ONE-DIMENSIONAL LONG-RANGE DIFFUSION-LIMITED AGGREGATION I

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We examine diffusion-limited aggregation generated by a random walk on $\mathbb{Z}$ with long jumps. We derive upper and lower bounds on the growth rate of the aggregate as a function of the number of moments a single step of the walk has. Under various regularity conditions on the tail of the step distribution, we prove that the diameter grows as $n^{\beta+o(1)}$, with an explicitly given β. The growth rate of the aggregate is shown to have three phase transitions, when the walk steps have finite third moment, finite variance, and conjecturally, finite half moment.

1. Introduction. Start with a single seed particle fixed in space. Bring a second particle from infinity, doing a random walk. Once it hits the first particle, freeze it at the last place it visited before hitting the first particle. Bring a third particle and freeze it when it hits the existing particles. Repeat, and watch the aggregate grow. This process, known as diffusion-limited aggregation, DLA for short, was suggested by physicists Witten and Sander [37] when the space is $\mathbb{Z}^2$. They ran simulations with several thousand particles and discovered that a random fractal ensues. The elegance of the model immediately caught the eyes of both physicists and mathematicians.

However, very little has been proved about this model rigorously. Kesten [22–24] proved nontrivial upper bounds for the growth rate, but these do not demonstrate the fractal nature of the model. Eberz-Wagner [16] has some results about local statistics of the aggregate. Various simplified models have been suggested, but the fractal nature of the aggregate is at best partially replicated. DLA on a cylinder was shown to have a fingering phenomenon, when the base of the cylinder mixes sufficiently rapidly [12] (see also [10]). In the superficially similar *internal* DLA, a process where the particles start from 0, walk on the aggregate and are glued

at their point of departure, the limiting shape is a ball [4, 5, 20, 21, 26, 27, 29, 35]. A similar phenomenon happens for the Richardson model, where the position of the glued particle is picked from the uniform measure on the boundary of the aggregate. Here, the limit shape is some (unknown) convex shape which is a far cry from being a fractal [30, 33]. DLA on trees requires one to adjust the parameters in order to get a "fingering" phenomenon [8]. See also [6, 7, 24, 32] and the fascinating deterministic analog, the Hele–Shaw flow [14, 17].

In this paper, we study one-dimensional *long-range* DLA. The random walk of the particles has unbounded long jumps. When such a long jump lands on a site already in the aggregate the jump is not performed and the particle is glued in its current position. Thus, we deviate from the view of DLA as a connected aggregate, but that is of course necessary to have an interesting aggregate in one dimension. (As a particle system, there are interesting problems even in the connected one-dimensional case; see [25].) One-dimensional long-range models have been studied for various questions, for example, for percolation, Ising, and others [1, 9, 13, 15, 19, 31, 34]. Such models frequently exhibit interesting phenomenology, reminiscent of the behaviour in $\mathbb{Z}^d$ but different from it. In particular, there is no canonical correspondence between the dimension d and the strength of the long-range interactions.

It is time to state our results (precise definitions will be given in Section 2). We say that a random variable ξ has α moments if

$$\alpha := \sup\{a \geq 0 : \mathbb{E}|\xi|^a < \infty\}.$$

A random walk $\{R_n\}$ has α moments if its step distribution does. In particular, if $\mathbb{P}(|R_1 - R_0| = k) = k^{-1-\alpha+o(1)}$, then R has α moments. Our results focus on the effect of α on the growth rate of the DLA generated by the random walk. We need our walk to be irreducible in order to have a reasonable definition of "coming from infinity", and assume this implicitly throughout the paper.

THEOREM 1.   *Let R be a symmetric random walk on $\mathbb{Z}$ with step distribution satisfying $\mathbb{P}(|R_1 - R_0| = k) = (c + o(1))k^{-1-\alpha}$. Let D_n be the diameter of the n particle aggregate. Then almost surely*:

- *If $\alpha > 3$, then $n - 1 \leq D_n \leq Cn + o(n)$, where C is a constant depending only on the random walk.*
- *If $2 < \alpha \leq 3$, then $D_n = n^{\beta+o(1)}$, where $\beta = \frac{2}{\alpha-1}$.*
- *If $1 < \alpha < 2$ then $D_n = n^{2+o(1)}$.*
- *If $\frac{1}{3} < \alpha < 1$ then*

$$n^{\beta+o(1)} \leq D_n \leq n^{\beta'+o(1)},$$

  where $\beta = \max(2, \alpha^{-1})$ and $\beta' = \frac{2}{\alpha(2-\alpha)}$.
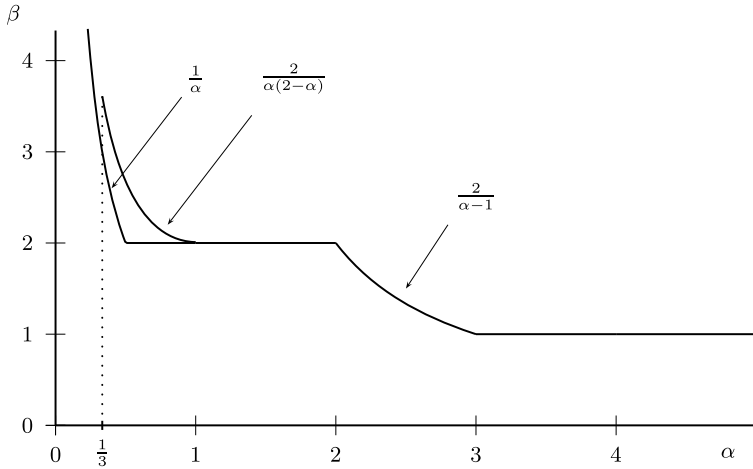- *If $0 < \alpha < \frac{1}{3}$ then $D_n = n^{\beta+o(1)}$, where $\beta = \alpha^{-1}$.*

FIG. 1.   *If the random walk R has α finite moments, then the diameter of the resulting n-particle aggregate grows as n^β. For $\frac{1}{3} < \alpha < 1$, our lower and upper bounds for β differ, and we believe the lower bound is correct.*

Figure 1 depicts the various regimes described in Theorem 1. Not all of Theorem 1 is proved in this paper—the cases $\alpha < 1$ are delegated to part II [3]. Let us remark that this formulation is significantly weaker than our results below for each regime. The theorems dealing with the various ranges of α apply to much more general random walks, and give more precise estimates on the diameter D_n. The exact requirements and resulting estimates vary, and the above formulation lies in their intersection. See the statement of Theorems 4.1, 5.1, 5.3 and 6.1 throughout the text, and the results in part II. While our results as stated do not cover the "critical" cases $\alpha = 1, 2$, the reasons are mainly simplicity of presentation. The proofs, generally speaking, can be extended to the boundary case with additional effort (and sometimes with additional regularity conditions).

The most interesting feature of Theorem 1, is of course the multiple phase transitions—as seen from Figure 1—at 3, 2 and at an (as yet) unknown place in $[\frac{1}{3}, 1]$. We feel compelled to discuss them on a heuristic level. Before we consider the transitions at 2 and 3, there is a point about the regime $\alpha > 2$ that should be made.

When $\alpha > 2$, R has a finite second moment and the large scale behaviour of R is similar to that of the simple random walk on $\mathbb{Z}$. In particular, the random walk has Brownian motion as its scaling limit. This suggests that all walks with finite step variation will give rise to similar DLA aggregates. As already stated, this is not the case. While the Green's function and the potential kernel of any such walk have linear asymptotics (see Section 5), the growth rate of the DLA diameter can differ. The basic reason is that the walker is more likely to discover new territory when making a large jump: A jump of size k takes the walker out of an interval where it has typically spent the past k^2 steps. Such a jump is therefore roughly k^2

times more likely to reach previously unvisited vertices. This causes large jumps to contribute disproportionately to the aggregate growth. A similar effect is exhibited by the ladder process corresponding to the random walk (see [36], Section 18): A large jump is more likely to bring the walk to a new maximum, and consequently the ladder steps have a thicker tail than the walk itself.

Throughout the regime $\alpha > 2$, the process is directed: particles coming from $+\infty$ have a bigger probability of hitting the right side and particles coming from $-\infty$ will have a bigger probability of hitting the left side. Further, each particle has probability bounded away from 0 of hitting the extreme particle on its side and increase the aggregate's diameter. The diameter can now be compared to a sum of i.i.d. variables (though the increments are, of course, dependent) where if the expectation is finite then the sum increases linearly whereas if the expectation is infinite then the largest contribution dominates all the rest. The phase transition at 3 reflects a transition between a regime of incremental additions and a regime of large jumps.

In the regime $2 < \alpha < 3$, a calculation shows that $\mathbb{P}(\Delta D_n > m) \approx nm^{1-\alpha}$, where here and below, the notation $\approx$ denotes that the ratio between the two sides are bounded between two positive constants that depend only on the walk. When $m = n^{2/(\alpha-1)}$ this probability is $\approx 1/n$ and there is probability bounded away from 0 that at least one such event occurs in the first n particles. As explained above, such an event dominates all the rest, and hence $D_n \approx n^{2/(\alpha-1)}$. For this reason, the proof of the lower bound is the easier (the calculation of the probability above, once justified, yields it immediately). The upper bound requires to bound the contribution of the smaller jumps, which turns out to be trickier and requires some insight into the structure of the aggregate.

The next phase transition is at $\alpha = 2$. This corresponds to the transition from Gaussian behaviour of the random walk to stable behaviour: for $\alpha < 2$ the walk scales to an α-stable process, and the Green's function grows like $n^{\alpha-1}$. In the recurrent stable regime, $1 < \alpha < 2$, the calculation is quite similar to the one for the case $2 < \alpha < 3$, but the result is that the additional contribution from the fatter tail of the walk is exactly canceled by the slower growth of the Green's function and the growth of the aggregate is always n^2.

Let us first state what we believe is the true behaviour in the regime $0 < \alpha < 1$, which is that the lower bound of Theorem 1 is sharp:

CONJECTURE. *Let R be a symmetric random walk on $\mathbb{Z}$ with step distribution satisfying $\mathbb{P}(|R_1 - R_0| = k) \approx ck^{-1-\alpha}$ for some $0 < \alpha < 1$. Then $D_n = n^{\beta+o(1)}$ with $\beta = \max(2, \frac{1}{\alpha})$.*

The reason for this conjecture will be discussed in more detail in part II, but for now let us remark that we can prove this conjecture in the regime $0 < \alpha < \frac{1}{3}$, and the same argument used there would show that same in the regime $\frac{1}{3} < \alpha < \frac{1}{2}$

under very reasonable assumptions on the amount far away parts of the aggregate influence one another. Thus, we have a sound basis to believe that at $\alpha = \frac{1}{2}$, the aggregate grows like $n^{2+o(1)}$. But this is exactly the growth rate at $\alpha = 1$. It is reasonable to believe that β is decreasing as a function of α (though, again, we have no proof of that either), and hence the exponent should be 2 throughout the interval $[\frac{1}{2}, 1]$.

This conjecture raises two questions. The first: why is there a transition at $\frac{1}{2}$? One may point at a certain transition in the behaviour of a certain bound on the capacity of a fractal, but that is not much different than saying "because that is what the calculation shows". But a more fundamental question is: why is there no transition at 1? After all, 1 is the location of the most dramatic transition in our picture, the transition between the recurrent and transient regimes (see, e.g., [36], E8.2). In the transient regime, one needs to modify the definition of the process: we cannot simply have a particle "coming from infinity". Instead one must condition on the particle ever hitting the aggregate. Put differently, even though the processes at both sides of 1 are different processes with only some kind of heuristic connection, they still seem to grow at the same rate.

We remark that even if our conjecture is false and there is a phase transition at 1, it must be very weak. Indeed, the upper bound of $n^{2/(\alpha(2-\alpha))}$ in Theorem 1 shows that $\beta(\alpha)$, assuming it exists, must be differentiable at 1, and $\beta'(1) = 0$. Thus, despite the fundamental difference between the process for $\alpha < 1$ and $\alpha > 1$, the effect on the behaviour of D_n is not so great. Let us stress again that for all we know the growth rate of the aggregate at $\frac{1}{3} < \alpha < 1$ might be undefined, or depend on the particular walk.

We should caution that there are difficulties in simulating the process to get good numerical support for the conjecture. There are heuristic reasons to believe (see Section 7) that the growth rate is not quite $n^{\max(2, 1/\alpha)}$, but that there are corrections which are at least logarithmic in size.

While Theorem 1 and the bulk of our results describe only the behaviour of the diameter of the aggregates, they give reason to believe that rescaling the process might yield an interesting process. One could ask, does the random set A_n have a scaling limit? Note that there several interpretations to this question. The scaling could be by some deterministic factor, or by normalizing the set A_n to the interval $[0, 1]$. There are also several topologies under which this question is interesting, including the Hausdorff topology on subsets of $[0, 1]$, and weak convergence of the uniform measure on (the rescaled) A_n. A natural topology to consider might be weak convergence of the rescaled harmonic measure.

Last but not least, let us discuss A_∞, the infinite aggregate defined as the union of the aggregates at all finite times. The natural expectation is that the density of A_∞ should reflect the growth rate of D_n, at least to order of magnitude, that is if $D_n = n^{\beta+o(1)}$ then

$$(1.1) \qquad\qquad \left| A_\infty \cap [-n, n] \right| = n^{1/\beta+o(1)}.$$

Indeed, in part III [2] we give a proof of (1.1) for the case $\alpha > 2$. When $\alpha > 3$ we show that the process has renewal times, at which the subsequent growth of the aggregate is independent of the structure of the aggregate. Equation (1.1) is a direct consequence. When $2 < \alpha < 3$, these renewal times no longer exist. However, we show that it is still hard for particles to penetrate deep into the aggregate, and derive (1.1) in this case as well. The case $\alpha < 2$ has other difficulties and at present we are not ready to speculate on the validity of (1.1). However, in Section 7 we give an example of a walk with $\alpha = 0$, "the $\mathbb{Z}^3$ restricted walk" for which, despite the fact that D_n grows super-exponentially, $A_\infty = \mathbb{Z}$. We do not know if such examples exist for $0 < \alpha < 2$, as the construction we use is somewhat special.

Roadmap. In Section 2, we derive a general formula for the gluing measure in the recurrent case. This section is a prerequisite for the rest of the paper. We then highly recommend reading Section 3 in which we analyze one specific case, the $\mathbb{Z}^2$ restricted walk (this walk has $\alpha = 1$). The proof in this case is much easier than the other cases, and does not require any knowledge of stable random variables. The next sections are arranged by α; Section 4 for $\alpha > 3$, Section 5 for $2 < \alpha < 3$, Section 6 for $1 < \alpha < 2$. Finally, Section 7 describes the aforementioned example with $\alpha = 0$.

A nontrivial portion of the paper (and of part II) is dedicated to discrete potential theory, both general and that of stable walks (e.g., Lemma 6.4). We expected to find many of these results in standard references, and did not.

2. Preliminaries.

2.1. *Notation.* For a subset $A \subset \mathbb{Z}$ we will denote by diam A the diameter of A, namely $\max A - \min A$. For $x \in \mathbb{Z}$ we will denote by $d(x, A)$ the point-to-set distance, namely $\min_{y \in A} |x - y|$. Throughout we let A_n be the n point aggregate, and denote $D_n = \text{diam } A_n$, $\Delta D_n = D_{n+1} - D_n$. Let $\mathcal{F}_n$ be the minimal σ-field generated by $A_0, \ldots, A_n$.

We denote a single step of the random walk by ξ, and the random walk itself by $R = (R_0, R_1, \ldots)$. We denote by $\mathbb{P}_x$ the probability measure of the random walk started at x. The transition probabilities of the random walk are denoted by $p_{x,y} = \mathbb{P}(\xi = y - x)$. For a given set A, define

$$p(x, A) = \sum_{a \in A} p_{x,a}.$$

We denote by T_A be the hitting time of A, defined as

$$T_A = \min\{n > 0 \text{ s.t. } R_n \in A\}.$$

Note that $T_A > 0$ even if the random walks starts in A. For a set $\{x\}$ with a single member we also write T_x for $T_{\{x\}}$. Denote by $g(x, y)$ the Green's function of R defined by

$$g(x, y) = \sum_{n=0}^{\infty} \mathbb{P}_x(R_n = y).$$

If $A \subset \mathbb{Z}$, then we define the relative Green's function (a.k.a. the Green's function for the walk killed on A) by

$$g_A(x, y) = \sum_{n=0}^{\infty} \mathbb{P}_x(R_n = y, T_A > n)$$

and the hitting measure by

$$H_A(x, a) = \begin{cases} \mathbb{P}_x(R_{T_A} = a), & x \notin A, \\ \delta_{x,a}, & x \in A, \end{cases} \qquad H_A(\pm\infty, a) = \lim_{x \to \pm\infty} H_A(x, a)$$

by [36], T30.1, the limit on the right-hand side exists for any irreducible random walk and any finite A (recall that a random walk is irreducible if for any $x, y \in \mathbb{Z}$ there exists some n such that $\mathbb{P}_x(R_n = y) \neq 0$—note that Spitzer uses "aperiodic" for what we call "irreducible"—see [36], D2.2).

By C and c, we denote constants which depend only on the law of ξ but not on any other parameter involved. The same holds for the constants hidden in the $o(\cdot)$ notation, except when it is used in estimates for D_n (as in Theorem 1 above and other results below) where the factor $o(\cdot)$ is random (this should always be clear from the context). Generally, C and c might take different values at different places, even within the same formula. C will usually pertain to constants which are "big enough" and c to constants which are "small enough".

$X \lesssim Y$ denotes that $X < CY$. By $X \approx Y$ we mean $cX < Y < CX$ (that is, $X \lesssim Y \lesssim X$). By $X \asymp Y$ we mean that X/Y is a slowly varying function—see Section 6 for details. $\lfloor x \rfloor$ denotes the integer value of x.

2.2. *Gluing measures.* Let R be a recurrent irreducible random walk on $\mathbb{Z}$. We will assume implicitly throughout the paper that all our random walks are irreducible. Let $A \subset \mathbb{Z}$ be some finite set, and T_A the (a.s. finite) hitting time of A by R. We would like to define the measure $\mathbb{P}_{\infty} = \lim_{y \to \infty} \mathbb{P}_y$. However, care must be taken here, since the limit is not a probability measure using the natural σ-algebra of the random walks, and the laws of natural quantities such as R_n or T_A do not have a limit. However, the law of R_{T_A}—the point at which A is hit—does have a limit, as do the probabilities of events like $\{T_a < T_b\}$.

We define the measure $\mathbb{P}_{+\infty}$, depending implicitly on A, as follows. This measure is supported on paths $\{\gamma_i\}_{i \leq 0}$, that is, paths with no beginning but a last step. It is defined as the limit as $y \to \infty$ of the law of $\{R_{T_A+i}\}_{i \leq 0}$. Informally, $\mathbb{P}_{+\infty}$ is interpreted as the random walk started at $+\infty$, and stopped when it hits A. Clearly, it is supported on paths in $\mathbb{Z} \setminus A$, except for $R_0 \in A$. The measure $\mathbb{P}_{-\infty}$ is defined similarly using $y \to -\infty$. We define the measure $\mathbb{P}_{\infty} = \frac{1}{2}(\mathbb{P}_{+\infty} + \mathbb{P}_{-\infty})$. Finally, let

$$\mu(x, a) = \mu(x, a; A) = \mathbb{P}_{\infty}(R_{-1} = x, R_0 = a)$$

be the probability that the random walk hits A by making a step from x to a.

LEMMA 2.1.   *For any recurrent random walk and any finite set A, the limits $\mathbb{P}_{\pm\infty}$ exist and are probability measures. Further, for any $x_0 \in A$ and $x_{-1}, \ldots, x_{-n} \notin A$*

$$(2.1) \quad \mathbb{P}_{\pm\infty}(R_i = x_i \text{ for } -n \le i \le 0) = \frac{\mathbb{P}_{\pm\infty}(T_{x_{-n}} < T_A)}{\mathbb{P}_{x_{-n}}(T_A < T_{x_{-n}})} \prod_{i=-n}^{-1} p_{x_i, x_{i+1}}$$

and in particular

$$(2.2) \quad \mu(x, a) = \mu(x, a; A) = \frac{p_{x,a} \mathbb{P}_{\infty}(T_x < T_A)}{\mathbb{P}_x(T_A < T_x)}.$$

PROOF.   Fix a starting point y and denote

$$\mathbb{P}_y(x_0, x_{-1}, \ldots, x_{-n}) = \mathbb{P}_y(R_{T_A - i} = x_i \text{ for } -n \le i \le 0)$$

(where $R_{T_A - k}$ is undefined if the walk hits A in less than k steps). For clarity, write $z = x_{-n}$. Now, in order for the event on the right-hand side to happen, the walk must first hit z, which happens with probability $\mathbb{P}_y(T_z < T_A)$. By the strong Markov property at T_z, with probability $\mathbb{P}_z(T_A < T_z)$ the walk will hit A before its next return to z. Thus, the expected number of visits to z before T_A is

$$\frac{\mathbb{P}_y(T_z < T_A)}{\mathbb{P}_z(T_A < T_z)}.$$

At each of these visits, there is probability $\prod_{i=-n}^{-1} p_{x_i, x_{i+1}}$ of making the prescribed sequence of jumps ending at $x_0 \in A$. Since the walk is stopped once such a sequence of jumps is made, the events of making these jumps after the i'th visit to z are disjoint (for different i's). Hence,

$$\mathbb{P}_y(x_0, x_{-1}, \ldots, x_{-n}) = \frac{\mathbb{P}_y(T_z < T_A)}{\mathbb{P}_z(T_A < T_z)} \prod_{i=-n}^{-1} p_{x_i, x_{i+1}}.$$

Thus, to see that $\lim_{y \to \pm\infty} \mathbb{P}_y(x_0, x_{-1}, \ldots, x_{-n})$ exists, it suffices to show that $\lim \mathbb{P}_y(T_z < T_A)$ exists. Recall that the harmonic measure from infinity on a finite set A is defined by

$$H_A(\pm\infty, a) = \lim_{y \to \pm\infty} H_A(y, a) = \lim_{y \to \pm\infty} \mathbb{P}_y(R_{T_A} = a).$$

By [36], T30.1, this limit always exists. Note that

$$\mathbb{P}_y(T_z < T_A) = \mathbb{P}_y(R_{T_{A \cup \{z\}}} = z) = H_{A \cup \{z\}}(y, z).$$

Existence of $\lim_{y \to \pm\infty} \mathbb{P}_y(x_0, x_{-1}, \ldots, x_{-n})$ follows.

It remains to show that the limit is a probability measure, that is,

$$\sum_{\substack{x_0 \in A \\ x_{-1}, \ldots, x_{-n} \notin A}} \mathbb{P}_{\pm\infty}(x_0, x_{-1}, \ldots, x_{-n}) = 1.$$

For any finite starting point y, this sum is 1 by recurrence. The problem is that as $y \to \pm\infty$, the walk might be have a high probability of hitting A by a large jump, so that for some i, the law of x_i is not tight as $y \to \infty$. However, if we show that the law of x_{-n} is tight, then $\lim_{y \to \pm\infty} \mathbb{P}_y(x_0, x_{-1}, \ldots, x_{-n})$ will be a probability measure.

CLAIM 2.2. *For any finite $A \subset Z$,*

$$\lim_{m \to \infty} H_{[-m,m]}(\pm\infty, A) = 0.$$

PROOF. For clarity, we use $H(x, y; A)$ in place of with $H_A(x, y)$. It suffices to prove the claim for a singleton $A = \{a\}$. We may assume $a \geq 0$. In this case, we write

$$1 = \sum_{|x| \leq m} H(\pm\infty, x; [-m, m])$$

$$\text{by monotonicity} \quad \geq \sum_{|x| \leq m} H(\pm\infty, x; [x - 2a - 2m, x + 2m])$$

$$\text{by translation invariance} \quad = \sum_{|x| \leq m} H(\pm\infty, a; [-a - 2m, a + 2m]).$$

Hence, $H(\pm\infty, a; [-(a + 2m), a + 2m]) \leq 1/(2m + 1)$. $\square$

Returning to the proof of Lemma 2.1, fix $\varepsilon > 0$. For any finite set A and any n, we can pick a sequence of finite intervals $A \subset I_0 \subset I_1 \subset \cdots \subset I_n$ so that for any $k < n$ and any $y \notin I_k$, the probability from y of hitting I_k at a point of I_{k-1} is at most ε/n. We get

$$\mathbb{P}_y(T_{I_n} < T_A - n) < \varepsilon \qquad \forall y \notin I_n$$

and, therefore, by the strong Markov property at the stopping time T_{I_n},

$$\mathbb{P}_y(|R_{T_A-n}| > M) < \mathbb{E}_y \mathbb{P}_{T_{I_n}}(|R_{T_A-n}| > M) + \varepsilon \qquad \forall M \ \forall y \notin I_n.$$

Now, the law of R_{T_A-n} with respect to any starting point in I_n is tight (since these are just $|I_n|$ distributions). Hence, we get

$$\lim_{M \to \infty} \max_{y \in \mathbb{Z}} \mathbb{P}_y(|R_{T_A-n}| > M) = 0,$$

which is the required tightness. $\square$

DEFINITION 2.3. Let R be a random walk on $\mathbb{Z}$. The *DLA process with respect to R* is a sequence of random sets $A_0 = \{0\} \subset A_1 \subset \cdots$ such that for any $A \subset \mathbb{Z}$, and $x \in \mathbb{Z} \setminus A$ and any $n > 0$,

$$(2.3) \qquad \mathbb{P}(A_{n+1} = A \cup \{x\} | A_n = A) = \sum_{a \in A} \mu(x, a; A),$$

where μ is defined by (2.2).

When ξ has infinite variance, $\mathbb{P}_{+\infty}(T_x < T_A) = \mathbb{P}_{-\infty}(T_x < T_A)$ for any x and A and indeed $\mathbb{P}_{+\infty} = \mathbb{P}_{-\infty}$ (see [36], T30.1 (1)), but otherwise $\mathbb{P}_{+\infty}$ and $\mathbb{P}_{-\infty}$ differ. It is possible to define the DLA using walks that start only at $+\infty$ or $-\infty$. This leads to minor variations on our results, and the proofs remain valid with minimal modification.

Since by the right-hand side of (2.3) the probability of adding a point x to A_n can be interpreted as a measure over infinite paths ending at A_n, we will say that x is "glued" to A_n at a if the last two steps of the path of the added particle are x and a. The measure μ is thus called the "gluing" measure.

3. The restricted $\mathbb{Z}^2$ walk.
In this section, we discuss a special random walk on $\mathbb{Z}$ resulting from an embedding of $\mathbb{Z}$ as a sub-group of $\mathbb{Z}^2$, say as the diagonal $\{(x, x)\}_{x \in \mathbb{Z}}$. Consider the sequence of vertices of $\mathbb{Z}$ visited by a simple random walk on $\mathbb{Z}^2$, that is, the restriction of the random walk to $\mathbb{Z}$. This sequence of vertices forms a random walk on $\mathbb{Z}$. It is well known that this walk has $\alpha = 1$, and more precisely that the steps of this random walk have approximately the Cauchy distribution, that is, $\mathbb{P}(\xi = k) = (c + o(1))|k|^{-2}$ (for the special case of the diagonal embedding, there is even a precise formula (see [36], E8.3), $\mathbb{P}(\xi = 0) = 1 - \frac{2}{\pi}$, $\mathbb{P}(\xi = k) = \frac{2}{\pi(4k^2-1)}$ but we do not use this extra precision). The fact that $\mathbb{Z}^2$ is recurrent immediately implies that the restricted $\mathbb{Z}^2$ walk is recurrent as well, hence we may consider the DLA formed by this walk.

While the restricted $\mathbb{Z}^2$ walk is a very special example, its study has merit. The proofs are simpler, but the general ideas are the basis for the proofs in more general cases. The reason the proofs are simpler is the vast and very precise knowledge concerning the behaviour of the simple random walk in $\mathbb{Z}^2$. This allows us to get sharp bounds for various quantities. Addition of a vertex to the DLA in $\mathbb{Z}$ may be studied by examining a simple random walk on $\mathbb{Z}^2$ and considering the last visit to $\mathbb{Z}$ before hitting A.

THEOREM 3.1.   *Consider the DLA generated by the $\mathbb{Z}^2$ restricted walk. For some $c > 0$, we have almost surely*

$$\liminf \frac{D_n \log \log n}{n^2} > c \quad and \quad \limsup \frac{D_n}{n^2} = \infty.$$

Theorem 3.4 below gives a matching upper bound for D_n, up to logarithmic factors. Together we find that the diameter grows essentially quadratically. It is reasonable to believe that $\{n^{-2} D_{nt}\}_t$ converges to some random process, though it is not even proved that the law of $n^{-2} D_n$ converges. To prove Theorem 3.1, we argue as follows: if D_n is small then there is some probability that D_{n+1} is large. We estimate this probability for a suitable threshold for being "large". We then bound this probability uniformly in A_n. By Borel–Cantelli, it follows that D_n is large for infinitely many n. To make this precise, suppose $D_n > m$. Then
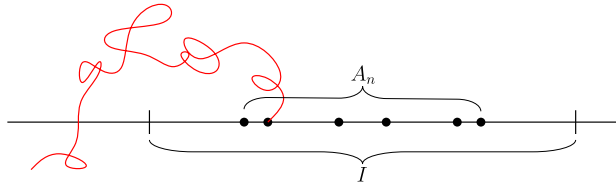
FIG. 2.   *The event that the random walk hits I at a point of A_n.*

$D_{n+1} > m$ as well. On the other hand, for any set A_n with $D_n \leq m$ we have the following. (Recall that $\Delta D_n = D_{n+1} - D_n$.)

LEMMA 3.2.   *In the DLA generated by the $\mathbb{Z}^2$ restricted walk, there is a constant $c > 0$ so that for any A and m with $\mathrm{diam}(A) < m$ we have*

$$\mathbb{P}(\Delta D_n > m | A_n = A) \geq \frac{cn}{m}.$$

PROOF.   Define the interval $I \subset \mathbb{Z}$ to be the m-neighbourhood of A_n. [This is an interval since $\mathrm{diam}(A) < m$.] Consider a random walk in $\mathbb{Z}^2$ used for a DLA step, and consider the first time it hits I. If it hits I at one of the points of A_n, then the previous visit to $\mathbb{Z}$ must have been at distance more than m from A_n. See Figure 2. In that case, a far point is added to A_n and $\Delta D_n > m$. Hence,

$$\mathbb{P}(\Delta D_n > m) \geq H_I(\infty, A_n),$$

where H_I is the harmonic measure from infinity on I for a random walk in $\mathbb{Z}^2$. We now use the well-known fact[5] that the harmonic measure satisfies the bound $H_I(x) \geq c/|I|$ for some universal c and any $x \in I$ (near the ends of the interval the harmonic measure is much larger).

It follows that

$$H_I(A_n) \geq \frac{c}{|I|} |A_n| = \frac{cn}{D_n + 2m + 1} \geq \frac{cn}{3m}$$

as required. $\square$

Recall that $\mathcal{F}_n$ is the σ-algebra spanned by $A_1, \ldots, A_n$. In preparation for the treatment of more general walks, we prove the following lemma. Theorem 3.1 follows by applying the following to $M_n = D_n$ with $\beta = 2$.

---

[5]This follows from translation invariance and the Skorokhod invariance principle: By translation invariance, it suffices to show this for $I = [-m, m]$. Let $J = [-3m, 3m]$. By the invariance principle, $H_J(I)$ is bounded below by some constant c independent of m. This implies that $H_J(x) > c/m$ for some point $x \in I$. Translation invariance and monotonicity of the harmonic measure now imply (as in the proof of Claim 2.2) that for any point $y \in I$, $H_I(y) = H_{I+(x-y)}(x) \geq H_J(x) \geq c/m$ as required.

LEMMA 3.3.   *Let $\{M_n\}$ be a nondecreasing sequence adapted to a filtration $\{\mathcal{F}_n\}$, and suppose $\mathbb{P}(M_{n+1} > m | \mathcal{F}_n) \geq \min\{c_1 n m^{-2/\beta}, 1\}$ for some $c_1 > 0$ and all $m, n > 0$. Then there is some deterministic value $K > 0$ such that a.s.*

$$\limsup n^{-\beta} M_n = \infty \quad and \quad \liminf n^{-\beta} (\log\log n)^{\beta/2} M_n > K.$$

PROOF.   Take $m = an^\beta$. By the conditions of the lemma,

$$\mathbb{P}(M_{n+1} \geq an^\beta | \mathcal{F}_n) \geq c_1 n (an^\beta)^{-2/\beta} \geq cn^{-1},$$

uniformly in $\mathcal{F}_n$. Consequently, $M_n \geq an^\beta$ infinitely often.

To estimate $\liminf \frac{M_n (\log\log n)^{\beta/2}}{n^\beta}$, take $m = \frac{c_1^{\beta/2} n^\beta}{(4\log\log n)^{\beta/2}}$. It follows that

$$\mathbb{P}\left(M_{n+1} \geq \frac{c_1^{\beta/2} n^\beta}{(4\log\log n)^{\beta/2}} \Big| \mathcal{F}_n \right) \geq \frac{4\log\log n}{n}.$$

Consequently, the probability that $M_{n+1} \leq \frac{c_1^{\beta/2} n^\beta}{(4\log\log n)^{\beta/2}}$ for all $n \in [N, 2N)$ is at most

$$\prod_{n=N}^{2N-1} \left(1 - \frac{4\log\log n}{n}\right) \leq \left(1 - \frac{4\log\log N}{2N}\right)^N \leq e^{-2\log\log N} = \frac{1}{\log^2 N}.$$

Considering only N of the form 2^k, we find that a.s. for all large k there is a some $n_k \in [2^k, 2^{k+1})$ such that $M_{n_k} \geq \frac{c_1^{\beta/2} n_k^2}{(4\log\log n_k)^{\beta/2}}$. For any other n, we argue, using the monotonicity of the sequence $\{M_n\}$, that if $n \in [2^{k+1}, 2^{k+2}]$ then

$$M_n \geq M_{n_k} \geq \frac{c_1^{\beta/2} n_k^\beta}{(4\log\log n_k)^{\beta/2}} \geq \frac{c_1^{\beta/2} (n/4)^\beta}{(4\log\log n)^{\beta/2}}.$$

Thus, $\liminf \frac{M_n (\log\log n)^{\beta/2}}{n^\beta} > \frac{c_1^{\beta/2}}{2^{3\beta}}$ a.s.   □

Theorem 3.1 now follows from Lemmas 3.2 and 3.3. As promised, we have a matching upper bound, up to logarithmic factors.

THEOREM 3.4.   *For the DLA generated by the $\mathbb{Z}^2$ restricted walk, a.s. for any $\varepsilon > 0$ and all large enough n, $D_n \leq n^2 (\log n)^{3+\varepsilon}$.*

We begin by bounding the probability of a large increment in D_n.

LEMMA 3.5.   *In the DLA generated by the $\mathbb{Z}^2$ restricted walk,*

$$\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) \lesssim \frac{n\log m}{m}.$$

PROOF. We use the asymptotics

$$\mathbb{P}_x(T_A < T_x) \approx \frac{1}{\log d(x, A)}$$

(assuming $d(x, A) > 1$). This follows from asymptotics of the two-dimensional random walk: The probability of reaching (in $\mathbb{Z}^2$) distance $\frac{1}{2}d(x, A)$ before returning to x is of order $\log^{-1} d(x, A)$ (this is a well-known fact; see, e.g., [11], Lemma 9). On this event, the probability of hitting A before returning to x is bounded away from 0 (even if A contains a single point).

Since $\mathbb{P}_\infty(T_x < T_A) \leq 1$, the gluing formula (2.2) implies

$$\mu(x, a) \lesssim p_{x,a} \log d(x, A).$$

Summing over all x with $d(x, A) > m$, we get

$$\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) \lesssim \sum_{\substack{a \in A \\ x: d(x, A) > m}} p_{x,a} \log d(x, A)$$

$$\leq \sum_{\substack{a \in A \\ |x-a| > m}} p_{x,a} \log d(x, a)$$

$$\approx \frac{n \log m}{m}.$$

The last estimate comes from the fact that the restricted $\mathbb{Z}^2$ walk satisfies $\mathbb{P}(\xi = k) \approx |k|^{-2}$. □

This allows us to bound the probability of D_n being large: We will also need the following lemma, which translates upper bounds on the probability of making large jumps into upper bounds on D_n

LEMMA 3.6. *Assume that*

$$\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) \lesssim \frac{n \log m}{m} \qquad \forall n, m.$$

Then we get, for all γ and N,

$$\mathbb{P}(D_N \geq \gamma N^2 \log^2 N) \lesssim \frac{1}{\gamma}.$$

If one has the weaker $\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) \leq nm^{-1+o(1)}$ then one gets $\mathbb{P}(D_N \geq \gamma \phi(N)) \lesssim 1/\gamma$ for some deterministic $\phi(N) = N^{2+o(1)}$.

PROOF. Fix N and some $M > N^2$, and set for $0 \leq k \leq \log_2 M$

$$B_k = \left\{ n \leq N : \Delta D_n \in \left(\frac{M}{2^{k+1}}, \frac{M}{2^k} \right] \right\}$$

and

$$B_{-\infty} = \{n \le N : \Delta D_n > M\}.$$

We argue that with high probability the contribution to D_N from increments in each of the B_k's is at most M. The event $\{D_N \ge M(2 + \log_2 M)\}$ is a subset of the event

$$\exists k \in \{-\infty, 0, \ldots, \log_2 M\} \qquad \text{such that} \sum_{n \in B_k} \Delta D_n > M.$$

By a union bound,

$$(3.1) \qquad \mathbb{P}(D_N > M(2 + \log M)) \le \mathbb{P}(B_{-\infty} \ne \varnothing) + \sum_k \mathbb{P}(|B_k| > 2^k).$$

By the conditions of the lemma,

$$\mathbb{P}(B_{-\infty} \ne \varnothing) \le N \cdot C \frac{N \log M}{M}.$$

Using also the bound $\binom{N}{a} \le (\frac{eN}{a})^a$,

$$\mathbb{P}(|B_k| > 2^k) \le \binom{N}{2^k} \left(CN 2^k \frac{\log M}{M} \right)^{2^k}$$

$$\le \left(CN^2 \frac{\log M}{M} \right)^{2^k}.$$

Setting $M = \gamma N^2 \log N$ and using the above bounds in (3.1) we get

$$\mathbb{P}(D_N > M(\log_2 M + 2)) \le \frac{C}{\gamma} + \sum_{k \ge 0} \left(\frac{C_1}{\gamma} \right)^{2^k}.$$

Clearly, we may assume γ is sufficiently large (by enlarging the constant implicit in the $\lesssim$ in the statement of the lemma, if necessary), and we assume $\gamma > 2C_1$. Now the sum is comparable to the first term. Since $M(\log_2 M + 2) \le C\gamma N^2 \log^2 N$, we are done.

The proof of the second part of the lemma is similar, and we omit it. $\quad\square$

PROOF OF THEOREM 3.4. By Lemmas 3.5 and 3.6, in the DLA generated by the $\mathbb{Z}^2$ restricted walk, for all γ,

$$\mathbb{P}(D_N \ge \gamma N^2 \log^2 N) \lesssim \frac{1}{\gamma}.$$

Now take $N = 2^k$ and $\gamma = k^{1+\varepsilon}$ for some $\varepsilon > 0$. By Borel–Cantelli only a finite number of these events happen, and we get that $D_N < N^2 \log^{3+\varepsilon} N$ for $N = 2^k$ sufficiently large. The bound for other N's follows by monotonicity of D_N. Since ε was arbitrary, the theorem follows. $\quad\square$

4. Walks with finite third moment.   As discussed in the Introduction, despite the fact that all symmetric walks with finite variance scale to Brownian motion, the growth rates of the aggregates resulting from such walks vary. In this section and the next, we analyze this phenomenon. We first consider the simpler case, where ξ has a finite third moment. In that case, the behaviour is similar to the behaviour when the walk has bounded steps, that is, the diameter grows linearly. The case of walks with finite variance and infinite third moment is more complex and is dealt with in Section 5.

THEOREM 4.1.   *If* $\mathbb{E}|\xi|^3 < \infty$ *and* $\mathbb{E}\xi = 0$, *then there is some* C *so that* $\limsup \frac{D_n}{n} < C$ *a.s.*

Thus, in the case $\alpha > 3$ (and sometimes when $\alpha = 3$), the diameter grows linearly. Of course, the diameter D_n must be at least $n - 1$, so the theorem gives the correct rate of growth. This suggests that the process behaves much as in the case where the jumps are bounded: only a few particles in the extremes of A_n affect subsequent growth, and the limit aggregate will have some positive density. In part III, we shall discuss existence of the limit.

We start with a technical lemma regarding hitting probabilities of our walks, which will be useful also in Section 5.

LEMMA 4.2.   *Let* R *be a random walk on* $\mathbb{Z}$ *with steps of mean* 0 *and variance* $\sigma^2 < \infty$. *Then there are* $c, C > 0$ *such that for any* $A \subset \mathbb{Z}$, $A \neq \varnothing$:

  (i) *If* $x > \max A$, *then* $\lim_{y \to \infty} \mathbb{P}_y(T_x < T_A) > c$.
  (ii) *If* $d(x, A)$ *is large enough then* $c < d(x, A)\mathbb{P}_x(T_A < T_x) < C$.

Note that the limit in clause (i) exists since the random walk is recurrent and the harmonic measure on the set $A \cup \{x\}$ exists. If A lies to one side of x, then in clause (ii) c and C can be arbitrarily close (this follows from the proof below). The lemma is related to the asymptotic linearity of the harmonic potential for the random walk (see [36], T28.1), P29.2, but we chose a somewhat different path for the proof.

PROOF OF LEMMA 4.2.   Let T_- denote the hitting time of $\mathbb{Z}^-$. Define the half-line Green's function

$$g(y, x) = \sum_{n \geq 0} \mathbb{P}_y\big(R(n) = x, n < T_-\big),$$

that is, the mean time spent at x before T_-. By [36], P19.3, P18.8, T18.1, g has a representation using two auxiliary functions u and v,

$$g(y, x) = \sum_{n=0}^{\min(x,y)} u(x - n)v(y - n).$$

The functions u, v are defined in terms of their generating functions, which are the anti-analytic and analytic parts of the Fourier transform of ξ—see [36] for the details. For our purposes, all we need to know about them is that the limits $\lim_{n\to\infty} u(n)$ and $\lim_{n\to\infty} v(n)$ both exist, and their product is $\lim_{n\to\infty} u(n)v(n) = 2/\sigma^2$. It follows that for any $\varepsilon > 0$, there is an $x_0(\varepsilon)$ such that for any $y \geq x \geq x_0$ we have

$$(4.1) \qquad \left| g(y, x) - \frac{2}{\sigma^2} x \right| \leq \varepsilon x.$$

By the strong Markov property for the hitting time of $\mathbb{Z}^- \cup \{x\}$, we have

$$g(y, x) = \delta_{y,x} + \mathbb{P}_y(T_x < T_-) g(x, x),$$

where δ is the Kronecker delta function. In particular, we find for $y \geq x \geq x_0$ that

$$\mathbb{P}_y(T_x < T_-) = \frac{g(y, x) - \delta_{y,x}}{g(x, x)}.$$

Applying (4.1), we find that

$$(4.2) \qquad \inf_{y \geq x} \mathbb{P}_y(T_x < T_-) \xrightarrow[x\to\infty]{} 1,$$

thus if x is far from $\mathbb{Z}^-$ and the walk starts to the right of x, it is likely to hit x before $\mathbb{Z}^-$. If the walk starts at x, we have the asymptotics

$$(4.3) \qquad \mathbb{P}_x(T_- < T_x) = \frac{1}{g(x, x)} = \frac{\sigma^2}{2x}(1 + o(1))$$

as $x \to \infty$.

To prove (i), note that by translation we may assume $\max A = 0$. By monotonicity in A, it suffices to prove the bound for $A = \mathbb{Z}^-$. In that case, the limit is just $\lim_{y\to\infty} \mathbb{P}_y(T_x < T_-)$. This limit is positive for any x (since the random walk is irreducible), and by (4.2) it is close to 1 for large x. In particular, it is always greater than some c.

To prove the upper bound in (ii), note that by monotonicity in A, given $d = d(x, A)$ it suffices to prove the bound for $A = \mathbb{Z} \setminus (x - d, x + d)$. Using (4.3), by translation we find that

$$\mathbb{P}_x(T_{(-\infty, x-d]} < T_x) = \frac{\sigma^2}{2d}(1 + o(1))$$

as $d \to \infty$. By symmetry, $\mathbb{P}_x(T_{[x+d,\infty)} < T_x)$ has the same asymptotics. A union bound gives for any A

$$\mathbb{P}_x(T_A < T_x) \leq \frac{\sigma^2 + o(1)}{d(x, A)}.$$

It remains to prove the lower bound of (ii). By monotonicity in the set A, it suffices to prove the bound for A that consist of a single point y, so that $d =$

$d(x, A) = |x - y|$. We consider here only the case $y = x - d$ as the case of $y = x + d$ is symmetric. Define the interval $B = (-\infty, y]$. In order to hit y before returning to x, the random walk must hit B (possibly at y) before returning to x. We have

$$\mathbb{P}_x(T_y < T_x) = \mathbb{P}_x(T_B < T_x) \cdot \mathbb{P}(T_y < T_x | T_B < T_x).$$

The second term on the right-hand side is an average over starting points in B of the probability that y is hit before x. When d is large, these probabilities are close to 1, uniformly over B, since (4.2) estimates the probability of hitting y before hitting $[x, \infty)$. Thus, the weighted average is also close to 1. As $d \to \infty$, we find

$$\mathbb{P}_x(T_y < T_x) = \mathbb{P}_x(T_B < T_x)(1 + o(1)) \sim \frac{\sigma^2 + o(1)}{2d}. \qquad \square$$

PROOF OF THEOREM 4.1.   Since $\mathbb{P}_{\pm\infty}(T_x < T_A) \leq 1$ by Lemma 4.2 and the gluing formula (2.2), we have

$$\mu(x) = \sum_a \mu(x, a) \leq \frac{p(x, A)}{c/d(x, A)} \leq Cd(x, A)\mathbb{P}(\xi \geq d(x, A)).$$

Summing over all x with $d(x, A) > t$, we get

$$\mathbb{P}(\Delta D_n > t) \leq C \sum_{k > t} k\mathbb{P}(\xi \geq k).$$

Thus, we have the stochastic domination $\Delta D_n \preceq Y$ with

$$\mathbb{P}(Y > t) = 1 \wedge C \sum_{k > t} k\mathbb{P}(\xi \geq k).$$

However,

$$\mathbb{E}Y = \sum_t \mathbb{P}(Y > t) \leq C \sum_{\substack{t,k \\ t < k}} k\mathbb{P}(\xi \geq k)$$

$$\leq C \sum_k k^2\mathbb{P}(\xi \geq k) \leq C\mathbb{E}|\xi|^3 < \infty.$$

We find that D_n is dominated by a sum of n independent copies of Y. By the law of large numbers, $\limsup \frac{D_n}{n} \leq \mathbb{E}Y < \infty$. $\square$

5. Walks with finite variance.
In this section, we will analyze random walks with $2 < \alpha < 3$ and show that the aggregate grows like $n^{2/(\alpha-1)+o(1)}$. We prove the lower bound in Section 5.1 and the upper bound in Section 5.2. The upper bound is the harder of the two.

We remark that the upper bound on the growth of the aggregate requires only an upper bound on $\mathbb{P}(|\xi| > t)$, while the lower bound on the aggregate requires a lower bound on $\mathbb{P}(|\xi| > t)$ and the assumption of a finite second moment. Hence,

we get information also on the case that $\mathbb{P}(|\xi| > k)$ decays irregularly, and satisfies only $ck^{\alpha_1} < \mathbb{P}(|\xi| > k) < Ck^{\alpha_2}$. Theorems 5.1 and 5.3 still apply, and give upper and lower bounds on the diameter (in terms of α_1 and α_2, respectively). In such cases, the bounds leave a polynomial gap, and it is reasonable to believe that $\frac{\log D_n}{\log n}$ also fluctuates between the corresponding β's.

5.1. *Lower bound.*

THEOREM 5.1. *Assume* $\mathbb{E}(|\xi|^2) < \infty$ *and* $\mathbb{E}\xi = 0$. *Fix* $\alpha \in (2, 3]$, *and let* $\beta = \beta(\alpha) = \frac{2}{\alpha-1}$.

(i) *If* $\mathbb{P}(|\xi| > t) \geq ct^{-\alpha}$, *then a.s.* $\limsup n^{-\beta} D_n = \infty$, *and* $D_n \geq \frac{n^\beta}{\log \log n}$ *for all large enough* n.

(ii) *If one has only* $\mathbb{P}(|\xi| > t) \geq t^{-\alpha+o(1)}$ *for* $2 < \alpha < 3$ *then a.s.* $D_n \geq n^{\beta+o(1)}$.

Let $D_n^+ = \max A_n$ and $D_n^- = -\min A_n$, so that $D_n = D_n^+ + D_n^-$. With subsequent papers in mind, we work with $D_n^\pm$ instead of D_n. One small argument that is needed to work with D_n^+ is the following lemma. Let $A_n^+ = A_n \cap \mathbb{Z}_+$, be the positive elements of A_n.

LEMMA 5.2. *There exists* $c_0 > 0$ *such that a.s.* $\liminf |A_n^+|/n > c_0$.

PROOF. Fix $k > 0$ such that $P(\xi = -k) > 0$. Fix n, and consider the probability of gluing $x_n = D_n^+ + k$ to A_n. By Lemma 4.2, $\mathbb{P}_\infty(T_{x_n} < T_{A_n}) > c$ and, therefore, $\mu(x_n; A_n) \geq c\mathbb{P}(\xi = -k)$ so is uniformly bounded away from zero. Since each time this happens a positive point is added to A_n^+, we find that $|A_n^+|$ dominates a sum of n i.i.d. Bernoulli variables. $\square$

PROOF OF THEOREM 5.1. Consider some x to the right of $A = A_n$. Since $\mathbb{E}|\xi|^2 < \infty$ Lemma 4.2 applies, giving the bounds $\mathbb{P}_{\pm\infty}(T_x < T_A) > c > 0$ and $\mathbb{P}_x(T_A < T_x) \approx 1/d(x, A)$. Therefore,

$$(5.1) \qquad \mu(x, a) \gtrsim d(x, A) p_{x,a}.$$

Summing over positive $x > D_n^+ + m$ and all $a \in A_n^+$ we get

$$\mathbb{P}\big(\Delta(\max A_n) > m | \mathcal{F}_n\big) \gtrsim \sum_{\substack{x>a\geq 0 \\ x>D_n^++m}} d(x, A_n^+) p_{x,a}$$

$$\geq m \sum_{\substack{x>a\geq 0 \\ x-a>m+D_n^+}} p_{x,a}$$

$$= m |A_n^+| \cdot \mathbb{P}(\xi > m + D_n^+).$$

It follows that on the event $D_n^+ < m$ we have

$$\mathbb{P}(\Delta D_n^+ > m | \mathcal{F}_n) \gtrsim m |A_n^+| \cdot \mathbb{P}(\xi > 2m).$$

Consider the events

$$E_n(m) = \{D_{n+1}^+ \geq m\} \cup \{|A_n^+| < c_0 n\},$$

where c_0 is from Lemma 5.2. If A_n^+ is small, or if $\max A_n > m$ then $E_n(m)$ occurs. Hence, we have the uniform bound

$$\mathbb{P}(E_n(m) | \mathcal{F}_n) \gtrsim mn\mathbb{P}(\xi > 2m).$$

Note that this bound is uniform in $\mathcal{F}_n$, and hence the events $E_n(m)$ stochastically dominate independent events with the above probabilities.

Applying the tail estimate of part (i), one finds

$$\mathbb{P}(E_n(m) | \mathcal{F}_n) \gtrsim nm^{1-\alpha}.$$

Taking $m = m(n) = an^\beta$ yields the bound $\mathbb{P}(E_n(an^\beta) | \mathcal{F}_n) \gtrsim n^{-1}$. By Borel–Cantelli, a.s. infinitely many of the $E_n(an^\beta)$ occur for any $a > 0$. By Lemma 5.2 $\{|A_n^+| > c_0 n\}$ for all but finitely many n. This implies the $\limsup$ bound on D_n^+.

For the uniform lower bound part of (i), take $E_n = E_n(\frac{n^\beta}{\log\log n})$ to find

$$\mathbb{P}(E_n | \mathcal{F}_n) \geq \frac{c(\log\log n)^{\alpha-1}}{n}.$$

Consequently, the probability that E_n fails to occur for all $n \in [N, 2N)$ is at most

$$\prod_{n=N}^{2N} \left(1 - \frac{c(\log\log n)^{\alpha-1}}{n}\right) \leq e^{-c(\log\log N)^{\alpha-1}} \ll \frac{1}{\log N}.$$

Looking at an exponential scale $N = 2^k$ one finds that a.s. only finitely many scales are bad. On this event, $D_n^+ \geq \frac{n^\beta}{\log\log n}$ for all large enough n.

Given the weaker tail estimate $\mathbb{P}(X > t) \geq t^{-\alpha+o(1)}$, we have for any $\alpha' > \alpha$ for some c, $\mathbb{P}(X > t) \geq ct^{-\alpha'}$. Thus, part (i) implies that a.s. eventually $D_n \geq \frac{n^{\beta'}}{\log n}$. As α' decreases to α, we can get β' close to β. $\quad\square$

5.2. *Infinite third moment*: *Upper bound*.

THEOREM 5.3. *Fix $\alpha \in (2, 3]$ and let $\beta = \frac{2}{\alpha-1}$. If the random walk is such that $\mathbb{P}(|\xi| > t) \leq ct^{-\alpha}$ and $\mathbb{E}\xi = 0$, then a.s. $D_n \leq n^{\beta+o(1)}$.*

The proof below gives $D_n \lesssim n^\beta (\log n)^2$, and with minimal modification one $\log n$ factor can be removed.

We analyze only $\max A_n$, noting that $\min A_n$ behaves identically. This yields bounds on $D_N = \max A_N - \min A_N$. Let $J(n, m)$ be the event that

$\Delta(\max A_n) \geq m$. This will be referred to as "making a large jump to the right" at time n. We treat $\max A_N$ as the sum of all jumps made to the right. The key idea is that if many large jumps to the right were already made, the probability of additional ones is smaller. This analysis is carried out for multiple scales of jumps.

The crux of the proof is the following estimate.

LEMMA 5.4.   *Assume* $\mathbb{P}(\xi < -t) \leq ct^{-\alpha}$ *for some* $\alpha \in (2, 3)$, *then*

$$\sum_{n \leq N} \mathbb{P}(J(n, m)) \leq CNm^{(1-\alpha)/2} = CNm^{-1/\beta}.$$

PROOF.   Define for $a \in A_n$

$$W(a) = W_n(a) = \max A_n - a$$

to be the distance from a to the rightmost point of A_n. Using Lemma 4.2 and the gluing formula (2.2), we have the bound

$$\mathbb{P}(J(n, m)|\mathcal{F}_n) \lesssim \sum_{a \in A_n} \sum_{x \geq m + \max A_n} p_{x,a} d(x, A)$$

$$= \sum_{a \in A_n} \sum_{d \geq m} d\mathbb{P}(\xi = -W(a) - d)$$

$$= \sum_{a \in A_n} \left(m\mathbb{P}(\xi \leq -m - W(a)) + \sum_{d > m} \mathbb{P}(\xi \leq -d - W(a)) \right)$$

$$\lesssim \sum_{a \in A_n} (m + W(a))^{1-\alpha}.$$

We proceed to use this bound to estimate the total expected number of such jumps up to time N. The idea is that if jumps are frequent then the maximum of A_n quickly moves away from any fixed $a \in A_n$, and so $W(a)$ is large and the probability of additional jumps is small.

Fix some L (to be determined later) and define $\hat{J}(n, m)$ to be the event that $J(n, m)$ occurs and that either $n \leq L$ or $J(n', m)$ occurs for some $n' \in [n - L, n)$. Thus, $\hat{J}(n, m)$ denotes the event that there is a large jump at time n and the process has waited at most L steps since the previous large jump (or from the beginning). In particular, $J(n, m)$ and $\hat{J}(n, m)$ can differ only once in any L consecutive n's. Thus, when L is large, $\hat{J}(n, m)$ is typically the same as $J(n, m)$, and there are at most $\lfloor N/L \rfloor$ different $n \leq N$ when J occurs and $\hat{J}$ does not.

Let $\{t_i\}$ be the set of times n at which $J(n, m)$ occurs, including (for notational convenience) $t_0 = 0$ and $t_{k+1} = N$, where k is the number of large jumps that occur. Let $s_i = t_i - t_{i-1}$ be the times spent waiting for large jumps. Finally, let $\hat{s}_i = \min(s_i, L)$.

Consider a particle at position a that has been added in the time interval $(t_i, t_{i+1}]$. At any later time $n \in (t_j, t_{j+1}]$, we have

$$W_n(a) \geq (j - i)m$$

since there have been at least $j - i$ large jumps to the right after the particle was added to the aggregate. We now have

$$\sum_{n \leq N} \mathbb{P}(\hat{J}(n, m)|\mathcal{F}_n) \lesssim \sum_{j=0}^{k} \sum_{n=t_j}^{t_j+\hat{s}_j} \sum_{a \in A_n} (m + W_n(a))^{1-\alpha}$$

$$\leq \sum_{j=0}^{k} \sum_{n=t_j}^{t_j+\hat{s}_j} \sum_{i=0}^{j} s_i (m(1 + j - i))^{1-\alpha}$$

$$= m^{1-\alpha} \sum_{j=0}^{k} \sum_{i=0}^{j} s_i \hat{s}_j (1 + j - i)^{1-\alpha}$$

and since $\hat{s}_j \leq L$

$$\leq m^{1-\alpha} L \sum_{i=0}^{k} s_i \sum_{j=i}^{k} (1 + j - i)^{1-\alpha}$$

$$\lesssim m^{1-\alpha} L \sum_{i=0}^{k} s_i = m^{1-\alpha} LN.$$

We now integrate over $\mathcal{F}_n$ to get

$$\sum_{n \leq N} \mathbb{P}(\hat{J}(n, m)) \leq C m^{1-\alpha} LN.$$

Since the difference between $\sum \mathbb{P}(J)$ and $\sum \mathbb{P}(\hat{J})$ is bounded by $\lfloor N/L \rfloor$, we get

$$\sum_{n \leq N} \mathbb{P}(J(n, m)) \leq \lfloor N/L \rfloor + C m^{1-\alpha} NL.$$

Setting $L = m^{(\alpha-1)/2}$ completes the proof. $\square$

PROOF OF THEOREM 5.3.   Given $n \leq N$, let $\ell = \log N$, and let $\tau_n = \tau_{n,N}$ be the sum of all jumps to the right of size at most $N^\beta \ell^2$ up to time n ("τ" for "truncated"). By Lemma 5.4, the probability that by time N there is some jump to the right of size at least $N^\beta \ell^2$ is at most $CN(N^\beta \ell^2)^{-1/\beta} = C\ell^{-2/\beta}$. Considering a geometric sequence of N's, since $\beta < 2$, we find that $\max A_n = \tau_n$ for all large enough N, and all $n \leq N$.

Truncating jumps at $N^\beta \ell^2$, we have that

$$\mathbb{E}\tau_n = \sum_{n=0}^{N-1} \sum_{m=1}^{N^\beta \ell^2} m\mathbb{P}(\max A_{n+1} = \max A_n + m)$$

by Abel resummation $\leq \sum_{m=1}^{N^\beta \ell^2} \sum_{n \leq N} \mathbb{P}(J(n,m))$

by Lemma 5.4 $\lesssim \sum_{m=1}^{N^\beta \ell^2} N m^{-1/\beta}$

$$\lesssim N(N^\beta \ell^2)^{1-1/\beta} = N^\beta \ell^{2(1-1/\beta)}.$$

By Markov's inequality, $\mathbb{P}(\tau_n > N^\beta \ell^2) < \ell^{-2/\beta}$. Considering a geometric sequence of N's, we find that a.s. $\tau_n \leq cN^\beta \ell^2$ for all large enough N. $\square$

REMARK.   Suppose one tries to prove Theorem 5.3 like before, that is, like Lemma 3.6, Theorem 4.1 or Claim 6.7 below. In other words, one looks for *uniform* estimates for $\mathbb{P}(D_N > m | \mathcal{F}_n)$. The best we could find was

$$\mathbb{P}(D_N > m | \mathcal{F}_n) \overset{(5.1)}{\lesssim} \min(nm^{1-\alpha}, m^{2-\alpha}).$$

This only gives an upper bound of $D_n \leq n^{4-\alpha}$ which is not sharp at any $\alpha \in (2,3)$. The failure of this uniform estimate approach means that one must use some information about the structure of the aggregate. However, the proof of Lemma 5.4 demonstrates that we do not need to know too much about the structure of A_n— only that it is not too concentrated near its right (or left) extremal points.

6. Walks with infinite variance.

6.1. *Preliminaries.*   Walks with $\alpha \in (1,2)$ all fall into this category. Any walk with mean 0 is recurrent (see [36], P2.8), and in particular so is any symmetric walk with finite mean. Thus, we can use the gluing formula (2.2) to calculate gluing probabilities. At the moment, our techniques do not work for completely general walks in this regime, but only for walks with sufficiently nice tail behaviour. Specifically, we focus on walks that are in the domain of attraction of a stable process. In particular, our results apply to any walk with $\mathbb{P}(\xi > t) = (c + o(1))t^{-\alpha}$. Our main result here is the following.

THEOREM 6.1.   *If ξ is a symmetric variable satisfying $\mathbb{P}(\xi > t) \asymp t^{-\alpha}$ with $1 < \alpha < 2$, then a.s. $D_n = n^{2+o(1)}$.*

Here and below we write $f \asymp g$ if f/g is a slowly varying function, as in the following definition:

DEFINITION 6.2. A function h is *slowly varying* at 0 (resp., at ∞) if for any $x > 0$,

$$\lim_{t \to 0} \frac{h(tx)}{h(t)} = 1$$

(resp., $\lim_{t \to \infty}$) and the limit is uniform on any compact set of x's.

Note that a common definition of slowly varying (see, e.g., [18]) requires only that the limit exists for all x. Since this is almost impossible to use, one then applies Karamata's theorem (see [18], Appendix 1), to show that any locally integrable function which is slowly varying in the weaker sense, is also slowly varying in the stronger (uniform) sense stated above. Occasionally, when we quote results from [18], we implicitly use Karamata's theorem to translate from the weaker to the stronger sense of slowly varying.

A simple consequence of the definition of a slowly varying function at 0 is that for any ε there are K, δ so that for $x < y < \varepsilon$

$$\frac{1}{K}\left(\frac{x}{y}\right)^{\delta} < \frac{h(x)}{h(y)} < K\left(\frac{y}{x}\right)^{\delta}$$

and $K \to 1$ and $\delta \to 0$ as $\varepsilon \to 0$. If the function is slowly varying at ∞, the same bounds hold for $y > x > \varepsilon^{-1}$ instead.

Following [28, 36], we define the harmonic potential by

$$a(n) = \sum_{t} \big(\mathbb{P}_0(R_t = 0) - \mathbb{P}_0(R_t = n) \big).$$

The harmonic potential is closely related to the Green's function, and the first stage is establishing its asymptotics. [36], T28.1, ensures us that the sum indeed converges.

LEMMA 6.3. *Assume* $\mathbb{P}(\xi > n) \asymp n^{-\alpha}$ *for some* $1 < \alpha < 2$. *Then the harmonic potential satisfies*

$$a(n) \asymp n^{\alpha - 1}.$$

PROOF. Given the tail of the step distribution we know from [18], Theorem 2.6.1, that ξ belongs to the domain of attraction of a (symmetric) stable random variable with exponent α. Denote by $\phi(\zeta)$ the Fourier transform of ξ. By [18], Theorem 2.6.5, we have as $\zeta \to 0$ for some real β, γ

$$\log \phi(\zeta) - i\gamma\zeta \asymp -|\zeta|^{\alpha}\left(1 - i\beta \frac{\zeta}{|\zeta|} \tan\left(\frac{\pi}{2}\alpha\right)\right)$$

(β is the skewness of the stable limit, γ corresponds to drift). In our case, ξ is symmetric so ϕ is real valued and so $\beta = \gamma = 0$ and

$$\log \phi(\zeta) \asymp -|\zeta|^{\alpha}.$$

This is the most essential use we make of the symmetry of ξ. In effect, if one only assumes that the drift γ is zero then the proof of the lemma follows through. However, for $\gamma \neq 0$ the conclusion of the lemma does not hold.

Now, write

$$\sum_{t \leq T} (\mathbb{P}_0(R_t = 0) - \mathbb{P}_0(R_t = n)) = \int_{-\pi}^{\pi} (1 - e^{in\zeta}) \sum_{t \leq T} \phi^t(\zeta) \, d\zeta.$$

Irreducibility gives that $\phi = 1$ only at $\zeta = 0$ and, therefore, (since $\alpha < 2$), $\frac{1 - e^{in\zeta}}{1 - \phi(\zeta)}$ is integrable. Hence, by dominated convergence,

$$a(n) = \int_{-\pi}^{\pi} \frac{1 - e^{in\zeta}}{1 - \phi(\zeta)} \, d\zeta = 2\mathrm{Re} \int_0^{\pi} \frac{1 - e^{in\zeta}}{|\zeta|^{\alpha}} h(\zeta) \, d\zeta,$$

where h is slowly varying at 0. Our plan is to use the fact that h is slowly varying and that bulk of the contribution to the last integral comes from $\zeta \in [\varepsilon/n, 1/(\varepsilon n)]$ to compare this integral to $K_{\alpha} n^{\alpha-1} h(n^{-1})$. We begin with the constant and work backward toward $a(n)$.

We begin with

$$\int_{\varepsilon}^{1/\varepsilon} \frac{1 - e^{ix}}{x^{\alpha}} \, dx = K_{\alpha} + \eta_1(\varepsilon),$$

where K_{α} is the integral from 0 to ∞ and where $\eta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ (since $1 < \alpha < 2$). This integral may be calculated explicitly. For example, one may change the path of integration to the imaginary line (so that e^{ix} is transformed into e^{-x}) and integrate by parts to get the integral defining the Gamma function. The result is that $K_{\alpha} = \Gamma(1 - \alpha) e^{i\pi(1-\alpha)/2}$ and in particular $\mathrm{Re} K_{\alpha}$ is nontrivial.

Since h is slowly varying, using the compactness of the interval $[\varepsilon, 1/\varepsilon]$:

$$\int_{\varepsilon}^{1/\varepsilon} \frac{1 - e^{ix}}{x^{\alpha}} \frac{h(xn^{-1})}{h(n^{-1})} \, dx = K_{\alpha} + \eta_1(\varepsilon) + \eta_2(\varepsilon, n),$$

where for any fixed ε we have $\eta_2(\varepsilon, n) \to 0$ as $n \to \infty$.

On the interval $[0, \varepsilon]$, we have

$$\left| \frac{h(xn^{-1})}{h(n^{-1})} \right| < C x^{-\delta},$$

where δ can be made arbitrarily small as $\varepsilon \to 0$ uniformly in n. Since $\alpha < 2$ we have

$$\left| \int_0^{\varepsilon} \frac{1 - e^{ix}}{x^{\alpha}} \frac{h(xn^{-1})}{h(n^{-1})} \, dx \right| \leq \int_0^{\varepsilon} C x^{1-\alpha-\delta} \leq C' \varepsilon^{2-\alpha-\delta}.$$

Similarly, on the interval $[1/\varepsilon, \varepsilon n]$ we have

$$\left| \frac{h(xn^{-1})}{h(n^{-1})} \right| < C x^{\delta},$$

where δ can be made small provided n^{-1} and $xn^{-1} \leq \varepsilon$ are both small. Since $\alpha < 2$, we have

$$\left| \int_{1/\varepsilon}^{\varepsilon n} \frac{1 - e^{ix}}{x^\alpha} \frac{h(xn^{-1})}{h(n^{-1})} \, dx \right| \leq \int_{1/\varepsilon}^{\infty} 2C x^{-\alpha+\delta} \leq C' \varepsilon^{\alpha-1-\delta}.$$

Since $1 < \alpha < 2$, both these bounds vanish as $\varepsilon \to 0$, and so we get

$$\int_0^{\varepsilon n} \frac{1 - e^{ix}}{x^\alpha} \frac{h(xn^{-1})}{h(n^{-1})} \, dx = K_\alpha + \eta_1(\varepsilon) + \eta_2(\varepsilon, n) + \eta_3(\varepsilon, n),$$

where $\limsup_{n\to\infty} |\eta_3(\varepsilon, n)| \to 0$ as $\varepsilon \to 0$. (Since for the lim sup it suffices to consider $n > 1/\varepsilon$.)

Now we are ready to consider $a(n)$. By a change of variable,

$$\int_0^{\varepsilon} \frac{1 - e^{in\zeta}}{|\zeta|^\alpha} h(\zeta) \, d\zeta = n^{\alpha-1} \int_0^{\varepsilon n} \frac{1 - e^{ix}}{x^\alpha} h(xn^{-1}) \, d\zeta.$$

Finally,

$$\left| \int_{\varepsilon}^{\pi} \frac{1 - e^{in\zeta}}{|\zeta|^\alpha} h(\zeta) \, d\zeta \right| \leq \int_{\varepsilon}^{\pi} \frac{2}{|\zeta|^\alpha} h(\zeta) \, d\zeta,$$

which is finite and independent of n.

Combining these identities, we get

$$a(n) = n^{\alpha-1} h(n^{-1}) \mathrm{Re}[K_\alpha + \eta_1 + \eta_2 + \eta_3] + \eta_4(\varepsilon),$$

with $\eta_4(\varepsilon)$ bounded. Using the estimates on the η_i's, we find

$$\lim_{\varepsilon \to 0} \limsup_{n\to\infty} \left| \frac{a(n)}{n^{\alpha-1} h(n^{-1})} - \mathrm{Re} K_\alpha \right|$$

$$\leq \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \left| \eta_1 + \eta_2 + \eta_3 + \frac{\eta_4}{n^{\alpha-1} h(n^{-1})} \right| = 0.$$

Since $a(n)$ does not depend on ε, this in fact means that

$$\limsup_{n\to\infty} \left| \frac{a(n)}{n^{\alpha-1} h(n^{-1})} - \mathrm{Re} K_\alpha \right| = 0$$

and since $\mathrm{Re} K_\alpha \neq 0$, this means that $a(n) \asymp n^{\alpha-1}$. $\quad\square$

Asymptotics of the potential kernel allow us to derive the following two estimates for the hitting probabilities.

LEMMA 6.4. *Assume* $\mathbb{P}(\xi > n) \asymp n^{-\alpha}$ *for some* $1 < \alpha < 2$, *and let* I *be the interval* $[-n, 0]$, *and* $k \in [n, 2n]$. *Then*

(6.1) $$\mathbb{P}_k(T_I < T_k) \approx n^{1-\alpha} h(n),$$

(6.2) $$\mathbb{P}_\infty(T_k < T_I) > c,$$

for some $c > 0$ *and slowly varying function* h.

PROOF.   Let $g(x, y) = g_I(x, y)$ be the Green's function with respect to I, namely

$$g(x, y) = \sum_{t=0}^{\infty} \mathbb{P}_x(R(t) = y; T_I > t),$$

and let $H(x, \cdot) = H_I(x, \cdot)$ be the hitting measures on I, namely

$$H(x, i) = \mathbb{P}_x(R(T_I) = i).$$

Finally, let $H(\infty, i) = \lim H(x, i)$ be the harmonic measure on I.

We use Lemma 6.3 and get that

(6.3)
$$a(n) = n^{\alpha-1} h(n),$$

where h is a slowly varying function. In particular, $a(n)$ is unbounded so R is recurrent (this can also be inferred directly from [36], P2.8). Hence, Theorem T30.2 of [36] applies to R (the condition that the walk is not "left- or right-continuous", to use Spitzer's terminology, is satisfied because R is symmetric). Combining (c) and (d) of that theorem, we get for every $x, y \in \mathbb{Z} \setminus I$

(6.4) $$g(x, y) = -a(y - x) + \kappa + \sum_{i \in I} H(\infty, i)a(x - i) + \sum_{i \in I} H(x, i)a(y - i),$$

where $\kappa = \kappa_I$ is some number. As a first step to understanding (6.4), let $y \to \infty$. Since ξ has infinite second moment, we may apply [36], T29.1(1), which states that

$$\lim_{|y| \to \infty} a(y - x) - a(y) = 0 \qquad \forall x$$

and hence

$$\lim_{|y| \to \infty} -a(y - x) + \sum_{i \in I} H(x, i)a(y - i) = 0 \qquad \forall x$$

or

(6.5)
$$g(x, \infty) := \lim_{|y| \to \infty} g(x, y) = \kappa + \sum_{i \in I} H(\infty, i)a(x - i).$$

Setting $x = 1$, we get

$$\kappa + \sum_{i \in I} H(\infty, i)a(1 - i) \geq 0$$

(this is not obvious because κ is a negative constant which is difficult to estimate directly from its definition). Consequently, for $x = k \in [n, 2n]$ we get

(6.6)
$$\begin{aligned}
g(k, \infty) &\geq \sum_{i=-n}^{0} (a(k - i) - a(1 - i))H(\infty, i) \\
&\geq \min_{i=-n,\dots,0} a(k - i) - a(1 - i) \\
&\geq n^{\alpha-1}h(n)(2^{\alpha-1} - 1)(1 + o(1)).
\end{aligned}$$

The last inequality requires clarification. Roughly, the minimum in the left-hand side is achieved when $i = -n$ and $k = n$. Other $i \in I$ and other $k \in [n, 2n]$ give larger values. This involves some simple playing around with the definition of a slowly varying function, in the spirit of the previous lemma which we shall omit.

On the other hand, it is easy to see that $g(k, \infty)/g(k, k)$ is the harmonic measure of k in the set $I \cup \{k\}$. Because the walk is symmetric and this set has more than 1 point the harmonic measure of any point is at most $1/2$, and hence $g(k, \infty) \leq \frac{1}{2} g(k, k)$. With (6.6), this implies

$$g(k, k) \geq cn^{\alpha-1} h(n).$$

However, we can also write, using (6.4) and (6.5),

$$g(k, k) = g(k, \infty) + \sum_{i \in I} H(k, i) a(k - i) \leq \frac{1}{2} g(k, k) + \sum_{i \in I} H(k, i) a(k - i)$$

or

$$
\begin{aligned}
g(k, k) &\leq 2 \sum_{i \in I} H(k, i) a(k - i) \leq 2 \max_{i \in I} a(k - i) \\
&\leq 2(3n)^{\alpha-1} h(n)(1 + o(1)) \\
&\leq C n^{\alpha-1} h(n).
\end{aligned}
$$

(6.7)

Since we have both upper and lower bounds, we find $g(k, k) \approx n^{\alpha-1} h(n)$. This implies our first goal (6.1) since $\mathbb{P}_k(T_k < T_I) = g(k, k)^{-1}$.

Similarly, we get (6.2) from

$$\mathbb{P}_\infty(T_k < T_I) = \frac{g(k, \infty)}{g(k, k)} \geq \frac{2^{\alpha-1} - 1}{2 \cdot 3^{\alpha-1}}(1 + o(1)). \qquad \square$$

While Lemma 6.4 talks about hitting an interval, by translation invariance and by monotonicity of T_A in A, it implies similar bounds for any set A and sufficiently far point x.

COROLLARY 6.5. *Assume* $\mathbb{P}(\xi > n) \asymp n^{-\alpha}$ *for some* $1 < \alpha < 2$, *and let* x, A *satisfy* $x \geq \max A + \operatorname{diam} A$. *Then*

$$\mathbb{P}_x(T_A < T_x) \leq Cd(x, A)^{1-\alpha} h\big(d(x, A)\big),$$
$$\mathbb{P}_\infty(T_x < T_A) > c,$$

for some $c > 0$, C *and slowly varying function* h.

6.2. *Proof of the lower bound.* We begin by proving a uniform lower bound on the probability of making a large jump. Its use is analogous to the role Lemma 3.2 plays in the restricted $\mathbb{Z}^2$ case.

LEMMA 6.6. *Under the assumptions of Theorem* 6.1, *uniformly in* $m \geq n$,

$$\mathbb{P}(D_{n+1} \geq m | \mathcal{F}_n) \geq \frac{n}{m^{1+o(1)}}.$$

PROOF. On the event $D_n \geq m$ the claim is trivial, so assume $D_n < m$. Consider some x such that $d(x, A) > m$. By Corollary 6.5, for some slowly varying function h

$$\mathbb{P}_x(T_A < T_x) \leq Cd(x, A)^{1-\alpha} h(d(x, A)) \leq Cd(x, A)^{1-\alpha+\varepsilon} \leq Cm^{1-\alpha+\varepsilon}.$$

[Since any slowly varying positive function h has $h(k) \lesssim k^\varepsilon$ for any $\varepsilon > 0$—the constant C and all constants below may depend on ε.]

By the second part of Corollary 6.5, if $m \geq D_n$ then $\mathbb{P}_\infty(T_x < T_A) > c$. Using these bounds in the gluing formula (2.2), we find that for any x with $d(x, A) \geq m$

$$\mu(x, a) = \frac{p_{x,a} \mathbb{P}_\infty(T_x < T_A)}{\mathbb{P}_x(T_A < T_x)} \geq c p_{x,a} m^{\alpha-1-\varepsilon}.$$

Summing over all $a \in A$ and x with $d(x, A) \geq m$, we get

$$\mathbb{P}(\Delta D_n \geq m | \mathcal{F}_n) \geq \sum_a \sum_{x \geq a+m+D_n} \mu(x, a)$$

$$\geq cm^{\alpha-1-\varepsilon} \sum_a \sum_{x \geq a+m+D_n} p_{x,a}$$

$$= cnm^{\alpha-1-\varepsilon} \mathbb{P}(\xi \geq m + D_n).$$

It follows that on the event $D_n \leq m$, we have

$$\mathbb{P}(\Delta D_n \geq m | \mathcal{F}_n) \geq cnm^{\alpha-1-\varepsilon} \mathbb{P}(\xi \geq 2m)$$

(6.8)

$$\geq cnm^{\alpha-1-\varepsilon} m^{-\alpha-\varepsilon} = cnm^{-1-2\varepsilon}.$$

Since ε was arbitrary, the proof of the lemma is complete. □

PROOF OF THEOREM 6.1 (LOWER BOUND). This follows from the last lemma and Lemma 3.3. □

6.3. *Proof of the upper bound.* Once again, we first prove a uniform bound on the probability of making a large jump. The theorem then follows from the bound the same way the upper bound for the $\mathbb{Z}^2$ case (Theorem 3.4) follows from Lemma 3.5.

CLAIM 6.7. *Under the conditions of Theorem 6.1,*

$$\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) \leq \frac{n}{m^{1-o(1)}}.$$

PROOF. Set $A = A_n$, and consider x outside the convex hull of A (so that its addition will increase the diameter). We have $\mathbb{P}_x(T_A < T_x) \geq \mathbb{P}_x(T_y < T_x)$, where y is an arbitrary point in A. By [36], P11.5 or T30.2, we have $\mathbb{P}_x(T_y < T_x) = (2a(x-y))^{-1}$. Using this in the gluing formula (2.2) gives

$$\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) = \sum_{y \in A} \sum_{x: d(x,A) > m} \frac{p_{x,y} \mathbb{P}_\infty(T_x < T_A)}{\mathbb{P}_x(T_A < T_x)}$$

$$\lesssim \sum_{y \in A} \sum_{|x-y| > m} p_{x,y} a(x-y).$$

Now, by Lemma 6.3, $a(x-y) \leq |x-y|^{\alpha-1} h(|x-y|)$ for some slowly varying h, hence $a(x-y) \leq C|x-y|^{\alpha-1+\varepsilon}$, for some $C = C(\varepsilon)$. This yields

$$\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) \lesssim \sum_{y \in A} \sum_{|x-y| > m} p_{x,y} |x-y|^{\alpha-1+\varepsilon}$$

(6.9)

$$\lesssim n \sum_{k > m} k^{\alpha-1+\varepsilon} \mathbb{P}(\xi = k).$$

(All y's give the same contribution, and there are two x's at any distance k.) To estimate the last sum, we use the following Abel-type summation formula: Suppose $\{a_n\}$, $\{b_n\}$ are two sequences, such that $\{b_n\}$ is summable, and $a_n B_{n+1} \to 0$, where $B_s = \sum_{k=s}^\infty b_k$, then

$$\sum_{n \geq m} a_n b_n = a_m B_m + \sum_{n > m} (a_n - a_{n-1}) B_n.$$

(Restricting the sums to $n \leq N$ gives a discrepancy of $a_N B_{N+1}$, so if one series converges so does the other and the identity holds.) Setting $a_n = n^{\alpha-1+\varepsilon}$ and $b_n = \mathbb{P}(\xi = n)$, we get

$$\sum_{k \geq m} k^{\alpha-1+\varepsilon} \mathbb{P}(\xi = k)$$

$$= m^{\alpha-1+\varepsilon} \mathbb{P}(\xi \geq m) + \sum_{k > m} (k^{\alpha-1+\varepsilon} - (k-1)^{\alpha-1+\varepsilon}) \mathbb{P}(\xi \geq k)$$

$$\leq m^{\alpha-1+\varepsilon} \mathbb{P}(\xi \geq m) + C \sum_{k > m} k^{\alpha-2+\varepsilon} \mathbb{P}(\xi \geq k)$$

$$\leq C m^{\alpha-1+\varepsilon} m^{-\alpha+\varepsilon} + C \sum_{k > m} k^{\alpha-2+\varepsilon} k^{-\alpha+\varepsilon}$$

$$\leq C m^{2\varepsilon-1}.$$

The penultimate inequality follows from the conditions on ξ together with the fact that a slowly varying function grows slower than any power. Combining this with (6.9), we get

$$\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) \leq \frac{Cn}{m^{1-2\varepsilon}}.$$

Since ε is arbitrary, the proof is complete. $\square$

PROOF OF THEOREM 6.1 (UPPER BOUND).   Claim 6.7 implies $\mathbb{P}(\Delta D_n > m | \mathcal{F}_n) \leq nm^{-1-o(1)}$. By the second part of Lemma 3.6, this implies that $\mathbb{P}(D_n \geq \gamma \phi(n)) \lesssim 1/\gamma$ for any n and γ, with $\phi(n) = n^{2+o(1)}$. Set $\gamma = \log^2 n$ and consider only the geometric sequence $n = 2^k$. It follows by Borel–Cantelli that a.s. $D_n \leq \phi(n) \log^2 n = n^{2+o(1)}$ for all large enough $n = 2^k$, and by monotonicity this holds for any n, as needed. $\square$

7. Hyper-transient: The restricted $\mathbb{Z}^3$ walk.

We consider here the restriction to $\mathbb{Z}$ of the simple random walk on $\mathbb{Z}^3$, where $\mathbb{Z}$ is embedded in $\mathbb{Z}^3$, say as $\mathbb{Z} \times \{(0, 0)\}$. Because simple random walk on $\mathbb{Z}^3$ is transient, so is our induced process on $\mathbb{Z}$. This means that the gluing formula (2.2) is no longer valid, nor is our definition of DLA. Let us therefore start by stating the analog of $\mu(x, a)$ in the transient case. For a set A and an element x (possibly in A), we define the escape probability $E_A(x)$ and the capacity $\mathrm{Cap}(A)$ by

$$E_A(x) = \mathbb{P}_x(T_A = \infty), \qquad \mathrm{Cap}(A) = \sum_{a \in A} E_A(a).$$

Now define the transient gluing measure by

$$(7.1) \qquad \mu(x, a) = \mu(x, a; A) = \frac{p_{x,a} E_A(x)}{\mathrm{Cap}(A)}.$$

In part II, we explain why (7.1) is the natural analog of (2.2) in the transient settings, but for now we take it as the definition [note that in part II, (7.1) contains $E_A^*(x)$, the escape probabilities for the reversed walk, but here our walk is symmetric]. With $\mu(x, a)$ defined the aggregate is defined exactly as in Definition 2.3. We keep the notation of R and ξ for the walk, A_n for the aggregate and $D_n = \mathrm{diam}\, A_n$.

We now consider the specific case of the $\mathbb{Z}^3$ restricted walk. It turns out that we only need the following property.

DEFINITION 7.1.   A random walk on $\mathbb{Z}$ is said to be *log-avoiding* if for some $c > 0$ and any finite $A \subset \mathbb{Z}$, and any x we have $E_A(x) \geq \frac{c}{\log |A|}$.

PROPOSITION 7.2.   *The restricted $\mathbb{Z}^3$ random walk is log-avoiding.*

PROOF. Since $\mathbb{Z}$ is embedded in $\mathbb{Z}^3$, take a cylinder of radius $|A|^2$ around $\mathbb{Z}$, and let B be the vertex boundary of the cylinder. Since the random walk projected orthogonally to the embedded copy of $\mathbb{Z}$ is a two-dimensional random walk, for any $x \in \mathbb{Z}$ we have $\mathbb{P}_x(T_B < T_{\mathbb{Z}}) \geq \frac{c}{\log |A|}$ (see, e.g., [11], Lemma 9).

The probability in $\mathbb{Z}^3$ of ever hitting a given point at distance d is or order c/d (see [36], T26.1 or [28], Theorem 4.3.1). Thus, for any point in B, the probability of ever hitting A is at most $c/|A|$. Combining the two,

$$E_A(x) \geq \mathbb{P}_x(T_B < T_{\mathbb{Z}})\mathbb{P}_B(T_A = \infty) \geq \frac{c}{\log |A|}\big(1 - c/|A|\big) \geq \frac{c'}{\log |A|}. \qquad \square$$

With this property, we have super-exponential growth.

THEOREM 7.3. *Let R be a log-avoiding random walk. Then a.s. for any C, $D_n > C^n$ infinitely often.*

PROOF. For any walk $\mathrm{Cap}(A_n) \leq n$, and by log-avoidance $E_A(x) \geq \frac{c}{\log n}$ and putting this into (7.1) gives

(7.2) $$\mu(x, a; A_n) \geq \frac{cp_{x,a}}{n \log n}.$$

Now, gluing any x with $|x - a| > m$ will imply $D_{n+1} > m$, and so since there are n possible a's,

$$\mathbb{P}(D_{n+1} > m | \mathcal{F}_n) \geq n\frac{c\mathbb{P}(\xi > m)}{n \log n}.$$

Next, we note that for any log-avoiding random walk with step ξ we have $\mathbb{P}(\xi > m) \geq \frac{c}{\log m}$, since this is larger than the probability of escaping the interval $[-m, m]$. Therefore,

$$\mathbb{P}(D_{n+1} > m | \mathcal{F}_n) \geq \frac{c}{\log m \log n}.$$

Taking $m = C^n$, we see that a.s. $D_{n+1} > C^n$ for infinitely many n. $\quad\square$

In light of this very fast growth, the following result is somewhat surprising.

THEOREM 7.4. *If R be a log-avoiding irreducible random walk, then a.s. $A_\infty = \mathbb{Z}$ (where $A_\infty = \cup A_n$.)*

PROOF. Fix some point $x \in \mathbb{Z}$ with $p_{0,x} > 0$. Taking $a = 0$ in (7.2), we get

$$\mu(x, 0) \geq \frac{cp_{0,x}}{n \log n}.$$

It follows that a.s. $x \in A_\infty$. If $p_{0,x} = 0$ we use the irreducibility of the walk to find $0 = a_1, a_2, \ldots, a_k = x$ such that $p_{a_i, a_j} > 0$. Since the same argument works with any $a_i \in A_n$ in place of 0, we can show inductively that a.s. all a_i are in A_∞, and in particular x. $\square$

Let us conclude with a related conjecture.

CONJECTURE. *For any transient random walk on $\mathbb{Z}$, $\mathrm{Cap}(A_n) = o(n)$ a.s.*

Our basis for this conjecture is similar to the argument for Theorem 7.4: If the capacity grows linearly, then $A_\infty = \mathbb{Z}$ which (morally) implies that A_n should have particles clumped into small intervals. However, that contradicts the assumption of linear capacity.

If this conjecture holds, then one may show that $D_n = o(n^{1/\alpha})$ for any $0 < \alpha < \frac{1}{2}$, so the aggregate does not grow in a precisely polynomial fashion, but rather has some sub-polynomial correction.

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