LÉVY PROCESSES AND LÉVY WHITE NOISE AS TEMPERED DISTRIBUTIONS[1]

BY ROBERT C. DALANG AND THOMAS HUMEAU

École Polytechnique Fédérale de Lausanne

We identify a necessary and sufficient condition for a Lévy white noise to be a tempered distribution. More precisely, we show that if the Lévy measure associated with this noise has a positive absolute moment, then the Lévy white noise almost surely takes values in the space of tempered distributions. If the Lévy measure does not have a positive absolute moment of any order, then the event on which the Lévy white noise is a tempered distribution has probability zero.

1. Introduction. It is well known that Gaussian white noise in $\mathbb{R}^d$ is a generalized random field that can be viewed as a random element of the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered (Schwartz) distributions [13, 27]. It is natural to ask whether the same is true of Lévy white noise.

This abstract mathematical question was posed to the authors by M. Unser and J. Fageot, who were interested in developing sparse statistical models for signal and image processing [11]. For this, they considered generalized random fields with values in $\mathcal{S}'(\mathbb{R}^d)$. Lévy white noises provide interesting examples of generalized random fields, and together with A. Amini, they showed in [11], Theorem 3, that a sufficient condition for Lévy white noise to take values in $\mathcal{S}'(\mathbb{R}^d)$ is that the associated Lévy measure has a positive absolute moment. The main result of this paper is that this condition is, in fact, necessary and sufficient.

The result of Unser and Fageot improves several other partial results that appear in the mathematical literature. In [19], Lévy white noise is studied as a natural generalization of Gaussian white noise, and the authors showed that this process takes values in $\mathcal{S}'(\mathbb{R}^d)$ if the associated Lévy measure has a first absolute moment. In order to develop a white noise theory for Lévy noise, Di Nunno et al. [8] consider Lévy white noise with a Lévy measure that has a finite second moment. In [20], Theorem 4.1, Y.-L. Lee and H.-H. Shih give a necessary and sufficient condition for Lévy white noise to take values in $\mathcal{S}'(\mathbb{R}^d)$; however, this condition involves checking the continuity of a functional and does not translate directly into a condition on the Lévy measure. Finally, knowing that Lévy white noise takes values in

$\mathcal{S}'(\mathbb{R}^d)$ is useful in the study of stochastic partial differential equations driven by
Lévy noise, as in [21], which again considers the case where the Lévy measure is
square integrable.

In order to address the question of Unser and Fageot, we first consider in Section 2 the case of dimension $d = 1$. In this case, Lévy white noise can be viewed
as the derivative of a Lévy process, for which there is a large literature (see [2,
4, 23], for instance). The question of whether or not sample paths of a Lévy process belong to $\mathcal{S}'(\mathbb{R}^d)$ reduces essentially to whether or not this process is slowly
growing (i.e., has no more than polynomial growth: see Remark 1.1). We make
use of the Lévy–Itô decomposition $X_t = \gamma t + \sigma W_t + X_t^P + X_t^M$ of a Lévy process $(X_t)_{t \in \mathbb{R}_+}$, in which (W_t) is a standard Brownian motion, (X_t^P) is a compound
Poisson process (term containing the large jumps of X), and (X_t^M) is a square integrable pure-jump martingale (term containing the small jumps of X). Using the
strong law of large numbers for Lévy processes with a first moment, we show that
the Lévy process $(\gamma t + \sigma W_t + X_t^M)$ is always slowly growing, so the question
reduces to the study of the process (X_t^P).

A first result (Proposition 2.2) is that a compound Poisson process can belong to
$\mathcal{S}'(\mathbb{R}^d)$ if and only if it is slowly growing. The question now reduces to determining
when a compound Poisson process is slowly growing, which is addressed in the
literature (see [23], Section 48), and we give a precise statement in Proposition 2.4.

With these results for Lévy processes in hand, we then easily deduce the corresponding result for Lévy white noise (see Theorem 2.7). Since [11] constructs
Lévy white noise as a measure on the cylinder σ-field of $\mathcal{S}'(\mathbb{R}^d)$ via the Bochner–
Minlos theorem [13], we relate our result to that of [11] by taking care to show
that Lévy noise actually defines a random variable with values in $\mathcal{S}'(\mathbb{R}^d)$ equipped
with its Borel σ-field (which is in fact equal to the cylinder σ-field, see the proof
of Corollary 2.9).

In Section 3, we turn to Lévy random fields and Lévy noise on $\mathbb{R}^d$, with $d \geq 1$.
Again, in the case of a Lévy random field, the Lévy–Itô decomposition applies
(see [1, 6]), and the three terms with moments greater than 1 always have sample
paths with values in $\mathcal{S}'(\mathbb{R}^d)$ (see Proposition 3.7). For the term containing the large
jumps, which is a compound Poisson sheet X^P, we are hampered by the fact that
even if there are multiparameter analogues of the law of large numbers of Kolmogorov, Marcinkiewicz and Zygmund (see [18]), multiparameter random walks
cannot be easily used to represent compound Poisson sheets. Therefore, we make
use of our study in dimension 1 by considering the Lévy random field X^P along
a line parallel to a coordinate axis. This defines a (one-parameter) Lévy process
L. A key technical step is to identify (in Lemma 3.12) a sequence of test functions $(\varphi_n) \subset \mathcal{S}(\mathbb{R}^d)$ with polynomially growing norms such that $\langle X^P, \varphi_n \rangle$ gives
precisely the value of L at the time of its nth jump. This leads to the characterization of Lévy white noises and random fields that take values in $\mathcal{S}'(\mathbb{R}^d)$ (see
Theorem 3.13).

We now introduce the main notation that will be used throughout the paper. Let $d \in \mathbb{N} \setminus \{0\}$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, a smooth function $\varphi : \mathbb{R}^d \to \mathbb{R}$, and $t \in \mathbb{R}^d$, we write

$$t^\alpha = \prod_{i=1}^{d} t_i^{\alpha_i} \quad \text{and} \quad \varphi^{(\alpha)} = \frac{\partial^{|\alpha|} \varphi}{\partial t_1^{\alpha_1} \cdots \partial t_d^{\alpha_d}} \qquad \text{where } |\alpha| = \sum_{i=1}^{d} \alpha_i.$$

For $\beta, \gamma \in \mathbb{N}^d$, we write $\binom{\beta}{\gamma} = \frac{\beta!}{\gamma!(\beta-\gamma)!}$, where $\beta! = \prod_{i=1}^{d}(\beta_i!)$. When $t \in \mathbb{R}^d$, we also write $|t|$ for the Euclidean norm and the meaning should be clear from the context. The Schwartz space is denoted $\mathcal{S}(\mathbb{R}^d)$ and is the space of all smooth functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that, for all multi-indexes $\alpha, \beta \in \mathbb{N}^d$, we have

$$\sup_{t \in \mathbb{R}^d} |t^\alpha \varphi^{(\beta)}(t)| < +\infty.$$

This space is equipped with the topology defined by the family of norms $\mathcal{N}_p$, where, for all $p \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{N}_p(\varphi) = \sum_{|\alpha|, |\beta| \leq p} \sup_{t \in \mathbb{R}^d} |t^\alpha \varphi^{(\beta)}(t)|.$$

A basis of neighborhoods of the origin for this topology is given by the family:

$$(1.1) \qquad \left(\{\varphi \in \mathcal{S}(\mathbb{R}^d) : \mathcal{N}_p(\varphi) < \varepsilon\}\right)_{p \in \mathbb{N}, \varepsilon > 0},$$

since such a basis is usually given by finite intersections of sets of this form, and for all $p \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\mathcal{N}_p(\varphi) \leq \mathcal{N}_{p+1}(\varphi)$. A sequence $(\varphi_n)_n$ converges to zero in $\mathcal{S}(\mathbb{R}^d)$ if for all $p \in \mathbb{N}$, $\mathcal{N}_p(\varphi_n) \to 0$ as $n \to +\infty$. The space of tempered distributions is denoted $\mathcal{S}'(\mathbb{R}^d)$ and is the space of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$. Equivalently, $u \in \mathcal{S}'(\mathbb{R}^d)$ if and only if there is an integer $p \geq 0$ and a constant C such that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$|\langle u, \varphi \rangle| \leq C \mathcal{N}_p(\varphi).$$

REMARK 1.1. We say that a Borel function $f : \mathbb{R}^d \to \mathbb{R}$ is slowly growing if $\sup_{t \in \mathbb{R}^d} |f(t)|(1 + |t|)^{-\alpha} < \infty$ for some $\alpha \geq 0$. In this case, f defines a tempered distribution by the formula $\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(t) \varphi(t) \, dt$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

We will also consider the space $\mathcal{D}(\mathbb{R}^d)$ of smooth compactly supported functions and its topological dual $\mathcal{D}'(\mathbb{R}^d)$, the space of distributions (see, e.g., [24] or [26]).

2. Lévy processes and Lévy white noise in $\mathcal{S}'(\mathbb{R})$. A Lévy process $(X_t)_{t \in \mathbb{R}_+}$ is a real valued stochastic process such that $X_0 = 0$ almost surely, X has stationary and independent increments and X is stochastically continuous (i.e., for any $s \geq 0$, $|X_t - X_s| \to 0$ in probability as $t \to s$). Every Lévy process has a *càdlàg* (right continuous with left limits) modification by [23], Theorem 11.5, and we

will always consider such a modification in the following. An important feature of Lévy processes is the Lévy–Itô decomposition: for a Lévy process X there exists a unique triplet (γ, σ, ν), where $\sigma \geq 0$, $\gamma \in \mathbb{R}$, and ν is a Lévy measure [in particular, ν is nonnegative and $\int_{\mathbb{R} \setminus \{0\}} (1 \wedge |x|^2) \nu(\mathrm{d}x) < +\infty$], such that the jump measure of X (denoted by J_X) is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$ with intensity $\mathrm{d}t\,\nu(\mathrm{d}x)$ and X has the decomposition $X_t = \gamma t + \sigma W_t + X_t^P + X_t^M$. In this decomposition, W is a standard Brownian motion,

$$X_t^P = \int_{s \in [0,t], |x| > 1} x\, J_X(\mathrm{d}s, \mathrm{d}x)$$

is a compound Poisson process (the term containing the large jumps of X), and

$$X_t^M = \int_{s \in [0,t], |x| \leq 1} x\big(J_X(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\,\nu(\mathrm{d}x)\big)$$

is a square integrable martingale (the term containing the small jumps of X).

Since a Lévy process is *càdlàg*, it is locally Lebesgue integrable, and defines almost surely an element of $\mathcal{D}'(\mathbb{R})$ via the L^2-inner product

$$\langle X, \varphi \rangle = \int_{\mathbb{R}_+} X_t \varphi(t)\, \mathrm{d}t \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}).$$

For any *càdlàg* process L, we define the following subset of Ω:

(2.1) $\Omega_L = \{\omega \in \Omega : L(\omega) \in \mathcal{S}'(\mathbb{R})\}$,

with the understanding that when $L(\omega) \in \mathcal{S}'(\mathbb{R})$, the continuous linear functional associated with $L(\omega)$ is given by $\langle L(\omega), \varphi \rangle = \int_{\mathbb{R}_+} L_t(\omega) \varphi(t)\, \mathrm{d}t$, for all $\varphi \in \mathcal{S}(\mathbb{R})$.

2.1. *The case of a Lévy process with integer moments.* Three terms in the Lévy–Itô decomposition have absolute moments of any order, and this leads to the following proposition.

PROPOSITION 2.1. *Let X be a Lévy process with characteristic triplet (γ, σ, ν) and Lévy–Itô decomposition $X_t = \gamma t + \sigma W_t + X_t^P + X_t^M$. Let $Y_t = \gamma t + \sigma W_t + X_t^M$. Then Y is slowly growing a.s., and the set Ω_Y defined as in* (2.1) *(with L there replaced by Y) has probability one.*

PROOF.   The process Y is a sum of a linear deterministic function, and of two independent square integrable Lévy processes. In particular, $\mathbb{E}(|Y_1|) < +\infty$. By the strong law of large numbers for Lévy processes in [23], Theorem 36.5, $t^{-1} Y_t \to \mathbb{E}(Y_1) = \gamma$ almost surely as $t \to +\infty$. It follows that Y is sublinear and locally bounded (by the *càdlàg* property) almost surely, so it is slowly growing. We deduce that Y is a tempered distribution almost surely by Remark 1.1.   □

2.2. *Growth of a compound Poisson process.* In view of Corollary 2.1, it remains to determine when a compound Poisson process belongs to $\mathcal{S}'(\mathbb{R})$. We begin with two key results on the growth of a compound Poisson process. Let $X_t = \sum_{i=1}^{N_t} Y_i$ be a compound Poisson process, where N is a Poisson process with parameter λ that is independent of the sequence $(Y_i)_{i \geq 1}$ of i.i.d. random variables. Let $S_0 = 0$ and $(S_n)_{n \geq 1}$ be the sequence of jump times of X and let $T_n = S_n - S_{n-1}$. Also, let $Z_n = X_{S_n} = \sum_{i=1}^{n} Y_i$. We first show that on the set Ω_X, the compound Poisson process is slowly growing.

PROPOSITION 2.2. *Let X be the compound Poisson process defined above and Ω_X the set defined in* (2.1). *There is a set A of probability one such that for all $\omega \in \Omega_X \cap A$, the function $t \mapsto X_t(\omega)$ is slowly growing.*

REMARK 2.3. We point out that this result relies on more than the piecewise constancy of a compound Poisson process. Indeed, there exist *càdlàg* piecewise constant functions in $\mathcal{S}'(\mathbb{R})$ which are not slowly growing. For example, consider the function f that is equal to zero except on intervals of the form $[n, n + 2^{-n}[$ where it is constant equal to $2^{\frac{n}{2}}$ for all $n \in \mathbb{N}$. Then $f \in L^1(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, but f is clearly not slowly growing.

PROOF OF PROPOSITION 2.2. The main idea is the following. Since X is constant on the interval $[S_n, S_{n+1}[$ and the jump times are rarely close together, we can build a sequence of random test functions φ_n supported just to the right of S_n, such that $\langle X, \varphi_n \rangle = X_{S_n}$ for large enough n. The control of $|\langle X, \varphi_n \rangle|$ by a norm $\mathcal{N}_p(\varphi_n)$ leads to a bound on X_{S_n}, and then on X_t since X is piecewise constant.

For $n \geq 1$, the jump time S_n has Gamma distribution with parameters n and λ. For $k \geq 1$ to be chosen later, and $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[0, 1]$, $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi = 1$, we consider the sequence φ_n defined by

$$(2.2) \qquad \varphi_n(t) = S_n^k \varphi\big((t - S_n) S_n^k\big).$$

Then

$$(2.3) \qquad \text{supp}(\varphi_n) \subset \left[S_n, S_n + \frac{1}{S_n^k} \right],$$

and $\int_{\mathbb{R}} \varphi_n = 1$. Furthermore, for any nonnegative integers p and $\alpha, \beta \leq p$,

$$\sup_{x \in \mathbb{R}} \big| x^\alpha \varphi_n^{(\beta)}(x) \big| = \sup_{x \in [S_n, S_n + \frac{1}{S_n^k}]} \big| x^\alpha \varphi_n^{(\beta)}(x) \big|$$

$$\leq \left(S_n + \frac{1}{S_n^k} \right)^\alpha S_n^{k(\beta+1)} \sup_{x \in \mathbb{R}} \big| \varphi^{(\beta)}(x) \big|,$$

hence,

$$(2.4) \qquad \mathcal{N}_p(\varphi_n) \mathbb{1}_{S_n \geq 1} \leq C \mathcal{N}_p(\varphi) S_n^{(p+1)k+p} \mathbb{1}_{S_n \geq 1},$$

where $C \in \mathbb{R}$ is deterministic, nonnegative, and depends only on p. We define the events

$$(2.5) \qquad A_{n,k} = \left\{ X \text{ does not jump in the interval } \left] S_n, S_n + \frac{1}{S_n^k} \right[\right\}.$$

Using the fact that T_{n+1} has exponential distribution with parameter λ and that T_{n+1} and S_n are independent, we have

$$\mathbb{P}(A_{n,k}^c) = \mathbb{P}\{N_{S_n + \frac{1}{S_n^k}} - N_{S_n} \geq 1\}$$

$$= \mathbb{P}\left\{ T_{n+1} < \frac{1}{S_n^k} \right\} = \mathbb{E}(1 - e^{-\frac{\lambda}{S_n^k}}) \leq \mathbb{E}\left(\frac{\lambda}{S_n^k} \right).$$

The Laplace transform of S_n is $\mathbb{E}(e^{-tS_n}) = \lambda^n (t + \lambda)^{-n}$, for $t \geq 0$. For $n \geq 3$, integrating twice from t to $+\infty$, we obtain

$$(2.6) \qquad \mathbb{E}\left(\frac{1}{S_n^2} \right) = \frac{\lambda^2}{(n-1)(n-2)}, \qquad n \geq 3.$$

We deduce that $\sum_n \mathbb{E}(S_n^{-2}) < +\infty$. Therefore, $\sum_n \mathbb{P}(A_{n,2}^c) < +\infty$ and by the Borel–Cantelli lemma,

$$\mathbb{P}\left(\limsup_{n \to +\infty} A_{n,2}^c \right) = 0,$$

and the set $A = \liminf_{n \to +\infty} A_{n,2}$ has probability one. Let $\omega \in A \cap \Omega_X$, and $N(\omega)$ be such that for all $n \geq N(\omega)$, $\omega \in A_{n,2}$. Then for $n \geq N(\omega)$, because of (2.3) and (2.5),

$$(2.7) \qquad \langle X, \varphi_n \rangle(\omega) = X_{S_n}(\omega) \mathbb{1}_{A_{n,2}}(\omega) + \langle X, \varphi_n \rangle(\omega) \mathbb{1}_{A_{n,2}^c}(\omega) = X_{S_n}(\omega).$$

Since $X(\omega)$ is a tempered distribution by definition of Ω_X, there is $p(\omega) \in \mathbb{N}$ and $C(\omega) \in \mathbb{R}$ such that

$$(2.8) \qquad \begin{aligned} |\langle X, \varphi_n \rangle(\omega)| \mathbb{1}_{S_n(\omega) \geq 1} &\leq C(\omega) \mathcal{N}_{p(\omega)}(\varphi_n) \mathbb{1}_{S_n(\omega) \geq 1} \\ &\leq C'(\omega) S_n^{3p(\omega)+2}(\omega) \mathbb{1}_{S_n(\omega) \geq 1}, \end{aligned}$$

by (2.4) with $k = 2$. Because $S_n \to +\infty$ a.s., we can choose $N(\omega)$ such that $S_n(\omega) \geq 1$ for all integers $n \geq N(\omega)$ (replacing A by another almost sure set). From (2.7) and (2.8), we deduce that for all $\omega \in A \cap \Omega_X$,

$$\frac{|X_{S_n}(\omega)|}{S_n^{3p(\omega)+2}(\omega)} \leq C'(\omega) < +\infty \qquad \text{for all } n \geq N(\omega).$$

Let $n \geq N(\omega)$ and let $t \geq S_n(\omega)$. There is an integer $j \geq n$ such that $t \in [S_j(\omega), S_{j+1}(\omega)[$. Then

$$|X_t(\omega)| = |X_{S_j}(\omega)| \leq C'(\omega) S_j^{3p(\omega)+2}(\omega) \leq C'(\omega) t^{3p(\omega)+2}.$$

We deduce that

$$\limsup_{t \to +\infty} \frac{|X_t(\omega)|}{1 + t^{3p(\omega)+2}} \le C'(\omega) < +\infty$$

on the set $A \cap \Omega_X$. This completes the proof. $\square$

The next proposition recalls properties of the long term behavior of a compound Poisson process.

PROPOSITION 2.4. *Let X be the compound Poisson process with jump heights* $(Y_i)_{i \ge 1}$ *defined at the beginning of this section*:

(i) *Suppose that there is a real number $p > 0$ such that $\mathbb{E}(|Y_1|^p) < +\infty$. Then there is $\alpha > 0$ such that*

$$\limsup_{t \to +\infty} \frac{|X_t|}{1 + t^\alpha} < +\infty \qquad a.s.$$

(ii) *Suppose that $\mathbb{E}(|Y_1|^p) = +\infty$ for every $p > 0$. Then for any $\alpha > 0$,*

$$\limsup_{t \to +\infty} \frac{|X_t|}{1 + t^\alpha} = +\infty \qquad a.s.$$

PROOF. This result is a straightforward consequence of [23], Proposition 48.10. $\square$

2.3. *Lévy white noise*: *The general case.* Let X be a Lévy process. We can define the derivative of X in the sense of distributions as follows.

DEFINITION 2.5. Let X be a Lévy process with characteristic triplet (γ, σ, ν). The Lévy white noise $\dot{X}$ is the derivative of X in $\mathcal{D}'(\mathbb{R})$: for $\omega \in \Omega$ and $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle \dot{X}(\omega), \varphi \rangle := -\langle X(\omega), \varphi' \rangle := -\int_{\mathbb{R}_+} X_t(\omega) \varphi'(t) \, dt.$$

Notice that the law of the Lévy white noise $\dot{X}$ is entirely characterized by the triplet (γ, σ, ν) (given that we use the truncation function $\mathbb{1}_{|x| \le 1}$ in the Lévy–Itô decomposition).

REMARK 2.6. Our definition of Lévy white noise is equivalent to other definitions such as the one found in [14], Definition 5.4.1, and in [11], Definition 3. We postpone the discussion of this issue to the multiparameter case: see Proposition 3.16.

We now turn to the question of whether or not a Lévy white noise is a tempered distribution. Similar to (2.1), for any Lévy noise $\dot{X}$, we define the set

$$(2.9) \qquad \Omega_{\dot{X}} = \{\omega \in \Omega : \dot{X}(\omega) \in \mathcal{S}'(\mathbb{R})\},$$

and we have the following characterization.

THEOREM 2.7. *Let X be a Lévy process with characteristic triplet* (γ, σ, ν), *and* $\dot{X}$ *the associated Lévy white noise. Then* $\Omega_X = \Omega_{\dot{X}}$ (*defined respectively in* (2.1) *and* (2.9)), *and the following holds*:

(i) *If there exists* $\eta > 0$ *such that* $\mathbb{E}(|X_1|^\eta) < +\infty$, *then* $\mathbb{P}(\Omega_X) = \mathbb{P}(\Omega_{\dot{X}}) = 1$.
(ii) *If* $\mathbb{E}(|X_1|^\eta) = +\infty$ *for all* $\eta > 0$, *then* $\mathbb{P}(\Omega_X) = \mathbb{P}(\Omega_{\dot{X}}) = 0$.

REMARK 2.8. If $\mathbb{E}(|X_1|^\eta) < +\infty$ for some $\eta > 0$, then we say that X has a positive absolute moment (PAM). Recall that for $\eta > 0$, $\mathbb{E}(|X_1|^\eta) < +\infty$ if and only if $\int_{|x|>1} |x|^\eta \nu(dx) < +\infty$ (see [23], Theorem 25.3). Hence, the condition PAM can be equivalently expressed in terms of the Lévy measure ν.

PROOF OF THEOREM 2.7. Differentiation maps $\mathcal{S}'(\mathbb{R})$ to itself, hence on Ω_X, the Lévy noise $\dot{X}$ is a tempered distribution: $\Omega_X \subset \Omega_{\dot{X}}$. We now show that $\Omega_{\dot{X}} \subset \Omega_X$. Let $\omega \in \Omega_{\dot{X}}$. Two solutions in $\mathcal{D}'(\mathbb{R})$ of the equation $u' = \dot{X}(\omega)$ differ by a constant (see [24], Théorème I, Chapter II, Section 4, page 51) and $X(\omega)$ is obviously one of them. Therefore, if there is a solution to this equation in $\mathcal{S}'(\mathbb{R})$, then $\omega \in \Omega_X$. To show that such a solution u exists, recall that a distribution is an element of $\mathcal{S}'(\mathbb{R})$ if and only if it is the derivative of some order of a slowly growing continuous function (see [24], Théorème VI, Chapter VII, Section 4, page 239): $\dot{X}(\omega) = g^{(n)}$ for some continuous slowly growing function g and some integer n. If $n \geq 1$, then $u = g^{(n-1)}$ is a solution in $\mathcal{S}'(\mathbb{R})$ of $u' = \dot{X}(\omega)$. If $n = 0$, then $u(t) = \int_0^t g(s)\,ds$ is a slowly growing solution, therefore, $u \in \mathcal{S}'(\mathbb{R})$.

To prove (i), it suffices to show that $\mathbb{P}(\Omega_X) = 1$. Let $X_t = \gamma t + \sigma W_t + X_t^M + X_t^P$ be the Lévy–Itô decomposition of X. Since $\mathbb{E}(|X_1|^\eta) < +\infty$, we have $\int_{|x|>1} |x|^\eta \nu(dx) < +\infty$ (see Remark 2.8). The jump heights $(Y_i)_{i \geq 1}$ of the compound Poisson part X^P are i.i.d., with law $\lambda^{-1} \mathbb{1}_{|x|>1} \nu(dx)$ (where λ is a normalizing constant), therefore, $\mathbb{E}(|Y_1|^\eta) < +\infty$. Then we can use Proposition 2.1 for the continuous and small jumps terms of the Lévy–Itô decomposition of X, and Proposition 2.4(i) for the large jumps term, to deduce that X is slowly growing. By the *càdlàg* property of X and Remark 1.1, we conclude that $\mathbb{P}(\Omega_X) = 1$.

To prove (ii), it suffices to show that $\mathbb{P}(\Omega_X) = 0$. By Proposition 2.1, $\Omega_X = \Omega_{X^P}$. Also, since

$$\{\omega : t \mapsto X_t^P(\omega) \text{ is slowly growing}\}$$
$$\cap \Big\{\omega : \forall \alpha > 0, \limsup_{t \to +\infty} (1 + t^\alpha)^{-1} |X_t^P| = +\infty \Big\} = \varnothing,$$

and under (ii) the second set has probability one by Proposition 2.4(ii), we deduce from Proposition 2.2 that $\mathbb{P}(\Omega_{\dot{X}^p} \cap A) = 0$, where A is the almost-sure set defined in Proposition 2.2. Therefore, $\mathbb{P}(\Omega_{\dot{X}^p}) = \mathbb{P}(\Omega_{\dot{X}}) = 0$. $\square$

COROLLARY 2.9. *Let X be a Lévy process with characteristic triplet (γ, σ, ν), let $\dot{X}$ be the associated Lévy noise and suppose it has a* PAM. *Then there is a random tempered distribution S, that is, a measurable map from $(\Omega, \mathcal{F})$ to $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-field for the weak-$*$ topology, such that almost surely, for all $\varphi \in \mathcal{S}(\mathbb{R})$,*

$$\langle S, \varphi \rangle = \langle \dot{X}, \varphi \rangle = - \int_{\mathbb{R}_+} X_t \varphi'(t) \, dt.$$

In addition, the maps $C : \omega \mapsto C(\omega)$ and $p : \omega \mapsto p(\omega)$ such that for all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\left| \langle S, \varphi \rangle \right| \le C \mathcal{N}_p(\varphi) \qquad a.s.$$

can be chosen to be $\mathcal{F}$-measurable.

PROOF. We already know from Theorem 2.7 that $\mathbb{P}(\Omega_{\dot{X}}) = 1$. We define S to be equal to $\dot{X}$ [in $\mathcal{S}'(\mathbb{R})$] on $\Omega_{\dot{X}}$ and zero elsewhere. We want to be able to consider S as a measurable map with values in $\mathcal{S}'(\mathbb{R})$. More precisely, we equip $\mathcal{S}'(\mathbb{R})$ with the weak-$*$ topology. A basis for this topology is given by cylinder sets of the form

$$O = \bigcap_{i=1}^{n} \{ u \in \mathcal{S}'(\mathbb{R}) : \langle u, \varphi_i \rangle \in A_i \},$$

where, for all $i \le n$, φ_i is an element of $\mathcal{S}(\mathbb{R})$, n is an integer and A_i is an open set in $\mathbb{R}$. The σ-field generated by all cylinder sets is called the cylinder σ-algebra and is denoted by $\mathcal{C}$. We first show that $S : (\Omega, \mathcal{F}) \longrightarrow (\mathcal{S}'(\mathbb{R}), \mathcal{C})$ is measurable. For this, clearly, it suffices to show that for all cylinder sets O as above, the set $S^{-1}(O) = \{ \omega \in \Omega : S(\omega) \in O \}$ belongs to $\mathcal{F}$. Clearly,

$$S^{-1}(O) = \bigcap_{i=1}^{n} \{ \omega \in \Omega : \langle S(\omega), \varphi_i \rangle \in A_i \}.$$

The map $(t, \omega) \to X_t(\omega)$ is jointly measurable so by Fubini's theorem, the map $\langle S, \varphi_i \rangle : \Omega \longrightarrow \mathbb{R}$ is $\mathcal{F}$-measurable and, therefore, $S^{-1}(O) \in \mathcal{F}$. The Borel σ-field $\mathcal{B}$ contains $\mathcal{C}$ since every cylinder set is an open set. The converse inclusion is not immediate; see [9], Proposition 2.1, for a proof of the equality $\mathcal{B} = \mathcal{C}$. This fact is also mentioned in [12], page 41.

The space $\mathcal{S}(\mathbb{R})$ is separable (see [22], 10.3.4, page 176) and we let A be a countable dense subset. Then the measurability of the maps C and p comes from the fact that we can choose

$$p(\omega) = \min \left\{ p \in \mathbb{N} : \sup_{\varphi \in A} \frac{|\langle S, \varphi \rangle|}{\mathcal{N}_p(\varphi)}(\omega) < +\infty \right\}$$

and

$$C(\omega) = \sup_{\varphi \in A} \frac{|\langle S, \varphi \rangle|}{\mathcal{N}_{p(\omega)}(\varphi)}(\omega). \qquad \square$$

REMARK 2.10. An alternate proof of the fact that $\Omega_X \supset \Omega_{\dot{X}}$ is as follows. We can restrict to the case where X is a compound Poisson process. We construct here a solution to the equation $u'(\omega) = \dot{X}(\omega)$ such that $u(\omega) \in \mathcal{S}'(\mathbb{R})$. Let $\theta \in \mathcal{D}(\mathbb{R})$ be such that $\theta \geq 0$, $\int_{\mathbb{R}} \theta = 1$ and $\operatorname{supp} \theta \subset [0, 1]$. Then let $\varphi \in \mathcal{S}(\mathbb{R})$. There exists a function $\Phi \in \mathcal{S}(\mathbb{R})$ such that $\varphi = \Phi'$ if and only if $\int_{\mathbb{R}} \varphi = 0$ [consider $\Phi(x) = \int_{-\infty}^{x} \varphi(t) \, dt$ for the if part, the other direction is obvious]. Then consider the linear functional I on $\mathcal{S}(\mathbb{R})$ defined by

$$(2.10) \qquad I\varphi(t) = \int_{-\infty}^{t} \left(\varphi(s) - \theta(s) \int_{\mathbb{R}} \varphi \right) ds.$$

This functional defines an antiderivative on $\mathcal{S}(\mathbb{R})$: for any $\varphi \in \mathcal{S}(\mathbb{R})$, $I(\varphi') = \varphi$. Also, the reader can easily check that, for all $p \in \mathbb{N}$,

$$\sup_{t \in \mathbb{R}} |t|^p |I\varphi(t)| \leq C_p \mathcal{N}_{p+2}(\varphi),$$

for some constant C_p depending only on p and, therefore, I is a continuous linear functional with values in $\mathcal{S}(\mathbb{R})$. This implies that for $\omega \in \Omega_{\dot{X}}$, we can define a tempered distribution $u(\omega)$ by

$$\langle u(\omega), \varphi \rangle = -\langle \dot{X}(\omega), I\varphi \rangle \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}).$$

This tempered distribution clearly satisfies $u'(\omega) = \dot{X}(\omega)$. By definition of $\dot{X}$, u and X only differ by a (random) constant, and so $X(\omega) \in \mathcal{S}'(\mathbb{R})$. Therefore, $\Omega_{\dot{X}} \subset \Omega_X$.

3. Lévy fields and Lévy noise in $\mathcal{S}'(\mathbb{R}^d)$. In this section, we consider the same questions as in Section 2, but for a generalization of the notion of Lévy process, where the "time" parameter is in $\mathbb{R}_+^d$, with $d \geq 1$. A general presentation of this theory of *multiparameter Lévy fields* can be found in [1]; see also [6].

In the following, for any $k \in \mathbb{N}$, $\mathbf{1}_k$ (resp., $\mathbf{0}_k$, $\mathbf{2}_k$) denotes the k-dimensional vector with coordinates all equal to 1 (resp., to 0, 2). We recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. Let $(X_t)_{t \in \mathbb{R}_+^d}$ be a d-parameter random field. For $s, t \in \mathbb{R}_+^d$ with $s = (s_1, \ldots, s_d)$, $t = (t_1, \ldots, t_d)$, we say that $s \leq t$ if $s_i \leq t_i$ for all $1 \leq i \leq d$, and $s < t$ if $s_i < t_i$ for all $1 \leq i \leq d$. For $a \leq b \in \mathbb{R}_+^d$, we define the box $]a, b] = \{t \in \mathbb{R}_+^d : a < t \leq b\}$, and the increment $\Delta_a^b X$ of X over the box $]a, b]$ by

$$(3.1) \qquad \Delta_a^b X = \sum_{\varepsilon \in \{0,1\}^d} (-1)^{|\varepsilon|} X_{c_\varepsilon(a,b)},$$

where for any $\varepsilon \in \{0, 1\}^d$, we write $|\varepsilon| = \sum_{i=1}^{d} \varepsilon_i$ and $c_\varepsilon(a, b) \in \mathbb{R}_+^d$ is defined by $c_\varepsilon(a, b)_i = a_i \mathbb{1}_{\{\varepsilon_i = 1\}} + b_i \mathbb{1}_{\{\varepsilon_i = 0\}}$, for all $1 \leq i \leq d$. We can check that when $d = 1$,

then $\Delta_a^b X = X_b - X_a$. In fact, for all $d \geq 1$, $\int_{[a,b]} \varphi^{(\mathbf{1}_d)}(t)\,dt = \Delta_a^b \varphi$. The next definition is a generalization of the *càdlàg* property to processes indexed by $\mathbb{R}_+^d$. We define the relations $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_d)$, where $\mathcal{R}_i$ is either $\leq$ or $>$, and $a\mathcal{R}b$ if and only if $a_i \mathcal{R}_i b_i$ for all $1 \leq i \leq d$.

DEFINITION 3.1. Using the terminology in [1] and [25], we say that X is *lamp* (for limit along monotone paths) if we have the following: (i) For all 2^d relations $\mathcal{R}$, $\lim_{u \to t, t\mathcal{R}u} X_u$ exists; (ii) If $\mathcal{R} = (\leq, \ldots, \leq)$ then $X_t = \lim_{u \to t, t\mathcal{R}u} X_u$; and (iii) $X_t = 0$ if $t_i = 0$ for some $1 \leq i \leq d$.

We are now ready to give the definition of a Lévy field in $\mathbb{R}_+^d$.

DEFINITION 3.2. $X = (X_t)_{t \in \mathbb{R}_+^d}$ is a d-parameter Lévy field if it has the following properties:

   (i) X is continuous in probability.
   (ii) X is *lamp* almost surely.
   (iii) For any sequence of disjoint boxes $]a_k, b_k]$, $1 \leq k \leq n$, the random variables $\Delta_{a_k}^{b_k} X$ are independent.
   (iv) Given two boxes $]a, b]$ and $]c, d]$ in $\mathbb{R}_+^d$ such that $]a, b] + t =]c, d]$ for some $t \in \mathbb{R}^d$, the increments $\Delta_a^b X$ and $\Delta_c^d X$ are identically distributed.

The jump $\Delta_t X$ of X at time t is defined by $\Delta_t X = \lim_{u \to t, u < t} \Delta_u^t X$.

This definition coincides with the notion of Lévy process when $d = 1$. In addition, for all $t = (t_1, \ldots, t_d) \in \mathbb{R}_+^d$, and for all $1 \leq i \leq d$, the process $X^{i,t} = X_{(t_1, \ldots, t_{i-1}, \cdot, t_{i+1}, \ldots, t_d)}$ is a Lévy process (the notation here means that it is the process in one parameter obtained by fixing all the coordinates of t except the ith).

The Brownian sheet is an example of such a d-parameter Lévy field. It is the analog in this framework of Brownian motion and further properties of this field are detailed in [5, 7, 16] or [27].

For all $t \in \mathbb{R}_+^d$, X_t is an infinitely divisible random variable, and by the Lévy–Khintchine formula ([23], Chapter 2, Theorem 8.1, page 37), there exist real numbers γ_t, σ_t and a Lévy measure ν_t such that $\mathbb{E}(e^{iuX_t}) = \exp[iu\gamma_t - \frac{1}{2}\sigma_t^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu_t(dx)]$. The triplet $(\gamma_t, \sigma_t, \nu_t)$ is called the characteristic triplet of X_t. Since for all $1 \leq i \leq d$ and $t \in \mathbb{R}_+^d$, the process $X^{i,t}$ defined above is a Lévy process, we deduce that there exists a triplet (γ, σ, ν) where $\gamma, \sigma \in \mathbb{R}$ and ν is a Lévy measure such that $(\gamma_t, \sigma_t, \nu_t) = (\gamma, \sigma, \nu)\,\mathrm{Leb}_d([0, t])$, where $\mathrm{Leb}_d(dx)$ is d-dimensional Lebesgue measure. We call (γ, σ, ν) the characteristic triplet of the Lévy field X. We can now state the multidimensional analog of the Lévy–Itô decomposition, taken from [1], Theorem 4.6, particularized to the case of stationary increments (see also [6]).

THEOREM 3.3.   *Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν). The following hold*:

(i) *The jump measure J_X defined by $J_X(B) = \#\{(t, \Delta_t X) \in B\}$, for B in the Borel σ-algebra of $\mathbb{R}^d_+ \times (\mathbb{R} \setminus \{0\})$, is a Poisson random measure with intensity $\mathrm{Leb}_d \times \nu$.*

(ii) *For all $t \in \mathbb{R}^d_+$, we have the decomposition*

$$X_t = \gamma \, \mathrm{Leb}_d\big([0, t]\big) + \sigma \, W_t + \int_{[0,t]} \int_{|x|>1} x \, J_X(\mathrm{d}s, \mathrm{d}x)$$

$$+ \int_{[0,t]} \int_{|x|\leq 1} x \, \tilde{J}_X(\mathrm{d}s, \mathrm{d}x),$$

where W is a Brownian sheet, $\tilde{J}_X = J_X - \mathrm{Leb}_d \times \nu$ is the compensated jump measure, and the equality holds almost surely. In addition, the terms of the decomposition are independent random fields.

If X is a d-parameter Lévy field, by the *lamp* property of its sample paths, it is locally bounded and defines almost surely an element of $\mathcal{D}'(\mathbb{R}^d)$ via the L^2-inner product. Similar to the one-dimensional case (see Definition 2.5), we now define the d-dimensional Lévy white noise.

DEFINITION 3.4.   Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν). The Lévy white noise $\dot{X}$ is the dth cross-derivative of X in the sense of Schwartz distributions: for $\omega \in \Omega$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle \dot{X}, \varphi \rangle(\omega) := (-1)^d \langle X, \varphi^{(\mathbf{1}_d)} \rangle(\omega) := (-1)^d \int_{\mathbb{R}^d_+} X_t(\omega) \varphi^{(\mathbf{1}_d)}(t) \, \mathrm{d}t,$$

where $\varphi^{(\mathbf{1}_d)} = \frac{\partial^d}{\partial t_1 \cdots \partial t_d} \varphi$.

As in Section 2.3, note that the law of the multidimensional Lévy white noise $\dot{X}$ is entirely characterized by the triplet (γ, σ, ν) (given that we use the truncation function $\mathbb{1}_{|x|\leq 1}$ in the Lévy–Itô decomposition). We will show in Proposition 3.16 that this definition is equivalent to other definitions of Lévy white noise.

REMARK 3.5.   Given a d-parameter Lévy field X with characteristic triplet (γ, σ, ν) and jump measure J_X, for a suitable class of functions $\varphi : \mathbb{R}^d_+ \to \mathbb{R}$, we can define the stochastic integral

$$\int_{\mathbb{R}^d_+} \varphi(s) \, \mathrm{d}X_s := \gamma \int_{\mathbb{R}^d_+} \varphi(s) \, \mathrm{d}s + \sigma \int_{\mathbb{R}^d_+} \varphi(s) \, \mathrm{d}W_s$$

$$+ \int_{\mathbb{R}^d_+} \int_{|x|>1} x \varphi(s) \, J_X(\mathrm{d}s, \mathrm{d}x)$$

(3.2)

$$+ \int_{\mathbb{R}_+^d} \int_{|x| \leq 1} x \varphi(s) \tilde{J}_X(\mathrm{d}s, \mathrm{d}x)$$

$$= \gamma A_1(\varphi) + \sigma A_2(\varphi) + A_3(\varphi) + A_4(\varphi),$$

where the first integral is a Lebesgue integral, the second integral is a Wiener integral (see [17], Chapter 2) and the last two integrals are Poisson integrals (see [15], Lemma 12.13) with the space $S = \mathbb{R}_+^d \times (\mathbb{R} \setminus \{0\})$.

The next lemma relates the definition of Lévy white noise above with the mapping $\varphi \to \int_{\mathbb{R}_+^d} \varphi(s) \, \mathrm{d}X_s$.

LEMMA 3.6. *Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν) and jump measure J_X. Then, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,*

$$(3.3) \qquad \langle \dot{X}, \varphi \rangle = \int_{\mathbb{R}_+^d} \varphi(s) \, \mathrm{d}X_s.$$

PROOF. Generically, if μ is a measure on $\mathbb{R}_+^d$ and if $x(t) := \mu([0, t])$, then $\frac{\partial^d}{\partial t_1 \cdots \partial t_d} x = \mu$ in $\mathcal{D}'(\mathbb{R}^d)$. Indeed, by (3.1), for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}_+^d} \varphi(s) \mu(\mathrm{d}s) = (-1)^d \int_{\mathbb{R}_+^d} \mu(\mathrm{d}s) \int_{\mathbb{R}_+^d} \mathrm{d}t \varphi^{(\mathbf{1}_d)}(t) \mathbb{1}_{t \geq s}$$

$$(3.4) \qquad = (-1)^d \int_{\mathbb{R}_+^d} \mathrm{d}t \varphi^{(\mathbf{1}_d)}(t) \int_{\mathbb{R}_+^d} \mu(\mathrm{d}s) \mathbb{1}_{t \geq s}$$

$$= (-1)^d \int_{\mathbb{R}_+^d} \varphi^{(\mathbf{1}_d)}(t) x(t) \, \mathrm{d}t,$$

where the second equality requires a Fubini-type theorem.

Notice that for bounded Borel sets, the set function

$$B \mapsto \tilde{X}(B) := \int_{\mathbb{R}_+^d} \mathbb{1}_B(s) \, \mathrm{d}X_s$$

defines an $L^0(\Omega, \mathcal{F}, \mathbb{P})$-valued measure (see, e.g., [6], Theorem 2.6), and $X_t = \tilde{X}([0, t])$ a.s. We shall apply the argument in (3.4) separately to the four integrals in (3.2). For the first integral, the standard Fubini's theorem applies. For the second integral, since $\varphi \in L^2(\mathbb{R}^d)$, it is well defined, and since it has compact support, we use the stochastic Fubini's theorem ([27], Theorem 2.6). For the third integral, let $J_{X^P}(\mathrm{d}s, \mathrm{d}x) = \mathbb{1}_{|x|>1} J_X(\mathrm{d}s, \mathrm{d}x)$ be the jump measure of the compound Poisson part X^P of the Lévy–Itô decomposition of X. Then $J_{X^P} = \sum_{i \geq 1} \delta_{\tau_i} \delta_{Y_i}$, where (τ_i, Y_i) are random elements of $\mathbb{R}_+^d \times (\mathbb{R} \setminus \{0\})$, and $A_3(\varphi) = \sum_{i \geq 1} Y_i \varphi(\tau_i)$. For a fixed φ with compact support, this is a finite sum, so Fubini's theorem applies

trivially. For the term $A_4(\varphi)$, the integral is a compensated Poisson integral, and we know that it exists (see [15], Lemma 12.13) if and only if

$$(3.5) \qquad \int_{\mathbb{R}_+^d} \int_{|x| \leq 1} \left(|x\varphi(s)|^2 \wedge |x\varphi(s)| \right) \mathrm{d}s \, \nu(\mathrm{d}x) < +\infty.$$

Since $\varphi \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}_+^d} \int_{|x| \leq 1} \left(|x\varphi(s)|^2 \wedge |x\varphi(s)| \right) \mathrm{d}s \, \nu(\mathrm{d}x) \leq \|\varphi\|_{L^2}^2 \int_{|x| \leq 1} x^2 \nu(\mathrm{d}x) < +\infty.$$

For $n \in \mathbb{N}$, define

$$A_{4,n}(\varphi) := \int_{\mathbb{R}_+^d} \int_{\frac{1}{2^{n+1}} < |x| \leq \frac{1}{2^n}} x\varphi(t) \tilde{J}_X(\mathrm{d}x, \mathrm{d}t)$$

$$= \int_{\mathbb{R}_+^d} \int_{\frac{1}{2^{n+1}} < |x| \leq \frac{1}{2^n}} x\varphi(t) J_X(\mathrm{d}x, \mathrm{d}t)$$

$$- \int_{\mathbb{R}_+^d} \int_{\frac{1}{2^{n+1}} < |x| \leq \frac{1}{2^n}} x\varphi(t) \nu(\mathrm{d}x) \, \mathrm{d}t.$$

Then $A_{4,n}(\varphi)$ is a sequence of centered independent random variables (the compensated Poisson integrals are over disjoint sets) in $L^2(\Omega)$, and we know that

$$\mathbb{E}\left(\left(A_{4,n}^2(\varphi) \right) \right) = \int_{\mathbb{R}_+^d} \varphi(s)^2 \, \mathrm{d}s \int_{\frac{1}{2^{n+1}} \leq |x| < \frac{1}{2^n}} x^2 \nu(\mathrm{d}x).$$

Since ν is a Lévy measure, we see that $\sum_n \mathbb{E}((A_{4,n}^2(\varphi))) < \infty$ and by Kolmogorov's convergence criterion (see [10], Theorem 2.5.3) we deduce that as $n \to +\infty$,

$$(3.6) \qquad \sum_{0 \leq k \leq n} A_{4,k}(\varphi) \to \int_{\mathbb{R}_+^d} \int_{|x| \leq 1} x\varphi(s) \tilde{J}_X(\mathrm{d}x, \mathrm{d}s) = A_4(\varphi) \qquad \text{a.s.}$$

For each $n \in \mathbb{N}$, since the Lévy measure ν is finite on compact subsets of $\mathbb{R}_+^d \times [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, Fubini's theorem applies to the set function $B \mapsto A_{4,n}(\mathbb{1}_B)$ in the same way it did for A_3 and A_1. Therefore, letting

$$X_t^{M,n} = \int_{\mathbb{R}_+^d} \int_{\frac{1}{2^{n+1}} < |x| \leq \frac{1}{2^n}} x \mathbb{1}_{t \geq s} \tilde{J}_X(\mathrm{d}s, \mathrm{d}x),$$

the argument in (3.4) implies that

$$A_{4,n}(\varphi) = (-1)^d \int_{\mathbb{R}_+^d} \varphi^{(\mathbf{1}_d)}(t) X_t^{M,n} \, \mathrm{d}t.$$

By [1], Theorem 4.6 (see also [6], Theorem 2.3), $\sum_{0 \leq k \leq n} X_t^{M,k} \to X_t^M$, where X^M is the small jumps part of X, and the convergence is a.s., uniformly on compact subsets of $\mathbb{R}_+^d$. Since φ has compact support, (3.6) implies that

$$A_4(\varphi) = (-1)^d \int_{\mathbb{R}_+^d} \varphi^{(\mathbf{1}_d)}(s) X_s^M \, \mathrm{d}s.$$

$\square$

3.1. *The case of a p-integrable martingale* ($p > 1$). We say that a random field M is a multiparameter martingale with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+^d}$ (see [16], Chapter 7, Section 2, page 233) if M is $\mathbb{F}$-adapted, integrable, and for all $s \leq t \in \mathbb{R}_+^d$, then $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$. We will also need the notion of commuting filtration (see [16], Chapter 7, Section 2, Definition, page 233). By [16], Theorem 2.1.1 in Chapter 7, to show that $\mathbb{F}$ is commuting, it suffices to show that for any $s, t \in \mathbb{R}_+^d$, $\mathcal{F}_s$ and $\mathcal{F}_t$ are conditionally independent given $\mathcal{F}_{s \wedge t}$, where $(s \wedge t)_i = s_i \wedge t_i$. In particular, if X is a d-parameter Lévy field and $\mathcal{F}_t$ is the σ-algebra generated by the family $(X_s)_{s \leq t}$, then $\mathbb{F}$ is commuting by the independence of the increments of X.

For any *lamp* random field L, we consider, similar to (2.1), the event

(3.7) $$\Omega_L = \{\omega \in \Omega : L(\omega) \in \mathcal{S}'(\mathbb{R}^d)\},$$

with the understanding that when $L(\omega) \in \mathcal{S}'(\mathbb{R}^d)$, the continuous linear functional associated with $L(\omega)$ is $\langle L(\omega), \varphi \rangle = \int_{\mathbb{R}_+^d} L_t(\omega)\varphi(t)\,dt$, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

PROPOSITION 3.7. *Fix $p > 1$ and let $(M_t)_{t \in \mathbb{R}_+^d}$ be a multiparameter martingale with respect to a commuting filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+^d}$, such that for all $t \in \mathbb{R}_+^d$,*

$$\mathbb{E}(|M_t|^p) \leq \left(c\operatorname{Leb}_d([0, t])\right)^{\frac{p}{2}}$$

for some constant c. Then the set Ω_M defined as in (3.7) has probability one.

PROOF. Similar to the one-dimensional case, we control the supremum of $|t|^{-\alpha}|M_t|$ as $|t| \to +\infty$, or, equivalently, the supremum of $|s|^{-\alpha}|M_s|$ for $s \in \mathbb{R}_+^d \setminus [0, t]$ as $\min_{i=1,\ldots,d} t_i \to +\infty$, and prove that the limit in probability of this supremum, as all the coordinates of t go to $+\infty$, is zero. The proof uses the multidimensional analog of Doob's L^p inequality: Cairoli's strong (p, p) inequality (see [16], Chapter 7, Theorem 2.3.2). For all $i \in \mathbb{N} \setminus \{0\}$, let $x_i = 2^{i-1}$ and $x_0 = 0$. For $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, let $a_k = (x_{k_1}, \ldots, x_{k_d})$, and let $b_k = (2^{k_1}, \ldots, 2^{k_d})$. We fix $k \in \mathbb{N}^d$, $k \neq (0, \ldots, 0)$. By using successively Jensen's inequality and Cairoli's inequality, for any $\alpha > 0$, we have

$$\mathbb{E}\left(\sup_{s \in [a_k, b_k]} \frac{|M_s|}{|s|^\alpha}\right) \leq \frac{1}{|a_k|^\alpha} \mathbb{E}\left(\sup_{s \leq b_k} |M_s|^p\right)^{\frac{1}{p}}$$

$$\leq \frac{c_p}{|a_k|^\alpha} \mathbb{E}(|M_{b_k}|^p)^{\frac{1}{p}} \leq \frac{c_p\sqrt{c\operatorname{Leb}_d([0, b_k])}}{|a_k|^\alpha},$$

for some constant c_p depending only on p and the dimension d, where $|a_k|$ and $|s|$ denote here the Euclidean norm. Since $k_1 \vee \cdots \vee k_d \geq 1$, we have $|a_k| \geq 2^{k_1 \vee \cdots \vee k_d - 1}$, hence

(3.8)
$$\mathbb{E}\left(\sup_{s \in [a_k, b_k]} \frac{|M_s|}{|s|^\alpha}\right) \leq c_p\sqrt{c}2^{\frac{1}{2}\sum_{i=1}^d k_i}2^{-\alpha(k_1 \vee \cdots \vee k_d - 1)}$$

$$\leq c_p\sqrt{c}2^\alpha 2^{-(\frac{\alpha}{d} - \frac{1}{2})\sum_{i=1}^d k_i}.$$

We choose $\alpha = \lfloor \frac{d}{2} \rfloor + 1$. Let $t \in \mathbb{R}_+^d$ be far enough from the origin (we will consider the limit as all the coordinates of t go to $+\infty$), and for all $1 \le i \le d$, let n_i be the largest integer such that $2^{n_i} \le t_i$ and let $n = (n_1, \ldots, n_d)$. We can suppose that $n_i \ge 2$ for all $1 \le i \le n$. We write Ξ for the set of all relations $\mathcal{R}$ of the form $(r_1, \ldots, r_d)$, where for all $i \in \{1, \ldots, d\}$, $r_i \in \{\le, \ge\}$ and $\mathcal{R} \ne (\le, \ldots, \le)$. Then $[0, t_n] \subset [0, t]$, where $t_n = (2^{n_1}, \ldots, 2^{n_d})$. The complement of the box $[0, t_n]$ in $\mathbb{R}_+^d$ is covered by boxes of the form $[a_k, b_k]$, where $k \in \mathbb{N}^d$ and $k\mathcal{R}n$ for some $\mathcal{R} \in \Xi$. Therefore,

$$\mathbb{P}\left(\sup_{s \notin [0,t]} \frac{|M_s|}{|s|^\alpha} > \varepsilon \right) \le \mathbb{P}\left(\sup_{s \notin [0,t_n]} \frac{|M_s|}{|s|^\alpha} > \varepsilon \right)$$

$$\le \sum_{\mathcal{R} \in \Xi} \sum_{\substack{k \in \mathbb{N}^d \\ k\mathcal{R}n}} \mathbb{P}\left(\sup_{s \in [a_k, b_k]} \frac{|M_s|}{|s|^\alpha} > \varepsilon \right)$$

$$\le \frac{c_p \sqrt{c} 2^\alpha}{\varepsilon} \sum_{\mathcal{R} \in \Xi} \sum_{\substack{k \in \mathbb{N}^d \\ k\mathcal{R}n}} 2^{-(\frac{\alpha}{d} - \frac{1}{2}) \sum_{i=1}^d k_i} \xrightarrow{t \twoheadrightarrow +\infty} 0,$$

where $t \twoheadrightarrow +\infty$ means that $t_1 \wedge \cdots \wedge t_d \to +\infty$. To check that the limit is indeed zero, one has that for any fixed $\mathcal{R} \in \Xi$, at least one of the inequalities in $\mathcal{R}$ is $\ge$. By symmetry, we can suppose that it is the first inequality. Then

$$\sum_{\substack{k \in \mathbb{N}^d \\ k\mathcal{R}n}} 2^{-(\frac{\alpha}{d} - \frac{1}{2}) \sum_{i=1}^d k_i} \le C_{\alpha,d} \sum_{k_1 \ge n_1} 2^{-(\frac{\alpha}{d} - \frac{1}{2}) k_1} \xrightarrow{n_1 \to +\infty} 0.$$

The result follows since Ξ is a finite set. Then $\sup_{s \notin [0,t]} |s|^{-\alpha} |X_s^M| \to 0$ in probability as $t \twoheadrightarrow +\infty$, therefore $|t|^{-\alpha} |M_t| \to 0$ a.s as $|t| \to +\infty$. By the *lamp* property of M, we deduce that M is slowly growing, and by Remark 1.1 we deduce that $\mathbb{P}(\Omega_M) = 1$. $\square$

COROLLARY 3.8. *Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν) and Lévy–Itô decomposition $X_t = \gamma \operatorname{Leb}_d([0, t]) + \sigma W_t + X_t^P + X_t^M$ where X^P is the large jump part of the decomposition and X^M is the compensated small jumps part. Let $Y_t = \gamma \operatorname{Leb}_d([0, t]) + \sigma W_t + X_t^M$. Then the set Ω_Y defined in (3.7) has probability one.*

PROOF. The random field $\tilde{Y} = \sigma W + X^M$ is a sum of two independent square integrable martingales and by a classical result on compensated Poisson integrals and Brownian sheets,

$$\mathbb{E}(\tilde{Y}_t^2) = \left(\sigma^2 + \int_{|x| \le 1} x^2 \nu(dx) \right) \operatorname{Leb}_d([0, t]),$$

where the multiplicative constant is finite since ν is a Lévy measure. Hence, $\tilde{Y}$ verifies the hypothesis of the Proposition 3.7 with $p = 2$, therefore, it defines a tempered distribution a.s. Since $\tilde{Y}$ and Y differ by a slowly growing function, we deduce that Y is a tempered distribution almost surely. $\square$

3.2. *The compound Poisson sheet.* By Corollary 3.8, for any d-parameter Lévy field X, we have $\Omega_X \cap \Omega_Y = \Omega_{X^P} \cap \Omega_Y$. We shall prove that Ω_{X^P} has probability 0 or 1. In the one-dimensional setting, we used the fact that a compound Poisson process with a PAM is slowly growing a.s. [see Proposition 2.4(i)]. As mentioned in the Introduction, the same results in a d-dimensional setting are to the best of our knowledge unavailable, which leads us to find another approach. In the multiparameter case, we will use properties of stochastic integrals with respect to a Poisson random measure to show that under a moment condition, a compound Poisson sheet and its associated white noise define tempered distributions. While this is in principle a special case of [11], Theorem 3, in view of Corollary 3.8, the two statements are in fact equivalent.

LEMMA 3.9. *Let ν be a Lévy measure and M be a Poisson random measure on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}_+^d$ with intensity measure $\mathbb{1}_{|x|>\eta} \nu(dx)\, dt$, where $\eta > 0$. Suppose that $\int_{|x|>\eta} |x|^\alpha \nu(dx) < +\infty$ for some $\alpha > 0$ (PAM) and consider the compound Poisson sheet $P_t = \int_{[0,t]} \int_{|x|>\eta} x M(dt, dx)$. Then:*

(i) *M almost surely defines a tempered distribution via the formula*

$$(3.9) \qquad \langle M, \varphi \rangle = \int_{\mathbb{R}_+^d} \int_{|x|>\eta} M(dt, dx) \varphi(t) x, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

(ii) *$\mathbb{P}(\Omega_P) = 1$ and for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,*

$$(3.10) \qquad \langle P, \varphi \rangle := \int_{\mathbb{R}_+^d} P_t \varphi(t)\, dt = \int_{\mathbb{R}_+^d} \int_{|x|>\eta} M(dt, dx) \int_{[t,+\infty[} ds \varphi(s) x,$$

(iii) *$M = P^{(\mathbf{1}_d)}$ in $\mathcal{S}'(\mathbb{R}^d)$, where we recall that $P^{(\mathbf{1}_d)} = \frac{\partial^d}{\partial t_1 \cdots \partial t_d} P$.*

PROOF. Since M is a Poisson random measure on $\mathbb{R}_+^d \times (\mathbb{R} \setminus \{0\})$ with jumps of size larger than η, there are (random) points $(\tau_i, Y_i)_{i \geq 1} \in \mathbb{R}_+^d \times (\mathbb{R} \setminus [-1, 1])$ such that $M = \sum_{i \geq 1} \delta_{\tau_i} \delta_{Y_i}$. To prove (i), we first need to check that the integral in (3.9) is well defined. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The stochastic integral is a Poisson integral, and it is well defined (as the limit in probability of Poisson integrals of elementary functions) if and only if (see [15], Lemma 12.13)

$$(3.11) \qquad \int_{|x|>\eta} \int_{\mathbb{R}_+^d} (|x\varphi(t)| \wedge 1)\, dt \nu(dx) < +\infty.$$

Let $r \in \mathbb{N}$. There is a constant $C > 1$ such that $\sup_{t \in \mathbb{R}_+^d} (1 + |t|^r)|\varphi(t)| \leq C < +\infty$. Then $|x\varphi(t)| \wedge 1 \leq \frac{C|x|}{1 + |t|^r} \wedge 1$. We write V_d for the volume of the d-dimensional unit sphere. Then, for $|x| > 1$,

$$
\begin{aligned}
\int_{\mathbb{R}_+^d} \left(|x\varphi(t)| \wedge 1 \right) dt &\leq \int_{\mathbb{R}_+^d} \left(\frac{C|x|}{1 + |t|^r} \wedge 1 \right) dt \\
&\leq dV_d \int_{\mathbb{R}_+} \left(\frac{C|x|}{1 + u^r} \wedge 1 \right) u^{d-1} \, du \\
&\leq dV_d \left(\int_0^{(C|x|-1)^{\frac{1}{r}}} u^{d-1} \, du + C|x| \int_{(C|x|-1)^{\frac{1}{r}}}^{+\infty} \frac{u^{d-1}}{1 + u^r} \, du \right) \\
&\leq V_d (C|x| - 1)^{\frac{d}{r}} + dV_d C|x| \int_{(C|x|-1)^{\frac{1}{r}}}^{+\infty} \frac{u^{d-1}}{1 + u^r} \, du.
\end{aligned}
$$

The last integral has to be well defined so we take $r > d$, and then

$$
\int_{(C|x|-1)^{\frac{1}{r}}}^{+\infty} \frac{u^{d-1}}{1 + u^r} \, du \leq \int_{(C|x|-1)^{\frac{1}{r}}}^{+\infty} u^{d-1-r} \, du = \frac{1}{r - d} (C|x| - 1)^{\frac{d-r}{r}},
$$

so

$$
\int_{\mathbb{R}_+^d} \left(|x\varphi(t)| \wedge 1 \right) dt \leq V_d (C|x| - 1)^{\frac{d}{r}} + \frac{dV_d C|x|}{r - d} (C|x| - 1)^{\frac{d-r}{r}}.
$$

We deduce that there exists a constant C' such that, for $|x| > 1$,

$$
\tag{3.12}
\int_{\mathbb{R}_+^d} \left(|x\varphi(t)| \wedge 1 \right) dt \leq C' |x|^{\frac{d}{r}}.
$$

We then choose r large enough so that $\frac{d}{r} \leq \alpha \wedge \frac{1}{2}$, in which case the moment condition on ν gives us (3.11) and, therefore, the Poisson integral is well defined and a.s. finite. Set $g_r(t) = \frac{1}{1 + |t|^r}$, $t \in \mathbb{R}_+^d$. Then for r sufficiently large,

$$
\int_{\mathbb{R}_+^d} \int_{|x| > \eta} M(dt, dx) g_r(t) |x|
$$

is well defined since, by (3.12) and PAM,

$$
\int_{|x| > \eta} \int_{\mathbb{R}_+^d} \left(|x g_r(t)| \wedge 1 \right) dt \, \nu(dx) < +\infty.
$$

Since $M = \sum_i \delta_{\tau_i} \delta_{Y_i}$,

$$
\langle M, \varphi \rangle = \sum_i Y_i \varphi(\tau_i).
$$

Now suppose $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R}^d)$. Then for large n, $|\varphi_n| \le g_r$, and

$$\big|\langle M, \varphi_n \rangle\big| = \bigg|\sum_i \varphi_n(\tau_i) Y_i\bigg|$$

$$\le \sum_i |Y_i| g_r(\tau_i) = \int_{\mathbb{R}_+^d} \int_{|x|>\eta} M(\mathrm{d}t, \mathrm{d}x) g_r(t)|x| < +\infty \qquad \text{a.s.}$$

For a.a. fixed $\omega \in \Omega$, $\varphi_n(\tau_i(\omega)) \to 0$ as $n \to +\infty$, $|\varphi_n(\tau_i(\omega))| \le g_r(\tau_i(\omega))$ and $\sum_i g_r(\tau_i(\omega))|Y_i(\omega)| < +\infty$. By the dominated convergence theorem,

$$\langle M(\omega), \varphi_n \rangle = \sum_i \varphi_n\big(\tau_i(\omega)\big) Y_i(\omega) \to 0 \qquad \text{as } n \to +\infty.$$

Therefore, the linear functional $\varphi_n \mapsto \langle M(\omega), \varphi_n \rangle$ is continuous on $\mathcal{S}(\mathbb{R}^d)$, and so $M(\omega) \in \mathcal{S}'(\mathbb{R}^d)$ for a.a. $\omega \in \Omega$.

To prove (ii), we first prove that the Poisson integral on the right-hand side of (3.10) is well defined, and we will need the PAM condition. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and let $\Phi(t) = \int_{[t,+\infty[} \varphi(s)\,\mathrm{d}s$. Then (3.10) is well defined if

$$(3.13) \qquad \int_{|x|>\eta} \int_{\mathbb{R}_+^d} \big(|x\,\Phi(t)| \wedge 1\big)\,\mathrm{d}t\,\nu(\mathrm{d}x) < +\infty.$$

Using (3.15) in Lemma 3.10 below, property (3.13) is established in the same way as (3.11) and, as above, the right-hand side of (3.10) defines almost surely a tempered distribution. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$(3.14) \quad \int_{\mathbb{R}_+^d} \int_{|x|>\eta} M(\mathrm{d}t, \mathrm{d}x) \int_{[t,+\infty[} \mathrm{d}s\,\varphi(s)x = \sum_{i\ge1} \int_{\mathbb{R}_+^d} Y_i \mathbb{1}_{\tau_i\in[0,s]}\varphi(s)\,\mathrm{d}s.$$

Following the argument in (3.4), we want to be able to use Fubini's theorem to exchange the sum and the integral in the last expression. For any $\alpha \in \mathbb{N}$, by the same argument as in the proof of Lemma 3.10 below with $\beta = 0$,

$$\sup_{t\in\mathbb{R}_+^d} \big(1 + |t|^\alpha\big) \int_{[t,+\infty[} \big|\varphi(s)\big|\,\mathrm{d}s \le C\mathcal{N}_{|\alpha|+2d}(\varphi).$$

As in the proof of (3.11), we deduce that

$$\int_{|x|>\eta} \int_{\mathbb{R}_+^d} \bigg(\bigg|x\int_{[t,+\infty[} |\varphi(s)|\,\mathrm{d}s\bigg| \wedge 1\bigg)\,\mathrm{d}t\,\nu(\mathrm{d}x) < +\infty.$$

Then $\sum_{i\ge1} \int_{\mathbb{R}_+^d} |Y_i|\mathbb{1}_{\tau_i\in[0,s]}|\varphi(s)|\,\mathrm{d}s < +\infty$, and by (3.14) and Fubini's theorem,

$$\int_{\mathbb{R}_+^d} \int_{|x|>\eta} M(\mathrm{d}t, \mathrm{d}x) \int_{[t,+\infty[} \mathrm{d}s\,\varphi(s)x = \int_{\mathbb{R}_+^d} \sum_{i\ge1} Y_i \mathbb{1}_{\tau_i\in[0,s]}\varphi(s)\,\mathrm{d}s$$

$$= \int_{\mathbb{R}_+^d} P_s\varphi(s)\,\mathrm{d}s.$$

This establishes (3.10). Property (iii) now follows by replacing φ by $\varphi^{(\mathbf{1}_d)}$ in (3.10).
□

LEMMA 3.10.   *For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, let Φ be the function defined by $\Phi(t) = \int_{[t,+\infty)} \varphi(s)\,ds$. Let $p \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^d$, such that $|\alpha|, |\beta| \leq p$. Then for all $a \in \mathbb{R}^d$, there is $C = C(p,d,a) < +\infty$, such that, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,*

$$(3.15) \qquad \sup_{t \geq a} |(1 + |t^\alpha|)\Phi^{(\beta)}(t)| \leq C' \mathcal{N}_{p+2d}(\varphi).$$

PROOF.   Let $t \in \mathbb{R}^d$. Then $\Phi(t) = \int_{\mathbb{R}^d_+} \varphi(s+t)\,ds$, so

$$\Phi^{(\beta)}(t) = \int_{\mathbb{R}^d_+} \varphi^{(\beta)}(s+t)\,ds.$$

Therefore,

$$
\begin{aligned}
|(1 + |t^\alpha|)\Phi^{(\beta)}(t)| &\leq (1 + |t^\alpha|)\int_{\mathbb{R}^d_+} |\varphi^{(\beta)}(s+t)|\,ds \\
&= (1 + |t^\alpha|)\int_{\mathbb{R}^d_+} \frac{|\varphi^{(\beta)}(s+t)|(1 + |(t+s)^{\alpha+2_d}|)}{1 + |(t+s)^{\alpha+2_d}|}\,ds \\
&\leq \mathcal{N}_{p+2d}(\varphi)(1 + |t^\alpha|)\int_{\mathbb{R}^d_+} \frac{1}{1 + |(t+s)^{\alpha+2_d}|}\,ds \\
&\leq C\mathcal{N}_{p+2d}(\varphi)
\end{aligned}
$$

for $t \geq a$, where C is a constant depending only on p, d and a.   □

3.3. *Multidimensional Lévy white noise*: *The general case.*   The following lemma extends to d-parameter Lévy fields the property recalled in Remark 2.8.

LEMMA 3.11.   *Let X be a d-parameter Lévy field with characteristic triplet (γ, σ, ν) and let $\alpha > 0$. The following are equivalent*: (i) $\forall t \in \mathbb{R}^d_+, \mathbb{E}(|X_t|^\alpha) < +\infty$; (ii) $\exists t \in (\mathbb{R}_+ \setminus \{0\})^d : \mathbb{E}(|X_t|^\alpha) < +\infty$; (iii) $\int_{|x|>1} |x|^\alpha \nu(dx) < +\infty$.

PROOF.   Clearly, (i) implies (ii). Suppose that (ii) is true for some t in $(\mathbb{R}_+ \setminus \{0\})^d$. As discussed just after Definition 3.2, the process $X^{i,t}$ obtained by fixing all coordinates of the parameter t except the ith is again a Lévy process with characteristic triplet $(\gamma, \sigma, \nu)\prod_{j \neq i} t_j$. By an application of [23], Theorem 25.3, we deduce that $(\prod_{j \neq i} t_i)\int_{|x|>1} |x|^\alpha \nu(dx) < +\infty$ and then (iii) is verified. Suppose now that (iii) is true. Let $t \in \mathbb{R}^d_+$, and $1 \leq i \leq d$. Since $(\prod_{j \neq i} t_i)\int_{|x|>1} |x|^\alpha \nu(dx) < +\infty$, another application [23], Theorem 25.3, gives us $\mathbb{E}(|X_s^{i,t}|^\alpha) < +\infty$ for all $s \in \mathbb{R}_+$. Since i and t are taken arbitrarily, we deduce (i).   □

We need a technical lemma that essentially states that for a compound Poisson sheet X^P, there is a well chosen sequence $(\varphi_n)_{n \geq 1}$ of test-functions with suitably decreasing compact support such that X^P is constant on $\text{supp}(\varphi_n)$ for n large enough (this was established in dimension one during the proof of Proposition 2.2).

LEMMA 3.12. *Let X^P be a d-parameter Lévy field with jump measure J_X and characteristic triplet $(0, 0, \mathbb{1}_{|x| \geq 1} \nu)$, where $\lambda := \int_{|x| \geq 1} \nu(\mathrm{d}x) < +\infty$. Let L be the compound Poisson process defined by $L_t = X^P_{(\mathbf{1}_{d-1}, t)}$, and let $(S_n)_{n \geq 1}$ denote its sequence of jump times. Then for all $p \in \mathbb{N}$, there exists a finite non random constant C_p with the following property: for all $\omega \in \Omega$, there exists a sequence $(\varphi_n)_{n \geq 1}$ of functions (depending on ω) in $\mathcal{D}(\mathbb{R}^d)$ such that*

$$(3.16) \qquad \mathcal{N}_p(\varphi_n) \mathbb{1}_{S_n \geq 1} \leq C_p S_n^{3d+4p} \mathbb{1}_{S_n \geq 1},$$

and there exists an event Ω' such that $\mathbb{P}(\Omega') = 1$ and for all $\omega \in \Omega'$, there exists an integer $N(\omega)$ such that, for all $n \geq N(\omega)$, X^P is constant on the support of φ_n and

$$(3.17) \qquad \langle X^P, \varphi_n \rangle(\omega) = L_{S_n}(\omega).$$

PROOF. As in the proof of Proposition 2.2, we will construct a sequence $(\varphi_n)_{n \geq 1}$ of functions with suitably decreasing compact support, and then use a Borel–Cantelli argument to show that X^P is constant on this support. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\text{supp}\,\varphi \subset [0, \mathbf{1}_d]$ and $\int_{\mathbb{R}^d} \varphi = 1$. Similar to (2.2), the sequence $(\varphi_n)_{n \geq 1}$ is defined by

$$\varphi_n(t) = S_n^{3d} \varphi((t_1 - 1)S_n^3, \ldots, (t_{d-1} - 1)S_n^3, (t_d - S_n)S_n^3), \qquad t \in \mathbb{R}^d,$$

so that $\text{supp}\,\varphi_n \subset [(\mathbf{1}_{d-1}, S_n), (1 + \frac{1}{S_n^3}, \ldots, 1 + \frac{1}{S_n^3}, S_n + \frac{1}{S_n^3})]$ and $\int_{\mathbb{R}^d} \varphi_n = 1$. Let $p \in \mathbb{N}$. Then

$$\begin{aligned}
\mathcal{N}_p(\varphi_n) \mathbb{1}_{S_n \geq 1} &= \sum_{|\alpha|, |\beta| \leq p} \sup_{t \in \mathbb{R}^d} |t^\alpha \varphi_n^{(\beta)}(t)| \mathbb{1}_{S_n \geq 1} \\
&= \sum_{|\alpha|, |\beta| \leq p} \sup_{t \in [0, (2, \ldots, 2, S_n+1))} t^\alpha |\varphi_n^{(\beta)}(t)| \mathbb{1}_{S_n \geq 1} \\
&\leq \sum_{|\alpha|, |\beta| \leq p} 2^{\sum_{i=1}^{d-1} \alpha_i} (S_n + 1)^{\alpha_d} \sup_{t \in \mathbb{R}^d} |\varphi_n^{(\beta)}(t)| \mathbb{1}_{S_n \geq 1} \\
&\leq \sum_{|\alpha|, |\beta| \leq p} 2^{\sum_{i=1}^{d-1} \alpha_i} (S_n + 1)^{\alpha_d} S_n^{3(d + \sum_{i=1}^d \beta_i)} \mathcal{N}_p(\varphi) \mathbb{1}_{S_n \geq 1} \\
&\leq C_p' \mathcal{N}_p(\varphi) S_n^{3d+4p} \mathbb{1}_{S_n \geq 1},
\end{aligned}$$

for some finite nonrandom constant C_p'. Therefore, (3.16) holds and $C_p := C_p' \mathcal{N}_p(\varphi)$ depends only on φ and p. Let

$$I_{n,k} = \left] (\mathbf{1}_{d-1}, S_n), \left(1 + \frac{1}{S_n^k}, \ldots, 1 + \frac{1}{S_n^k}, S_n + \frac{1}{S_n^k}\right) \right[,$$

and let $A_{n,k}$ be the event "X^P is constant in the box $I_{n,k}$". Clearly, (3.17) holds on $A_{n,k}$.

Observe that

$$\mathbb{P}(A_{n,k}^c) = \mathbb{P}\{X^P \text{ has at least one jump time in the set } J_{n,k}\}$$
$$= \mathbb{P}\{J_{X^P}((\mathbb{R} \setminus [-1,1]) \times J_{n,k}) \geq 1\},$$

where $J_{n,k}$ is defined as the following set:

$$J_{n,k} = \left[\mathbf{0}_d, \left(1 + \frac{1}{S_n^k}, \ldots, 1 + \frac{1}{S_n^k}, S_n + \frac{1}{S_n^k}\right)\right[\setminus [\mathbf{0}_d, (\mathbf{1}_{d-1}, S_n)] = J_{n,k}^1 \cup J_{n,k}^2,$$

where $J_{n,k}^1$ and $J_{n,k}^2$ are disjoint sets defined by

$$J_{n,k}^1 = \left\{x \in \mathbb{R}_+^d : \forall 1 \leq i \leq d-1, x_i < 1 + \frac{1}{S_n^k}, x_d \leq S_n,\right.$$

$$\left. \text{and } \exists i_0 \in \{1, \ldots, d-1\} \text{ s.t. } x_{i_0} > 1\right\},$$

$$J_{n,k}^2 =](\mathbf{0}_{d-1}, S_n), \left(1 + \frac{1}{S_n^k}, \ldots, 1 + \frac{1}{S_n^k}, S_n + \frac{1}{S_n^k}\right)[.$$

Therefore, we can write

$$\mathbb{P}(A_{n,k}^c) = \mathbb{P}\{J_{X^P}((\mathbb{R} \setminus [-1,1]) \times J_{n,k}^1)$$
$$+ J_{X^P}((\mathbb{R} \setminus [-1,1]) \times J_{n,k}^2) \geq 1\}$$
(3.18)
$$\leq \mathbb{P}\{J_{X^P}((\mathbb{R} \setminus [-1,1]) \times J_{n,k}^1) \geq 1\}$$
$$+ \mathbb{P}\{J_{X^P}((\mathbb{R} \setminus [-1,1]) \times J_{n,k}^2) \geq 1\}.$$

Let $\mathcal{F}_{(\mathbf{1}_{d-1},t)} = \sigma(X_s, s \in [\mathbf{0}_d, (\mathbf{1}_{d-1}, t)])$ and $\mathcal{F}_{(\mathbf{1}_{d-1},\infty)} = \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_{(\mathbf{1}_{d-1},t)}$. We also write $H_1 = \{x \in \mathbb{R}_+^d : x_1 \leq 1, \ldots, x_{d-1} \leq 1\}$. Then, due to the independence of the increments of X^P, the collection of random variables $(J_{X^P}((\mathbb{R} \setminus [-1,1]) \times A))_{A \subset \mathbb{R}_+^d \setminus H_1}$ is independent of $\mathcal{F}_{(\mathbf{1}_{d-1},\infty)}$. Since S_n is $\mathcal{F}_{(\mathbf{1}_{d-1},\infty)}$-measurable, we deduce that conditionally on S_n, the random variable $J_{X^P}((\mathbb{R} \setminus [-1,1]) \times J_{n,k}^1)$ has a Poisson law with parameter $\lambda \operatorname{Leb}_d(J_{n,k}^1)$, where $\lambda := \int_{|x|>1} \nu(dx)$. Further, on the event $\{S_n \geq 1\}$,

$$\operatorname{Leb}_d(J_{n,k}^1) = \sum_{j=1}^{d-1} \binom{d-1}{j} S_n \left(\frac{1}{S_n^k}\right)^j \left(1 + \frac{1}{S_n^k}\right)^{d-1-j} \leq 3^{d-1} S_n^{-(k-1)}.$$

Indeed, the Lebesgue measure of a subset of $J_{n,k}^1$ of vectors with exactly j components strictly greater than one is $S_n(\frac{1}{S_n^k})^j(1 + \frac{1}{S_n^k})^{d-1-j}$, and there are $\binom{d-1}{j}$ such

subsets. We deduce that

$$\mathbb{P}\{J_{X^P}\big((\mathbb{R} \setminus [-1, 1]) \times J_{n,k}^1\big) \geq 1\}$$

$$(3.19) \qquad \leq \mathbb{P}\{S_n \leq 1\} + \mathbb{E}\big(\mathbb{1}_{S_n > 1}\big(1 - e^{-\lambda \operatorname{Leb}_d(J_{n,k}^1)}\big)\big)$$

$$\leq \mathbb{P}\{S_n \leq 1\} + \lambda 3^{d-1} \mathbb{E}\big(S_n^{-(k-1)}\big).$$

We also define a process $\tilde{L}_t = X_{(\mathbf{2}_{d-1}, t)}^P$. It is a Lévy process with Lévy measure $\mu(\mathrm{d}x) = 2^{d-1} \mathbb{1}_{|x|>1} \nu(\mathrm{d}x)$. Since X^P is piecewise constant, $\tilde{L}$ is a piecewise constant Lévy process, therefore, a compound Poisson process (see [23], Theorem 21.2). On the event $\{S_n > 1\}$, we have

$$J_{n,k}^2 \subset \big[(\mathbf{0}_{d-1}, S_n), (\mathbf{2}_{d-1}, S_n + S_n^{-k})\big].$$

Therefore, if X^P has a jump point in $J_{n,k}^2$, then $\tilde{L}$ has a jump in $]S_n, S_n + S_n^{-k}[$. Let $\mathcal{G}_t = \sigma(X_u : u \in [0, (\mathbf{2}_{d-1}, t)])$. Then S_n is a $\mathcal{G}$-stopping time and $\tilde{L}$ is a Lévy process adapted to the filtration $\mathcal{G}$, so by the strong Markov property, the number of jumps of the process $\hat{L}_\cdot = \tilde{L}_{\cdot + S_n} - \tilde{L}_{S_n}$ is independent of S_n and has Poisson distribution of parameter $2^{d-1} \lambda t$. Therefore, we can write

$$\mathbb{P}\{J_{X^P}\big((\mathbb{R} \setminus [-1, 1]) \times J_{n,k}^2\big) \geq 1\}$$

$$\leq \mathbb{P}\{S_n \leq 1\} + \mathbb{P}\big(\{J_{X^P}\big((\mathbb{R} \setminus [-1, 1]) \times J_{n,k}^2\big) \geq 1\} \cap \{S_n > 1\}\big)$$

$$\leq \mathbb{P}\{S_n \leq 1\} + \mathbb{P}\Big\{\tilde{L} \text{ has a jump in } \Big(S_n, S_n + \frac{1}{S_n^k}\Big)\Big\}$$

$$(3.20) \qquad = \mathbb{P}\{S_n \leq 1\} + \mathbb{P}\Big\{\hat{L} \text{ has a jump in } \Big(0, \frac{1}{S_n^k}\Big)\Big\}$$

$$= \mathbb{P}\{S_n \leq 1\} + \mathbb{E}\Big(1 - \exp\Big[-\frac{2^{d-1}\lambda}{S_n^k}\Big]\Big)$$

$$\leq \mathbb{P}\{S_n \leq 1\} + \mathbb{E}\Big(\frac{2^{d-1}\lambda}{S_n^k}\Big).$$

Using the density of the Gamma distribution, we see that

$$(3.21) \qquad \mathbb{P}\{S_n \leq 1\} = \int_0^1 \frac{\lambda^n}{(n-1)!} e^{-\lambda x} x^{n-1} \, \mathrm{d}x \leq \frac{\lambda^n}{(n-1)!}.$$

Integrating the Laplace transform of S_n as in (2.6), for $n \geq 4$, we see that

$$(3.22) \quad \mathbb{E}\big(S_n^{-3}\big) = \frac{\lambda^3}{(n-1)(n-2)(n-3)} \quad \text{and} \quad \mathbb{E}\big(S_n^{-2}\big) = \frac{\lambda^2}{(n-1)(n-2)}.$$

Then we get from (3.18), (3.19), (3.20) with $k = 3$, (3.21) and (3.22), that for $n \geq 4$,

$$\mathbb{P}(A_{n,3}^c) \leq \frac{2\lambda^n}{(n-1)!} + \lambda 3^{d-1}\mathbb{E}\left(\frac{1}{S_n^2}\right) + \lambda 2^{d-1}\mathbb{E}\left(\frac{1}{S_n^3}\right)$$

$$= \frac{2\lambda^n}{(n-1)!} + \frac{\lambda^2 3^{d-1}}{(n-1)(n-2)} + \frac{\lambda^3 2^{d-1}}{(n-1)(n-2)(n-3)},$$

and we deduce that $\sum_{n\geq 1} \mathbb{P}(A_{n,3}^c) < \infty$. By the Borel–Cantelli lemma,

$$(3.23) \qquad\qquad \mathbb{P}\left(\limsup_{n\to+\infty} A_{n,3}^c\right) = 0,$$

and the set $\Omega' = \liminf_{n\to+\infty} A_{n,3}$ has probability one. This completes the proof. $\qquad\square$

We now return to the question of whether or not a Lévy white noise is a tempered distribution. Similar to (2.9), for any d-dimensional Lévy noise $\dot{X}$, we define the set $\Omega_{\dot{X}}$ by

$$(3.24) \qquad\qquad \Omega_{\dot{X}} = \{\omega \in \Omega : \dot{X}(\omega) \in \mathcal{S}'(\mathbb{R}^d)\},$$

and we have the following characterization.

THEOREM 3.13. *Let X be a d-parameter Lévy field with jump measure J_X and characteristic triplet (γ, σ, ν) and $\dot{X}$ the associated Lévy white noise. Then the following holds for the set $\Omega_{\dot{X}}$ defined in (3.24) and the set Ω_X defined as in (3.7):*

(i) *If there exists $\eta > 0$ such that $\mathbb{E}(|X_{\mathbf{1}_d}|^\eta) < +\infty$, then $\mathbb{P}(\Omega_X) = \mathbb{P}(\Omega_{\dot{X}}) = 1$.*
(ii) *If for all $\eta > 0$, $\mathbb{E}(|X_{\mathbf{1}_d}|^\eta) = +\infty$, then $\mathbb{P}(\Omega_X) = \mathbb{P}(\Omega_{\dot{X}}) = 0$.*

REMARK 3.14. By Lemma 3.11, the equivalent condition mentioned in Remark 2.8 remains valid in the d-parameter case.

As mentioned in the Introduction, the first assertion of Theorem 3.13 was established in [11], Theorem 3, using a different definition of Lévy white noise. In Proposition 3.16 below, we show that the two definitions are equivalent.

PROOF OF THEOREM 3.13. To prove (i), by the Lévy–Itô decomposition (Theorem 3.3), Corollary 3.8 and Lemma 3.9(ii), we have $\mathbb{P}(\Omega_X) = 1$. Since derivation maps $\mathcal{S}'(\mathbb{R}^d)$ to itself, we deduce that $\mathbb{P}(\Omega_{\dot{X}}) = 1$.

To prove (ii), suppose that $\dot{X}$ does not have a PAM. We can use Theorem 3.3 to decompose X into the sum of a continuous part C, a small jumps part X^M and a compound Poisson part X^P. By Corollary 3.8, $\mathbb{P}(\Omega_{C+X^M}) = 1$. Then we deduce that, for all $\omega \in \Omega_{\dot{X}} \cap \Omega_{C+X^M}$,

$$\dot{X}^P(\omega) = \dot{X}(\omega) - \dot{C}(\omega) - \dot{X}^M(\omega) = \dot{X}(\omega) - \left(C(\omega) + X^M(\omega)\right)^{(\mathbf{1}_d)}$$

belongs to $\mathcal{S}'(\mathbb{R}^d)$. The general strategy of the proof is to construct, from the compound Poisson sheet X^P, a compound Poisson process that has the same moment properties, and show that when $\dot{X}^P \in \mathcal{S}'(\mathbb{R}^d)$, this process has polynomial growth at infinity, and this occurs with probability zero by Proposition 2.4(ii).

We first examine the noise $\dot{X}^P$ associated with the compound Poisson part. The jump measure $J_{X^P}(\mathrm{d}s, \mathrm{d}x) = \mathbb{1}_{|x|>1} J_X(\mathrm{d}s, \mathrm{d}x)$ of X^P is a Poisson random measure on $\mathbb{R}_+^d \times (\mathbb{R} \setminus \{0\})$ and $J_{X^P} = \sum_{i \geq 1} \delta_{\tau_i} \delta_{Y_i}$, where $\tau_i \in \mathbb{R}_+^d$ and $|Y_i| \geq 1$. By Lemma 3.6, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$(3.25) \qquad \langle \dot{X}^P, \varphi \rangle = \int_{\mathbb{R}_+^d} \int_{|x|>1} x\varphi(t) J_X(\mathrm{d}t, \mathrm{d}x) = \sum_{i \geq 1} Y_i \varphi(\tau_i).$$

By Lemma 3.12, for all $\omega \in \Omega$, there exists a sequence $(\varphi_n)_{n \geq 1}(\omega)$ of smooth compactly supported functions such that (3.16) holds. Furthermore, there is an event $\Omega' \subset \Omega$ with probability one such that there is an integer $N(\omega)$ with the property that for all $n \geq N(\omega)$, X^P is constant on the support of $\varphi_n(\omega)$, and (3.17) holds. Let L be the compound Poisson process defined in Lemma 3.12 by $L_t = X^P_{(\mathbf{1}_{d-1}, t)}$. We restrict ourselves to $\omega \in \Omega_{\dot{X}} \cap \Omega_{C+X^M} \cap \Omega'$, but we drop the dependence on ω in the following for simplicity of notation. We write $\Phi_n(t) = \int_{[t, +\infty)} \varphi_n(s) \, \mathrm{d}s$. Let $\theta \in C^\infty(\mathbb{R}^d)$ be such that $\theta = 0$ on the set $\{t \in \mathbb{R}^d : t_1 \wedge \cdots \wedge t_d \leq -1\}$ and $\theta = 1$ on the set $\{t \in \mathbb{R}^d : t_1 \wedge \cdots \wedge t_d \geq -\frac{1}{2}\}$ and such that all its derivatives are bounded. Then for all $n \geq 1, \theta \Phi_n \in \mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. So, in particular, for all $n \geq 1$, since θ is constant on $\mathbb{R}_+^d$,

$$\langle \dot{X}^P, \theta \Phi_n \rangle = (-1)^d \langle X^P, (\theta \Phi_n)^{(\mathbf{1}_d)} \rangle$$
$$= (-1)^d \langle X^P, (\Phi_n)^{(\mathbf{1}_d)} \rangle = \langle X^P, \varphi_n \rangle = L_{S_n},$$

by (3.17), and since $\Omega_{\dot{X}} \cap \Omega_{C+X^M} \subset \Omega_{\dot{X}^P}$, we deduce that

$$(3.26) \qquad |L_{S_n}| \leq C \mathcal{N}_p(\theta \Phi_n),$$

for some real number C and integer p (both depending on ω). For $\alpha, \beta \in \mathbb{N}^d$, with $|\alpha|, |\beta| \leq p$, we estimate $\sup_{t \in \mathbb{R}^d} |t^\alpha (\theta \Phi_n)^{(\beta)}|$. Since all the derivatives of θ are bounded,

$$\sup_{t \in \mathbb{R}^d} |t^\alpha (\theta \Phi_n)^{(\beta)}(t)| = \sup_{t \geq -\mathbf{1}_d} |t^\alpha (\theta \Phi_n)^{(\beta)}(t)|$$

$$= \sup_{t \geq -\mathbf{1}_d} \left| t^\alpha \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \Phi_n^{(\gamma)}(t) \theta^{(\beta-\gamma)}(t) \right|$$

$$\leq C_1 \sum_{\gamma \leq \beta} \sup_{t \geq -\mathbf{1}_d} |t^\alpha \Phi_n^{(\gamma)}(t)|,$$

for some constant C_1 depending only on p and θ. By (3.15), for some constant C_2,

$$\sup_{t \geq -\mathbf{1}_d} \left| t^\alpha \Phi_n^{(\gamma)}(t) \right| \mathbb{1}_{S_n \geq 1} \leq C_2 \mathcal{N}_{p+2d}(\varphi_n) \mathbb{1}_{S_n \geq 1} \leq C_3 S_n^{\tilde{p}} \mathbb{1}_{S_n \geq 1},$$

by (3.16), for some constant C_3 and $\tilde{p}$ independent of n. Therefore, for any integer p, there is an integer $\tilde{p}$ and a constant C depending only p and d, such that

$$(3.27) \qquad \mathcal{N}_p(\theta \Phi_n) \mathbb{1}_{S_n \geq 1} \leq C S_n^{\tilde{p}} \mathbb{1}_{S_n \geq 1}.$$

We deduce from (3.26) and (3.27) that

$$\left| \frac{L_{S_n}}{S_n^{\tilde{p}}} \right| \mathbb{1}_{S_n \geq 1} \leq C \mathbb{1}_{S_n \geq 1} < +\infty.$$

As in the proof of Proposition 2.2, we deduce that for all $\omega \in \Omega_{\dot{X}} \cap \Omega_{C+X^M} \cap \Omega'$, there exists $p(\omega) \in \mathbb{N}$ and $C(\omega) \in \mathbb{R}_+$ such that

$$(3.28) \qquad \limsup_{t \to +\infty} \frac{|L_t|(\omega)}{1 + t^{\tilde{p}(\omega)}} \leq C(\omega) < +\infty.$$

Since L is a compound Poisson process with no absolute moment of any positive order (it has the same Lévy measure as X^P), we can now conclude by Proposition 2.4(ii) that $\Omega_{\dot{X}} \cap \Omega_{C+X^M} \cap \Omega'$ is contained in a set of probability zero. Since $\mathbb{P}(\Omega_{C+X^M} \cap \Omega') = 1$, we deduce that $\mathbb{P}(\Omega_{\dot{X}}) = 0$.

By the fact that the derivative of a tempered distribution is a tempered distribution, $\Omega_X \subset \Omega_{\dot{X}}$. Therefore, $\mathbb{P}(\Omega_X) = 0$. $\quad\square$

REMARK 3.15. The statement of Corollary 2.9 extends directly to d-dimensional Lévy white noise, with the same proof.

We now relate our definition of Lévy white noise (Definition 3.4) to stochastic integrals, and to [11], Theorem 3.

PROPOSITION 3.16. *Let $\dot{X}$ be a Lévy white noise with jump measure J_X and characteristic triplet (γ, σ, ν) that has a* PAM:

(i) *For all functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have the following equality*:

$$\langle \dot{X}, \varphi \rangle = \int_{\mathbb{R}_+^d} \varphi(t) \, dX_t$$

$$:= \gamma \int_{\mathbb{R}_+^d} \varphi(t) \, dt + \sigma \int_{\mathbb{R}_+^d} \varphi(t) \, dW_t$$

$$(3.29) \qquad + \int_{\mathbb{R}_+^d} \int_{|x| \geq 1} x \varphi(t) J_X(dx, dt)$$

$$+ \int_{\mathbb{R}_+^d} \int_{|x| < 1} x \varphi(t) \big(J_X(dx, dt) - \nu(dx) \, dt \big)$$

$$= \gamma A_1(\varphi) + \sigma A_2(\varphi) + A_3(\varphi) + A_4(\varphi),$$

where the second integral is a Wiener integral (cf. Remark 3.5), and the last two are Poisson integrals as defined in [15], Lemma 12.13.

(ii) *The characteristic functional of the Lévy white noise is given, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, by*

$$\mathbb{E}\big(e^{i\langle \dot{X}, \varphi \rangle}\big) = \exp\bigg[\int_{\mathbb{R}_+^d} \psi\big(\varphi(t)\big)\, dt\bigg],$$

where ψ is the Lévy symbol of X:

$$\psi(z) = i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \big(e^{ixz} - 1 - izx \mathbb{1}_{|x|\le 1}\big)\nu(dx).$$

PROOF. Even without PAM, equality (3.29) has already been proven in Lemma 3.6 when $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We now assume that X has a PAM and check first that for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the right-hand side of (3.29) is well defined. Since $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, this is clearly the case for $A_1(\varphi)$ and $A_2(\varphi)$. For $A_3(\varphi)$, using PAM, one checks condition (3.11) as in the proof of Lemma 3.9. For $A_4(\varphi)$, one checks condition (3.5) using the same proof as when $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

We now deduce (3.29) for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (assuming PAM). By definition,

$$\langle \dot{X}, \varphi \rangle = (-1)^d \bigg[\gamma \int_{\mathbb{R}_+^d} \bigg(\prod_{i=1}^d t_i\bigg) \varphi^{(\mathbf{1}_d)}(t)\, dt + \sigma \int_{\mathbb{R}_+^d} W_t \varphi^{(\mathbf{1}_d)}(t)\, dt$$

$$+ \int_{\mathbb{R}_+^d} X_t^P \varphi^{(\mathbf{1}_d)}(t)\, dt + \int_{\mathbb{R}_+^d} X_t^M \varphi^{(\mathbf{1}_d)}(t)\, dt \bigg]$$

$$= \gamma \tilde{A}_1(\varphi) + \sigma \tilde{A}_2(\varphi) + \tilde{A}_3(\varphi) + \tilde{A}_4(\varphi).$$

The equality $A_3(\varphi) = \tilde{A}_3(\varphi)$ comes from Lemma 3.9. For the other three terms, since $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, it suffices to check that $\varphi \mapsto \gamma \tilde{A}_1(\varphi) + \sigma \tilde{A}_2(\varphi) + \tilde{A}_4(\varphi)$ and $\varphi \mapsto \gamma A_1(\varphi) + \sigma A_2(\varphi) + A_4(\varphi)$ define continuous (in probability) linear functionals of $\varphi \in \mathcal{S}(\mathbb{R}^d)$. For the first, this is obvious because $\dot{X} \in \mathcal{S}'(\mathbb{R}^d)$ by Theorem 3.13. For the second, consider $(\varphi_n) \subset \mathcal{S}(\mathbb{R}^d)$ such that $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R}^d)$, hence in $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$. Then $A_1(\varphi_n) \to 0$ and $A_2(\varphi_n) \to 0$ in probability. According to Cohen and Istas ([3], (2.34), page 27),

$$\mathbb{E}\big(\exp(i A_4(\varphi))\big) = \exp\bigg[\int_{\mathbb{R}_+^d} \int_{|x|<1} \big(e^{ix\varphi(t)} - 1 - ix\varphi(t)\big)\, dt\, \nu(dx)\bigg],$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. By the inequality in [15], Lemma 5.14,

$$\bigg|\int_{\mathbb{R}_+^d} \int_{|x|<1} \big(e^{ix\varphi_n(t)} - 1 - ix\varphi_n(t)\big)\, dt\, \nu(dx)\bigg|$$

$$\le \frac{1}{2} \int_{\mathbb{R}_+^d} \varphi_n(t)^2\, dt \int_{|x|<1} x^2 \nu(dx) \to 0.$$

So A_4 defines a linear functional on $\mathcal{S}(\mathbb{R}^d)$ that is continuous in law at 0, hence continuous in probability. This completes the proof of (i).

To prove (ii), we use (i) and standard results on the characteristic function of Poisson and Wiener integrals. See [15], Lemma 12.2, and [3], (2.34), page 27, for the Poisson integrals and [16], Theorem 1.4.1, for the Wiener integral. □

REMARK 3.17. In dimension one, we used the map I in Remark 2.10 to give an alternate proof of the inclusion $\Omega_{\dot{X}} \subset \Omega_X$. The analog of this map I in higher dimensions also exists. Let $\theta \in \mathcal{D}(\mathbb{R})$ such that $\theta \geq 0$, $\operatorname{supp}\theta \subset [0, 1]$ and $\int_{\mathbb{R}} \theta = 1$. We write $\tilde{\theta} = \theta \otimes \cdots \otimes \theta$ the dth-order tensor product of θ with itself: $\tilde{\theta}(s_1, \ldots, s_d) = \theta(s_1) \cdots \theta(s_d)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Define

$$(3.30) \qquad I_d\varphi(t) = \int_{(-\infty,t]} ds \int_{\mathbb{R}^d} dr\, \tilde{\Delta}_r^s(\varphi, \tilde{\theta}),$$

where

$$\tilde{\Delta}_r^s(\varphi, \theta) = \sum_{\varepsilon \in \{0,1\}^d} (-1)^{|\varepsilon|} \varphi\big(c_\varepsilon(r, s)\big)\tilde{\theta}\big(c_{1-\varepsilon}(r, s)\big),$$

and $c_\varepsilon(r, s)$ was defined just after (3.1). It is easy to see that if $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_d$, where $\varphi_1, \ldots, \varphi_d \in \mathcal{S}(\mathbb{R})$, then $I_d\varphi = (I_1\varphi_1) \otimes \cdots \otimes (I_1\varphi_d)$, where I_1 coincides with the map I of Remark 2.10. Then, since I was built as an antiderivative, for such φ,

$$(3.31) \qquad I_d\left(\frac{\partial^d\varphi}{\partial t_1 \cdots \partial t_d}\right) = \varphi.$$

We have already shown that I_1 maps continuously $\mathcal{S}(\mathbb{R})$ to itself. We equip $\mathcal{S}(\mathbb{R}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R})$ with the topology π generated by the family of semi-norms $\mathcal{N}_{p_1,\ldots,p_d}(\varphi_1 \otimes \cdots \otimes \varphi_d) = \prod_{i=1}^d \mathcal{N}_{p_i}(\varphi_i)$. Then $I_d : \mathcal{S}(\mathbb{R}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R})$ is continuous (and then uniformly continuous by linearity). We denote $\mathcal{S}(\mathbb{R}) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{S}(\mathbb{R})$ the completion of $\mathcal{S}(\mathbb{R}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R})$. By [26], Theorem 51.6, $\mathcal{S}(\mathbb{R}) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \mathcal{S}(\mathbb{R}) \simeq \mathcal{S}(\mathbb{R}^d)$, therefore I_d extends (by uniform continuity) to a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to itself. Formula (3.31) is true by linearity for $\varphi \in \mathcal{S}(\mathbb{R}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R})$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. There is a sequence $(\varphi_n)_{n \geq 1}$ of elements of $\mathcal{S}(\mathbb{R}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R})$ such that $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$. Since derivation is a continuous map from $\mathcal{S}(\mathbb{R}^d)$ to itself, we deduce that (3.31) holds for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

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Institut de mathématiques
École Polytechnique Fédérale de Lausanne
Station 8
CH-1015 Lausanne
Switzerland
E-mail: robert.dalang@epfl.ch
thomas.humeau@epfl.ch