)|^2|z|^{2H-2}\,dy\,dz\,ds.$$

Now for every fixed $\varepsilon>0$ we choose $R>0$ sufficiently large such that

$$\int_0^t\int_{|y|>R}\big[|p_{t-s}(y)|^2+[\mathcal{N}_{\frac12-H}p_{t-s}(y)]^2\big]\,dy\,ds<\varepsilon.$$

With this choice of R, we choose n so that

$$\sup_{s\in[0,T],|y|\le R}\mathbf{E}f_n^2(s,y)+\sup_{s\in[0,T],|y|\le R}\int_{\mathbb{R}}\mathbf{E}|f_n(s,y+z)-f_n(s,y)|^2|y|^{2H-2}\,dy<\varepsilon.$$

By making a shift in y, we end up with

$$\begin{aligned}J_1(t)&=\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}|p_{t-s}(x-y)-p_{t-s}(x-y+z)|^2\mathbf{E}f_n^2(s,y)|z|^{2H-2}\,dy\,dz\,ds\\
&\le\int_0^t\sup_{|y|\le R}\mathbf{E}f_n^2(s,y)\int_{|y-x|\le R}[\mathcal{N}_{\frac12-H}p_{t-s}(x-y)]^2\,dy\,ds\\
&\quad+\sup_{r\in[0,T],w\in\mathbb{R}}\mathbf{E}f_n^2(r,w)\int_0^t\int_{|y-x|>R}[\mathcal{N}_{\frac12-H}p_{t-s}(x-y)]^2\,dy\,ds\\
&\le C\varepsilon+CM\int_0^t\int_{|y|>R}[\mathcal{N}_{\frac12-H}p_{t-s}(y)]^2\,dy\,ds.\end{aligned}$$

Similarly,

$$J_2(t) \leq C\varepsilon + CM \int_0^t \int_{|y|>R} p_{t-s}^2(y)\, dy\, ds.$$

Then $\mathbf{E}|Y_n(t,x)|^2 \leq C\varepsilon$ for n sufficiently large. This implies the result. $\square$

4.5. *Proof of the moment bounds.* In this subsection, we prove Theorem 4.7 and Proposition 4.8.

PROOF OF THEOREM 4.7. We will apply Proposition 3.5, in particular, the estimate (3.12) by taking f to be the solution u to equation (1.1), and combine it with the mild formulation of the solution. For every fixed $\varepsilon > 0$, by noticing that $\|p_t * u_0\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}} \leq \|u_0\|_\varepsilon$, we get the following bound:

$$\begin{aligned} &\|u\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}} \\ &\quad \leq \|u_0\|_\varepsilon \\ &\qquad + C_0\|\sigma\|_{\mathrm{Lip}}\sqrt{p}\|u\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}}\big(\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}} + \varepsilon^{-1}\kappa^{-\frac{1}{4}}\theta^{-\frac{1}{4}} + \varepsilon\kappa^{H-\frac{3}{4}}\theta^{\frac{1}{4}-H}\big). \end{aligned}$$

We optimize the formula above by choosing $\varepsilon = \kappa^{\frac{1}{4}-\frac{H}{2}}\theta^{-\frac{1}{4}+\frac{H}{2}}$, in order to obtain

$$\|u\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}} \leq \|u_0\|_\varepsilon + 3C_0\|\sigma\|_{\mathrm{Lip}}\sqrt{p}\|u\|_{\mathfrak{X}^p_{T,\theta,\varepsilon}}\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}},$$

then choose $\theta = \theta_0$ so that $3C_0\|\sigma\|_{\mathrm{Lip}}\sqrt{p}\kappa^{\frac{H}{2}-\frac{1}{2}}\theta^{-\frac{H}{2}} = \frac{1}{2}$, that is,

$$\theta_0 = (6C_0)^{\frac{2}{H}} p^{\frac{1}{H}} \kappa^{1-\frac{1}{H}} \|\sigma\|_{\mathrm{Lip}}^{\frac{2}{H}} \quad \text{and take}$$

$$\varepsilon = \varepsilon_0 := (6C_0)^{1-\frac{1}{2H}} \kappa^{\frac{1}{4H}-\frac{1}{2}} p^{\frac{1}{2}-\frac{1}{4H}} \|\sigma\|_{\mathrm{Lip}}^{1-\frac{1}{2H}}.$$

Plugging this choice into the above inequality gives the bound

$$\|u\|_{\mathfrak{X}^p_{T,\theta_0,\varepsilon_0}} \leq 2\|u_0\|_{\varepsilon_0}$$

from which our claims are easily deduced by noticing that the constant C_0 does not depend on T. $\square$

PROOF OF PROPOSITION 4.8. Applying Itô isometry to equation (2.12), we see that

$$\mathbf{E}\big|u(t,x)\big|^2 = \big|p_t * u_0(x)\big|^2 + c_{1,H}\mathbf{E}\int_0^t \big\|p_{t-s}(x-y)\sigma\big(u(s,y)\big)\big\|^2_{\dot{H}^{\frac{1}{2}-H}}\, ds. \tag{4.31}$$

Let us recall the well-known Sobolev embedding inequality

$$\|g\|_{\dot{H}^{\frac{1}{2}-H}} \geq c\|g\|_{L^{\frac{1}{H}}}, \qquad \forall g \in \dot{H}^{\frac{1}{2}-H}(\mathbb{R}).$$

Hence, together with our assumption on σ, it follows that there exists some positive constant b such that

$$\mathbf{E}|u(t,x)|^2 \geq |p_t * u_0(x)|^2 + b\sigma_*^2 \mathbf{E}\int_0^t \|p_{t-s}(x-\cdot)u(s,\cdot)\|^2_{L^{\frac{1}{H}}(\mathbb{R})}\,ds.$$

Since $2H < 1$, applying Jensen's inequality we see that

$$\begin{aligned}
&\|p_{t-s}(x-\cdot)u(s,\cdot)\|^2_{L^{\frac{1}{H}}(\mathbb{R})} \\
&\quad = \left(\int_{\mathbb{R}} p_{t-s}^{\frac{1}{H}-1}(x-y)|u(s,y)|^{\frac{1}{H}} p_{t-s}(x-y)\,dy\right)^{2H} \\
&\quad \geq \int_{\mathbb{R}} p_{t-s}^{3-2H}(x-y)|u(s,y)|^2\,dy.
\end{aligned}$$

It follows that

$$\mathbf{E}|u(t,x)|^2 \geq |p_t * u_0(x)|^2 + b\sigma_*^2 \int_0^t \int_{\mathbb{R}} p_{t-s}^{3-2H}(x-y)\mathbf{E}|u(s,y)|^2\,dy\,ds.$$

Iterating the previous inequality yields

$$\mathbf{E}|u(t,x)|^2 \geq |p_t * u_0(x)|^2 + \sum_{n=1}^{\infty}(b\sigma_*^2)^n I_n(t,x). \tag{4.32}$$

In the above, we have adopted the notation

$$I_n(t,x) = \int_{T_n(t)}\int_{\mathbb{R}^n} p_{t-s_n}^{3-2H}(x-y_n)\cdots p_{s_2-s_1}^{3-2H}(y_2-y_1)|p_{s_1}u_0(y_1)|^2\,d\bar{y}\,d\bar{s},$$

where $T_n(t) = \{(s_1,\dots,s_n) \in [0,t]^n : 0 < s_1 < \cdots < s_n < t\}$ and $d\bar{y} = dy_1 \cdots dy_n$, $d\bar{s} = ds_1 \cdots ds_n$. Note that for every $x, z \in \mathbb{R}$ and $a, b > 0$, the following identity holds:

$$\int_{\mathbb{R}} p_a^{3-2H}(x-y)p_b^{3-2H}(y-z)\,dy = (3-2H)^{-\frac{1}{2}}\left(\frac{2\pi\kappa ab}{a+b}\right)^{H-1} p_{a+b}^{3-2H}(x-z).$$

We thus can compute $I_n(t,x)$ by integrating y_j's in descending order starting from y_n. This procedure yields

$$\begin{aligned}
I_n(t,x) &= (3-2H)^{-\frac{n-1}{2}} \\
&\quad \times \int_{T_n(t)}\left(\frac{t-s_n}{t-s_1}\prod_{j=2}^{n} 2\pi\kappa(s_j-s_{j-1})\right)^{H-1} \\
&\quad \times \int_{\mathbb{R}} p_{t-s_1}^{3-2H}(x-y_1)|p_{s_1}u_0(y_1)|^2\,dy_1\,d\bar{s}.
\end{aligned} \tag{4.33}$$

On the other hand, for every fixed $R > 0$, applying Jensen's inequality, we see that

$$
\begin{aligned}
&\int_{\mathbb{R}} p_{t-s_1}^{3-2H}(x-y_1)\big|p_{s_1}u_0(y_1)\big|^2\,dy_1 \\
&\quad\geq p_{t-s_1}^{1-2H}(R)\int_{|x-y_1|<R} p_{t-s_1}^2(x-y_1)\big|p_{s_1}u_0(y_1)\big|^2\,dy_1 \qquad (4.34)\\
&\quad\geq p_{t-s_1}^{1-2H}(R)R^{-1}\biggl(\int_{|x-y_1|<R} p_{t-s_1}(x-y_1)p_{s_1}*u_0(y_1)\,dy_1\biggr)^2.
\end{aligned}
$$

The integral on the right-hand side can be rewritten as

$$
p_t * u_0(x) - \int_{|x-y_1|\geq R} p_{t-s_1}(x-y_1)p_{s_1}*u_0(y_1)\,dy_1.
$$

Since u_0 is bounded, we see that $|p_{s_1}*u_0(y_1)|\leq \|u_0\|_{L^\infty}$, and hence

$$
\begin{aligned}
&\biggl|\int_{|x-y_1|\geq R} p_{t-s_1}(x-y_1)p_{s_1}*u_0(y_1)\,dy_1\biggr| \\
&\qquad\leq \|u_0\|_{L^\infty}\int_{|y|>R} p_{t-s_1}(y)\,dy \\
&\qquad= \|u_0\|_{L^\infty}\pi^{-\frac{1}{2}}\int_{|z|>\frac{R}{\sqrt{2\kappa(t-s_1)}}} e^{-z^2}\,dz.
\end{aligned}
$$

For every fixed ε in $(0,1)$, we now choose $R = M\sqrt{2\kappa(t-s_1)}$ where M is such that $e^{-(1-2H)M^2}M^{-1} = \varepsilon$. It follows that

$$
p_{t-s_1}^{1-2H}(R)R^{-1} = \pi^{H-\frac{1}{2}}\bigl(2\kappa(t-s_1)\bigr)^{H-1}e^{-(1-2H)M^2}M^{-1}
$$

and

$$
\biggl|\int_{|x-y_1|<R} p_{t-s_1}(x-y_1)p_{s_1}*u_0(y_1)\,dy_1\biggr| \geq \big|p_t*u_0(x)\big| - \|u\|_\infty e^{-M^2}M^{-1}.
$$

Together with (4.34), we see that

$$
\begin{aligned}
&\int_{\mathbb{R}} p_{t-s_1}^{3-2H}(x-y_1)\big|p_{s_1}u_0(y_1)\big|^2\,dy_1 \\
&\qquad\geq ce^{-M^2}M^{-1}\bigl(\kappa(t-s_1)\bigr)^{H-1}\bigl(\big|p_t*u_0(x)\big| - e^{-M^2}M^{-1}\|u_0\|_{L^\infty}\bigr)^2
\end{aligned}
$$

for some universal constant c. Hence, upon combining the previous estimate and (4.33), we arrive at

$$
I_n(t,x) \geq \varepsilon c^n\kappa^{(H-1)n}\int_{T_n(t)}\prod_{j=2}^{n+1}(s_j - s_{j-1})^{H-1}\,d\bar{s}\bigl(\big|p_t*u_0(x)\big| - \varepsilon\|u_0\|_{L^\infty}\bigr)^2,
$$

where $s_{n+1} = t$ and c is some universal constant. It is elementary to compute

$$\int_{T_n(t)} \prod_{j=2}^{n+1} (s_j - s_{j-1})^{H-1} \, d\bar{s} = \frac{\Gamma(H)^n t^{nH}}{\Gamma(nH+1)}.$$

Therefore, together with (4.32), we obtain

$$\mathbf{E}\big|u(t,x)\big|^2 \geq \varepsilon\big(\big|p_t * u_0(x)\big| - \varepsilon\|u_0\|_{L^\infty}\big)^2 \sum_{n=0}^{\infty} \big(cb\Gamma(H)\big)^n \frac{(\sigma_*^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} t)^{nH}}{\Gamma(nH+1)}.$$

We now recall the asymptotic

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+an)} = \frac{1}{a} \exp\big(z^{\frac{1}{a}}\big) + O\big(|z|^{-1}\big) \qquad \text{as } z \to \infty,$$

which can be found, for example, in [8], page 208. Together with the previous estimate, this yields:

$$\mathbf{E}\big|u(t,x)\big|^2 \geq C\varepsilon\big(p_t * u_0(x) - \varepsilon\|u_0\|_{L^\infty}\big)^2 e^{L\sigma_*^{\frac{2}{H}} \kappa^{1-\frac{1}{H}} t}. \tag{4.35}$$

By choosing $\varepsilon = \frac{|p_t * u_0(x)|}{3\|u_0\|_{L^\infty}}$, we conclude the proof. □

APPENDIX

In this Appendix we gather the results about the space–time function spaces and probability measures on X_T^β. These function spaces are defined in Sections 3.1 and 4.2.

A.1. Space–time function spaces. Let us start by noting that the notation $\mathcal{N}_\beta^{B,(\delta)} f(x)$, defined in (4.3), gives information of the Hölder continuity of the function f. Indeed, suppose, for instance, that a function f has modulus of continuity $|h|^\beta \omega(h)$ at x, for any $|h| \leq \delta$. Then $[\mathcal{N}_\beta^{B,(\delta)} f(x)]^2$ is bounded above by $2\int_0^\delta \omega^2(h) h^{-1} \, dh$. Thus, for $\mathcal{N}_\beta^{B,(\delta)} f(x)$ to be finite, it is sufficient that $\omega^2(h)h^{-1}$ is integrable near 0. On the other hand, if $\mathcal{N}_\beta^{B,(\delta)} f$ is bounded over a domain, the following proposition asserts that f is necessarily Hölder continuous.

PROPOSITION A.1. *Let I be a nonempty open interval of $\mathbb{R}$ and $\delta \in (0, \infty]$. Let f be a function on $\mathbb{R}$ such that $\sup_{x \in \bar{I}} \mathcal{N}_\beta^{B,(\delta)} f(x)$ is finite. Then*

$$\sup_{x \in I, |y| \leq \frac{\delta}{3} \wedge \operatorname{dist}(x, \partial I)} \frac{\|f(x+y) - f(x)\|}{|y|^\beta} \leq c(\beta) \sup_{x \in \bar{I}} \mathcal{N}_\beta^{B,(\delta)} f(x) \tag{A.1}$$

for some finite constant $c(\beta)$ which depends only on β.

PROOF. For every $x \in I$ and positive R, $R \le \delta$, we denote $f_{x,R} = \frac{1}{2R}\int_{-R}^{R} f(y+x)\,dy$. We first estimate $\|f(x) - f_{x,R}\|$ as follows:

$$\begin{aligned}
&\|f(x) - f_{x,R}\| \\
&\quad\le \frac{1}{2R}\int_{-R}^{R}\|f(x) - f(x+y)\|\,dy \\
&\quad\le \frac{1}{2R}\left(\int_{-R}^{R}\|f(x) - f(x+y)\|^2|y|^{-1-2\beta}\,dy\right)^{\frac{1}{2}}\left(\int_{-R}^{R}|y|^{1+2\beta}\,dy\right)^{\frac{1}{2}} \\
&\quad\le \frac{R^\beta}{2\sqrt{1+\beta}}\sup_{x\in\bar{I}}\mathcal{N}_\beta^{B,(\delta)}f(x).
\end{aligned}\tag{A.2}$$

Let us now fix $x \in I$ and $y \in \mathbb{R}$ such that $|y| \le \delta/3 \wedge \operatorname{dist}(x, \partial I)$. We also choose $R = |y|$. It follows from triangle inequality that

$$\begin{aligned}
&\|f(x+y) - f(x)\| \\
&\qquad\le \|f(x+y) - f_{x+y,R}\| + \|f_{x+y,R} - f_{x,R}\| + \|f(x) - f_{x,R}\|.
\end{aligned}\tag{A.3}$$

For the second term, we apply Minkowski's inequality to get

$$\|f_{x+y,R} - f_{x,R}\| \le \frac{1}{4R^2}\int_{-R}^{R}\int_{-R}^{R}\|f(x+y+z) - f(x+w)\|\,dz\,dw,$$

and invoking Cauchy–Schwarz's inequality this yields

$$\begin{aligned}
&\|f_{x+y,R} - f_{x,R}\| \\
&\quad\le \frac{1}{4R^2}\int_{-R}^{R}\left(\int_{-R}^{R}\|f(x+y+z) - f(x+w)\|^2|y+z-w|^{-2\beta-1}\,dz\right)^{\frac{1}{2}} \\
&\quad\quad\times\left(\int_{-R}^{R}|y+z-w|^{2\beta+1}\,dz\right)^{\frac{1}{2}}dw.
\end{aligned}$$

Notice that because of the restrictions on the variables, the domain of integration above satisfies $|y+z-w| \le 3R \le \delta$ and $x + w \in \bar{I}$. Hence,

$$\|f_{x+y,R} - f_{x,R}\| \le C_\beta \sup_{y\in\bar{I}}\mathcal{N}_\beta^{B,(\delta)}f(y)R^\beta.$$

We can now conclude our proof as follows: the first and third terms on the right-hand side of (A.3) are estimated in (A.2). Combining these estimates within (A.3) yields (A.1). $\square$

Now we turn to the results about the function spaces $\mathfrak{X}_T^\beta(B)$ and X_T^β, which are defined in Definitions 3.3 and 4.2, respectively.

PROPOSITION A.2. *$\mathfrak{X}_T^\beta(B)$ is a Banach space.*

PROOF. Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{X}_T^\beta(B)$. Since the space $C_b([0,T]\times\mathbb{R};B)$ of bounded continuous functions from $[0,T]\times\mathbb{R}$ to B is complete, there exists a bounded continuous function $f:[0,T]\times\mathbb{R}\to B$ such that

$$\lim_{n\to\infty}\sup_{t\in[0,T],x\in\mathbb{R}}\|f_n(t,x)-f(t,x)\|=0.$$

For any $\varepsilon>0$, there exists $n_0>0$ such that

$$\sup_{x\in\mathbb{R}}\mathcal{N}_\beta^B(f_n-f_m)(t,x)<\varepsilon$$

for all $m,n\ge n_0$. It follows from Fatou's lemma that

$$\mathcal{N}_\beta^B(f_n-f)(t,x)\le\liminf_{m\to\infty}\mathcal{N}_\beta^B(f_n-f_m)(t,x)\le\varepsilon$$

for every $t\in[0,T]$, $x\in\mathbb{R}$ and $n\ge n_0$. This implies that

$$\lim_{n\to\infty}\sup_{t\le T,x\in\mathbb{R}}\mathcal{N}_\beta^B(f_n-f)(t,x)=0$$

which means f_n converges to f in $\mathfrak{X}_T^\beta(B)$. $\square$

PROPOSITION A.3. *X_T^β is a complete metric space.*

PROOF. Let $\{f_n\}$ be a Cauchy sequence in X_T^β. Since the space $C_{\mathrm{uc}}([0,T]\times\mathbb{R})$ is complete, there exists a continuous function $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ such that for all compact intervals I,

$$\lim_{n\to\infty}\sup_{t\in[0,T],x\in I}|f_n(t,x)-f(t,x)|=0.$$

Let us fix a compact interval $I=[-N,N]$, and $\varepsilon>0$. There exists $n_0>0$ such that

$$\sup_{t\in[0,T],x\in I}\mathcal{N}_\beta^{(1)}(f_n-f_m)(t,x)<\varepsilon$$

for all $m,n\ge n_0$. It follows from Fatou's lemma that

$$\mathcal{N}_\beta^{(1)}(f_n-f)(t,x)\le\liminf_{m\to\infty}\mathcal{N}_\beta^{(1)}(f_n-f_m)(t,x)\le\varepsilon,$$

for every $t\in[0,T]$, $x\in I$ and $n\ge n_0$. This implies that $\mathcal{N}_\beta^{(1)}(f_n-f)$ converges to 0 uniformly on $[0,T]\times I$. In addition, from (4.4), it follows that $\mathcal{N}_\beta^{(1)}f_n$ converges to $\mathcal{N}_\beta^{(1)}f$ uniformly on $[0,T]\times I$, thus the continuity of $\mathcal{N}_\beta^{(1)}f_n$ implies that of $\mathcal{N}_\beta^{(1)}f$.

It remains to check that f satisfies the condition (ii) of Definition 4.2. For every $\varepsilon>0$ and $|h|\le 1$, choose n sufficiently large so that

$$\sup_{t\in[0,T],x\in[N-1,N+1]}\mathcal{N}_\beta^{(1)}(f_n-f)(t,x)<\varepsilon.$$

Applying Minkowski's inequality, for every $(t,x) \in [0,T] \times [-N,N]$, we have

$$\begin{aligned}
&\mathcal{N}_\beta^{(1)}(\tau_h f - f)(t,x) \\
&\quad \le \mathcal{N}_\beta^{(1)}(\tau_h f - \tau_h f_n)(t,x) + \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t,x) + \mathcal{N}_\beta^{(1)}(f_n - f)(t,x) \\
&\quad \le 2\varepsilon + \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t,x).
\end{aligned}$$

Since f_n belongs to X_T^β, $\lim_{h\to 0} \sup_{t\in[0,T],x\in[-N,N]} \mathcal{N}_\beta^{(1)}(\tau_h f_n - f_n)(t,x) = 0$ which implies f belongs to X_T^β. □

The next results give some characterizations of the space X_T^β.

LEMMA A.4. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $t \mapsto \mathcal{N}_\beta^{(1)} f(t,x)$ is continuous for every fixed x. Suppose in addition that, for every $R > 0$,*

$$\lim_{\delta\downarrow 0} \sup_{t\in[0,T],x\in[-R,R]} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1}\, dy = 0.$$

Then $\mathcal{N}_\beta^{(1)} f$ is continuous and f belongs to X_T^β.

PROOF. Fix $R, \varepsilon > 0$, and choose δ such that

$$\sup_{t\in[0,T],x\in[-R-1,R+1]} \int_{-\delta}^{\delta} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1}\, dy < \varepsilon.$$

Then for every $t \in [0,T]$, $x \in [-R,R]$ and $|h| \le 1$,

$$\begin{aligned}
&[\mathcal{N}_\beta^{(1)}(\tau_h f - f)(t,x)]^2 \\
&\quad \le 2\varepsilon + \sup_{t\in[0,T],x\in[-R-1,R+1]} 2|\tau_h f(t,x) - f(t,x)|^2 \int_{|y|>\delta} |y|^{-2\beta-1}\, dy.
\end{aligned}$$

Since f is continuous, $\lim_{h\to 0} \sup_{t\in[0,T],x\in[-R-1,R+1]} |\tau_h f(t,x) - f(t,x)| = 0$. Together with the previous estimate, this yields $\lim_{h\to 0} \sup_{t\in[0,T],x\in[-R,R]} \mathcal{N}_\beta^{(1)} \times (\tau_h f - f)(t,x) = 0$ which on one hand, together with (4.4) implies the continuity of $\mathcal{N}_\beta^{(1)} f$. On the other hand, it obviously implies $f \in X_T^\beta$. □

PROPOSITION A.5. *Let $\phi \in C^\infty(\mathbb{R})$ be supported in $[-1,1]$, such that $\int_{\mathbb{R}} \phi(x)\, dx = 1$ and $0 \le \phi \le 1$. Set $\phi_n(x) = n\phi(nx)$. Then:*

(1) *If $f \in X_T^\beta$, then $f * \phi_n \to f$ in X_T^β as $n \to \infty$, where $*$ denotes the convolution with respect to the space variable.*

(2) *$C^{0,1}([0,T] \times \mathbb{R})$, that is, the space of functions which are continuous in time and continuously differentiable in space, is dense in X_T^β.*

(3) *Suppose that f is a continuous function on $[0,T]\times\mathbb{R}$ such that $t\mapsto \mathcal{N}_\beta^{(1)}f(t,x)$ is finite and continuous in time for every fixed $x\in\mathbb{R}$. Then f belongs to X_T^β if and only if for every $R>0$*

$$\lim_{\delta\downarrow 0}\sup_{t\in[0,T],x\in[-R,R]}\int_{-\delta}^{\delta}\big|f(t,x+y)-f(t,x)\big|^2|y|^{-2\beta-1}\,dy=0. \tag{A.4}$$

PROOF. We denote $f_n=f*\phi_n$. To show (1), we observe that

$$\begin{aligned}&f_n(t,x+y)-f_n(t,x)-f(t,x+y)+f(t,x)\\&\quad=\int_{\mathbb{R}}\big[\tau_h f(t,x+y)-\tau_h f(t,x)-f(t,x+y)+f(t,x)\big]\phi_n(h)\,dh\end{aligned}$$

and hence, for every $x\in[-R,R]$, applying Jensen's inequality, we get

$$\begin{aligned}&\int_{-1}^{1}\big|f_n(t,x+y)-f_n(t,x)-f(t,x+y)+f(t,x)\big|^2|y|^{-2\beta-1}\,dy\\&\qquad\le\int_{\mathbb{R}}\int_{-1}^{1}\big|\tau_h f(t,x+y)-\tau_h f(t,x)-f(t,x+y)\\&\qquad\quad+f(t,x)\big|^2|y|^{-2\beta-1}\phi_n(h)\,dh\,dy\\&\qquad\le\sup_{r\in[0,T],z\in[-R-1,R+1]}\sup_{|h|\le\frac{1}{n}}\big[\mathcal{N}_\beta^{(1)}(\tau_h f-f)(r,z)\big]^2.\end{aligned}$$

By assumption f belongs to X_T^β. Therefore, owing to condition (ii) in Definition 4.2, this integral converges to 0 when $n\to\infty$. This proves item (1).

To show (2), we first prove that X_T^β contains $C^{0,1}([0,T]\times\mathbb{R})$. Indeed, if g is a function in $C^{0,1}([0,T]\times\mathbb{R})$, by dominated convergence theorem, it is easy to show that $\mathcal{N}_\beta^{(1)}g(t,x)$ is finite and continuous in time for every fixed x. Moreover, for every $R>0$, we have

$$\begin{aligned}&\sup_{t\in[0,T],x\in[-R,R]}\int_{-\delta}^{\delta}\big|g(t,x+y)-g(t,x)\big|^2|y|^{-2\beta-1}\,dy\\&\qquad\le\sup_{x\in[-R,R],t\in[0,T]}\big|\partial_x g(t,x)\big|^2\int_{|y|\le\delta}|y|^{1-2\beta}\,dy.\end{aligned} \tag{A.5}$$

Since $\lim_{\delta\to 0}\int_{|y|\le\delta}|y|^{1-2\beta}\,dy=0$, Lemma A.4 implies that g belongs to X_T^β. We have thus proved that $C^{0,1}\subset X_T^\beta$. Together with item (1), this yields item (2).

The sufficiency of (3) is in fact the content of Lemma A.4. We focus on the necessity of (A.4). Assume that f belongs to X_T^β. Fix $R>0$, $\varepsilon>0$ and choose g in $C^{0,1}$ so that

$$\sup_{t\in[0,T],x\in[-R,R]}\mathcal{N}_\beta^{(1)}(f-g)(t,x)<\varepsilon.$$

Then for every $\delta > 0$ we have

$$\begin{aligned}
&\sup_{t\in[0,T],|x|\le R} \int_{-\delta}^{\delta} |f(t,x+y)-f(t,x)|^2 |y|^{-2\beta-1}\,dy \\
&\quad \le 2\varepsilon^2 + 2 \sup_{t\in[0,T],|x|\le R} \int_{-\delta}^{\delta} |g(t,x+y)-g(t,x)|^2 |y|^{-2\beta-1}\,dy.
\end{aligned} \tag{A.6}$$

Since g is $C^{0,1}$, the last term converges to 0 when $\delta \downarrow 0$ [see relation (A.5)]. Due to the fact that ε can be chosen arbitrarily small, this implies that f satisfies the condition (A.4). $\square$

COROLLARY A.6. *X_T^β is a Polish (complete and separable) space.*

PROOF. Completeness comes from Proposition A.3. For separability, we invoke Proposition A.5(2) and the fact that the functions in $C^{0,1}([0,T]\times\mathbb{R})$ can be approximated by polynomials with rational coefficients, using a truncation argument. $\square$

REMARK A.7. The space which satisfies only condition (i) in Definition 4.2 would be too big and fails to be separable. Analogous situations occur frequently in analysis. In the study of Morrey spaces, this fact was first observed by Zorko in [16]. The continuity of spatial translations with respect to a norm is sometimes called Zorko condition.

PROPOSITION A.8. *The inclusion $X_T^\beta \subset X_T^\alpha$ holds continuously for $\beta > \alpha$.*

PROOF. Suppose f belongs to X_T^β. Fix $n \ge 1$. By Proposition A.1, we see that

$$\sup_{t\in[0,T],|x|\le n} |f(t,x+y)-f(t,x)| \le C \sup_{t\in[0,T],|x|\le n+1} \mathcal{N}_\beta^{(3)} f(t,x)|y|^\beta$$

for every $|y| \le 1$. Hence, for every $t \le T$, $|x| \le n$ and $\alpha < \beta$ we have

$$\int_{|y|\le 1} |f(t,x+y)-f(t,x)|^2 |y|^{-2\alpha-1}\,dy \le C \sup_{t\in[0,T],|x|\le n+1} \mathcal{N}_\beta^{(3)} f(t,x),$$

which is a finite quantity. The continuity of $(t,x) \mapsto \int_{|y|\le 1} |f(t,x+y) - f(t,x)|^2 |y|^{-2\alpha-1}\,dy$ follows at once from dominated convergence theorem. $\square$

We state an analogous result for $\mathfrak{X}_T^{\beta,p}$ without proof.

PROPOSITION A.9. *The inclusion $\mathfrak{X}_T^{\beta,p} \subset \mathfrak{X}_T^{\alpha,q}$ holds continuously for $\beta > \alpha$ and $p \ge q$.*

A.2. Compactness criteria. In the current subsection, we derive compactness criteria for X_T^β and $\mathfrak{X}_T^\beta(B)$. We first recall some well-known definitions and facts. An *ε-cover* of a metric space is a cover of the space consisting of sets of diameter at most ε. A metric space is called *totally bounded* if it admits a finite ε-cover for every $\varepsilon > 0$. It is well known that a metric space is compact if and only if it is complete and totally bounded. The following lemma is the key ingredient for many compactness results.

LEMMA A.10. *Let X be a metric space. Assume that, for every $\varepsilon > 0$, there exists a $\delta > 0$, a metric space W, and a mapping $\Phi : X \to W$ such that $\Phi(X)$ is totally bounded, and for all $x, y \in X$ with $d(\Phi(x), \Phi(y)) < \delta$, we have $d(x, y) < \varepsilon$. Then X is totally bounded.*

The proof of this lemma is elementary; we refer readers to Lemma 1 in [12] for details. The following result provides sufficient conditions for relative compactness in X_T^β.

PROPOSITION A.11. *A set $\mathfrak{F}$ in X_T^β is relatively compact if*:

[A1] $\sup_{f\in\mathfrak{F}} |f(0,0)|$ *is finite.*
[A2] *For every fixed $x \in \mathbb{R}$, $\{f(\cdot, x) : f \in \mathfrak{F}\}$ is equicontinuous in time.*
[A3] *For every $R > 0$,*

$$\lim_{\delta\downarrow 0} \sup_{f\in\mathfrak{F}} \sup_{t\in[0,T]} \sup_{x\in[-R,R]} \int_{-\delta}^{\delta} \frac{|f(t, x + y) - f(t, x)|^2}{|y|^{1+2\beta}} dy = 0.$$

PROOF. Suppose that $\mathfrak{F}$ satisfies the three conditions. We first observe that condition [A3] together with (A.1) implies the following equicontinuity property. For every $R > 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t\in[0,T]} \bigl|f(t, x) - f(t, y)\bigr| < \varepsilon$$

whenever $f \in \mathfrak{F}$ and $x, y \in [-R, R]$ satisfy $|x - y| < \eta$. Together with [A2], this implies equicontinuity for $\mathfrak{F}$ in $(t, x) \in [0, T] \times [-R, R]$. Indeed, take N to be a sufficiently large integer, and set $x_i = -R + \frac{j}{N} R$, $j = 0, 1, \dots, 2N$. According to [A2], $\{f(\cdot, x_i) : f \in \mathfrak{F}\}$ is equicontinuous in time, uniformly for $j = 0, 1, \dots, 2N$. By writing

$$\begin{aligned}&\bigl|f(t, x) - f(s, x)\bigr| \\ &\quad\le \bigl|f(t, x) - f(t, x_i)\bigr| + \bigl|f(t, x_i) - f(s, x_i)\bigr| + \bigl|f(s, x_i) - f(s, x)\bigr|,\end{aligned}$$

where x_i is chosen in such a way that $|x - x_i| < \eta$, this shows the uniformity in x.

Fix now $R > 0$ and $\varepsilon > 0$. From [A3], we can choose a positive number $\delta_1 = \delta_1(\varepsilon)$, such that $\delta_1 < 1$ and

$$2 \sup_{f \in \mathfrak{F}} \sup_{t \in [0,T], x \in [-R,R]} \int_{-\delta_1}^{\delta_1} \frac{|f(t,x+y) - f(t,x)|^2}{|y|^{1+2\beta}} \, dy < \varepsilon^2.$$

We now choose $\delta_2 \le \varepsilon$ satisfying

$$2(3\delta_2)^2 \int_{|y|>\delta_1} \frac{dy}{|y|^{1+2\beta}} < \varepsilon^2.$$

By the equicontinuity, we can also choose a positive number $\eta = \eta(\varepsilon)$, $\eta < 1$, such that

$$\|f(t,x) - f(s,y)\| < \delta_2, \tag{A.7}$$

whenever $f \in \mathfrak{F}$ and $(t,x), (s,y) \in [0,T] \times [-R-2, R+2]$ satisfy $|t-s| + |x-y| < \eta$. Since $[0,T] \times [-R-2, R+2]$ is compact, we can find a finite set of points $\{(t_a, x_i) : 1 \le a, i \le n\}$ in $[0,T] \times [-R-2, R+2]$ such that for every $(t,x) \in [0,T] \times [-R-1, R+1]$, there is some (t_a, x_j) so that $|t-t_a| + |x-x_j| < \eta$ and $[x_j - 1, x_j + 1] \subset [-R-2, R+2]$.

Define $\Phi : \mathfrak{F} \to \mathbb{R}^{n^2}$ by

$$\Phi(f) = \big(f(t_a, x_i) : 1 \le a, i \le n\big).$$

Condition [A1] and equicontinuity imply that the image $\Phi(\mathfrak{F})$ is bounded, and thus totally bounded in $\mathbb{R}^{n^2}$. Furthermore, consider $f, g \in \mathfrak{F}$ with $\|\Phi(f) - \Phi(g)\|_\infty < \delta_2$. Resorting to the fact that for any $(t,x) \in [0,T] \times [-R-1, R+1]$ there are some a, j so that $|t-t_a| + |x-x_j| < \eta$, we can write

$$\begin{aligned}
&\big|f(t,x) - g(t,x)\big| \\
&\quad \le \big|f(t,x) - f(t_a,x_j)\big| + \big|f(t_a,x_j) - g(t_a,x_j)\big| + \big|g(t_a,x_j) - g(t,x)\big| \\
&\quad \le 3\delta_2,
\end{aligned}$$

where we bounded the first and third term on the right-hand side thanks to (A.7), and the second one according to the fact that $\|\Phi(f) - \Phi(g)\|_\infty < \delta_2$. We end up with

$$\sup_{t \in [0,T], x \in [-R-1, R+1]} \big|f(t,x) - g(t,x)\big| \le 3\delta_2 \le 3\varepsilon.$$

In addition, for every $(t,x) \in [0,T] \times [-R,R]$ we have

$$\begin{aligned}
&\big[\mathcal{N}_\beta(f-g)(t,x)\big]^2 \\
&\quad \le 2 \sup_{h \in \{f,g\}} \int_{|y| \le \delta_1} \big|h(t,x+y) - h(t,x)\big|^2 \frac{dy}{|y|^{1+2\beta}} \\
&\qquad + 2 \sup_{r \in [0,T], z \in [-R-1, R+1]} \big|f(r,z) - g(r,z)\big|^2 \int_{|y|>\delta_1} \frac{dy}{|y|^{1+2\beta}} \le 2\varepsilon^2.
\end{aligned}$$

Therefore, by the definition of the metric on X_T^β [see (4.5)] and Lemma A.10, the set $\mathfrak{F}$ is totally bounded in X_T^β. $\square$

A useful consequence of the previous proposition is the following corollary.

COROLLARY A.12. *Suppose $\alpha > \beta$. Let $\mathfrak{F}$ be a subset of X_T^α such that $\mathfrak{F}$ is equicontinuous in time for every fixed x, $\sup_{f\in\mathfrak{F}} |f(0,0)| < \infty$ and $\sup_{f\in\mathfrak{F}} \sup_{t\in[0,T],|x|\le R} \mathcal{N}_\alpha^{(1)} f(t,x) < \infty$ for every positive R. Then $\mathfrak{F}$ is relatively compact in X_T^β.*

PROOF. It suffices to check that $\mathfrak{F}$ satisfies condition [A3]. Applying (A.1), for δ small enough, the assumption on $\mathfrak{F}$ implies

$$\sup_{f\in\mathcal{F}} \sup_{t\in[0,T],|x|\le R} \bigl|f(t,x+y) - f(t,x)\bigr| \le C|y|^\alpha,$$

for all $|y| \le \delta$. Hence,

$$\begin{aligned}
&\sup_{f\in\mathcal{F}} \sup_{t\in[0,T],|x|\le R} \int_{|y|\le\delta} \bigl|f(t,x+y) - f(t,x)\bigr|^2 |y|^{-2\beta-1}\,dy \\
&\qquad \le C\int_{|y|\le\delta} |y|^{2(\alpha-\beta)-1}\,dy,
\end{aligned}$$

which clearly implies [A3] since $\alpha > \beta$. $\square$

The following result provides sufficient conditions for relative compactness in $\mathfrak{X}_T^\beta(B)$. Its proof is completely analogous to that of Proposition A.11 and is omitted for the sake of conciseness.

PROPOSITION A.13. *Suppose that a set $\mathfrak{F}$ in $\mathfrak{X}_T^\beta(B)$ satisfies the following properties*:

(1) *For every $t \in [0,T]$ and $x \in \mathbb{R}$, $\mathfrak{F}(t,x) := \{f(t,x) : f \in \mathfrak{F}\}$ is relatively compact in the Banach space B.*

(2) *For every fixed $x \in \mathbb{R}$, $\{f(\cdot,x) : f \in \mathfrak{F}\}$ is equicontinuous in time.*

(3) *For every $R > 0$, we have*

$$\lim_{\delta\downarrow 0} \sup_{f\in\mathfrak{F}} \sup_{t\in[0,T],x\in[-R,R]} \int_{-\delta}^{\delta} \frac{\|f(t,x+y) - f(t,x)\|^2}{|y|^{1+2\beta}}\,dy = 0.$$

Then $\mathfrak{F}$ is relatively compact in $\mathfrak{X}_T^\beta(B)$.

In order to handle the nonlinearity in equation (1.1), the following composition rule is crucial.

PROPOSITION A.14 (Left composition). *Let σ be a Lipschitz function on $\mathbb{R}$ and let f be a function in X_T^β. Suppose that for every fixed x, the map $t \mapsto \mathcal{N}_\beta^{(1)}\sigma(f)(t,x)$ is continuous. Then $\sigma(f)$ belongs to X_T^β. Furthermore, if f_n is a sequence converging to f in X_T^β, then for every positive R and for any $\delta > 0$ we have*

$$\lim_{n\to\infty} \sup_{t\in[0,T],|x|\le R} \mathcal{N}_\beta^{(\delta)}\big(\sigma(f_n)-\sigma(f)\big)(t,x) = 0.$$

PROOF. We first show that $\sigma(f)$ belongs to X_T^β. For any $\delta > 0$, we have

$$\int_{|y|\le\delta} \big|\sigma\big(f(t,x+y)\big)-\sigma\big(f(t,x)\big)\big|^2 |y|^{-2\beta-1}\,dy \le \|\sigma\|_{\mathrm{Lip}}^2 \big[\mathcal{N}_\beta^{(\delta)} f(t,x)\big]^2,$$

which together with the criterion (3) in Proposition A.5 implies that $\sigma(f)$ belongs to X_T^β.

For the second assertion, for every positive R and any $\varepsilon > 0$, we can choose $\delta_0 > 0$ and $n_0 > 0$, so that, for any $n \ge n_0$,

$$\sup_{t\in[0,T],|x|\le R} \mathcal{N}_\beta^{(\delta_0)}\big(\sigma(f_n)-\sigma(f)\big)(t,x) \le \varepsilon. \tag{A.8}$$

Indeed, it is easily seen that

$$\begin{aligned}
\mathcal{N}_\beta^{(\delta_0)}\big(\sigma(f_n)-\sigma(f)\big)(t,x) &\le \mathcal{N}_\beta^{(\delta_0)}\sigma(f_n)(t,x) + \mathcal{N}_\beta^{(\delta)}\sigma(f)(t,x) \\
&\le \|\sigma\|_{\mathrm{Lip}}\big(\mathcal{N}_\beta^{(\delta_0)} f_n(t,x) + \mathcal{N}_\beta^{(\delta_0)} f(t,x)\big) \\
&\le \|\sigma\|_{\mathrm{Lip}}\big(\mathcal{N}_\beta^{(\delta_0)}(f_n-f)(t,x) + 2\mathcal{N}_\beta^{(\delta_0)} f(t,x)\big),
\end{aligned}$$

and the last term is readily bounded by ε if δ_0 is chosen small enough. Now with (A.8) in hand we obtain, for any $\delta > 0$,

$$\begin{aligned}
&\sup_{t\in[0,T],|x|\le R} \mathcal{N}_\beta^{(\delta)}\big(\sigma(f_n)-\sigma(f)\big)(t,x) \\
&\quad \le C\varepsilon + C\|\sigma\|_{\mathrm{Lip}} \sup_{t\in[0,T],|x|\le R+1} \big|f_n(t,x)-f(t,x)\big| \bigg(\int_{|y|>\delta_0} |y|^{-2\beta-1}\,dy\bigg)^{\frac{1}{2}}.
\end{aligned}$$

We conclude the proof by taking the limit as n tends to infinity. $\square$

The next lemma gives a criterion for a process in $\mathfrak{X}_T^{\alpha,p}$ to have its paths almost surely lie in the space X_T^β for a certain value of β.

LEMMA A.15. *Let f be a stochastic process in $\mathfrak{X}_T^{\alpha,p}$ with $p\alpha > 1$. Assume that for any $R > 0$*

$$\sup_{s,t\in[0,T]} \sup_{|x|\le R} \big\|f(t,x)-f(s,x)\big\|_{L^p(\Omega)} \le C_R|t-s|^\lambda, \tag{A.9}$$

where $\lambda > p^{-1}$. *Then* f *has a version* $\tilde{f}$ *such that with probability one*, $\tilde{f}$ *belongs to* X_T^{β} *for every* $\beta < \alpha - \frac{1}{p}$.

PROOF. Since f belongs to $\mathfrak{X}_T^{\alpha,p}$, inequality (A.1) implies

$$\sup_{t\in[0,T]}\sup_{x,y\in\mathbb{R}}\frac{\|f(t,x+y)-f(t,x)\|_{L^p(\Omega)}}{|y|^{\alpha}} \\ \le C\sup_{t\in[0,T],x\in\mathbb{R}}\int_{\mathbb{R}}\|f(t,x+y)-f(t,x)\|_{L^p(\Omega)}^2|y|^{-2\alpha-1}\,dy.$$

Then by the Kolmogorov continuity criterion, f has a version $\tilde{f}$ such that with probability one, $\tilde{f}$ satisfies

$$\sup_{s,t\in[0,T],|x|\le R}\big|\tilde{f}(t,x+y)-\tilde{f}(s,x)\big| \le C\big(|y|^{\beta'}+|t-s|^{\lambda'}\big)$$

for every R and $|y|\le 1$, where β' and λ' are fixed and such that $\beta<\beta'<\alpha-1/p$ and $\lambda<\lambda'<\lambda-1/p$. This implies that a.s. $\mathcal{N}_{\beta}^{(1)}\tilde{f}(t,x)$ is finite and a.s. $\tilde{f}(t,x)$ satisfies condition (A.4). The continuity of $\mathcal{N}_{\beta}^{(1)}\tilde{f}$ follows from the dominated convergence theorem. These facts imply that $\tilde{f}$ belongs to X_T^{β} almost surely. $\square$

A.3. Probability measures on X_T^{β}. This subsection is devoted to study tightness of probability measures defined on X_T^{β}. These properties are needed in Section 4 to show the existence of solution to equation (1.1). We have the following result toward this aim.

THEOREM A.16. *Let* $\{\mathbf{P}_n, n\ge 1\}$ *be a sequence of probability measures on* X_T^{β}. *This sequence is tight if the following three conditions hold*:

(1) *For each positive* η, *there exist* a *and* n_0 *such that for all* $n\ge n_0$:

$$\mathbf{P}_n\big(f\in X_T^{\beta}:\big|f(0,0)\big|\ge a\big)\le\eta. \tag{A.10}$$

(2) *For every* $x\in\mathbb{R}$, *and every positive* ε *and* η, *there exist* δ *satisfying* $0<\delta<1$, *and* n_0 *such that for all* $n\ge n_0$

$$\mathbf{P}_n\Big(f\in X_T^{\beta}:\sup_{s,t\le T,|t-s|<\delta}\big|f(t,x)-f(s,x)\big|\ge\varepsilon\Big)\le\eta. \tag{A.11}$$

(3) *For every* $R>0$, *for each positive* ε *and* η, *there exist* $\delta\in(0,1)$ *and* n_0 *such that for all* $n\ge n_0$

$$\mathbf{P}_n\Big(f\in X_T^{\beta}:\sup_{t\in[0,T],|x|\le R}\int_{-\delta}^{\delta}\big|f(t,x+y)-f(t,x)\big|^2|y|^{-2\beta-1}\,dy\ge\varepsilon\Big) \\ \le\eta. \tag{A.12}$$

PROOF. Without loss of generality we assume $n_0 = 1$. For a given $\eta > 0$, we choose a so that $\mathbf{P}_n(B^c) \le \eta$ for all $n \ge 1$, where

$$B = \{ f \in X_T^\beta : |f(0,0)| < a \}.$$

According to condition (3), for any integer k, N, we also choose and fix $\delta_{k,N}$ such that $\mathbf{P}_n(A_{k,N}^c) \le \eta 2^{-k-N}$ for all $n \ge 1$, where

$$A_{k,N} = \left\{ f \in X_T^\beta : \sup_{t \in [0,T], |x| \le N} \int_{-\delta_{k,N}}^{\delta_{k,N}} |f(t,x+y) - f(t,x)|^2 |y|^{-2\beta-1} \, dy \le \frac{1}{k^2} \right\}.$$

Then for each $\tilde{x} \in [-N,N] \cap \frac{\delta_{k,N}}{3}\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers (note that the number of such $\tilde{x}$ has order $\frac{N}{\delta_{k,N}}$), we choose $\delta'_{k,N}(\tilde{x})$ according to condition (2) such that $\mathbf{P}_n(B_{k,N}^c(\tilde{x})) \le \delta_{k,N} \eta 2^{-k-N}$, where

$$B_{k,N}(\tilde{x}) = \left\{ f \in X_T^\beta : \sup_{t,s,\le T, |t-s| \le \delta'_{k,N}(\tilde{x})} |f(t,\tilde{x}) - f(s,\tilde{x})| \le \frac{1}{k^2} \right\}.$$

Consider now $B_{k,N} = \bigcap_{\tilde{x} \in [-N,N] \cap \frac{\delta_{k,N}}{3}\mathbb{Z}} B_{k,N}(\tilde{x})$. It is easy to see that

$$\mathbf{P}_n(B_{k,N}^c) \le \sum_{\tilde{x} \in [-N,N] \cap \frac{\delta_{k,N}}{3}\mathbb{Z}} \mathbf{P}_n(B_{k,N}^c(\tilde{x})) \le C \frac{N}{\delta_{k,N}} \eta \delta_{k,N} 2^{-k-N} = C\eta 2^{-k-N} N.$$

We thus set $A = \bigcap_{k,N} (A_{k,N} \cap B_{k,N}) \cap B$. Then according to Proposition A.11 we see that the closure of A is compact in X_T^β, and $\mathbf{P}_n(A) \ge 1 - C\eta$. This shows the tightness of $\mathbf{P}_n$. $\square$

The following proposition states that under some moment conditions, a sequence of processes $\{u_n\}$ can be regarded as a tight sequence of probability measures on the space X_T^β.

PROPOSITION A.17. *Assume that $\alpha, \lambda \in (0,1)$ and $p \ge 1$ satisfy $p\alpha > 1$, $p\lambda > 1$ and $\beta < \alpha - 1/p$. Let $\{u_n, n \ge 1\}$ be a sequence of stochastic processes such that*:

(1) $\lim_{\delta \to \infty} \limsup_n \mathbf{P}(|u_n(0,0)| > \delta) = 0$,

(2) *for every* $R > 0$, $\sup_n \sup_{s,t \in [0,T], |x| \le R} \|u_n(t,x) - u_n(s,x)\|_{L^p(\Omega)} \le C_R |t-s|^\lambda$,

(3) $\sup_n \|u_n\|_{\mathfrak{X}_T^{\alpha,p}}$ *is finite.*

From Lemma A.15, the law of u_n can be considered as a probability measure on X_T^β. In addition, as probability measures on X_T^β, the sequence $\{u_n, n \ge 1\}$ is tight.

PROOF. This proposition can be easily proved using the same ideas as in the proof of Lemma A.15 and Theorem A.16. We omit the details. $\square$

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REFERENCES

[1] BAHOURI, H., CHEMIN, J.-Y. and DANCHIN, R. (2011). *Fourier Analysis and Nonlinear Partial Differential Equations*. *Grundlehren der Mathematischen Wissenschaften* **343**. Springer, Heidelberg. MR2768550

[2] BALAN, R. M., JOLIS, M. and QUER-SARDANYONS, L. (2015). SPDEs with affine multiplicative fractional noise in space with index $H < 1/2$. *Electron*. *J*. *Probab*. **20** 36 pp. MR3354614

[3] BRZEŹNIAK, Z. and PESZAT, S. (1999). Space–time continuous solutions to SPDE's driven by a homogeneous Wiener process. *Studia Math*. **137** 261–299. MR1736012

[4] DALANG, R. (1999). Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron*. *J*. *Probab*. **4** 29 pp.

[5] DALANG, R. C. (2001). Corrections to: "Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s" [*Electron J. Probab.* **4** (1999) 29 pp. (electronic); MR1684157 (2000b:60132)]. *Electron*. *J*. *Probab*. **6** 5 pp. MR1825714

[6] DALANG, R. C. and QUER-SARDANYONS, L. (2011). Stochastic integrals for spde's: A comparison. *Expo*. *Math*. **29** 67–109. MR2785545

[7] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136

[8] ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. and TRICOMI, F. G. (1981). *Higher Transcendental Functions*. *Vol*. *III*. *Based on Notes Left by Harry Bateman*. Robert E. Krieger Publishing, Melbourne, FL. MR0698781

[9] GYÖNGY, I. (1998). Existence and uniqueness results for semilinear stochastic partial differential equations. *Stochastic Process*. *Appl*. **73** 271–299. MR1608641

[10] GYÖNGY, I. and KRYLOV, N. (1996). Existence of strong solutions for Itô's stochastic equations via approximations. *Probab*. *Theory Related Fields* **105** 143–158. MR1392450

[11] GYÖNGY, I. and NUALART, D. (1999). On the stochastic Burgers' equation in the real line. *Ann*. *Probab*. **27** 782–802. MR1698967

[12] HANCHE-OLSEN, H. and HOLDEN, H. (2010). The Kolmogorov–Riesz compactness theorem. *Expo*. *Math*. **28** 385–394. MR2734454

[13] PESZAT, S. and ZABCZYK, J. (1997). Stochastic evolution equations with a spatially homogeneous Wiener process. *Stochastic Process*. *Appl*. **72** 187–204.

[14] PIPIRAS, V. and TAQQU, M. S. (2000). Integration questions related to fractional Brownian motion. *Probab*. *Theory Related Fields* **118** 251–291. MR1790083

[15] SAMKO, S., KILBAS, A. and MARICHEV, O. (1993). *Fractional Integrals and Derivatives*. *Theory and Applications*. Gordon and Breach Science Publishers, Yverdon.

[16] ZORKO, C. T. (1986). Morrey space. *Proc*. *Amer*. *Math*. *Soc*. **98** 586–592.

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