RANDOM WALKS AND ISOPERIMETRIC PROFILES UNDER MOMENT CONDITIONS

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Let G be a finitely generated group equipped with a finite symmetric generating set and the associated word length function $|\cdot|$. We study the behavior of the probability of return for random walks driven by symmetric measures μ that are such that $\sum \rho(|x|)\mu(x) < \infty$ for increasing regularly varying or slowly varying functions ρ, for instance, $s \mapsto (1+s)^\alpha$, $\alpha \in (0, 2]$, or $s \mapsto (1 + \log(1 + s))^\varepsilon$, $\varepsilon > 0$. For this purpose, we develop new relations between the isoperimetric profiles associated with different symmetric probability measures. These techniques allow us to obtain a sharp L^2-version of Erschler's inequality concerning the Følner functions of wreath products. Examples and assorted applications are included.

CONTENTS

1. Introduction. Let G be a finitely generated group. The following nota-
tion will be used throughout this work. Let $S = (s_1, \ldots, s_k)$ be a fixed generating
k-tuple and $S^* = \{e, s_1^{\pm 1}, \ldots, s_k^{\pm 1}\}$ be the associated symmetric generating set. Let
$|\cdot|$ be the associated word-length so that $|g|$ is the least m such that $g = \sigma_1 \cdots \sigma_m$
with $\sigma_i \in \mathcal{S}^*$ (and the convention that $|e| = 0$, where e is the identity element in G).
Let $B(r) = \{g \in G : |g| \leq r\}$ and let V be the associated volume growth function
defined by

$$V(r) = \big|\{g \in G : |g| \leq r\}\big|,$$

where $|\Omega| = \#\Omega$ is the number of elements in $\Omega \subset G$. For $r \geq 1$, let $\mathbf{u}_r$ be the
uniform probability measure on $B(r)$ and set $\mathbf{u} = \mathbf{u}_1$, that is,

$$(1.1) \qquad \mathbf{u}_r = \frac{1}{|B(r)|}\mathbf{1}_{B(r)} \quad \text{and} \quad \mathbf{u} = \mathbf{u}_1 = \frac{1}{|S^*|}\mathbf{1}_{S^*}.$$

Given two functions f_1, f_2 taking real values but defined on an arbitrary domain
(not necessarily a subset of $\mathbb{R}$), we write $f \asymp g$ to signify that there are constants
$c_1, c_2 \in (0, \infty)$ such that $c_1 f_1 \leq f_2 \leq c_2 f_1$. Given two monotone real functions
f_1, f_2, write $f_1 \simeq f_2$ if there exists $c_i \in (0, \infty)$ such that $c_1 f_1(c_2 t) \leq f_2(t) \leq
c_3 f_1(c_4 t)$ on the domain of definition of f_1, f_2 (usually, f_1, f_2 will be defined
on a neighborhood of 0 or infinity and tend to 0 or infinity at either 0 or infin-
ity. In some cases, one or both functions are defined only on a countable set such
as $\mathbb{N}$). We denote the associated order by $\lesssim$. Note that the equivalence relation $\simeq$
distinguishes between power functions of different degrees and between stretched
exponentials $\exp(-t^\alpha)$ of different exponent $\alpha > 0$ but does not distinguish be-
tween different rates of exponential growth or decay (e.g., $2^n \simeq 5^n$). It is not hard
to verify that the volume growth functions associated with two finite symmetric
generating sets of a given group G are $\simeq$-equivalent.

Given an arbitrary probability measure ϕ on a group G, we let $(S_n)_0^\infty$ denote the
trajectory of the random walk driven by ϕ (often started at the identity element e).
We let $\mathbf{P}_\phi$ be the associated measure on $G^{\mathbf{N}}$ with $S_0 = e$ and $\mathbf{E}_\phi$ the corresponding
expectation $\mathbf{E}_\phi^x(F) = \int_{G^{\mathbf{N}}} F(\omega)\, d\mathbf{P}_\phi^x(\omega)$. In particular,

$$\mathbf{P}_\phi(S_n = x) = \mathbf{E}_\phi\big(\mathbf{1}_x(S_n)\big) = \phi^{(n)}(x).$$

Here and in the rest of the paper, $\phi^{(n)}$ denotes the nth convolution power of ϕ.

1.1. *The random walk invariants* $\Phi_{G,\rho}$ *and* $\widetilde{\Phi}_{G,\rho}$. In [21], it is proved that, for
any finitely generated group G, there exists a function $\Phi_G : \mathbb{N} \to (0, \infty)$ such that,

if μ is a symmetric probability measure with generating support and finite second moment, that is $\sum |g|^2 \mu(g) < \infty$, then

$$\mu^{(2n)}(e) \simeq \Phi_G(n).$$

Further, [21] proves that Φ_G is an invariant of quasi-isometry. Throughout this paper and referring to definition (1.1), we will use $n \mapsto \mathbf{u}^{(n)}(e)$ as our favorite representative for Φ_G.

In [3], A. Bendikov and the first author considered the question of finding lower bounds for the probability of return $\mu^{(2n)}(e)$ when μ is symmetric and is only known to have a finite moment of some given exponent lower than 2. Very generally, let $\rho : [0, \infty) \to [1, \infty)$ be given. We say that a measure μ has finite ρ-moment if $\sum \rho(|g|)\mu(g) < \infty$. We say that μ has finite weak-ρ-moment if

$$(1.2) \qquad W(\rho, \mu) := \sup_{s>0}\{s\mu(\{g : \rho(|g|) > s\})\} < \infty.$$

DEFINITION 1.1 (Fastest decay under ρ-moment). Let G be a countable group. Fix a function $\rho : [0, \infty) \to [1, \infty)$. Let $\mathcal{S}_{G,\rho}$ be the set of all symmetric probability ϕ on G with the property that $\sum \rho(|g|)\phi(g) \le 2\rho(0)$. Set

$$\Phi_{G,\rho} : n \mapsto \Phi_{G,\rho}(n) = \inf\{\phi^{(2n)}(e) : \phi \in \mathcal{S}_{G,\rho}\}.$$

In words, $\Phi_{G,\rho}$ provides the best lower bound valid for all convolution powers of probability measures in $\mathcal{S}_{G,\rho}$. The following variant deals with finite weak-moments and will be key for our purpose.

DEFINITION 1.2 (Fastest decay under weak-ρ-moment). Let G be a countable group. Fix a function $\rho : [0, \infty) \to [1, \infty)$. Let $\widetilde{\mathcal{S}}_{G,\rho}$ be the set of all symmetric probability ϕ on G with the properties that $W(\rho, \mu) \le 2\rho(0)$. Set

$$\widetilde{\Phi}_{G,\rho} : n \mapsto \widetilde{\Phi}_{G,\rho}(n) = \inf\{\phi^{(2n)}(e) : \phi \in \widetilde{\mathcal{S}}_{G,\rho}\}.$$

REMARK 1.3. Assume that ρ has the property that $\rho(s+t) \le C(\rho(s)+\rho(t))$, $0 \le s, t < \infty$. Under this natural condition ([3], Corollary 2.3) shows that $\Phi_{G,\rho}$ and $\widetilde{\Phi}_{G,\rho}$ stay strictly positive. Further, ([3], Proposition 2.4) shows that, for any symmetric probability measure μ on G such that $\sum \rho(|g|)\mu(g) < \infty$ [resp., $W(\rho, \mu) < \infty$], there exist constants c_1, c_2 (depending on μ) such that

$$\mu^{(2n)}(e) \ge c_1 \Phi_{G,\rho}(c_2 n)$$

[resp., $\mu^{(2n)}(e) \ge c_1\widetilde{\Phi}_{G,\rho}(c_2 n)$]. This makes it very natural to consider not the functions $\Phi_{G,\rho}$ and $\widetilde{\Phi}_{G,\rho}$ themselves but their equivalence classes under the equivalence relation $\simeq$.

REMARK 1.4. The reader may wonder why we are considering the weak-moment variant $\widetilde{\Phi}_{G,\rho}$. The reason is that it appears to be the more natural version of the two variants. For instance, when $G = \mathbb{Z}$ and $\rho_\alpha(s) = (1+s)^\alpha$, we do not know what the behavior of $\Phi_{\mathbb{Z},\rho_\alpha}$ is for $\alpha \in (0,2)$ whereas it is very well known and easy to show that $\widetilde{\Phi}_{\mathbb{Z},\rho_\alpha}(n) \simeq n^{-1/\alpha}$, $\alpha \in (0,2)$.

When $\rho_\alpha(s) = (1+s)^\alpha$ with $\alpha > 0$, the main result of [21] implies that

$$\Phi_G \simeq \Phi_{G,\rho_2} \simeq \Phi_{G,\rho_\alpha} \simeq \widetilde{\Phi}_{G,\rho_\alpha} \qquad \text{for all } \alpha > 2.$$

This holds even so there are great many finitely generated groups G (indeed, uncountably many), for which we do not know how Φ_G behaves. For a group G of polynomial volume growth with $V(r) \simeq r^D$, we know that (see [3, 16, 23] and the references therein)

$$\Phi_G(n) \simeq n^{-D/2}, \qquad \widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq n^{-D/\alpha}, \qquad \alpha \in (0,2)$$

showing that the condition $\alpha > 2$ cannot be relaxed.

Definitions 1.1–1.2 lead to the following problems.

PROBLEM 1. Let G, H be two finitely generated groups. Let ρ, θ be two (nice) increasing functions with $\theta \le \rho$.

(1) Does $\Phi_G \simeq \Phi_H$ imply $\widetilde{\Phi}_{G,\rho} \simeq \widetilde{\Phi}_{H,\rho}$?
(2) Does $\widetilde{\Phi}_{G,\rho} \simeq \widetilde{\Phi}_{H,\rho}$ imply $\widetilde{\Phi}_{G,\theta} \simeq \widetilde{\Phi}_{H,\theta}$?
(3) Fix $\alpha \in (0,2)$. Is it true that $\widetilde{\Phi}_{G,\rho_\alpha} \simeq \widetilde{\Phi}_{H,\rho_\alpha}$ is equivalent to $\Phi_G \simeq \Phi_H$?
(4) What is the behavior of $\widetilde{\Phi}_{G,\rho_2}$ and, more generally, of $\widetilde{\Phi}_{G,\rho}$ when ρ is close to $t \mapsto t^2$?

In contemplating these questions, it is reasonable to make additional assumptions on the functions ρ, θ, for instance, one may want to assume that ρ, θ are continuous increasing functions satisfying the doubling condition $\exists C > 0$, $\forall t > 0$, $f(2t) \le Cf(t)$. Or one may even want to assume that ρ, θ are taken from a given list of functions such a

$$s \mapsto (1+s)^{\alpha_0} \prod_{k=1}^m \log_{[k]}^{\alpha_k}(s)$$

with $\alpha_0 \ge 0$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ (the first nonzero α_i should be positive). Here and in the rest of this paper, $\log_{[k]}$ is defined inductively by $\log_{[k]}(s) = 1 + \log(\log_{[k-1]}(s))$, $\log_{[1]}(s) = 1 + \log(1+s)$. For instance, an interesting restricted version of the first question is concerned with the case when $\rho \in \{\rho_\alpha : \alpha \in (0,2)\}$.

If G has polynomial volume growth of degree $D > 0$, then $\widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq n^{-D/\alpha}$, $\alpha \in (0,2)$ while [23, 25]

$$\widetilde{\Phi}_{G,\log_{[1]}^\varepsilon}(n) \simeq \exp(-n^{1/(1+\varepsilon)}).$$

So, $\widetilde{\Phi}_{G,\alpha}$ distinguishes between different degrees of growth whereas $\widetilde{\Phi}_{G,\log^{\varepsilon}_{[1]}}$ does not (except between $D = 0$ and $D > 0$).

From a heuristic point of view, there are reasons to believe that the slower the function ρ grows, the coarser the group invariant $\widetilde{\Phi}_{G,\rho}$ is (modulo the equivalence relation $\simeq$). The first two questions stated in Problem 1 relate to this heuristic and ask if this conjectural picture is correct. Namely, if $\theta \leq \rho$, is it correct that the partition one obtains by considering the classes of groups sharing the same $\widetilde{\Phi}_{G,\theta}$ are obtained by lumping together classes corresponding to $\widetilde{\Phi}_{G,\rho}$? The third question in Problem 1 asks whether the classes of groups one obtains by considering Φ_G and $\widetilde{\Phi}_{G,\rho_\alpha}$ [for some/any fixed $\alpha \in (0, 2)$] are all exactly the same. These questions are difficult and none of them are solved in this paper although we make significant progress in describing the behavior of $\widetilde{\Phi}_{G,\rho}$ in a number of cases.

Question 4 is technically interesting because we do not have good techniques to understand the subtle difference of behavior between $\widetilde{\Phi}_{G,\rho_2}$ and Φ_G. We obtain a sharp answer for groups of polynomial volume growth (see Corollary 3.3) and for some wreath products (see Theorem 5.3).

1.2. *The spectral profiles* Λ_G *and* $\widetilde{\Lambda}_{G,\rho}$. Given a symmetric probability measure ϕ, consider the associated Dirichlet form

$$\mathcal{E}_\phi(f_1, f_2) = \frac{1}{2} \sum_{x,y \in G} (f_1(xy) - f_1(x))(f_2(xy) - f_2(x))\phi(y)$$

and set

$$\Lambda_{G,\phi}(v) = \Lambda_\phi(v) = \inf\{\lambda_\phi(\Omega) : \Omega \subset G, |\Omega| \leq v\},$$

where

(1.3) $$\lambda_\phi(\Omega) = \inf\{\mathcal{E}_\phi(f, f) : \text{support}(f) \subset \Omega, \|f\|_2 = 1\}.$$

In words, $\lambda_\phi(\Omega)$ is the lowest eigenvalue of the operator of convolution by $\delta_e - \phi$ with Dirichlet boundary condition in Ω. This operator is associated with the discrete time Markov process corresponding to the ϕ-random walk killed outside Ω. The function $v \mapsto \Lambda_\phi(v)$ is called the L^2-isoperimetric profile or spectral profile of ϕ. See [6], Section 2.5, where the notion of L^p-isoperimetric is introduced with slightly different notation and where earlier references are discussed.

The L^2-isoperimetric profile of a group G is defined as the $\simeq$-equivalence class Λ_G of the functions Λ_ϕ associated to any symmetric probability measure ϕ with finite generating support. In Section 2.1, we give a short review of the well-known relations that exist between the behavior of $n \mapsto \phi^{(2n)}(e)$ and $v \mapsto \Lambda_\phi(v)$. It will be useful to introduce the following definition analogous to Definition 1.2.

DEFINITION 1.5. Let $\rho, \widetilde{\mathcal{S}}_{G,\rho}$ be as in Definitions 1.1–1.2. Set

$$\widetilde{\mathbf{\Lambda}}_{G,\rho}(v) = \sup\{\Lambda_\phi(v) : \phi \in \widetilde{\mathcal{S}}_{G,\rho}\}, \qquad v > 0.$$

In words, $\widetilde{\boldsymbol{\Lambda}}_{G,\rho}$ is the extremal spectral profile under the weak ρ-moment condition. Upper bounds on $\widetilde{\boldsymbol{\Lambda}}_{G,\rho}$ are tightly related to lower bounds on $\widetilde{\Phi}_{G,\rho}$ and vice versa. See Section 2.1.

The Appendix provides examples of the computation Λ_ϕ (and assorted L^p-variants) for radial stable-like probability measures defined in terms of the word distance.

1.3. *Main results.* The goal of this work is twofold. First, we develop a new approach to obtain lower bounds on $\Phi_{G,\rho}$ and $\widetilde{\Phi}_{G,\rho}$. This method is simpler than the technique developed in [3] and is more generally applicable. In particular, the technique in [3] fails badly when the function ρ grows too slowly (e.g., logarithmically). In contrast, the approach developed below provides good lower bounds on $\Phi_{G,\rho}$ for any increasing slowly varying function ρ and any group G for which one has a lower bound on Φ_G. Second, we develop a method that allows us to obtain sharp upper bounds on $\widetilde{\Phi}_{G,\rho}$ in the context of wreath products. Here, we make essential use of earlier work of A. Erschler [13]. Our contribution is to develop a technique that allows us to harvest the L^1-isoperimetric results of Erschler in order to bound the random walk invariants $\widetilde{\Phi}_{G,\rho}$. Both goals are attained by focusing on the notion of isoperimetric profile (the L^2-isoperimetric profile but also the L^p versions, $p \geq 1$, especially $p = 1$).

Our main results regarding the spectral profile Λ_ϕ and the extremal profile $\widetilde{\boldsymbol{\Lambda}}_{G,\rho}$ are stated in Theorems 2.13–2.15. Theorem 2.13 gives a general and easily applicable upper bound on Λ_ϕ in terms of Λ_G under weak-moment conditions on ϕ. Theorem 2.15 gives a completely satisfactory positive answer to a spectral profile version of Problem 1(1) for a large class of slowly varying functions ρ including all moment conditions of logarithm or iterated logarithm type. The following statement captures the nature of these results.

THEOREM 1.6. *Let G be a finitely generated group. Let $\rho : [0, \infty) \to [1, \infty)$ be a continuous increasing function. The spectral profile functions Λ_G and $\widetilde{\boldsymbol{\Lambda}}_{G,\rho}$ satisfy*

$$\widetilde{\boldsymbol{\Lambda}}_{G,\rho} \lesssim \Lambda_G \int_0^{1/\Lambda_G^{1/2}} \frac{s\,ds}{\rho(s)}.$$

Further, if $\rho(t) \simeq (\int_t^\infty \frac{ds}{(1+s)\ell(s)})^{-1}$ where ℓ is a slowly varying function satisfying $\ell(t^a) \simeq \ell(t)$ for all $a > 0$, we have

$$\widetilde{\boldsymbol{\Lambda}}_{G,\rho} \simeq \frac{1}{\rho(1/\Lambda_G)}.$$

In particular, for any $k \geq 1$ and $\varepsilon > 0$, $\widetilde{\boldsymbol{\Lambda}}_{G,\log_{[k]}^\varepsilon}(v) \simeq 1/\log_{[k]}^\varepsilon(1/\Lambda_G)$.

The second part of this theorem indicates that, for slowly varying functions ρ of the type described above, $\widetilde{\Lambda}_{G,\rho}$ is determined by Λ_G. Given the tight connections between $n \mapsto \phi^{(2n)}(e)$ and $v \mapsto \Lambda_\phi(v)$, this means that Φ_G determines $\widetilde{\Phi}_{G,\rho}$ under some a priori regularity conditions on these functions.

In Section 2.3, we derive a sublinear upper bound for the entropy $H_\mu(n) = \sum_{g \in G}(-\log \mu^{(n)}(g))\mu^{(n)}(g)$ when the symmetric probability measure μ has finite p-moment and satisfies an appropriate L^p-isoperimetric profile condition. For a significant discussion of entropy in the context of random walk theory, see [17] which contains earlier references, and also [11, 14].

The displacement $L_\mu(n) = \mu^{(n)}(|\cdot|)$ is the average distance traveled by the random walk at time n when distance is measure using the word length $|\cdot|$. See, for example, [4] and the references therein. For a discussions of some key relations between entropy and displacement, see [11, 14]. We obtain the following interesting result regarding entropy and displacement. Compare to [15], Theorem 1.4 and Conjecture 1.5.

THEOREM 1.7. *Let G be a finitely generated infinite group such that*

$$\Phi_G(n) \gtrsim \exp(-n^\gamma)$$

for some $\gamma \in (0, \frac{1}{2})$. Let μ be a symmetric probability measure on G with finite p-moment where $p > 2\gamma/(1-\gamma)$ and $p > 1$. Then, for any fixed $\varepsilon > 0$,

$$H_\mu(n) \lesssim \left(n(\log n)^{1+\varepsilon}\right)^{2\gamma/(p(1-\gamma))}.$$

In particular, if $p = 2$, we have $H_\mu(n) \lesssim (n(\log n)^{1+\varepsilon})^{\gamma/(1-\gamma)}$ and

$$L_\mu(n) \lesssim n^{1/(2(1-\gamma))}(\log n)^{(1+\varepsilon)\gamma/(2(1-\gamma))}.$$

The last conclusion follows from the entropy bound by [14], Corollary 1.1, which gives the bound $L_\mu(n) \lesssim \sqrt{nH_\mu(n)}$ assuming that μ is symmetric and has finite second moment.

Section 3 describes applications of the spectral profile upper bound provided by Theorem 2.13 to the problem of bounding $\widetilde{\Phi}_{G,\rho}$ from below. The main result is Theorem 3.2 which gives sharp lower bounds on $\widetilde{\Phi}_{G,\rho}$ in terms of a lower bound on Φ_G for a wide variety of weak-moment conditions. One important feature of this result (which distinguishes it from the results obtained in [3]) is that it is just as effective around the critical weak-moment condition of order 2 than for power weak-moment conditions in the classical range $(0, 2)$ (stable like moment-conditions) and for moment conditions associated with slowly varying functions (including positive powers of any iterated logarithms). Explicit statements are given in Corollaries 3.3, 3.4 and 3.6.

Section 4 is devoted to wreath products. These groups are important for many reasons including the fact that they provide a class of groups of exponential volume growth in which a rich variety of different behaviors of Φ_G occurs. Here, we

provide sharp upper bounds on $\widetilde{\Phi}_{G,\rho}$. More generally, we provide sharp two-sided bounds on $n \mapsto \phi^{(2n)}(e)$ for a wide variety of measures ϕ on wreath products and iterated wreath products. For instance, let $G = \mathbb{Z}_2 \wr H$ be the lamplighter group with the usual binary lamps over a based group H which has polynomial volume growth of degree d. In this simple case, we obtain the following estimates:

$$\widetilde{\Phi}_{G,\rho_2}(n) \simeq \exp(-(n \log n)^{d/(d+2)}),$$

$$\widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq \exp(-n^{d/(d+\alpha)}), \qquad \alpha \in (0,2),$$

$$\widetilde{\Phi}_{G,\log_{[k]}^\varepsilon}(n) \simeq \exp(-n/\log_{[k]}^\varepsilon(n)), \qquad k = 1, 2, \ldots, \varepsilon > 0.$$

The first and last estimates appear to be new even for $H = \mathbb{Z}$. When $H = \mathbb{Z}^d$, the second estimate can be derived from the celebrated Donsker–Varadhan large deviation theorem on the number of visited point by a random walk that belongs to the domain of attraction of a stable law. These results follow from the techniques developed in Section 4 and are part of a large collection of illustrative examples described in Section 5.

A key result concerning wreath products is Erschler's isoperimetric inequality [12, 13] which gives a lower bound on the Følner function of $G = H_1 \wr H_2$ in terms of the Følner functions of H_1 and H_2. Here, we use Erschler's inequality and a new comparison idea to obtain the following L^2-version where $\Lambda_{H,\mu}^{-1}$ denotes the right-continuous inverse of the L^2-isoperimetric profile $\Lambda_{H,\mu}$.

THEOREM 1.8. *There exists a constant $K > 0$ such that, for any symmetric probability measures μ_{H_1} and μ_{H_2} on two finitely generated groups H_1 and H_2, the switch or walk measure $q = \frac{1}{2}(\mu_{H_1} + \mu_{H_2})$ on the wreath product $H_1 \wr H_2$ satisfies*

$$\Lambda_{H_1 \wr H_2, q}(v) \geq s/K \qquad \textit{for any } v \leq \Lambda_{H_1, \mu_{H_1}}^{-1}(s)^{\Lambda_{H_2, \mu_{H_2}}^{-1}(s)/K}.$$

In the other direction,

$$\Lambda_{H_1 \wr H_2, q}(v) \leq s \qquad \textit{for } v \geq \Lambda_{H_1, \mu_{H_1}}^{-1}(s)^{\Lambda_{H_2, \mu_{H_2}}^{-1}(s)} \Lambda_{H_2, \mu_{H_2}}^{-1}(s).$$

The only case where this result is far from sharp is when either $H_1 = \{e\}$ is trivial or H_2 is finite. In those cases, it is a simple matter to obtain the desired sharp results by different arguments. Because of the detailed relations between the L^2-isoperimetric profile Λ_ϕ and the behavior of $n \mapsto \phi^{(2n)}(e)$ (see Section 2.1), the above theorem typically yields good bounds on $q^{(2n)}(e)$ in terms of bounds on $n \mapsto \mu_{H_1}^{(2n)}(e)$ and $n \mapsto \mu_{H_2}^{(2n)}(e)$.

2. The isoperimetric functions $\Lambda_{p,\phi}$, $p \geq 1$.

2.1. Φ, Λ *and the Nash profile.* In this section, we quickly review Coulhon's results from [5] which, in the present context, relate the behavior of $n \mapsto \phi^{(2n)}(e)$

to that of the spectral profile $v \mapsto \Lambda_\phi(v)$. We refer the reader to [5] for references to earlier related works, in particular, work by Grigor'yan in which the spectral profile play a key role.

It is convenient to introduce the notion of Nash profile. Namely, define the Nash profile $\mathcal{N}_A$ of a symmetric Markov generator A with associated Dirichlet form $\mathcal{E}_A$ by

$$\mathcal{N}_A(t) = \sup\left\{\frac{\|f\|_2^2}{\mathcal{E}_A(f,f)} : f \in \mathrm{Dom}(\mathcal{E}_A) \text{ with } 0 < \|f\|_1^2 \le t\|f\|_2^2\right\}$$

so that, for all f in the domain of the Dirichlet form $\mathcal{E}_A$,

$$\|f\|_2^2 \le \mathcal{N}_A(\|f\|_1^2/\|f\|_2^2)\mathcal{E}_A(f,f).$$

For our purpose, we can restrict ourselves to the case when A is convolution by $\delta_e - \phi$, for some symmetric probability measure ϕ on G. In this case, with some abuse of notation, $\mathcal{E}_A = \mathcal{E}_\phi$, $\mathrm{Dom}(\mathcal{E}_A) = L^2(G)$ and we will write $\mathcal{N}_\phi$ for the Nash profile of $A = \cdot * (\delta_e - \phi)$. The following lemma relates the L^2-isoperimetric profile and the Nash profile.

LEMMA 2.1 (Folklore). *For any symmetric probability measure ϕ on a (countable) group G, we have*

$$\forall v > 0, \qquad \frac{1}{\Lambda_\phi(v)} \le \mathcal{N}_\phi(v) \le \frac{2}{\Lambda_\phi(4v)}.$$

PROOF. For any finite set Ω and any function f with support in Ω,

$$\|f\|_1^2 \le |\Omega| \|f\|_2^2.$$

Hence, the lower bound on $\mathcal{N}_\phi$ follows easily from the definitions of $\mathcal{N}_\phi$ and Λ_ϕ. Conversely, the definition of Λ_ϕ gives

$$\|f\|_2^2 \le \Lambda_\phi(|\mathrm{support}(f)|)^{-1}\mathcal{E}_\phi(f,f).$$

For any $t \ge 0$, set $f_t = \max\{f - t, 0\}$ and observe that, for any nonnegative f, $|f|^2 \le (f_t)^2 + 2tf$ and $\mathcal{E}(f_t, f_t) \le \mathcal{E}(f,f)$. It follows that, for any t and $f \ge 0$,

$$\|f\|_2^2 \le \Lambda_\phi(|\{f \ge t\}|)^{-1}\mathcal{E}(f,f) + 2t\|f\|_1.$$

Picking t such that $4t = \|f\|_2^2/\|f\|_1$ and using $|\{f \ge t\}| \le t^{-1}\|f\|_1$, we obtain

$$\|f\|_2^2 \le 2\Lambda_\phi(4\|f\|_1^2/\|f\|_2^2)^{-1}\mathcal{E}(f,f).$$

The upper bound on $\mathcal{N}_\phi$ follows. $\square$

THEOREM 2.2 (Essentially, [5], Proposition II.1).   *We have*

$$\phi^{(2n+2)}(e) \le 2\psi(2n),$$

where $\psi : [0, +\infty) \to [1, +\infty)$ *is defined implicitly by*

$$t = \int_1^{1/\psi(t)} \frac{ds}{2s\,\Lambda_\phi(4s)}.$$

PROOF.   It is convenient to observe that

$$\phi^{(2n+2)}(e) \le 2h_{2n}^\phi(e) \quad \text{and} \quad h_{4n}^\phi(e) \le e^{-2n} + \phi^{(2n)}(e),$$

where

(2.1) $$h_t^\phi = e^{-t} \sum_0^\infty \frac{t^k}{k!}\phi^{(k)}.$$

See, for example, [21], Section 3.2. Convolution by h_t^ϕ defines the continuous time semigroup associated with the continuous time random walk driven by ϕ. Lemma 2.1 gives us the Nash inequality

$$\|f\|_2^2 \le 2\Lambda_\phi\big(4\|f\|_1^2/\|f\|_2^2\big)^{-1}\mathcal{E}(f, f).$$

Using this inequality in the proof of [5], Proposition II.1, gives $h_t^\phi(e) \le \psi(t)$.   □

The following is a sort of converse of Theorem 2.2.

THEOREM 2.3 ([5], Proposition II.2).   *For* $v \ge 1$,

$$\Lambda_\phi(v) \ge \sup_{t>0}\left\{\frac{1}{2t}\log\frac{1}{vh_t^\phi(e)}\right\}.$$

REMARK 2.4.   Assume that ψ is a continuous decreasing function with continuous derivative with the property that there exists $\varepsilon >$ such that for all $t > 0$ and all $s \in (t, 2t)$ we have

$$\frac{-\psi'(s)}{\psi(s)} \ge \varepsilon\frac{\psi'(t)}{\psi(t)}.$$

As noted in [5] and elsewhere, under this condition the functions

$$x \mapsto \Lambda(x) = \sup_{t>0}\left\{\frac{1}{2t}\log\frac{1}{x\psi(t)}\right\} \quad \text{and} \quad x \mapsto -\frac{1}{x}\psi' \circ \psi^{-1}(x)$$

are $\simeq$-equivalent. Hence, under this regularity condition on ψ, $n \mapsto \phi^{(2n)}(e) \lesssim \psi$ is equivalent to $\Lambda \lesssim \Lambda_\phi$.

The following lemma will be useful later.

LEMMA 2.5. *Assume that $\phi^{(n)}(e) \geq \exp(-n/\pi(n))$ where $\pi : (0, \infty) \to (0, \infty)$ is an increasing function satisfying $\pi(t) \leq ct$. Then there exists A such that for all n we have*

$$\Lambda_\phi(\exp(An/\pi(n)) \leq \frac{A}{2\pi(n)}.$$

PROOF. Let ψ be defined in terms of Λ_ϕ as in Theorem 2.2. By definition and since Λ_ϕ is a nonincreasing function, we have

$$t \leq \frac{\log(1/\psi(t))}{2\Lambda(1/\psi(t))}$$

which we rewrite as

$$\Lambda_\phi(1/\psi(t)) \leq \frac{\log(1/\psi(t))}{2t}.$$

By Theorem 2.2 and the hypothesis, for A large enough,

$$\exp(-An/\pi(n)) \leq \psi(n).$$

Hence,

$$\Lambda_\phi(\exp(An/\pi(n))) \leq \Lambda_\phi(1/\psi(n)) \leq \frac{A}{2\pi(n)}. \qquad \square$$

REMARK 2.6. In most cases, $n \to An/\pi(n)$ is invertible and the lemma gives an upper-bound on Λ_ϕ.

COROLLARY 2.7 (Folklore). *Let ϕ be a symmetric probability measure on G.*

- *If $\phi^{(n)}(e) \geq \exp(-n^\gamma)$. Then $\Lambda_\phi(v) \leq C[\log(e + v)]^{-(1-\gamma)/\gamma}$.*
- *If $\phi^{(n)}(e) \geq \exp(-n/[\log n]^\gamma)$. Then $\Lambda_\phi(v) \leq C[\log(e + \log(e + v))]^{-\gamma}$.*

2.2. *The profiles $\Lambda_{p,\phi}$.* The L^2-isoperimetric profile $\Lambda_{2,\phi} = \Lambda_\phi$ is naturally related to the analogous L^1-profile

$$\Lambda_{1,\phi}(v) = \inf\left\{\frac{1}{2}\sum_{x,y}|f(xy) - f(x)|\phi(y) : |\text{support}(f)| \leq v, \|f\|_1 = 1\right\}.$$

Using an appropriate discrete co-area formula, $\Lambda_{1,\phi}$ can equivalently be defined by

$$\Lambda_{1,\phi}(v) = \inf\left\{|\Omega|^{-1}\sum_{x,y}\mathbf{1}_\Omega(x)\mathbf{1}_{G\setminus\Omega}(xy)\phi(y) : |\Omega| \leq v\right\}.$$

If we define the boundary of Ω to be the set

$$\partial\Omega = \{(x, y) \in G \times G : x \in \Omega, y \in G \setminus \Omega\}$$

and set $\phi(\partial\Omega) = \sum_{x\in\Omega, xy\in G\setminus\Omega} \phi(y)$ then $\Lambda_{1,\phi}(v) = \inf\{\phi(\partial\Omega)/|\Omega| : |\Omega| \leq v\}$.

From these definitions and remarks, it follows that

$$(2.2) \qquad \frac{1}{2}\Lambda_{1,\phi}^2 \leq \Lambda_{2,\phi} \leq \Lambda_{1,\phi}.$$

The upper bound is very straightforward since it suffices to test the definition of $\Lambda_{2,\phi}$ on functions of the type $\mathbf{1}_\Omega$ to obtain it. The lower bound is obtained by testing the definition of $\Lambda_{1,\phi}$ on functions of the form f^2, $f \geq 0$, and using the Cauchy–Schwarz inequality. In fact, for any $p \geq 1$, set

$$\mathcal{E}_{p,\phi}(f) = \frac{1}{2}\sum_{x,y}|f(xy) - f(x)|^p\phi(y)$$

and

$$(2.3) \qquad \Lambda_{p,\phi}(v) = \inf\{\mathcal{E}_{p,\phi}(f) : |\text{support}(f)| \leq v, \|f\|_p = 1\}.$$

PROPOSITION 2.8 (Folklore). *For $1 \leq p \leq q < \infty$, we have*

$$(2.4) \qquad c(p,q)\Lambda_{p,\phi}^{q/p} \leq \Lambda_{q,\phi} \leq C(p,q)\Lambda_{p,\phi}.$$

PROOF. This is closely related but different from [9], Corollaire 3.2. The inequality $c(p,q)\Lambda_{p,\phi}^{q/p} \leq \Lambda_{q,\phi}$, $1 \leq p \leq q$, which is a form of Cheeger's inequality, is obtained by testing $\Lambda_{p,\phi}$ on functions of the form $f^{q/p}$, $f \geq 0$, and using Hölder inequality. The inequality $\Lambda_{q,\phi} \leq C(p,q)\Lambda_{p,\phi}$ can be proved as follows.

For any function $f \geq 0$, set $f_k = (f - 2^k)^+ \wedge 2^k$, $k \in \mathbb{Z}$. By [1], Section 6, we have

$$(2.5) \qquad \left(\sum_k \mathcal{E}_{p,\phi}(f_k)^{\alpha/p}\right)^{1/\alpha} \leq 2(1+p)\mathcal{E}_{p,\phi}(f).$$

This should be understood as an L^p substitute for the L^1 co-area formula.

Now, if we assume that $|\text{support}(f)| \leq v$, we have

$$\Lambda_{q,\phi}(v)\|f_k\|_q^q \leq \mathcal{E}_{q,\phi}(f_k).$$

Noting that $f_k \geq 2^k$ on $\{f \geq 2^{k+1}\}$ and that $0 \leq f_k/2^k \leq 1$, we obtain

$$\Lambda_{q,\phi}(v)2^{kq}|\{f \geq 2^{k+1}\}| \leq 2^{k(q-p)}\mathcal{E}_{p,\phi}(f_k).$$

This gives

$$(2.6) \qquad \Lambda_{q,\phi}(v)2^{(k+1)p}|\{f \geq 2^{k+1}\}| \leq 2^p\mathcal{E}_{p,\phi}(f_k).$$

It is easy to check that (see, e.g., [1], (4.2))

$$\|f\|_p^p \leq 2^p\sum_k 2^{(k+1)p}|\{f \geq 2^{k+1}\}|.$$

Using (2.6) and (2.5), this yields

$$\Lambda_{q,\phi}(v)\|f\|_p^p \leq 2(1+p)4^p\mathcal{E}_{p,\phi}(f).$$

Optimizing over all f implies that $\Lambda_{q,\phi}(v) \leq 2(1+p)4^p\Lambda_{p,\phi}(v)$ as desired. $\qquad\square$

2.3. *Entropy upper bounds using* $\Lambda_{p,G}$ *upper bounds.* Given a probability measure μ on G, its entropy function H_μ is defined by

$$H_\mu(n) = \sum_{g \in G} -\big(\log \mu^{(n)}(g)\big)\mu^{(n)}(g).$$

See, for example, [10, 11, 17]. Recall that **u** denotes the uniform probability measure on the symmetric generating set S^* (by definition, S^* contains the identity element). Also, consider the displacement function

$$n \mapsto L_\mu(n) = \sum_{g \in G} |g|\mu^{(n)}(g).$$

THEOREM 2.9. *Assume that there exist $p \in (1, 2]$, $\alpha \in (0, 1)$ and an increasing slowly varying function ℓ such that*

$$\Lambda_{p,\mathbf{u}}(v) \le \frac{\ell(\log(e + v))}{\log(e + v)^{1/\alpha}}.$$

For any symmetric probability measure μ with a finite p-moment $\sum |g|^p \mu(g) < \infty$, we have

$$(2.7) \qquad H_\mu(n) \le C(\mu, p, \omega)n^\alpha \omega(n)$$

for any increasing slowly varying function ω such that

$$\ell(y^{\alpha\eta})^\alpha \big[\log(e + y^{\alpha\eta})\big]^{\alpha(1+\varepsilon)} \le \omega(y)$$

for some $\eta > 1$ and $\varepsilon > 0$. Further

$$(2.8) \qquad L_\mu(n) \le C(\mu, p, \omega)n^{[(3-p)+\alpha(p-1)]/2}\omega(n)^{(p-1)/2}.$$

REMARK 2.10. Assume that the group G satisfies $\Phi_G(n) \ge \exp(-n^\gamma)$ where $\gamma \in (0, 1)$. By Lemma 2.5, we have $\Lambda_{2,G}(v) \le C[\log(e + v)]^{-(1-\gamma)/\gamma}$ and

$$\Lambda_{p,G}(v) \le C\big[\log(e + v)\big]^{-\alpha_p}, \qquad \alpha_p = \frac{2\gamma}{p(1-\gamma)}, p \in [1, 2].$$

If $\gamma \in (0, 1/2)$, then $2\gamma/(1-\gamma) < 2$. For any $p > 1$, such that $2\gamma/(1-\gamma) < p \le 2$, we have $\alpha_p = 2\gamma/(p(1-\gamma)) \in (0, 1)$. Under these hypotheses, for any symmetric measure μ with finite p-moment, Theorem 2.9 gives

$$H_\mu(n) \le C_\mu n^{\alpha_p}(\log n)^{\alpha_p(1+\varepsilon)}$$

for any $\varepsilon > 0$. In particular, the entropy of μ is sublinear and the entropy criteria ([17], Theorem 1.1) implies that bounded μ-harmonic functions must be constant.

REMARK 2.11. The lamplighter group $G = \mathbb{Z}_2 \wr \mathbb{Z}^2$ satisfies $\Phi_G(n) \simeq \exp(-n^{1/2})$ [equivalently, $\Lambda_G(v) \simeq \log(e + v)^{-1}$] and $H_{\mathbf{u}}(n) \simeq n/\log n$. See [11, 13, 22]. This example is just beyond the limit of application of our result. Kotowski and Virág [18] describes a group G for which $\Phi_G(n) \gtrsim \exp(-n^{1/2+o(1)})$ and for which simple random walk has linearly growing entropy (the group has nontrivial bounded harmonic functions).

REMARK 2.12. Theorem 2.9 is related to ([15], Theorem 1.4) and ([15], Conjecture 1.5) and some of the ingredients of the proof given below are similar to those used in [15]. The hypothesis (OD) appearing in [15], Theorem 1.4, plays no role in Theorem 2.9.

PROOF. The proof of (2.7) uses the embedding of G into a L^p space introduced in [30] together with [19], Theorem 2.1.

For each k, let ϕ_k be a function supported in a set U_k of size 2^{2^k} and such that

$$\Lambda_{p,\mathbf{u}}(2^{2^k}) = \inf\left\{ \frac{1}{2} \sum_{x,y} |f(x) - f(xy)|^p \mathbf{u}(y) : |\text{support}(f)| \leq 2^{2^k}, \|f\|_p = 1 \right\}$$

is greater than

$$\frac{1}{4} \frac{\sum_{x,y} |\phi_k(x) - \phi_k(xy)|^p \mathbf{u}(y)}{\sum_x |\phi_k(x)|^p}.$$

Let $\mathbf{B}_p(G)$ be the Banach space of sequences $(w_k)_1^\infty$ of elements of $\ell^p(G)$ such that $\sum_k \|w_k\|_p^p < \infty$ equipped with the norm $\|w\|_p = (\sum_k \|w_k\|_p^p)^{1/p}$.

Consider the embedding b of the group G into $\mathbf{B}_p(G)$ defined by

$$b(g) = \left(c_k \frac{\phi_k - \tau_r(g)\phi_k}{\mathcal{E}_p(\phi_k)^{1/p}} \right)_1^\infty, \qquad c_k^p = \frac{1}{(1+k)^{1+\varepsilon}},$$

where $\tau_r(g)f : x \mapsto f(xg)$ is right translation by g and

$$\mathcal{E}_p(f) = \frac{1}{2} \sum_{x,y} |f(xy) - f(x)|^p \mathbf{u}(y).$$

By construction, this is a 1-cocycle, more precisely, an element of $Z^1(G, \tau_r, \mathbf{B}_p(G))$. Indeed, for each $g \in G$, $b(g)$ belongs to $\mathbf{B}_p(G)$ because

$$(2.9) \qquad \|f - \tau_r(g)f\|_p^p \leq |S| |g|^p \sum_{x,y} |f(xy) - f(x)|^p \mathbf{u}(y)$$

and $\sum c_k^p < \infty$.

Set $\Omega_0 = \varnothing$ and $\Omega_k = [\bigcup_1^k \tilde{U}_i]^{-1}[\bigcup_1^k \tilde{U}_i]$ where $\tilde{U}_i = U_i \cup (S^*)^i$. Note that for $g \notin \Omega_k$, the functions ϕ_k and $\tau_r(g)\phi_k$ have disjoint supports and write

$$\|b(g)\|_p^p \geq \sum_1^\infty c_k^p \frac{2\|\phi_k\|_p^p}{\mathcal{E}_p(\phi_k)} \mathbf{1}_{\{G \setminus \Omega_k\}}(g)$$

(2.10)

$$\geq \sum_1^\infty c_k^p \frac{1}{\Lambda_{p,\mathbf{u}}(2^{2^k})} \mathbf{1}_{\{\Omega_{k+1} \setminus \Omega_k\}}(g).$$

Set

$$Z_n = \sum_1^\infty k \mathbf{1}_{\Omega_k \setminus \Omega_{k-1}}(S_n)$$

and write $H_\mu(n) \leq H_\mu(S_n|Z_n) + H_\mu(Z_n)$. We have $\mathbf{E}_\mu(Z_n) \leq n(\sum |g|\mu(g))$ which easily gives $H_\mu(Z_n) \leq C(1 + \log n)$. It will suffice to bound $H_\mu(S_n|Z_n)$. By a well-known convexity argument (see, e.g., [4], page 1148)

$$H_\mu(S_n|Z_n) \leq \mathbf{E}_\mu\left(\sum_1^\infty \log|\Omega_k \setminus \Omega_{k-1}| \mathbf{1}_{\Omega_k \setminus \Omega_{k-1}}(S_n)\right).$$

By hypothesis,

$$\log(e + |\Omega_{k+1}|) \simeq 2^k \simeq \log(e + |\Omega_k|) \leq C \frac{\ell(2^k)}{\Lambda_{p,\mathbf{u}}(2^{2^k})^\alpha}.$$

Let $F : [0, \infty) \to [0, \infty)$ be a concave increasing function with $F(0) = 0$ and such that

$$at^\alpha \omega(t) \leq F(t) \leq At^\alpha \omega(t),$$

where ω is as in the statement of the theorem. As $c_k^p = (1+k)^{-1-\varepsilon}$, one can check that

$$\frac{\ell(2^k)}{\Lambda_{p,\mathbf{u}}(2^{2^k})^\alpha} \leq CF\left(\frac{c_k^p}{\Lambda_{p,\mathbf{u}}(2^{2^k})}\right)$$

from which it follows that (with a different constant C)

$$\log|\Omega_{k+1}| \leq CF\left(\frac{c_k^p}{\Lambda_{p,\mathbf{u}}(2^{2^k})}\right).$$

Now, we have

$$\mathbf{E}_\mu\left(\sum_1^\infty \log|\Omega_{k+1}| \mathbf{1}_{\Omega_{k+1} \setminus \Omega_k}(S_n)\right)$$

$$\leq C\mathbf{E}_\mu\left(\sum_1^\infty F(c_k^p/\Lambda_{p,G}(2^{2^k})) \mathbf{1}_{\Omega_{k+1} \setminus \Omega_k}(S_n)\right)$$

$$= C\mathbf{E}_\mu\left(F\left(\sum_1^\infty c_k^p/\Lambda_{p,G}(2^{2^k})\mathbf{1}_{\Omega_{k+1}\setminus\Omega_k}(S_n)\right)\right)$$

$$\leq CF\left(\mathbf{E}_\mu\left(\sum_1^\infty \frac{c_k^p}{\Lambda_{p,G}(2^{2^k})}\mathbf{1}_{\Omega_{k+1}\setminus\Omega_k}(S_n)\right)\right)$$

$$\leq CF(\mathbf{E}_\mu(\|b(S_n)\|_p^p)),$$

where the second to last inequality is Jensen's inequality applied to the concave function F. Finally, we appeal to [19], Theorem 2.1 and (2.9) to conclude that (since $1 < p \leq 2$)

$$\mathbf{E}_\mu(\|b(S_n)\|_p^p) \leq C_p n\mathbf{E}_\mu(|b(S_1)|^p) \leq C(p, S)n\sum_x |x|^p\mu(x).$$

The statement in [19] is for simple random walk but the proof works for an arbitrary symmetric measure μ with finite p-moment. Note that $p > 1$ is essential here. This completes the proof of the entropy bound (2.7).

We now explain how (2.8) follows from (2.7). The statement in [14], Corollary 5.2(i), gives the bound

$$L_\mu(n) \leq C\sqrt{nH_\mu(n)}$$

under the assumption that the symmetric probability measure μ has finite second moment. This follows from two bounds:

(a) $|L_\mu(n+1) - L_\mu(n)| \leq C\beta(n)$ ([14], Corollary 5.2(i)),
(b) $\beta(n) \leq C\sqrt{H_\mu(n+1) - H_\mu(n)}$ ([14], Lemma 5.1(ii))

where

$$\beta(n) = \sup_{s\in S}\left\{\sum_{g\in G}|\mu^{(n)}(gs) - \mu^{(n)}(g)|\right\}.$$

The hypothesis that μ has second moment enters (a) but is not necessary for (b). If we replace the hypothesis that μ has finite second moment by the hypothesis that μ as finite weak-p-moment $W(\rho_p, \mu) < \infty$ for some $p \in (1, 2]$, an easy modification of the proof of (a) given in [14] gives:

(a′) $|L_\mu(n+1) - L_\mu(n)| \leq C\beta(n)^{p-1}$.

Set $\theta = (p-1)/2 \in (0, 1/2]$ and write

$$L_\mu(n) = \sum_1^n \frac{L_\mu(j) - L_\mu(j-1)}{\beta(j)^{p-1}}\beta(j)^{p-1}$$

$$\leq \left(\sum_1^n\left(\frac{L_\mu(j) - L_\mu(j-1)}{\beta(j)^{p-1}}\right)^{1/(1-\theta)}\right)^{1-\theta}\left(\sum_1^n\beta(j)^2\right)^\theta$$

$$\leq Cn^{1-\theta}H_\mu(n)^\theta = Cn^{(3-p)/2}H_\mu(n)^{(p-1)/2}.$$

This shows that (2.8) follows from (2.7). □

2.4. *Comparison of* $\Lambda_{p,\phi}$ *with* $\Lambda_{p,\mathbf{u}}$. By definition, we let $\Lambda_{p,G}$ be the $\simeq$-equivalence class of $\Lambda_{p,\phi}$ when ϕ is a fixed symmetric measure with finite generating support on G. Note that $\Lambda_{p,G}$ does not depend on the choice of ϕ. We refer to this case as the classical case.

This subsection is devoted to a simple yet very useful result that provides upper bounds for $\Lambda_{p,\phi}$, $p \geq 1$ in terms of $\Lambda_{p,G}$ and basic information on the probability measure ϕ. We can represent $\Lambda_{p,G}$ by $\Lambda_{p,\mathbf{u}}$ where $\mathbf{u}$ is the uniform measure on the fixed generating finite symmetric set S^*.

For any increasing continuous function $\rho : [0, \infty) \to [1, \infty)$, set

$$(2.11) \qquad M_{p,\rho}(t) = t^p \left(\int_0^t \frac{s^{p-1}}{\rho(s)} \, ds \right)^{-1}.$$

Note that we always have

$$M_{p,\rho}(t) \leq p\rho(t).$$

Further, when ρ is regularly varying of index $\alpha \in [0, \infty)$, we have $M_{p,\rho} \simeq \rho$ if $\alpha \in [0, p)$ and $M_{p,\rho}(t) \simeq t^p$ if $\alpha > p$. In the case $\alpha = p$, explicit computations are necessary. For instance, when $\rho(t) = (1+t)^p$, $M_{p,\rho}(t) \simeq 1 + \log(1+t)$.

The following theorem will be used to obtain good lower bounds on $\widetilde{\Phi}_{G,\rho}$, in particular, when ρ is a slowly growing function.

THEOREM 2.13. *Let ϕ be a symmetric probability measure satisfying the weak moment condition*

$$W(\rho, \phi) = \sup_{s>0}\{s\phi(\{x : \rho(|x|) > s\})\} \leq K.$$

Then for any $v > 0$ and $p \in [1, \infty)$, we have

$$\Lambda_{p,\phi}(v) \leq C(p, \rho)K \inf_{s>0}\left\{ \frac{1}{\rho(s)} + \frac{|S^*|s^p}{M_{p,\rho}(s)}\Lambda_{p,\mathbf{u}}(v) \right\}.$$

In particular,

$$\Lambda_{p,\phi}(v) \leq \frac{C(p, \rho, |S^*|, K)}{M_{p,\rho}(\Lambda_{p,\mathbf{u}}(v)^{-1/p})}.$$

PROOF. Recall that

$$\Lambda_{p,\phi}(v) = \inf\left\{ \frac{1}{2}\sum_{x,y}|f(xy) - f(x)|^p\phi(y) : |\text{support}(f)| \leq v, \|f\|_p = 1 \right\}.$$

For any function f, write

$$
\sum_{x,y} |f(xy) - f(x)|^p \phi(y) = \sum_x \sum_{|y| \le s} |f(xy) - f(x)|^p \phi(y)
$$

(2.12)

$$
+ \sum_x \sum_{|y| > s} |f(xy) - f(x)|^p \phi(y).
$$

Making use of the well-known (pseudo-Poincaré) inequality [9]

$$
\forall y \in G, \qquad \sum_x |f(xy) - f(x)|^p \le |S^*| |y|^p \sum_{x,z} |f(xz) - f(x)|^p \mathbf{u}(z)
$$

the first right-hand term is bounded by

$$
|S^*| \left(\sum_{|y| \le s} |y|^p \phi(y) \right) \sum_{x,y} |f(xy) - f(x)|^p \mathbf{u}(y).
$$

Further,

$$
\sum_{|y| \le s} |y|^p \phi(y) = p \sum_{0 \le k \le s} (k+1)^{p-1} \phi(\{x : |x| > k\})
$$

$$
\le pK \sum_{0 \le k \le s} \frac{(1+k)^p}{\rho(k)} \le C(p, \rho) K s^p / M_{p,\rho}(s).
$$

To bound the second term on the right-hand side of (2.12), we let

$$
\phi_s'(y) = (\phi(\{x : |x| > s\}))^{-1} \phi(y) \mathbf{1}_{\{|\cdot| > s\}}(y)
$$

and write

$$
\sum_x \sum_{|y| > s} |f(xy) - f(x)|^p \phi(y) \le \phi(\{x : |x| > s\}) \mathcal{E}_{\phi_s'}(f, f) \le \frac{2K}{\rho(s)} \|f\|_2^2.
$$

The desired inequality follows [with an adjusted constant $C(p, \rho)$]. $\quad\square$

2.5. *Subordination.* This section introduces notation and results regarding the notion of subordination. We will use this notion in several important ways. For more background and further references to the literature, see [2, 3].

Recall that a Bernstein function is a function $f : (0, \infty) \to \mathbb{R}$ such that

(2.13)
$$
f(s) = a + bs + \int_{(0,\infty)} (1 - e^{-st}) \nu(dt),
$$

where $a, b \ge 0$ and ν is a measure satisfying $\int_{(0,2)} d\nu(dt) + \int_{(1,\infty)} \nu(dt) < \infty$. The measure ν is called the Lévy measure of f. See [28] for details. For our purpose, it suffices to consider the case $a = b = 0$. The most classic example of Bernstein function is $s \mapsto s^\alpha$, $\alpha \in (0, 1)$, which has $\nu(dt) = \alpha \Gamma(\alpha - 1)^{-1} t^{-1-\alpha} \, dt$.

A complete Bernstein function is a function f of the form

$$f(s) = s^2 \int_0^\infty e^{-ts} g(t) \, dt,$$

where g is a Bernstein function. These are Bernstein functions and they are also called operator monotone functions. See [28], Chapter 6.

Given a Bernstein function f, and a reversible Markov generator A, we can always form the operator $f(A)$ which is also the generator of a reversible Markov semigroup $e^{-tF(A)}$, $t \geq 0$. In the case of interest to us here, A is the operator of right-convolution by $\delta_e - \phi$ on a group G where ϕ is a symmetric probability measure which is (minus) the generator of the continuous time semigroup $e^{-tA} = H_t^\phi = \cdot * h_t^\phi$ with h_t^ϕ defined at (2.1). Similarly, assuming $f(0) = a = 0$ and $f(1) = 1$, we have

$$f(A) = \cdot * (\delta_e - \phi_f)$$

with

$$\phi_f = \sum_1^\infty c(f, n) \phi^{(n)},$$

where the coefficients $c(f, n)$ are given by the Taylor series $1 - f(1 - x) = \sum_1^\infty c(f, n) x^n$ at $x \sim 0$. Equivalently and more explicitly (see [2]),

(2.14)
$$c(f, 1) = b + \int_0^\infty t e^{-t} v(dt),$$

$$c(f, n) = \int_0^\infty t^n e^{-t} v(dt), \qquad n > 1.$$

Obviously, the continuous time semigroup $e^{-tf(A)}$ is also the semigroup of right-convolution by $h_t^{\phi_f}$. Further, because of the representation of f using the measure v (see the definition of Bernstein function), assuming that $a = 0$, we have

$$f(A) = bA + \int_{(0,\infty)} (I - e^{-tA}) v(dt).$$

The following elegant result gives a sharp inequality for the Nash profile of $f(A)$, that is, in our setting, the Nash profile of ϕ_f.

THEOREM 2.14 ([29], Theorem 1). *Let f be a Bernstein function with $f(0) = 0$, $f(1) = 1$, and Lévy measure v. Referring to the above setting and notation, for any symmetric probability measure ϕ on G, the Nash profile $\mathcal{N}_{\phi_f}$ satisfies*

$$\forall v > 0, \qquad \mathcal{N}_{\phi_f}(v) \leq \frac{2}{f(1/\mathcal{N}_\phi(2v))}.$$

Further, for any function u such that $\|u\|_1^2/\|u\|_2^2 \leq v$,

$$(2.15) \qquad \frac{\mathcal{E}_{\phi_f}(u,u)}{\|u\|_2^2} \geq \frac{1}{2\mathcal{N}_\phi(2v)} \int_0^{\mathcal{N}_\phi(2v)} \nu(s,\infty)\,ds.$$

The second statement is obtained in the proof of the first inequality provided in [29]. By Lemma 2.1, the Nash profile inequality stated above translates into the L^2-isoperimetric profile inequality

$$(2.16) \qquad f\big(\Lambda_\phi(8v)/2\big) \leq 2\Lambda_{\phi_f}(v).$$

2.6. *Extremal profile under a moment condition.* In this subsection, we focus on the L^2-profile $\Lambda_{2,\phi} = \Lambda_\phi$ and on symmetric probability measures ϕ with a finite weak moment $W(\rho, \phi)$ relative to a natural class of slowly varying functions ρ. We show that, in this context, the upper bound of Theorem 2.13 is sharp for any (amenable) group G. To make this important result precise, we need the following notation.

Consider the set of all continuous increasing functions $\rho : [0, \infty) \to [1, \infty)$ such that

$$(2.17) \qquad \rho(t) \simeq \left(\int_t^\infty \frac{ds}{(1+s)\ell(s)} \right)^{-1},$$

where ℓ is a continuous increasing regularly varying function $\ell : [0, \infty) \to [1, \infty)$ of index $\alpha \geq 0$ and such that $\int_0^\infty \frac{ds}{(1+s)\ell(s)} < \infty$. Under the condition $\alpha \in [0, 1)$, [2], Theorems 2.5–2.6, shows that $\rho(t) \asymp 1/\psi(1/t)$ where ψ is a complete Bernstein function satisfying $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(s) \sim c \int_0^s \frac{ds}{s\ell(1/s)}$ for some $c > 0$. Further, $1 - \psi(1-x) = \sum_1^\infty c(\psi, n)x^n$ with $0 \geq c(\psi, n) \sim \frac{1}{n\ell(n)}$.

Now, referring to (2.17), assume that $\alpha = 0$ and that the slowly varying function ℓ satisfies $\ell(t^a) \simeq \ell(t)$ for any $a > 0$. Proposition 4.2 and Remark 4.4 of [2] show that, on any group G, the symmetric probability measure

$$\mathbf{u}_\psi = \sum_1^\infty c(\psi, n)\mathbf{u}^{(n)}$$

obtained by ψ-subordination of $\mathbf{u}$ (recall that $\mathbf{u}$ is uniform on the fixed generating set S^* of G) satisfies

$$W(\rho, \mathbf{u}_\psi) \simeq \sup_{n \geq 1}\left\{ \rho(n) \sum_n^\infty \frac{1}{k\ell(k)} \right\} < +\infty.$$

That is, $\mathbf{u}_\psi$ has finite weak ρ-moment.

THEOREM 2.15. *Let G be a finitely generated amenable group. Let $\rho :$ $[0, \infty) \to [1, \infty)$ be of the type* (2.17) *with ℓ slowly varying and satisfying*

$\ell(t^a) \simeq \ell(t)$ *for all $a > 0$. Let ψ be the associated complete Bernstein function described above. There are constants $c = c(G, |S^*|, \rho)$, $C = C(G, |S^*|, \rho) \in (0, \infty)$ such that $W(\rho, \mathbf{u}_\psi) \le C <$ and*

$$\frac{c}{\rho(1/\Lambda_{\mathbf{u}}(v))} \le \Lambda_{\mathbf{u}_\psi}(v) \le \frac{C}{\rho(1/\Lambda_{\mathbf{u}}(v))}.$$

Further, for any symmetric probability measure ϕ with $W(\rho, \phi) \le K$

$$\Lambda_\phi(v) \le \frac{CK}{\rho(1/\Lambda_{\mathbf{u}}(v))}.$$

In particular, the extremal profile $\widetilde{\boldsymbol{\Lambda}}_{G,\rho}$ satisfies

$$\widetilde{\boldsymbol{\Lambda}}_{G,\rho}(v) \simeq \frac{1}{\rho(1/\Lambda_G(v))}.$$

PROOF.   Since the upper bounds are given by Theorem lower on $\Lambda_{\mathbf{u}_\psi}$. This follows easily from (2.16), that is, from Lemma 2.1 and Theorem 2.14 (i.e., [29], Theorem 1).   □

REMARK 2.16.   Theorem 2.15 can be interpreted as an "almost positive" answer to Problem 1(1) in the case where ρ is of the type (2.17) with ℓ slowly varying and satisfying $\ell(t^a) \simeq \ell(t)$ for all $a > 0$ (e.g., $\rho(t) = [1 + \log(1 + t)]^\alpha$, $\alpha > 0$). Indeed, Theorem 2.15 says that Λ_G determines $\widetilde{\Lambda}_{G,\rho}$ for such ρ and this result can be transferred to the relation between Φ_G and $\widetilde{\Phi}_{G,\rho}$ to the extend that Theorems 2.2–2.3 give tight relations between the Λ's and the Φ's. See the next section for more explicit statements.

REMARK 2.17.   On $\mathbb{Z}$ or $\mathbb{Z}^d$, if $\psi_\beta(s) = s^\beta$, $\beta \in (0, 1)$, $\rho_\alpha(s) = (1 + s)^\alpha$, $\alpha > 0$, then $W(\rho_\alpha, \mathbf{u}_{\psi_{\alpha/2}}) < \infty$ since, in fact, $\mathbf{u}_{\psi_\beta}(x) \asymp (1 + |x|)^{-\beta - d}$. However, on a general amenable group G, it is not true that $W(\rho_\alpha, \mathbf{u}_{\psi_{\alpha/2}}) < \infty$. Indeed, the optimal moment condition one should expect from $\mathbf{u}_{\psi_\beta}$ is a weak $\rho_{\beta\gamma}$-moment were $\gamma \in [1/2, 1]$ is the displacement exponent of simple random walk on G. See [2] for details. Because of this, it is an open question whether Λ_G determines $\widetilde{\Lambda}_{G,\rho_\alpha}$ for $\alpha \in (0, 2)$ and, in fact, the authors believe the answer to this open question is likely to be negative.

3. Lower bounds on Φ_ρ.   Together, Lemma 2.5 and Theorem 2.13 provide an excellent way to obtain lower bounds on convolution powers of measures with a given moment condition, that is, on the group invariants $\Phi_{G,\rho}$, $\widetilde{\Phi}_{G,\rho}$ of Definitions 1.1–1.2. This method is simpler than that of [3] and applies much more generally (the techniques developed in [3] provides additional insight and complementary results when they apply).

LEMMA 3.1 (see, e.g., [7], Proposition 2.3).   *Referring to notation* (1.3), *there are constants* $C, c \in (0, \infty)$ *such that for any symmetric probability measure* μ *on* G, *any finite subset* $U \subset G$, *and any* $n = 1, 2, \ldots$, *we have*

$$(3.1) \qquad \mu^{(2n)}(e) \geq \frac{ce^{-Cn\lambda_\mu(U)}}{|U|}.$$

PROOF.   Inspection indicates that $\lambda_\mu(U)$ is the lowest eigenvalue of the continuous time semigroup

$$H_t^{U,\mu} = e^{-t} \sum_0^\infty \frac{t^n}{n!} K_{U,\mu}^n,$$

where $K_{U,\mu}(x, y) = \mu(x^{-1}y)\mathbf{1}_U(x)\mathbf{1}_U(y)$. Let $h_t^{U,\mu}(x, y)$ be the kernel of this semigroup, that is,

$$h_t^{U,\mu}(x, y) = e^{-t} \sum_0^\infty \frac{t^n}{n!} K_{U,\mu}^n(x, y).$$

By elementary spectral theory,

$$e^{-t\lambda_\phi(U)} = \sup\{\|H_t^U f\|_2 : \text{support}(f) \subset U, \|f\|_2 = 1\}.$$

Note also that $h_t^{U,\mu}(x, y) \leq h_t^\mu(x, y)$ for all $x, y \in U$. Now, we have

$$\mu^{(2n)}(e) \gtrsim h_n^\mu(e) \quad \text{and} \quad h_t^\mu(e) = \sup\{\|H_{t/2}^\mu f\|_2^2 : \|f\|_1 = 1\}.$$

It follows that (see [7], Proposition 2.3), for any finite set U and f supported in U,

$$h_t^\mu(e) \geq \frac{\|f\|_2^2}{\|f\|_1^2} \frac{\|H_{t/2}^{U,\mu} f\|_2^2}{\|f\|_2^2} \geq \frac{1}{|U|} \frac{\|H_{t/2}^{U,\mu} f\|_2^2}{\|f\|_2^2}.$$

Taking the supremum of all $f \neq 0$ with support in U, we obtain that there are constants $c, C \in (0, \infty)$ such that for any finite set $U \subset G$ and any n, $\mu^{(2n)}(e) \geq \frac{ce^{-Cn\lambda_\mu(U)}}{|U|}$.   □

THEOREM 3.2.   *Let* G *be a finitely generated group equipped with a finite symmetric generating set and the associated word-length. Let* $\rho : [0, \infty) \to [1, \infty)$ *be an nondecreasing continuous function and set* $M(t) = t^2 / \int_0^t \frac{s\,ds}{\rho(s)}$. *Let* μ *be a symmetric probability measure on* G *satisfying the weak moment condition*

$$W(\rho, \mu) = \sup_{s>0}\{s\mu(\{x : \rho(|x|) > s\})\} \leq K.$$

• *Assume that* $\Lambda_G(v) \simeq v^{-2/D}$ [*equivalently*, $\Phi_G(n) \simeq n^{-D/2}$]. *Then*

$$\mu^{(2n)}(e) \gtrsim \frac{1}{(M^{-1}(n))^D},$$

where M^{-1} *is the inverse function of* M.

• *Let $\pi : [0, \infty) \to [1, \infty)$ be an nondecreasing function such that $\pi(t) \le ct$ for some c and assume that*

$$\Phi_G(n) \ge \exp(-n/\pi(n)).$$

Then there exist $a, A \in (0, \infty)$ such that for any integers k, n we have

$$\mu^{(2n)}(e) \ge \exp\left(-A\left(\frac{k}{\pi(k)} + \frac{n}{M(a\pi(k)^{1/2})}\right)\right).$$

PROOF. The first case follows straightforwardly from Lemma 3.1, Theorem 2.13 and elementary computations. It is useful to note here that the first stated estimate is not sharp when ρ is a slowly varying function. In this particular context (polynomial volume growth and ρ slowly varying), the second stated result provides a sharp estimate. See Corollary 3.3. Many of the results provided by this first case are already covered in [3, 23, 25] by different methods but the case when ρ is regularly varying of index 2 is new.

In the second case, referring to Lemma 2.5 applied to the measure **u**, that is, the uniform measure of the generating set S, for any natural integer k, let U be a set of volume $\simeq$ to $\exp(Ak\pi(k))$. By Lemma 2.5, we then have

$$\Lambda_{\mathbf{u}}\big(\exp(Ak/\pi(k))\big) \le \frac{A}{2\pi(k)}.$$

By Theorem 2.13, this gives

$$\Lambda_\mu\big(\exp(Ak/\pi(k))\big) \le \frac{C(\mu, \rho, |S^*|)}{M(a\pi(k)^{1/2})}$$

for some constant $a > 0$. Putting these estimates together yields

$$\mu^{(2n)}(e) \gtrsim \exp\left(-\left(A'\frac{k}{\pi(k)} + \frac{n}{M(a\pi(k)^{1/2})}\right)\right)$$

for some $A' \in (0, \infty)$. □

The following corollaries of Theorem 3.2 illustrate the wide applicability and the sharpness of the results this theorem provides. To state these results, let us consider the set of all continuous increasing functions $\rho : [0, \infty) \to [1, \infty)$ satisfying (2.17), that is, such that $\rho(t) \simeq (\int_t^\infty \frac{ds}{(1+s)\ell(s)})^{-1}$ where ℓ is a regularly varying function $\ell : [0, \infty) \to [1, \infty)$ of index $\alpha \ge 0$ with $\int_0^\infty \frac{ds}{(1+s)\ell(s)} < \infty$. Under this hypothesis, the function ρ is regularly varying (at infinity) of index $\alpha \in [0, \infty)$ and the probability measure

$$\phi_\ell(g) = c_\ell \sum_1^\infty \frac{1}{\ell(4^k)} \mathbf{u}_{4^k}$$

is well defined because $\sum \frac{1}{\ell(4^k)} \simeq \int_0^\infty \frac{ds}{s\ell(s)}$ and satisfies

$$W(\rho, \phi_\ell) = \sup_{s>0}\{s\phi_\ell(\{g : \rho(|g|) > s\})\}$$

$$\leq C \sup_k \left\{\rho(4^k) \int_{4^k}^\infty \frac{ds}{s\ell(s)}\right\} < +\infty.$$

This makes ϕ_ℓ a potential witness for the behavior of $\widetilde{\Phi}_{G,\rho}$.

COROLLARY 3.3. *Let G be a finitely generated group with polynomial volume growth of degree D. Let ρ be as in* (2.17) *and set $M(t) = t^2/\int_0^t \frac{s\,ds}{\rho(s)}$.*

1. *Assume that $\alpha > 0$. In this case,*

$$\widetilde{\Phi}_{G,\rho}(n) \simeq 1/\big(M^{-1}(n)\big)^D.$$

2. *Assume that ρ is slowly varying and satisfies*

$$\rho \circ \exp(u) \simeq u^{1/\gamma}\kappa(u)$$

   with $\gamma \in (0, \infty)$, κ slowly varying at infinity and $\kappa(t^a) \simeq \kappa(t)$ for any $a > 0$. Then

$$\widetilde{\Phi}_{G,\rho}(n) \simeq \exp\big(-[n/\kappa(n)]^{\gamma/(1+\gamma)}\big).$$

3. *Assume that the function $\kappa = \rho \circ \exp$ is slowly varying and satisfies $s\kappa^{-1}(s) \simeq \kappa^{-1}(s)$ at infinity. Then*

$$\widetilde{\Phi}_{G,\rho}(n) \simeq \exp\big(-n/\kappa(n)\big).$$

PROOF. For statement 1, the lower bound follows obviously from the first statement in Theorem 3.2. The upper bound is provided by [23], Theorem 1.5.

The proofs of the last two statements are similar and we give the details only for statement (2). By the second statement in Theorem 3.2, we have

$$\phi_{G,\rho}(n) \geq \exp\big(-A\big(\log k + n/\rho(k)\big)\big)$$

because the hypotheses on ρ implies in particular that $\rho(k/\log k) \simeq \rho(k)$. Pick k as a function of n so that $\log k\rho(k) \simeq n$. We then have $\phi_{G,\rho}(n) \geq \exp(-Ct)$ with $t = \log k$ with $t\rho \circ \exp(t) \simeq n$, that is, $t^{(1+\gamma)/\gamma}\kappa(t) \simeq n$. Because of the assumed property of κ, this yields

$$t \simeq \big(n/\kappa(t)\big)^{\gamma/(1+\gamma)} \simeq \big(n/\kappa(n)\big)^{\gamma/(1+\gamma)}.$$

The matching upper bound can be derived from [25], Theorem 2.4, of by using the subordination results of [2]. □

EXAMPLE 3.1. To illustrate case 1, consider the case when $\rho(s) = (1+s)^2$. Corollary 3.3 states implies that on a group with polynomial volume growth of degree D, any symmetric measure μ with finite second weak-moment satisfies $\mu^{(2n)}(e) \gtrsim [n \log n]^{-D/2}$. This was not known and could not be proved by the techniques of [3]. In [23], the authors prove that the measure $\phi_2(x) = \frac{c}{(1+|x|)^{2+D}}$ (which has finite second weak-moment) satisfies $\phi_2^{(n)}(e) \simeq [n \log n]^{-D/2}$. Hence, ϕ_2 provides a witness to the behavior of $\widetilde{\Phi}_{G,2}$.

The simplest illustration of case 2 is when $\rho(s) = (1 + \log(1+s))^\alpha$. In this case, the result reads

$$\widetilde{\Phi}_{G,\log_{[1]}^\alpha}(n) \simeq \exp(-n^{1/(1+\alpha)}).$$

This was derived by a different method in [25].

The last case, case 3, is illustrated by taking ρ to be the power of an iterated logarithm, $\rho(s) = [\log_{[k]}(s)]^\alpha$, $k \geq 2$, $\alpha > 0$, in which case we obtain

$$\widetilde{\Phi}_{G,\log_{[k]}^\alpha}(n) \simeq \exp(-n/[\log_{[k-1]} n]^\alpha).$$

This result was also derived by a different method in [25].

COROLLARY 3.4. *Assume that G is a finitely generated group with exponential volume growth and such that $\Phi_G(n) \simeq \exp(-n^{1/3})$. Let ρ be as in (2.17).*

1. *Assume that ρ is regularly varying of index 2. In this case*

$$\widetilde{\Phi}_{G,\rho}(n) \gtrsim \exp\left(-n^{1/3} \int_0^{n^{1/3}} \frac{s\,ds}{\rho(s)}\right).$$

2. *Assume that $\rho(s) = (1+s)^\alpha$, $\alpha \in (0,2)$, Then*

$$\widetilde{\Phi}_{G,\rho}(n) \simeq \exp(-n^{1/(1+\alpha)}).$$

3. *Assume that ρ is slowly varying and satisfies $\rho(s^a) \simeq \rho(s)$ for any $a > 0$. Then*

$$\widetilde{\Phi}_{G,\rho}(n) \simeq \exp(-n/\rho(n)).$$

PROOF. Each lower bound follows easily from Theorem 3.2. In cases 2 and 3, the upper bound can be obtained by the simple method of [3], Section 4.2. In case 3, the upper bound can also be obtain by the subordination technique of [2]. □

REMARK 3.5. We note that [3] contains a complete proof of both the upper and lower bound for case 2 but that it completely fails to cover the lower bound in cases 1 and 3. These lower bounds (cases 1 and 3) are new. Proving a matching upper bound in case 1 under the same hypotheses is an interesting open question. It is proved below that the lower bound in case 1 is sharp in the case of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. A matching upper bound for polycyclic groups of exponential growth will be given elsewhere.

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COROLLARY 3.6. *Assume that G is a finitely generated group and that there exist $0 < \gamma_1 \leq \gamma_2 < 1$ such that*

$$\exp(-n^{\gamma_2}) \lesssim \Phi_G(n) \lesssim \exp(-n^{\gamma_1}).$$

Let ρ be as in (2.17) and assume that ρ is slowly varying function satisfying $\rho(t^a) \simeq \rho(t)$ for any $a > 0$. Then

$$\widetilde{\Phi}_{G,\rho}(n) \simeq \exp(-n/\rho(n)).$$

EXAMPLE 3.2. For any group in the large class described in Corollary 3.6, we have

$$\widetilde{\Phi}_{G,\log_{[k]}^\alpha}(n) \simeq \exp(-n/[\log_{[k]} n]^\alpha)$$

for each $k = 1, 2, \ldots$ and $\alpha > 0$.

PROOF. The lower bounds follows from Theorem 3.2 by inspection. The upper bound follows from the subordination technique in [2]. □

The same proof gives the following complementary result.

COROLLARY 3.7. *Assume that G is a finitely generated group and that there exist continuous positive increasing functions of slow variation $\pi_1 \leq \pi_2$ such that $\pi_i(t^a) \simeq \pi_i(t)$ for all $a > 0$ and*

$$\exp(-n/\pi_1(n)) \lesssim \Phi_G(n) \lesssim \exp(-n/\pi_2(n)).$$

Let ρ be as in (2.17) and assume that ρ is slowly varying function satisfying $\rho(t^a) \simeq \rho(t)$ for any $a > 0$. Then

$$\exp(-n/\rho(\pi_1(n))) \lesssim \widetilde{\Phi}_{G,\rho}(n) \lesssim \exp(-n/\rho(\pi_2(n))).$$

EXAMPLE 3.3. Let $\mathbf{S}_{d,r}$ be the free solvable group of solvable length d on r generators. The behavior of $\Phi_{\mathbf{S}_{d,r}}$ is described in [26]. In particular, for $d > 2$,

$$\Phi_{\mathbf{S}_{d,r}}(n) \simeq \exp\left(-n\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n}\right)^{2/r}\right).$$

For ρ as in (2.17), slowly varying and satisfying $\rho(t^a) \simeq \rho(t)$ for all $a > 0$, we have

$$\widetilde{\Phi}_{\mathbf{S}_{d,r},\rho}(n) \simeq \exp(-n/\rho(\log_{[d-2]} n)), \qquad d > 2, r \geq 1.$$

EXAMPLE 3.4. Consider the iterated wreath products (the factor $\mathbb{Z}$ is repeated k times)

$$W_k(\mathbb{Z}, \mathbb{Z}^d) = \mathbb{Z} \wr (\mathbb{Z} \wr (\cdots \mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z}^d) \cdots))$$

and

$$W^k(\mathbb{Z}, \mathbb{Z}^d) = \left(\cdots\left((\mathbb{Z} \wr \mathbb{Z}) \wr \cdots\right) \wr \mathbb{Z}\right) \wr \mathbb{Z}^d.$$

From [13], we know that

$$\Phi_{W_k(\mathbb{Z}, \mathbb{Z}^d)}(n) \simeq \exp\left(-n\left(\frac{\log_{[k]} n}{\log_{[k-1]} n}\right)^{2/d}\right)$$

and

$$\Phi_{W^k(\mathbb{Z}, \mathbb{Z}^d)}(n) \simeq \exp\left(-n^{(k+d)/(2+k+d)}(\log n)^{2/(2+k+d)}\right).$$

If ρ at (2.17) is slowly varying and satisfies $\rho(t^a) \simeq \rho(t)$ for all $a > 0$, Corollaries 3.6–3.7 give

$$\Phi_{W_k(\mathbb{Z}, \mathbb{Z}^d), \rho}(n) \simeq \exp\left(-n/\rho(\log_{[k-1]} n)\right)$$

and

$$\Phi_{W^k(\mathbb{Z}, \mathbb{Z}^d), \rho}(n) \simeq \exp\left(-n/\rho(n)\right).$$

4. Random walks on wreath products.

This section is devoted to the computations of the behavior of a variety random walks on wreath products.

First, we briefly review the definition of wreath products. Our notation follows [22] and [24]. Let H, K be two finitely generated groups. Denote the identity element of K by e_K and identity element of H by e_H. Let K_H denote the direct sum:

$$K_H = \sum_{h \in H} K_h.$$

The elements of K_H are functions $f : H \to K$, $h \mapsto f(h) = k_h$, which have finite support in the sense that $\{h \in H : f(h) = k_h \neq e_K\}$ is finite. Multiplication on K_H is simply coordinate-wise multiplication. The identity element of K_H is the constant function $e_K : h \mapsto e_K$ which, abusing notation, we denote by e_K. The group H acts on K_H by left translation:

$$\tau_l(h) f(h') = f(h^{-1} h'), \qquad h, h' \in H.$$

The wreath product $K \wr H$ is defined to be semidirect product

$$K \wr H = K_H \rtimes_\tau H,$$

$$(f, h)(f', h') = (f \cdot \tau_l(h) f', hh').$$

In the lamplighter interpretation of wreath products, H corresponds to the base on which the lamplighter lives and K corresponds to the lamp. We embed K and H naturally in $K \wr H$ via the injective homomorphisms

$$k \longmapsto \underline{k} = (\boldsymbol{k}_{e_H}, e_H), \qquad \boldsymbol{k}_{e_H}(e_H) = k, \qquad \boldsymbol{k}_{e_H}(h) = e_K \qquad \text{if } h \neq e_H,$$

$$h \longmapsto \underline{h} = (\boldsymbol{e}_K, h).$$

Let μ_H and μ_K be probability measures on H and K, respectively. Through the embedding, μ_H and μ_K can be viewed as probability measures on $K \wr H$. Consider the measure

$$\nu = \mu_K * \mu_H * \mu_K$$

on $K \wr H$. This is called the switch-walk-switch measure on $K \wr H$ with switch-measure μ_K and walk-measure μ_H.

We can also consider the measure (again, on $K \wr H$)

$$\mu = \tfrac{1}{2}(\mu_K + \mu_H).$$

We will mostly work with this type of measure which is better adapted to the techniques developed below. We note that it is obvious that

$$\mathcal{E}_{p,\nu} \leq C(\mu_H, \mu_K)\mathcal{E}_{p,\mu}.$$

Conversely, if μ_K, μ_H are symmetric and $\mu_K(e_K) > 0$, we also have

$$\mathcal{E}_{p,\nu} \leq C'(\mu_H, \mu_K)\mathcal{E}_{p,\mu}.$$

So, for symmetric measures μ_H, μ_K with $\mu_K(e) > 0$, we have $\Lambda_{p,\mu} \simeq \Lambda_{p,\nu}$ on $K \wr H$.

4.1. *Upper bounds for Λ on wreath products.* We describe a general upper bound on $\Lambda_{p,H_2 \wr H_1, \mu}$ in terms of Λ_{p,H_i,μ_i}, $i = 1, 2$ when $\mu = \tfrac{1}{2}(\mu_1 + \mu_2)$. Throughout this work, $\Lambda_{p,H,\mu}^{-1}$ denotes the right-continuous inverse of the nondecreasing function $v \mapsto \Lambda_{p,H,\mu}(v)$.

THEOREM 4.1. *Let μ_i be a symmetric probability measures on H_i, $i = 1, 2$. The measure $\mu = \tfrac{1}{2}(\mu_1 + \mu_2)$ defined on $H_2 \wr H_1$ satisfies*

$$\Lambda_{p,H_2 \wr H_1, \mu}(v) \leq s$$

for all $s, v > 0$ such that

$$v \geq \left(\Lambda_{p,H_2,\mu_2}^{-1}(s)\right)^{\Lambda_{p,H_1,\mu_1}^{-1}(s)} \Lambda_{p,H_1,\mu_1}^{-1}(s),$$

where

$$\Lambda_{p,H_i,\mu_i}^{-1}(s) = \inf\{v : \Lambda_{p,H_i,\mu_i}(v) \leq s\}.$$

PROOF. For each s and $i = 1, 2$, let v_i be the smallest v such that $\Lambda_{p,H_i,\mu_i}(v) \leq s$. Let ϕ_i be a test function on H_i such that $|\operatorname{support}(\phi_i)| \leq v_i$ and

$$\frac{\mathcal{E}_{p,\mu_i}(\phi_i)}{\|\phi_i\|_p^p} = \Lambda_{p,H_i,\mu_i}(v_i).$$

Let U_1 be the support of ϕ_1. Let W be the set of functions $\eta : H_1 \to H_2$ whose support is contained in U_1 [i.e., $\eta(y)$ is equal to the identity element in H_2 when $y \notin U_1$]. On $H_2 \wr H_1$, consider the function

$$H_2 \wr H_1 \ni (\eta, x) \mapsto \phi(\eta, x) = \left(\prod_{y \in U_1} \phi_2\big(\eta(y)\big) \right) \phi_1(x) \mathbf{1}_W(\eta).$$

This function is supported on a set of size

$$|W||U_1| \le v_2^{v_1} v_1$$

and its ℓ^p-norm on $H_2 \wr H_1$ is given by

$$\|\phi\|_{\ell^p(H_2 \wr H_1)} = \|\phi_1\|_{\ell^p(H_1)}^p \|\phi_2\|_{\ell^p(H_2)}^{p|U_1|}.$$

Next, we have

$$\phi\big((\eta, x)(\mathbf{e}, z)\big) - \phi\big((\eta, x)\big) = \phi\big((\eta, xz)\big) - \phi\big((\eta, x)\big)$$
$$= \left(\prod_{y \in U_1} \phi_2\big(\eta(y)\big) \right) \big(\phi_1(xz) - \phi_1(x)\big) \mathbf{1}_W(\eta)$$

and

$$\phi\big((\eta, x)(\mathbf{1}_t^{e_1}, e_1)\big) - \phi\big((\eta, x)\big)$$
$$= \phi\big((\eta \mathbf{1}_t^x, x)\big) - \phi\big((\eta, x)\big)$$
$$= \left(\prod_{y \in U_1 \setminus \{x\}} \phi_2\big(\eta(y)\big) \right) \big(\phi_2\big(\eta(x)t\big) - \phi_2\big(\eta(x)\big)\phi_1(x)\big) \mathbf{1}_W(\eta).$$

This gives

$$\mathcal{E}_{p,\mu}(\phi) = \frac{1}{2} \left(\frac{\mathcal{E}_{p,\mu_1}(\phi_1)}{\|\phi_1\|_{\ell^p(H_1)}^p} + \frac{\mathcal{E}_{p,\mu_2}(\phi_2)}{\|\phi_2\|_{\ell^p(H_2)}^p} \right) \|\phi_2\|_{\ell^p(H_2)}^{p|U_1|} \|\phi_1\|_{\ell^p(H_1)}^p$$
$$\le s \|\phi\|_{\ell^p(H_1 \wr H_2)}^p.$$

This is the desired result. $\square$

4.2. Lower bounds on Λ on wreath products.

In [12, 13], Erschler developed a method to bound the Følner functions of the wreath product $H_2 \wr H_1$ from below in terms of the Følner functions of H_1 and H_2. This can be expressed as lower bounds on $\Lambda_{1, H_2 \wr H_1}$ and yields good lower bounds on $\Lambda_{2, H_2 \wr H_1}$ via the Cheeger inequality (2.2). We generalize this in order to study spread-out measures on wreath product. Erschler [13], Theorem 1, which we recall below in a less general form, provides good lower bound for $\Lambda_{1, H_2 \wr H_1, \mu}$ but, for spread-out measures, the Cheeger inequality might fail to provide good lower bound on $\Lambda_{2, H_2 \wr H_1, \mu}$. We combine comparison arguments with the results of Erschler to provide a widely applicable method to obtain satisfactory lower bounds on $\Lambda_{2, H_2 \wr H_1, \mu}$.

Define the Følner function $\text{Føl}_{G,\mu}$ by

$$\text{Føl}(t) = \inf\{v : \Lambda_{1,G,\mu}(v) \leq 1/t\}.$$

Note that $\text{Føl}_{G,\mu}(t) = \Lambda_{1,G,\mu}^{-1}(1/t)$ in the notation of Theorem 4.1. In the context of random walks on groups, [13], Theorem 1, can be stated as follows.

THEOREM 4.2 ([13], Theorem 1). *There exists a constant $K \geq 1$ such that for any countable groups H_i and symmetric probability measures μ_i, $i = 1, 2$, the measure $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ defined on $H_2 \wr H_1$ satisfies*

$$\Lambda_{1,H_2 \wr H_1, \mu}(v) \geq s/K$$

for all $s, v > 0$ such that

$$v \leq \left(\Lambda_{1,H_2,\mu_2}^{-1}(s)\right)^{\Lambda_{1,H_1,\mu_1}^{-1}(s)/K}.$$

Consider the following problem. On a finitely generated group G, given a volume v, find a symmetric probability measure $\zeta_{G,v}$ such that $\Lambda_{1,G,\zeta_{G,v}}(v) \simeq 1$. For instance, on any group G, if we let $r(v)$ be the smallest radius of a ball of volume greater than v, the uniform probability measure $\mathbf{u}_{r(2v)}$ on the ball of radius $r(2v)$ satisfies

$$\Lambda_{1,G,\mathbf{u}_{r(2v)}}(v) \geq 1/2.$$

For our purpose, we will need to consider the following question. Fix a symmetric probability measure μ and fix $t > 0$. Given a solution $\zeta_{G,v}$ to the previous problem, what is the largest volume $v(t)$ such that

$$t\mathcal{E}_{G,\mu} \geq \mathcal{E}_{G,\zeta_{G,v(t)}}?$$

Solution to this problem can be obtained by using pseudo-Poincaré inequalities involving $\mathcal{E}_{G,\mu}$. For example, if $\mu = \mathbf{u}$ is the uniform measure on our generating set S^*, we have the pseudo-Poincaré inequality

$$\sum_x |f(xy) - f(x)|^2 \leq |S^*| |y|^2 \mathcal{E}_{G,\mathbf{u}}(f, f).$$

It follows that for a given t we can choose $v(t) \simeq V_G(\sqrt{t})$ to achieve

$$t\mathcal{E}_{G,\mathbf{u}} \geq \mathcal{E}_{G,\mathbf{u}_{r(2v(t))}}.$$

The following proposition is based on this circle of ideas and is stated in a form that is suitable to treat iterated wreath products. See Theorems 4.5 and 5.1 below.

PROPOSITION 4.3. *Let μ_i be a symmetric probability measures on H_i, $i = 1, 2$. Fix $\delta > 0$. Assume that for each $t > 0$ we can find $v_i^{\delta}(t) > 0$ and symmetric probability measures ζ_{i,v_i} on H_i, $i = 1, 2$, such that*

(4.1) $$t\mathcal{E}_{H_i,\mu_i} \geq \mathcal{E}_{H_i,\zeta_{i,v_i^{\delta}(t)}} \quad and \quad \Lambda_{1,H_i,\zeta_{i,v_i^{\delta}(t)}}(v_i^{\delta}(t)) \geq \delta.$$

Then, for the measure $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ on $G = H_2 \wr H_1$ and any $t > 0$, we have

$$t\mathcal{E}_{G,\mu} \geq \mathcal{E}_{G,\zeta_{v(t)}} \quad and \quad \Lambda_{1,G,\zeta_{v(t)}}(v(t)) \geq \delta/K,$$

where $t \mapsto v(t)$ and the probability measure $\zeta_{v(t)}$ on G are given by

$$v(t) = \left[v_2^\delta(t)\right]^{v_1^\delta(t)/K} \quad and \quad \zeta_{v(t)} = \tfrac{1}{2}(\zeta_{1,v_1^\delta(t)} + \zeta_{2,v_2^\delta(t)}).$$

In particular,

$$\Lambda_{G,\mu}(v(t)) \geq \frac{c}{t}\left(\frac{\delta}{K}\right)^2.$$

PROOF. The hypotheses on $\mathcal{E}_{H_i,\mu_i}$ immediately imply that $t\mathcal{E}_{G,\mu} \geq \mathcal{E}_{G,\zeta_{v(t)}}$. The lower bound on $\Lambda_{1,G,\zeta_{v(t)}}$ for the given volume $v(t)$ follows from Erschler's result stated in Theorem 4.2. $\square$

This proposition will allow us to treat a great variety of examples. To illustrate how this proposition works, we treat two simple examples.

EXAMPLE 4.1. On $\mathbb{Z}$, consider the measure μ_α given by

$$\mu_\alpha(z) = c_\alpha(1 + |z|)^{-\alpha-1}.$$

What is the behavior of $\mu^{(n)}(e)$ if $\mu = \frac{1}{2}(\mu_{\alpha_1} + \mu_{\alpha_2})$ on $\mathbb{Z} \wr \mathbb{Z}$ where μ_{α_1} is supported on the base and μ_{α_2} on the lamp above the identity of the base?

As noted in the Appendix section, on $\mathbb{Z}$ and for any $r > 0$ we have

$$\mathcal{E}_{\mathbf{u}_r} \leq C_\alpha r^\alpha \mathcal{E}_{\mu_\alpha} \quad and \quad \Lambda_{1,\mathbb{Z},\mathbf{u}_r}(r) \geq 1/2.$$

In other words, for any $t > 0$ and $v_i(t) \simeq t^{1/\alpha_i}$, we have

$$\mathcal{E}_{\mathbf{u}_{v_i(t)}} \leq t\mathcal{E}_{\mu_{\alpha_i}} \quad and \quad \Lambda_{1,\mathbb{Z},\mathbf{u}_{v_i(t)}}(v_i(t)) \geq 1/2.$$

By Proposition 4.3, on $G = \mathbb{Z} \wr \mathbb{Z}$,

$$t\mathcal{E}_{G,\mu} \geq \mathcal{E}_{G,\zeta_{v(t)}} \quad and \quad \Lambda_{1,G,\zeta_{v(t)}}(v(t)) \geq \frac{1}{2K},$$

where

$$v(t) \simeq \exp(t^{1/\alpha} \log t) \quad and \quad \zeta_{G,v(t)} = \tfrac{1}{2}(\mathbf{u}_{v_1(t)} + \mathbf{u}_{v_2(t)}).$$

It follows that

$$\Lambda_{2,\mathbb{Z}\wr\mathbb{Z},\mu}(\exp(at^{1/\alpha_1} \log t)) \geq 1/t,$$

equivalently

$$\Lambda_{2,\mathbb{Z}\wr\mathbb{Z},\mu}(v) \gtrsim \left(\frac{\log \log t}{\log t}\right)^{\alpha_1}.$$

By Theorem 2.2, this gives

$$\mu^{(n)}(e) \leq \exp\left(-n^{1/(1+\alpha_1)}(\log n)^{\alpha_1/(1+\alpha_1)}\right).$$

Theorem 4.1 provides a matching lower bound (see also [24]).

It is instructive to see what happens in this example if one applies directly Erschler Følner function results and Cheeger's inequality to obtain lower bound on Λ_2 and an upper bound on $\mu^{(n)}(e)$. By Theorem A.7, we know that at least for $\alpha \neq 1$

$$\Lambda_{1,\mathbb{Z},\mu_\alpha}(v) \simeq \begin{cases} v^{-1}, & \text{if } \alpha \in (1,2), \\ v^{-\alpha}, & \text{if } \alpha \in (0,1). \end{cases}$$

By Theorem 4.2, if $0 < \alpha_1 \neq 1$, this implies [the value of $\alpha_2 \in (0,2)$ does not matter]

$$\Lambda_{1,\mathbb{Z}\wr\mathbb{Z},\mu}(v) \gtrsim \begin{cases} \dfrac{\log\log v}{\log v}, & \text{if } \alpha_1 \in (1,2), \\[2ex] \left(\dfrac{\log\log v}{\log v}\right)^{\alpha_1}, & \text{if } \alpha_1 \in (0,1). \end{cases}$$

In fact, these lower bounds admit matching upper bounds. Now, a lower bound on $\Lambda_{2,\mathbb{Z}\wr\mathbb{Z},\mu}$ can be derived since $\Lambda_{2,\mathbb{Z}\wr\mathbb{Z},\mu} \gtrsim \Lambda_{1,\mathbb{Z}\wr\mathbb{Z},\mu}^2$. However, this produces a lower bound that is significantly weaker than the one obtained above using Proposition 4.3.

EXAMPLE 4.2. Consider the wreath product $G = H \wr \mathbb{Z}$ where H is a polycyclic group of exponential volume growth. On this group, we consider the measure $\mu = \frac{1}{2}(\phi + \mathbf{u})$ where ϕ is the measure on $\mathbb{Z}$ given by $\phi(z) = c(1 + |z|)^{-3}$ and $\mathbf{u}$ is the uniform measure on a finite symmetric generating set in H containing the identity. Recall that $\Phi_H(n) \simeq \exp(-n^{1/3})$, $\Lambda_{2,H}(v) \simeq (\log(e+v))^{-2}$ and $\Lambda_{1,H}(v) \simeq (\log(e+v))^{-1}$. Further, $\Lambda_{1,H,\mathbf{u}_{r(2v)}}(v) \geq 1/2$ and $r(v) \simeq \log v$ since the volume function on H has exponential growth. Also, by the universal pseudo-Poincaré inequality for finitely supported symmetric measure and associated word-length, we have

$$Cr(2v)^2 \mathcal{E}_{H,\mathbf{u}_H} \geq \mathcal{E}_{H,\mathbf{u}_{r(2v)}}.$$

Next, note that the measure ϕ on $\mathbb{Z}$ sits in between the domains of attraction of symmetric stable law with parameter $\alpha \in (0,2)$ and the classical Gaussian domain of attraction. It is well known (and it follows from Theorems 2.2–A.7) that $\phi^{(n)}(0) \simeq (n \log n)^{-1/2}$, $\Lambda_{2,\mathbb{Z},\phi}(v) \simeq (1+v)^{-2}\log(e+v)$ and $\Lambda_{1,\mathbb{Z},\phi}(v) \simeq (1+v)^{-1}$. We also have $\Lambda_{1,\mathbb{Z},\mathbf{u}_{r(2v)}}(v) \geq 1/2$. Further, we have the pseudo-Poincaré inequality

$$\sum_{x\in\mathbb{Z}} |f(xy) - f(x)|^2 \phi(y) \leq C|y|^2 (\log(e+|y|))^{-1} \mathcal{E}_{\mathbb{Z},\phi}(f,f).$$

Applying Proposition 4.3 with $H_1 = \mathbb{Z}$, $\mu_1 = \phi$, $H_2 = H$, $\mu_2 = \mathbf{u}$, the above data leads to

$$v_1(t) \simeq (t \log t)^{-1/2}, \qquad v_2(t) \simeq \exp(t^{1/2})$$

and

$$v(t) \simeq \exp(t (\log t)^{1/2}).$$

This gives

$$\Lambda_{2,G,\mu}(v) \gtrsim \frac{(\log \log v)^{1/2}}{\log v}.$$

For comparison, we note that Theorem 4.2, gives

$$\Lambda_{1,G,\mu}(v) \gtrsim \frac{1}{(\log v)^{1/2}}.$$

These two lower bounds can be complemented by matching upper bounds using Theorem 4.1.

It is worth noting that Proposition 4.3 admits a version that leads to good lower bound for the p-isoperimetric profile Λ_p on wreath products. The proof is the same.

PROPOSITION 4.4. *Let μ_i be a symmetric probability measures on H_i, $i = 1, 2$. Fix $\delta > 0$ and $p \in [1, \infty)$. Assume that for each $t > 0$ we can find $v_i^\delta(t) > 0$ and symmetric probability measures ζ_{i,v_i} on H_i, $i = 1, 2$, such that*

$$(4.2) \qquad t \mathcal{E}_{p,H_i,\mu_i} \geq \mathcal{E}_{p,H_i,\zeta_{i,v_i^\delta(t)}} \quad and \quad \Lambda_{1,H_i,\zeta_{i,v_i^\delta(t)}}\big(v_i^\delta(t)\big) \geq \delta.$$

Then, for the measure $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ on $G = H_2 \wr H_1$ and any $t > 0$, we have

$$t \mathcal{E}_{p,G,\mu} \geq \mathcal{E}_{p,G,\zeta_{v(t)}} \quad and \quad \Lambda_{1,G,\zeta_{v(t)}}\big(v(t)\big) \geq \delta/K,$$

where $t \mapsto v(t)$ and the probability measure $\zeta_{v(t)}$ on G are given by

$$v(t) = \big[v_2^\delta(t)\big]^{v_1^\delta(t)/K} \quad and \quad \zeta_{v(t)} = \frac{1}{2}(\zeta_{1,v_1^\delta(t)} + \zeta_{2,v_2^\delta(t)}).$$

We now state a theorem that uses the iterative nature of Proposition 4.3. Consider a sequence $(H_i)_1^m$ of finitely generated groups. Since taking wreath product is neither commutative nor associative, this sequence gives rise to many different iterated wreath product including $H_m \wr (H_{m-1} \wr (\cdots (H_2 \wr H_1) \cdots))$ and $(\cdots (H_m \wr H_{m-1}) \wr \cdots H_2) \wr H_1$. Let $\mathfrak{B}$ be a symbol of length m describing a possible bracketing and $W_{\mathfrak{B}}(H_m, \ldots, H_1)$ be the corresponding wreath product. This can be define inductively with $(\cdot)$ representing the bracketing of one single group, $(\wr)$,

representing the bracketing of groups (i.e., gives $H_2 \wr H_1$). Inductively, if $\mathfrak{B}_1, \mathfrak{B}_2$ are such symbols, then $\mathfrak{B} = (\mathfrak{B}_2 \wr \mathfrak{B}_1)$ is also such a symbol and

$$W_{\mathfrak{B}}(H_m, \ldots, H_1) = W_{\mathfrak{B}_2}(H_m, \ldots, H_{m_1+1}) \wr W_{\mathfrak{B}_1}(H_{m_1}, \ldots, H_1).$$

Note that the length of $\mathfrak{B}$ is defined inductively as the sum of the length of $\mathfrak{B}_1, \mathfrak{B}_2$ and length of $(\cdot)$ equal 1. We can now introduce a similar operation on sequences of numbers $(v_1, \ldots, v_m)$ by setting

$$W_{(\cdot)}(v) = v, \qquad W_{(\cdot \wr \cdot)}(v_2, v_1) = v_2^{v_1/K}$$

and, if $\mathfrak{B} = (\mathfrak{B}_2 \wr \mathfrak{B}_1)$ as above,

$$W_{\mathfrak{B}}(v_m, \ldots, v_1) = W_{\mathfrak{B}_2}(v_m, \ldots, v_{m_1+1})^{W_{\mathfrak{B}_1}(v_{m_1}, \ldots, v_1)/K}.$$

Here, K is the constant provided by Erschler's theorem, that is, Theorem 4.2.

Similarly, given probability measures μ_i on H_i, $1 \le i \le m$, and $\mathfrak{B} = (\mathfrak{B}_2 \wr \mathfrak{B}_1)$ define $\mu_{\mathfrak{B}}$ to be the probability measure on $W_{\mathfrak{B}}(H_m, \ldots, H_1)$ define inductively by

$$\mu_{\mathfrak{B}} = \tfrac{1}{2}(\mu_{\mathfrak{B}_2} + \mu_{\mathfrak{B}_1}),$$

where $\mu_{\mathfrak{B}_2}$ is understood as a probability measure on $W_{\mathfrak{B}}(H_m, \ldots, H_1)$ supported on the copy of $W_{\mathfrak{B}_2}(H_m, \ldots, H_{m_1+1})$ above the identity element of $W_{\mathfrak{B}_1}(H_{m_1}, \ldots, H_1)$ and $\mu_{\mathfrak{B}_2}$ is a probability measure on $W_{\mathfrak{B}}(H_m, \ldots, H_1)$ supported on $W_{\mathfrak{B}_1}(H_{m_1}, \ldots, H_1)$. For instance, given μ_i, H_i, $1 \le i \le 3$ and $\mathfrak{B} = ((\cdot \wr \cdot) \wr (\cdot))$, $\mu_{\mathfrak{B}}$ is a measure on $(H_3 \wr H_2) \wr H_1$ and is equal to

$$\mu_{\mathfrak{B}} = \tfrac{1}{2}\big(\tfrac{1}{2}(\mu_3 + \mu_2) + \mu_1\big),$$

where $\tfrac{1}{2}(\mu_3 + \mu_2)$ is the measure on $H_3 \wr H_2$ considered in Theorem 4.1 and Proposition 4.3. In some instance, it is useful to write

$$(4.3) \qquad \mu_{\mathfrak{B}} = \nu_{\mathfrak{B}, \mu_m, \ldots, \mu_1}$$

to specify the measures used in the construction.

THEOREM 4.5. *Let μ_i be a symmetric probability measures on H_i, $1 \le i \le m$. Fix $\delta > 0$ and $p \ge 1$. assume that for each $t > 0$ we can find $v_i^\delta(t) > 0$ and symmetric probability measures ζ_{i,v_i} on H_i, $1 \le i \le m$, such that*

$$(4.4) \qquad t\mathcal{E}_{p,H_i,\mu_i} \ge \mathcal{E}_{p,H_i,\zeta_{i,v_i^\delta(t)}} \quad and \quad \Lambda_{1,H_i,\zeta_{i,v_i^\delta(t)}}\big(v_i^\delta(t)\big) \ge \delta.$$

Fix a symbol $\mathfrak{B}$ of length m as above. Then, for any $t > 0$, the measure $\mu = \mu_{\mathfrak{B}}$ on $G = W_{\mathfrak{B}}(H_m, \ldots, H_1)$ satisfies

$$t\mathcal{E}_{p,G,\mu} \ge \mathcal{E}_{p,G,\zeta_{v(t)}} \quad and \quad \Lambda_{1,G,\zeta_{v(t)}}\big(v(t)\big) \ge \delta/K^m,$$

where $t \mapsto v(t)$ and the probability measure $\zeta_{v(t)}$ on G are given by

$$v(t) = W_{\mathfrak{B}}\big(v_m^\delta(t), \ldots, v_1^\delta(t)\big) \quad and \quad \zeta_{v(t)} = \nu_{\mathfrak{B}, \zeta_{m,v_m^\delta(t)}, \ldots, \zeta_{1,v_1^\delta(t)}}.$$

In particular,

$$\Lambda_{p,G,\mu}\big(v(t)\big) \geq \frac{c(1,p)}{t}\left(\frac{\delta}{K^m}\right)^p.$$

EXAMPLE 4.3. Let $p = 2$. Assume that H_i is a group of polynomial volume growth of degree d_i, $1 \leq i \leq 4$. On H_i, consider the measure $\mu_i(h) \asymp (1 + |h|)^{-\alpha_i - d_i}$, $\alpha_i \in (0, 2)$, $1 \leq i \leq 4$. The symbols $\mathfrak{B}$ of length four are $\mathfrak{B}_1 = (((\cdot \wr \cdot) \wr \cdot) \wr \cdot)$, $\mathfrak{B}_2 = ((\cdot \wr (\cdot \wr \cdot)) \wr \cdot)$, $\mathfrak{B}_3 = ((\cdot \wr \cdot) \wr (\cdot \wr \cdot))$, $\mathfrak{B}_4 = (\cdot \wr ((\cdot \wr \cdot) \wr \cdot))$ and $\mathfrak{B}_5 = (\cdot \wr (\cdot \wr (\cdot \wr \cdot)))$. Set $v_{\mathfrak{B}} = W_{\mathfrak{B}}(v_4, \ldots, v_1)$. By inspection, we have

$$v_{\mathfrak{B}_1}(t) \simeq \exp\big(t^{d_1/\alpha_1 + d_2/\alpha_2 + d_3/\alpha_3}\log t\big), \qquad v_{\mathfrak{B}_2}(t) \simeq \exp_{[2]}\big(t^{d_2/\alpha_2}\log t\big),$$

$$v_{\mathfrak{B}_3}(t) \simeq \exp_{[2]}\big(t^{d_1/\alpha_1}\log t\big), \qquad v_{\mathfrak{B}_4}(t) \simeq \exp_{[2]}\big(t^{d_1/\alpha_1 + d_2/\alpha_2}\log t\big),$$

and

$$v_{\mathfrak{B}_5}(t) \simeq \exp_{[3]}\big(t^{d_1/\alpha_1}\log t\big).$$

Here, $\exp_{[k]}$ denotes iterated exponentials so that $\exp_{[2]}(x) = \exp(\exp(x))$. This gives

$$\Lambda_{W_{\mathfrak{B}_1}, \mu_{\mathfrak{B}_1}}(v) \simeq \left(\frac{\log \log v}{\log v}\right)^{1/(d_2/\alpha_2 + d_3/\alpha_3)},$$

$$\Lambda_{W_{\mathfrak{B}_2}, \mu_{\mathfrak{B}_2}}(v) \simeq \left(\frac{\log \log \log v}{\log \log v}\right)^{\alpha_2/d_2},$$

$$\Lambda_{W_{\mathfrak{B}_3}, \mu_{\mathfrak{B}_3}}(v) \simeq \left(\frac{\log \log \log v}{\log \log v}\right)^{\alpha_1/d_1},$$

$$\Lambda_{W_{\mathfrak{B}_4}, \mu_{\mathfrak{B}_4}}(v) \simeq \left(\frac{\log \log \log v}{\log \log v}\right)^{1/(d_1/\alpha_1 + d_2/\alpha_2)},$$

and

$$\Lambda_{W_{\mathfrak{B}_5}, \mu_{\mathfrak{B}_5}}(v) \simeq \left(\frac{\log \log \log \log v}{\log \log \log v}\right)^{\alpha_1/d_1}.$$

4.3. *Comparison measures and applications.* The main theorems stated in the previous sections require that, for any symmetric probability measure μ, we exhibit a collection ζ_v, $v > 0$, of spread-out symmetric probability measures with the property that

(4.5) $$\Lambda_{1, \zeta_v}(v) \geq \delta$$

for some fixed $\delta \in (0, 1)$ and such that we can control $\mathcal{E}_{\zeta_v}$ in terms of v and $\mathcal{E}_\phi$. The following two theorems show that we can always produce such a collection of measures.

The first of these two theorems applies to subordinated measures ϕ_f. Namely, given a Bernstein function with Lévy measure ν and $t > 0$, set

$$\nu_t(ds) = \left[\nu((t,\infty))\right]^{-1}\mathbf{1}_{(t,\infty)}\nu(ds).$$

That is, the measures ν_t, $t > 0$, are the normalized tail measures of ν. Let

$$(4.6) \qquad\qquad c(t, f, n) = c(f_t, n)$$

be the coefficients associated by (2.14) with the Bernstein function

$$f_t(s) = \left(\int_{(0,\infty)} e^{-\tau}\nu_t(d\tau)\right)s + \int_{(0,\infty)}(1 - e^{-s\tau})\nu_t(d\tau).$$

Note that $f_t(0) = 0$, $f_t(1) = 1$.

THEOREM 4.6 (Spread-out measures for subordinated measures). *Let ϕ be a symmetric probability measure on a countable group G with Nash profile $\mathcal{N}_\phi$. Let f be a Bernstein function with Lévy measure ν and such that $f(0) = 0$, $f(1) = 1$. Then the measures $\zeta_v^f = \phi_{f_t}$, $t = \mathcal{N}_\phi(2v)$, $v > 0$, satisfy*

$$(4.7) \qquad \Lambda_{1,\zeta_v^f}(v) \geq \Lambda_{2,\zeta_v^f}(v) \geq \tfrac{1}{2} \quad\text{and}\quad \mathcal{E}_{\phi_f} \geq \nu\left((\mathcal{N}_\phi(2v),\infty)\right)\mathcal{E}_{\zeta_v^f}.$$

EXAMPLE 4.4. Assume that $f(s) = bs + \int_{(0,\infty)}(1 - e^{-ts})\nu(ds)$ is a complete Bernstein with $\nu(ds) = g(s)\,ds$, $f(1) = 1$ and $f'(s) - b \sim \frac{s^\alpha}{s\ell(1/s)}$ at 0^+ where ℓ is slowly varying at infinity and $\alpha \in [0, 1]$. By [2], (2.13), we have $g(s) \sim 1/[\Gamma(1 - \alpha)s^{1+\alpha}\ell(s)]$ at infinity which implies $\nu((s,\infty)) \sim c_\alpha f(1/s)$. This and Lemma 2.1 means that (4.7) is equivalent to the more explicit statement

$$\Lambda_{1,\zeta_v^f}(v) \geq \Lambda_{2,\zeta_v^f}(v) \geq \tfrac{1}{2} \quad\text{and}\quad \mathcal{E}_{\phi_f} \geq c_\alpha f\left(\Lambda_\phi(8v)/2\right)\mathcal{E}_{\zeta_v^f}.$$

PROOF. Write $f(s) = s + \int_{(0,\infty)}(1 - e^{-st})\nu(dt)$. First, apply (2.15) to f_t. Since, by definition, $\zeta_v^f = \phi_{f_t}$ and $\nu_t(I) = 0$ if the interval I is contained in $(0, \mathcal{N}_\phi(2v))$, for any function u with finite support, we have

$$\mathcal{E}_{\zeta_v^f}(u, u) \geq \tfrac{1}{2}\|u\|_2^2.$$

That is $\Lambda_{\zeta_v^f}(v) \geq \tfrac{1}{2}$.

Next, write $A = \cdot * (\delta_e - \phi)$ (right-convolution by $\delta_e - \phi$) and

$$\mathcal{E}_{\zeta_v^f}(u, u) = \int_0^\infty \tau e^{-\tau}\nu_t(d\tau)\mathcal{E}_\phi(u, u) + \int_{(0,\infty)}\langle(I - e^{-\tau A})u, u\rangle\nu_t(d\tau).$$

Since

$$\int_0^\infty \tau e^{-\tau}\nu_t(d\tau) \leq \frac{1}{\nu((\mathcal{N}_\phi(2v),\infty))}\int_0^\infty \tau e^{-\tau}\nu(d\tau),$$

and

$$\int_{(0,\infty)} \langle (I - e^{-\tau A})u, u \rangle v_t(d\tau) \leq \frac{1}{v((\mathcal{N}_\phi(2v), \infty))} \int_{(0,\infty)} \langle (I - e^{-\tau A})u, u \rangle v(d\tau)$$

it follows that

$$\mathcal{E}_{\zeta_v^f}(u, u) \leq \frac{1}{v((\mathcal{N}_\phi(2v), \infty))} \mathcal{E}_{\phi_f}(u, u). \qquad \square$$

EXAMPLE 4.5. Consider the Bernstein function $f_\alpha(s) = s^\alpha$, $\alpha \in (0, 1]$. In this case, $v_\alpha(dt) = \frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha-1}$. The construction above yields spread-out measures $\{\zeta_v^{f_\alpha}\}$ such that

$$\Lambda_{2, \zeta_v^{f_\alpha}}(v) \geq \tfrac{1}{2}$$

and (using Lemma 2.1 for the last inequality)

$$\mathcal{E}_{\phi_{f_\alpha}} \geq \frac{\mathcal{N}_\phi(2v)^{-\alpha}}{\Gamma(1-\alpha)} \mathcal{E}_{\zeta_v} \geq c_\alpha \Lambda_{2,\phi}(8v)^\alpha \mathcal{E}_{\zeta_v}.$$

The next result apply to any symmetric probability measure ϕ. For any fixed $\alpha \in (0, 1)$ and $t > 0$, consider the Lévy measure

$$v_t^\alpha(ds) = \kappa(\alpha, t)\mathbf{1}_{(t,2t)}(s)s^{-\alpha-1}\,ds, \qquad \kappa(\alpha, t) = \alpha(1 - 2^{-\alpha})^{-1}t^\alpha$$

and $b_t^\alpha = \kappa(\alpha, t) \int_t^{2t} e^{-u}u^{-\alpha-1}\,du$. Denote by f_t^α the Bernstein function

$$f_t^\alpha(s) = b_t^\alpha s + \int_{(0,\infty)} (1 - e^{-su})v_t^\alpha(du)$$

and note that, by construction, $f_t^\alpha(0) = 0$, $f_t^\alpha(1) = 1$. The Bernstein function f_t^α is a localized version of the classical Bernstein function $s \mapsto s^\alpha$, $\alpha \in (0, 1)$. For the applications we have in mind, using any arbitrary fixed value of $\alpha \in (0, 1)$ in the following theorem will be adequate.

THEOREM 4.7 (Spread-out measures for ϕ). *Let ϕ be a symmetric probability measure on a countable group G with L^2-isoperimetric profile Λ_ϕ. Fix $\alpha \in (0, 1)$. Then the measures $\zeta_v^\alpha = \phi_{f_t^\alpha}$, $t = \mathcal{N}_\phi(2v)$, $v > 0$, satisfy*

$$(4.8) \qquad \Lambda_{1, \zeta_v^\alpha}(v) \geq \Lambda_{2, \zeta_v^\alpha}(v) \geq \tfrac{1}{2} \quad and \quad \mathcal{E}_\phi \geq c_\alpha \Lambda_\phi(8v)\mathcal{E}_{\zeta_v^\alpha}.$$

PROOF. The proof of the first inequality in (4.8) is the same as in the case of (4.7). For the Dirichlet form comparison, write again $A = \cdot * (\delta_e - \phi)$ and recall that $t^{-1}\langle(I - e^{-tA})u, u\rangle$ is an increasing function of t with limit $\mathcal{E}_\phi(u, u)$. It follows that

$$\mathcal{E}_{\zeta_v^\alpha}(u, u) = b_t^\alpha \mathcal{E}_\phi(u, u) + \int_{(0,\infty)} \langle(I - e^{-\tau A})u, u\rangle v_t^\alpha(d\tau)$$

$$\leq \left(b_t^\alpha + \int_{(0,\infty)} \tau v_t^\alpha(d\tau)\right)\mathcal{E}_\phi(u, u).$$

Since $b_t^\alpha \le e^{-t}$ and $\int_0^\infty \tau v_t^\alpha (d\tau) = \frac{\alpha(2^{1-\alpha})}{(1-\alpha)(1-2^{-\alpha})} t$ and $t \ge 1$, this gives

$$\mathcal{E}_{\zeta_v^\alpha} \le c_\alpha t \mathcal{E}_\phi.$$

The desired result (with a different c_α) follows since, by Lemma 2.1,

$$t = \mathcal{N}_\phi(2v) \le 2/\Lambda_\phi(8v). \qquad \square$$

Theorem 4.7 turns the statements of Section 4.2 into very effective results by providing the needed hypotheses. In particular, Theorem 1.8 stated in the Introduction follows immediately from Proposition 4.3 and Theorem 4.7. Similarly, the case $p = 2$ of Theorem 4.5 and Theorem 4.7 yields the following statement.

THEOREM 4.8. *Let μ_i be symmetric probability measures on H_i, $1 \le i \le m$. Fix a symbol $\mathfrak{B}$ of length m as in Theorem 4.5. Then, for any $v, s > 0$, the measure $\mu = \mu_{\mathfrak{B}}$ on $G = W_{\mathfrak{B}}(H_m, \ldots, H_1)$ satisfies*

$$\Lambda_{2,G,\mu}(v) \ge s/K^m \qquad \textit{for any } v \le W_{\mathfrak{B}}\big(\Lambda_{2,H_1,\mu_1}^{-1}(s), \ldots, \Lambda_{2,H_m,\mu_m}^{-1}(s)\big).$$

5. Spread-out random walks on wreath products. This section provides a host of explicit examples where the behavior of random walks associated with spread-out measures on wreath products can be computed. In particular, we obtain a variety of sharp estimates for $\Phi_{G,\rho}$ when G is a wreath product (or an iterated wreath product) and ρ is a moment function.

5.1. *Groups where Λ_G is controlled by volume growth.* We say that Λ_G is controlled by volume growth if $\Lambda_G \simeq \mathcal{W}_G^{-2}$ where $\mathcal{W}_G(v) = \inf\{r : V_G(r) > v\}$. It is always true that $\Lambda_G(v) \gtrsim \mathcal{W}_G^{-2}$ (this follows from the L^2-version of the argument in [9]; see the Appendix for variations). Groups quasi-isometric to polycyclic groups satisfy $\Lambda_G \simeq \mathcal{W}_G^{-2}$ and Tessera ([31], Theorem 4) describes a large class of groups of exponential volume growth (Geometrically Elementary Solvable or GES groups) which satisfy $\Lambda_G \simeq \mathcal{W}_G^{-2}$. In all these cases, the volume growth function is of type $V_G(r) \simeq r^d$ or $V_G(r) \simeq \exp(r)$ and $\Lambda_G(r) \simeq v^{-2/d}$ [equivalently $\Phi_G(n) \simeq n^{-d/2}$] or $\Lambda_G(v) \simeq (\log(1+v))^{-2}$ [equivalently $\Phi_G(n) \simeq \exp(-n^{1/3})$], respectively.

THEOREM 5.1. *Let (H_i), $1 \le i \le m$ be groups for which $\Lambda_{H_i} \simeq \mathcal{W}_{H_i}^{-2}$. For each i, $1 \le i \le m$, let $\mu_i(h) = c_i \sum_1^\infty 4^{\alpha_i k} \mathbf{u}_{H_i}(4^k)$ with $\alpha_i \in (0, 2)$. Referring to the notation of Theorem 4.5, fix a wreath product symbol $\mathfrak{B}$ of length m and consider the measure $\mu_{\mathfrak{B}} = v_{\mathfrak{B}, \mu_m, \ldots, \mu_1}$ defined at (4.3) on the wreath product $W_{\mathfrak{B}}(H_m, \ldots, H_1)$. Then the $\simeq$-class of $\Lambda_{\mu_{\mathfrak{B}}}$ can be computed and is described by Theorem 4.8. In particular, when $m = 2$ and $\mu = \mu_{(\cdot \wr \cdot)}$ on $H_2 \wr H_1$:*

- *If V_{H_1} is exponential and H_2 nontrivial, $\mu^{(2n)}(e) \simeq \exp(-n/[\log n]^{\alpha_1})$.*

- If $V_{H_1}(r) \simeq r^{d_1}$ and $V_{H_2}(r) \simeq r^{d_2}$,
$$\mu^{(2n)}(e) \simeq \exp\left(-n^{d_1/(\alpha_1+d_1)}[\log n]^{\alpha_1/(\alpha_1+d_1)}\right).$$

- If $V_{H_1}(r) \simeq r^{d_1}$ and $V_{H_2}(r) \simeq \exp(r)$,
$$\mu^{(2n)}(e) \simeq \exp\left(-n^{(\alpha_1+\alpha_2 d_1)/(\alpha_1+\alpha_2 d_1+\alpha_1\alpha_2)}\right).$$

Let K be a finitely generated group which will be either finite, of polynomial volume growth or of exponential volume growth and such that $\Phi_K(n) \simeq \exp(-n^{1/3})$. For instance, K could be any polycyclic group. Let H be a group of polynomial volume growth. In any of these cases, $\Phi_{K \wr H}$ is known (thanks to the results of [13, 22]). In the first case (K finite) $\Phi_{K \wr H}(n) \simeq \exp(-n^{d/(2+d)})$. In the second case, $\Phi_{K \wr H}(n) \simeq \exp(-n^{d/(2+d)}(\log n)^{2/(2+d)})$ and in the third case, $\Phi_{K \wr H}(n) \simeq \exp(-n^{(1+d)/(3+d)})$. In particular, Corollary 3.6 applies to these groups and gives that for any slowly varying function ρ as in (2.17) such that $\rho(t^a) \simeq \rho(t)$ for each $a > 0$, we have
$$\Phi_{K \wr H, \rho}(n) \simeq \exp(-n/\rho(n)).$$

The following two theorems provide the behavior of $\widetilde{\Phi}_{K \wr H, \rho}$ for $\rho(s) = \rho_\alpha(s) = (1+s)^\alpha$, $\alpha \in (0, 2)$ and for $\rho(s)$ regularly varying of index 2.

THEOREM 5.2. *Let H be a group of polynomial volume growth of degree d.*

1. *If $K \neq \{e_K\}$ is finite, we have*
$$\widetilde{\Phi}_{K \wr H, \rho_\alpha}(n) \simeq \exp(-n^{d/(\alpha+d)}).$$

2. *If K is not finite and has polynomial volume growth, we have*
$$\widetilde{\Phi}_{K \wr H, \rho_\alpha}(n) \simeq \exp(-n^{d/(\alpha+d)}(\log n)^{\alpha/(\alpha+d)}).$$

3. *If K has exponential growth and satisfies $\Phi_K(n) \simeq \exp(-n^{1/3})$, we have*
$$\widetilde{\Phi}_{K \wr H, \rho_\alpha}(n) \simeq \exp(-n^{(d+1)/(\alpha+d+1)}).$$

PROOF.   The lower bounds are already derived in [3]. They also follow from Theorem 3.2. The upper bounds follow from Theorem 1.8 and known results on K, H. Consider, for instance, the case when K has exponential volume growth. To obtain an upper bound on $\Phi_{K \wr H, \rho_\alpha}$, consider the measures

$$\mu_{H,\alpha}(h) \asymp (1+|h|)^{-\alpha-d} \quad \text{on } H \quad \text{and} \quad \mu_{K,\alpha}(k) \asymp \sum_1^\infty 4^{-\alpha k}\mathbf{u}_{4^k} \quad \text{on } K.$$

They satisfy $W(\rho_\alpha, \mu_{H,\alpha}) < \infty$ and $W(\rho_\alpha, \mu_{K,\alpha}) < \infty$ and this immediately implies $W(\rho_\alpha, \mu) < \infty$ where $\mu = \frac{1}{2}(\mu_{H,\alpha} + \mu_{K,\alpha})$ is understood as a probability measure on $K \wr H$. By Theorems A.2–A.7, $\Lambda_{2,K,\mu_{K,\alpha}}(v) \simeq (\log(e+v))^{-\alpha}$ and $\Lambda_{2,H,\mu_{H,\alpha}}(v) \simeq v^{-\alpha/d}$. Theorem 1.8, Lemma 2.1 and Theorem 2.2 give the desired result. □

THEOREM 5.3. *Let H be a group of polynomial volume growth of degree d. Let ρ be a regularly varying function of index 2 and set $M(t) = s^2 / \int_0^t \frac{s\,ds}{\rho(s)}$. Assume that $\theta(t) = \int_0^t \frac{s\,ds}{\rho(s)}$ satisfies $\theta(t^a) \simeq \theta(t)$ for each $a > 0$.*

1. *If $K \neq \{e_K\}$ is finite, we have*

$$\widetilde{\Phi}_{K \wr H, \rho}(n) \simeq \exp\left(-\left(n \int_0^n \frac{s\,ds}{\rho(s)}\right)^{d/(2+d)}\right).$$

2. *If K is not finite and has polynomial volume growth, we have*

$$\widetilde{\Phi}_{K \wr H, \rho}(n) \simeq \exp\left(-\left(n(\log n)^{2/d} \int_0^n \frac{s\,ds}{\rho(s)}\right)^{d/(2+d)}\right).$$

3. *If K satisfies $\Phi_K(n) \gtrsim \exp(-n^{1/3})$, we have*

$$\widetilde{\Phi}_{K \wr H, \rho}(n) \gtrsim \exp\left(-\left(n(\log n)^{2/(d+1)} \int_0^n \frac{s\,ds}{\rho(s)}\right)^{(d+1)/(d+3)}\right).$$

PROOF. The lower bounds follow from Theorem 3.2. The upper bounds follow from Proposition 4.3. Note that the upper bound is missing in the last case. We outline the upper-bound argument in case 2. Consider the measures

$$\mu_{G,\rho}(g) \asymp \frac{1}{(1 + |g|)^{2+d}\ell(1 + |g|)}$$

for $G = H$ and $G = K$. By Proposition A.4, we have

$$\mathcal{E}_{G,\mathbf{u}_r} \leq C \frac{r^2}{\theta(r)} \mathcal{E}_{\mu_{G,\rho}}$$

for $G = H, K$. This allows us to verify the hypotheses of Proposition 4.3 with $H_1 = H$, $H_2 = K$, $\zeta_{i,v_i(t)} = \mathbf{u}_{r(2v_i(t))}$, $r(v_i) \simeq v^{1/d_i}$ and $v_i(t) \simeq (t\theta(t))^{d_i/2}$, where $d_1 = d$ and d_2 is the degree of polynomial volume growth of K. In the notation of Proposition 4.3, this gives $v(t) \simeq \exp([t\theta(t)]^{d/2} \log t)$ which translates into

$$\Lambda_{K \wr H, \mu}(v) \gtrsim \frac{(\log \log v)^{2/d} \theta(\log v)}{(\log v)^{2/d}},$$

where the measure μ on $K \wr H$ is given by $\mu = \frac{1}{2}(\mu_{H,\rho} + \mu_{K,\rho})$. With this estimate in hand, Theorem 2.2 gives $\mu^{(2n)}(e) \lesssim \psi(n)$ where ψ is given implicitly as a function of t by

$$t = \int_1^{1/\psi} \frac{(\log v)^{2/d}}{[\log(e + \log v)]^{2/d} \theta(\log v)}.$$

A somewhat tedious computation shows that this equality gives

$$t \simeq \frac{[\log(1/\psi)]^{(2+d)/d}}{[\log \log(1/\psi)]^{2/d} \theta(\log(1/\psi))}$$

or, equivalently,

$$\log(1/\psi) \simeq \big(t(\log t)^{2/d}\theta(t)\big)^{d/(2+d)}.$$

Note that the assumed property that $\theta(t^a) \simeq \theta(t)$ for $a > 0$ has been repeatedly used in these computations. This gives the desired upper bound on $\mu^{(2n)}(e)$ and thus on $\Phi_{K \wr H,\rho}$ as well. $\square$

REMARK 5.4. In the third statement of Theorem 5.3, even if we assume in addition that K has exponential volume growth [in which case $\Phi_K(n) \simeq \exp(-n^{1/3})$], we would still not be able to state a matching upper bound. The reason is that we do not have at our disposal the appropriate pseudo-Poincaré inequality on K (in the case when K has polynomial volume growth, we used Proposition A.4). However, consider the special case when $K = F \wr \mathbb{Z}$ with $F \neq \{e\}$ finite. This group has exponential volume growth and satisfies $\Phi_K(n) \simeq \exp(-n^{1/3})$. Further, Proposition 4.3 applied with $H_1 = K$, $H_2 = F$ provides us with a measure $\mu_{K,\rho}$ on K (and accompanying measures ζ_v) which is a good witness for $\widetilde{\Phi}_{K,\rho}$ and can be used to apply Proposition 4.3 with $H_1 = H$, $H_2 = K = F \wr \mathbb{Z}$ $\mu_1 = \mu_{H,\rho}$ as above and $\mu_2 = \mu_{K,\rho}$ (the measure just obtained on $K = F \wr \mathbb{Z}$). After elementary but tedious computations, Proposition 4.3 implies that the measure $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ on $K \wr H = (F \wr \mathbb{Z}) \wr H$ satisfies

$$\mu^{(2n)}(e) \leq \exp\left(-\left(n(\log n)^{2/(d+1)}\int_0^n \frac{s\,ds}{\rho(s)}\right)^{(d+1)/(d+3)}\right).$$

This shows that

$$\Phi_{(F\wr\mathbb{Z})\wr H,\rho}(n) \simeq \exp\left(-\left(n(\log n)^{2/(d+1)}\int_0^n \frac{s\,ds}{\rho(s)}\right)^{(d+1)/(d+3)}\right).$$

In particular, the lower bound stated in Theorem 5.3 is sharp in this case. We conjecture that it is also sharp when K is polycyclic of exponential volume growth.

5.2. *Anisotropic measures on nilpotent groups.* This section is concerned with special cases of the following problem raised and studied in [27]. Given a group G generated by a k-tuple $S = (s_1, \ldots, s_k)$, study the behavior of the random walks driven by the measures

$$\mu_{S,a}(g) = \frac{1}{k}\sum_1^k \sum_{m\in\mathbb{Z}} \frac{c_i}{(1+|m|)^{1+\alpha_i}}\mathbf{1}_{s_i^m}(g),$$

where $a = (\alpha_1, \ldots, \alpha_k) \in (0,\infty)^k$ and $c_i = (\sum_{\mathbb{Z}}(1+|m|)^{-1-\alpha_i})^{-1}$. In words, to take a step according to $\mu_{S,a}$, pick one of the k generators, say s_i, uniformly at random. Independently, pick an integer $m \in \mathbb{Z}$ according to the power law giving probability $c_i(1+|m|)^{-1-\alpha_i}$ to m. Then multiply the present position (on the right) by s_i^m.

It is of interest to investigate the behavior of $\mu_{S,a}^{(n)}(e)$ and to understand how, given the generating k-tuple S, this behavior depends on the choice of $a = (\alpha_1, \ldots, \alpha_k)$. Here, we use the results of [27] and the technique of the present paper to prove the following result.

We consider two groups $H = H_1$ and $K = H_2$. The group H is assumed to be nilpotent generated by the $S = (s_1, \ldots, s_p)$. On this nilpotent group, we considered the measures $\mu_{H,S,a}$ with $a = (\alpha_1, \ldots, \alpha_p) \in (0, 2)^p$. The group K will be either finite or nilpotent. If it is finite, we let μ_K be the uniform measure on K. If K is nilpotent, generated by a given tuple $T = (t_1, \ldots, t_q)$, we consider the measures $\mu_{K,T,b}$ with $b = (\beta_1, \ldots, \beta_q) \in (0, 2)^q$.

Next, we consider the wreath product $G = K \wr H$. When K is nilpotent, the generating sets S and T (for H and K, resp.) together produce a generating set $\Sigma = (\sigma_1, \ldots, \sigma_k)$, $k = p + q$, of $K \wr H$ where $\sigma_1, \ldots, \sigma_p$ corresponds to $s_1, \ldots, s_p$ and generates H inside $K \wr H$ and the generators $\sigma_{p+1}, \ldots, \sigma_{p+q}$ correspond to $t_1, \ldots, t_q$ and generate the copy of K in $K \wr H$ which seats at e_H. Similarly, set $c = (\gamma_1, \ldots, \gamma_k)$ with $\gamma_i = \alpha_i$, $1 \le i \le p$ and $\gamma_i = \beta_{i-p}$, $i = p + 1, \ldots, p + q = k$. By elementary Dirichlet form comparison arguments, we know that the measures

$$\mu = \tfrac{1}{2}(\mu_{H,S,a} + \mu_{K,T,b}) \quad \text{and} \quad \mu_{K \wr H, \Sigma, c} \qquad \text{on } G = K \wr H$$

satisfy $\mu^{(2n)}(e) \simeq \mu_{\Sigma,c}^{(2n)}(e)$.

THEOREM 5.5. *Let H, S, p and $a \in (0, 2)^p$ be as above. Let $d(a)$ be the real given by* [27], *Theorem* 1.8, *and such that $\mu_{S,a}^{(n)}(e) \simeq n^{-d(a)}$.*

1. *Assume that $K \ne \{e\}$ is finite. Then the measure $\mu = \tfrac{1}{2}(\mu_{S,a} + \mu_K)$ on $K \wr H$ satisfies*

$$\mu^{(n)}(e) \simeq \exp(-n^{d(a)/(1+d(a))}).$$

2. *Assume that K is nilpotent (infinite) and T, q and $b \in (0, 2)^q$ are as described above. Then, on $G = K \wr H$ equipped with the generating set Σ define above, the measure $\mu_{\Sigma,c}$ satisfies*

$$\mu_{\Sigma,c}^{(n)}(e) \simeq \exp(-n^{d(a)/(1+d(a))}(\log n)^{1/(1+d(a))}).$$

PROOF. This follows from Theorem 1.8 because [27], Theorem 1.8 shows that the measures $\mu_1 = \mu_{H,S,a}$ and $\mu_2 = \mu_{K,T,b}$ satisfy $\mu_i^{(n)}(e) \simeq n^{-d_i}$ where $d_1 = d(a)$, $d_2 = d(b)$ are the real described in [27], Theorem 1.8 [recall that these estimates are equivalent to $\Lambda_{2,\mu_i}(v) \simeq v^{-2/d_i}$]. □

5.3. *Local time functionals.* Let H be a group equipped with a symmetric measure μ. Let $\ell(x, n)$ be the number of visits to x up to time n. More precisely, let (X_n) denotes the trajectory of a random walk driven by μ on H and set

$$l(n, x) = \#\{0 < k \le n : X_k = x\}.$$

It is well known that the behavior of the probability of return of the switch-walk-switch random walk on the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr H$ is related to certain functionals of the local times $(\ell(x, n))_{x \in H}$. More precisely and more generally, let K be a finitely generated group (possibly finite). Let μ_K be a symmetric measure on K satisfying $\mu_K(e_K) > 0$. Let $q = \mu_K * \mu * \mu_K$ be the switch-walk-switch measure on $K \wr H$ (see, e.g., [24] for details). With this notation, we have

$$q^{(n)}\big((e_K, h)\big) \asymp \mathbf{E}_\mu\bigg(\prod_{x \in H} \nu^{(2l(n,x))}(e_K)\mathbf{1}_{\{X_n=h\}} \bigg),$$

where $\mathbf{E}_\mu$ and (X_n) refers to the random walk on H driven by μ.

Set

$$F_K(n) := -\log \nu^{(2n)}(e_K)$$

so that, for any $h \in H$,

$$(5.1) \qquad q^{(n)}\big((e_K, h)\big) \simeq \mathbf{E}\big(e^{-\sum_{x \in H} F_K(l(n,x))}\mathbf{1}_{\{X_n=h\}}\big).$$

Assume next that, for each $R > 0$ there is a set $U_R \subset of\ H$ and $\kappa \geq 1$ such that

$$(5.2) \qquad |U_R| \leq R^\kappa \quad \text{and} \quad \mu^{(n)}(H \setminus U_R) \leq Cn^\kappa(1 + R/n)^{-1/\kappa}.$$

We note that the second condition follows easily from the tail condition

$$(5.3) \qquad \mu(H \setminus U_R) \leq C(1 + R)^{-1/\kappa}$$

when $U_R = \{h : N(h) > R\}$ where $N : H \to [0, \infty)$ satisfies $N(h_1h_2) \leq N(h_1) + N(h_2)$. Indeed, under such circumstances, we have

$$\mu^{(n)}(H \setminus U_R) \leq n\mu(H \setminus U_{R/n}) \leq Cn(1 + R/n)^{-1/\kappa}.$$

Writing

$$\mathbf{E}_\mu\big(e^{-\sum_{x \in H} F_K(l(n,x))}\big)$$

$$= \mathbf{E}_\mu\bigg(\sum_{h \in U_R} e^{-\sum_{x \in H} F_K(l(n,x))}\mathbf{1}_{\{X_n=h\}} \bigg)$$

$$+ \mathbf{E}_\mu\bigg(\sum_{h \in H \setminus U_R} e^{-\sum_{x \in H} F_K(l(n,x))}\mathbf{1}_{\{X_n=h\}} \bigg)$$

$$\leq |U_R|q^{(2[n/2])}(e_{K \wr H}) + \mu^{(n)}(H \setminus U_R)$$

shows that, under assumption (5.2) and assuming that $q^{(n)}(e_{K \wr H}) \simeq \exp(-\omega(n))$ with $\omega(n)$ regularly varying of index in $(0, 1]$, we can conclude that

$$(5.4) \qquad \mathbf{E}_\mu\big(e^{-\sum_{x \in H} F_K(l(n,x))}\big) \simeq \exp\big(-\omega(n)\big)$$

as well.

This technique and remarks, together with Theorems 5.2–5.3, suffice to prove the following results.

COROLLARY 5.6. *Let H be a group of polynomial volume growth of degree d. Let $\mu_\alpha(h) \asymp (1 + |h|)^{-\alpha-d}$, $\alpha > 0$. Let R_n be the number of visited point up to time n. For any fixed $\kappa > 0$:*

- *If $\alpha > 2$ then $\mathbf{E}_{\mu_\alpha}(e^{-\kappa R_n}) \simeq \exp(-n^{d/(2+d)})$.*
- *If $\alpha = 2$ then $\mathbf{E}_{\mu_2}(e^{-\kappa R_n}) \simeq \exp(-(n \log n)^{d/(2+d)})$.*
- *If $\alpha \in (0, 2)$ then $\mathbf{E}_{\mu_\alpha}(e^{-\kappa R_n}) \simeq \exp(-n^{d/(\alpha+d)})$.*

REMARK 5.7. Note that the second case, $\alpha = 2$, may be new even in the case when $H = \mathbb{Z}$. It gives the behavior of $\mathbf{E}(e^{-\kappa R_n})$ for the walk on $\mathbb{Z}$ driven by the measure $\mu_2(z) = c(1 + |z|)^{-2}$ for which there is no classical local limit theorem and to which the classical Donsker–Varadhan theorem does not apply.

COROLLARY 5.8. *Let H be a group of polynomial volume growth of degree d. Let R_n be the number of visited point up to time n. Consider the random walk driven by $\mu_{S,a}$ with $a = (\alpha_1, \ldots, \alpha_k) \in (0, 2)^k$. Let $d(a)$ be the real given by [27], Theorem 1.8, and such that $\mu_{S,a}^{(n)}(e) \simeq n^{-d(a)}$. For any fixed $\kappa > 0$, we have*

$$\mathbf{E}_{\mu_{S,a}}(e^{-\kappa R_n}) \simeq \exp(-n^{d(a)/(1+d(a))}).$$

Given a measure μ such as μ_α or $\mu_{S,a}$ on H, and fixed $\kappa > 0$ and $\gamma \in (0, 1)$, we can determine the behavior of

$$n \mapsto \mathbf{E}(e^{-\kappa \sum_H \ell(n,h)^\gamma}).$$

Indeed, it suffices consider the wreath product $\mathbb{Z} \wr H$ with a measure ϕ on $\mathbb{Z}$ such that $\nu^{(2n)}(0) \simeq \exp(-n^\gamma)$. The choice $\phi(x) \asymp (1 + |x|)^{-1}[1 + \log(1 + |x|)]^{-1/\gamma}$ fulfills these requirements (see [25]).

COROLLARY 5.9. *Let H be a group of polynomial volume growth of degree d. Fix $\kappa > 0$ and $\gamma \in (0, 1)$ Let $\ell(n, x)$ be the number of visits to $x \in H$ up to time n. Let $\mu_\alpha(h) \asymp (1 + |h|)^{-\alpha-d}$, $\alpha \in (0, 2)$, or $\mu = \mu_{S,a}$ with $a = (\alpha_1, \ldots, \alpha_k) \in (0, 2)^k$. In the second case, let $d(a)$ be the real given by [27], Theorem 1.8, and such that $\mu_{S,a}^{(n)}(e) \simeq n^{-d(a)}$ and set $\frac{d}{\alpha} = d(a)$. We have*

$$\mathbf{E}_\mu(e^{-\kappa \sum_H \ell(n,h)^\gamma}) \simeq \exp(-n^{(\alpha\gamma+d(1-\gamma))/(\alpha+d(1-\gamma))}).$$

REMARK 5.10. If $\mu(h) \asymp (1+|h|)^{-2-d}$ or $\mu = \mu_{S,a}$ with $a = (2, \ldots, 2)$, one gets

$$\mathbf{E}_\mu(e^{-\kappa \sum_H \ell(n,h)^\gamma}) \simeq \exp(-n^{(2\gamma+d(1-\gamma))/(2+d(1-\gamma))}[\log n]^{d(1-\gamma)/(2+d(1-\gamma))}).$$

APPENDIX: RADIAL POWER LAWS ON GROUPS

In this Appendix, we compute the L^p-profiles $\Lambda_{p,\phi}$ for radial "power law" probability measures on certain groups.

A.1. Norm-radial power laws. Let G be a countable group. Let $N : G \rightarrow [0, \infty)$ be such that $N(e) = 0$, $N(x^{-1}) = N(x)$ and $N(xy) \leq C_N(N(x) + N(y))$. Set

$$V_N(r) = \left|\{x \in G : N(x) \leq r\}\right|, \qquad B_N(m) = \{x \in G : N(x) \leq m\}$$

and $\boldsymbol{v}_m = V_N(m)^{-1}\mathbf{1}_{B(m)}$. For $\alpha > 0$, set

$$\phi_\alpha = c_\alpha \sum_1^\infty 4^{-\alpha k}\boldsymbol{v}_{4^k}, \qquad c_\alpha = (4^\alpha - 1)/(4^\alpha - 2).$$

It is obvious that

$$\forall r = 4^k,$$

(A.1)
$$\sum_{x,y}|f(xy) - f(x)|^p v_r(y) \leq c_\alpha r^\alpha \sum_{x,y}|f(x) - f(xy)|^p \phi_\alpha(y).$$

Let W_N be the inverse function of the modified volume function defined by $\mathfrak{V}_N(r) = V_N(4^k)$ if $4^k \leq r < 4^{k+1}$, that is, $\mathfrak{W}_N(t) = \inf\{s : \mathfrak{V}_N(s) > t\}$. Note that $\mathfrak{V}_N \simeq V_N$.

PROPOSITION A.1. *Referring to the setup introduced above, for any $\alpha > 0$ and $p \in [1, \infty)$, we have*

$$\Lambda_{p,\phi_\alpha}(v) \geq \frac{1}{c_\alpha 8^p \mathfrak{W}_N(2^p v)^\alpha}.$$

PROOF. The argument is well known and is given here for convenience of the reader. See also [9, 20]. Consider a function $f \geq 0$ with $|\operatorname{support}(f)| \leq v$. For any $\lambda > 0$, write

$$\left|\{f \geq \lambda\}\right| \leq \left|\{|f - f * v_r| \geq \lambda/2\}\right| + \left|\{|f * v_r| \geq \lambda/2\}\right|$$

and note that $\|f * v_r\|_\infty \leq V_N(r)^{-1/p}\|f\|_p$.

Recall the notation $f_k = (f - 2^k)^+ \wedge 2^k$ and observe that $\|f_k\|_p \leq 2^k v^{1/p}$. It follows that $\|f_k * v_r\|_\infty \leq 2^k V_N(r)^{-1/p}v^{1/p}$. Pick r so that $V_N(r) > 2^p v$ and pick $\lambda = 2^k$. We have

$$\left|\{f_k \geq 2^k\}\right| \leq \left|\{|f - f * v_r| \geq \lambda/2\}\right| \leq 2^{-p(k-1)}r^\alpha \mathcal{E}_{p,\phi_\alpha}(f_k).$$

Recall that $|\{f \geq 2^{k+1}\}| = |\{f \geq 2^k\}|$ and write

$$\|f\|_p^p \leq 8^p \sum_k 2^{p(k-1)}\left|\{f \geq 2^{k+1}\}\right|$$

$$\leq c_\alpha 8^p r^\alpha \sum_k \mathcal{E}_{p,\phi_\alpha}(f_k) \leq c_\alpha 8^p r^\alpha E_{p,\phi_\alpha}(f),$$

where, for the last step, we have used (2.5). Given the choice of r as a function of v, this gives

$$\Lambda_{p,\phi_\alpha}(v) \geq c_\alpha^{-1} 8^{-p} \mathfrak{W}_N (2^p v)^{-\alpha}$$

which is the desired inequality. $\square$

In the case when $N = |\cdot|_N$ is the word-length associated with a finite symmetric generating set S, write $\mathcal{W}$ for the inverse function of the volume growth $V = V_S$. Proposition A.1 gives

$$\Lambda_{p,\phi_\alpha} \gtrsim \mathcal{W}^{-\alpha}, \qquad p \geq 1, \alpha > 0.$$

However, these inequalities compete with those deduced in a similar way from

$$(A.2) \qquad \forall r > 0, \qquad \|f - f_r\|_p^p \leq C(p, S, \alpha) r^p \sum_{x,y} |f(x) - f(xy)|^p \phi_\alpha(y).$$

This inequality is an immediate consequence of the well-known pseudo-Poincaé inequality

$$\forall r > 0, \qquad \|f - f_r\|_p^p \leq C(p, S) r^p \sum_{x,y} |f(x) - f(xy)|^p \mathbf{u}(y)$$

which follows from the definition of the word length and a simple telescoping sum argument. See, for example, [9, 20].

It follows that we have

$$\Lambda_{p,\phi_\alpha} \gtrsim \begin{cases} \mathcal{W}^{-\alpha}, & \text{if } \alpha \in (0, p], \\ \mathcal{W}^{-p}, & \text{if } \alpha > p. \end{cases}$$

In fact, because of the Dirichlet form comparison $\mathcal{E}_{\phi_\alpha} \simeq \mathcal{E}_{\mathbf{u}}$ which holds for $\alpha > 2$ (see, e.g., [21]), we must have $\Lambda_{\phi_\alpha} \simeq \Lambda_G$ for $\alpha > 2$. Similarly, for $\alpha > p$, we have

$$\forall f, \qquad \sum_{x,y} |f(xy) - f(x)|^p \phi_\alpha(y) \asymp \sum_{x,y} |f(xy) - f(x)|^p \mathbf{u}(y)$$

and thus $\Lambda_{p,\phi_\alpha} \simeq \Lambda_{p,G}$. In the case $p = 1$, this implies that $J_{\phi_\alpha} \simeq J_G$ for all $\alpha > 1$. This discussion is captured in the following result.

THEOREM A.2. *Let G be a finitely generated group equipped with a finite symmetric generating set and associated word-length. Set $\phi_\alpha = \sum_1^\infty 4^{-k\alpha} \mathbf{u}_{4^k}$ where $\mathbf{u}_r$ is the uniform measure on the ball of radius r in G. Let W be the inverse function of the volume growth function of G.*

- *For $1 \leq p < \alpha < \infty$, $\Lambda_{p,\phi_\alpha} \simeq \Lambda_{p,G}$.*
- *For $\alpha \in (0, p)$, we always have $\mathcal{W}^{-\alpha} \lesssim \Lambda_{p,\phi_\alpha} \lesssim \Lambda_{p,G}^{\alpha/p}$.*
- *If for a given $p \in [1, \infty)$ we have $\Lambda_{p,G} \simeq \mathcal{W}^{-p}$ then*

$$\forall \alpha \in (0, p), \qquad \Lambda_{p,\phi_\alpha} \simeq \mathcal{W}^{-\alpha}.$$

Note that the case $\alpha = p$ is excluded from this statement. Note also that the wreath product construction provides many examples of groups for which $\Lambda_{p,G} \not\simeq \mathcal{W}^{-p}$.

PROOF OF THEOREM A.2.   The case when $\alpha > p$ is explained above as well as the lower bounds when $\alpha \in (0, p]$. The upper bound for $\alpha \in (0, p)$ follows from Theorem 2.13.   □

EXAMPLE A.1.   Polycyclic groups satisfy $\Lambda_{p,G} \simeq \mathcal{W}^{-p}$ for each $p \in [1, \infty)$. The lower bound follows from the argument of [9] as explained above. The upper bound is best derived from the existence of adapted Følner couples, a technique developed and explained in [8]. Other groups for which $\Lambda_{p,G} \simeq \mathcal{W}^{-p}$ include the Baumslag solitar groups $\mathbf{BS}(1, m) = \langle a, b : aba^{-1} = b^m \rangle$ and the lamplighter groups $F \wr \mathbb{Z}$ with F finite. Romain Tessera ([31], Theorem 4) describes a large class of groups of exponential volume growth (Geometrically Elementary Solvable or GES groups) which satisfy $\Lambda_{p,G} \simeq \mathcal{W}^{-p}$. Note that what Tessera denotes by $j_{p,G}$ is $1/\Lambda_{p,G}^{1/p}$.

REMARK A.3.   Recall the two-sided Cheeger inequality (2.4), that is,

$$c(p, q)\Lambda_{p,\phi}^{q/p} \leq \Lambda_{q,\phi} \leq C(p, q)\Lambda_{p,\phi}, \qquad 1 \leq p \leq q < \infty.$$

Let G be a group such that $\Lambda_{p,G} \simeq \mathcal{W}^{-p}$, $p \geq 1$ and fix $\alpha \in (0, \infty)$. By Theorem A.2, if $p \in [1, \alpha)$, $\Lambda_{p,\phi_\alpha} \simeq \mathcal{W}^{-p}$ but if $p > \alpha$, $\Lambda_{p,\phi_\alpha} \simeq \mathcal{W}^{-\alpha}$. In particular, if $p, q > \alpha$, then $\Lambda_{p,\phi_\alpha} \simeq \Lambda_{q,\phi_\alpha}$ but, if $p, q \in [1, \alpha)$ then $\Lambda_{p,G}^{q/p} \simeq \Lambda_{p,G}$. In the case $1 \leq p < \alpha < q < \infty$, neither of the two sides of the Cheeger inequality is optimal.

A.2. Word-length power laws on group with polynomial volume growth.

We now focus on the case when $N(x) = |x|_S$ is the word-length of x with respect to a finite symmetric generating set S on a group of polynomial volume growth. Dropping the reference to the set S, we set $V(r) = |\{x : |x| \leq r\}|$ and assume that $V(r) \simeq r^D$, that is, we assume that the group G has polynomial volume growth of degree D. In this case, we can use a more refined version of the measure ϕ_α by setting

$$\phi_\alpha = c_\alpha \sum_1^\infty k^{-\alpha-1}\mathbf{u}_k, \qquad c_\alpha^{-1} = \sum_1^\infty k^{-1-\alpha}.$$

It is easy to use an Abel summation argument to check that

$$\forall x \in G, \qquad \phi_\alpha(x) \asymp (1 + |x|)^{-\alpha-D}$$

(the same holds true for the measure $c_\alpha' \sum_1^\infty 4^{-\alpha k}\mathbf{u}_{4^k}$).

PROPOSITION A.4. *Let G be a group with polynomial volume growth. Then, for each $p \geq 1$ and $r > s \geq 1$, we have*

$$\sum_{x,y} |f(xy) - f(x)|^p \mathbf{u}_r(y) \leq C(G, p)(r/s)^p \sum_{x,y} |f(xy) - f(x)|^p \mathbf{u}_s(y).$$

PROOF. For any $1 \leq s \leq s_0$, this follows from the usual Dirichlet argument using paths. So, we can assume $s > 3$ and let s' be the largest integer smaller than $s/3$. For any $y \in G$, write $y = y_0 y_1 \cdots y_k$ with $y_0 = e$, $|y_i| \leq s'$, $1 \leq i \leq k$, and $k \leq 9|y|/s$. For any finite supported function f, $\xi_1, \ldots, \xi_{k-1} \in G$ and $|y| \leq r$, we have

$$|f(xy) - f(x)|^p \leq 9(r/s)^{p-1} \sum_1^k |f(xz_0 \cdots z_i) - f(xz_0 \cdots z_{i-1})|^p,$$

where $z_i = \xi_{i-1}^{-1} y_i \xi_i$ with $\xi_0 = \xi_k = e$. Summing over $x \in G$ gives

$$\sum_x |f(xy) - f(x)|^p \leq 9(r/s)^{p-1} \sum_1^k \sum_x |f(x\xi_{i-1}^{-1} y_i \xi_i) - f(x)|^p.$$

We now average this inequality over

$$(\xi_0, \xi_1, \ldots, \xi_{k-1}, \xi_k) \in \{e\} \times B(s') \times \cdots \times B(s') \times \{e\}.$$

This gives

$$\sum_x |f(xy) - f(x)|^p \leq 9(r/s)^{p-1} \left(V(s')^{-1} \sum_x \sum_{|\xi| \leq s'} |f(xy_1 \xi) - f(x)|^p \right.$$

$$+ V(s')^{-2} \sum_{i=2}^{k-1} \sum_x \sum_{\xi, \zeta \in B(s')} |f(x\xi^{-1} y_i \zeta) - f(x)|^p$$

$$\left. + V(s')^{-1} \sum_x \sum_{|\xi| \leq s'} |f(x\xi^{-1} y_k) - f(x)|^p \right).$$

Obviously, we have

$$\sum_x \sum_{|\xi| \leq s'} |f(xy_1 \xi) - f(x)|^p \leq V(s) \sum_{x,z} |f(xz) - f(x)|^p \mathbf{u}_s(z)$$

and

$$\sum_x \sum_{|\xi| \leq s'} |f(x\xi^{-1} y_k) - f(x)|^p \leq V(s) \sum_{x,z} |f(xz) - f(x)|^p \mathbf{u}_s(z).$$

Similarly, we have

$$\sum_x \sum_{\xi,\zeta \in B(s')} |f(x\xi^{-1}y_i\zeta) - f(x)|^p \le V(s) \sum_{\xi \in B(s')} \sum_{x,z} |f(xz) - f(x)|^p \mathbf{u}_s(z)$$

$$\le V(s)V(s') \sum_{x,z} |f(xz) - f(x)|^p \mathbf{u}_s(z).$$

Since $k \le 9(r/s)$, putting these inequalities together yields

$$\sum_x |f(xy) - f(x)|^p \le 9[V(s)/V(s')](r/s)^p \sum_x |f(xz) - f(x)|^p \mathbf{u}_s(z).$$

Averaging over $y \in B(r)$ gives the desired inequality. □

COROLLARY A.5. *On group with polynomial volume growth, for $p \ge 1$ and $\alpha > 0$ there exists a constant $C(G, p, \alpha)$ such that*

$$\|f - f * \mathbf{u}_r\|_p^p \le C(G, p, \alpha) Q_{p,\alpha}(r) \sum_{x,z} |f(xy) - f(x)|^p \phi_\alpha(z),$$

where

$$Q_{p,\alpha}(r) = \begin{cases} r^p, & \text{if } \alpha > p, \\ r^p/\log(e+r), & \text{if } \alpha = p, \\ r^\alpha, & \text{if } \alpha < p. \end{cases}$$

PROOF. Only the case $\alpha = p$ needs a proof. This case follows immediately from the previous proposition and the definition of ϕ_α. □

REMARK A.6. Fix a continuous increasing function ℓ such that $\ell(2t) \le C\ell(t)$ and $\int^\infty \frac{ds}{s\ell(s)} < \infty$. Let ϕ be a symmetric probability measure on the group G (which we assume to have polynomial volume growth of degree D) and such that

$$\phi \asymp \sum_1^\infty \frac{1}{k\ell(k)} \mathbf{u}_k.$$

Proposition A.4 immediately gives

$$\mathcal{E}_{\mathbf{u}_r} \le C(G, p, \ell) r^p \left(\int_r^\infty \frac{s^{p-1}}{\ell(s)} \, ds \right)^{-1} \mathcal{E}_\phi.$$

This covers the different cases of Corollary A.5. When ℓ is slowly varying, this estimate is often not sharp and a sharp version is provided in [25].

THEOREM A.7. *On a group with polynomial volume growth of degree D and for any $1 \le p < \infty$, we have*

$$\Lambda_{p,\phi_\alpha}(v) \simeq \begin{cases} v^{-p/D}, & \text{if } \alpha > p, \\ v^{-p/D}\log(e+v), & \text{if } \alpha = p, \\ v^{-\alpha/D}, & \text{if } 0 < \alpha < p. \end{cases}$$

PROOF. The lower bounds on Λ_{p,ϕ_α} follow easily from the previous corollary and the argument in [9] as explained in the previous section. The upper bounds on Λ_{p,ϕ_α} follows from Theorem 2.13. For instance, in the case $\alpha = 1$, Theorem 2.13 gives

$$\Lambda_{1,\phi_1}(v) \lesssim \big(\log(e+s)\big)\Lambda_{1,G}(v) + \frac{1}{s}.$$

This is optimized by the choice $s = 1/\Lambda_{1,G}(v)$ and we know that $\Lambda_{1,G}(v) \simeq (v)^{-1/D}$. Hence, $\Lambda_{1,\phi_1}(v) \lesssim v^{-1/D} \log(e+v)$. □

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