*The Annals of Probability*
2017, Vol. 45, No. 6A, 3481–3534
DOI: 10.1214/16-AOP1141


# ON THE BOUNDARY OF THE SUPPORT OF SUPER-BROWNIAN MOTION

BY CARL MUELLER[1,*] LEONID MYTNIK[2,†] AND EDWIN PERKINS[3,‡]

*University of Rochester*, Technion—Israel Institute of Technology† and University of British Columbia‡*

We study the density $X(t,x)$ of one-dimensional super-Brownian motion and find the asymptotic behaviour of $P(0 < X(t,x) \le a)$ as $a \downarrow 0$ as well as the Hausdorff dimension of the boundary of the support of $X(t,\cdot)$. The answers are in terms of the leading eigenvalue of the Ornstein–Uhlenbeck generator with a particular killing term. This work is motivated in part by questions of pathwise uniqueness for associated stochastic partial differential equations.

CONTENTS



## 1. Introduction.

We consider the jointly continuous density $X(t,x)$ ($t > 0, x \in \mathbb{R}$) of one-dimensional super-Brownian motion given by the unique in law solution of

$$\frac{\partial X(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 X(t,x)}{\partial x^2} + \sqrt{X(t,x)}\dot{W}(t,x), \qquad X \ge 0. \tag{1.1}$$

Here, $\dot{W}$ is a space–time white noise, $X_0$ is in the space $M_F(\mathbb{R})$ of finite measures on the line and $(X_t, t > 0)$ is a continuous process taking values in the space

Received December 2015; revised August 2016.

[1]Supported by an NSF grant.

[2]Supported by the Israel Science Foundation Grant 1325/14.

[3]Supported by an NSERC Discovery Grant.

*MSC2010 subject classifications.* Primary 60H15, 60J68; secondary 35K15.

*Key words and phrases.* Super-Brownian motion, Hausdorff dimension, stochastic partial differential equations.

$C_K(\mathbb{R})$ of continuous functions with compact support in $\mathbb{R}$ [see, e.g., Section III.4 of [18] for these results and the meaning of (1.1)]. We abuse notation slightly and also let $X_t(A) = \int_A X(t,x)\,dx$ denote the continuous $M_F(\mathbb{R})$-valued process with density $X(t,\cdot)$ for $t > 0$, that is, the associated super-Brownian motion.

Our goal is to study the boundary of the zero set of $X_t$, or equivalently the boundary of the Borel support $\{x : X(t,x) > 0\}$, given by

$$
\begin{aligned}
BZ_t &= \partial\big(\{x : X(t,x) = 0\}\big) \\
&= \big\{x : X(t,x) = 0, \forall\delta > 0, X_t\big((x-\delta, x+\delta)\big) > 0\big\}.
\end{aligned}
\tag{1.2}
$$

The two related questions we consider are:

1. How large is $BZ_t$? For example, what is its Hausdorff dimension?
2. What is the asymptotic behaviour of $P(0 < X(t,x) < a)$ as $a \downarrow 0$?

As super-Brownian motion models a population undergoing random motion and critical reproduction, a detailed understanding of the interface between the population and empty space gives a snapshot of how the population ebbs and flows. Moreover, the answers we found are not what was originally expected. Standard estimates show that $X(t,\cdot)$ is locally Hölder continuous of index $1/2 - \varepsilon$ for any $\varepsilon > 0$ (see Proposition 5.6 below). But near the zero set of $X(t,\cdot)$, one can expect more regular behavior as the noise in (1.1) is mollified. In fact, [17] essentially showed that near the zero set of $X(t,\cdot)$ the density is locally Hölder continuous of any index less than one (see Proposition 5.7 below for a precise statement). The increased regularity led to independent conjectures 14 years ago (one by Carl Mueller and Roger Tribe and the other by Ed Perkins and Yongjin Wang) of the following.

CONJECTURE A. *The Hausdorff dimension of $BZ_t$ is zero a.s.*

This also was spurred on by wishful thinking as such a result would help prove pathwise uniqueness in equations such as (1.1), as we explain next.

The connection with pathwise uniqueness in stochastic pde's with non-Lipschitz coefficients is one reason for our interest in these questions. Mass moves with a uniform modulus of continuity in equations such as (1.1) and more generally in

$$
\frac{\partial X(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 X(t,x)}{\partial x^2} + X(t,x)^\gamma \dot{W}(t,x), \qquad X \geq 0,
\tag{1.3}
$$

for $0 < \gamma < 1$ (see Theorem 3.5 of [13]). This means one can localize the evolution of solutions to (1.3) in space and so if pathwise uniqueness fails then one expects that the solutions $X, Y$ which separate at time $T$ say, will initially separate at points in $BZ_T(X) \cap BZ_T(Y)$, where we have introduced dependence on the particular process. This is because in the interior of the support, say where $X_T \geq \eta > 0$, we have Lipschitz continuous coefficients and so solutions should coincide for a positive time due to the uniform modulus of continuity, and in the interior of the zero set solutions will remain at zero for some positive length of time thanks to the same reasoning (and lack of any immigration terms). As a result, one expects that the larger $BZ_t$ is, the easier it is for solutions to separate, and so the less likely pathwise uniqueness is. This reasoning is of course heuristic but here are some precise illustrations of the principle.

THEOREM 1.1 ([2]). *Let* $b : \mathbb{R} \to \mathbb{R}_+$ *be a smooth function with support* $[0, 1]$. *Then pathwise uniqueness fails in*

$$\frac{\partial X(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 X(t,x)}{\partial x^2} + \sqrt{X(t,x)}\dot{W}(t,x) + b(x). \tag{1.4}$$

One step in the proof is to show that $BZ_t \cap [0, 1]$ has positive Lebesgue measure with positive probability. This proceeds by first fixing $x \in (0, 1)$ and using a Poisson point process calculation to show that $P(X(t,x) = 0) > 0$. It is then easy to see that the $b$-immigration forces $X_t((x-\delta, x+\delta)) > 0$ a.s. for each $\delta > 0$. The proof of the theorem then goes on to show that solutions $X, Y$ can separate in $\bigcup_{t\in[0,1]} BZ_t(X) \cap BZ_t(Y)$. In short, the presence of the immigration $b$ changes the nature of the boundary of the support and makes it possible to establish pathwise nonuniqueness.

On the uniqueness side of things, two of us conjectured that the methods of [16] would allow one to establish:

$$\begin{aligned} &\text{If for some } \varepsilon > 0,\ P\big(0 < X(t,x) \le a\big) \le Ca^{1+\varepsilon}, \\ &\quad \text{then pathwise uniqueness would hold in (1.1).} \end{aligned} \tag{1.5}$$

We never tried to write out a careful proof of the implication in part because we believed the correct answer to Question 2 above was the following.

CONJECTURE B. $P(0 < X(t,x) \le a) = O(a)$ (*which is consistent with Conjecture* A *above*).

Nonetheless, this would be a rather nice state of affairs as it would suggest that the $\gamma = 1/2$ case of (1.3) is critical and one could expect pathwise uniqueness to hold for $\gamma > 1/2$. It is natural to expect that the $\gamma = 1/2$ case would then require additional work just as in the classical SDE counterpart resolved by Yamada and Watanabe 45 years ago (and unlike the signed case for general SPDEs where 3/4-Hölder continuity in the solution variable is critical by [16] and [11]).

Our main results on Questions 1 and 2 will show both Conjectures A and B are in fact false. To describe them, for $\lambda > 0$, let $V(t,x) = V^\lambda(t,x)$ be the unique solution of

$$\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{2}V^2, \qquad V(0,x) = \lambda\delta_0(x), \tag{1.6}$$

where $V$ is $C^{1,2}$ on $[0,\infty)\times\mathbb{R}\setminus\{(0,0)\}$. (See [1, 15] and the references therein.) A simple scaling argument shows that

$$V^{\lambda r}(s,x)=\lambda^2 V^r(\lambda^2 s,\lambda x) \qquad \forall r,\lambda,s>0, x\in\mathbb{R}. \tag{1.7}$$

If $E_\mu$ denotes expectation for $X$ starting at $X_0=\mu$, then we have (see, e.g., Theorem 1.1 in [15] and the references there)

$$E_{\delta_0}(e^{-\lambda X(t,x)})=E_{\delta_x}(e^{-\lambda X(t,0)})=e^{-V^\lambda(t,x)}, \tag{1.8}$$

and by the multiplicative property,

$$E_{X_0}(e^{-\lambda X(t,0)})=\exp\biggl(-\int V^\lambda(t,y)\,dX_0(y)\biggr). \tag{1.9}$$

So from the above we see that $V^\lambda(t,x)\uparrow V^\infty(t,x)$ as $\lambda\uparrow\infty$, where

$$P_{\delta_0}(X(t,x)=0)=P_{\delta_x}(X(t,0)=0)=e^{-V^\infty(t,x)} \tag{1.10}$$

and so

$$V^\infty(t,x)\le 2/t<\infty \tag{1.11}$$

because $P_{\delta_0}(X_t\equiv 0)=\exp(-2/t)$ (see, e.g., (II.5.12) in [18]). If $r\to\infty$ in (1.7) with $\lambda^2=s^{-1}$, we get

$$V^\infty(s,x)=s^{-1}F(xs^{-1/2}), \tag{1.12}$$

where $F(x)=V^\infty(1,x)$. The function $F$ has been studied in the PDE literature and can be intrinsically characterized as the solution of an ode. This, and other properties of $F$, are recalled in Section 3. For now, we only need to know that it is a symmetric $C^2$ function on the line which vanishes at infinity. Let $Lh(x)=\frac{h''}{2}-\frac{x}{2}h'(x)$ be the generator of the Ornstein–Uhlenbeck process, let $m$ be the standard normal distribution on the line and set $L^F(h)=Lh-Fh$. By standard Sturm–Liouville theory (see Theorem 2.3), there is a complete orthonormal system for $L^2(m)$ consisting of $C^2$ eigenfunctions for $L^F$, $\{\psi_n:n\in\mathbb{Z}_+\}$, with corresponding negative eigenvalues $\{-\lambda_n\}$ where $\{\lambda_n\}$ is nondecreasing. The largest eigenvalue $\lambda_0$ is simple and satisfies $1/2<\lambda_0<1$. The latter and a bit more is proved in Proposition 3.4.

Here are our answers to the above questions. $\dim(A)$ denotes the Hausdorff dimension of a set $A\subset\mathbb{R}$. In the next two results, $X(t,x)$ is the density of super-Brownian motion satisfying (1.1) starting with finite initial measure $X_0$ and $BZ_t$ is defined as in (1.2).

THEOREM 1.2. (a) *For some* $C_{1.2}$, *for all* $a,t>0$, *and* $x\in\mathbb{R}$,

$$P_{X_0}(0<X(t,x)\le a)\le C_{1.2}X_0(\mathbb{R})t^{-(1/2)-\lambda_0}a^{2\lambda_0-1}.$$

(b) *For all* $K\in\mathbb{N}$ *there is a* $C(K)>0$ *so that if* $X_0(\mathbb{R})\le K$, $X_0([-K,K])\ge K^{-1}$, $t\ge K^{-1}$ *and* $|x|\le K$, *then*

$$P_{X_0}(0<X(t,x)\le a)\ge C(K)t^{-(1/2)-\lambda_0}a^{2\lambda_0-1} \qquad \text{for all } 0<a\le\sqrt{t}.$$

THEOREM 1.3. *For all $X_0 \neq 0$ and $t > 0$:*

(a) $\dim(BZ_t) \le 2 - 2\lambda_0$ $P_{X_0}$*-a.s.*
(b) $\dim(BZ_t) = 2 - 2\lambda_0$ *with positive* $P_{X_0}$*-probability.*

In the above results, both $2\lambda_0 - 1$ and $2 - 2\lambda_0$ are in (0, 1) and so these results do disprove Conjectures A and B. In Theorem 1.3(b), one expects that $\dim(BZ_t) = 2 - 2\lambda_0$ a.s. on $\{X_t \neq 0\}$ but the proof does not show this.

REMARK 1.4. Both of the above results extend immediately to solutions of

$$\frac{\partial X(t,x)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 X(t,x)}{\partial x^2} + \sqrt{\gamma X(t,x)}\dot{W}(t,x), \tag{1.13}$$

where the constants in Theorem 1.2 now may depend on $\gamma, \sigma^2 > 0$. This is clear since a simple scaling result shows that if $X$ is the solution of (1.1), then $\gamma\sigma^{-2}X(\sigma^2 t, x)$ has the unique law of any solution of (1.13).

Theorem 1.2 is contained in Theorem 4.8 below. A Tauberian theorem will show that $P_{\delta_0}(0 < X(t,x) \le a) \sim a^\alpha$ as $a \downarrow 0$ if and only if

$$E_{\delta_0}\big(e^{-\lambda X(t,x)}1\big(X(t,x) > 0\big)\big) \sim \lambda^{-\alpha} \qquad \text{as } \lambda \uparrow \infty.$$

Here, $\sim$ means bounded above and below by positive constants and $\alpha = 2\lambda_0 - 1$. If we use (1.9) and (1.10), this becomes

$$e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} \sim \lambda^{-\alpha} \qquad \text{as } \lambda \uparrow \infty,$$

and using (1.11) this reduces to

$$V^\infty(t,x) - V^\lambda(t,x) \sim \lambda^{-\alpha} \qquad \text{as } \lambda \uparrow \infty. \tag{1.14}$$

We have not been careful with dependence on $t$ or $x$ but by Dini's theorem we know that $\lim_{\lambda\to\infty} V^\lambda(t,x) = V^\infty(t,x)$ uniformly for $(t,x)$ in compact subsets of $[0,\infty) \times \mathbb{R} \setminus \{(0,0)\}$—this is Theorem 1 of [9]. Evidently to prove Theorem 1.2, we need a rate of convergence in the Kamin and Peletier result, and Proposition 4.6 will give the following which may be of interest to PDE specialists and will be used to complete the proof of Theorem 1.2.

THEOREM 1.5. *There are positive constants $\bar{C}$ and for each $K \ge 1$, $\underline{C}(K)$, such that for all $t > 0$,*

$$\sup_x V^\infty(t,x) - V^\lambda(t,x) \le \bar{C} t^{-\frac{1}{2}-\lambda_0}\lambda^{-(2\lambda_0-1)} \qquad \forall \lambda > 0 \tag{1.15}$$

*and*

$$\underline{C}(K) t^{-\frac{1}{2}-\lambda_0}\lambda^{-(2\lambda_0-1)} \le \inf_{|x|\le K\sqrt{t}} V^\infty(t,x) - V^\lambda(t,x) \qquad \forall \lambda \ge t^{-1/2}. \tag{1.16}$$

The lower bound in Theorem 1.3(b) is established in Section 5.1 (see Theorem 5.5) by first establishing a capacity condition for a set to be nonpolar for $BZ_t$ (Corollary 5.3) through a Frostman-type argument, and then taking the above set to be the range of an appropriate subordinator and using the potential theory for the subordinator (a well-known trick). The key to the above capacity condition is a second moment bound (Proposition 5.1) which is established in Section 6. The corresponding upper bound is proved in Theorem 5.9 in Section 5.2 by first modifying the proof of Theorem 1.2 to get a bound on $P_{X_0}(0 < X_t([x, x+\varepsilon]) \le \varepsilon^2 M)$ (Theorem 5.8 which is proved in Section 7). One then uses this and the improved modulus of continuity for $X$ near its zero set (Proposition 5.7) to carry out a standard covering argument which in fact bounds the box dimension.

Although the failure of Conjectures A and B indicate that our approach may not shed light on the pathwise of uniqueness of solutions to (1.1), we remain optimistic that progress can be made on (1.3) for some values of $\gamma$ in $(1/2, 3/4]$. In fact, (1.5) was a special case of the conjecture

$$\text{if for some } \alpha > 3-4\gamma,\ P\bigl(0 < X(t,x) \le a\bigr) \le C a^{\alpha}, \quad \text{then pathwise uniqueness would hold in (1.3).} \tag{1.17}$$

Note for $\gamma > 3/4$ one can take $\alpha = 0$ but in this case pathwise uniqueness is a special case of the main result in [16] whose ideas underly our heuristic proof of (1.17). There is also an exponential dual to solutions of (1.3) for $\gamma \in [1/2, 1)$ (see [14]) and we believe the methods of this paper can be used to resolve the left-hand tail asymptotics for solutions of (1.3) as well. We expect power tail behaviour for all $\gamma$, and so by (1.17), for $\gamma < 3/4$ but close to 3/4, we conjecture that pathwise uniqueness will hold in (1.3). Hence, although the 3/4-Hölder condition in [16] is sharp for pathwise uniqueness in general (by [11]), it would not be sharp for the family of nonnegative solutions to (1.3).

CONVENTION. We will use $E_a^Z$ to denote expectation for the Markov process $Z$ starting at a point $a$ and (abusing the notation slightly) use $E_\mu^Z$ to denote the corresponding expectation where $Z_0$ now has law $\mu$—sometimes $\mu$ will be only a finite measure.

We close this section with a heuristic explanation for the connection between our problem and the largest eigenvalue of $L^F$. Let $U^\lambda(t,x) = \frac{d}{d\lambda} V^\lambda(t,x)$. Then (1.14) with $\alpha = 2\lambda_0 - 1$ (as is required) would clearly follow from

$$U^\lambda(t,x) \sim \lambda^{-2\lambda_0} \qquad \text{as } \lambda \uparrow \infty. \tag{1.18}$$

A formal differentiation of (1.6) shows that $U = U^\lambda$ satisfies

$$\frac{\partial U}{\partial t} = \frac{1}{2} U'' - VU, \qquad U_0 = \delta_0,$$

which has Feynmann–Kac representation

$$
\begin{aligned}
U^\lambda(t,x) &= E_x^B\left(\delta_0(B_t)\exp\left\{-\int_0^t V^\lambda(t-s,B_s)\,ds\right\}\right) \\
&= E_0^{(\lambda^2 t,\lambda x)}\left(\exp\left\{-\int_0^{\lambda^2 t} V^1(s,B_s)\,ds\right\}\right)p_t(x).
\end{aligned}
\tag{1.19}
$$

The last line follows by time reversal and the scaling relation (1.7). Here, $B$ under $E_0^{(\lambda^2 t,\lambda x)}$ is a Brownian motion starting at 0 conditioned to equal $\lambda x$ at time $\lambda^2 t$. A further use of (1.7) shows that

$$
H(u,x) = uV^1(u,\sqrt{u}x) = V^{\sqrt{u}}(1,x) \uparrow F(x) \qquad \text{as } u \uparrow \infty. \tag{1.20}
$$

If $Y(u) = B(e^u - 1)e^{-u/2}$, $u \ge 0$, then $Y$ is an Ornstein–Uhlenbeck process with $Y_0 = B_0$. If we ignore the conditioning in (1.19), and use the Markov property for $B$ at $t = 1$, we obtain as $\lambda \to \infty$ [ignoring dependence on $(t,x)$]

$$
\begin{aligned}
U^\lambda(t,x) &\sim E_0^B\left(\exp\left\{-\int_0^{\lambda^2 t} V^1(s,B_s)\,ds\right\}\right) \\
&\sim E_m\left(\exp\left\{-\int_0^{\lambda^2 t-1}(s+1)V^1\left(s+1,\frac{\sqrt{s+1}\,B_s}{\sqrt{s+1}}\right)(s+1)^{-1}\,ds\right\}\right) \\
&= E_m^Y\left(\exp\left\{-\int_0^{\log(\lambda^2 t)} H(e^u,Y_u)\,du\right\}\right) \qquad (s = e^u - 1) \\
&\sim E_m\left(\exp\left\{-\int_0^{\log(\lambda^2 t)} F(Y_u)\,du\right\}\right) \qquad \text{[by (1.20) and a bit of work]} \\
&\sim C\exp\{-\lambda_0\log(\lambda^2 t)\} \\
&\sim C\lambda^{-2\lambda_0},
\end{aligned}
$$

giving us the required (1.18). The next to last line follows by a standard eigenfunction expansion recalled in Section 2. There are of course a number of nonrigorous steps in the above. Some of them [like the use of (1.20)] will be verified in this work, but our basic approach will not follow this plan but rather depend on a Campbell measure formula (see the proof of Lemma 4.1 in Section 4).

**2. Eigenfunction expansions.** Let $Y_t$ denote the Ornstein–Uhlenbeck process associated with the infinitesimal generator $L$. In this section, we usually will drop the $Y$ in the notation $E_x^Y$. Its semigroup is denoted by $P_t$ and its resolvent by $R_\lambda$. If $\phi \in C[-\infty,\infty]$ (the space of continuous functions with finite limits at $\pm\infty$) with $\phi \ge 0$, we let $L^\phi h = Lh - \phi h$, the generator associated with the diffusion, $Y^\phi$, obtained by killing $Y$ at time $\rho_\phi = \inf\{t : \int_0^t \phi(Y_s)\,ds > e\}$, where $e$ is an independent exponential variable. We denote its semigroup and resolvent by $P_t^\phi$ and $R_\lambda^\phi$

($\lambda > 0$), respectively. The above semigroups are strongly continuous contraction semigroups on $L^2(m)$. The contraction part is elementary as $m$ is stationary for $Y$. For the strong continuity, see Lemma 2.1 below for $P_t$ and it is easy to check that $\lim_{t\to 0} \|P_t f - P_t^\phi f\|_2 = 0$ for all $f \in L^2(m)$. For now, we will consider $L$ and $L^\phi$ defined on $D = \{h \in C^2 \cap L^2(m) : Lh \in L^2(m)\}$.

LEMMA 2.1. *For all* $f \in L^2(m)$, $\lim_{t\downarrow 0} E_m((f(Y_t) - f(Y_0))^2) = 0$.

PROOF. As $Y_t$ is stationary under $P_m$, a standard approximation argument allows us to assume $f$ is continuous with compact support. The result now follows by dominated convergence. □

We let $L_0^\phi$ and $L_0$ denote the infinitesimal generators of the $L^2(m) \equiv L^2$-semigroups $P_t^\phi$ and $P_t$, respectively, on their domains $D(L_0^\phi)$ and $D(L_0)$, respectively. So, for example,

$$D(L_0) = \Big\{ f \in L^2 : \exists L_0 f \in L^2 \text{ such that } \lim_{t\downarrow 0} \big\| (P_t f - f)/t - L_0 f \big\|_2 = 0 \Big\}.$$

The subscript 0 is a temporary measure to avoid confusion which we address now.

LEMMA 2.2. (a) $D(L_0^\phi) = D(L_0)$ *and* $L_0^\phi f = L_0 f - \phi f$ *for all* $f \in D(L_0)$.
(b) $L_0^\phi$ *is an extension of the differential operator* $L^\phi$, *the latter on* $D$.

PROOF. (a) This is a routine calculation. The fact that $\phi \in C[-\infty, \infty]$ helps here.

(b) By the above, we may assume $\phi = 0$. If $f \in D$, then $M_f(t) = f(Y_t) - f(Y_0) - \int_0^t Lf(Y_s)\,ds$ is a square integrable martingale under $P_m$. Therefore,

$$\begin{aligned} \|(P_t f - f)/t - Lf\|_2^2 &= \int \left( E_x\left( \int_0^t Lf(Y_s)\,ds \right)/t - Lf(x) \right)^2 dm(x) \\ &\le \int_0^t E_m\big((Lf(Y_s) - Lf(Y_0))^2\big)\,ds/t. \end{aligned}$$

The last expression approaches 0 as $t \to 0+$ by the previous lemma, and the result follows. □

Henceforth, we will drop the subscript 0's on $L_0^\phi$ (in view of the above result this should cause no confusion).

Here is the result we will need to describe our main results. In (e), $C([0,\infty), \mathbb{R})$ is the usual space of continuous paths with the topology of uniform convergence on bounded time sets.

THEOREM 2.3. (a) *There is a complete orthonormal system* (*cons*), $\{\psi_n : n \in \mathbb{Z}_+\}$, *of* $C^2$ *eigenfunctions in* $L^2(m)$ *for* $L^\phi$ *satisfying* $L^\phi \psi_n = -\lambda_n \psi_n$, *where* $\{\lambda_n\}$ *is a nondecreasing nonnegative sequence diverging to* $\infty$, $-\lambda_0$ *is a simple eigenvalue and* $\psi_0 > 0$.

(b) $R_\lambda^\phi$ *is a symmetric Hilbert–Schmidt integral operator on* $L^2(m)$. *There is a jointly continuous symmetric kernel* $G_\lambda^\phi : \mathbb{R}^2 \to [0, \infty)$ *such that*

$$R_\lambda^\phi h(x) = \int G_\lambda^\phi(x, y) h(y)\, dm(y)$$

*and*

$$G_\lambda^\phi(x, y) = \sum_{n=0}^{\infty} \frac{1}{\lambda + \lambda_n} \psi_n(x)\psi_n(y),$$

*where the series converges in* $L^2(m \times m)$ *and uniformly absolutely on compacts.*

(c) *The killed diffusion* $Y^\phi$ *has a jointly* [*in* $(t, x, y)$] *continuous transition density*, $q^\phi(t, x, y) \equiv q(t, x, y)$, *for* $t > 0$, *given by*

$$q(t, x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \psi_n(x)\psi_n(y),$$

*where the convergence is in* $Ł^2(m \times m)$ *and uniformly absolutely for* $(t, x, y) \in [\varepsilon, \infty) \times [\varepsilon, \varepsilon^{-1}]^2$ *for any* $\varepsilon > 0$. *Moreover, if* $0 < \delta < 1/4$, *and* $s^* = s^*(\delta) > 0$ *satisfies*

$$2\delta = \frac{e^{-s^*/2} - e^{-s^*}}{1 - e^{-s^*}} \tag{2.1}$$

($s^*$ *will increase to* $\infty$ *as* $\delta \downarrow 0$), *then there is a* $c(\delta)$ *such that*

$$q(t, x, y) \le c(\delta) e^{-\lambda_0 t} \exp\bigl(\delta\bigl(x^2 + y^2\bigr)\bigr) \qquad \text{for all } t \ge s^*(\delta). \tag{2.2}$$

(d) *If* $\theta = \int \psi_0\, dm$, *then for any* $\delta > 0$ *there is a* $c_\delta > 0$ *such that for all* $t \ge 0$ *and* $x \in \mathbb{R}$,

$$e^{\lambda_0 t} P_x(\rho_\phi > t) = \theta \psi_0(x) + r(t, x), \tag{2.3}$$

*where*

$$\bigl|r(t, x)\bigr| \le c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)t}, \tag{2.4}$$

$$\psi_0(x) \le c_\delta e^{\delta x^2}, \tag{2.5}$$

*and for* $t \ge s^*(\delta)$,

$$\bigl|r(t, x)\bigr| \le \sum_{1}^{\infty} e^{-(\lambda_n - \lambda_0)t} \bigl|\psi_n(x)\bigr| \int |\psi_n|\, dm \le c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)t}. \tag{2.6}$$

(e) *As* $T \to \infty$, $P_x(Y \in \cdot \mid \rho_\phi > T) \to P_x^\infty$ *weakly on* $C([0,\infty),\mathbb{R})$ *where* $P_x^\infty$ *is the law of the diffusion with transition density* (*with respect to* $m$),

$$\tilde{q}(t,x,y) \equiv q(t,x,y)\frac{\psi_0(y)}{\psi_0(x)}e^{\lambda_0 t}. \tag{2.7}$$

PROOF. Parts (a), (b) and the first equation in (c) follow from standard Sturm–Liouville theory. Here, note that multiplication by the square root of the normal density converts $L^\phi$ into the operator

$$A^\phi g = g''/2 - (x^2/8 - 1/4 + \phi)g,$$

now acting on $L^2(dx)$, and so one can proceed as in Example 2 in Section 9.5 of [3]. One applies a minor variant of Mercer's theorem on page 245 of [19] for the uniform convergence; only the continuity of $G_\lambda^\phi$ takes a bit of work. The bound (2.2) and part (d) can be proved through minor modifications of the arguments in the proof of Theorem 1.1 of [21]. Details of these arguments may be found in the Appendix of [12].

(e) Fix $x \in \mathbb{R}$. If $0 \le t_1 < t_2 < T$ and $\phi_i$ are bounded measurable functions, then

$$\begin{aligned}
&E_x\left(\prod_{i=1}^2 \phi_i(Y_{t_i})1(\rho_\phi > T)\right)\Big/ P_x(\rho_\phi > T)\\
&\quad= E_x\left(\prod_{i=1}^2 \phi_i(Y_{t_i})1(\rho_\phi > t_2)P_{Y_{t_2}}(\rho_\phi > T - t_2)\right)\Big/ P_x(\rho_\phi > T)\\
&\quad= \left[E_x\left(\prod_{i=1}^2 \phi_i(Y_{t_i})1(\rho_\phi > t_2)\frac{\theta\psi_0(Y_{t_2})e^{-\lambda_0(T-t_2)}}{\theta\psi_0(x)e^{-\lambda_0 T}}\right)\right.\\
&\qquad\left. + E_x\left(\prod_{i=1}^2 \phi_i(Y_{t_i})1(\rho_\phi > t_2)\frac{e^{-\lambda_0(T-t_2)}r(T-t_2, Y_{t_2})}{\theta\psi_0(x)e^{-\lambda_0 T}}\right)\right]\\
&\qquad\times \frac{\theta\psi_0(x)e^{-\lambda_0 T}}{\theta\psi_0(x)e^{-\lambda_0 T} + r(T,x)e^{-\lambda_0 T}}\\
&\quad\equiv [T_1 + T_2] \times T_3,
\end{aligned}$$

where (2.3) is used in the second equality. By using (2.4) with $\delta = 1/4$, we have

$$\begin{aligned}
|T_2| &\le \frac{\|\phi_1\|_\infty\|\phi_2\|_\infty}{\theta\psi_0(x)}e^{\lambda_0 t_2}E_x\big(c_\delta e^{\delta Y_{t_2}^2}\big)e^{-(\lambda_1-\lambda_0)(T-t_2)}\\
&\le c(x)e^{\lambda_0 t_2}e^{-(\lambda_1-\lambda_0)(T-t_2)} \to 0 \qquad \text{as } T \to \infty.
\end{aligned}$$

By (2.4), we also have

$$|T_3 - 1| \le \frac{|r(T,x)|e^{-\lambda_0 T}}{\theta\psi_0(x)e^{-\lambda_0 T} - |r(T,x)|e^{-\lambda_0 T}}$$
$$\le \frac{c_\delta e^{\delta x^2} e^{-(\lambda_1-\lambda_0)T}}{\theta\psi_0(x) - c_\delta e^{\delta x^2} e^{-(\lambda_1-\lambda_0)T}} \to 0 \qquad \text{as } T \to \infty.$$

Use these last results in (2.8) to conclude that

$$\lim_{T\to\infty} E_x\left(\prod_{i=1}^{2} \phi_i(Y_{t_i}) \mid \rho_\phi > T\right)$$
$$= E_x\left(\prod_{i=1}^{2} \phi_i(Y_{t_i}) 1(\rho_\phi > t_2)\psi_0(Y_{t_2})e^{\lambda_0 t_2}\right)\Big/\psi_0(x)$$
$$= \iint q(t_1, x, x_1) q(t_2 - t_1, x_1, x_2) \prod_{i=1}^{2} \phi_i(x_i) \frac{\psi_0(x_2)}{\psi_0(x)}$$
$$\times e^{\lambda_0 t_1} e^{\lambda_0(t_2 - t_1)}\, dm(x_1)\, dm(x_2)$$

which by definition equals

$$\iint \tilde{q}(t_1, x, x_1)\tilde{q}(t_2 - t_1, x_1, x_2) \prod_{i=1}^{2} \phi_i(x_i)\, dm(x_1)\, dm(x_2).$$

Similar reasoning gives the convergence of the $k$-dimensional distributions for all $k$.

It remains to establish tightness. If $0 \le s < t \le t_0 < T$ with $t - s \le 1$, then

$$E_x\big((Y_t - Y_s)^4 \mid \rho_\phi > T\big) \le E_x\big((Y_t - Y_s)^4 P_{Y_t}(\rho_\phi > T - t)\big)/P_x(\rho_\phi > T). \tag{2.8}$$

It is now easy to use (2.3), (2.4) and (2.5), together with Cauchy–Schwarz to bound the right-hand side of (2.8) by $c'(x, t_0)(t - s)^2$ at least for $T > T_0(x, t_0)$. This gives the required tightness and the proof of (e) is complete. □

**3. A nonlinear differential equation and some associated eigenvalues.** Recall that $F(x) = V^\infty(1, x)$. We start by recording the convergence results from Theorems 1 and 2 of [9] which were discussed in Section 1. Part (b) in fact is immediate from (a) and (1.7).

PROPOSITION 3.1. (a) $\lim_{\lambda\to\infty} V^\lambda(t, x) = V^\infty(t, x)$, *where the convergence is uniform on compact subsets of* $S$.

(b) *For any* $\lambda > 0$ *and* $a > 0$,

$$\lim_{t\to\infty} \sup_{|x|\le a} \big|tV^\lambda(t, xt^{1/2}) - F(x)\big| = 0.$$

In the PDE literature, $F : \mathbb{R} \to [0, \infty)$ is characterized as the unique solution of the following differential equation:

$$
\text{(3.1)} \qquad
\begin{aligned}
&\text{(i)}\ \ \frac{F''(y)}{2} + \frac{y}{2}F'(y) + F(y) - \frac{F^2(y)}{2} = 0,\\
&\text{(ii)}\ \ F > 0, \qquad F \text{ is } C^2 \text{ on } \mathbb{R},\\
&\text{(iii)}\ \ F'(0) = 0, \qquad F(y) \sim c_0 y e^{-y^2/2} \qquad \text{as } y \to \infty.
\end{aligned}
$$

Here, $\sim$ means the ratio goes to one as $y \to \infty$ and $c_0 > 0$. This result follows from [1], with $f : [0, \infty) \to [0, \infty)$ satisfying equations (1.7)–(1.9) of that reference (with $N = 1$ and $p = 2$ in our setting), where $F(y) = 2f(\sqrt{2}y)$ for $y \geq 0$ and we extend $F$ to the line by symmetry. The above ODE is then immediate from the theorem following (1.9) in [1] and the trivial fact that the condition $F'(0) = 0$ and fact that $F$ is $C^2$ on the positive half-line ensures the symmetric extension is $C^2$ on the line. In fact (see the the aforementioned theorem of Brezis, Peletier and Terman [1]), uniqueness holds if the strong asymptotic in condition (iii) is replaced with

$$
\text{(iii)}'\ F'(0) = 0, \qquad \lim_{y \to \infty} y^2 F(y) = 0.
$$

Here are some additional properties of $F$.

LEMMA 3.2. (a) *For all* $y \geq x_0 \geq 0$,

$$
\begin{aligned}
F'(y) = {} & \exp\{-y^2/2 + x_0^2/2\} F'(x_0)\\
& + \int_{x_0}^{y} \exp\{-y^2/2 + z^2/2\} F(z)\big(F(z) - 2\big)\, dz.
\end{aligned}
$$

(b) $\lim_{y \to \infty} e^{y^2/2} \frac{F'(y)}{y^2} = -c_0$, *where* $c_0$ *is as in* (3.1)(iii).

(c) $1 < F(0) < 2$.

(d) $F$ *is strictly decreasing on* $[0, \infty)$.

PROOF. The differential equation (3.1)(i) may be rewritten as

$$
\big(e^{z^2/2} F'(z)\big)' = e^{z^2/2} F(z)\big(F(z) - 2\big).
$$

(a) follows easily. To derive (b), take $x_0$ large and then use the asymptotics from (3.1)(iii). For (c), note by (a) with $x_0 = 0$, $F$ is increasing until $F \leq 2$. So if $F(0) > 2$, it can never pass below 2, a contradiction. If $F(0) = 2$, then by uniqueness to the initial value problem, $F \equiv 2$, another contradiction. It now follows from (a) with $x_0 = 0$ that $F' < 0$ for positive values until $F$ hits 2 but evidently this can therefore never happen. This proves (d). It remains to prove $F(0) > 1$. A simple calculation using (3.1)(i) gives $(yF + F')' = F(F - 1)$. Integrating over the line, we get $\int_{\mathbb{R}} F^2\, dy = \int_{\mathbb{R}} F\, dy$. If $F(0) \leq 1$, then by (d), $0 < F < 1$ on $(0, \infty)$

which contradicts the above equality of integrals. (Note that the Remark prior to Lemma 11 in [1] gives $F(0) \geq 1$.) □

In the PDE literature, $V^\infty(t,x)$ given by (1.12) is called a very singular solution of the heat equation with absorption. One can easily check (or see Section 1 of [1]) that $V = V^\infty$ is a $C^{1,2}$ (on $S = [0,\infty) \times \mathbb{R} - \{(0,0)\}$) solution of:

$$\text{(3.2)} \qquad \begin{aligned} &\text{(i)} \quad \frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{2}V^2, \\ &\text{(ii)} \quad V(0,x) = 0 \qquad \text{for all } x \neq 0; \qquad \lim_{t \to 0} \int_{\mathbb{R}} V(t,x)\,dx = \infty. \end{aligned}$$

Recall that $X_t(dx) = X(t,x)\,dx$, where $X$ solves (1.1). Translation invariance and (1.9) imply that

$$E^X_{X_0}(e^{-\lambda X(t,x)}) = \exp\biggl(-\int V^\lambda(t, y-x)\,dX_0(y)\biggr). \tag{3.3}$$

We let $\mathbb{N}_x$ denote the canonical measure associated with $X$ starting at $\delta_x$ (see Section II.7 of [18]). It is an easy consequence of Theorem II.7.2 of the latter reference that

$$\exp\biggl(-\int (1 - e^{-\lambda X(t,0)})\,d\mathbb{N}_x(X)\biggr) = e^{-V^\lambda(t,x)}. \tag{3.4}$$

PROPOSITION 3.3. *For all* $x \in \mathbb{R}$, $e^{-F(x)} = P_{\delta_x}(X(1,0) = 0)$ *and* $F(x) = \mathbb{N}_x(X(1,0) > 0)$.

PROOF. The first equality is immediate from (1.10). Let $t = 1$ and $\lambda \uparrow \infty$ in (3.4) to derive the second. □

We recall the general exponential duality which underlies the above (see, e.g., Theorem II.5.11 of [18]). For $\phi$ nonnegative, bounded and measurable, let $V(t,x) = V(\phi)(t,x)$ be the unique mild solution of

$$\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{2}V^2, \qquad V_0 = \phi. \tag{3.5}$$

If $X_t(\phi) = \int \phi\,dX_t$, then

$$E^X_{X_0}\bigl(\exp(-X_t(\phi))\bigr) = \exp\bigl(-X_0(V_t(\phi))\bigr). \tag{3.6}$$

If $\phi \in C^2_b(\mathbb{R})$ (functions with continuous bounded partials of order up to 2), then $V(t,x)$ has continuous bounded derivatives of order up to 1 in $t$ and 2 in $x$, and (3.5) holds in the classical (i.e., pointwise) sense.

Now return to the eigenfunction expansions of Section 2 in the case where $\phi = F$ or $F/2$. We denote dependence on $\phi$ by $\lambda_0(\phi)$ and $\psi_0^\phi$, and $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(m)$.

PROPOSITION 3.4. (a) $\lambda_0(F/2) = \frac{1}{2}$ *and the corresponding eigenfunction is* $\psi_0^{F/2}(x) = c_F e^{x^2/2} F(x)$, *where* $c_F > 0$ *is a normalizing constant*.

(b) $\frac{1}{2} < \lambda_0(F) < 1$. *More precisely*,

$$\frac{1}{2} + \frac{1}{2}\int F(x)\big(\psi_0^F\big)^2(x)\,dm(x) \le \lambda_0(F) \le 1 - \frac{1}{2}\int \big(c_F e^{x^2/2}F(x)\big)'(x)^2\,dm(x).$$

PROOF. Let $\psi(x) = e^{x^2/2}F(x) \in C^2 \cap L^2(m)$ [the latter by (3.1)(iii)]. Then

$$\begin{aligned} L^{F/2}\psi &= \frac{1}{2}\big(\psi' e^{-x^2/2}\big)' e^{x^2/2} - \frac{F}{2}\psi \\ &= e^{x^2/2}\left[\frac{F''}{2} + \frac{xF'}{2} + \frac{F}{2} - \frac{F^2}{2}\right] \\ &= -e^{x^2/2}\left(\frac{F}{2}\right), \end{aligned}$$

the last by (3.1)(i). This shows $\psi \in D \subset D(L)$ and $L^{F/2}\psi = -\frac{1}{2}\psi$. Recall by Theorem 2.3(a), the eigenfunction corresponding to the simple eigenvalue $-\lambda_0(F/2)$ is positive, as is $F$. By orthogonality of eigenfunctions corresponding to distinct eigenvalues we must therefore have $\lambda_0(F/2) = \frac{1}{2}$ and $\psi_0^{F/2} = c\psi$, for some normalizing constant $c > 0$.

(b) The variational characterization of $\lambda_0$ gives

$$\lambda_0(F) = \min\big\{\big\langle -L^F\psi, \psi\big\rangle : \psi \in D(L), \|\psi\|_2 = 1\big\}, \tag{3.7}$$

where the minimum is attained at $\psi = \psi_0^F$. (The latter is clear and to see the former one can set $\psi = R_\lambda^F \phi$ and expand $\phi$ in terms of the basis $\psi_n$.) If we set $\psi = \psi_0^{F/2}$ we therefore get

$$\begin{aligned} \lambda_0(F) &\le \big\langle -L^F\psi, \psi\big\rangle = 2\big\langle -L^{F/2}\psi, \psi\big\rangle + \langle L\psi, \psi\rangle \\ &= 2\lambda_0(F/2) - \frac{1}{2}\int_{-\infty}^{\infty} \psi'(x)^2\,dm \\ &= 1 - \frac{1}{2}\|\psi'\|_2^2 < 1, \end{aligned}$$

where the next-to-last equality holds by an integration by parts [Lemma 3.2(b) handles the boundary terms], and the last equality holds by (a). Turning next to the lower bound on $\lambda_0(F)$, if $\psi_0 = \psi_0^F$, we have, using the variational characterization of $\lambda_0(F/2)$,

$$\begin{aligned} \lambda_0(F) &= \big\langle -L^F\psi_0, \psi_0\big\rangle = \big\langle -L^{F/2}\psi_0, \psi_0\big\rangle + \frac{1}{2}\int F\psi_0^2\,dm \\ &\ge \lambda_0(F/2) + \frac{1}{2}\int F\psi_0^2\,dm = \frac{1}{2} + \frac{1}{2}\int F\psi_0^2\,dm. \end{aligned}$$

□

*Henceforth, we will write* $\psi_0$, $\lambda_n$ *and* $\rho$ *for* $\psi_0^F$, $\lambda_n(F)$ *and* $\rho_F$, *respectively*.

**4. Asymptotics for super-Brownian motion at the boundary of its support.** Recall that $X(t,x)$ is the density of super-Brownian motion which solves (1.1). Define

$$
\begin{aligned}
H_u(x) &= uV^1(u,\sqrt{u}x) \\
&= V^{\sqrt{u}}(1,x) \uparrow V^{\infty}(1,x) = F(x), \qquad \text{as } u \uparrow \infty,
\end{aligned}
\tag{4.1}
$$

uniformly on compacts, where we have used (1.7) in the second equation and Proposition 3.1 in the convergence. By (1.11), one obtains the elementary inequality

$$
\begin{aligned}
e^{-V^{\lambda}(t,x)} - e^{-V^{\infty}(t,x)} &\le V^{\infty}(t,x) - V^{\lambda}(t,x) \\
&\le e^{2/t}\bigl(e^{-V^{\lambda}(t,x)} - e^{-V^{\infty}(t,x)}\bigr).
\end{aligned}
\tag{4.2}
$$

NOTATION. We let $p(t,x) = p_t(x)$ denote the standard Brownian density.

In the following lemma, recall that $Y$ is the Ornstein–Uhlenbeck process starting at $x$ under $P_x$.

LEMMA 4.1. *Let $h \ge 0$ be a bounded Borel measurable function on the real line, let $B$ be a standard Brownian motion starting at 0 under $P_0^B$, and set $T = \log(\lambda^2 t)$. Then for $\lambda^2 t \ge 1$ and any finite initial measure $X_0$,*

$$
\begin{aligned}
&E^X_{X_0}\left(\int e^{-\lambda X(t,x)} h(x) X(t,x)\,dx\right) \\
&\quad = E_0^B\biggl(\exp\biggl(-\int_0^1 V^1\bigl(u,B(u)\bigr)\,du\biggr) \\
&\qquad \times E^Y_{B_1}\biggl(\exp\biggl(-\int_0^T H_{e^s}(Y_s)\,ds\biggr) \\
&\qquad \times \int\biggl[h(w_0+\sqrt{t}Y_T) \\
&\qquad \times \exp\biggl(-\frac{1}{t}\int H_{e^T}\biggl(Y_T + \frac{w_0-x_0}{\sqrt{t}}\biggr)\,dX_0(x_0)\biggr)\biggr]\,dX_0(w_0)\biggr)\biggr).
\end{aligned}
\tag{4.3}
$$

PROOF. Let $W_t$ be a Brownian motion starting with initial "law" $X_0$ under the finite measure $E^W_{X_0}$. Apply the Campbell measure formula for $X_t$, or more specifically use Theorem 4.1.1 and then Theorem 4.1.3 of [4] with $\beta = 1$ and

$\gamma = 1/2$ to see that

$$\begin{aligned} &E^X_{X_0}\Big(\int e^{-\lambda X(t,x)} h(x) X(t,x)\,dx\Big) \\ &\quad = E^W_{X_0} \times E^X_{X_0}\Big(h(W_t) \\ &\qquad \times \exp(-\lambda X(t,W_t)) \exp\Big(-\int_0^t V^\lambda(t-s, W_s - W_t)\,ds\Big)\Big). \end{aligned} \tag{4.4}$$

In the above, we approximate $X(t,x)$ by $\int p_\varepsilon(x-y) X_t(dy)$ and let $\varepsilon \downarrow 0$ in order to apply Theorem 4.1.3 in [4]. This limiting argument is easy to justify; use (3.6) with $\phi^{\varepsilon,\lambda} = \lambda p_\varepsilon$ and the bound $V(\phi^{\lambda,\varepsilon})(t-s,x) \le \lambda p_{\varepsilon+t-s}(x) \le \lambda(t-s)^{-1/2}$ to take the limit through the Lebesgue integral in $s$. Now use (3.3) and then the scaling (1.7) to conclude that

$$\begin{aligned} &E^X_{X_0}\Big(\int e^{-\lambda X(t,x)} h(x) X(t,x)\,dx\Big) \\ &\quad = E^W_{X_0}\Big(h(W_t) \exp\Big(-\int_0^t V^\lambda(t-s, W_t - W_s)\,ds \\ &\qquad - \int V^\lambda(t, W_t - x_0)\,dX_0(x_0)\Big)\Big) \\ &\quad = E^W_{X_0}\Big(h(W_t) \exp\Big(-\int_0^t V^1(\lambda^2(t-s), \lambda(W_t - W_s))\lambda^2\,ds \\ &\qquad - \lambda^2 \int V^1(\lambda^2 t, \lambda(W_t - x_0))\,dX_0(x_0)\Big)\Big). \end{aligned} \tag{4.5}$$

If $\hat{W}_s = W_t - W_{t-s}$ and $B_u = \lambda \hat{W}_{\lambda^{-2}u}$ for $u \le \lambda^2 t$, then under $E^W_{X_0}$ and conditional on $W_0$, $B$ is a Brownian motion starting from 0. Noting that $W_t = W_0 + \lambda^{-1} B_{\lambda^2 t}$, we may rewrite (4.5) as

$$\begin{aligned} &E^W_{X_0}\Big(h\Big(W_0 + \frac{1}{\lambda} B_{\lambda^2 t}\Big) \exp\Big(-\int_0^{t\lambda^2} V^1(u, B_u)\,du \\ &\quad - \lambda^2 \int V^1(\lambda^2 t, B_{\lambda^2 t} + \lambda(W_0 - x_0))\,dX_0(x_0)\Big)\Big) \\ &\quad = E^W_{X_0}\Big(\exp\Big(-\int_0^1 V^1(u, B_u)\,du\Big) E^B_{B_1}\Big[h(W_0 + \lambda^{-1} B_{\lambda^2 t - 1}) \\ &\qquad \times \exp\Big(-\int_0^{\lambda^2 t - 1} V^1(1+u, B_u)\,du \\ &\qquad - \lambda^2 \int V^1(\lambda^2 t, B_{\lambda^2 t - 1} + \lambda(W_0 - x_0))\,dX_0(x_0)\Big)\Big]\Big). \end{aligned}$$

Set

$$Y_s = B(e^s - 1)e^{-s/2},$$

which is an Ornstein–Uhlenbeck process starting at $B_1$ under $E^B_{B_1}$. Then (4.5) equals

$$\begin{aligned}
&E^W_{X_0}\biggl(\exp\biggl(-\int_0^1 V^1(u, B_u)\,du\biggr)E^Y_{B_1}\biggl[\exp\biggl(-\int_0^T e^s V^1(e^s, e^{s/2}Y_s)\,ds\biggr) \\
&\quad \times h(W_0 + \sqrt{t}Y_T)\exp\biggl(-\int \frac{e^T}{t} \\
&\quad \times V^1\biggl(e^T, e^{T/2}Y_T + \frac{e^{T/2}}{\sqrt{t}}(W_0 - x_0)\biggr)\,dX_0(x_0)\biggr)\biggr]\biggr) \\
&\quad = E^W_{X_0}\biggl(\exp\biggl(-\int_0^1 V^1(u, B_u)\,du\biggr) \\
&\qquad \times E^Y_{B_1}\biggl[\exp\biggl(-\int_0^T H_{e^s}(Y_s)\,ds\biggr)h(W_0 + \sqrt{t}Y_T) \\
&\qquad \times \exp\biggl(-t^{-1}\int H_{e^T}\biggl(Y_T + \frac{W_0 - x_0}{\sqrt{t}}\biggr)\,dX_0(x_0)\biggr)\biggr]\biggr).
\end{aligned}$$

Recalling that $W_0$ is independent of $B$ under $E^W_{X_0}$, we see that the above equals the required expression. □

We will use the following lemma which will allow us to apply Lemma 4.1 to first get a preliminary bound on $V^\infty - V^\lambda$ for large $\lambda$, and then reapply these results to get exact asymptotics.

LEMMA 4.2. *Assume for some $r > 1$ and $\underline{\lambda} \geq 0$*

$$E^X_{\delta_0}\biggl(\int e^{-\lambda X(t,x)}X(t,x)\,dx\biggr) \leq C(t)\lambda^{-r} \qquad \forall \lambda > \underline{\lambda}.$$

*Then*

$$\sup_x\bigl[V^\infty(t,x) - V^\lambda(t,x)\bigr] \leq e^{6/t}t^{-1/2}C(t/2)(r-1)^{-1}\lambda^{1-r} \qquad \forall \lambda > \underline{\lambda}.$$

PROOF. Recalling (1.8), we have

$$e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} = E^X_{\delta_0}\bigl(e^{-\lambda X(t,x)}1\bigl(X(t,x) > 0\bigr)\bigr).$$

The left-hand side is Lebesgue integrable in $x$ [e.g., it is bounded by $V^\infty(t,x)$ which is Lebesgue integrable by (1.12) and the asymptotics for $F$] and so

$$\int e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)}\,dx = E^X_{\delta_0}\biggl(\int e^{-\lambda X(t,x)}1\bigl(X(t,x) > 0\bigr)\,dx\biggr).$$

It is easy to differentiate with respect to $\lambda > 0$ through the integrals on the right-hand side and so conclude, for any $\lambda > \underline{\lambda}$,

$$-\frac{d}{d\lambda}\left(\int e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)}\,dx\right) = E^X_{\delta_0}\left(\int e^{-\lambda X(t,x)} X(t,x)\,dx\right) \le C(t)\lambda^{-r}. \tag{4.6}$$

For $\lambda > \underline{\lambda}$, integrate the above over $[\lambda, \infty)$ and so deduce from (4.2) that for $\lambda > \underline{\lambda}$,

$$\int V^\infty(t,x) - V^\lambda(t,x)\,dx \le e^{2/t}\int e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)}\,dx \le \frac{e^{2/t}C(t)}{r-1}\lambda^{1-r}. \tag{4.7}$$

Next, use the Markov property of $X$ to see that, for $\lambda > \underline{\lambda}$,

$$\begin{aligned} e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} &= E^X_{\delta_x}\big(e^{-\lambda X(t,0)} 1(X(t,0)>0)\big) \\ &= E^X_{\delta_x}\big(E^X_{X(t/2)}\big(e^{-\lambda X(t/2,0)} 1(X(t/2,0)>0)\big)\big) \\ &= E^X_{\delta_x}\big(e^{-X_{t/2}(V^\lambda_{t/2})} - e^{-X_{t/2}(V^\infty_{t/2})}\big) \\ &\le E^X_{\delta_x}\big(X_{t/2}(V^\infty_{t/2} - V^\lambda_{t/2})\big) \\ &= \int p(t/2, y-x)\big(V^\infty(t/2,y) - V^\lambda(t/2,y)\big)\,dy \\ &\le t^{-1/2}\frac{e^{4/t}C(t/2)}{r-1}\lambda^{1-r}, \end{aligned} \tag{4.8}$$

where we used (4.7) in the last line. Finally, use (4.2) again to obtain the required bound. □

The critical term in (4.3) is $\exp(-\int_0^T H_{e^s}(Y_s)\,ds)$. To estimate its decay rate, we introduce

$$\begin{aligned} Z_T = Z_T(Y) &= \exp\left(\int_0^T F(Y_s) - H_{e^s}(Y_s)\,ds\right) \\ &= \exp\left(\int_0^T V^\infty(1,Y_s) - V^{e^{s/2}}(1,Y_s)\,ds\right) \uparrow Z_\infty(Y) \le \infty \\ &\quad \text{as } T \to \infty. \end{aligned} \tag{4.9}$$

Let $\lambda_0 = \lambda_0(F) \in (\frac{1}{2}, 1)$ be as in Proposition 3.4. Choose $\varepsilon \in (0, 2\lambda_0 - 1)$ and set

$$\delta_{(4.10)} = 2\lambda_0 - \varepsilon > 1. \tag{4.10}$$

LEMMA 4.3. (a) *For all* $t>0$, *there is a* $C(t)$, *nonincreasing in* $t$, *such that* $\sup_x V^\infty(t,x)-V^\lambda(t,x)\le C(t)\lambda^{1-\delta_{(4.10)}}$ *for all* $\lambda>0$.

(b) *There is a constant* $C$ *so that* $Z_\infty - Z_T \le Ce^{-T(\delta_{(4.10)}-1)/2}$ *for all* $T\ge 0$. *In particular,* $Z_\infty$ *is uniformly bounded by some constant* $C_Z$.

PROOF. By Lemma 3.2(b) and (4.1), we may first choose $K$ and then $T_0$ so that

$$\sup_{|x|\ge K} F(x)<\varepsilon/2, \qquad \sup_{s\ge T_0}\sup_{|x|\le K} F(x)-H_{e^s}(x)<\varepsilon/2, \tag{4.11}$$

which in turn implies

$$\sup_{s\ge T_0}\sup_x F(x)-H_{e^s}(x)<\varepsilon/2. \tag{4.12}$$

Now take $h\equiv 1$ in Lemma 4.1, recall that $m$ is the invariant law for $Y$ and use the Markov property at $T_0$ to conclude that, for $T\equiv \log(\lambda^2 t)\ge T_0$,

$$\begin{aligned}
&E^X_{X_0}\left(\int e^{-\lambda X(t,x)}X(t,x)\,dx\right)\Big/X_0(1)\\
&\quad\le E^Y_m\left(\exp\left(-\int_0^T H_{e^s}(Y_s)\,ds\right)\right)\\
&\quad\le E^Y_m\left(E^Y_{Y_{T_0}}\left(\exp\left(-\int_0^{T-T_0} H_{e^{s+T_0}}(Y_s)\,ds\right)\right)\right)\\
&\quad\le E^Y_m\left(\exp\left(-\int_0^{T-T_0} F(Y_s)\,ds\right)\right)\\
&\qquad\times \exp\big((\varepsilon/2)(T-T_0)\big) \qquad \text{by (4.12)}\\
&\quad\le (t\lambda^2)^{\varepsilon/2}P^Y_m(\rho_F>T-T_0),
\end{aligned} \tag{4.13}$$

where we recall that $\rho_F$ is the lifetime of the killed Ornstein–Uhlenbeck process $Y^F$. Now use Theorem 2.3(d) to see that, for $\lambda\ge\lambda'(\varepsilon)$, the far right side of (4.13) is at most

$$(t\lambda^2)^{\varepsilon/2}e^{-\lambda_0(T-T_0)}\left[\theta\int\psi_0\,dm+c'e^{-(\lambda_1-\lambda_0)(T-T_0)}\right]\le c\cdot(\sqrt{t}\lambda)^{-\delta_{(4.10)}},$$

for some universal constant $c$. We now may apply Lemma 4.2 to conclude that

$$\sup_x V^\infty(t,x)-V^\lambda(t,x)\le C(t)\lambda^{1-\delta_{(4.10)}},$$

first for $\lambda>\lambda(\varepsilon)$, and then for all $\lambda>0$, the latter using (1.11) and by increasing $C(t)$. It is easy to use the explicit form for the constant in Lemma 4.2 to see that we may take $C(t)$ to be nonincreasing in $t$.

Turning next to (b), we have, from (4.9) and (a),

$$Z_T\le\exp\left(C(1)\int_0^\infty e^{-(\delta_{(4.10)}-1)s/2}\,ds\right)\equiv c_0.$$

This bound and (a) imply that

$$
\begin{aligned}
Z_\infty - Z_T &\le Z_\infty\Big[1-\exp\Big(-\int_T^\infty V^\infty(1,Y_s) - V^{e^{s/2}}(1,Y_s)\,ds\Big)\Big]\\
&\le c_0 C(1)\int_T^\infty e^{-(\delta_{(4.10)}-1)s/2}\,ds\\
&= \frac{2c_0C(1)}{\delta_{(4.10)}-1}e^{-(\delta_{(4.10)}-1)T/2}.
\end{aligned}
$$

□

Recall the law $P_x^\infty$ from Theorem 2.3(e). We need a slight extension of the latter result.

LEMMA 4.4. *For any $\phi:\mathbb{R}\to\mathbb{R}$ bounded and measurable*,

$$\lim_{T\to\infty} E_x^Y\big(Z_T\phi(Y_T)\mid \rho>T\big) = E_x^\infty(Z_\infty)\int\phi\psi_0\,dm/\theta.$$

PROOF. If $T>T_1>0$, then the monotonicity of $T\to Z_T$ and Lemma 4.3(b) imply that

$$E_x^Y\big(|Z_T - Z_{T_1}||\phi|(Y_T)\mid \rho>T\big)\le C\|\phi\|_\infty e^{-T_1(\delta_{(4.10)}-1)/2}. \tag{4.14}$$

So using this and the bound in Lemma 4.3(b), it clearly suffices to show that, for each $T_1>0$,

$$\lim_{T\to\infty} E_x^Y\big(Z_{T_1}\phi(Y_T)\mid \rho>T\big) = E_x^\infty(Z_{T_1})\int\phi\psi_0\,dm/\theta. \tag{4.15}$$

By the Markov property and the eigenfunction expansion in Theorem 2.3(c), the left-hand side of the above is

$$
\begin{aligned}
&\lim_{T\to\infty} E_x^Y\Big(Z_{T_1}1(\rho>T_1)\int q(T-T_1,Y_{T_1},z)\phi(z)\,dm(z)\Big)\Big/P_x^Y(\rho>T)\\
&\quad= \lim_{T\to\infty} E_x^Y\Big(Z_{T_1}1(\rho>T_1)e^{-\lambda_0(T-T_1)}\\
&\qquad\times\int\phi\psi_0\,dm\,\psi_0(Y_{T_1})\Big)\Big/P_x^Y(\rho>T)\\
&\qquad+\lim_{T\to\infty}\delta(x,T),
\end{aligned} \tag{4.16}
$$

where (recall that $Z_\infty$ is uniformly bounded)

$$
\begin{aligned}
|\delta(x,T)| \le C\|\phi\|_\infty E_x^Y\Big(\int\sum_{n=1}^\infty|\psi_n(Y_{T_1})|\\
\times e^{-\lambda_n(T-T_1)}|\psi_n(z)|\,dm(z)\Big)\Big/P_x^Y(\rho>T).
\end{aligned}
$$

Now use the second inequality in (2.6) with $\delta = 1/8$ to deduce that for $T - T_1 \geq s^*(1/8)$,

$$\begin{aligned}|\delta(x, T)| &\leq C\|\phi\|_\infty e^{-\lambda_1 T} c_\delta E_x^Y\bigl(e^{Y_{T_1}^2/8}\bigr)/P_x^Y(\rho > T)\\ &\leq C'(x)e^{-\lambda_1 T}/P_x^Y(\rho > T) \to 0 \qquad \text{as } T \to \infty,\end{aligned}$$

the last convergence by (2.3) and (2.4). (2.3) also shows that the first term in (4.16) is

$$\begin{aligned}&\lim_{T\to\infty} E_x^Y\bigl(Z_{T_1}\psi_0(Y_{T_1})1(\rho > T_1)\bigr)\frac{e^{\lambda_0 T_1}}{\psi_0(x)\theta + r(T, x)}\int \phi\psi_0\, dm\\ &\quad = E_x^Y\biggl(Z_{T_1}\frac{\psi_0(Y_{T_1})}{\psi_0(x)}1(\rho > T_1)\biggr)e^{\lambda_0 T_1}\int \phi\psi_0\, dm/\theta \qquad \text{[by (2.4)]}\\ &\quad = E_x^\infty(Z_{T_1})\int \phi\psi_0\, dm/\theta.\end{aligned}$$

This establishes (4.15) and so completes the proof. □

PROPOSITION 4.5. *Assume $h \geq 0$ is a bounded Borel function on the line.*

(a) *There is a universal constant $C_{4.5} > 0$ such that, for any $t > 0$,*

$$\begin{aligned}&\lim_{\lambda\to\infty}(\lambda^2 t)^{\lambda_0} E_{X_0}^X\biggl(\int e^{-\lambda X(t,x)}h(x)X(t, x)\, dx\biggr)\\ &\qquad = C_{4.5}\int\int h(w_0 + \sqrt{t}z)\\ &\qquad\quad \times \exp\biggl(-t^{-1}\int F\bigl(z + t^{-1/2}(w_0 - x_0)\bigr)\, dX_0(x_0)\biggr)\\ &\qquad\quad \times \psi_0(z)\, dm(z)\, dX_0(w_0).\end{aligned}$$

(b) *There is a constant $C$ such that, for all $\lambda, t > 0$,*

$$(\lambda^2 t)^{\lambda_0} E_{X_0}^X\biggl(\int e^{-\lambda X(t,x)}h(x)X(t, x)\, dx\biggr) \leq C\|h\|_\infty X_0(1).$$

PROOF. Set $T = \log(\lambda^2 t)$. To simplify the expression obtained in Lemma 4.1, introduce

$$\begin{aligned}I_T(w_0, y) &= \exp\biggl(-t^{-1}\int H_{e^T}\bigl(y + t^{-1/2}(w_0 - x_0)\bigr)\, dX_0(x_0)\biggr),\\ I_\infty(w_0, y) &= \exp\biggl(-t^{-1}\int F\bigl(y + t^{-1/2}(w_0 - x_0)\bigr)\, dX_0(x_0)\biggr)\end{aligned}$$

and

$$\Psi(w_0, x, T) = E_x^Y\biggl(\exp\biggl(-\int_0^T F(Y_s)\, ds\biggr)Z_T h(w_0 + \sqrt{t}Y_T)I_T(w_0, Y_T)\biggr),$$

then Lemma 4.1 states that, for $\lambda^2 t \geq 1$,

$$\begin{aligned} & E^X_{X_0}\left(\int e^{-\lambda X(t,x)} h(x) X(t,x)\,dx\right) \\ & \quad = E^B_0\left(\exp\left(-\int_0^1 V^1(u, B(u))\,du\right) \int \Psi(w_0, B_1, T)\,dX_0(w_0)\right). \end{aligned} \tag{4.17}$$

It follows from (4.1) and Lemma 4.3(a) that

$$0 \leq F(x) - H_u(x) \leq c(1) u^{(1-\delta_{(4.10)})/2} \qquad \forall u > 0, x \in \mathbb{R},$$

and so

$$\begin{aligned} 0 & \leq (I_T - I_\infty)(w_0, Y_T) \\ & \leq \frac{X_0(1)}{t} c(1) e^{T(1-\delta_{(4.10)})/2} \\ & = c(1) X_0(1) t^{-(1+\delta_{(4.10)})/2} \lambda^{1-\delta_{(4.10)}}. \end{aligned} \tag{4.18}$$

Therefore,

$$\begin{aligned} & \frac{\Psi(w_0, x, T)}{P^Y_x(\rho > T)} \\ & \quad = E^Y_x\big(Z_T(Y) h(w_0 + \sqrt{t} Y_T) I_T(w_0, Y_T) \mid \rho > T\big) \\ & \quad = E^Y_x\big(Z_T(Y) h(w_0 + \sqrt{t} Y_T) I_\infty(w_0, Y_T) \mid \rho > T\big) + \delta(T, x, w_0) \end{aligned} \tag{4.19}$$

where, by (4.18),

$$\begin{aligned} & \big|\delta(T, x, w_0)\big| \\ & \quad \leq \|h\|_\infty C_Z c(1) X_0(1) t^{-(1+\delta_{(4.10)})/2} \lambda^{1-\delta_{(4.10)}} \to 0 \qquad \text{as } \lambda \to \infty. \end{aligned} \tag{4.20}$$

Lemma 4.4, together with (4.19) and (4.20), shows that

$$\begin{aligned} & \lim_{\lambda\to\infty} \frac{\Psi(w_0, x, T)}{P^Y_x(\rho > T)} \\ & \quad = E^\infty_x(Z_\infty) \int h(w_0 + \sqrt{t} z) I_\infty(w_0, z) \psi_0(z)\,dm(z)/\theta, \end{aligned} \tag{4.21}$$

and the first line in (4.19) implies

$$\frac{|\Psi(w_0, x, T)|}{P^Y_x(\rho > T)} \leq C_Z \|h\|_\infty. \tag{4.22}$$

Apply (2.3) and (2.4) with $\delta = 1/8$ to conclude

$$\big|e^{\lambda_0 T} P^Y_x(\rho > T) - \theta \psi_0(x)\big| \leq c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0) T}. \tag{4.23}$$

Returning to (4.17), we have for $\lambda^2 t \geq 1$ (assumed henceforth)

$$
\begin{aligned}
&(\lambda^2 t)^{\lambda_0} E_{X_0}^X\Big(\int e^{-\lambda X(t,x)} X(t,x)\,dx\Big)\\
&\quad = E_0^B\Big[\exp\Big(-\int_0^1 V^1\big(u, B(u)\big)\,du\Big)\\
&\qquad \times \int \frac{\Psi(w_0, B_1, T)}{P_{B_1}^Y(\rho > T)}\,dX(w_0) e^{\lambda_0 T} P_{B_1}^Y(\rho > T)\Big] \qquad (4.24)\\
&\quad \to E_0^B\Big(\exp\Big(-\int_0^1 V^1\big(u, B(u)\big)\,du\Big) E_{B_1}^\infty(Z_\infty)\\
&\qquad \times \int h(w_0 + \sqrt{t}z) I_\infty(w_0, z)\psi_0(z)\,dm(z)\,dX_0(w_0)\psi_0(B_1)\Big)
\end{aligned}
$$

as $\lambda \to \infty$, where (4.21) is used in the last and dominated convergence may be applied thanks to (4.22), (4.23) and (2.5). This gives (a) with

$$C_{4.5} = E_0^B\Big(\exp\Big(-\int_0^1 V^1(u, B_u)\,du\Big) E_{B_1}^\infty\big(Z_\infty(Y)\big)\psi_0(B_1)\Big). \tag{4.25}$$

For (b), if $\lambda^2 t \geq 1$, use (4.22), (4.23) and (2.5) to bound the second line in the display (4.24) by

$$c\|h\|_\infty X_0(1) E_0^B\big(e^{B(1)^2/8}\big) \leq c\|h\|_\infty X_0(1).$$

If $\lambda^2 t < 1$, the expression to be bounded is at most $\|h\|_\infty E_{X_0}^X(X_t(1)) = \|h\|_\infty \times X_0(1)$. $\square$

Here is the promised refinement of Lemma 4.3(a) giving the exact rate of convergence in Proposition 3.1(a).

PROPOSITION 4.6. (a) *There is a constant* $\bar{C}_{4.6}$ *such that*

$$\sup_x V^\infty(t,x) - V^\lambda(t,x) \leq \bar{C}_{4.6} t^{-\frac{1}{2}-\lambda_0} \lambda^{-(2\lambda_0 - 1)} \qquad \forall \lambda > 0.$$

(b) *For any* $K \geq 1$ *there is a* $\underline{C}_{4.6}(K) > 0$ *which is nonincreasing in* $K$ *such that, for any* $t > 0$,

$$\inf_{|x| \leq K\sqrt{t}} V^\infty(t,x) - V^\lambda(t,x) \geq \underline{C}_{4.6}(K) t^{-\frac{1}{2}-\lambda_0} \lambda^{-(2\lambda_0 - 1)} \qquad \forall \lambda \geq t^{-1/2} \tag{4.26}$$

*and*

$$
\begin{aligned}
&\inf_{|x| \leq K\sqrt{t}} V^\infty(t,x) - V^\lambda(t,x)\\
&\qquad \geq \underline{C}_{4.6}(K)\big(t^{-1} \wedge t^{-\frac{1}{2}-\lambda_0}\big)\lambda^{-(2\lambda_0 - 1)} \qquad \forall \lambda \geq 1.
\end{aligned} \tag{4.27}
$$

PROOF. (a) Apply Proposition 4.5(b) and Lemma 4.2 to see that, for $\lambda^2 t \geq 1$,

$$\begin{aligned}\sup_x V^\infty(t,x) - V^\lambda(t,x) &\leq e^{6/t} t^{-1/2-\lambda_0} C(2\lambda_0-1)^{-1}\lambda^{-(2\lambda_0-1)}\\ &= C' e^{6/t} t^{-1}(\sqrt{t}\lambda)^{-(2\lambda_0-1)}.\end{aligned}$$

For $\lambda^2 t < 1$, by (1.11) the left-hand side of the above is at most

$$\sup_x V^\infty(t,x) \leq 2t^{-1} \leq 2e^{6/t} t^{-1}(\sqrt{t}\lambda)^{-(2\lambda_0-1)}.$$

This proves (a) but with an additional factor of $e^{6/t}$. This can be removed by applying the scaling result (1.7) and the above bound for $t = 1$ to conclude that, for all $\lambda > 0$,

$$\begin{aligned}V^\infty(t,x) - V^\lambda(t,x) &= t^{-1}\bigl(V^\infty - V^{\sqrt{t}\lambda}\bigr)(1, x/\sqrt{t})\\ &\leq \bar{C} t^{-1}(\sqrt{t}\lambda)^{-(2\lambda_0-1)}.\end{aligned}\tag{4.28}$$

(b) Set $h = 1_{[-K,K]}$ in Proposition 4.5 with $X_0 = \delta_0$ and argue as in the first line of (4.6) to see that, for $\lambda \geq \underline{\lambda}(t,K)$ and a universal positive constant $c$,

$$\begin{aligned}&-\frac{d}{d\lambda}\int_{-K}^{K} e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)}\,dx\\ &\quad= E_{\delta_0}\biggl(\int e^{-\lambda X(t,x)} h(x) X(t,x)\,dx\biggr)\\ &\quad\geq c\int_{-K/\sqrt{t}}^{K/\sqrt{t}} \exp\bigl(-t^{-1}F(z)\bigr)\psi_0(z)\,dm(z) t^{-\lambda_0}\lambda^{-2\lambda_0}\\ &\quad= c_0(t,K)\lambda^{-2\lambda_0}.\end{aligned}\tag{4.29}$$

Integrate out $\lambda$ to conclude that, for $\lambda \geq \underline{\lambda}(t,K)$,

$$\begin{aligned}\int_{-K}^{K} V^\infty(t,x) - V^\lambda(t,x)\,dx &\geq \int_{-K}^{K} e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)}\,dx\\ &\geq \frac{c_0(t,K)}{2\lambda_0 - 1}\lambda^{1-2\lambda_0}.\end{aligned}\tag{4.30}$$

If $0 < \lambda < \underline{\lambda}(t,K)$, then

$$\begin{aligned}\int_{-K}^{K}\bigl(V^\infty - V^\lambda\bigr)(t,x)\,dx &\geq \int_{-K}^{K}\bigl(V^\infty - V^{\underline{\lambda}(t,K)}\bigr)(t,x)\,dx\\ &\equiv c_1(t,K) > 0.\end{aligned}\tag{4.31}$$

The last inequality holds since the first line of (4.8) shows strict positivity of $V^\infty(t,x) - V^\lambda(t,x)$ for all $t > 0$ and $x$. (4.30) and (4.31) imply that, for some $c_2(t,K) > 0$,

$$\int_{-K}^{K}\bigl(V^\infty - V^\lambda\bigr)(t,x)\,dx \geq c_2(t,K)\lambda^{1-2\lambda_0} \qquad \forall \lambda \geq 1.\tag{4.32}$$

From the third line of (4.8), we have

$$\begin{aligned}
&e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} \\
&\quad = E_{\delta_x}\big(e^{-X_{t/2}(V^\lambda_{t/2})} - e^{-X_{t/2}(V^\infty_{t/2})}\big) \\
&\quad \geq E_{\delta_x}\big(e^{-X_{t/2}(V^\infty_{t/2})} X_{t/2}\big(V^\infty_{t/2} - V^\lambda_{t/2}\big)\big) \\
&\quad \geq E_{\delta_x}\big(e^{-(4/t)X_{t/2}(1)} X_{t/2}\big(V^\infty_{t/2} - V^\lambda_{t/2}\big)\big) \qquad \text{[by (1.11)]}.
\end{aligned}$$

So if $r = r(t) = 4/t$ and $G = G(t) = V^\infty_{t/2} - V^\lambda_{t/2}$, we seek a lower bound on

$$E^X_{\delta_x}\big(e^{-rX_{t/2}(1)} X_{t/2}(G)\big). \tag{4.33}$$

If $V_t(\phi)$ denotes the nonlinear semigroup associated with $X$ (see Section II.5 of [18]), we may use the Campbell measure formula for $X_t$ as in (4.4) along with $V_s(r) = \frac{2r}{2+rs}$ (i.e., the nonlinear semigroup with $\phi \equiv r$ constant) to see that if $W$ is a Brownian motion starting at $x$, then for $|x| \leq K$, (4.33) equals

$$\begin{aligned}
&E^W_x \times E^X_{\delta_x}\left(e^{-rX_{t/2}(1)} \exp\left(-\int_0^{t/2} \frac{2r}{2+rs}\,ds\right) G(W_{t/2})\right) \\
&\quad = \exp\left(\frac{-2r}{2+(rt/2)}\right)\left(1+\frac{rt}{4}\right)^{-2} \int p_{t/2}(y-x) G(y)\,dy \\
&\quad \geq c(t,K) \int_{-K}^{K} G(y)\,dy \\
&\quad \geq c'(t,K)\lambda^{-(2\lambda_0 - 1)} \qquad \text{for all } \lambda \geq 1,
\end{aligned} \tag{4.34}$$

the last by (4.32). Now use the first inequality in (4.2) and the above to derive

$$\inf_{|x|\leq K} V^\infty(t,x) - V^\lambda(t,x) \geq c'(t,K)\lambda^{-(2\lambda_0-1)} \qquad \text{for all } \lambda \geq 1,$$

and where we may assume $c'(t,K) > 0$ is nonincreasing in $K$ for each $t$.

The scaling relation in (4.28) and the above bound for $t = 1$ shows that for all $\lambda \geq t^{-1/2}$,

$$\begin{aligned}
\inf_{|x|\leq K\sqrt{t}} V^\infty(t,x) - V^\lambda(t,x) &= \inf_{|x|\leq K} t^{-1}\big(V^\infty - V^{\sqrt{t}\lambda}\big)(1,x) \\
&\geq c'(1,K) t^{-\frac{1}{2}-\lambda_0} \lambda^{-(2\lambda_0-1)}.
\end{aligned} \tag{4.35}$$

This gives (4.26) and it remains to prove (4.27). By (4.26), we may assume $t < 1$ and $1 \leq \lambda \leq t^{-1/2}$. Then $c(K) \equiv \inf_{|x|\leq K} V^\infty(1,x) - V^1(1,x) > 0$, where the last inequality holds by the strict positivity of the difference for each $x$ as noted above. By scaling, as in the above display, the left-hand side of (4.35) is at least $t^{-1}c(K)$ which implies (4.27). $\square$

The following Tauberian theorem is implicit in Theorem 1 of [5] (see especially page 350). The explicit constants below are not given there but follow from an elementary argument which can be found in an Appendix of [12].

LEMMA 4.7. *Let $U$ be the distribution function of a sub-probability on* $(0,\infty)$, *set* $\hat U(\lambda)=\int_0^\infty e^{-\lambda x}\,dU(x)$ *and let* $p>0$.

(a) *Assume for some* $C_2>0$

$$\hat U(\lambda)\le C_2\lambda^{-p}\qquad \text{for all } \lambda>0. \tag{4.36}$$

*Then* $U(a)\le eC_2a^p$ *for all* $a>0$.

(b) *Assume* (4.36) *and, for some* $C_1>0$, $\underline{\lambda}\ge 0$,

$$\hat U(\lambda)\ge C_1\lambda^{-p}\qquad \text{for all } \lambda>\underline{\lambda}. \tag{4.37}$$

*If* $d_1=\frac{C_1}{2}(2\log((\frac{2p}{e})^p\frac{4eC_2}{C_1})\vee 2p\vee\underline{\lambda})^{-p}$, *then*

$$U(a)\ge d_1a^p\qquad \text{for all } a\in[0,1]. \tag{4.38}$$

*In particular if* $p\le 1$ *and* $\underline{\lambda}\le 4$, *then*

$$U(a)\ge\frac{C_1}{4}\left(\log\left(\frac{4eC_2}{C_1}\right)\right)^{-1}a^p\qquad \text{for all } a\in[0,1]. \tag{4.39}$$

THEOREM 4.8. *Let* $X(t,x)$ *be the density of super-Brownian motion satisfying* (1.1) *with finite initial measure* $X_0$.

(a) $P^X_{X_0}(0<X(t,x)\le a)\le e\bar C_{4.6}X_0(1)t^{-(1/2)-\lambda_0}a^{2\lambda_0-1}$ $\forall a,t>0, x\in\mathbb{R}$.

(b) *For all* $K\ge 1$, *there is a* $\underline{C}_{4.8}(K)>0$ *such that if* $X_0(1)\le Kt$ *and* $X_0([x-K\sqrt{t},x+K\sqrt{t}])/X_0(1)\ge K^{-2}$, *then*

$$\begin{aligned}&P^X_{X_0}(0<X(t,x)\le a)\\&\quad\ge\underline{C}_{4.8}(K)X_0(1)t^{-(1/2)-\lambda_0}a^{2\lambda_0-1}\qquad\forall 0\le a\le\sqrt{t}.\end{aligned}\tag{4.40}$$

*In particular if* $|x-x_0|\le K\sqrt{t}$ *and* $t\ge K^{-1}$, *then*

$$P^X_{\delta_{x_0}}(0<X(t,x)\le a)\ge\underline{C}_{4.8}(K)t^{-(1/2)-\lambda_0}a^{2\lambda_0-1}\qquad\forall 0\le a\le\sqrt{t}, \tag{4.41}$$

*and if* $X_0(1)\le K$, $X_0([-K,K])\ge K^{-1}$, $t\ge K^{-1}$, *and* $|x|\le K$, *then*

$$\begin{aligned}&P^X_{X_0}(0<X(t,x)\le a)\\&\quad\ge\underline{C}_{4.8}(K^2)K^{-1}t^{-(1/2)-\lambda_0}a^{2\lambda_0-1}\qquad\forall 0\le a\le\sqrt{t}.\end{aligned}\tag{4.42}$$

PROOF. We will apply Lemma 4.7 to $U(a) = P(0 < X(t,x) \le a)$. By (1.9), (1.10) and translation invariance,

$$\begin{aligned}\hat{U}(\lambda) &= E^X_{X_0}\big(e^{-\lambda X(t,x)} 1\big(X(t,x) > 0\big)\big)\\ &= \exp\Big(-\int V^\lambda(t, y-x)\, dX_0(y)\Big)\\ &\quad - \exp\Big(-\int V^\infty(t, y-x)\, dX_0(y)\Big).\end{aligned} \tag{4.43}$$

(a) Use Proposition 4.6(a) to see that, for $\lambda > 0$,

$$\begin{aligned}\hat{U}(\lambda) &\le \int (V^\infty - V^\lambda)(t, y-x)\, dX_0(y)\\ &\le \bar{C}_{4.6} X_0(1) t^{-(1/2)-\lambda_0} \lambda^{-(2\lambda_0 - 1)},\end{aligned} \tag{4.44}$$

and so (a) is immediate from Lemma 4.7(a).

(b) Consider first $t = 1$. By (4.43), (1.11) and (4.26) for $\lambda \ge 1$,

$$\begin{aligned}\hat{U}(\lambda) &\ge \exp\Big(-\int V^\infty(1, y-x)\, dX_0(y)\Big) \int (V^\infty - V^\lambda)(1, y-x)\, dX_0(y)\\ &\ge \exp(-2X_0(1)) \underline{C}_{4.6}(K) X_0\big([x-K, x+K]\big) \lambda^{-(2\lambda_0 - 1)}\\ &\equiv C_1(K, x, X_0) \lambda^{-(2\lambda_0 - 1)}.\end{aligned}$$

So by this and (4.44), we may use (4.39) in Lemma 4.7 (with $p = 2\lambda_0 - 1 < 1$) to see that

$$\begin{aligned}&P^X_{X_0}\big(0 < X(1,x) \le a\big)\\ &\quad \ge \frac{C_1(K, x, X_0)}{4} \Big(\log\Big(\frac{4e\bar{C}_{4.6} X_0(1)}{C_1(K, x, X_0)}\Big)\Big)^{-1} a^{2\lambda_0 - 1} \qquad \forall a \in [0,1].\end{aligned}$$

For general $t$, we may use the scaling relation (4.28) and (4.43) to see that if $X_0^t(A) = t^{-1} X_0(\sqrt{t} A)$, then by the above for $0 \le a \le \sqrt{t}$,

$$\begin{aligned}&P^X_{X_0}\big(0 < X(t,x) \le a\big)\\ &\quad = P^X_{X_0^t}\big(0 < X(1, x/\sqrt{t}) \le a/\sqrt{t}\big)\\ &\quad \ge \frac{C_1(K, x/\sqrt{t}, X_0^t)}{4} \Big(\log\Big(\frac{4e\bar{C}_{4.6} X_0(1)/t}{C_1(K, x/\sqrt{t}, X_0^t)}\Big)\Big)^{-1} (a/\sqrt{t})^{2\lambda_0 - 1}.\end{aligned} \tag{4.45}$$

A simple calculation, using the definition of $C_1$, shows that there is a $C_0(K) > 0$ so that if $X_0$ is as in (b), then

$$C_1\big(K, x/\sqrt{t}, X_0^t\big) \ge C_0(K) X_0(1)/t,$$

and so

$$\log\left(\frac{4e\bar{C}_{4.6}X_0(1)/t}{C_1(K,x,X_0)}\right) \le C_3(K).$$

Use the above in (4.45) to conclude that

$$P^X_{X_0}\big(0 < X(t,x) \le a\big) \ge C(K)X_0(1)t^{-1}(a/\sqrt{t})^{2\lambda_0-1}.$$

This proves (4.40), and the last two assertions follow by elementary reasoning. (The last follows first for $K \ge 4$, and hence for all $K \ge 1$.) $\square$

## 5. Proof of Theorem 1.3.

5.1. *Lower bound on the Hausdorff dimension.* Recall that $X$ is as in (1.1) and the boundary of the zero set is

$$\begin{aligned} BZ_t &= \partial\big(\{x : X(t,x) = 0\}\big) \\ &= \big\{x : X(t,x) = 0, \forall\delta > 0, X_t\big((x-\delta, x+\delta)\big) > 0\big\}. \end{aligned} \tag{5.1}$$

The key step in our lower bound on the Hausdorff dimension of the boundary of the zero set, $\dim(BZ_t)$, is the following second moment bound. The bound will depend on a diffusion parameter $\sigma_0^2 > 1$, whose exact value is not important (but $\sigma_0^2 = 6$ will work). For a finite initial measure $X_0$ and $t > 0$, define $X_0 p_u(x) = \int p_u(x - w_0)X_0(dw_0)$ and

$$\begin{aligned} h_{t,X_0}(z_1,z_2) &= t^{-2\lambda_0}\prod_{i=1}^{2} X_0 p_{\sigma_0^2 t}(z_i) \\ &\quad + t^{-\lambda_0}\int_0^t (t-s)^{-\lambda_0} p_{8\sigma_0^2(t-s)}(z_1 - z_2) X_0 p_{4\sigma_0^2 t}(z_1)\,ds. \end{aligned} \tag{5.2}$$

PROPOSITION 5.1. *There is a constant $C_{5.1}$ such that for all $\lambda^2 \ge (9/t)$, and all $z_1, z_2$:*

(a) $\lambda^{4\lambda_0} E^X_{X_0}(X_t(z_1)X_t(z_2)e^{-\lambda X_t(z_1) - \lambda X_t(z_2)}) \le C_{5.1} h_{t,X_0}(z_1,z_2)$.
(b) $h_{t,X_0}(z_1,z_2) \le C_{5.1}(t^{-\lambda_0-1/2}X_0(1)|z_1 - z_2|^{1-2\lambda_0} + (t^{-\lambda_0-1/2}X_0(1))^2)$.

We will prove this result in Section 6 below, but first show how it can be used to obtain lower bounds on $\dim(BZ_t)$. The lack of symmetry between $z_1$ and $z_2$ in the definition of $h_{t,X_0}$ indicates that our bound is not optimal, but it is the negative power along the diagonal which will be important for us, and our results suggest that this is optimal.

If we introduce random measures

$$L^\lambda_t(\phi) = \big(\lambda^2 t\big)^{\lambda_0}\int \phi(x)X(t,x)e^{-\lambda X(t,x)}\,dx, \tag{5.3}$$

then Proposition 4.5 shows that for some finite measure $\ell_t$ and any bounded Borel function $\phi$,

$$\lim_{\lambda\to\infty} E\big(L_t^\lambda(\phi)\big) = \ell_t(\phi),$$

and the above result shows that $E(L_t^\lambda(1)^2)$ remains bounded as $\lambda \to \infty$.

CONJECTURE. *There is a random finite nontrivial measure $L_t$ on $\mathbb{R}$ such that for any bounded continuous $\phi$,*

$$L_t^\lambda(\phi) \to L_t(\phi) \qquad \text{in } L^2 \text{ as } \lambda \to \infty.$$

*Assuming this, it is then not hard to show that for some sequence $\lambda_n \uparrow \infty$, $L_t^{\lambda_n}$ approaches $L_t$ weakly on the space of measures a.s. and that $L_t$ is supported by $BZ_t$. We further conjecture that $L_t(1) > 0$ a.s. on $\{X_t(1) > 0\}$.*

If $g_\beta(r) = r^{-\beta}$ for $\beta > 0$, and $\mu$ is a finite measure on $\mathbb{R}$, and $A$ is an analytic subset of $\mathbb{R}$, let

$$\langle \mu, \mu\rangle_{g_\beta} = \int\int g_\beta\big(|x-y|\big)\,d\mu(x)\,d\mu(y),$$

$$I(g_\beta)(A) = \inf\big\{\langle \mu, \mu\rangle_{g_\beta} : \mu \text{ a probability supported by } A\big\}.$$

Then the $g_\beta$-capacity of $A$ is $C(g_\beta)(A) = I(g_\beta)(A)^{-1}$ (see, e.g., [8], Section 3).

THEOREM 5.2. *For every $K \ge 1$, there is a positive constant $C_{5.2}(K)$, nonincreasing in $K$, so that for any analytic subset $A$ of $[-K, K]$, initial measure, $X_0$, satisfying $X_0(1) \le K$ and $X_0([-K, K]) \ge 1/K$, and $t \in [K^{-1}, K]$,*

$$P^X_{X_0}(A \cap BZ_t \ne \varnothing) \ge C_{5.2}(K)C(g_{2\lambda_0-1})(A).$$

PROOF. Let $0 < \delta_0 < e^{-1}$, and let $0 < k_1 < 1 < k_2$ be the solutions of $\delta_0 = k_i e^{-k_i}$. We will choose $\delta_0$ small enough below, noting that as $\delta \downarrow 0$, $k_1(\delta) \downarrow 0$ and $k_2(\delta) \uparrow \infty$. We approximate $BZ_t$ by

$$BZ(\varepsilon) \equiv \big\{x : X(t,x)e^{-X(t,x)/\varepsilon} \ge \delta_0\varepsilon\big\} = \big\{x : k_1\varepsilon \le X(t,x) \le k_2\varepsilon\big\},$$

where the second equality is by an elementary calculus argument and we have suppressed dependence on $t > 0$. Now fix $K \ge 1$ and assume $X_0$ and $t$ are is in the statement of the theorem. Let $F$ be a compact subset of $[-K, K]$. If $I(A) = I(g_{2\lambda_0-1})(A)$ and $C(A) = C(g_{2\lambda_0-1})(A)$, we may choose $\{x_i^N : 1 \le i \le N\} \subset F$ so that (suppressing the superscript $N$) as $N \to \infty$,

$$I_N \equiv \frac{1}{N(N-1)}\sum_i\sum_{j\ne i}|x_i - x_j|^{-(2\lambda_0-1)} \to I(F) = 1/C(F). \tag{5.4}$$

(See, e.g., Lemma A of [20].) By Theorem 4.8, there are constants $\bar{C}(K) > \underline{C}(K) > 0$ so that for $0 < \varepsilon < \varepsilon_0(K)$,

$$P\big(k_1\varepsilon \le X(t, x_i) \le k_2\varepsilon\big) \ge \big[\underline{C}(K)k_2^{2\lambda_0-1} - \bar{C}(K)k_1^{2\lambda_0-1}\big]\varepsilon^{2\lambda_0-1} \ge \varepsilon^{2\lambda_0-1},$$

where in the last line we have chosen $\delta_0 = \delta_0(K)$ sufficiently small so that $k_2$ is very large and $k_1$ is close to 0. Therefore, by inclusion-exclusion and Proposition 5.1, for $\varepsilon < \varepsilon_0(K)$,

$$\begin{aligned}
&P^X_{X_0}\big(F \cap BZ(\varepsilon) \ne \varnothing\big)\\
&\quad\ge \sum_{i=1}^{N} P^X_{X_0}\big(x_j \in BZ(\varepsilon)\big) - \sum\sum\nolimits_{i\ne j} P^X_{X_0}\big(x_i, x_j \in BZ(\varepsilon)\big)\\
&\quad\ge N\varepsilon^{2\lambda_0-1} - \sum_{i\ne j} \frac{E^X_{X_0}(X(t, x_i)X(t, x_j)e^{-X(t,x_i)/\varepsilon - X(t,x_j)/\varepsilon})}{(\delta_0\varepsilon)^2}\\
&\quad\ge N\varepsilon^{2\lambda_0-1} - c'(K)\sum_{i\ne j}\big(1 + |x_i - x_j|^{1-2\lambda_0}\big)\varepsilon^{4\lambda_0-2}\\
&\quad\ge N\varepsilon^{2\lambda_0-1} - c(K)\big[N\varepsilon^{2\lambda_0-1}\big]^2 I_N.
\end{aligned}$$

Now choose $\varepsilon_N \to 0$ so that $N\varepsilon_N^{2\lambda_0-1} = \frac{1}{2I_N c(K)}$. Therefore,

$$P^X_{X_0}\big(F \cap BZ(\varepsilon_N) \ne \varnothing\big) \ge \frac{1}{4c(K)I_N} \to \big(4c(K)\big)^{-1}C(F) \qquad \text{as } N \to \infty.$$

This implies that

$$P^X_{X_0}\big(F \cap BZ(\varepsilon_N) \ne \varnothing \text{ infinitely often}\big) \ge \big(4c(K)\big)^{-1}C(F).$$

An elementary argument shows that the event on the left-hand side implies that $F \cap BZ_t \ne \varnothing$ and so the proof is complete for $A = F$ compact. Use the inner regularity of capacity to now extend the result to analytic subsets of $[-K, K]$. □

COROLLARY 5.3. *Let $A$ be an analytic set such that $C(g_{2\lambda_0-1})(A) > 0$. Then for any nonzero $X_0$ and any $t > 0$, $P^X_{X_0}(A \cap BZ_t \ne \varnothing) > 0$.*

PROOF. Choose $K$ large enough so that $C(g_{2\lambda_0-1})(A \cap [-K, K]) > 0$ (inner regularity of capacity), $X_0([-K, K]) \ge 1/K$, and $K > t \vee t^{-1} \vee X_0(1)$. The result is then immediate from the above theorem. □

LEMMA 5.4. *Let $\alpha = 2\lambda_0 - 1$ and let $Z$ be the subordinator starting at zero with Lévy measure $\nu$ where*

$$H(x) = \nu\big([x, \infty)\big) = x^{-\alpha}\big(\log\big((1/x) + 1\big)\big)^2.$$

*Then*

(5.5) $$C(g_\alpha)\big(\{Z_s : s \in (0,1)\}\big) > 0 \qquad a.s.$$

*and*

(5.6) *any analytic set $A$ satisfying* $\dim(A) < 2 - 2\lambda_0$ *is polar for $Z$, that is*, $P(Z_t \in A$ *for some* $t > 0) = 0$.

PROOF. If

$$g(\lambda) = \int_0^\infty 1 - e^{-\lambda u}\, d\nu(u) = \lambda \int_0^\infty H(u) e^{-\lambda u}\, du,$$

then Karamata's Abelian–Tauberian theorem and a short calculation shows that for some $c > 0$, $\lim_{\lambda\to\infty} g(\lambda)/(\lambda^\alpha(\log\lambda)^2) = c$. If

$$f(r) = r^\alpha\big(\log(1/r)\big)^{-2}\big(\log\log(1/r)\big)^{1-\alpha}$$

and $f - m(A)$ is the $f$-Hausdorff measure of $A$, then [6] (see also Lemma 2.1 of [7]) shows that

$$f - m\big(\{Z_s : s \le t\}\big) = c_H t \qquad \text{for some positive } c_H.$$

By [20], this implies (5.5). Note that $\lim_{t\to 0+} \frac{tg(1/t)}{t^{1-\alpha}\log(1/t)^2} = c$, and so by Theorem 4.4(iii) of [7], (5.6) also holds. (A short calculus argument shows that $(\log H)'' \ge 0$ and so by Theorem 2.1 of [7], the condition B of Theorem 4.4(iii) is valid.) $\square$

Recall that $\dim(B)$ is the Hausdorff dimension of a set $B \subset \mathbb{R}$.

THEOREM 5.5. *If $X_0 \ne 0$ and $t > 0$, then* $P^X_{X_0}(\dim(BZ_t) \ge 2 - 2\lambda_0) > 0$.

PROOF. Let $Z_t$ be as in Lemma 5.4 and set $F = \{Z_s : s \in (0,1)\}$. We assume $Z$ is independent of the super-Brownian motion $X$ and so work on the product space $(\Omega, \mathcal{F}, P) = (\Omega_X, \mathcal{F}_X, P^X_{X_0}) \times (\Omega_Z, \mathcal{F}_Z, P^Z_0)$. By (5.5) and Corollary 5.3, we have $P(BZ_t(\omega_1) \cap F(\omega_2) \ne \varnothing) > 0$. This implies that

$$P^X_{X_0}\big(\big\{\omega_1 : P^Z_0\big(\{\omega_2 : F(\omega_2) \cap BZ_t(\omega_1) \ne \varnothing\}\big) > 0\big\}\big) > 0.$$

By (5.6), this in turn implies that $P^X_{X_0}(\dim(BZ_t) \ge 2 - 2\lambda_0) > 0$, as required. $\square$

5.2. *Upper bound on the Hausdorff dimension.* We begin with a classical result on the modulus of continuity of the density $X(t,x)$, $t > 0$, of super-Brownian motion, the solution of (1.1), where the initial condition is an arbitrary finite measure $X_0$.

NOTATION. If $(t_i, x_i) \in \mathbb{R}_+ \times \mathbb{R}$ for $i = 1, 2$, let

$$d\big((t_1, x_1), (t_2, x_2)\big) = \sqrt{|t_1 - t_2|} + |x_1 - x_2|.$$

The following is an easy consequence of Theorem II.4.2 (and its proof) of [18] and standard consequences of Kolmogorov's continuity criteria. (One should use the decomposition (III.4.11) in [18] for $t \geq 2t_0$.)

PROPOSITION 5.6. *If $\xi \in (0, 1/2)$, then for any $K \in \mathbb{N}$ there is a $\rho(K, \xi, \omega) > 0$ a.s. such that*

$$\begin{aligned} &\forall (t, x) \in [K^{-1}, K] \times [-K, K], \\ &\forall (t', x') \in [0, \infty) \times \mathbb{R},\ d\big((t, x), (t', x')\big) \leq \rho \\ &\textit{implies } \big|X(t', x') - X(t, x)\big| \leq d\big((t', x'), (t, x)\big)^{\xi}. \end{aligned} \tag{5.7}$$

*Moreover, there is a $\delta_{5.6} > 0$, depending only on $\xi$, and a constant $C(X_0(1), K, \xi)$ so that*

$$P^X_{X_0}\big(\rho(K, \xi) \leq r\big) \leq C\big(X_0(1), K, \xi\big) r^{\delta_{5.6}} \qquad \textit{for all } r > 0. \tag{5.8}$$

Near the zero set $Z = \{(t, x) : X(t, x) = 0\}$ one can improve the above modulus since the noise term in (1.1) will be mollified. This idea plays a central role in the pathwise uniqueness arguments in [17] and [16]. The following result can be derived using the same proof as that of Theorem 2.3 in [16] (see also Corollary 4.2 of [17]). There are a few minor changes as these references study the difference of two solutions as opposed to the solution itself. The minor changes that are required are outlined in an Appendix of [12].

THEOREM 5.7. *If $\xi \in (0, 1)$, then for any $K \in \mathbb{N}$ there is a $\rho_{5.7}(K, \xi, \omega) > 0$ a.s. such that*

$$\begin{aligned} &\forall (t, x) \in Z \textit{ such that } t \geq K^{-1}, \\ &\forall (t', x') \in [0, \infty) \times \mathbb{R}, d\big((t, x), (t', x')\big) \leq \rho_{5.7} \\ &\textit{implies } X(t', x') \leq d\big((t', x'), (t, x)\big)^{\xi}. \end{aligned} \tag{5.9}$$

In order to get a good cover of $BZ_t$, we need a version of our low density bound Theorem 4.8(a) for small intervals. (Set $M = 1$ and $\varepsilon = a$ in the following result to compare.)

THEOREM 5.8. *There is a $C_{5.8} > 0$ so that for all $t > 0$, $M \geq 1$, $0 < \varepsilon \leq \sqrt{t}$, $x \in \mathbb{R}$ and $X_0$,*

$$P_{X_0}\big(0 < X_t([x, x + \varepsilon]) \leq \varepsilon^2 M\big) \leq C_{5.8} X_0(1) t^{-(1/2)-\lambda_0} M^{19} \varepsilon^{2\lambda_0 - 1}.$$

This will be proved in Section 7 below. We now show how it gives an upper bound on $\dim(BZ_t)$.

THEOREM 5.9. *For all* $t > 0$, $\dim(BZ_t) \le 2 - 2\lambda_0$ $P^X_{X_0}$-*a.s.*

PROOF. By scaling we may take $t = 1$. By translation invariance, it suffices to show

$$\dim\big(BZ_1 \cap [0, 1]\big) \le 2 - 2\lambda_0 P^X_{X_0}\text{-a.s.} \tag{5.10}$$

Fix $\delta > 0$ and choose $\xi \in (0, 1)$ so that $19(1 - \xi) < \delta$. Let $\rho_{5.7}(\xi, \omega)$ be as in Theorem 5.7 with $K = 2$. Then by Theorems 5.8 and 5.7 for any $x \in [0, 1]$ and $0 < \varepsilon \le 1$,

$$\begin{aligned}
&P^X_{X_0}\big([x, x + \varepsilon] \cap BZ_1 \ne \varnothing, 3\varepsilon < \rho_{5.7}\big)\\
&\qquad \le P^X_{X_0}\big(0 < X_1\big((x - \varepsilon, x + 2\varepsilon)\big) \le 6\varepsilon^{\xi+1}\big)\\
&\qquad \le C X_0(1)\varepsilon^{2\lambda_0 - 1 + 19(\xi - 1)}\\
&\qquad \le C X_0(1)\varepsilon^{2\lambda_0 - 1 - \delta}.
\end{aligned}$$

A standard covering argument using intervals of the form $[i/N, (i + 1)/N]$ now gives $\dim(BZ_1 \cap [0, 1]) \le 2 - 2\lambda_0 + \delta$ and (5.10) follows. □

Theorem 1.3 is now immediate from Theorems 5.5 and 5.9.

**6. Proof of Proposition 5.1.** We now consider the proof of Proposition 5.1. As before, $Y$ is the Ornstein–Uhlenbeck process starting with law $\mu$ under $P^Y_\mu$ and we enlarge this space to include an independent random variable $W_0$ with "law" $X_0$. The same convention is in place on the space carrying a standard Brownian motion starting at 0 under $P^B_0$, and $P'_t$ denotes the Brownian semigroup. We fix a pair of bounded nonnegative continuous functions on the line, $\phi_i$, $i = 1, 2$ and define

$$\psi^\lambda(u, x) = E^B_0\left(\phi_2(\lambda^{-1} B_{u\lambda^2} + x) \exp\left\{-\int_0^{\lambda^2 u} V^1(r, B_r)\, dr\right\}\right). \tag{6.1}$$

Recall from the end of Section 3 that $\rho = \rho_F$. We note that the parameters $\lambda_i \ge 0$, $i = 1, 2$, in the following result are not the eigenvalues $\lambda_i(F)$ from Section 3.

LEMMA 6.1. *There is a constant* $C_{6.1}$ *so that for all* $\lambda_i \ge 0$, $\lambda_i^2 t \ge 1$, *if* $T^i = \log(\lambda_i^2 t)$, *then*

$$
\begin{aligned}
&E^X_{X_0}\left(\iint \phi_1(x_1)\phi(x_2)\exp\bigl(-\lambda_1 X_t(x_1) - \lambda_2 X_t(x_2)\bigr)\,dX_t(x_1)\,dX_t(x_2)\right) \\
&\quad \le C_{6.1}\prod_{i=1}^{2} E^Y_m\bigl(\phi_i(W_0 + \sqrt{t}Y_{T^i})1(\rho > T^i)\bigr) \\
&\qquad + \int_0^t E^B_0\left(\int \phi_1(W_0 + B_t)\exp\left(-\int_0^t V^{\lambda_1}_{t-r}(B_t - B_r)\,dr\right)\right. \\
&\qquad\qquad \left. \times\, \psi^{\lambda_2}(t-s, W_0 + B_s)\right) ds.
\end{aligned} \tag{6.2}
$$

PROOF. For $\lambda_i \ge 0$, $x_i \in \mathbb{R}$ $(i = 1, 2)$ and $\delta > 0$ let $\vec{\lambda} = (\lambda_1, \lambda_2)$, $\vec{x} = (x_1, x_2)$ and $V_t(x) = V_t^{\delta,\vec{\lambda},\vec{x}}(x)$ be the unique smooth solution of

$$
\frac{\partial V}{\partial t} = \frac{\Delta V}{2} - \frac{V^2}{2}, \qquad V_0(\cdot) = \lambda_1 p_\delta(\cdot - x_1) + \lambda_2 p_\delta(\cdot - x_2), \tag{6.3}
$$

so that

$$
E^X_{X_0}\bigl(e^{-\lambda_1 P'_\delta X_t(x_1) - \lambda_2 P'_\delta X_t(x_2)}\bigr) = \exp\bigl(-X_0(V_t)\bigr) \tag{6.4}
$$

[recall (3.5) and (3.6)]. Let $U_t^{(j)}(x) \equiv U_t^{(j),\delta,\vec{\lambda},\vec{x}}(x)$ denote the unique solution of

$$
\frac{\partial U_t^{(j)}}{\partial t} = \frac{\Delta U_t^{(j)}}{2} - V_t^{\delta,\vec{\lambda},\vec{x}} U_t^{(j)}, \qquad U_0^{(j)}(\cdot) = p_\delta(\cdot - x_j), \tag{6.5}
$$

so that by Feynmann–Kac (see page 268 of [10]),

$$
U_t^{(j)}(x) = E^B_x\left(p_\delta(B_t - x_j)\exp\left(-\int_0^t V_{t-s}^{\delta,\vec{\lambda},\vec{x}}(B_s)\,ds\right)\right). \tag{6.6}
$$

Next, define

$$
\tilde{V}_t^{\delta,\vec{\lambda},\vec{x}}(x) = V_t^{\delta,0,\lambda_2,\vec{x}}(x) + \int_0^{\lambda_1} U_t^{(1),\delta,\lambda,\lambda_2,\vec{x}}(x)\,d\lambda. \tag{6.7}
$$

In the above, it is easy to justify differentiation with respect to $t$ and $x$ through the integral and so by (6.5) we have

$$
\frac{\partial \tilde{V}}{\partial t} - \frac{\Delta \tilde{V}}{2} = -\frac{(V_t^{\delta,0,\lambda_2,\vec{x}})^2}{2} - \int_0^{\lambda_1} V_t^{\delta,\lambda,\lambda_2,\vec{x}} U_t^{(1),\delta,\lambda,\lambda_2,\vec{x}}\,d\lambda, \tag{6.8}
$$

and by integration by parts,

$$
\frac{(\tilde{V}_t^{\delta,\vec{\lambda},\vec{x}})^2}{2} = \frac{(V_t^{\delta,0,\lambda_2,\vec{x}})^2}{2} + \int_0^{\lambda_1} V_t^{\delta,\lambda,\lambda_2,\vec{x}} U_t^{(1),\delta,\lambda,\lambda_2,\vec{x}}\,d\lambda. \tag{6.9}
$$

A quick check of the initial condition at $t=0$ and a comparison of (6.8) and (6.9) show that $\tilde{V}$ satisfies (6.3) and so $\tilde{V}=V$. Continuity of $V$ in $\vec{\lambda}$ is clear from (6.4), and hence continuity of $U^{(j)}$ in $\vec{\lambda}$ follows from (6.6). This allows us to differentiate (6.7) with respect to $\lambda_1$ and conclude

$$\frac{\partial V^{\delta,\vec{\lambda},\vec{x}}}{\partial \lambda_1} = U^{(1),\delta,\vec{\lambda},\vec{x}}, \quad \text{and symmetrically,} \quad \frac{\partial V^{\delta,\vec{\lambda},\vec{x}}}{\partial \lambda_2} = U^{(2),\delta,\vec{\lambda},\vec{x}}. \tag{6.10}$$

We can differentiate the left-hand side of (6.4) (set $X_0=\delta_x$) with respect to $\lambda_1$ and then $\lambda_2$ under the integral. This shows that $U_t^{\delta,\vec{\lambda},\vec{x}}(x) = \frac{\partial^2}{\partial\lambda_2\partial\lambda_1}(V^{\delta,\vec{\lambda},\vec{x}})$ exists and is continuous in $\vec{\lambda}$, and the differentiation yields [use (6.10)]

$$\begin{aligned} &E_{X_0}^X\biggl(\prod_{i=1}^2 \bigl(P_\delta' X_t(x_i) e^{-\lambda_i P_\delta' X_t(x_i)}\bigr)\biggr) \\ &\quad = \exp\bigl(-X_0\bigl(V_t^{\delta,\vec{\lambda},\vec{x}}\bigr)\bigr)\bigl[X_0\bigl(U^{(1),\delta,\vec{\lambda},\vec{x}}\bigr)X_0\bigl(U^{(2),\delta,\vec{\lambda},\vec{x}}\bigr) - X_0\bigl(U^{\delta,\vec{\lambda},\vec{x}}\bigr)\bigr] \\ &\quad \equiv T_1(\delta,\vec{\lambda},\vec{x}) - T_2(\delta,\vec{\lambda},\vec{x}). \end{aligned} \tag{6.11}$$

It is clear from (3.6) that if $V^{\delta,\lambda_i,x_i}$ is the solution to (6.3) with initial condition $V_0^{\delta,\lambda_i,x_i}(\cdot) = \lambda_i p_\delta(\cdot - x_i)$, then

$$V^{\delta,\vec{\lambda},\vec{x}} \geq V^{\delta,\lambda_i,x_i} \qquad \text{for } i=1,2. \tag{6.12}$$

Using the above and (6.6), we see that

$$\begin{aligned} &\iint \phi_1(x_1)\phi_2(x_2) T_1(\delta,\vec{\lambda},x_1,x_2)\,dx_1\,dx_2 \\ &\quad \leq \iint \prod_{i=1}^2 \phi_i(x_i) E_0^B\biggl(p_\delta(W_0+B_t-x_i) \\ &\qquad \times \exp\biggl(-\int_0^t V_{t-s}^{\delta,\lambda_i,x_i}(W_0+B_s)\,ds\biggr)\biggr)\,dx_1\,dx_2. \end{aligned}$$

The above equals

$$\begin{aligned} &\iint \prod_{i=1}^2 \phi_i(x_i) E_0^B\biggl(p_\delta(W_0+B_t-x_i) \\ &\quad \times \exp\biggl(-\int_0^t V_{t-s}^{\delta,\lambda_i,0}(W_0+B_s-x_i)\,ds\biggr)\biggr)\,dx_1\,dx_2 \\ &\quad = \prod_{i=1}^2 E_0^B\biggl(\phi_i(W_0+B_{t+\delta})\exp\biggl(-\int_0^t V_{t-s}^{\delta,\lambda_i,0}(B_s-B_{t+\delta})\,ds\biggr)\biggr). \end{aligned} \tag{6.13}$$

An elementary argument using (3.6) shows that

$$\lim_{\delta\downarrow 0}\exp\bigl(-V_{t-s}^{\delta,\lambda_i,0}(B_s-B_{t-\delta})\bigr)=\exp\bigl(-V_{t-s}^{\lambda_i}(B_s-B_t)\bigr). \tag{6.14}$$

The elementary bound $V(\phi)(t,x)\le P_t'(\phi)(x)$ for $\phi$ nonnegative, bounded and measurable shows that

$$V_{t-s}^{\delta,\lambda_i,0}(B_s-B_{t+\delta})\le P_{t-s}'(\lambda_i p_\delta)(B_s-B_{t+\delta})\le \lambda_i(t-s)^{-1/2}. \tag{6.15}$$

The above two results allow us to apply dominated convergence in (6.13) and conclude that

$$\begin{aligned}
&\limsup_{\delta\downarrow 0}\iint \phi_1(x_1)\phi_2(x_2)T_1(\delta,\vec{\lambda},x_1,x_2)\,dx_1\,dx_2\\
&\quad\le\prod_{i=1}^2 E_0^B\Bigl(\phi_i(W_0+B_t)\exp\Bigl(-\int_0^t V^{\lambda_i}(t-s,B_t-B_s)\,ds\Bigr)\Bigr)\\
&\quad=\prod_{i=1}^2 E_0^B\Bigl(\phi_i(W_0+B_t)\\
&\qquad\times\exp\Bigl(-\int_0^t V^{\lambda_i}(s,B_s)\,ds\Bigr)\Bigr)\qquad\text{(time reversal)}\\
&\quad=\prod_{i=1}^2 E_0^B\Bigl(\phi_i(W_0+\lambda_i^{-1}B_{\lambda_i^2 t})\\
&\qquad\times\exp\Bigl(-\int_0^{\lambda_i^2 t} V^1(u,B_u)\,du\Bigr)\Bigr)\qquad\text{[by the scaling (1.7)]}\\
&\quad\le\prod_{i=1}^2 E_m^B\Bigl(\phi_i(W_0+\lambda_i^{-1}B_{\lambda_i^2 t-1})\\
&\qquad\times\exp\Bigl(-\int_0^{\lambda_i^2 t-1}(1+u)V^1(1+u,B_u)(1+u)^{-1}\,du\Bigr)\Bigr),
\end{aligned}$$

where we have used the Markov property at $t=1$ in the last line. Now proceed as in the proof of Proposition 4.5, using Lemma 4.3(b), to conclude that

$$\begin{aligned}
&\limsup_{\delta\downarrow 0}\iint \phi_1(x_1)\phi_2(x_2)T_1(\delta,\vec{\lambda},x_1,x_2)\,dx_1\,dx_2\\
&\qquad\le C\prod_{i=1}^2 E_m^Y\bigl(\phi_i(W_0+\sqrt{t}Y_{T^i})1(\rho>T^i)\bigr).
\end{aligned}\tag{6.16}$$

Consider next the contribution from $T_2$ in (6.11). By (6.10) and (6.6), we have

$$\frac{\partial}{\partial\lambda_2}V_{t-s}^{\delta,\vec{\lambda},\vec{x}}(x)\le E_x^B\bigl(p_\delta(B_{t-s}-x_2)\bigr)\le (t-s)^{-1/2}.$$

As the above bound is Lebesgue integrable, the dominated convergence theorem and (6.10) give

$$\frac{\partial}{\partial\lambda_2}\int_0^t V_{t-s}^{\delta,\vec{\lambda},\vec{x}}(B_s)\,ds=\int_0^t U_{t-s}^{(2),\delta,\vec{\lambda},\vec{x}}(B_s)\,ds.$$

This in turn allows us to differentiate (6.6) (with $j=1$) with respect to $\lambda_2$ and conclude

$$\begin{aligned}U_t^{\delta,\vec{\lambda},\vec{x}}(x)=-E_x^B\biggl(&p_\delta(B_t-x_1)\exp\biggl(-\int_0^t V_{t-r}^{\delta,\vec{\lambda},\vec{x}}(B_r)\,dr\biggr)\\&\times\int_0^t U_{t-s}^{(2),\delta,\vec{\lambda},\vec{x}}(B_s)\,ds\biggr).\end{aligned}$$

Therefore, $-T_2(\delta,\vec{\lambda},\vec{x})\geq 0$ and

$$\begin{aligned}-\iint &T_2(\delta,\vec{\lambda},\vec{x})\phi_1(x_1)\phi_2(x_2)\,dx_1\,dx_2\\
&\leq\iint E_0^B\biggl(p_\delta(W_0+B_t-x_1)\\
&\qquad\times\exp\biggl(-\int_0^t V_{t-r}^{\delta,\vec{\lambda},\vec{x}}(W_0+B_r)\,dr\biggr)\phi_1(x_1)\\
&\qquad\times\int_0^t U_{t-s}^{(2),\delta,\vec{\lambda},\vec{x}}(W_0+B_s)\,ds\biggr)\phi_2(x_2)\,dx_1\,dx_2\\
&\leq\int_0^t E_0^B\biggl(\int p_\delta(W_0+B_t-x_1)\\
&\qquad\times\exp\biggl(-\int_0^t V_{t-r}^{\delta,\lambda_1,x_1}(W_0+B_r)\,dr\biggr)\phi_1(x_1)\\
&\qquad\times\int E_{B(s)+W_0}^{\hat{B}}\biggl(p_\delta(\hat{B}_{t-s}-x_2)\\
&\qquad\times\exp\biggl(-\int_0^{t-s} V_{t-s-u}^{\delta,\lambda_2,x_2}(\hat{B}_u)\,du\biggr)\biggr)\phi_2(x_2)\,dx_2\,dx_1\biggr)\,ds.\end{aligned}$$

In the last line, $\hat{B}$ is a Brownian motion and we have used (6.12) and (6.6). The above equals

$$\begin{aligned}&\int_0^t E_0^B\biggl(\phi_1(W_0+B_{t+\delta})\exp\biggl(-\int_0^t V_{t-r}^{\delta,\lambda_1,0}(B_r-B_{t+\delta})\,dr\biggr)\\
&\quad\times E_{B_s+W_0}^{\hat{B}}\biggl(\phi_2(\hat{B}_{t-s+\delta})\exp\biggl(-\int_0^{t-s} V_{t-s-u}^{\delta,\lambda_2,0}(\hat{B}_u-\hat{B}_{t-s})\,du\biggr)\biggr)\biggr)\,ds\\
&\quad\to\int_0^t E_0^B\biggl(\phi_1(W_0+B_t)\exp\biggl(-\int_0^t V_{t-r}^{\lambda_1}(B_t-B_r)\,dr\biggr)\\
&\qquad\times E_{B_s+W_0}^{\hat{B}}\biggl(\phi_2(\hat{B}_{t-s})\exp\biggl(-\int_0^{t-s} V_{t-s-u}^{\lambda_2}(\hat{B}_{t-s}-\hat{B}_u)\,du\biggr)\biggr)\biggr)\,ds,\end{aligned}$$

as $\delta \downarrow 0$. Here, we used dominated convergence and (6.15) as in (6.14). If $B'_u = \hat{B}_{t-s} - \hat{B}_{t-s-u}$ (a Brownian motion starting at 0), then the above shows that

$$\limsup_{\delta\downarrow 0}\Big[-\iint T_2(\delta,\vec{\lambda},\vec{x})\phi_1(x_1)\phi_2(x_2)\,dx_1\,dx_2\Big]$$

$$\begin{aligned} &\le \int_0^t E_0^B\Big(\phi_1(W_0+B_t)\exp\Big(-\int_0^t V^{\lambda_1}_{t-r}(B_t-B_r)\,dr\Big) \\ &\qquad\times E_0^{B'}\Big(\phi_2(W_0+B_s+B'_{t-s})\exp\Big(-\int_0^{t-s} V_r^{\lambda_2}(B'_r)\,dr\Big)\Big)\Big)\,ds \\ &= \int_0^t E_0^B\Big(\phi_1(W_0+B_t)\exp\Big(-\int_0^t V^{\lambda_1}(t-r,B_t-B_r)\,dr\Big) \\ &\qquad\times \psi^{\lambda_2}(t-s,W_0+B_s)\Big)\,ds. \end{aligned} \tag{6.17}$$

The last line is an easy consequence of the scaling relation (1.7). Now use (6.11) and Fatou's lemma to see that

$$\begin{aligned} &E^X_{X_0}\Big(\iint \phi_1(x_1)\phi_2(x_2)X_t(x_1)X_t(x_2)e^{-\lambda_1 X_t(x_1)-\lambda_2 X_t(x_2)}\,dx_1\,dx_2\Big) \\ &\quad\le \liminf_{\delta\downarrow 0}\Big[\iint \prod_{i=1}^2 \phi_i(x_i)T_1(\delta,\vec{\lambda},\vec{x})\,dx_1\,dx_2 \\ &\qquad + \iint \prod_{i=1}^2 \phi_i(x_i)\big(-T_2(\delta,\vec{\lambda},\vec{x})\big)\,dx_1\,dx_2\Big]. \end{aligned}$$

Finally, apply (6.16) and (6.17) to bound the above by the required expression. □

PROOF OF PROPOSITION 5.1. Let $T_1$ and $T_2$ denote the first and second terms, respectively on the right-hand side of (6.2), where $t$ and $\lambda = \lambda_1 = \lambda_2$ are fixed as in Proposition 5.1, and let $T = \log(\lambda^2 t)$. Consider first the much easier $T_1$. Recall from (2.3)–(2.5) we have, for any $\delta > 0$,

$$P_x^Y(\rho > T) \le c_\delta e^{\delta x^2} e^{-\lambda_0 T}. \tag{6.18}$$

Therefore,

$$\begin{aligned} &\lambda^{2\lambda_0} E_m^Y(\phi_i(W_0+\sqrt{t}Y_T)1(\rho > T)) \\ &\quad= \lambda^{2\lambda_0}\iiint \phi_i(w_0+\sqrt{t}y)q_T(y_0,y)\,dm(y_0)\,dm(y)\,dX_0(w_0) \\ &\quad\le \iint \phi_i(w_0+\sqrt{t}y)\lambda^{2\lambda_0}P_y^Y(\rho > T)\,dm(y)\,dX_0(w_0) \\ &\quad\le \iint \phi_i(w_0+\sqrt{t}y)t^{-\lambda_0}c_\delta e^{\delta y^2}\,dm(y)\,dX_0(w_0), \end{aligned}$$

the last by (6.18). Now let $\sigma^2 = (1-2\delta)^{-1}$. A simple substitution now shows the above is at most

$$ct^{-\lambda_0} \int \phi_i(x_i) X_0 p_{t\sigma^2}(x_i)\, dx_i.$$

This in turn implies

$$\lambda^{4\lambda_0} T_1 \le c_{\sigma^2} t^{-2\lambda_0} \iint \prod_{i=1}^{2} [\phi_i(x_i) X_0 p_{t\sigma^2}(x_i)]\, dx_1\, dx_2, \tag{6.19}$$

where here $\sigma^2$ is any number greater than 1. Below we will choose a convenient value of $\sigma^2$ when doing the $T_2$ bound.

Turning now to $T_2$, we can write $T_2 = \int_0^t T_2(s)\, ds$, where $T_2(s)$ is the integrand on the right-hand side of (6.2). We may replace the Brownian motion $B_t$ with $\lambda^{-1} B_{t\lambda^2}$ (the new $B$ is still a Brownian motion starting at 0) and use the scaling relation (1.7) to conclude after a short and familiar argument that

$$\begin{aligned} T_2(s) = E_0^B\Big(&\phi_1(W_0 + \lambda^{-1} B_{t\lambda^2}) \psi^\lambda\big(t-s, W_0 + \lambda^{-1}(B_{\lambda^2 t} - B_{\lambda^2(t-s)})\big) \\ &\times \exp\Big(-\int_0^{\lambda^2 t} V^1(r, B_r)\, dr\Big)\Big). \end{aligned} \tag{6.20}$$

*Case* 1. Assume $\lambda^2(t-s) > 1$.

Apply the Markov property of $B$ at time 1 to the right-hand side of (6.20) and conclude that

$$\begin{aligned} T_2(s) \le E_m^B\Big(&\phi_1(W_0 + \lambda^{-1} B_{t\lambda^2 - 1}) \psi^\lambda\big(t-s, \lambda^{-1}(B_{t\lambda^2-1} - B_{\lambda^2(t-s)-1})\big) \\ &\times \exp\Big(-\int_0^{\lambda^2 t - 1} V^1(r+1, B_r)\, dr\Big)\Big), \end{aligned} \tag{6.21}$$

where $B$ remains independent of $W_0$. As before, $Y(u) = B(e^u - 1)e^{-u/2}$ is a stationary Ornstein–Uhlenbeck process. If $U = \log(\lambda^2(t-s))$, then arguing as in the proof of Lemma 4.1, we may re-express (6.21) as

$$\begin{aligned} T_2(s) &\le E_m^Y\Big(\phi_1(W_0 + \sqrt{t} Y_T) \psi^\lambda(t-s, W_0 + \sqrt{t} Y_T - \sqrt{t-s} Y_U) \\ &\qquad \times \exp\Big(-\int_0^T H_{e^u}(Y_u)\, du\Big)\Big) \\ &\le e^{Cz} E_m^Y\Big(\phi_1(W_0 + \sqrt{t} Y_T) \psi^\lambda(t-s, W_0 + \sqrt{t} Y_T - \sqrt{t-s} Y_U) \\ &\qquad \times \exp\Big(-\int_0^T F(Y_u)\, du\Big)\Big), \end{aligned} \tag{6.22}$$

where we used Lemma 4.3 in the last inequality. The above equals

$$
\begin{aligned}
&e^{Cz}\int \phi_1(w_0+\sqrt{t}y_2)\psi^\lambda(t-s,w_0+\sqrt{t}y_2-\sqrt{t-s}\,y_1)q_U(y_0,y_1)\\
&\quad\times q_{T-U}(y_1,y_2)\,dm(y_0)\,dm(y_1)\,dm(y_2)\,dX_0(w_0)\\
&\quad\le C_{\delta_1}\int \phi_1(w_0+\sqrt{t}y_2)\psi^\lambda(t-s,w_0+\sqrt{t}y_2-\sqrt{t-s}\,y_1)\\
&\qquad\times e^{\delta_1 y_1^2}(t-s)^{-\lambda_0}\lambda^{-2\lambda_0}q_{T-U}(y_1,y_2)\,dm(y_1)\,dm(y_2)\,dX_0(w_0),
\end{aligned}
\tag{6.23}
$$

where in the last line, $1/2>\delta_1>0$, and we used (6.18) and the symmetry of $q_U$ in integrating out $y_0$.

At this point, we take a break from the long proof and obtain a bound on $\psi^\lambda$.

LEMMA 6.2. *For any $\sigma_0^2>1$, there is a $C_{6.2}(\sigma_0^2)$ such that for all $x$ and all $0\le s<t$,*

$$\psi^\lambda(t-s,x)\le C_{6.2}\lambda^{-2\lambda_0}(t-s)^{-\lambda_0}P'_{\sigma_0^2(t-s)}\phi_2(x).$$

PROOF. If $\lambda^2(t-s)\le 1$, we can drop the negative exponential in the definition of $\psi^\lambda$ and note that $\lambda^{-2\lambda_0}(t-s)^{-\lambda_0}\ge 1$ to conclude that

$$\psi^\lambda(t-s,x)\le \lambda^{-2\lambda_0}(t-s)^{-\lambda_0}P'_{t-s}\phi_2(x),$$

from which the required bound follows easily.

Assume now that $\lambda^2(t-s)>1$. If $Y$ and $U$ are as above, then the same reasoning leading to (6.22) and (6.23) above [compare the definition of $\psi^\lambda$ with the right-hand side of (6.20) without the $\psi^\lambda$] leads to (for any $0<\delta<1/2$)

$$
\begin{aligned}
\psi^\lambda(t-s,x)&\le CE_m^Y\Big(\phi_2(\sqrt{t-s}Y_U+x)\exp\Big(-\int_0^U F(Y_u)\,du\Big)\Big)\\
&\le C_\delta(t-s)^{-\lambda_0}\lambda^{-2\lambda_0}\int \phi_2(x+\sqrt{t-s}y)e^{\delta y^2}\,dm(y).
\end{aligned}
\tag{6.24}
$$

An elementary argument now gives the required bound with $\sigma_0^2=(1-2\delta)^{-1}$. □

Returning to the proof of Proposition 5.1 in Case 1, we use the above lemma in (6.23) and then the substitution $z_1=w_0+\sqrt{t}y_2$, to obtain

$$
\begin{aligned}
T_2(s)&\le C(\sigma_0^2,\delta_1)\lambda^{-4\lambda_0}(t-s)^{-2\lambda_0}\\
&\quad\times\iiint \phi_1(w_0+\sqrt{t}y_2)\\
&\qquad\times P'_{\sigma_0^2(t-s)}\phi_2(w_0+\sqrt{t}y_2-\sqrt{t-s}\,y_1)e^{\delta_1 y_1^2}\\
&\qquad\times q_{T-U}(y_1,y_2)\,dm(y_1)\,dm(y_2)\,dX_0(w_0)
\end{aligned}
\tag{6.25}
$$

$$
\begin{aligned}
&\le C(\sigma_0^2,\delta_1)\lambda^{-4\lambda_0}(t-s)^{-2\lambda_0}\iint \phi_1(z_1)\phi_2(z_2)\\
&\quad\times\biggl[\iint p_{\sigma_0^2(t-s)}(z_2-z_1+\sqrt{t-s}\,y_1)\\
&\quad\times q_{\log(t/t-s)}\bigl(y_1,(z_1-w_0)t^{-1/2}\bigr)\\
&\quad\times e^{\delta_1 y_1^2}p_t(z_1-w_0)\,dm(y_1)\,dX_0(w_0)\biggr]\,dz_1\,dz_2.
\end{aligned}
$$

*Case* 1a. Assume also $t/2\le s$.

Let $\delta=\frac{\sqrt{2}-1}{2}$ for which $e^{-s^*(\delta)}=1/2$ [recall that $s^*(\delta)$ is as in (2.1)]. Therefore, $\log(t/t-s)\ge\log 2=s^*(\delta)$, and so by (2.2),

$$
q_{\log(t/t-s)}\bigl(y_1,(z_1-w_0)t^{-1/2}\bigr)\le ce^{-\lambda_0\log(t/t-s)}\exp\bigl(\delta\bigl(y_1^2+(z_1-w_0)^2t^{-1}\bigr)\bigr).
$$

If $\delta_1=\delta$ and $\sigma_0^2=(1-4\delta)^{-1}=(3-2\sqrt{2})^{-1}\le 6$, this implies that the expression in square brackets in (6.25) is at most

$$
\begin{aligned}
&ct^{-\lambda_0}(t-s)^{\lambda_0}\iint p_{\sigma_0^2(t-s)}(z_2-z_1+\sqrt{t-s}\,y_1)e^{2\delta y_1^2}\,dm(y_1)\\
&\quad\times\exp\bigl(\delta(z_1-w_0)^2/t\bigr)p_t(z_1-w_0)\,dX_0(w_0)\\
&\quad\le ct^{-\lambda_0}(t-s)^{\lambda_0}\iint p_{\sigma_0^2(t-s)}(z_2-z_1+w)\\
&\qquad\times\exp\bigl(-(1-4\delta)w^2/\bigl(2(t-s)\bigr)\bigr)\bigl(2\pi(t-s)\bigr)^{-1/2}\,dw\\
&\qquad\times\exp\bigl(-(1-2\delta)(z_1-w_0)^2/2t\bigr)(2\pi t)^{-1/2}\,dX_0(w_0)\\
&\quad\le ct^{-\lambda_0}(t-s)^{\lambda_0}\int p_{\sigma_0^2(t-s)}(z_2-z_1+w)p_{\sigma_0^2(t-s)}(w)\,dw\,X_0p_{\sigma_0^2t}(z_1)\\
&\quad= ct^{-\lambda_0}(t-s)^{\lambda_0}p_{2\sigma_0^2(t-s)}(z_2-z_1)X_0p_{\sigma_0^2t}(z_1).
\end{aligned}
$$

Use the above in (6.25) to conclude that in Case 1a,

$$
\begin{aligned}
T_2(s)&\le C\lambda^{-4\lambda_0}(t-s)^{-\lambda_0}t^{-\lambda_0}\\
&\quad\times\int\phi_1(z_1)\phi_2(z_2)p_{2\sigma_0^2(t-s)}(z_2-z_1)X_0p_{\sigma_0^2t}(z_1)\,dz_1\,dz_2.
\end{aligned}
\tag{6.26}
$$

*Case* 1b. Assume $0\le s<t/2$ $(\le t-\lambda^{-2})$.

The last inequality is immediate by our hypothesis that $\lambda^2t\ge 9>2$. Return to (6.25) with the choices of $\delta_1$ and $\sigma_0^2$ made in the previous case and let $R=\log(t/t-s)$. Bounding the transition density (with respect to Lebesgue measure) of the killed Ornstein–Uhlenbeck process starting at $y_1$ by the unkilled process and noting the latter has a normal density with mean $y_1e^{-R/2}$ and variance $1-e^{-R}$, we get

$$q_R(y_1, z)e^{-z^2/2}(2\pi)^{-1/2} \\ \le \exp\left(\frac{-(z - y_1e^{-R/2})^2}{2(1 - e^{-R})}\right)(2\pi)^{-1/2}(1 - e^{-R})^{-1/2}.$$

Setting $z = \frac{z_1 - w_0}{\sqrt{t}}$ and simplifying, this becomes

$$q_R(y_1, (z_1 - w_0)/\sqrt{t})p_t(z_1 - w_0) \le p_s(z_1 - w_0 - \sqrt{t-s}\,y_1).$$

Now use this in (6.25) to conclude

$$\begin{aligned} T_2(s) \le C\lambda^{-4\lambda_0}(t-s)^{-2\lambda_0} \iint \phi_1(z_1)\phi_2(z_2) \\ \times \int\left[\int p_{\sigma_0^2(t-s)}(z_2 - z_1 + \sqrt{t-s}\,y_1)e^{\delta_1 y_1^2}\right. \\ \left.\times\, p_s(z_1 - w_0 - \sqrt{t-s}\,y_1)\,dm(y_1)\right]dX_0(w_0)\,dz_1\,dz_2. \end{aligned} \tag{6.27}$$

The fact that $\sigma_0^2 = (1-4\delta_1)^{-1} > (1-2\delta_1)^{-1} > 1$ and a simple substitution shows that the term in square brackets is at most

$$\begin{aligned} f_{s,t}(z_1, z_2, w_0) \\ = \int p_{\sigma_0^2(t-s)}(z_2 - z_1 + w)p_{\sigma_0^2(t-s)}(w)p_{\sigma_0^2 s}(z_1 - w_0 - w)\,dw. \end{aligned} \tag{6.28}$$

We claim that

$$f_{s,t}(z_1, z_2, w_0) \le cp_{8\sigma_0^2(t-s)}(z_2 - z_1)p_{4\sigma_0^2 t}(z_1 - w_0). \tag{6.29}$$

By scaling, it suffices to obtain the above for $\sigma_0 = 1$. Set $a = z_1 - z_2$ and $b = z_1 - w_0$, so that

$$f_{s,t}(z_1, s_2, w_0) = \int p_{t-s}(w-a)p_{t-s}(w)p_s(b-w)\,dw.$$

A simple calculation (complete the square) shows that

$$f_{s,t}(z_1, s_2, w_0) = (2\pi)^{-1}(t^2 - s^2)^{-1/2}\exp\left(-\frac{a^2t + 2b^2(t-s) - 2ab(t-s)}{2(t^2 - s^2)}\right).$$

Now use $ab \le \alpha a^2 + (4\alpha)^{-1}b^2$ with $\alpha = 5/16$ to see that

$$f_{s,t}(z_1, s_2, w_0) \\ \le (2\pi)^{-1}(t^2 - s^2)^{-1/2}\exp\left(-\frac{a^2(1-2\alpha)}{2(t+s)}\right)\exp\left(-\frac{b^2(2-(2\alpha)^{-1})}{2(t+s)}\right)$$

$$
\begin{aligned}
&\le (2\pi)^{-1}(t^2-s^2)^{-1/2}\exp\left(-\frac{a^2(1-2\alpha)}{6(t-s)}\right)\exp\left(-\frac{b^2(2-(2\alpha)^{-1})}{3t}\right)\\
&\le (2\pi)^{-1}(t-s)^{-1/2}\exp\left(-\frac{a^2}{16(t-s)}\right)t^{-1/2}\exp\left(-\frac{b^2}{8t}\right),
\end{aligned}
$$

where we use $s < t/2$ in the next to last line and the value of $\alpha$ in the last line. This completes the proof of (6.29).

Now insert (6.29) into (6.27), noting that $(t-s)^{-\lambda_0} \le ct^{-\lambda_0}$, to conclude that in Case 1b,

$$
\begin{aligned}
T_2(s) &\le C\lambda^{-4\lambda_0}(t-s)^{-\lambda_0}t^{-\lambda_0}\\
&\quad\times \iint \phi_1(z_1)\phi_2(z_2)p_{8\sigma_0^2(t-s)}(z_2-z_1)X_0p_{4\sigma_0^2 t}(z_1)\,dz_1\,dz_2.
\end{aligned} \tag{6.30}
$$

*Case* 2. Assume $\lambda^2(t-s) \le 1$.

Use (6.20), the Markov property at time 1 and then argue as in (6.23) to see that

$$
\begin{aligned}
T_2(s) &\le E_0^B\biggl(E^B_{B_1(\omega)}\biggl(\phi_1(W_0+\lambda^{-1}B_{\lambda^2 t-1})\\
&\qquad\times \psi^\lambda(t-s, W_0+\lambda^{-1}(B_{\lambda^2 t-1}-B_{\lambda^2(t-s)}(\omega)))\\
&\qquad\times \exp\left(-\int_0^{\lambda^2 t-1} V^1(r+1,B_r)\,dr\right)\biggr)\biggr)\\
&\le C\iint p_{\lambda^2(t-s)}(x_0)p_{1-\lambda^2(t-s)}(x_1-x_0)\\
&\qquad\times E^Y_{x_1}\Bigl(\phi_1(W_0+\sqrt{t}Y_T)\psi^\lambda(t-s, W_0+\sqrt{t}Y_T-x_0\lambda^{-1})\\
&\qquad\times \exp\left(-\int_0^T F(Y_u)\,du\right)\Bigr)\,dx_1\,dx_0.
\end{aligned}
$$

By definition, $\psi^\lambda(t-s,x) \le P'_{t-s}\phi_2(x)$, and so arguing as in the derivation of (6.25) we get

$$
\begin{aligned}
T_2(s) &\le C\iint \phi_1(z_1)\phi_2(z_2)\biggl[\iiint p_{t-s}\left(z_2-z_1+\frac{x_0}{\lambda}\right)p_{\lambda^2(t-s)}(x_0)\\
&\qquad\times p_{1-\lambda^2(t-s)}(x_1-x_0)q_T(x_1,(z_1-w_0)t^{-1/2})\\
&\qquad\times p_t(z_1-w_0)\,dx_1\,dx_0\,dX_0(w_0)\biggr]\,dz_1\,dz_2.
\end{aligned} \tag{6.31}
$$

Let $g_{s,t}(z_1,z_2)$ denote the expression in square brackets. A simple calculation shows that our condition $\lambda^2 t \ge 9$ implies $T \ge s^*(1/8)$, and so, by (2.2),

$$
q_T(x_1,(z_1-w_0)t^{-1/2}) \le C(\lambda^2 t)^{-\lambda_0}\exp\left(\frac{1}{8}\left(x_1^2+\frac{(z_1-x_0)^2}{t}\right)\right).
$$

First, use this in (6.31), and then set $\sigma_1^2 = 4/3$ and use $1 - \lambda^2(t-s) \le 1$ and an easy calculation to obtain

$$\begin{aligned} &g_{s,t}(z_1, z_2) \\ &\quad \le C\lambda^{-2\lambda_0} t^{-\lambda_0} \iint p_{t-s}\big(z_2 - z_1 + (x_0/\lambda)\big) p_{\lambda^2(t-s)}(x_0) \\ &\qquad \times p_{1-\lambda^2(t-s)}(x_1 - x_0) e^{x_1^2/8}\, dx_1\, dx_0\, p_{\sigma^2 t}(z_1 - w_0)\, dX_0(w_0) \\ &\quad \le C\lambda^{-2\lambda_0} t^{-\lambda_0} \int p_{t-s}\big(z_2 - z_1 + (x_0/\lambda)\big) p_{\lambda^2(t-s)}(x_0) \\ &\qquad \times e^{x_0^2/4}\, dx_0\, X_0 p_{\sigma_1^2 t}(z_1). \end{aligned}$$

If $\sigma_2^2 = \frac{\lambda^2(t-s)}{1-\lambda^2(t-s)/2}$, then

$$\lambda^2(t-s) \le \sigma_2^2 \le 2\lambda^2(t-s) \tag{6.32}$$

and so

$$e^{x_0^2/4} p_{\lambda^2(t-s)}(x_0) = p_{\sigma_2^2}(x_0) \sqrt{\frac{\sigma_2^2}{\lambda^2(t-s)}} \le \sqrt{2}\, p_{\sigma_2^2}(x_0),$$

which in turn implies

$$\begin{aligned} g_{s,t}(z_1, z_2) &\le C\lambda^{-2\lambda_0} t^{-\lambda_0} \int p_{t-s}\big(z_2 - z_1 + (x_0/\lambda)\big) p_{\sigma_2^2}(x_0)\, dx_0\, X_0 p_{\sigma_1^2}(z_1) \\ &= C\lambda^{-2\lambda_0} t^{-\lambda_0} p_{(\sigma_2^2/\lambda^2)+(t-s)}(z_2 - z_1) X_0 p_{\sigma_1^2}(z_1) \\ &\le C\lambda^{-2\lambda_0} t^{-\lambda_0} p_{3(t-s)}(z_2 - z_1) X_0 p_{\sigma_1^2}(z_1), \end{aligned}$$

the last by (6.32). Use this in (6.31) to see that, in Case 2,

$$\begin{aligned} T_2(s) &\le C\lambda^{-2\lambda_0} t^{-\lambda_0} \iint \phi_1(z_1)\phi_2(z_2) \\ &\qquad \times p_{3(t-s)}(z_2 - z_1) X_0 p_{\sigma_1^2 t}(z_1)\, dz_1\, dz_2 \\ &\le C\lambda^{-2\lambda_0} t^{-\lambda_0} \iint \phi_1(z_1)\phi_2(z_2) \\ &\qquad \times p_{2\sigma_0^2(t-s)}(z_2 - z_1) X_0 p_{\sigma_0^2 t}(z_1)\, dz_1\, dz_2, \end{aligned} \tag{6.33}$$

the last using $3 < 2\sigma_0^2$ and $\sigma_1^2 < \sigma_0^2$.

Now combine (6.26), (6.30) and (6.33) to see that (6.30) holds for all $0 < s \le t$ and conclude that

$$\begin{aligned} T_2 \le \iint \phi_1(z_1)\phi_2(z_2) \Big[ & C\lambda^{-4\lambda_0} t^{-\lambda_0} \\ & \times \int_0^t (t-s)^{-\lambda_0} p_{8\sigma_0^2(t-s)}(z_1 - z_2) X_0 p_{4\sigma_0^2 t}(z_1)\, ds \Big] dz_1\, dz_2. \end{aligned}$$

Combine this with (6.19) to see that if

$$f_{t,\lambda}(z_1, z_2) = \lambda^{4\lambda_0} E\big(X_t(z_1)X_t(z_2)e^{-\lambda X_t(z_1)-\lambda X_t(z_2)}\big),$$

then for $\lambda^2 t \geq 9$, and $\sigma_0^2$ as above,

$$\int \phi_1(z_1)\phi_2(z_2) f_{t,\lambda}(z_1, z_2)\, dz_1\, dz_2 \leq \iint \phi_1(z_1)\phi_2(z_2) C h_{t,X_0}(z_1, z_2)\, dz_1\, dz_2.$$

This implies that $f_{t,\lambda}(z) \leq C h_{t,X_0}(z)$ for Lebesgue a.a. $z = (z_1, z_2)$. Note that Fatou's lemma shows that $h_{t,X_0}(\cdot)$ is lower semicontinuous while $f_{t,\lambda}(\cdot)$ is continuous by dominated convergence. Therefore,

$$f_{t,\lambda}(z) \leq C h_{t,X_0}(z) \qquad \text{for all } z, \tag{6.34}$$

proving (a).

The first term in the definition of $h_{t,X_0}(z)$ is trivially bounded by $Ct^{-2\lambda_0-1} \times X_0(1)^2$. A simple substitution in the integral shows that the second term is bounded by

$$Ct^{-\lambda_0-1/2} X_0(1)|z_1 - z_2|^{1-2\lambda_0}. \tag{6.35}$$

This proves (b), and so the proof of Proposition 5.1 is complete. □

**7. Proof of Theorem 5.8.** Let $\tilde{V}^{b,\varepsilon} = V(b1_{[0,\varepsilon]})$ and let $\tilde{V}^{\infty,\varepsilon} = \lim_{b\to\infty} \tilde{V}^{b,\varepsilon}$, where the pointwise finite limit exists by monotonicity [from (3.6)] and the bound

$$\tilde{V}^{b,\varepsilon}(t, x) \leq V(b)(t, x) = \frac{b}{1 + (bt/2)} \leq \frac{2}{t}. \tag{7.1}$$

The scaling relation for $\tilde{V}^{b,\varepsilon}$, is

$$\text{for each } r > 0, \qquad \tilde{V}^{b,\varepsilon}(s, x) = r^2 \tilde{V}^{br^{-2},\varepsilon r}\big(r^2 s, rx\big). \tag{7.2}$$

We state two lemmas which will be used to prove Theorem 5.8 and prove them after establishing the theorem.

LEMMA 7.1. *Let $\mu_1$ denote the uniform law on* $[0, 1]$. *For any $\delta_0 \in (0, 1)$ there is a $C_{7.1}(\delta_0) > 1$ so that for all $\lambda > 0$,*

$$E^B_{\mu_1}\bigg(\exp\bigg(-\int_0^1 \tilde{V}^{\lambda,1}(u, B_u)\, du\bigg)\bigg) \leq C_{7.1}\lambda^{-((1/2)+\lambda_0)+\delta_0}.$$

The analogue of $H_u$ in (4.1) is (for any $b > 0$)

$$H^b(u, x) = u\tilde{V}^{b,1}(u, \sqrt{u}x) = \tilde{V}^{bu,u^{-1/2}}(1, x), \tag{7.3}$$

where we used (7.2) with $r = u^{-1/2}$. The latter expression suggests that $\lim_{u\to\infty} H^b(u, x) = V^\infty(1, x) = F(x)$. Let $Z^b_T = \exp(\int_0^T F(Y_s) - H^b(e^s, Y_s)\, ds)$. With Lemma 4.3(b) in mind, we have the following.

LEMMA 7.2. *There is a constant $C_{7.2}$ such that for all $b, T > 0$,*

$$Z_T^b \le C_{7.2}(b \wedge 1)^{-20}.$$

PROOF OF THEOREM 5.8. Let $t, \varepsilon$ and $M$ be as in the theorem and set $\lambda = (\varepsilon M)^{-1}$. An elementary argument, partitioning $[x, x+\varepsilon]$ into two nonoverlapping intervals of length $\varepsilon/2$ and replacing $M$ by $4M$, will in fact allow us to assume

$$\varepsilon \le \sqrt{t}/2 < \sqrt{t/2}. \tag{7.4}$$

By (3.6) and translation invariance,

$$\begin{aligned}
&P^X_{X_0}\big(0 < X_t\big([x, x+\varepsilon]\big)/\varepsilon \le \varepsilon M\big)\\
&\quad\le e E^X_{X_0}\big(1\big(X_t\big([x, x+\varepsilon] > 0\big)\exp\big(-\lambda X_t\big([x, x+\varepsilon]\big)/\varepsilon\big)\big)\big)\\
&\quad= e\Big[\exp\Big(-\int \tilde V^{\lambda/\varepsilon,\varepsilon}(t, y-x)\,dX_0(y)\Big)\\
&\qquad - \exp\Big(-\int \tilde V^{\infty,\varepsilon}(t, y-x)\,dX_0(y)\Big)\Big].
\end{aligned} \tag{7.5}$$

Differentiate the last inequality (with $x = 0$ and $X_0$ a point mass) with respect to $\lambda$, using dominated convergence to take the derivative through the integral, to conclude for $\lambda' > 0$,

$$\begin{aligned}
&\frac{d}{d\lambda'}\big(\exp\big(-\tilde V^{(\lambda'/\varepsilon),\varepsilon}(t,x)\big) - \exp\big(-\tilde V^{\infty,\varepsilon}(t,x)\big)\big)\\
&\quad= -\frac{d}{d\lambda'}E^X_{\delta_x}\big(\exp\big(-(\lambda'/\varepsilon)X_t\big([0,\varepsilon]\big)\big)1\big(X_t\big([0,\varepsilon) > 0\big)\big)\big)\\
&\quad= -E^X_{\delta_x}\big(\exp\big(-(\lambda'/\varepsilon)X_t\big([0,\varepsilon]\big)\big)X_t\big([0,\varepsilon]\big)/\varepsilon\big).
\end{aligned}$$

Integrate both sides from $\lambda$ to $\infty$ and then integrate out $x$ to conclude (by Fubini)

$$\begin{aligned}
&\int e^{-\tilde V^{\lambda/\varepsilon,\varepsilon}(t,x)} - e^{-\tilde V^{\infty,\varepsilon}(t,x)}\,dx\\
&\quad= \int_\lambda^\infty \int E^X_{\delta_x}\big(e^{-(\lambda'/\varepsilon)X_t([0,\varepsilon])}X_t\big([0,\varepsilon]\big)/\varepsilon\big)\,dx\,d\lambda'.
\end{aligned}$$

The bound (7.1) and the above show that

$$\begin{aligned}
&\int \big(\tilde V^{\infty,\varepsilon} - \tilde V^{\lambda/\varepsilon,\varepsilon}\big)(t,x)\,dx\\
&\quad\le e^{2/t}\int e^{-\tilde V^{\lambda/\varepsilon,\varepsilon}(t,x)} - e^{-\tilde V^{\infty,\varepsilon}(t,x)}\,dx\\
&\quad\le e^{2/t}\int_\lambda^\infty \int E^X_{\delta_x}\big(e^{-(\lambda'/\varepsilon)X_t([0,\varepsilon])}X_t\big([0,\varepsilon]\big)/\varepsilon\big)\,dx\,d\lambda'\\
&\quad= e^{2/t}\int_\lambda^\infty \int E^X_{\delta_0}\big(e^{-(\lambda'/\varepsilon)X_t([x,x+\varepsilon])}X_t\big([x,x+\varepsilon]\big)/\varepsilon\big)\,dx\,d\lambda'.
\end{aligned} \tag{7.6}$$

Now use the Markov property at time $t/2$ just as in (4.8) and then apply the above to see that

$$\begin{aligned} &e^{-\tilde{V}^{\lambda/\varepsilon,\varepsilon}(t,x)} - e^{-\tilde{V}^{\infty,\varepsilon}(t,x)} \\ &\qquad \le \int p_{t/2}(y-x)\big(\tilde{V}^{\infty,\varepsilon}(t/2,y) - \tilde{V}^{\lambda/\varepsilon,\varepsilon}(t/2,y)\big)\,dy \\ &\qquad \le e^{4/t}t^{-1/2}\int_\lambda^\infty \int E^X_{\delta_0}\big(e^{-(\lambda'/\varepsilon)X_{t/2}([y,y+\varepsilon])}X_{t/2}([y,y+\varepsilon])/\varepsilon\big)\,dy\,d\lambda', \end{aligned}$$

and so, by (7.1), if $f_{\varepsilon,t}(\lambda') = \int E^X_{\delta_0}(e^{-(\lambda'/\varepsilon)X_t([y,y+\varepsilon])}X_t([y,y+\varepsilon])/\varepsilon)\,dy$, then

$$\sup_x[\tilde{V}^{\infty,\varepsilon} - \tilde{V}^{\lambda/\varepsilon,\varepsilon}](t,x) \le c(t)\int_\lambda^\infty f_{\varepsilon,t/2}(\lambda')\,d\lambda'.$$

Returning to (7.5), we therefore see that [recall $\lambda = (\varepsilon M)^{-1}$]

$$\begin{aligned} P^X_{X_0}\big(0 < X_t([x,x+\varepsilon])/\varepsilon \le \varepsilon M\big) &\le eX_0(1)\sup_y[\tilde{V}^{\infty,\varepsilon} - \tilde{V}^{\lambda/\varepsilon,\varepsilon}](t,y) \\ &\le c(t)X_0(1)\int_{(\varepsilon M)^{-1}}^\infty f_{\varepsilon,t/2}(\lambda')\,d\lambda'. \end{aligned} \tag{7.7}$$

Let $\mu_\varepsilon$ denote the uniform distribution on $[0,\varepsilon]$. Argue as in the derivation of (4.4) to see that

$$\begin{aligned} f_{\varepsilon,t}(\lambda') &= E^X_{\delta_0}\bigg(\int e^{-(\lambda'/\varepsilon)X_t([y,y+\varepsilon])}\int 1_{[0,\varepsilon]}(x-y)X(t,x)\,dx\,dy/\varepsilon\bigg) \\ &= \varepsilon^{-1}\int_0^\varepsilon E^X_{\delta_0}\bigg(\int X(t,x)\exp\big(-(\lambda'/\varepsilon)X_t(x-z+[0,\varepsilon])\big)\,dx\bigg)\,dz \\ &\le \varepsilon^{-1}\int_0^\varepsilon E^B_0\bigg(\exp\bigg(-\int_0^t \tilde{V}^{\lambda'/\varepsilon,\varepsilon}(t-s,B_s-B_t+z)\,ds\bigg)\bigg)\,dz \\ &= E^B_{\mu_\varepsilon}\bigg(\exp\bigg(-\int_0^t \tilde{V}^{\lambda'/\varepsilon,\varepsilon}(s,B_s)\,ds\bigg)\bigg). \end{aligned} \tag{7.8}$$

Now apply the scaling relation (7.2) with $r = \varepsilon^{-1}$ to see that, for $\varepsilon^{-2}t \ge 1$,

$$\begin{aligned} f_{\varepsilon,t}(\lambda') &\le E^B_{\mu_\varepsilon}\bigg(\exp\bigg(-\int_0^t \tilde{V}^{\lambda'\varepsilon,1}(\varepsilon^{-2}s,\varepsilon^{-1}B_s)\varepsilon^{-2}\,ds\bigg)\bigg) \\ &= E^B_{\mu_1}\bigg(\exp\bigg(-\int_0^{t\varepsilon^{-2}} \tilde{V}^{\lambda'\varepsilon,1}(u,B_u)\,du\bigg)\bigg) \\ &= E^B_{\mu_1}\bigg(\exp\bigg(-\int_0^1 \tilde{V}^{\lambda'\varepsilon,1}(u,B_u)\,du\bigg)\psi(B_1)\bigg), \end{aligned} \tag{7.9}$$

where

$$\psi(x) = E^B_x\bigg(\exp\bigg(-\int_0^{t\varepsilon^{-2}-1} \tilde{V}^{\lambda'\varepsilon,1}(u+1,B_u)\,du\bigg)\bigg),$$

and we have used $\varepsilon^{-2}t \geq 1$. So if $Y_s$ is the Ornstein–Uhlenbeck process $B(e^s - 1)e^{-s/2}$, $T = \log(t\varepsilon^{-2})$ and $b = \lambda'\varepsilon$, then, as in the proof of Lemma 4.1,

$$\psi(x) = E_x^Y\left(\exp\left(-\int_0^T H^b(e^s, Y_s)\,ds\right)\right)$$
$$= E_x^Y\left(Z_T^b \exp\left(-\int_0^T F(Y_s)\,ds\right)\right).$$

By Lemma 7.2 and then (6.18),

$$\begin{aligned}\psi(x) &\leq C_{7.2}(b \wedge 1)^{-20} P_x^Y(\rho > T)\\ &\leq C_{7.2}(b \wedge 1)^{-20} c_\delta e^{\delta x^2} \varepsilon^{2\lambda_0} t^{-\lambda_0}.\end{aligned} \tag{7.10}$$

Take $0 < \delta < 1/4$ and use the above in (7.9) and then Hölder's inequality with $q = (4\delta)^{-1}$ and $p = (1 - 4\delta)^{-1}$, to get (with $b = \lambda'\varepsilon$)

$$f_{\varepsilon,t}(\lambda') \leq c_\delta (b \wedge 1)^{-20} \varepsilon^{2\lambda_0} t^{-\lambda_0} E_{\mu_1}^B\left(\exp\left(-\int_0^1 \tilde{V}^{b,1}(u, B_u)\,du\right)\right)^{1-4\delta}.$$

So recalling (7.7), applying the above found with $t/2$ in place of $t$ [by (7.4) we have $\varepsilon^{-2}\frac{t}{2} \geq 1$] and using Lemma 7.1, we have

$$\begin{aligned}
&P_{X_0}^X\big(0 < X_t([x, x+\varepsilon]) \leq \varepsilon^2 M\big)\\
&\quad\leq c(t) X_0(1) \varepsilon^{2\lambda_0} c'_\delta \int_{(\varepsilon M)^{-1}}^\infty (\varepsilon\lambda' \wedge 1)^{-20}\\
&\qquad\times \left[E_{\mu_1}^B\left(\exp\left(-\int_0^1 \tilde{V}^{\lambda'\varepsilon,1}(u, B_u)\,du\right)\right)\right]^{1-4\delta} d\lambda'\\
&\quad\leq c(t) X_0(1) \varepsilon^{2\lambda_0 - 1} c'_\delta \left[\int_{M^{-1}}^1 w^{-20}\,dw\right.\\
&\qquad\left. + \int_1^\infty \left[E_{\mu_1}^B\left(\exp\left(-\int_0^1 \tilde{V}^{w,1}(u, B_u)\,du\right)\right)\right]^{1-4\delta} dw\right]\\
&\quad\leq c(t) X_0(1) \varepsilon^{2\lambda_0 - 1} c'_\delta \left[M^{19}\right.\\
&\qquad\left. + \int_1^\infty C_{7.1}(\delta_0) w^{(-((1/2)+\lambda_0)+\delta_0)(1-4\delta)}\,dw\right]\\
&\quad\leq c(t) X_0(1) \varepsilon^{2\lambda_0 - 1} M^{19},
\end{aligned}$$

by choosing $\delta$ and $\delta_0$ sufficiently small since $\lambda_0 > 1/2$. This gives the required result for $t = 1$. By scaling (see, e.g., Exercise II.5.5 in [18]) it follows for general $t > 0$. □

PROOF OF LEMMA 7.2. Lemma 3.2(c) and (7.1) show that $F, H^b \le 2$ and so by (3.6) and (1.8),

$$
\begin{aligned}
F(x) &- H^b(u,x) \\
&\le e^2\big[e^{-H^b(u,x)} - e^{-F(x)}\big] \\
&\le e^2\biggl[E^X_{\delta_x}\biggl(\exp\biggl(-bu\int_0^{u^{-1/2}} X(1,y)\,dy\biggr) - 1\bigl(X(1,0)=0\bigr)\biggr)\biggr] \\
&\le e^2\biggl[E^X_{\delta_x}\bigl(1\bigl(0 < X(1,0) < u^{-1/8}\bigr)\bigr) \qquad (7.11)\\
&\qquad + E^X_{\delta_x}\biggl(1\bigl(X(1,0)\ge u^{-1/8}\bigr)\exp\biggl(-bu^{1/2}\frac{\int_0^{u^{-1/2}} X(1,y)\,dy}{u^{-1/2}}\biggr)\biggr)\biggr] \\
&\equiv e^2(T_1+T_2).
\end{aligned}
$$

Theorem 4.8(a) implies that

$$T_1 \le Cu^{-(2\lambda_0-1)/8}. \tag{7.12}$$

We apply the modulus of continuity in Proposition 5.6 with $\xi = 3/8$ and $K = 2$ and so set $\rho_0(\omega) = \rho(2, 3/8, \omega)$. Choose $u_0$ so that

$$u^{-((\xi/2)-(1/8))} \le 1/2 \qquad \text{for all } u \ge u_0. \tag{7.13}$$

If $u \ge u_0$, $\rho_0(\omega) \ge u^{-1/2}$ and $y \in [0, u^{-1/2}]$, then on $\{X(1,0) \ge u^{-1/8}\}$,

$$X(1,y) \ge X(1,0) - |y|^\xi \ge u^{-1/8} - u^{-\xi/2} \ge u^{-1/8}/2 \qquad \text{[by (7.13)]}.$$

Therefore, for $u \ge u_0$,

$$
\begin{aligned}
T_2 &\le P^X_{X_0}\bigl(\rho_0 < u^{-1/2}\bigr) + \exp\biggl(-\frac{b}{2}u^{(1/2)-(1/8)}\biggr) \\
&\le Cu^{-\delta_{5.6}/2} + \exp\biggl(-\frac{b}{2}u^{3/8}\biggr).
\end{aligned}
\tag{7.14}
$$

Combining (7.12) and (7.14), we have

$$F(x) - H^b(u,x) \le C\bigl(u^{-(2\lambda_0-1)/8} + u^{-\delta_{5.6}/2}\bigr) + e^2\exp\biggl(-\frac{b}{2}u^{3/8}\biggr),$$

first for $u \ge u_0$, and then for all $u \ge 1$ by increasing the constant $C$. Therefore,

$$
\begin{aligned}
Z^b_T &\le C\exp\biggl(\int_0^T e^2\exp\biggl(-\frac{b}{2}e^{\frac{3s}{8}}\biggr)\,ds\biggr) \\
&= C\exp\biggl(\frac{8e^2}{3}\int_{b/2}^{(b/2)e^{3T/8}} e^{-w}w^{-1}\,dw\biggr)
\end{aligned}
$$

$$\leq C\exp\biggl(\frac{8e^2}{3}\int_{(b/2)\wedge 1}^1 w^{-1}\,dw\biggr)$$
$$\leq C[b\wedge 1]^{-20}. \qquad \square$$

PROOF OF LEMMA 7.1. Set $\gamma=\frac{1}{2}+\lambda_0\in(1,\frac{3}{2})$ and $\beta=\frac{1}{2}-\frac{\gamma}{4}\in(\frac{1}{8},\frac{1}{4})$. The exponent on $\lambda$ (in the statement of the Lemma) is negative and so we may assume without loss of generality that $\lambda\geq 1$. For $\lambda\geq 1$, use an obvious symmetry to see that

(7.15)

$$\begin{aligned}\psi(\lambda)&\equiv E_{\mu_1}^B\biggl(\exp\biggl(-\int_0^1\tilde V^{\lambda,1}(s,B_s)\,ds\biggr)\biggr)\\&=E_{\mu_1}^B\biggl(1\bigl(B_0\in[\lambda^{-\beta},1-\lambda^{-\beta}]\bigr)E_{B_0}^B\biggl(\exp\biggl(-\int_0^1\tilde V^{\lambda,1}(s,B_s)\,ds\biggr)\biggr)\biggr)\\&\quad+2E_{\mu_1}^B\biggl(1\bigl(B_0\in[0,\lambda^{-\beta}]\bigr)E_{B_0}^B\biggl(\exp\biggl(-\int_0^1\tilde V^{\lambda,1}(s,B_s)\,ds\biggr)\biggr)\biggr)\\&\equiv T_1+2T_2.\end{aligned}$$

By Feynman–Kac, we have

$$\tilde V^{\lambda,1}(t,x)=E_x^B\biggl(\lambda 1_{[0,1]}(B_t)\exp\biggl(-\int_0^t\frac{\tilde V^{\lambda,1}(t-s,B_s)}{2}\,ds\biggr)\biggr).\tag{7.16}$$

If $\tilde V_0^{\lambda,1}$ was nonnegative continuous, this would follow from [10] (page 268) and for general nonnegative Borel initial conditions it follows by taking bounded pointwise limits [use (3.6)]. Now bound $\tilde V^{\lambda,1}(u,x)$ above by $V(\lambda)(u,x)=\frac{2\lambda}{2+\lambda u}$ and use this bound in the exponent in (7.16) to conclude that

$$\begin{aligned}\tilde V^{\lambda,1}(t,x)&\geq\lambda P_x^B\bigl(B_t\in[0,1]\bigr)\exp\biggl(-\int_0^t\frac{2\lambda}{2(2+\lambda s)}\,ds\biggr)\\&=\frac{2\lambda}{2+\lambda t}P_x^B\bigl(B_t\in[0,1]\bigr).\end{aligned}\tag{7.17}$$

If $x\in[\lambda^{-\beta}/2,1-(\lambda^{-\beta}/2)]$ and $t\leq\lambda^{-2\beta(1+\varepsilon)}$, for some $\varepsilon>0$, then by (7.17),

$$\begin{aligned}\tilde V^{\lambda,1}(t,x)&\geq\frac{2\lambda}{2+\lambda t}\bigl(1-P_x^B\bigl(B_t\notin[0,1]\bigr)\bigr)\\&\geq\frac{\lambda}{1+(\lambda t/2)}(1-\eta_\lambda),\end{aligned}$$

where $\eta_\lambda = \exp(-\lambda^{2\beta\varepsilon/8})$. Therefore,

$$
\begin{aligned}
T_1 &\le \int_{\lambda^{-\beta}}^{1-\lambda^{-\beta}} E_x^B\biggl(\exp\biggl(-2\int_0^{\lambda^{-2\beta(1+\varepsilon)}} \frac{(1-\eta_\lambda)\lambda}{(1+(\lambda s/2))2}\,ds\biggr) \\
&\qquad \times 1\biggl(\sup_{s\le\lambda^{-2\beta(1+\varepsilon)}} |B_s - x| \le \frac{1}{2}\lambda^{-\beta}\biggr)\biggr)\,dx \\
&\qquad + P_0^B\biggl(\sup_{s\le\lambda^{-2\beta(1+\varepsilon)}} |B_s| > \frac{1}{2}\lambda^{-\beta}\biggr) \qquad (7.18)\\
&\le \biggl[1 + \frac{\lambda^{1-2\beta(1+\varepsilon)}}{2}\biggr]^{-2(1-\eta_\lambda)} + C\exp(-\lambda^{2\beta\varepsilon}/8) \\
&\le C_\varepsilon \lambda^{-(1-2\beta(1+\varepsilon))2(1-\eta_\lambda)} \\
&\le C_\varepsilon \lambda^{-2(1-2\beta(1+2\varepsilon))},
\end{aligned}
$$

where the last inequality holds first for $\lambda \ge \lambda(\varepsilon)$, and then for all $\lambda \ge 1$ by increasing $C_\varepsilon$.

For $T_2$, we use the scaling relation (7.2) with $r = \lambda^\beta$ to see that

$$
\tilde V^{\lambda,1}(s,x) = \lambda^{2\beta} \tilde V^{\lambda^{1-2\beta},\lambda^\beta}(\lambda^{2\beta}s, \lambda^\beta x),
$$

and so

$$
\begin{aligned}
T_2 &= \lambda^{-\beta} E^B_{\mu_{\lambda^{-\beta}}}\biggl(\exp\biggl(-\int_0^1 \tilde V^{\lambda^{1-2\beta},\lambda^\beta}(\lambda^{2\beta}s, \lambda^\beta B_s)\lambda^{2\beta}\,ds\biggr)\biggr) \\
&= \lambda^{-\beta} E^B_{\mu_1}\biggl(\exp\biggl(-\int_0^{\lambda^{2\beta}} \tilde V^{\lambda^{1-2\beta},\lambda^\beta}(u, B_u)\,du\biggr)\biggr) \qquad (7.19)\\
&\le \lambda^{-\beta} E^B_{\mu_1}\biggl(\exp\biggl(-\int_0^1 \tilde V^{\lambda^{1-2\beta},1}(u, B_u)\,du\biggr)\hat\psi(B_1)\biggr),
\end{aligned}
$$

where

$$
\hat\psi(x) = E_x^B\biggl(\exp\biggl(-\int_0^{\lambda^{2\beta}-1} \tilde V^{\lambda^{1-2\beta},1}(s+1, B_s)\,ds\biggr)\biggr).
$$

By (7.10) with $b = \lambda^{1-2\beta}$ and $\lambda^{2\beta}$ in place of $t\varepsilon^{-2}$, for any $\delta > 0$,

$$
\hat\psi(x) \le C_\delta e^{\delta x^2} \lambda^{-2\beta\lambda_0}.
$$

Use this in (7.19), and then for any $p > 1$ choose $\delta > 0$ so that $1 - 2\delta > p^{-1}$, and apply Hölder's inequality to see that

$$
T_2 \le c_p \lambda^{-\beta(1+2\lambda_0)} E^B_{\mu_1}\biggl(\exp\biggl(-\int_0^1 \tilde V^{\lambda^{1-2\beta,1}}(u, B_u)\,dy\biggr)\biggr)^{1/p}. \qquad (7.20)
$$

Combine (7.15), (7.18) and (7.20) to conclude that for any $\varepsilon > 0$ and $p > 1$ there are constants $C_\varepsilon$ and $c_p$ so that

$$\begin{aligned}\psi(\lambda) &\le C_\varepsilon \lambda^{-2(1-2\beta(1+2\varepsilon))} + c_p \lambda^{-\beta(1+2\lambda_0)} \psi\big(\lambda^{1-2\beta}\big)^{1/p} \\ &\le C_\varepsilon \lambda^{-\gamma+2\varepsilon} + c_p \lambda^{-2\beta\gamma} \psi\big(\lambda^{\gamma/2}\big)^{1/p},\end{aligned} \tag{7.21}$$

where the last line uses $\gamma > 1$. Now fix $r \in (1, \gamma)$, choose $\varepsilon > 0$ so that

$$\gamma - 2\varepsilon > r,$$

and then $p > 1$ so that

$$2\beta\gamma + \frac{\gamma r}{2p} > r.$$

The latter is easily seen to be possible by the choice of $r$ and a bit of arithmetic. With these choices of $\varepsilon$ and $p$, we then may choose $\lambda' = \lambda'(r)$ sufficiently large so that (by our choice of $\varepsilon$) for $C_\varepsilon$ and $c_p$ as in (7.21)

$$\lambda^{-\gamma+2\varepsilon} \le \frac{1}{2C_\varepsilon} \lambda^{-r} \qquad \text{for } \lambda \ge \lambda'$$

and (by our choice of $p$),

$$\lambda^{-2\beta\gamma} \lambda^{-\gamma r/(2p)} \le \frac{1}{2c_p} \lambda^{-r} \qquad \text{for } \lambda \ge \lambda'.$$

Using the two bounds above, we see that (7.21) becomes

$$\psi(\lambda) \le \frac{\lambda^{-r}}{2} + \frac{\lambda^{-r}}{2} \lambda^{\gamma r/(2p)} \psi\big(\lambda^{\gamma/2}\big)^{1/p} \quad \text{for } \lambda \ge \lambda'. \tag{7.22}$$

Let $N = N(r)$ be the minimal natural number such that $2^{(2/\gamma)^N} \ge \lambda'$. Now choose $C_0 = C_0(r) \ge 1$ so that for all $1 \le \lambda \le 2^{(2/\gamma)^N}$,

$$\psi(\lambda) \le C_0 \lambda^{-r}. \tag{7.23}$$

We now prove by induction on $n \ge N$ that (7.23) holds for $1 \le \lambda \le 2^{(2/\gamma)^n}$. It holds for $n = N$ by our choice of $C_0$, so assume it holds for $1 \le \lambda \le 2^{(2/\gamma)^n}$ $(n \ge N)$, and let $\lambda \in [2^{(2/\gamma)^n}, 2^{(2/\gamma)^{n+1}}]$. Then $\lambda \ge \lambda'$ and

$$\lambda^{\gamma/2} \le 2^{(2/\gamma)^n}. \tag{7.24}$$

Therefore, by (7.22), (7.24) and our induction hypothesis,

$$\begin{aligned}\psi(\lambda) &\le \frac{\lambda^{-r}}{2} + \frac{\lambda^{-r}}{2} \lambda^{\gamma r/(2p)} C_0^{1/p} \lambda^{-\gamma r/(2p)} \\ &\le C_0 \lambda^{-r}.\end{aligned}$$

This completes the induction and so for any $r < \frac{1}{2} + \lambda_0$, $\psi(\lambda) \le C_0(r)\lambda^{-r}$ for all $\lambda \ge 1$. $\square$

**Acknowledgments.** We thank Roger Tribe for sharing his thoughts and unpublished notes on this problem with us.

## REFERENCES

[1] BREZIS, H., PELETIER, L. A. and TERMAN, D. (1986). A very singular solution of the heat equation with absorption. *Arch. Ration. Mech. Anal.* **95** 185–209. MR0853963

[2] CHEN, Y. (2015). Pathwise uniqueness for the SPDE's of some super-Brownian motion with immigration. *Ann. Probab.* **43** 3359–3467. MR3433584

[3] CODDINGTON, E. A. and LEVINSON, N. (1955). *Theory of Ordinary Differential Equations*. McGraw-Hill, New York. MR0069338

[4] DAWSON, D. A. and PERKINS, E. A. (1991). Historical processes. *Mem. Amer. Math. Soc.* **93** iv+179. MR1079034

[5] DE HAAN, L. and STADTMÜLLER, U. (1985). Dominated variation and related concepts and Tauberian theorems for Laplace transforms. *J. Math. Anal. Appl.* **108** 344–365. MR0793651

[6] FRISTEDT, B. E. and PRUITT, W. E. (1971). Lower functions for increasing random walks and subordinators. *Z. Wahrsch. Verw. Gebiete* **18** 167–182. MR0292163

[7] HAWKES, J. (1975/76). On the potential theory of subordinators. *Z. Wahrsch. Verw. Gebiete* **33** 113–132. MR0388553

[8] HAWKES, J. (1979). Potential theory of Lévy processes. *Proc. Lond. Math. Soc.* (3) **38** 335–352. MR0531166

[9] KAMIN, S. and PELETIER, L. A. (1985). Singular solutions of the heat equation with absorption. *Proc. Amer. Math. Soc.* **95** 205–210. MR0801324

[10] KARATZAS, I. and SHREVE, S. E. (1979). *Brownian Motion and Stochastic Calculus*. Springer, New York.

[11] MUELLER, C., MYTNIK, L. and PERKINS, E. (2014). Nonuniqueness for a parabolic SPDE with $\frac{3}{4} - \varepsilon$-Hölder diffusion coefficients. *Ann. Probab.* **42** 2032–2112. MR3262498

[12] MUELLER, C., MYTNIK, L. and PERKINS, E. (2015). On the boundary of the support of super-Brownian motion. Preprint. Available at arXiv:1512.08610.

[13] MUELLER, C. and PERKINS, E. A. (1992). The compact support property for solutions to the heat equation with noise. *Probab. Theory Related Fields* **93** 325–358. MR1180704

[14] MYTNIK, L. (1998). Weak uniqueness for the heat equation with noise. *Ann. Probab.* **26** 968–984. MR1634410

[15] MYTNIK, L. and PERKINS, E. (2003). Regularity and irregularity of $(1 + \beta)$-stable super-Brownian motion. *Ann. Probab.* **31** 1413–1440. MR1989438

[16] MYTNIK, L. and PERKINS, E. (2011). Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients: The white noise case. *Probab. Theory Related Fields* **149** 1–96. MR2773025

[17] MYTNIK, L., PERKINS, E. and STURM, A. (2006). On pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients. *Ann. Probab.* **34** 1910–1959. MR2271487

[18] PERKINS, E. (2002). Dawson–Watanabe superprocesses and measure-valued diffusions. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1999). *Lecture Notes in Math.* **1781** 125–324. Springer, Berlin. MR1915445

[19] RIESZ, F. and SZ.-NAGY, B. (1955). *Functional Analysis*. Frederick Ungar, New York.

[20] TAYLOR, S. J. (1961). On the connexion between Hausdorff measures and generalized capacity. *Math. Proc. Cambridge Philos. Soc.* **57** 524–531. MR0133420

[21] UCHIYAMA, K. (1980). Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Z. Wahrsch. Verw. Gebiete* **54** 75–116.

C. MUELLER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROCHESTER
ROCHESTER, NEW YORK 14627
USA
E-MAIL: carl.2016@outlook.com

L. MYTNIK
FACULTY OF INDUSTRIAL ENGINEERING AND MANAGEMENT
TECHNION—ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA 32000
ISRAEL
E-MAIL: leonid@ie.technion.ac.il

E. PERKINS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BRITISH COLUMBIA V6T 1Z2
CANADA
E-MAIL: perkins@math.ubc.ca