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BULK UNIVERSALITY FOR DEFORMED WIGNER MATRICES

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We consider $N \times N$ random matrices of the form $H = W + V$ where W is a real symmetric or complex Hermitian Wigner matrix and V is a random or deterministic, real, diagonal matrix whose entries are independent of W. We assume subexponential decay for the matrix entries of W, and we choose V so that the eigenvalues of W and V are typically of the same order. For a large class of diagonal matrices V, we show that the local statistics in the bulk of the spectrum are universal in the limit of large N.

1. Introduction. A prominent class of random matrix models is the Wigner ensemble, consisting of $N \times N$ real symmetric or complex Hermitian matrices, $W = (w_{ij})$, whose matrix entries are random variables that are independent up to the symmetry constraint $W = W^*$. The first rigorous result about the spectrum of random matrices of this type is Wigner's global semicircle law [60], which states that the empirical distribution of the rescaled eigenvalues, (λ_i), of a Wigner matrix W is given by

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}(E) \longrightarrow \rho_{\mathrm{sc}}(E) := \frac{1}{2\pi}\sqrt{(4-E^2)_+} \qquad (E \in \mathbb{R}), \tag{1.1}$$

as $N \to \infty$, in the weak sense. The distribution ρ_{sc} is called the *semicircle law*.

Let $p_W^N(\lambda_1, \ldots, \lambda_N)$ denote the joint probability density of the (unordered) eigenvalues of W. If the entries of the Wigner matrix W are i.i.d. (independent and identically distributed) real or complex Gaussian random variables, the joint density of the eigenvalues, $p_W^N \equiv p_G^N$, is given by

$$p_G^N(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_G^N}\prod_{i<j}|\lambda_i - \lambda_j|^\beta \mathrm{e}^{-\beta N\sum_{i=1}^N \lambda_i^2/4}, \tag{1.2}$$

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with $\beta = 1, 2$, for the real, complex case, respectively. The normalization $Z_G^N \equiv Z_G^N(\beta)$ in (1.2) can be computed explicitly. The real and complex Gaussian matrix ensembles so defined are known as the Gaussian orthogonal ensemble (GOE, $\beta = 1$) and Gaussian unitary ensemble (GUE, $\beta = 2$), respectively, and as noted above we denote the corresponding joint densities as p_G^N instead of p_W^N.

The *n-point correlation functions* are defined by

$$\varrho_{W,n}^N(\lambda_1, \ldots, \lambda_n) := \int_{\mathbb{R}^{N-n}} p_W^N(\lambda_1, \lambda_2, \ldots, \lambda_N)\, \mathrm{d}\lambda_{n+1}\, \mathrm{d}\lambda_{n+2} \cdots \mathrm{d}\lambda_N,$$

$1 \le n \le N$. Using orthogonal polynomials the correlation functions of the GUE and GOE have been explicitly computed by Dyson, Gaudin and Mehta; see, for example, [44]. For the Gaussian unitary ensemble, their results assert that the limiting behavior on small scales at a fixed energy E in the bulk of the spectrum, that is, for $|E| < 2$, satisfies

$$\begin{aligned} &\frac{1}{[\rho_{\mathrm{sc}}(E)]^n} \varrho_{G,n}^N\left(E + \frac{\alpha_1}{\rho_{\mathrm{sc}}(E)N}, E + \frac{\alpha_2}{\rho_{\mathrm{sc}}(E)N}, \ldots, E + \frac{\alpha_n}{\rho_{\mathrm{sc}}(E)N}\right) \\ &\qquad \longrightarrow \det\bigl(K(\alpha_i - \alpha_j)\bigr)_{i,j=1}^n, \end{aligned} \tag{1.3}$$

as $N \to \infty$, where K is the sine-kernel

$$K(x, y) := \frac{\sin \pi(x - y)}{\pi(x - y)}.$$

Note that the limit in (1.3) is independent of the energy E as long as E is in the bulk of the spectrum. The rescaling by a factor $1/N$ of the correlation functions in (1.3) corresponds to the typical separation of consecutive eigenvalues, and we refer to the law under such a scaling as *local statistics*. Similar but more complicated formulas were also obtained for the Gaussian orthogonal ensemble; see, for example, [1, 44] for reviews. Note that the limiting correlation functions do not factorize, reflecting the fact that the eigenvalues remain strongly correlated in the limit of large N.

The Wigner–Dyson–Gaudin–Mehta conjecture, or bulk universality conjecture, states that the local eigenvalue statistics of Wigner matrices are universal in the sense that they depend only on the symmetry class of the matrix, but are otherwise independent of the details of the distribution of the matrix entries. The bulk universality can be formulated in terms of weak convergence of correlation functions or in terms of eigenvalue gap statistics. This conjecture for all symmetry classes has been established in a series of papers [22–24, 28, 31, 33]. After this work began, parallel results were obtained for complex Hermitian matrices and certain symmetric matrices in [55, 56]; see [30] for a more detailed review.

In the present paper, we consider deformed Wigner matrices. A deformed Wigner matrix, H, is an $N \times N$ random matrix of the form

$$H = V + W, \tag{1.4}$$

where V is a real, diagonal, random or deterministic matrix and W is a real symmetric or complex Hermitian Wigner matrix independent of V. The matrices are normalized so that the eigenvalues of V and W are order one. If the entries, (v_i), of V are random we may think of V as a "random potential"; if the entries of V are deterministic, matrices in the form of (1.4) are sometimes referred to as *Wigner matrices with external source*.

Assuming that the empirical eigenvalue distribution of V,

$$\widehat{\nu} := \frac{1}{N}\sum_{i=1}\delta_{v_i},$$

converges weakly, respectively, weakly in probability, to a nonrandom measure, ν, it was shown in [46] that the empirical distribution of the eigenvalues of H converges weakly in probability to a deterministic measure. This measure depends on ν and is thus in general distinct from ρ_{sc}. We refer to it as the *deformed semicircle law*, henceforth denoted by ρ_{fc}. There is no explicit formula for ρ_{fc} in terms of ν. Instead, ρ_{fc} is obtained as the solution of a functional equation for its Stieltjes transform; see (2.9) below. It is known that ρ_{fc} admits a density [6]. Depending on ν, ρ_{fc} may be supported on several disjoint intervals. For simplicity, we assume below that ν is such that ρ_{fc} is supported on a single bounded interval. Further, we choose $\widehat{\nu}$ such that all eigenvalues of H remain close to the support of ρ_{fc}; that is, there are no "outliers" for N sufficiently large.

If W belongs to the GUE, H is said to belong to the deformed GUE. The deformed GUE for the special case when V has two eigenvalues $\pm a$, each with equal multiplicity, has been treated in a series of papers [2, 8, 9]. In this setting the local eigenvalue statistics of H can be obtained via the solution to a Riemann–Hilbert problem; see also [17] for the case when V has equispaced eigenvalues. Bulk universality for correlation functions of the deformed GUE with rather general deterministic or random V has been proved in [51] by means of the Brezin–Hikami/Johansson integration formula.

In the present paper, we establish bulk universality of local averages of correlation functions for deformed Wigner matrices of the form $H = V + W$, where W is a real symmetric or complex Hermitian Wigner matrix and V is a deterministic or random real diagonal matrix. We assume that the entries of W are centered independent random variables with variance $1/N$ whose distributions decay subexponentially; see Definition 2.1. If V is random, we assume for simplicity that its entries (v_i) are i.i.d. random variables. We assume that $\widehat{\nu}$ converges weakly, respectively, weakly in probability, to a nonrandom measure ν; see Assumption 2.2. We further assume that the corresponding deformed semicircle law ρ_{fc} is supported on a single compact interval and has square root decay at both endpoints. Sufficient conditions for these assumptions to hold have appeared in [52] and are rephrased in Assumption 2.3. Under these assumptions, our main results in Theorem 2.5 and in Theorem 2.6 assert that the limiting correlation functions of the deformed Wigner

ensemble are universal when averaged over a small energy window. Note that our results hold for complex Hermitian and real symmetric deformed Wigner matrices.

Before we outline our proofs, we recall the notion of *β-ensemble* or *log-gas* which generalizes the measures in (1.2). Let U be a real-valued potential, and consider the measure on $\mathbb{R}^N$ defined by the density

$$\mu_U^N(\lambda_1, \dots, \lambda_N) := \frac{1}{Z_U^N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \mathrm{e}^{-\beta N \sum_{i=1}^N (\lambda_i^2/2 + U(\lambda_i))/2}, \tag{1.5}$$

where $\beta > 0$ and $Z_U^N \equiv Z_U^N(\beta)$ is a normalization. Bulk universality for β-ensembles asserts that the local correlation functions for measures in the form of (1.5) are universal (for sufficiently regular potentials U) in the sense that for each value of $\beta > 0$ they agree with the local correlation functions of the Gaussian ensemble with $U \equiv 0$.

For the classical values $\beta \in \{1, 2, 4\}$, the eigenvalue correlation functions of μ_U^N can be explicitly expressed in terms of polynomials orthogonal to the exponential weight in (1.5). Thus the analysis of the correlation functions relies on the asymptotic properties of the corresponding orthogonal polynomials. This approach, initiated by Dyson, Gaudin and Mehta (see [44] for a review), was the starting point for many results on the universality for β-ensemble with $\beta \in \{1, 2, 4\}$ [7, 18–20, 37, 42, 43, 49].

For general $\beta > 0$, bulk universality of β-ensembles has been established in [10–12] for potentials $U \in C^4$. Recently, alternative approaches to bulk universality for β-ensembles with general β have been presented in [50] and [4] under different conditions on U.

We emphasize at this point that the eigenvalue distributions of the deformed ensemble in (1.4) are in general not of the form (1.5), even when W belongs to the GUE or the GOE.

Returning to the random matrix setting, we recall that the general approach to bulk universality for (generalized) Wigner matrices in [24, 28, 33] consists of three steps:

(1) establish a local semicircle law for the density of eigenvalues;
(2) prove universality of Wigner matrices with a small Gaussian component by analyzing the convergence of Dyson Brownian motion to local equilibrium;
(3) compare the local statistics of Wigner ensembles with Gaussian divisible ensembles to remove the small Gaussian component of step (2).

For an overview of recent results and this three-step strategy, see [30]. Note that the "local equilibrium" in step (2) refers to measure (1.2), with $\beta = 1, 2$, respectively, in the real symmetric, complex Hermitian case.

For deformed Wigner matrices, the *local deformed semicircle law*, the analogue of step (1), was established in [39] for random V. However, when V is random, the eigenvalues of $V + W$ fluctuate on scale $N^{-1/2}$ in the bulk (see [39]), but their

gaps remain rigid on scale N^{-1}. To circumvent the mesoscopic fluctuations of the eigenvalue positions, we condition on V, considering its entries to be fixed. The methods of [39] can be extended, as outlined in Section 3, to prove a local law on the optimal scale for "typical" realizations of random as well as deterministic potentials V.

Our corresponding version of step (2), a proof of bulk universality for deformed Wigner ensembles with small Gaussian component, is the main novelty of this paper. The local equilibrium of Dyson Brownian motion in the deformed case is unknown but may effectively be approximated by a "reference" β-ensemble that we explicitly construct in Section 4. In Section 5, we analyze the convergence of the local distribution of the deformed Wigner ensemble under Dyson Brownian motion to the "reference" β-ensemble. However, since the "reference" β-ensemble is not given by the invariant GUE/GOE, it also evolves in time. Using the rigidity estimates for the deformed ensemble established in step (1) and the rigidity estimates for general β-ensembles established in [12], we obtain, in Section 5, bounds on the time evolution of the relative entropy between the two measures being compared. The idea to estimate the entropy flow of the Dyson Brownian motion with respect to the "global equilibrium state" given by the GUE/GOE was initiated in [28] and [29]. On the other hand, the idea to use "time dependent local equilibrium states" to control the entropy flow of hydrodynamical equations was introduced in [61]. There it is observed that the change of relative entropy is negligible provided that the time dependent local equilibrium is chosen in agreement with the density predicted by the hydrodynamical equations. In this paper, we combine both methods to yield an effective estimate on the entropy flow of the Dyson Brownian motion in the deformed case. This global entropy estimate is then used in Section 6 to conclude that the local statistics of the locally-constrained deformed ensemble with small Gaussian component agree with those of the locally-constrained reference β-ensemble. Relying on the main technical result of [31], we further conclude that the local statistics of the locally-constrained reference β-ensemble agrees with the local statistics of the GUE/GOE. Once this conclusion is obtained for the locally-constrained ensembles, it can be extended to the nonconstrained ensembles. This completes step (2) in the deformed case.

In Sections 7 and 8, we outline step (3) for deformed Wigner matrices; the proof is similar to the argument for Wigner matrices in [32]. The main technical input is a bound on the resolvent entries of H on scales $N^{-1-\varepsilon}$ that can be obtained from the local law in step (1). In Section 8, we then combine steps (1)–(3) to conclude the proof of our main results, Theorems 2.5 and 2.6.

We remark that our arguments in step (2) do not rely on V being diagonal. Step (3) depends only on the deformed local semicircle law of step (1); in principle, step (3) is independent of whether or not V is diagonal, as long as a deformed local semicircle law is given. Currently, our proof for the deformed local semicircle law uses that V is diagonal.

In Section 9, we prove that, in addition to bulk universality, the edge universality also holds for our model, that is, that the local statistics at the spectral edges are given by the Tracy–Widom–Airy statistics. From the main technical result of [12], the proof of the edge universality follows the same three-step program as the proof of bulk universality. A detailed discussion of our edge universality result, Theorem 2.10, and related results can be found in Section 2.4.

In the Appendix, we collect several technical results on the deformed semicircle law and its Stieltjes transform. Some of these results have previously appeared in [52] and [39, 40].

2. Assumptions and main results. In this section, we list our assumptions and our main results.

2.1. *Definition of the model.* We first introduce real symmetric and complex Hermitian Wigner matrices.

DEFINITION 2.1. A real symmetric Wigner matrix is an $N \times N$ random matrix, W, whose entries, (w_{ij}) $(1 \le i, j \le N)$, are independent (up to the symmetry constraint $w_{ij} = w_{ji}$) real centered random variables satisfying

$$\mathbb{E}w_{ii}^2 = \frac{2}{N}, \qquad \mathbb{E}w_{ij}^2 = \frac{1}{N} \qquad (i \ne j). \tag{2.1}$$

In case (w_{ij}) are Gaussian random variables, W belongs to the Gaussian orthogonal ensemble (GOE).

A complex Hermitian Wigner matrix is an $N \times N$ random matrix, W, whose entries, (w_{ij}) $(1 \le i, j \le N)$, are independent (up to the symmetry constraint $w_{ij} = \bar{w}_{ji}$) complex centered random variables satisfying

$$\mathbb{E}w_{ii}^2 = \frac{1}{N}, \qquad \mathbb{E}|w_{ij}|^2 = \frac{1}{N}, \qquad \mathbb{E}w_{ij}^2 = 0 \qquad (i \ne j). \tag{2.2}$$

For simplicity, we assume that the real and imaginary parts of (w_{ij}) are independent for all i, j. This ensures that $\mathbb{E}w_{ij}^2 = 0$ $(i \ne j)$. In case $(\operatorname{Re} w_{ij})$ and $(\operatorname{Im} w_{ij})$ are Gaussian random variables, W belongs to the Gaussian unitary ensemble (GUE).

Irrespective of the symmetry class of W, we assume that the entries (w_{ij}) have a subexponential decay, that is,

$$\mathbb{P}\big(\sqrt{N}|w_{ij}| > x\big) \le C_0 \mathrm{e}^{-x^{1/\theta}}, \tag{2.3}$$

for some positive constants C_0 and $\theta > 1$. In particular,

$$\mathbb{E}|w_{ij}|^p \le C \frac{(\theta p)^{\theta p}}{N^{p/2}} \qquad (p \ge 3). \tag{2.4}$$

Let $V = \operatorname{diag}(v_i)$ be an $N \times N$ diagonal, random or deterministic matrix, whose entries (v_i) are real-valued. We denote by $\widehat{\nu}$ the empirical eigenvalue distribution of the diagonal matrix $V = \operatorname{diag}(v_i)$,

$$\widehat{\nu} := \frac{1}{N} \sum_{i=1}^{N} \delta_{v_i}. \tag{2.5}$$

ASSUMPTION 2.2. There is a (nonrandom) centered, compactly supported probability measure ν such that the following holds:

(1) If V is a *random* matrix, we assume that (v_i) are independent and identically distributed real random variables with law ν. Further, we assume that (v_i) are independent of (w_{ij}).

(2) If V is a *deterministic* matrix, we assume that there is $\alpha_0 > 0$, such that for any fixed compact set $\mathcal{D} \subset \mathbb{C}^+$ (independent of N) with $\operatorname{dist}(\mathcal{D}, \operatorname{supp} \nu) > 0$, there is C such that

$$\max_{z \in \mathcal{D}} \left| \int \frac{\mathrm{d}\widehat{\nu}(v)}{v - z} - \int \frac{\mathrm{d}\nu(v)}{v - z} \right| \leq C N^{-\alpha_0}, \tag{2.6}$$

for N sufficiently large.

Note that (2.6) implies that $\widehat{\nu}$ converges to ν in the weak sense as $N \to \infty$. Also note that condition (2.6) holds for large N with high probability for $0 < \alpha_0 < 1/2$ if (v_i) are i.i.d. random variables.

2.2. *Deformed semicircle law.* The deformed semicircle can be described in terms of the Stieltjes transform: for a (probability) measure ω on the real line we define its Stieltjes transform, m_ω, by

$$m_\omega(z) := \int \frac{\mathrm{d}\omega(v)}{v - z} \qquad (z \in \mathbb{C}^+).$$

Note that m_ω is an analytic function in the upper half plane and that $\operatorname{Im} m_\omega(z) \geq 0$, $\operatorname{Im} z > 0$. Assuming that ω is absolutely continuous with respect to Lebesgue measure, we can recover the density of ω from m_ω by the inversion formula

$$\omega(E) = \lim_{\eta \searrow 0} \frac{1}{\pi} \operatorname{Im} m_\omega(E + \mathrm{i}\eta) \qquad (E \in \mathbb{R}). \tag{2.7}$$

We use the same symbols to denote measures and their densities. Moreover, we have

$$\lim_{\eta \searrow 0} \operatorname{Re} m_\omega(E + \mathrm{i}\eta) = \int \frac{\omega(v)\,\mathrm{d}v}{v - E} \qquad (E \in \mathbb{R}),$$

whenever the left-hand side exists. Here the integral on the right is understood as principal value integral. We denote in the following by $\operatorname{Re} m_\omega(E)$ and $\operatorname{Im} m_\omega(E)$ the limiting quantities

$$\begin{aligned} \operatorname{Re} m_\omega(E) &\equiv \lim_{\eta \searrow 0} \operatorname{Re} m_\omega(E + \mathrm{i}\eta), \\ \operatorname{Im} m_\omega(E) &\equiv \lim_{\eta \searrow 0} \operatorname{Im} m_\omega(E + \mathrm{i}\eta), \end{aligned} \tag{2.8}$$

$E \in \mathbb{R}$, whenever the limits exist.

Choosing ω to be the standard semicircular law ρ_{sc}, the Stieltjes transform $m_{\rho_{\mathrm{sc}}} \equiv m_{\mathrm{sc}}$ can be computed explicitly, and one checks that m_{sc} satisfies the relation

$$m_{\mathrm{sc}}(z) = \frac{-1}{m_{\mathrm{sc}}(z) + z}, \qquad \operatorname{Im} m_{\mathrm{sc}}(z) \geq 0 \qquad (z \in \mathbb{C}^+).$$

The deformed semicircle law is conveniently defined through its Stieltjes transform. Let ν be the limiting probability measure of Assumption 2.2. Then it is well known [46] that the functional equation

$$m_{\mathrm{fc}}(z) = \int \frac{\mathrm{d}\nu(v)}{v - z - m_{\mathrm{fc}}(z)}, \qquad \operatorname{Im} m_{\mathrm{fc}}(z) \geq 0 \qquad (z \in \mathbb{C}^+), \tag{2.9}$$

has a unique solution, also denoted by m_{fc}, that satisfies, for all $E \in \mathbb{R}$, $\limsup_{\eta \searrow 0} \operatorname{Im} m_{\mathrm{fc}}(E + \mathrm{i}\eta) < \infty$. Indeed, from (2.9), we obtain that

$$\int \frac{\mathrm{d}\nu(v)}{|v - z - m_{\mathrm{fc}}(z)|^2} = \frac{\operatorname{Im} m_{\mathrm{fc}}(z)}{\operatorname{Im} m_{\mathrm{fc}}(z) + \eta} \leq 1 \qquad (z \in \mathbb{C}^+), \tag{2.10}$$

thus $|m_{\mathrm{fc}}(z)| \leq 1$, for all $z \in \mathbb{C}^+$.

The deformed semicircle law, denoted by ρ_{fc}, is then defined through its density

$$\rho_{\mathrm{fc}}(E) := \lim_{\eta \searrow 0} \frac{1}{\pi} \operatorname{Im} m_{\mathrm{fc}}(E + \mathrm{i}\eta) \qquad (E \in \mathbb{R}).$$

The measure ρ_{fc} has been studied in detail in [6]. For example, it was shown there that the density ρ_{fc} is an analytic function inside the support of the measure.

The measure ρ_{fc} is also referred to as the additive free convolution of the semicircular law and the measure ν. More generally, the additive free convolution of two (probability) measures ω_1 and ω_2, usually denoted by $\omega_1 \boxplus \omega_2$, is defined as the distribution of the sum of two freely independent noncommutative random variables, having distributions ω_1, ω_2, respectively; we refer, for example, to [1, 59] for reviews. Similar to (2.9), the free convolution measure $\omega_1 \boxplus \omega_2$ can be described in terms of a set of functional equations for the Stieltjes transforms; see [5, 16].

Our second assumption on ν guarantees (see Lemma 3.5 below) that ρ_{fc} is supported on a single interval and that ρ_{fc} has a square root behavior at the two endpoints of its support. Sufficient conditions for this behavior have been presented

in [52]. The assumptions below also rule out the possibility that the matrix H has "outliers" in the limit of large N.

ASSUMPTION 2.3. Let I_ν be the smallest interval such that $\operatorname{supp}\nu \subseteq I_\nu$. Then there exists $\varpi > 0$ such that

$$\inf_{x\in I_\nu} \int \frac{\mathrm{d}\nu(v)}{(v-x)^2} \geq 1+\varpi. \tag{2.11}$$

Similarly, let $I_{\widehat{\nu}}$ be the smallest interval such that $\operatorname{supp}\widehat{\nu} \subseteq I_{\widehat{\nu}}$. Then:

(1) for *random* (v_i), there is a constant $\mathfrak{t} > 0$, such that

$$\mathbb{P}\Big(\inf_{x\in I_{\widehat{\nu}}} \int \frac{\mathrm{d}\widehat{\nu}(v)}{(v-x)^2} \geq 1+\varpi\Big) \geq 1-N^{-\mathfrak{t}}, \tag{2.12}$$

for N sufficiently large;

(2) for *deterministic* (v_i),

$$\inf_{x\in I_{\widehat{\nu}}} \int \frac{\mathrm{d}\widehat{\nu}(v)}{(v-x)^2} \geq 1+\varpi, \tag{2.13}$$

for N sufficiently large.

We give two examples for which (2.11) is satisfied:

(1) Choosing $\nu = \frac{1}{2}(\delta_{-a}+\delta_a)$, $a \geq 0$, we have $I_\nu = [-a, a]$. For $a < 1$, one checks that there is a $\varpi = \varpi(a)$ such that (2.11) is satisfied and that the deformed semicircle law is supported on a single interval with a square root type behavior at the edges. However, for $a > 1$, the deformed semicircle law is supported on two disjoint intervals; for further details, see [2, 8, 9].

(2) Let ν be a centered Jacobi measure of the form

$$\nu(v) = Z^{-1}(v-1)^a(1-v)^b d(v)\mathbb{1}_{[-1,1]}(v), \tag{2.14}$$

where $d \in C^1([-1,1])$, $d(v) > 0$, $-1 < a, b < \infty$ and Z, a normalization constant. Then for $a, b < 1$, there is $\varpi > 0$ such that (2.11) is satisfied with $I_\nu = [-1, 1]$. However, if $a > 1$ or $b > 1$, then (2.3) may not be satisfied. In this setting the deformed semicircle law is still supported on a single interval; however, the square root behavior at the edge may fail. We refer to [39, 40] for a detailed discussion.

LEMMA 2.4. *Let ν satisfy (2.11) for some $\varpi > 0$. Then there are L_-, L_+, with $L_- \leq -2, 2 \leq L_+$, such that* $\operatorname{supp}\rho_{\mathrm{fc}} = [L_-, L_+]$. *Moreover, ρ_{fc} has a strictly positive density in (L_-, L_+).*

Lemma 2.4 follows directly from Lemma 3.5 below.

2.3. *Results on bulk universality.* Recall that we denote by $\varrho^N_{H,n}$ the n-point correlation function of $H = V + W$, where V is either a real deterministic or real random diagonal matrix. We denote by $\varrho^N_{G,n}$ the n-point correlation function of the GUE, respectively, the GOE.

A function $O:\mathbb{R}^n \to \mathbb{R}$ is called an n-particle *observable* if O is symmetric, smooth and compactly supported. Recall from Lemma 2.4 that we denote by $L_\pm$ the endpoints of the support of the measure ρ_{fc}. For deterministic V we have the following result.

THEOREM 2.5. *Let W be a complex Hermitian or a real symmetric Wigner matrix satisfying the assumptions in Definition 2.1. Let V be a deterministic real diagonal matrix satisfying Assumptions 2.2 and 2.3. Set $H = V + W$. Let E, E' be two energies satisfying $E \in (L_-, L_+)$, $E' \in (-2, 2)$. Fix $n \in \mathbb{N}$, and let O be an n-particle observable. Let $\delta > 0$ be arbitrary, and choose $b \equiv b_N$ such that $N^{-\delta} \geq b_N \geq N^{-1+\delta}$. Then*

$$\begin{aligned}
&\lim_{N\to\infty} \int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n O(\alpha_1, \ldots, \alpha_n) \\
&\qquad \times \left[\frac{1}{2b} \int_{E-b}^{E+b} \frac{\mathrm{d}x}{[\rho_{\mathrm{fc}}(E)]^n} \varrho^N_{H,n}\left(x + \frac{\alpha_1}{\rho_{\mathrm{fc}}(E)N}, \ldots, x + \frac{\alpha_n}{\rho_{\mathrm{fc}}(E)N} \right) \right. \\
&\qquad \left. - \frac{1}{[\rho_{\mathrm{sc}}(E')]^n} \varrho^N_{G,n}\left(E' + \frac{\alpha_1}{\rho_{\mathrm{sc}}(E')N}, \ldots, E' + \frac{\alpha_n}{\rho_{\mathrm{sc}}(E')N} \right) \right] \\
&\quad = 0,
\end{aligned} \tag{2.15}$$

where ρ_{fc} denotes the density of the deformed semicircle law and ρ_{sc} denotes the density of the standard semicircle law. Here, $\varrho^N_{G,n}$ denotes the n-point correlation function of the GUE in case W is a complex Hermitian Wigner matrix, respectively, the n-point correlation function of the GOE in case W is a real symmetric Wigner matrix.

For random V we have the following result.

THEOREM 2.6. *Let W be a complex Hermitian or a real symmetric Wigner matrix satisfying the assumptions in Definition 2.1. Let V be a random real diagonal matrix whose entries are i.i.d. random variables that are independent of W and satisfy Assumptions 2.2 and 2.3. Set $H = V + W$. Let E, E' be two energies satisfying $E \in (L_-, L_+)$, $E' \in (-2, 2)$. Fix $n \in \mathbb{N}$, and let O be an n-particle observable. Let $\delta > 0$ be arbitrary, and choose $b \equiv b_N$ such that $N^{-\delta} \geq b_N \geq N^{-1/2+\delta}$. Then*

$$\begin{aligned}
&\lim_{N\to\infty} \int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n O(\alpha_1, \ldots, \alpha_n) \\
&\qquad \times \left[\frac{1}{2b} \int_{E-b}^{E+b} \frac{\mathrm{d}x}{[\rho_{\mathrm{fc}}(E)]^n} \varrho^N_{H,n}\left(x + \frac{\alpha_1}{\rho_{\mathrm{fc}}(E)N}, \ldots, x + \frac{\alpha_n}{\rho_{\mathrm{fc}}(E)N} \right) \right. \\
&\qquad \left. - \frac{1}{[\rho_{\mathrm{sc}}(E')]^n} \varrho^N_{G,n}\left(E' + \frac{\alpha_1}{\rho_{\mathrm{sc}}(E')N}, \ldots, E' + \frac{\alpha_n}{\rho_{\mathrm{sc}}(E')N} \right) \right] \\
&\quad = 0,
\end{aligned} \tag{2.16}$$

where ρ_{fc} denotes the density of the deformed semicircle law and ρ_{sc} denotes the density of the standard semicircle law. Here, $\varrho_{G,n}^N$ denotes the n-point correlation function of the GUE in case W is a complex Hermitian Wigner matrix, respectively, the n-point correlation function of the GOE in case W is a real symmetric Wigner matrix.

REMARK 2.7. Theorem 2.5 and Theorem 2.6 show that the averaged local correlation functions of $H = V + W$ are universal in the limit of large N in the sense that they are independent of the diagonal matrix V and also independent of the precise distribution of the entries of W. Both theorems hold for real symmetric and complex Hermitian matrices. For the former choice, $\varrho_{G,n}^N$ stands for the n-point correlation functions of the GOE. For the latter choice, $\varrho_{G,n}^N$ stands for the n-point correlation functions of the GUE.

Note that we can choose b_N of order $N^{-1+\delta}$, $\delta > 0$, for deterministic V in Theorem 2.5, while we have to choose b_N of order $N^{-1/2+\delta}$, $\delta > 0$, for random V in Theorem 2.6. The latter condition is technical and not optimal. It is related to our next comment.

For random V with (v_i) i.i.d. bounded random variables, the eigenvalues of H fluctuate on scale $N^{-1/2}$ in the bulk [39]. Yet, under the assumptions of Theorem 2.6, the eigenvalue gaps remain rigid over small scales so that the universality of local correlation functions, a statement about the eigenvalue gaps, is unaffected by these mesoscopic fluctuations. We thus expect Theorem 2.6 to hold with $b_N \gg N^{-1}$. Relying on explicit integration formulas in the complex Hermitian setting, we suppose that the averaging over an energy window can be dropped; cf. the results for the deformed GUE in [51].

REMARK 2.8. The main ingredient of our proofs of Theorem 2.5 and Theorem 2.6 is an entropy estimate; see Proposition 5.3. Once such an estimate is obtained, the method in [31] also implies the single gap universality in the sense that the distribution of any single gap in the bulk is the same (up to a scaling) as the one from the corresponding Gaussian case. More precisely, fix $\alpha > 0$, and let $k \in \mathbb{N}$ be such that $\alpha N \le k \le (1-\alpha)N$. Let O be an n-particle observable. Then there are $\chi > 0$ and C such that

$$\begin{aligned}\big|\mathbb{E}^H O\big((N\rho_{\mathrm{fc},k})(\lambda_k - \lambda_{k+1}), (N\rho_{\mathrm{fc},k})(\lambda_k - \lambda_{k+2}), \dots, (N\rho_{\mathrm{fc},k})(\lambda_k - \lambda_{k+n})\big)\\ - \mathbb{E}^{\mu_G} O\big((N\rho_{\mathrm{sc},k})(\lambda_k - \lambda_{k+1}), (N\rho_{\mathrm{sc},k})(\lambda_k - \lambda_{k+2}), \dots,\\ (N\rho_{\mathrm{sc},k})(\lambda_k - \lambda_{k+n})\big)\big|\\ \le C N^{-\chi},\end{aligned}$$

for N sufficiently large, where μ_G is the standard GOE or GUE ensemble, depending on the symmetry class of H. Here $\rho_{\mathrm{fc},k}$ stands for the density of the measure

ρ_{fc} at the classical location, γ_k, of the kth eigenvalue defined through

$$\int_{-\infty}^{\gamma_k} \rho_{\mathrm{fc}}(x)\,\mathrm{d}x = \frac{k-1/2}{N}. \tag{2.17}$$

Similarly, $\rho_{\mathrm{sc},k}$ stands for the density of the standard semicircle law ρ_{sc} at the classical location of the kth eigenvalue of the Gaussian ensembles.

REMARK 2.9. To conclude, we mention two extensions of the above results. In Theorem 2.6 we may relax the assumption that (v_i) are independent among themselves: our results can be extended to dependent random variables provided that (v_i) satisfy (2.6), (2.11) and (2.12) for some constants $\alpha_0, \varpi, \mathrm{t} > 0$, and provided that (v_i) are independent of (w_{ij}). In such a setting the required lower bound on b_N depends on α_0.

The assumption that V is diagonal can be relaxed by assuming in turn that W belongs to the GUE/GOE. Then using the invariance of W, we can diagonalize V and apply our approach for diagonal potentials. For W a Wigner matrix and V a nondiagonal matrix, we expect that similar results hold by slowly changing W to a GUE/GOE. This, however, involves many more technical steps.

2.4. *Results on edge universality.* In this subsection, we show that our model also satisfies the edge universality. Edge universality states that the statistics of the extremal eigenvalues of many random matrix ensembles are universal: let λ_N denote the largest eigenvalue of a Wigner matrix W. The limiting distribution of λ_N was identified for the Gaussian ensembles by Tracy and Widom [57, 58]. They proved that

$$\lim_{N\to\infty} \mathbb{P}\big(N^{2/3}(\lambda_N - 2) \le s\big) = F_\beta(s) \qquad \big(\beta \in \{1,2,4\}\big), \tag{2.18}$$

$s \in \mathbb{R}$, where the Tracy–Widom distribution functions F_β are described by Painlevé equations. The edge universality can also be extended to the k largest eigenvalues, where the joint distribution of the k largest eigenvalues can be written in terms of the Airy kernel, as first shown for the GUE/GOE in [34]. These results also hold for the k smallest eigenvalues.

Edge universality for Wigner matrices was first proved in [54] (see also [53]) for real symmetric and complex Hermitian ensembles with symmetric distributions. The symmetry assumption on the entries' distribution was partially removed in [47, 48]. Edge universality was proved in [55] under the condition that the distribution of the matrix elements has subexponential decay, and its first three moments match those of the Gaussian distribution; that is, the third moment of the entries vanish. The vanishing third moment condition was removed in [33]. Finally, edge universality for generalized Wigner matrices was proved only recently in [12].

Edge universality for the deformed GUE was obtained for the special case when V has two eigenvalues $\pm a$, each with equal multiplicity, via a Riemann–Hilbert approach in [2, 8]. For general V, the joint distribution of the eigenvalues

of the deformed GUE can be expressed explicitly by the Brezin–Hikami/Johansson formula that may be used to prove the edge universality various choices and ranges of V; see [14, 36, 51].

Our result on the edge universality for real symmetric and complex Hermitian deformed Wigner matrices is as follows.

THEOREM 2.10. *Let W be a complex Hermitian or a real symmetric Wigner matrix satisfying the assumptions in Definition 2.1. Let V be either a random real diagonal matrix whose entries are i.i.d. random variables that are independent of W, or a deterministic real diagonal matrix. Assume that V satisfies Assumptions 2.2 and 2.3. Set $H = V + W$.*

Then there are $\varkappa > 0$, $\chi > 0$, $c_0 > 0$ such that the following result holds for any fixed $n \in \mathbb{N}$. For any n-particle observable O and for $\Lambda \subset [\![1, N^{\varkappa}]\!]$, respectively, $\Lambda \subset [\![N - N^{\varkappa}, N]\!]$, with $|\Lambda| = n$, we have

$$\begin{aligned} &\big|\mathbb{E}^H O\big(\big(c_0 N^{2/3} j^{1/3}(\lambda_j - \widehat{\gamma}_j)\big)_{j\in\Lambda}\big) - \mathbb{E}^{\mu_G} O\big(\big(N^{2/3} j^{1/3}(\lambda_j - \gamma_j)\big)_{j\in\Lambda}\big)\big| \\ &\qquad \le C_O N^{-\chi}, \end{aligned} \tag{2.19}$$

for N sufficiently large, for some constant C_O (depending on O), where μ_G is the standard GUE/GOE, depending on the symmetry class of W. Here, the constant c_0 is a scaling factor so that the eigenvalue density at the edge of H can be compared with the Gaussian case. It only depends on ν. Further, $\widehat{\gamma}_j$, γ_j denote here the classical locations of the jth eigenvalue with respect to the measure $\widehat{\rho}_{\mathrm{fc}}$ introduced in (3.8) below, respectively, with respect to the standard semicircle law ρ_{sc}.

Theorem 2.10 shows that the local statistics of the k largest, respectively, smallest, eigenvalues of our model are given by the Tracy–Widom–Airy statistics.

The measure $\widehat{\varrho}_{\mathrm{fc}}$ depends solely on the empirical eigenvalue distribution, $\widehat{\nu}$, of V, and so do the classical locations $(\widehat{\gamma}_k)$. The scaling factor c_0 in (2.19) may be computed explicitly [51].

Theorem 2.10 is proved in a similar way to Theorems 2.5 and 2.6. Using the Dirichlet form bound obtained in Proposition 5.3 below, we invoke the edge universality result for localized β-ensembles, Theorem 3.3 of [12], and follow the same strategy as for the bulk universality. The proof of Theorem 2.10 is given in Section 9.

To conclude, we mention that Theorem 2.10 has recently been proved in [41] using a completely different approach based on the Green function comparison theorem; see, for example, [32] for earlier ideas of using the Green function comparison for edge universality.

2.5. *Notation and conventions.* In this subsection, we introduce some more notation and conventions used throughout the paper. For high probability estimates we use two parameters $\xi \equiv \xi_N$ and $\varphi \equiv \varphi_N$: we let

$$a_0 < \xi \le A_0 \log\log N, \qquad \varphi = (\log N)^{C_1}, \tag{2.20}$$

for some constants $a_0 > 2$, $A_0 \geq 10$, $C_1 > 1$.

DEFINITION 2.11. We say an event Ξ has (ξ, υ)-high probability if

$$\mathbb{P}(\Xi^c) \leq \mathrm{e}^{-\upsilon(\log N)^{\xi}} \qquad (\upsilon > 0),$$

for N sufficiently large. We say an event Ξ has ς-exponentially high probability if

$$\mathbb{P}(\Xi^c) \leq \mathrm{e}^{-N^{\varsigma}} \qquad (\varsigma > 0),$$

for N sufficiently large. Similarly, for a given event Ξ_0 we say an event Ξ holds with (ξ, υ)-high probability, respectively, ς-exponentially high probability, on Ξ_0, if

$$\mathbb{P}(\Xi^c \cap \Xi_0) \leq \mathrm{e}^{-\upsilon(\log N)^{\xi}} \qquad (\upsilon > 0), \qquad \mathbb{P}(\Xi^c \cap \Xi_0) \leq \mathrm{e}^{-N^{\varsigma}} \qquad (\varsigma > 0),$$

respectively, for N sufficiently large.

For brevity, we occasionally say an event holds with exponentially high probability, when we mean ς-exponentially high probability. We do not keep track of the explicit value of υ or ς in the following, allowing υ and ς to decrease from line to line such that $\upsilon, \varsigma > 0$.

We use the symbols $\mathcal{O}(\cdot)$ and $o(\cdot)$ for the standard big-O and little-o notation. The notation $\mathcal{O}, o, \ll, \gg$, refers to the limit $N \to \infty$, if not indicated otherwise. Here $a \ll b$ means $a = o(b)$. We use c and C to denote positive constants that do not depend on N. Their value may change from line to line. We write $a \sim b$ if there is $C \geq 1$ such that $C^{-1}|b| \leq |a| \leq C|b|$, and occasionally we write for N-dependent quantities $a_N \lesssim b_N$ if there exist constants $C, c > 0$ such that $|a_N| \leq C(\varphi_N)^{c\xi}|b_N|$.

Finally, we abbreviate

$$\sum_{j}^{(i)}(\cdot) \equiv \sum_{\substack{j=1 \\ j \neq i}}^{N}(\cdot),$$

and we use double brackets to denote index sets, that is,

$$[\![n_1, n_2]\!] := [n_1, n_2] \cap \mathbb{Z},$$

for $n_1, n_2 \in \mathbb{R}$.

3. Local law and rigidity estimates. Recall the constant $\varpi > 0$ in Assumption 2.3. Set $\varpi' := \varpi/10$. In this section we consider the family of interpolating random matrices

$$H^{\vartheta} := \vartheta V + W, \qquad \vartheta \in \Theta_{\varpi} := [0, 1 + \varpi'], \tag{3.1}$$

where V and W are chosen to satisfy Assumptions 2.2 and 2.3, respectively, the assumptions in Definition 2.1. Here ϑ has the interpretation of a possibly N-dependent positive "coupling parameter."

We define the resolvent or *Green function*, $G^{\vartheta}(z)$, and the averaged Green function, $m^{\vartheta}(z)$, of H^{ϑ} by

$$G^{\vartheta}(z) = \big(G^{\vartheta}_{ij}(z)\big) := \frac{1}{\vartheta V + W - z}, \qquad m^{\vartheta}_N(z) := \frac{1}{N}\operatorname{Tr} G^{\vartheta}(z), \tag{3.2}$$

$z \in \mathbb{C}^+$. Frequently, we abbreviate $G^{\vartheta} \equiv G^{\vartheta}(z)$, $m^{\vartheta}_N \equiv m^{\vartheta}_N(z)$, etc.

To conveniently cope with the cases when (v_i) are random, respectively, deterministic, we introduce an event Ω on which the random variables (v_i) exhibit "typical" behavior. Recall that we denote by $m_{\widehat{\nu}}$ and m_{ν} the Stieltjes transforms of $\widehat{\nu}$, respectively, ν.

DEFINITION 3.1. Let $\Omega \equiv \Omega(N)$ be an event on which the following holds:

(1) There is a constant $\alpha_0 > 0$ such that, for any fixed compact set $\mathcal{D} \subset \mathbb{C}^+$ (independent of N) with $\operatorname{dist}(\mathcal{D}, \operatorname{supp}\nu) > 0$, there is C such that

$$\big|m_{\widehat{\nu}}(z) - m_{\nu}(z)\big| \le CN^{-\alpha_0}, \tag{3.3}$$

for N sufficiently large.

(2) Recall the constant $\varpi > 0$ in Assumption 2.3. We have

$$\inf_{x \in I_{\widehat{\nu}}} \int \frac{\mathrm{d}\widehat{\nu}(v)}{(v-x)^2} \ge 1 + \varpi, \qquad \inf_{x \in I_{\nu}} \int \frac{\mathrm{d}\nu}{(v-x)^2} \ge 1 + \varpi, \tag{3.4}$$

for N sufficiently large.

In case (v_i) are deterministic, Ω has full probability for N sufficiently large by the Assumptions in 2.2.

Similar to the definition of m_{fc}, we define $m^{\vartheta}_{\mathrm{fc}}$ and $\widehat{m}^{\vartheta}_{\mathrm{fc}}$ as the solutions to the equations

$$m^{\vartheta}_{\mathrm{fc}}(z) = \int \frac{\mathrm{d}\nu(v)}{\vartheta v - z - m^{\vartheta}_{\mathrm{fc}}(z)}, \qquad \operatorname{Im} m^{\vartheta}_{\mathrm{fc}}(z) \ge 0 \qquad \big(z \in \mathbb{C}^+\big) \tag{3.5}$$

and

$$\widehat{m}^{\vartheta}_{\mathrm{fc}}(z) = \int \frac{\mathrm{d}\widehat{\nu}(v)}{\vartheta v - z - \widehat{m}^{\vartheta}_{\mathrm{fc}}(z)}, \qquad \operatorname{Im} \widehat{m}^{\vartheta}_{\mathrm{fc}}(z) \ge 0, \qquad \big(z \in \mathbb{C}^+\big), \tag{3.6}$$

respectively. Following the discussion of Section 2.2, $m^{\vartheta}_{\mathrm{fc}}$ and $\widehat{m}^{\vartheta}_{\mathrm{fc}}$ define two probability measures $\rho^{\vartheta}_{\mathrm{fc}}$ and $\widehat{\rho}^{\vartheta}_{\mathrm{fc}}$ through the densities

$$\rho^{\vartheta}_{\mathrm{fc}}(E) := \lim_{\eta \searrow 0} \frac{1}{\pi} \operatorname{Im} m^{\vartheta}_{\mathrm{fc}}(E + \mathrm{i}\eta) \qquad (E \in \mathbb{R}) \tag{3.7}$$

and

$$\widehat{\rho}^{\vartheta}_{\mathrm{fc}}(E) := \lim_{\eta \searrow 0} \frac{1}{\pi} \operatorname{Im} \widehat{m}^{\vartheta}_{\mathrm{fc}}(E + \mathrm{i}\eta) \qquad (E \in \mathbb{R}); \tag{3.8}$$

cf. (2.7). More precisely, we have the following result which follows directly from the proofs of Lemmas 3.5 and 3.6 below. Recall the definition of Θ_{ϖ} in (3.1).

LEMMA 3.2. *Let $\widehat{\nu}$ and ν satisfy the Assumptions 2.2 and 2.3. Then, for any $\vartheta \in \Theta_{\varpi}$ and $N \in \mathbb{N}$, equations (3.5) and (3.6) define, through the inversion formulas in (3.7) and (3.8), absolutely continuous measures $\rho^{\vartheta}_{\mathrm{fc}}$ and $\widehat{\rho}^{\vartheta}_{\mathrm{fc}}$. Moreover, the measure $\rho^{\vartheta}_{\mathrm{fc}}$ is supported on a single interval with strictly positive density inside this interval. The same holds true on Ω for the measures $\widehat{\rho}^{\vartheta}_{\mathrm{fc}}$, for N sufficiently large.*

Note that if (v_i) are random, then so are $\widehat{m}^{\vartheta}_{\mathrm{fc}}$, respectively, $\widehat{\rho}^{\vartheta}_{\mathrm{fc}}$. As noted above, we use the symbol $\widehat{\ }$ to denote quantities that depend on the empirical distribution $\widehat{\nu}$ of the (v_i), while we drop this symbol for quantities depending on the limiting distribution ν of (v_i).

We denote by $\widehat{L}^{\vartheta}_{\pm}$, respectively, $L^{\vartheta}_{\pm}$, the endpoints of the support of $\widehat{\rho}^{\vartheta}_{\mathrm{fc}}$, respectively, $\rho^{\vartheta}_{\mathrm{fc}}$. Let $E_0 \geq 1 + \max\{|L^1_-|, L^1_+\}$, and define the domain

$$\mathcal{D}_L := \big\{ z = E + \mathrm{i}\eta \in \mathbb{C} : |E| \leq E_0, (\varphi_N)^L \leq N\eta \leq 3N \big\}, \tag{3.9}$$

with $L \equiv L(N)$, such that $L \geq 12\xi$; see (2.20).

The following theorem is the main result of this section.

THEOREM 3.3 (Strong local deformed semicircle law). *Let $H^{\vartheta} = \vartheta V + W$, $\vartheta \in \Theta_{\varpi}$ [see (3.1)], where W is a real symmetric or complex Hermitian Wigner matrix satisfying the assumptions in Definition 2.1 and V is a deterministic or random real diagonal matrix satisfying Assumptions 2.2 and 2.3. Let*

$$\xi = \frac{A_0 + o(1)}{2} \log\log N. \tag{3.10}$$

Then there are constants $\upsilon > 0$ and c_1, depending on the constants E_0 in (3.9), α_0 in (3.3), A_0, a_0, C_1 in (2.20), θ, C_0 in (2.3) and the measure $\widehat{\nu}$ such that the following holds for $L \geq 40\xi$. For any $z \in \mathcal{D}_L$ and any $\vartheta \in \Theta_{\varpi}$, we have

$$\big| m^{\vartheta}_N(z) - \widehat{m}^{\vartheta}_{\mathrm{fc}}(z) \big| \leq (\varphi_N)^{c_1\xi} \frac{1}{N\eta}, \tag{3.11}$$

with (ξ, υ)-high probability on Ω.

Moreover, we have, for any $z \in \mathcal{D}_L$, any $\vartheta \in \Theta_{\varpi}$ and any $i, j \in [\![1, N]\!]$,

$$\big| G^{\vartheta}_{ij}(z) - \delta_{ij} \widehat{g}^{\vartheta}_i(z) \big| \leq (\varphi_N)^{c_1\xi} \bigg(\sqrt{\frac{\operatorname{Im} \widehat{m}^{\vartheta}_{\mathrm{fc}}(z)}{N\eta}} + \frac{1}{N\eta} \bigg), \tag{3.12}$$

with (ξ, υ)-high probability on Ω, where we have set

$$\widehat{g}_i^{\vartheta}(z) := \frac{1}{\vartheta v_i - z - \widehat{m}_{\mathrm{fc}}^{\vartheta}(z)}. \tag{3.13}$$

The study of local laws for Wigner matrices was initiated in [25–27]. For more recent results, we refer to [23]. For deformed Wigner matrices with random potential, a local law was obtained in [39].

Denote by $\boldsymbol{\lambda}^{\vartheta} = (\lambda_1^{\vartheta}, \lambda_2^{\vartheta}, \dots, \lambda_N^{\vartheta})$ the eigenvalues of the random matrix $H^{\vartheta} = \vartheta V + W$ arranged in ascending order. We define the classical location, $\widehat{\gamma}_i^{\vartheta}$, of the eigenvalue λ_i^{ϑ} by

$$\int_{-\infty}^{\widehat{\gamma}_i^{\vartheta}} \widehat{\rho}_{\mathrm{fc}}^{\vartheta}(x)\,\mathrm{d}x = \frac{i - (1/2)}{N} \qquad (1 \le i \le N). \tag{3.14}$$

Note that $(\widehat{\gamma}_i^{\vartheta})$ are random in case (v_i) are too. We have the following rigidity result on the eigenvalue locations of H^{ϑ}:

COROLLARY 3.4. *Let $H^{\vartheta} = \vartheta V + W$, $\vartheta \in \Theta_{\varpi}$, where W is a real symmetric or complex Hermitian Wigner matrix satisfying the assumptions in Definition 2.1, and V is a deterministic or random real diagonal matrix satisfying Assumptions 2.2 and 2.3. Let ξ satisfy (3.10). Then there are constants $\upsilon > 0$ and c_1, c_2, depending on the constants E_0 in (3.9), α_0 in (3.3), A_0, a_0, C_1 in (2.20), θ, C_0 in (2.3) and the measure $\widehat{\nu}$, such that*

$$|\lambda_i^{\vartheta} - \widehat{\gamma}_i^{\vartheta}| \le (\varphi_N)^{c_1\xi} \frac{1}{N^{2/3}\check{\alpha}_i^{1/3}} \qquad (1 \le i \le N), \tag{3.15}$$

$$\sum_{i=1}^{N} |\lambda_i^{\vartheta} - \widehat{\gamma}_i^{\vartheta}|^2 \le (\varphi_N)^{c_2\xi} \frac{1}{N}, \tag{3.16}$$

with (ξ, υ)-high probability on Ω, for all $\vartheta \in \Theta_{\varpi}$, where we have abbreviated $\check{\alpha}_i := \min\{i, N - i + 1\}$.

In the rest of this section we sum up the proofs of Theorem 3.3 and Corollary 3.4.

3.1. *Properties of $m_{\mathrm{fc}}^{\vartheta}$ and $\widehat{m}_{\mathrm{fc}}^{\vartheta}$.* In this subsection, we discuss properties of the Stieltjes transforms $m_{\mathrm{fc}}^{\vartheta}$ and $\widehat{m}_{\mathrm{fc}}^{\vartheta}$. We first derive the desired properties for $m_{\mathrm{fc}}^{\vartheta}$ (Lemma 3.5 and Corollary A.2 in the Appendix) and then show in a second step that $m_{\mathrm{fc}}^{\vartheta}$ is a good approximation to $\widehat{m}_{\mathrm{fc}}^{\vartheta}$ so that $\widehat{m}_{\mathrm{fc}}^{\vartheta}$ also shares these properties; see Lemma 3.6.

For E_0 as in (3.17), we define the domain, $\mathcal{D}'$, of the spectral parameter z by

$$\mathcal{D}' := \{z = E + \mathrm{i}\eta : E \in [-E_0, E_0], \eta \in (0, 3]\}. \tag{3.17}$$

The next lemma, whose proof is postponed to the Appendix, gives a qualitative description of the deformed semicircle law $\rho_{\mathrm{fc}}^{\vartheta}$ and its Stieltjes transform $m_{\mathrm{fc}}^{\vartheta}$.

LEMMA 3.5. *Let ν satisfy Assumption 2.3, for some $\varpi > 0$. Then the following holds true for any $\vartheta \in \Theta_{\varpi}$. There are $L_-^{\vartheta}, L_+^{\vartheta} \in \mathbb{R}$, with $L_-^{\vartheta} < 0 < L_+^{\vartheta}$, such that* $\operatorname{supp} \rho_{\mathrm{fc}}^{\vartheta} = [L_-^{\vartheta}, L_+^{\vartheta}]$, *and there exists a constant $C > 1$ such that, for all $\vartheta \in \Theta_{\varpi}$,*

$$C^{-1}\sqrt{\kappa_E} \le \rho_{\mathrm{fc}}^{\vartheta}(E) \le C\sqrt{\kappa_E} \qquad \big(E \in [L_-^{\vartheta}, L_+^{\vartheta}]\big), \tag{3.18}$$

where κ_E denotes the distance of E to the endpoints of the support of $\rho_{\mathrm{fc}}^{\vartheta}$, that is,

$$\kappa_E := \min\{|E - L_-^{\vartheta}|, |E - L_+^{\vartheta}|\}. \tag{3.19}$$

The Stieltjes transform, $m_{\mathrm{fc}}^{\vartheta}$, of $\rho_{\mathrm{fc}}^{\vartheta}$ has the following properties:

(1) *for all $z = E + \mathrm{i}\eta \in \mathcal{D}'$,*

$$\operatorname{Im} m_{\mathrm{fc}}^{\vartheta}(z) \sim \begin{cases} \sqrt{\kappa + \eta}, & E \in [L_-^{\vartheta}, L_+^{\vartheta}], \\ \dfrac{\eta}{\sqrt{\kappa + \eta}}, & E \in [L_-^{\vartheta}, L_+^{\vartheta}]^c; \end{cases} \tag{3.20}$$

(2) *there exists a constant $C > 1$ such that for all $z \in \mathcal{D}'$ and all $x \in I_{\nu}$,*

$$C^{-1} \le |\vartheta x - z - m_{\mathrm{fc}}^{\vartheta}(z)| \le C. \tag{3.21}$$

Moreover, the constants in (3.18), (3.20) *and* (3.21) *can be chosen uniformly in $\vartheta \in \Theta_{\varpi}$.*

Next, we argue that $\widehat{m}_{\mathrm{fc}}^{\vartheta}$ behaves qualitatively in the same way as $m_{\mathrm{fc}}^{\vartheta}$ on Ω for N sufficiently large. Lemma 3.6 below is proven in the Appendix.

LEMMA 3.6. *Let $\widehat{\nu}$ satisfy Assumptions 2.2 and 2.3, for some $\varpi > 0$. Then the following holds for all $\vartheta \in \Theta_{\varpi}$ and all sufficiently large N on Ω. There are $\widehat{L}_-^{\vartheta}, \widehat{L}_+^{\vartheta} \in \mathbb{R}$, with $\widehat{L}_-^{\vartheta} < 0 < \widehat{L}_+^{\vartheta}$, such that* $\operatorname{supp} \widehat{\rho}_{\mathrm{fc}}^{\vartheta} = [\widehat{L}_-^{\vartheta}, \widehat{L}_+^{\vartheta}]$. *Let $\widehat{\kappa}_E := \min\{|E - \widehat{L}_-^{\vartheta}|, |E - \widehat{L}_+^{\vartheta}|\}$. Then* (3.18), (3.20) *and* (3.21) *of Lemma 3.5, hold true on Ω, for N sufficiently large, with $m_{\mathrm{fc}}^{\vartheta}$ replaced by $\widehat{m}_{\mathrm{fc}}^{\vartheta}$, $\rho_{\mathrm{fc}}^{\vartheta}$ replaced by $\widehat{\rho}_{\mathrm{fc}}^{\vartheta}$, etc. Moreover, the constants in these inequalities can be chosen uniformly in $\vartheta \in \Theta_{\varpi}$ and N, for N sufficiently large.*

Further, there is $c > 0$ such that for all $z \in \mathcal{D}'$ we have

$$|\widehat{m}_{\mathrm{fc}}^{\vartheta}(z) - m_{\mathrm{fc}}^{\vartheta}(z)| \le N^{-c\alpha_0/2} \qquad |\widehat{L}_{\pm}^{\vartheta} - L_{\pm}^{\vartheta}| \le N^{-c\alpha_0}, \tag{3.22}$$

on Ω for N sufficiently large and all $\vartheta \in \Theta_{\varpi}$.

3.2. *Proof of Theorem 3.3 and Corollary 3.4.* The proof of Theorem 3.3 follows closely the proof of Theorem 2.10 in [39]. The difference between Theorem 3.3 of the present paper and Theorem 2.10 in [39] is that we presently condition on the diagonal entries (v_i); that is, we consider the entries of V as fixed. Accordingly, we compare [on the event Ω of typical (v_i)] the averaged Green function m^{ϑ} with $\widehat{m}^{\vartheta}_{\mathrm{fc}}$ [see (3.6)] instead of $m^{\vartheta}_{\mathrm{fc}}$; see (3.5). For consistency, we momentarily drop the ϑ dependence form our notation. To establish Theorem 3.3, we first derive a weak local deformed semicircle law (see Theorem 4.1 in [39]) by following the proof in [39]. Using the Lemma 3.5, Lemma 3.6 and the results in the Appendix, it is then straightforward to obtain the following result.

LEMMA 3.7. *Under the assumption of Theorem 3.3, there are c_1 and $\upsilon > 0$ such that*

$$\bigl|m_N(z) - \widehat{m}_{\mathrm{fc}}(z)\bigr| \le (\varphi_N)^{c_1\xi}\frac{1}{(N\eta)^{1/3}}, \qquad \bigl|G_{ij}(z)\bigr| \le (\varphi_N)^{c_1\xi}\frac{1}{\sqrt{N\eta}},$$

with (ξ, υ)-high probability on Ω, uniformly in $z \in \mathcal{D}_L$ and $\vartheta \in \Theta_{\varpi}$.

To prove Theorem 3.3 we follow mutatis mutandis the proof of Theorem 4.1 in [39]. But we note that in the corresponding equation to (5.25) in [39], we may set $\lambda = 0$ in the error term, at the cost of replacing m_{fc} by $\widehat{m}_{\mathrm{fc}}$. In the subsequent analysis, we can simply set $\lambda = 0$ in the error terms. In this way, one establishes the proof of Theorem 3.3. Similarly, Corollary 3.4 can be proven in the same way as is Theorem 2.21 in [39]. It suffices to set $\lambda = 0$ in the analysis in [39]. We leave the details aside.

4. Reference β-ensemble.

4.1. *Definition of β-ensemble and known results.* We first recall the notion of β-ensembles. Let $N \in \mathbb{N}$, and let $F^{(N)} \subset \mathbb{R}^N$ denote the set

$$F^{(N)} := \bigl\{\mathbf{x} = (x_1, x_2, \ldots, x_N) : x_1 \le x_2 \le \cdots \le x_N\bigr\}. \tag{4.1}$$

Consider the probability distribution, $\mu_U \equiv \mu_U^N$, on $F^{(N)}$ given by

$$\mu_U^N(\mathrm{d}\mathbf{x}) := \frac{1}{Z_U^N}\mathrm{e}^{-\beta N\mathcal{H}(\mathbf{x})}\,\mathrm{d}\mathbf{x}, \qquad \mathrm{d}\mathbf{x} := \mathbb{1}\bigl(\mathbf{x} \in F^{(N)}\bigr)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_N, \tag{4.2}$$

where $\beta > 0$,

$$\mathcal{H}(\mathbf{x}) := \sum_{i=1}^{N}\frac{1}{2}\biggl(U(x_i) + \frac{x_i^2}{2}\biggr) - \frac{1}{N}\sum_{1\le i<j\le N}\log(x_j - x_i) \tag{4.3}$$

and $Z_U^N \equiv Z_U^N(\beta)$ is a normalization. Here U is a potential, that is, a real-valued, sufficiently regular function on $\mathbb{R}$. In the following, we often omit the parameters

N and β from the notation. We use $\mathbb{P}^{\mu_U}$ and $\mathbb{E}^{\mu_U}$ to denote the probability and the expectation with respect to μ_U. We view μ_U as a Gibbs measure of N particles on $\mathbb{R}$ with a logarithmic interaction, where the parameter $\beta > 0$ may be interpreted as the inverse temperature. (For the results in the present paper, we choose $\beta = 2$ in case W is complex Hermitian Wigner matrix and $\beta = 1$ in case W is a real symmetric Wigner matrix.) We refer to the variables (x_i) as particles or points, and we call the system a log-gas or a β-ensemble. We assume that the potential U is a C^4 function on $\mathbb{R}$ such that its second derivative is bounded below; that is, we have

$$\inf_{x\in\mathbb{R}} U''(x) \geq -2C_U, \tag{4.4}$$

for some constant $C_U \geq 0$, and we further assume that

$$U(x) + \frac{x^2}{2} > (2+\varepsilon)\log\bigl(1+|x|\bigr) \qquad (x \in \mathbb{R}), \tag{4.5}$$

for some $\varepsilon > 0$, for large enough $|x|$. It is well known (see, e.g., [13]) that under these conditions the measure is normalizable, $Z_U^N < \infty$. Moreover, the averaged density of the empirical spectral measure, ρ_U^N, defined as

$$\rho_U^N := \mathbb{E}^{\mu_U} \frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \tag{4.6}$$

converges weakly in the limit $N \to \infty$ to a continuous function ρ_U, the equilibrium density, which is of compact support. It is well known that ρ_U can be obtained as the unique solution to the variational problem

$$\inf\biggl\{\int_{\mathbb{R}}\biggl(\frac{x^2}{2}+U(x)\biggr)\,\mathrm{d}\rho(x) - \int_{\mathbb{R}}\log|x-y|\,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y) : \rho \text{ is a probability measure}\biggr\} \tag{4.7}$$

and that the equilibrium density $\rho = \rho_U$ satisfies

$$U'(x) + x = -2\fint_{\mathbb{R}} \frac{\rho(y)\,\mathrm{d}y}{y-x} \qquad (x \in \operatorname{supp}\rho_U). \tag{4.8}$$

In fact, (4.8) holds if and only if $x \in \operatorname{supp}\rho_U$. We will assume in addition that the minimizer ρ_U is supported on a single interval $[A_-, A_+]$ and that U is "regular" in the sense of [38]; that is, the equilibrium density of U is positive on (A_-, A_+) and vanishes like a square root at each of the endpoints of $[A_-, A_+]$. Viewing the points $\mathbf{x} = (x_i)$ as points or particles on $\mathbb{R}$, we define the classical location of the kth particle, γ_k, under the β-ensemble μ_U by

$$\int_{-\infty}^{\gamma_k} \rho_U(x)\,\mathrm{d}x = \frac{k-(1/2)}{N}. \tag{4.9}$$

For a detailed discussion of general β-ensemble we refer, for example, to [1, 12].

For $U \equiv 0$, we write $\mu_G \equiv \mu_G^N$ instead of μ_0, since μ_0 is the equilibrium measure for the GUE ($\beta = 2$), respectively, the GOE ($\beta = 1$). More precisely, setting

$$\mathcal{H}_G(\mathbf{x}) := \sum_{i=1}^{N} \frac{1}{4} x_i^2 - \frac{1}{N} \sum_{1 \le i < j \le N} \log(x_j - x_i), \tag{4.10}$$

the GUE, respectively, GOE, distribution on $F^{(N)}$ are given by

$$\mu_G^N(\mathrm{d}\mathbf{x}) = \frac{1}{Z_G^N} \mathrm{e}^{-\beta N \mathcal{H}_G(\mathbf{x})} \, \mathrm{d}\mathbf{x}, \tag{4.11}$$

where $Z_G^N \equiv Z_G^N(\beta)$ is a normalization, and we either choose $\beta = 2$ or $\beta = 1$.

We are interested in the n-point correlation functions defined by

$$\varrho_{U,n}^N(x_1, \ldots, x_n) = \int_{\mathbb{R}^{N-n}} \mu_U^{\#}(\mathbf{x}) \, \mathrm{d}x_{n+1} \cdots \mathrm{d}x_N, \tag{4.12}$$

where $\mu_U^{\#}$ is the symmetrized version of μ_U given in (4.2) but defined on $\mathbb{R}^N$ instead of the simplex $F^{(N)}$,

$$\mu_U^{\#}(\mathrm{d}x) = \frac{1}{N!} \mu_U\big(\mathrm{d}\mathbf{x}^{(\sigma)}\big), \qquad \mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_N, \tag{4.13}$$

where $\mathbf{x}^{(\sigma)} = (x_{\sigma(1)}, \ldots, x_{\sigma(N)})$, with $x_{\sigma(1)} < \cdots < x_{\sigma(N)}$. The following universality result is proven in [12].

THEOREM 4.1 (Bulk universality for β-ensembles, Theorem 2.1 in [12]). *Let U be a C^4 regular potential with equilibrium density supported on a single interval $[A_-, A_+]$ that satisfies (4.4) and (4.5). Then the following result holds. For any fixed $\beta > 0$, $E \in (A_-, A_+)$, $|E'| < 2$, $n \in \mathbb{N}$, $0 < \delta \le \frac{1}{2}$ and any n-particle observable O, we have with $b := N^{-1+\delta}$,*

$$\begin{aligned} \lim_{N \to \infty} \int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \, O(\alpha_1, \ldots, \alpha_n) \\ \times \Bigg[\int_{E-b}^{E+b} \frac{\mathrm{d}x}{2b} \frac{1}{[\rho_U(E)]^n} \varrho_{U,n}^N\bigg(x + \frac{\alpha_1}{N\rho_U(E)}, \ldots, x + \frac{\alpha_n}{N\rho_U(E)}\bigg) \\ - \frac{1}{[\rho_{\mathrm{sc}}(E')]^n} \varrho_{G,n}^N\bigg(E' + \frac{\alpha_1}{N\rho_{\mathrm{sc}}(E')}, \ldots, E' + \frac{\alpha_n}{N\rho_{\mathrm{sc}}(E')}\bigg) \Bigg] = 0. \end{aligned}$$

Here, ρ_{sc} denotes the density of the semicircle law, and $\varrho_{G,n}^N$ is the n-point the correlation function of the Gaussian β-ensemble, that is, with $U \equiv 0$.

Theorem 4.1 was first proved in [11] under the assumption that U is analytic, a hypothesis that was only required for proving rigidity. The analyticity assumption has been removed in [12]. Recently, alternative proofs of bulk universality for

β-ensembles with general $\beta > 0$, that is, results similar to Theorem 4.1, have been obtained in [50] and [4]. In the present paper, we will not use Theorem 4.1; it is stated here for completeness.

To conclude this subsection, we recall an important tool in the study of β-ensembles, the "first order loop" equation. In the notation above it reads (in the limit $N \to \infty$)

$$m_U(z)^2 = \int \frac{x + U'(x)}{x - z} \rho_U(x)\,\mathrm{d}x \qquad (z \in \mathbb{C}^+), \tag{4.14}$$

where m_U denotes the Stieltjes transform of the equilibrium measure ρ_U, that is,

$$m_U(z) \equiv m_{\rho_U}(z) = \int \frac{\rho_U(x)}{x - z}\,\mathrm{d}x \qquad (z \in \mathbb{C}^+).$$

The loop equation (4.14) can be obtained by a change of variables in (4.2) (see [35]) or by integration by parts; see [49].

4.2. *Time-dependent modified β-ensemble.* In this subsection, we introduce a modified β-ensemble by specifying potentials $\widehat{U}$ and U that depend, among other things, on a parameter $t \geq 0$ which has the interpretation of a time. The potential $\widehat{U}$ also depends on N, the size of our original matrix $H = V + W$, yet the N dependence is only through the fixed random variables (v_i). Recall that we have defined $\widehat{m}^{\vartheta}_{\mathrm{fc}}$, respectively, $m^{\vartheta}_{\mathrm{fc}}$, as the solutions to the equations

$$\widehat{m}^{\vartheta}_{\mathrm{fc}}(z) = \int \frac{\mathrm{d}\widehat{\nu}(v)}{\vartheta v_i - z - \widehat{m}^{\vartheta}_{\mathrm{fc}}(z)}, \qquad m^{\vartheta}_{\mathrm{fc}}(z) = \int \frac{\mathrm{d}\nu(v)}{\vartheta v - z - m^{\vartheta}_{\mathrm{fc}}(z)}, \tag{4.15}$$

$z \in \mathbb{C}^+$, subject to the conditions $\operatorname{Im}\widehat{m}^{\vartheta}_{\mathrm{fc}}(z), \operatorname{Im} m^{\vartheta}_{\mathrm{fc}}(z) \geq 0$, for $\operatorname{Im} z > 0$. Recall from (3.1) that we denote $\Theta_{\varpi} = [0, 1 + \varpi']$, $\varpi' = \varpi/10$. We then fix some $t_0 \geq 0$ such that $\mathrm{e}^{t_0/2} \in \Theta_{\varpi}$ and let

$$\vartheta \equiv \vartheta(t) := \mathrm{e}^{-(t - t_0)/2} \qquad (t \geq 0). \tag{4.16}$$

In the following we consider $t \geq 0$ as time, and we henceforth abbreviate $m^{\vartheta(t)}_{\mathrm{fc}}(z) \equiv m_{\mathrm{fc}}(t, z)$, etc. Equation (4.15) defines time dependent measures $\widehat{\rho}_{\mathrm{fc}}(t)$, $\rho_{\mathrm{fc}}(t)$, respectively, whose densities at the point $x \in \mathbb{R}$ are denoted by $\widehat{\rho}_{\mathrm{fc}}(t, x)$, respectively, $\rho_{\mathrm{fc}}(t, x)$.

We denote by $\widehat{U}'(t, x)$, $\widehat{U}^{(n)}(t, x)$ the first, respectively, the nth derivative of $\widehat{U}(t, x)$ with respect to x, and we use the same notation for U. We define $\widehat{U}$ and U (up to finite additive constants that enter the formalism only in normalizations) through their derivatives $\widehat{U}'$ and U'. For $t \geq 0$, we set

$$\widehat{U}'(t, x) + x := -2 \fint_{\mathbb{R}} \frac{\widehat{\rho}_{\mathrm{fc}}(t, y)}{y - x}\,\mathrm{d}y, \tag{4.17}$$

for $x \in \operatorname{supp} \widehat{\rho}_{\mathrm{fc}}(t)$, respectively,

$$U'(t, x) + x := -2 \fint_{\mathbb{R}} \frac{\rho_{\mathrm{fc}}(t, y)}{y - x}\,\mathrm{d}y, \tag{4.18}$$

for $x \in \operatorname{supp} \rho_{\mathrm{fc}}(t)$. Outside the support of the measures $\widehat{\rho}_{\mathrm{fc}}(t)$ and $\rho_{\mathrm{fc}}(t)$, we define $\widehat{U}'$ and U' as C^3 extensions such that they are "regular" potentials satisfying (4.4) and (4.5) for all $t \geq 0$. The definitions of such potentials are obviously not unique. One possible construction is outlined in the Appendix in the form of the proof of the next lemma.

LEMMA 4.2. *There exist potentials* $\widehat{U}, U : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto \widehat{U}(t,x)$, $U(t,x)$ *such that for* $n \in [\![1,4]\!]$, $\widehat{U}^{(n)}(t,x)$, $U^{(n)}(t,x)$, $\partial_t \widehat{U}^{(n)}(t,x)$, $\partial_t U^{(n)}(t,x)$ *are continuous functions of* $x \in \mathbb{R}$ *and* $t \in \mathbb{R}^+$, *which can be uniformly bounded in* x *on compact sets, uniformly in* $t \in \mathbb{R}^+$ *and sufficiently large* N. *Moreover the following holds for all* $t \geq 0$ *on* Ω *for* N *sufficiently large*:

(1) $\widehat{U}'(t,x)$ *and* $U'(t,x)$ *satisfy* (4.17) *and* (4.18) *for* $x \in \operatorname{supp} \widehat{\rho}_{\mathrm{fc}}(t)$, *respectively,* $x \in \operatorname{supp} \rho_{\mathrm{fc}}(t)$. *For* $x \notin \operatorname{supp} \widehat{\rho}_{\mathrm{fc}}(t)$, *respectively,* $x \notin \operatorname{supp} \rho_{\mathrm{fc}}(t)$, *we have*

$$\big|\widehat{U}'(t,x) + x\big| > 2\big|\operatorname{Re} \widehat{m}_{\mathrm{fc}}(t,x)\big|, \qquad \big|U'(t,x) + x\big| > 2\big|\operatorname{Re} m_{\mathrm{fc}}(t,x)\big|.$$

(2) *There is a constant* $c > 0$ *such that for all* $x \in \mathbb{R}$ *and all* $t \geq 0$, *we have*

$$\big|\widehat{U}'(t,x) - U'(t,x)\big| \leq N^{-c\alpha_0/2}, \tag{4.19}$$

where $\alpha_0 > 0$ *is the constant in* (3.3).

(3) *The potentials* $\widehat{U}$ *and* U *satisfy* (4.4) *and* (4.5). *In particular, there is* $C_U \geq 0$ (*independent of* N), *such that*

$$\inf_{x \in \mathbb{R}, t \in \mathbb{R}^+} \widehat{U}''(t,x) \geq -2C_U, \qquad \inf_{x \in \mathbb{R}, t \in \mathbb{R}^+} U''(t,x) \geq -2C_U. \tag{4.20}$$

Moreover, $\widehat{U}$ *and* U *are "regular"*; *see the paragraph below* (4.8) *for the definition of "regular" potential.*

Below, we are mainly interested in β-ensembles determined by the potential $\widehat{U}$. For ease of notation, we thus limit the discussion to $\widehat{U}$.

For $N \in \mathbb{N}$ we define a measure on $I^{(N)}$ by setting

$$\widehat{\psi}_t(\mathbf{x}) \mu_G(\mathrm{d}\mathbf{x}) := \frac{1}{Z_{\widehat{\psi}_t}} \mathrm{e}^{-((\beta N)/2) \sum_{i=1}^N \widehat{U}(t,x_i)} \mu_G(\mathrm{d}\mathbf{x}) \qquad \big(\mathbf{x} \in I^{(N)}\big), \tag{4.21}$$

where $Z_{\widehat{\psi}_t} \equiv Z_{\widehat{\psi}_t}(\beta)$ is a normalization, and we usually choose $\beta = 1, 2$. By Lemma 4.2, $\widehat{\psi}_t \mu_G$ is a well-defined β-ensemble, and from the discussion in Section 4.1 we further infer that the equilibrium density of $\widehat{\psi}_t \mu_G$, that is, the unique measure solving the minimization problem in (4.7), is for any $t \geq 0$, $\widehat{\rho}_{\mathrm{fc}}(t)$. Viewing $\widehat{\psi}_t \mu_G$ as a Gibbs measure of N (ordered) particles (x_i) on the real line, we define the classical location of the ith particles, $\widehat{\gamma}_i(t)$, as in (4.9), that is,

$$\int_{-\infty}^{\widehat{\gamma}_i(t)} \widehat{\rho}_{\mathrm{fc}}(t,x)\, \mathrm{d}x = \frac{i - (1/2)}{N} \qquad \big(i \in [\![1,N]\!]\big). \tag{4.22}$$

From [12] we have the following rigidity result.

PROPOSITION 4.3. *Let $\widehat{U}(t,\cdot)$, with $t \geq 0$ and $N \in \mathbb{N}$, be given by Lemma 4.2. Then the following holds on Ω. For any $\delta > 0$, there is $\varsigma > 0$, such that for any $t \geq 0$,*

$$\mathbb{P}^{\widehat{\psi}_t \mu_G}\big(|x_i - \widehat{\gamma}_i(t)| > N^{-(2/3)+\delta} \breve{\alpha}_i^{-1/3}\big) \leq \mathrm{e}^{-N^{\varsigma}} \qquad (1 \leq i \leq N), \tag{4.23}$$

for N sufficiently large, where $\mathbb{P}^{\widehat{\psi}_t \mu_G}$ stands for the probability under $\widehat{\psi}_t \mu_G$ conditioned on V. Here, $\breve{\alpha}_i := \min\{i, N-i+1\}$.

PROOF. The rigidity estimate (4.23) is taken from Theorem 2.4 of [12]. To achieve uniformity in $t \geq 0$ and N sufficiently large, we note that estimate (4.23) depends on the potential mainly through the convexity bounds (4.4) and (4.5). Starting from the uniform bounds of Lemma 4.2, one checks that Proposition 4.3 holds uniformly in t and N large enough. □

In the rest of this section, we derive equations of motion for the potential $\widehat{U}(t,\cdot)$ and the classical locations $(\widehat{\gamma}_i(t))$. To derive these equations we observe that the Stieltjes transform $\widehat{m}_{\mathrm{fc}}(t,z)$ can be obtained from $\widehat{m}_{\mathrm{fc}}(t=0,z)$ as the solution to the following complex Burgers equation [46]:

$$\partial_t \widehat{m}_{\mathrm{fc}}(t,z) = \tfrac{1}{2}\partial_z\big[\widehat{m}_{\mathrm{fc}}(t,z)\big(\widehat{m}_{\mathrm{fc}}(t,z)+z\big)\big] \qquad \big(z \in \mathbb{C}^+, t \geq 0\big). \tag{4.24}$$

This can be checked by differentiating (4.15). Combining the complex Burgers equation (4.24) and the loop equation (4.14) we obtain the following result.

LEMMA 4.4. *Let $N \in \mathbb{N}$. Assume that $\widehat{v}$ satisfies the Assumptions 2.2 and 2.3. Then the following holds on Ω for N sufficiently large. For $t \geq 0$, we have*

$$\partial_t \widehat{\gamma}_i(t) = \tfrac{1}{2}\widehat{U}'\big(t, \widehat{\gamma}_i(t)\big), \tag{4.25}$$

respectively,

$$\partial_t \widehat{\gamma}_i(t) = -\fint_{\mathbb{R}} \frac{\widehat{\rho}_{\mathrm{fc}}(t,y)}{y - \widehat{\gamma}_i(t)}\,\mathrm{d}y - \frac{1}{2}\widehat{\gamma}_i(t) \qquad \big(i \in [\![1, N]\!]\big). \tag{4.26}$$

Further, the potential $\widehat{U}$ satisfies

$$\partial_t \widehat{U}(t,x) = \fint_{\mathbb{R}} \frac{\widehat{U}'(t,y)\widehat{\rho}_{\mathrm{fc}}(t,y)}{y-x}\,\mathrm{d}y \qquad \big(x \in \operatorname{supp}\widehat{\rho}_{\mathrm{fc}}(t)\big). \tag{4.27}$$

Moreover, there exist constants C, C' such that the following bounds hold on Ω:

$$\big|\partial_t \widehat{\gamma}_i(t)\big| \leq C, \qquad \big|\partial_t \widehat{U}(t,x)\big| \leq C', \tag{4.28}$$

for all $i \in [\![1, N]\!]$, uniformly in $t \geq 0$, $x \in \operatorname{supp}\widehat{\rho}_{\mathrm{fc}}(t)$ and N, for N sufficiently large.

Finally, $U(t,\cdot)$ and $(\gamma_i(t))$, share the same properties.

PROOF. Combining (4.24) and (4.14), we find, for $z \in \mathbb{C}^+$, $t \geq 0$,

$$\begin{aligned}\partial_t \widehat{m}_{\mathrm{fc}}(t,z) &= \frac{1}{2}\partial_z\Big(-\int \frac{v+\widehat{U}'(t,v)}{v-z}\widehat{\rho}_{\mathrm{fc}}(t,v)\,\mathrm{d}v + z\int \frac{\widehat{\rho}_{\mathrm{fc}}(t,v)}{v-z}\,\mathrm{d}v\Big)\\ &= \frac{1}{2}\partial_z\Big(-\int \frac{\widehat{U}'(t,v)}{v-z}\widehat{\rho}_{\mathrm{fc}}(t,v)\,\mathrm{d}v - 1\Big)\\ &= -\frac{1}{2}\partial_z\int \frac{\widehat{U}'(t,v)}{v-z}\widehat{\rho}_{\mathrm{fc}}(t,v)\,\mathrm{d}v.\end{aligned}$$

Hence, for $\operatorname{Im} z > 0$, we get

$$\partial_t \widehat{m}_{\mathrm{fc}}(t,z) = -\frac{1}{2}\int \frac{\widehat{U}'(t,v)}{(v-z)^2}\widehat{\rho}_{\mathrm{fc}}(t,v)\,\mathrm{d}v = -\frac{1}{2}\int \frac{(\widehat{U}'(t,v)\widehat{\rho}_{\mathrm{fc}}(t,v))'}{(v-z)}\,\mathrm{d}v.$$

Clearly $\widehat{U}'(t,v)\widehat{\rho}_{\mathrm{fc}}(t,v)$ is a C^3 function inside the support of $\widehat{\rho}_{\mathrm{fc}}(z)$ that has a square root behavior at the endpoints. Thus we obtain from the Stieltjes inversion formula that

$$\partial_t \widehat{\rho}_{\mathrm{fc}}(t,E) = \frac{1}{\pi}\lim_{\eta\searrow 0}\operatorname{Im}\partial_t \widehat{m}_{\mathrm{fc}}(t,z) = -\frac{1}{2}\big(\widehat{U}'(t,E)\widehat{\rho}_{\mathrm{fc}}(t,E)\big)', \tag{4.29}$$

for all $E \in (\widehat{L}_-(t), \widehat{L}_+(t))$, where $\widehat{L}_\pm(t)$ denote the endpoints of the support of $\widehat{\rho}_{\mathrm{fc}}(t)$.

On the other hand, differentiating (4.22) with respect to time, we obtain

$$\int_{-\infty}^{\widehat{\gamma}_i(t)} \partial_t \widehat{\rho}_{\mathrm{fc}}(t,v)\,\mathrm{d}v = -\widehat{\rho}_{\mathrm{fc}}\big(t,\widehat{\gamma}_i(t)\big)\partial_t\widehat{\gamma}_i(t).$$

Substituting from (4.29), we get

$$\partial_t\widehat{\gamma}_i(t) = \frac{1}{2}\frac{1}{\widehat{\rho}_{\mathrm{fc}}(t,\widehat{\gamma}_i(t))}\int_{-\infty}^{\widehat{\gamma}_i(t)} \mathrm{d}v\big(\widehat{U}'_{\mathrm{fc}}(t,v)\widehat{\rho}_{\mathrm{fc}}(t,v)\big)'.$$

Hence

$$\partial_t\widehat{\gamma}_i(t) = \frac{1}{2}\frac{1}{\widehat{\rho}_{\mathrm{fc}}(t,\widehat{\gamma}_i(t))}\widehat{U}'\big(t,\widehat{\gamma}_i(t)\big)\widehat{\rho}_{\mathrm{fc}}\big(t,\widehat{\gamma}_i(t)\big),$$

and (4.25) follows. Using that $\widehat{U}$ satisfies (4.17), we can recast this last equation as

$$\partial_t\widehat{\gamma}_i(t) = -\fint_{\mathbb{R}}\frac{\widehat{\rho}_{\mathrm{fc}}(t,y)}{y-\widehat{\gamma}_i(t)}\,\mathrm{d}y - \frac{1}{2}\widehat{\gamma}_i(t),$$

and we find (4.26). Equation (4.26) follows in a similar way by differentiating (4.17) with respect to time. By a similar computation we obtain (4.27). The bound in (4.28) follows from Lemma 4.2. □

Starting from the relations in (4.15), we derived via the time dependent potential $\widehat{U}$, an equation of motions for the classical locations $(\widehat{\gamma}_i(t))$. The points $(\widehat{\gamma}_i(t))$

may also be viewed as the classical locations of the eigenvalues of a family of random matrices which is parametrized by the times t_0 and t. This is the subject of the next section.

5. Dyson Brownian motion: Evolution of the entropy.

5.1. *Dyson Brownian motion.* Let $H_0 = (h_{ij,0})$ be the matrix

$$H_0 := \mathrm{e}^{t_0/2} V + W,$$

where V satisfies Assumptions 2.2 and 2.3, and W is real symmetric or complex Hermitian satisfying the assumptions in Definition 2.1. Here, $t_0 \geq 0$ is chosen such that $\vartheta = \mathrm{e}^{t_0/2} \in \Theta_{\varpi}$ [see (3.1)], and we consider ϑ as an a priori free "coupling parameter" that we fix in Section 8 below. Let $B = (b_{ij}) \equiv (b_{ij,t})$ be a real symmetric, respectively, a complex Hermitian, matrix whose entries are a collection of independent, up to the symmetry constraint, real (complex) Brownian motions, independent of $(h_{ij,0})$. More precisely, in case W is a complex Hermitian Wigner matrix, we choose the entries $(b_{ij,t})$ to have variance t; in case W is a real symmetric Wigner matrix, we choose the off-diagonal entries of $(b_{ij,t})$ to have variance t, while the diagonal entries are chosen to have variance $2t$. Let $H_t = (h_{ij,t})$ satisfy the stochastic differential equation

$$\mathrm{d}h_{ij} = \frac{\mathrm{d}b_{ij}}{\sqrt{N}} - \frac{1}{2} h_{ij}\,\mathrm{d}t \qquad (t \geq 0). \tag{5.1}$$

It is then easy to check that the distribution of H_t agrees with the distribution of the matrix

$$\mathrm{e}^{-(t-t_0)/2} V + \mathrm{e}^{-t/2} W + \left(1 - \mathrm{e}^{-t}\right)^{1/2} W', \tag{5.2}$$

where W' is, in case W is a complex Hermitian, a GUE matrix, independent of V and W, respectively, a GOE matrix, independent of V and W, in case W is a real symmetric Wigner matrix. The law of the eigenvalues of the matrix W' is explicitly given by (4.11) with $\beta = 2$, respectively, $\beta = 1$.

Denote by $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t))$ the ordered eigenvalues of H_t. It is well known that $\boldsymbol{\lambda}(t)$ satisfy the following stochastic differential equation:

$$\mathrm{d}\lambda_i = \frac{\sqrt{2}}{\sqrt{\beta N}}\,\mathrm{d}b_i + \left(-\frac{\lambda_i}{2} + \frac{1}{N} \sum_{j}^{(i)} \frac{1}{\lambda_i - \lambda_j} \right) \mathrm{d}t \qquad (i \in [\![1, N]\!]), \tag{5.3}$$

where (b_i) is a collection of real-valued, independent standard Brownian motions. If the matrix (b_{ij}) in (5.1) is real symmetric, we have $\beta = 1$ in (5.3), respectively, $\beta = 2$, if (b_{ij}) is complex Hermitian. The evolution of $\boldsymbol{\lambda}(t)$ is the celebrated Dyson Brownian motion [21].

For $t \geq 0$, we denote by $f_t \mu_G$ the distribution of $\boldsymbol{\lambda}(t)$. In particular, $\int f_t \,\mathrm{d}\mu_G \equiv \int f_t(\boldsymbol{\lambda}) \mu_G(\mathrm{d}\boldsymbol{\lambda}) = 1$. Note that $f_t \mu_G$ depends on V through the initial condition

$f_0\mu_G$. In the following we always keep the (v_i) fixed; that is, we condition on V. For simplicity, we omit this conditioning from our notation. The density f_t is the solution of the equation

$$\partial_t f_t = \mathcal{L} f_t \qquad (t \geq 0),$$

where the generator $\mathcal{L}$ is defined via the Dirichlet form

$$D_{\mu_G}(f) = -\int f \mathcal{L} f \, \mathrm{d}\mu_G = \sum_{i=1}^{N} \frac{1}{\beta N} \int (\partial_i f)^2 \, \mathrm{d}\mu_G \qquad (\partial_i \equiv \partial_{x_i}). \tag{5.4}$$

Formally, we have $\mathcal{L} = \frac{1}{\beta N}\Delta - (\nabla \mathcal{H}_G) \cdot \nabla$, that is,

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{\beta N} \partial_i^2 + \sum_{i=1}^{N} \left(-\frac{1}{2}\lambda_i + \frac{1}{N} \sum_{j}^{(i)} \frac{1}{\lambda_i - \lambda_j} \right) \partial_i. \tag{5.5}$$

We remark that we use a different normalization in the definition of the Dirichlet form $D_{\mu_G}(f)$ in (5.4) (and the generator $\mathcal{L}$) than in earlier works, as in, for example, [30], where the Dirichlet from is defined as $\sum_{i=1}^{N} \frac{1}{2N} \int (\partial_i f)^2 \, \mathrm{d}\mu_G$.

LEMMA 5.1 (Dyson Brownian motion). *The equation $\partial_t f_t = \mathcal{L} f_t$, with initial data $f_t|_{t=0} = f_0$ has a unique solution on $\mathrm{L}^1(\mu_G) \equiv \mathrm{L}^1(\mathbb{R}^N, \mu_G)$ for all $t \geq 0$. Moreover, the domain $\mathcal{F}^{(N)}$ is invariant under the dynamics; that is, if f_0 is supported in $\mathcal{F}^{(N)}$, then is f_t for all $t \geq 0$.*

(Strictly speaking, the eigenvalue distribution of H_0 may not allow a density f_0, but for $t > 0$, H_t admits a density f_t. Our proofs are not affected by this technicality.)

We refer, for example, to [1] for more details and proofs. To conclude, we record one of the technical tools used in the next sections.

LEMMA 5.2. *Denote by $f_t(\boldsymbol{\lambda})\mu_G(\mathrm{d}\boldsymbol{\lambda})$ the distribution of the eigenvalues of matrix (5.2) with $t \geq 0$. Then, for any $0 < \mathfrak{a} < 1/2$, we have*

$$\sup_{t \geq 0} \int \frac{1}{N} \sum_{i=1}^{N} \bigl(\lambda_i - \widehat{\gamma}_i(t)\bigr)^2 f_t(\boldsymbol{\lambda}) \, \mathrm{d}\mu(\boldsymbol{\lambda}) \leq N^{-1-2\mathfrak{a}}, \tag{5.6}$$

on Ω for N sufficiently large, where $(\widehat{\gamma}_i(t))$ denote the classical locations with respect to the measure $\widehat{\rho}_{\mathrm{fc}}(t)$; that is, they are defined through the relation

$$\int_{-\infty}^{\widehat{\gamma}_i(t)} \widehat{\rho}_{\mathrm{fc}}(t, x) \, \mathrm{d}x = \frac{i - (1/2)}{N} \qquad (1 \leq i \leq N). \tag{5.7}$$

[*They agree with the classical locations of* (4.22).]

PROOF. The random matrix $W_t \equiv (w_{ij,t}) := \mathrm{e}^{-t/2}W + (1-\mathrm{e}^{-t})^{1/2}W'$ satisfies the assumptions in Definition 2.1: the entries are centered and have variance $1/N$. Moreover, since the distributions of $(w_{ij,0})$, satisfies (2.3) and since (w'_{ij}) are real, respectively, complex, centered Gaussian random variables with variance $1/N$, respectively, $2/N$, the distributions of $(w_{ij,t})$ also satisfy (2.3). The claim now follows from (3.15) of Corollary 3.4 and the moment bounds $\mathbb{E}\operatorname{Tr} W_t^{2p} \le C_p$ (see, e.g., [1]), as well as the boundedness of (v_i). □

5.2. *Entropy decay estimates.* Let ω and ν be two (probability) measures on $\mathbb{R}^N$ that are absolutely continuous with respect to Lebesgue measure. We denote the Radon–Nikodym derivative of ν with respect to ω by $\frac{\mathrm{d}\nu}{\mathrm{d}\omega}$, define the relative entropy of ν with respect to ω by

$$S(\nu|\omega) := \int_{\mathbb{R}^N} \frac{\mathrm{d}\nu}{\mathrm{d}\omega} \log \frac{\mathrm{d}\nu}{\mathrm{d}\omega}\,\mathrm{d}\omega, \tag{5.8}$$

and, in case $\nu = f\omega$, $f \in \mathrm{L}^1(\mathbb{R}^N)$, abbreviate

$$S_\omega(f) = S(f\omega|\omega).$$

The entropy $S_\omega(f)$ controls the total variation norm of f through the inequality

$$\int |f-1|\,\mathrm{d}\omega \le \sqrt{2S_\omega(f)}, \tag{5.9}$$

a result we will use repeatedly in the next sections.

Besides the dynamics $(f_t)_{t\ge 0}$ generated by $\mathcal{L}$ introduced in Section 5.1, we also consider a (a priori undetermined) time dependent density, $(\tilde\psi_t)_{t\ge 0}$, with respect to μ_G. We assume that $\tilde\psi_t \ne 0$, almost everywhere with respect to μ_G and abbreviate $\tilde g_t := \frac{f_t}{\tilde\psi_t}$. Setting $\tilde\omega_t := \tilde\psi_t\mu_G$, we can write

$$f_t(\boldsymbol{\lambda})\mu_G(\mathrm{d}\boldsymbol{\lambda}) = \tilde g_t(\boldsymbol{\lambda})\tilde\omega_t(\mathrm{d}\boldsymbol{\lambda}).$$

A natural choice for $\tilde\psi_t\mu_G$ is the time dependent β-ensemble, $\widehat\psi_t\mu_G$, introduced in (4.21). Yet, following the arguments of Erdős et al. [29] we make a slightly different choice for $\tilde\psi_t$: for $\tau > 0$, we define a measure $\tilde\psi_t\mu_G$ on $\mathsf{\Gamma}^{(N)}$ by setting

$$\tilde\psi_t(\boldsymbol{\lambda})\mu_G(\mathrm{d}\boldsymbol{\lambda}) := \frac{1}{Z'_{\tilde\psi_t}} \mathrm{e}^{-N\beta\sum_{i=1}^N (\lambda_i - \widehat\gamma_i(t))^2/(2\tau)} \widehat\psi_t(\boldsymbol{\lambda})\mu_G(\mathrm{d}\boldsymbol{\lambda}), \tag{5.10}$$

where $Z'_{\tilde\psi_t} \equiv Z'_{\tilde\psi_t}(\beta)$ is chosen such that $\int \tilde\psi_t(\boldsymbol{\lambda})\mu_G(\mathrm{d}\boldsymbol{\lambda}) = 1$. In the following, we mostly choose τ to be N-dependent with $1 \gg \tau > 0$.

We call the measure $\tilde\psi_t\mu_G$ the instantaneous relaxation measure. The density $\tilde\psi_t$ depends on $V = \operatorname{diag}(v_i)$ via the initial condition $\tilde\psi_0$. As for the distribution f_t, we condition on V and omit this from the notation. We may write the measure $\tilde\psi_t\mu_G$ in the Gibbs form

$$\tilde\psi_t(\boldsymbol{\lambda})\mu_G(\mathrm{d}\boldsymbol{\lambda}) = \frac{1}{Z_{\tilde\psi_t}} \mathrm{e}^{-\beta N\widetilde{\mathcal{H}}_t(\boldsymbol{\lambda})}\,\mathrm{d}\boldsymbol{\lambda} \qquad \big(\boldsymbol{\lambda} \in \mathsf{\Gamma}^{(N)}\big),$$

with

$$\widetilde{\mathcal{H}}_t(\boldsymbol{\lambda}) = \mathcal{H}_G(\boldsymbol{\lambda}) + \sum_{i=1}^{N}\Big(\frac{(\lambda_i - \widehat{\gamma}_i(t))^2}{2\tau} + \frac{\widehat{U}(t,\lambda_i)}{2}\Big), \tag{5.11}$$

where $\mathcal{H}_G$ is defined in (4.10) and $Z_{\tilde{\psi}_t} \equiv Z_{\tilde{\psi}_t}(\beta)$ is a normalization. Then we compute

$$\begin{aligned}\nabla u \cdot (\nabla^2 \widetilde{\mathcal{H}}_t) \cdot \nabla u &\geq \sum_{i=1}^{N}(\partial_i u)^2\Big(\frac{1}{\tau} + \frac{\widehat{U}''(t,\lambda_i)}{2} + \frac{1}{2}\Big) \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}\sum_{j}^{(i)}\frac{1}{(x_i - x_j)^2}(\partial_i u - \partial_j u)^2 \\ &\geq \sum_{i=1}^{N}\frac{(\partial_i u)^2}{2\tau},\end{aligned} \tag{5.12}$$

for $u \in C^1(\mathbb{R}^N)$ and τ sufficiently small (independent of N), where we use that $\widehat{U}''(t,\cdot)$ is uniformly bounded below by Lemma 4.2. Then, by the Bakry–Émery criterion [3], there is a constant C such that the following logarithmic Sobolev inequality holds for all sufficiently small $\tau > 0$:

$$S_{\tilde{\omega}_t}(q) \leq C\tau D_{\tilde{\omega}_t}(\sqrt{q}) \qquad (t \geq 0), \tag{5.13}$$

where $q \in \mathrm{L}^\infty(\mathrm{d}\tilde{\omega}_t)$ is such that $\int q\,\mathrm{d}\tilde{\omega}_t = 1$. We refer, for example, to [28–30, 33] for more details.

Recall the definition of $\widehat{\psi}_t\mu_G$ in (4.21). Let $\widehat{\mathcal{L}}_t$ denote the generator defined by the natural Dirichlet form with respect to $\widehat{\omega}_t$, that is,

$$D_{\widehat{\omega}_t}(q) = \frac{1}{\beta N}\sum_{i=1}^{N}\int(\partial_i q)^2\,\mathrm{d}\widehat{\omega}_t = -\int q\widehat{\mathcal{L}}_t q\,\mathrm{d}\widehat{\omega}_t \qquad (t > 0). \tag{5.14}$$

The main result of this section is the following proposition.

PROPOSITION 5.3. *Let $\widehat{g}_t := f_t/\widehat{\psi}_t$, and set $\widehat{\omega}_t := \widehat{\psi}_t\mu_G$ such that*

$$S(f_t\mu_G|\widehat{\psi}_t\mu_G) = S_{\widehat{\omega}_t}(\widehat{g}_t).$$

Then there is a constant C (independent of t) such that, for all $0 < \mathfrak{a} < 1/2$, we have

$$\partial_t S_{\widehat{\omega}_t}(\widehat{g}_t) \leq -4D_{\widehat{\omega}_t}(\sqrt{\widehat{g}_t}) + CN^{1-2\mathfrak{a}} \qquad (t > 0), \tag{5.15}$$

for N sufficiently large on Ω.

The results of Proposition 5.3 resemble the relative entropy estimate of Theorem 2.5 in [30] for Wigner matrices. However, due to the fact that both distributions $f_t \mu_G$ and $\widehat{\psi}_t \mu_G$ are not close to the global equilibrium for the Dyson Brownian motion, μ_G, the reference ensemble $\widehat{\psi}_t \mu_G$ changes with time, too. Thus to establish (5.15), we need to include additional factors coming from time derivatives of $\widehat{\psi}_t \mu_G$. These can be controlled using the definition of the potential $\widehat{U}(t)$. The idea of choosing slowly varying time dependent approximation states and controlling the entropy flow goes back to the work [61].

The relative entropy $S_{\widehat{\omega}_t}$ and the Dirichlet form $D_{\widehat{\omega}_t}$ do not satisfy the logarithmic Sobolev inequality (5.13). However, we have for $t > 0$ the estimates

$$D_{\widehat{\omega}_t}(\sqrt{\widehat{g}_t}) \leq 2D_{\tilde{\omega}_t}(\sqrt{\tilde{g}_t}) + C\frac{\beta N^2 Q_t}{\tau^2} \tag{5.16}$$

and

$$D_{\tilde{\omega}_t}(\sqrt{\tilde{g}_t}) \leq 2D_{\widehat{\omega}_t}(\sqrt{\widehat{g}_t}) + C\frac{\beta N^2 Q_t}{\tau^2}, \tag{5.17}$$

respectively,

$$S_{\tilde{\omega}_t}(\tilde{g}_t) = S_{\widehat{\omega}_t}(\widehat{g}_t) + \mathcal{O}\Big(\frac{\beta N^2 Q_t}{\tau}\Big), \tag{5.18}$$

where we have set

$$Q_t := \mathbb{E}^{f_t \mu_G} \frac{1}{N} \sum_{i=1}^{N} \big(\lambda_i - \widehat{\gamma}_i(t)\big)^2. \tag{5.19}$$

Estimates (5.16), (5.17) and (5.18) can be checked by elementary computations, which we omit here. In the following we always bound $Q_t \leq CN^{-1-2\mathfrak{a}}$ [$t \geq 0$, $\mathfrak{a} \in (0, 1/2)$]; see Lemma 5.6. Using (5.16), (5.17) and (5.18) in combination with the logarithmic Sobolev inequality (5.13) and with Proposition 5.3, we can follow [30] to obtain a bound on the Dirichlet form $D_{\widehat{\omega}_t}(\sqrt{\widehat{g}_t})$.

COROLLARY 5.4. *Under the assumptions of Proposition 5.3, the following holds on Ω for N sufficiently large. For any $\varepsilon' > 0$ and $t \geq \tau N^{\varepsilon'}$ with $1 \gg \tau \geq N^{-2\mathfrak{a}}$, we have the entropy and Dirichlet form bounds*

$$S_{\widehat{\omega}_t}(\widehat{g}_t) \leq C\frac{N^{1-2\mathfrak{a}}}{\tau}, \qquad D_{\widehat{\omega}_t}(\sqrt{\widehat{g}_t}) \leq C\frac{N^{1-2\mathfrak{a}}}{\tau^2}, \tag{5.20}$$

where the constants depend on ε'.

Before we prove Proposition 5.3, we obtain rigidity estimates for the time dependent β-ensemble $\widehat{\psi}_t \mu_G$. Recall that we denote by $(\widehat{\gamma}_i(t))$ the classical locations with respect to the measure $\widehat{\rho}_{\mathrm{fc}}(t)$. Also recall the notation $\breve{\alpha}_i = \min\{i, N - i + 1\}$.

LEMMA 5.5. *Let $\widehat{U}(t,\cdot)$, $t \geq 0$ be as in Lemma 4.2. Then the following holds on Ω for N sufficiently large*:

For any $\delta > 0$, there is $\varsigma > 0$ such that

$$\mathbb{P}^{\widehat{\psi}_t \mu_G}\big(|\lambda_i - \widehat{\gamma}_i(t)| > N^{-(2/3)+\delta} \breve{\alpha}_i^{-1/3}\big) \leq \mathrm{e}^{-N^{\varsigma}}, \tag{5.21}$$

for all $t \geq 0$, $1 \leq i \leq N$, where $\mathbb{P}^{\widehat{\psi}_t \mu_G}$, stands for the probability under $\widehat{\psi}_t \mu_G$ conditioned on Ω. Moreover, for any $0 < \mathfrak{a} < 1/2$, we have

$$\sup_{t \geq 0} \int \frac{1}{N} \sum_{i=1}^{N} \big(\lambda_i - \widehat{\gamma}_i(t)\big)^2 \widehat{\psi}_t(\boldsymbol{\lambda}) \mu_G(\mathrm{d}\boldsymbol{\lambda}) \leq N^{-1-2\mathfrak{a}}, \tag{5.22}$$

for N sufficiently large.

PROOF. The rigidity estimate (5.21) follows from Proposition 4.3 by choosing $N \in \mathbb{N}$ sufficiently large. Estimate (5.22) is a direct consequence of (5.21) and the fast decay of the distribution $\widehat{\psi}_t(\boldsymbol{\lambda}) \mu_G(\boldsymbol{\lambda})$. $\square$

For brevity, we often drop the t-dependence of $\widehat{\gamma}_i(t)$ from the notation.

PROOF OF PROPOSITION 5.3. Recall that we have set $\widehat{g}_t = f_t / \widehat{\psi}_t$ and $\widehat{\omega}_t = \widehat{\psi}_t \mu_G$. The relative entropy $S(f_t \mu_G | \widehat{\psi}_t \mu_G) = S_{\widehat{\omega}_t}(\widehat{g}_t)$ satisfies [61],

$$\partial_t S(f_t \mu_G | \widehat{\psi}_t \mu_G) = -\frac{1}{\beta N} \int \frac{|\nabla \widehat{g}_t|^2}{g_t} \widehat{\psi}_t \, \mathrm{d}\mu_G + \int \frac{(\mathcal{L} - \partial_t)\widehat{\psi}_t}{\widehat{\psi}_t} f_t \, \mathrm{d}\mu_G. \tag{5.23}$$

We note that the first term on the right-hand side of (5.23) equals

$$-\frac{1}{\beta N} \int \frac{|\nabla \widehat{g}_t|^2}{\widehat{g}_t} \widehat{\psi}_t \, \mathrm{d}\mu_G = -4 D_{\widehat{\omega}_t}(\sqrt{\widehat{g}_t}). \tag{5.24}$$

To bound the second term on the right-hand side of (5.23), we write

$$\begin{aligned} &\int \frac{(\mathcal{L} - \partial_t)\widehat{\psi}_t}{\widehat{\psi}_t} f_t \, \mathrm{d}\mu_G \\ &\quad = \int (\widehat{\mathcal{L}}_t \widehat{g}_t) \, \mathrm{d}\widehat{\omega}_t + \frac{1}{2} \int \sum_{i=1}^{N} \widehat{U}'(t, \lambda_i) \big(\partial_i \widehat{g}_t(\boldsymbol{\lambda})\big) \, \mathrm{d}\widehat{\omega}_t(\boldsymbol{\lambda}) - \int \widehat{g}_t \partial_t \widehat{\psi}_t \, \mathrm{d}\mu_G, \end{aligned} \tag{5.25}$$

with $\widehat{\mathcal{L}}_t$ defined in (5.14).

Note that the first term on the right-hand side of (5.25) vanishes since, by construction, $\widehat{\omega}_t$ is the reversible measure for the instantaneous flow generated by $\widehat{\mathcal{L}}_t$. The last term on the right-hand side of (5.25) can be computed explicitly as (recall that the normalization $Z_{\widehat{\psi}_t}$ in the definition of $\widehat{\psi}_t \mu_G$ also depends on t),

$$-\int \widehat{g}_t \partial_t \widehat{\psi}_t \, \mathrm{d}\mu_G = \big[\mathbb{E}^{f_t \mu_G} - \mathbb{E}^{\widehat{\psi}_t \mu_G}\big] \left[\frac{\beta N}{2} \sum_{i=1}^{N} \partial_t \widehat{U}(t, \lambda_i) \right]. \tag{5.26}$$

To deal with the second term on the right-hand side of (5.25), we integrate by parts to find

$$\begin{aligned}
&\frac{1}{2}\int \sum_{i=1}^{N} \widehat{U}'(t,\lambda_i)\big(\partial_i \widehat{g}_t(\boldsymbol{\lambda})\big)\,\mathrm{d}\widehat{\omega}_t(\boldsymbol{\lambda}) \\
&\qquad = \mathbb{E}^{f_t\mu_G}\left[-\frac{1}{2}\sum_{i=1}^{N}\widehat{U}''(t,\lambda_i)\right] \\
&\qquad\quad + \mathbb{E}^{f_t\mu_G}\left[\frac{\beta N}{4}\sum_{i=1}^{N}\widehat{U}'(t,\lambda_i)\left(\widehat{U}'(t,\lambda_i)+\lambda_i-\frac{2}{N}\sum_{j}^{(i)}\frac{1}{\lambda_i-\lambda_j}\right)\right].
\end{aligned} \tag{5.27}$$

Setting $\widehat{g}_t \equiv 1$ in the above computation, we also obtain the identity

$$\begin{aligned}
0 &= \mathbb{E}^{\widehat{\psi}_t\mu_G}\left[-\frac{1}{2}\sum_{i=1}^{N}\widehat{U}''(t,\lambda_i)\right] \\
&\quad + \mathbb{E}^{\widehat{\psi}_t\mu_G}\left[\frac{\beta N}{4}\sum_{i=1}^{N}\widehat{U}'(t,\lambda_i)\left(\widehat{U}'(t,\lambda_i)+\lambda_i-\frac{2}{N}\sum_{j}^{(i)}\frac{1}{\lambda_i-\lambda_j}\right)\right].
\end{aligned} \tag{5.28}$$

Equation (5.28) may alternatively be derived from the "first order loop equation" for the β-ensemble $\widehat{\psi}_t\mu_G$. Equation (5.27) can thus be rewritten as

$$\begin{aligned}
&\frac{1}{2}\int \sum_{i=1}^{N} \widehat{U}'(t,\lambda_i)\big(\partial_i \widehat{g}_t(\boldsymbol{\lambda})\big)\,\mathrm{d}\widehat{\omega}_t(\boldsymbol{\lambda}) \\
&\qquad = \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\left[-\frac{1}{2}\sum_{i=1}^{N}\widehat{U}''(t,\lambda_i)\right] \\
&\qquad\quad + \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big] \\
&\qquad\qquad \times\left[\frac{\beta N}{4}\sum_{i=1}^{N}\widehat{U}'(t,\lambda_i)\left(\widehat{U}'(t,\lambda_i)+\lambda_i-\frac{2}{N}\sum_{j}^{(i)}\frac{1}{\lambda_i-\lambda_j}\right)\right].
\end{aligned} \tag{5.29}$$

Next, to control the second and third terms on the right-hand side of (5.25), respectively, the right-hand side of (5.29), we proceed as follows. We expand the potential terms $\widehat{U}'(t,\lambda_i)$, respectively, $\widehat{U}''(t,\lambda_i)$, in Taylor series in λ_i to second order around the classical location $\widehat{\gamma}_i$. The resulting zero order terms cancel exactly since the classical locations of the ensembles $f_t\mu_G$, and $\widehat{\psi}_t\mu_G$ agree by construction. The first order terms in the Taylor expansion can (1) either be bounded in terms of the expectations of $\sum_{i=1}^{N}(\lambda_i-\widehat{\gamma}_i)^2$ (which can be controlled with the rigidity estimates in Lemmas 5.5 and 5.2); or (2) they cancel exactly due to the definition of the potential $\widehat{U}(t,\cdot)$ and its equation of motion in (4.27). Finally, the second order terms in the Taylor expansion can be bounded by the rigidity estimates in Lemmas 5.5 and 5.2. The details are as follows.

Expanding $\partial_t \widehat{U}(t,\lambda_i)$ to second order around $\widehat{\gamma}_i$, we obtain from (5.26) that

$$\begin{aligned}-\int \widehat{g}_t \partial_t \widehat{\psi}_t \,\mathrm{d}\mu_G &= \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big] \\ &\quad\times \left[\frac{\beta N}{2}\sum_{i=1}^N \partial_t \widehat{U}(t,\widehat{\gamma}_i) + \frac{\beta N}{2}\sum_{i=1}^N \partial_t \widehat{U}'(t,\gamma_i)(\lambda_i-\widehat{\gamma}_i)\right] \\ &\quad + \mathcal{O}\big(N^{1-2\mathfrak{a}}\big),\end{aligned} \tag{5.30}$$

on Ω, where we use the rigidity estimates in Lemmas 5.5 and 5.2, and that $\partial_t \widehat{U}''(t,\cdot)$ is uniformly bounded on compact sets by Lemma 4.2.

To save notation, we introduce a function $G:\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{R}$ by setting

$$G(t;x,y) := \frac{\widehat{U}'(t,x)-\widehat{U}'(t,y)}{x-y}, \tag{5.31}$$

with $G(t;x,x) := \widehat{U}''(t,x)$. Note that $G(t;x,y) = G(t;y,x)$ and that G is C^2 in the spatial coordinates by Lemma 4.2. Recalling the equation of motion for $\partial_t \widehat{U}(t,\cdot)$ in (4.27), we can write

$$\begin{aligned}\partial_t \widehat{U}(t,x) &= \fint \frac{\widehat{U}'(t,y)\,\mathrm{d}\widehat{\rho}_{\mathrm{fc}}(t,y)}{y-x} \\ &= \int \frac{\widehat{U}'(t,y)-\widehat{U}'(t,x)}{y-x}\,\mathrm{d}\widehat{\rho}_{\mathrm{fc}}(t,y) + \widehat{U}'(t,x)\fint \frac{\mathrm{d}\widehat{\rho}_{\mathrm{fc}}(t,y)}{y-x},\end{aligned} \tag{5.32}$$

for x inside the support of the measure $\widehat{\rho}_{\mathrm{fc}}$. Thus, recalling (4.17) and (5.31), we obtain

$$\partial_t \widehat{U}(t,x) = \int G(t;x,y)\,\mathrm{d}\widehat{\rho}_{\mathrm{fc}}(y) - \frac{1}{2}\widehat{U}'(t,x)\big(\widehat{U}'(t,x)+x\big), \tag{5.33}$$

for x inside the support of the measure $\widehat{\rho}_{\mathrm{fc}}$.

We hence obtain from (5.30) that

$$\begin{aligned}-\int \widehat{g}_t \partial_t \widehat{\psi}_t \,\mathrm{d}\mu_G &= \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\left[\frac{\beta N}{2}\sum_{i=1}^N \int G'(t;\widehat{\gamma}_i,y)\,\mathrm{d}\widehat{\rho}_{\mathrm{fc}}(y)(\lambda_i-\widehat{\gamma}_i)\right] \\ &\quad - \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\left[\frac{\beta N}{4}\sum_{i=1}^N \widehat{U}''(t,\widehat{\gamma}_i)\big(\widehat{U}'(t,\widehat{\gamma}_i)+\widehat{\gamma}_i\big)(\lambda_i-\widehat{\gamma}_i)\right] \\ &\quad - \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\left[\frac{\beta N}{4}\sum_{i=1}^N \widehat{U}'(t,\widehat{\gamma}_i)\big(\widehat{U}''(t,\widehat{\gamma}_i)+1\big)(\lambda_i-\widehat{\gamma}_i)\right] \\ &\quad + \mathcal{O}\big(N^{1-2\mathfrak{a}}\big),\end{aligned} \tag{5.34}$$

on Ω, where we denote by $G'(t;x,y)$ the first derivative of $G(t;x,y)$ with respect to x.

Next we return to (5.29). Using the rigidity estimates of the Lemmas 5.5 and 5.2, we find

$$\begin{aligned}
&\frac{1}{2}\int \sum_{i=1}^{N} \widehat{U}'(t,\lambda_i)\big(\partial_i \widehat{g}_t(\boldsymbol{\lambda})\big)\,\mathrm{d}\widehat{\omega}_t(\boldsymbol{\lambda}) \\
&\quad = \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\bigg[-\frac{1}{2}\sum_{i=1}^{N}\widehat{U}''\big(t,\widehat{\gamma}_i(t)\big)\bigg] \\
&\quad\quad + \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big] \\
&\quad\quad\quad \times\bigg[\frac{\beta N}{4}\sum_{i=1}^{N}\widehat{U}'(t,\lambda_i)\bigg(\widehat{U}'(t,\lambda_i) + \lambda_i - \frac{2}{N}\sum_{j}^{(i)}\frac{1}{\lambda_i-\lambda_j}\bigg)\bigg] \\
&\quad\quad + \mathcal{O}\big(N^{1/2-\mathfrak{a}}\big),
\end{aligned} \tag{5.35}$$

on Ω, where we use a Taylor expansion of the first term on the right-hand side of (5.29). Here we also use that $\widehat{U}'$ is three times continuously differentiable with uniformly bounded derivatives on compact sets. Note that the first term on the right-hand side of (5.35) vanishes.

Using the definition of $G(t;\cdot,\cdot)$ in (5.31), we can recast (5.35) as

$$\begin{aligned}
&\frac{1}{2}\int \sum_{i=1}^{N} \widehat{U}'(t,\lambda_i)\big(\partial_i \widehat{g}_t(\boldsymbol{\lambda})\big)\,\mathrm{d}\widehat{\omega}_t(\boldsymbol{\lambda}) \\
&\quad = \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\bigg[\frac{\beta N}{4}\sum_{i=1}^{N}\widehat{U}'(t,\lambda_i)\big(\widehat{U}'(t,\lambda_i)+\lambda_i\big)\bigg] \\
&\quad\quad + \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\bigg[-\frac{\beta N}{4}\sum_{i=1}^{N}\frac{1}{N}\sum_{j}^{(i)}G(t;\lambda_i,\lambda_j)\bigg] \\
&\quad\quad + \mathcal{O}\big(N^{1/2-\mathfrak{a}}\big),
\end{aligned} \tag{5.36}$$

where we use the symmetry $G(t;x,y) = G(t;y,x)$. Expanding the second term on the right-hand side (5.36) to second order in (λ_i,λ_j) around $(\widehat{\gamma}_i,\widehat{\gamma}_j)$, we obtain

$$\begin{aligned}
&\big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\bigg[-\frac{\beta N}{4}\sum_{i=1}^{N}\frac{1}{N}\sum_{j}^{(i)}G(t;\lambda_i,\lambda_j)\bigg] \\
&\quad = \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\bigg[-\frac{\beta N}{2}\sum_{i=1}^{N}\bigg(\frac{1}{N}\sum_{j=1}^{N}G'(t;\widehat{\gamma}_i,\widehat{\gamma}_i)\bigg)(\lambda_i-\widehat{\gamma}_i)\bigg] \\
&\quad\quad + \mathcal{O}\big(N^{1/2-\mathfrak{a}}\big) + \mathcal{O}\big(N^{1-2\mathfrak{a}}\big),
\end{aligned} \tag{5.37}$$

on Ω, where we use $G(t;x,y) = G(t;y,x)$, $G(t;x,x) = \widehat{U}''(t,x)$ and that $G(t;x,y)$ is C^2 in the spatial variables. Thus, also expanding the first term on the right-hand side of (5.36) in λ_i around $\widehat{\gamma}_i$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int \sum_{i=1}^{N} \widehat{U}'(t,\lambda_i)\big(\partial_i \widehat{g}_t(\boldsymbol{\lambda})\big)\,\mathrm{d}\widehat{\omega}_t(\boldsymbol{\lambda}) \\
&\quad = \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\bigg[-\frac{\beta N}{2}\sum_{i=1}^{N}\bigg(\frac{1}{N}\sum_{j} G'(t;\widehat{\gamma}_i,\widehat{\gamma}_j)\bigg)(\lambda_i - \widehat{\gamma}_i)\bigg] \\
&\quad\quad + \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\bigg[\frac{\beta N}{4}\sum_{i=1}^{N}\widehat{U}''(t,\widehat{\gamma}_i)\big(\widehat{U}'(t,\widehat{\gamma}_i)+\widehat{\gamma}_i\big)(\lambda_i - \widehat{\gamma}_i)\bigg] \\
&\quad\quad + \big[\mathbb{E}^{f_t\mu_G} - \mathbb{E}^{\widehat{\psi}_t\mu_G}\big]\bigg[\frac{\beta N}{4}\sum_{i=1}^{N}\widehat{U}'(t,\widehat{\gamma}_i)\big(\widehat{U}''(t,\widehat{\gamma}_i)+1\big)(\lambda_i - \widehat{\gamma}_i)\bigg] \\
&\quad\quad + \mathcal{O}\big(N^{1/2-\mathfrak{a}}\big) + \mathcal{O}\big(N^{1-2\mathfrak{a}}\big),
\end{aligned} \tag{5.38}$$

on Ω, where we use the rigidity estimates in Lemmas 5.5 and 5.2.

Adding up (5.34) and (5.38), we hence obtain

$$\begin{aligned}
&\bigg|\int \frac{(\mathcal{L}-\partial_t)\widehat{\psi}_t}{\widehat{\psi}_t} f_t\,\mathrm{d}\mu_G\bigg| \\
&\quad \le \frac{\beta N}{2}\bigg|\mathbb{E}^{f_t\mu_G}\bigg[-\sum_{i=1}^{N}\bigg(\frac{1}{N}\sum_{j=1}^{N} G'(t;\widehat{\gamma}_i,\widehat{\gamma}_j) \\
&\qquad\qquad - \int G'(t;\widehat{\gamma}_i,y)\,\mathrm{d}\widehat{\rho}_{\mathrm{fc}}(y)\bigg)(\lambda_i - \widehat{\gamma}_i)\bigg] \\
&\qquad - \mathbb{E}^{\widehat{\psi}_t\mu_G}\bigg[-\sum_{i=1}^{N}\bigg(\frac{1}{N}\sum_{j=1}^{N} G'(t;\widehat{\gamma}_i,\widehat{\gamma}_j) \\
&\qquad\qquad - \int G'(t;\widehat{\gamma}_i,y)\,\mathrm{d}\widehat{\rho}_{\mathrm{fc}}(y)\bigg)(\lambda_i - \widehat{\gamma}_i)\bigg]\bigg| \\
&\quad + \mathcal{O}\big(N^{1/2-\mathfrak{a}}\big) + \mathcal{O}\big(N^{1-2\mathfrak{a}}\big),
\end{aligned}$$

on Ω. To finish the proof we observe that for all $\widehat{\gamma}_i$,

$$\frac{1}{N}\sum_{j=1}^{N} G'(t;\widehat{\gamma}_i,\widehat{\gamma}_j) = \int G'(t;\widehat{\gamma}_i,y)\,\mathrm{d}\widehat{\rho}_{\mathrm{fc}}(t,y) + \mathcal{O}\big(N^{-1}\big),$$

on Ω, where we use that $\widehat{\gamma}_{i+1} - \widehat{\gamma}_i \sim N^{-2/3}\check{\alpha}_i^{-1}$ ($\check{\alpha}_i = \min\{i, N-i+1\}$), and the square root decay of $\widehat{\rho}_{\mathrm{fc}}(t)$ at the edges of the support. Thus

$$\int \frac{(\mathcal{L} - \partial_t)\widehat{\psi}_t}{\widehat{\psi}_t} f_t \, \mathrm{d}\mu_G = \mathcal{O}(N^{1/2-\mathfrak{a}}) + \mathcal{O}(N^{1-\mathfrak{a}}), \tag{5.39}$$

for N sufficiently large on Ω, where we use one last time the rigidity estimates. Using that $N^{1/2-\mathfrak{a}} < N^{1-2\mathfrak{a}}$, $\mathfrak{a} \in (0, 1/2)$, we get from (5.23), (5.24) and (5.39) the desired estimate (5.15). □

Before we move on to the proof of Corollary 5.4, we give a rough estimate on $S_{\widehat{\omega}_t}(\widehat{g}_t)$ for $t > 0$.

LEMMA 5.6. *There is a constant m such that, for $\tau > 0$ and $t \geq \tau$, we have*

$$S_{\widehat{\omega}_t}(\widehat{g}_t) = S(f_t \mu_G | \widehat{\psi}_t \mu_G) \leq C N^m \tag{5.40}$$

on Ω, for N sufficiently large. Here the constant C depends on τ.

PROOF. From the definition of the relative entropy in (5.8), we have

$$\begin{aligned} &S(f_t \mu_G | \widehat{\psi}_t \mu_G) \\ &\quad \leq S(f_t \mu_G | \mu_G) + \left| \frac{\beta N}{2} \sum_{i=1}^{N} \int \widehat{U}(t, \lambda_i) f_t(\boldsymbol{\lambda}) \, \mathrm{d}\mu_G(\boldsymbol{\lambda}) \right| + \log Z_{\widehat{\psi}_t}. \end{aligned} \tag{5.41}$$

Since the potential $\widehat{U}(t)$ is bounded below, we have (for N sufficiently large on Ω) $\log Z_{\widehat{\psi}_t} \leq C\beta N^2$. Similarly, using the rigidity estimate (5.6), we can bound the second term on the right-hand side of (5.41) by CN^2. To bound the first term on the right of (5.41), we use that $S(f_t \mu_G | \mu_G) \leq S(H_t | W') \leq N^2 \max S(h_{ij,t} | w'_{ij}) + N \max S(h_{ii,t} | w'_{ii})$, where $(h_{ij,t})$ are the entries of the in (5.2) and w'_{ij} are the entries of the GOE, respectively, GUE, matrix W'. By explicit calculations, remembering that the diagonal entries (v_i) are fixed, one finds $\max S(h_{ij,t} | g_{ij}) \leq CN$ for $t \geq \tau$; see, for example, [22]. (Note that we choose $t > 0$; otherwise the relative entropy may be ill defined.) □

To complete the proof of Corollary 5.4 we follow the discussion in [30].

PROOF OF COROLLARY 5.4. Using an approximation argument, we can assume that $\tilde{g}_t \in \mathrm{L}^\infty(\mathrm{d}\tilde{\omega}_t)$. Using first the entropy bound (5.15) and then the Dirichlet form estimate in (5.17), we obtain

$$\begin{aligned} \partial_t S_{\widehat{\omega}_t}(\widehat{g}_t) &\leq -4 D_{\widehat{\omega}_t}(\sqrt{\widehat{g}_t}) + C N^{1-2\mathfrak{a}} \\ &\leq -2 D_{\tilde{\omega}_t}(\sqrt{\tilde{g}_t}) + C N^{1-2\mathfrak{a}} + C \frac{N^{1-2\mathfrak{a}}}{\tau^2} \\ &\leq -C\tau^{-1} S_{\tilde{\omega}_t}(\tilde{g}_t) + C \frac{N^{1-2\mathfrak{a}}}{\tau^2}, \end{aligned}$$

for N sufficiently large on Ω. To get the third line we use the logarithmic Sobolev inequality (5.13) and that, by assumption, $\tau < 1$. Using the entropy estimate (5.18), we thus obtain

$$\partial_t S_{\widehat{\omega}_t}(\widehat{g}_t) \le -C\tau^{-1} S_{\widehat{\omega}_t}(\widehat{g}_t) + C\frac{N^{1-2\mathfrak{a}}}{\tau^2}, \tag{5.42}$$

for N sufficiently large on Ω. Integrating (5.42) from τ to $t/2$, we infer

$$S_{\widehat{\omega}_{t/2}}(\widehat{g}_{t/2}) \le \mathrm{e}^{-C\tau^{-1}(t/2-\tau)} S_{\widehat{\omega}_\tau}(\widehat{g}_\tau) + C\frac{N^{1-2\mathfrak{a}}}{\tau},$$

for N sufficiently large on Ω. Bounding $S_{\widehat{\omega}_\tau}(\widehat{g}_\tau)$ by (5.41), we get

$$S_{\widehat{\omega}_{t/2}}(\widehat{g}_{t/2}) \le CN^m \mathrm{e}^{-C\tau^{-1}(t/2-\tau)} + C\frac{N^{1-2\mathfrak{a}}}{\tau},$$

for N sufficiently large on Ω. Recalling that $t \ge \tau_0 = \tau N^{\varepsilon'}$ and using the monotonicity of the relative entropy, we obtain the first inequality in (5.20).

Integrating (5.15) from $t/2$ to t, we obtain

$$\int_{t/2}^{t} D_{\widehat{\omega}_s}(\sqrt{\widehat{g}_s})\,\mathrm{d}s \le -\int_{t/2}^{t} \partial_s S_{\widehat{\omega}_s}(\widehat{g}_s)\,\mathrm{d}s + CtN^{1-2\mathfrak{a}}.$$

Thus, using the above estimate on the relative entropy and the monotonicity of the Dirichlet form,

$$D_{\widehat{\omega}_t}(\sqrt{\widehat{g}_t}) \le C\frac{N^{1-2\mathfrak{a}}}{t\tau} + CN^{1-2\mathfrak{a}}.$$

Recalling that $t \ge \tau_0 = N^{\varepsilon'}\tau$, we get the second inequality in (5.20). □

6. Local equilibrium measures. The estimates on the relative entropy and the Dirichlet form obtained in Corollary 5.4 do not directly imply that the local statistics of the measures $f_t\mu_G$ and $\widehat{\psi}_t\mu_G$ agree in the limit of large N. However, the averaged local gap statistics of $f_t\mu_G$, $\widehat{\psi}_t\mu_G$ and μ_G can be compared (for $1 \gg t \gg N^{-1/2}$) for large N as is asserted in the main theorems of this section, Theorem 6.1 and Theorem 6.2, below. We first state these results and give a short outline of their proofs in Section 6.1 before going into the details in Sections 6.2–6.5.

6.1. *Averaged local gap statistics for small times.* Recall that we call a symmetric function $O : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, an n-particle observable if O is smooth and compactly supported. For a given observable O, a time $t \ge 0$, a small constant $\alpha > 0$ and $j \in [\![\alpha N, (1-\alpha)N]\!]$, we define an observable $G_{j,n,t}(\mathbf{x}) \equiv G_{j,n}(\mathbf{x})$, by setting

$$\begin{aligned} G_{j,n}(\mathbf{x}) := O\big(&N\rho_j(x_{j+1} - x_j), N\rho_j(x_{j+2} - x_j), \ldots, \\ &N\rho_j(x_{j+n+1} - x_j)\big), \end{aligned} \tag{6.1}$$

$\mathbf{x} = (x_k)_{k=1}^N \in \mathcal{F}^{(N)}$, where we set $G_{j,n} = 0$ if $j + n > (1-\alpha)N$. Here ρ_j denotes the density of the measure $\widehat{\rho}_{\mathrm{fc}}(t)$ at the classical location of the jth particle at time t, that is, $\rho_j := \widehat{\rho}_{\mathrm{fc}}(t, \widehat{\gamma}_j(t))$. We also set

$$G_{j,n,\mathrm{sc}}(\mathbf{x}) := O\big(N\rho_{\mathrm{sc},j}(x_{j+1} - x_j), N\rho_{\mathrm{sc},j}(x_{j+2} - x_j), \ldots, N\rho_{\mathrm{sc},j}(x_{j+n+1} - x_j)\big), \tag{6.2}$$

$\mathbf{x} \in \mathcal{F}^{(N)}$, where $\rho_{\mathrm{sc},j}$ denotes the density of the semicircle law at the classical location of the jth particle with respect to the semicircle law.

In the following, we denote constants depending on O by C_O. Recall the definition of the density $\widehat{\psi}_t$ in (4.21). We have the following statement on the averaged local gap statistics.

THEOREM 6.1. *Let $n \in \mathbb{N}$ be fixed, and consider an n-particle observable O. Fix a small constant $\alpha > 0$, and consider an interval of consecutive integers $J \subset [\![\alpha N, (1-\alpha)N]\!]$ in the bulk. Then, for any small $\delta > 0$, there is a constant $\mathfrak{f} > 0$ such that, for $t \geq N^{-1/2+\delta}$,*

$$\left| \int \frac{1}{|J|} \sum_{j \in J} G_{j,n}(\mathbf{x}) f_t(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) - \int \frac{1}{|J|} \sum_{j \in J} G_{j,n}(\mathbf{x}) \widehat{\psi}_t(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) \right| \leq C_O N^{-\mathfrak{f}}, \tag{6.3}$$

for N sufficiently large on Ω. The constant C_O depends on α and O, and the constant $\mathfrak{f}$ depends on α and δ.

We can also compare the averaged local gap statistics of $f_t \mu_G$, with the averaged local gap statistics of the Gaussian unitary, respectively, orthogonal, ensemble.

THEOREM 6.2. *Under the same assumptions as in Theorem 6.1 and with similar constants, we have*

$$\left| \int \frac{1}{|J|} \sum_{j \in J} G_{j,n}(\mathbf{x}) f_t(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) - \int \frac{1}{|J|} \sum_{j \in J} G_{j,n,\mathrm{sc}}(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) \right| \leq C_O N^{-\mathfrak{f}},$$

for N sufficiently large on Ω.

The proofs of Theorem 6.1 and Theorem 6.2 proceed in two steps. We first localize the measures $f_t \mu_G$ and $\widehat{\psi}_t, \mu_G$; that is, we study the statistics of $\mathcal{K}$, $1 \ll \mathcal{K} \ll N$, consecutive particles inside the bulk—the interior particles—with the remaining particles—the exterior particles—being fixed; for details, see Section 6.2. For most configurations of the exterior particles (boundary conditions), we can compare the statistics of the localized versions of $f_t \mu_G$ and $\widehat{\psi}_t, \mu_G$. This

is accomplished in Proposition 6.4 of Section 6.3 by using that (1) the localized β-ensemble satisfies a logarithmic Sobolev inequality (6.23) with constant $C\mathcal{K}/N$ and that (2) the localized Dirichlet form can be controlled by the global Dirichlet form [see (6.24)], the latter being estimated in Corollary 5.4.

In a second step, we use Theorem 4.1 of [31] that, roughly speaking, assures that the local gap statistics of localized β-ensembles are essentially independent of the boundary conditions and indeed agree with the local gap statistics of the Gaussian ensembles. Putting this universality result to work in Section 6.4, we conclude that the local gap statistics of the localized version of the measure $f_t \mu_G$ are universal, for $1 \gg t \gg N^{-1/2}$ and for most boundary conditions. Theorems 6.1 and 6.2 are then proven in Section 6.5 by integrating out the boundary conditions.

We conclude this subsection with the following two remarks: once the entropy estimate of Proposition 5.3 has been established, one can apply the methods of [31] to prove the gap universality in the bulk for deformed Wigner matrices; see Remark 2.8 above for an explicit statement; we leave the details to the interested readers.

As an alternative to the approach outlined above, one could combine the approach from [30] with Theorem 2.1 in [12] (see Theorem 4.1 above), to prove Theorems 6.1 and 6.2.

6.2. *Preliminaries.* Let $\alpha, \sigma > 0$ be two small positive numbers, and choose two integer parameters L and K such that

$$L \in [\![\alpha N, (1-\alpha)N]\!], \qquad K \in [\![N^{\sigma}, N^{1/4}]\!]. \tag{6.4}$$

We denote by $I_{L,K} := [\![L-K, L+K]\!]$ a set of $\mathcal{K} := 2K+1$ consecutive indices in the bulk of the spectrum. Below we often abbreviate $I \equiv I_{L,K}$. Recall the definition of the set $\mathcal{F}^{(N)} \subset \mathbb{R}^N$ in (4.1). For $\boldsymbol{\lambda} \in \mathcal{F}^{(N)}$, we write

$$\boldsymbol{\lambda} = (y_1, \dots, y_{L-K-1}, x_{L-K}, \dots, x_{L+K}, y_{L+K+1}, \dots, y_N), \tag{6.5}$$

and we call $\boldsymbol{\lambda}$ a configuration (of N particles or points on the real line). Note that on the right-hand side of (6.5) the points keep their original indices and are in increasing order so that

$$\begin{aligned} \mathbf{x} &= (x_{L-K}, \dots, x_{K+L}) \in \mathcal{F}^{(\mathcal{K})}, \\ \mathbf{y} &= (y_1, \dots, y_{L-K-1}, y_{L+K+1}, \dots, y_N) \in \mathcal{F}^{(N-\mathcal{K})}. \end{aligned} \tag{6.6}$$

We refer to $\mathbf{x}$ as the interior points or particles and to $\mathbf{y}$ as the exterior points or particles.

In the following, we often fix the exterior points and consider the conditional measures on the interior points: let ω be a measure on $\mathcal{F}^{(N)}$ with a density. Then we denote by $\omega^{\mathbf{y}}$ the measure obtained by conditioning on $\mathbf{y}$; that is, for $\boldsymbol{\lambda}$ in the form of (6.5),

$$\omega^{\mathbf{y}}(d\mathbf{x}) \equiv \omega^{\mathbf{y}}(\mathbf{x})\, d\mathbf{x} := \frac{\omega(\boldsymbol{\lambda})\, d\mathbf{x}}{\int \omega(\boldsymbol{\lambda})\, d\mathbf{x}} = \frac{\omega(\mathbf{x}, \mathbf{y})\, d\mathbf{x}}{\int \omega(\mathbf{x}, \mathbf{y})\, d\mathbf{x}},$$

where, with slight abuse of notation, $\omega(\mathbf{x}, \mathbf{y})$ stands for $\omega(\boldsymbol{\lambda})$. We refer to the fixed exterior points $\mathbf{y}$ as boundary conditions of the measure $\omega^{\mathbf{y}}$. For fixed $\mathbf{y} \in F^{(N-\mathcal{K})}$, all (x_i) lie in the open configuration interval

$$\mathbf{I} \equiv \mathbf{I}_{L,K} := (y_{L-K-1}, y_{L+K+1}).$$

Set $\bar{y} := (y_{L-K-1} + y_{L+K+1})/2$, and let

$$\alpha_j := \bar{y} + \frac{j-L}{\mathcal{K}+1}|\mathbf{y}| \qquad (j \in I_{L,K}) \tag{6.7}$$

denote $\mathcal{K}$ equidistant points in the interval $\mathbf{I}$.

Let $U \in C^4(\mathbb{R})$ be a "regular" potential satisfying (4.4) and (4.5). We then consider the β-ensemble

$$\mu(\mathrm{d}\boldsymbol{\lambda}) \equiv \mu_U(\mathrm{d}\boldsymbol{\lambda}) := \frac{1}{Z_U} \mathrm{e}^{-\beta N \mathcal{H}(\boldsymbol{\lambda})}\, \mathrm{d}\boldsymbol{\lambda} \qquad (\beta > 0), \tag{6.8}$$

with [cf. (4.2)]

$$\mathcal{H}(\boldsymbol{\lambda}) := \sum_{i=1}^{N} \frac{1}{2}\Big(U(\lambda_i) + \frac{\lambda_i^2}{2}\Big) - \frac{1}{N} \sum_{1 \le i < j \le N} \log|\lambda_j - \lambda_i|, \tag{6.9}$$

and with $Z_U \equiv Z_U(\beta)$ a normalization. For K, L and $\mathbf{y}$ fixed, we can write $\mu^{\mathbf{y}}$ as the Gibbs measure

$$\mu^{\mathbf{y}}(\mathrm{d}\mathbf{x}) = \frac{1}{Z_U^{\mathbf{y}}} \mathrm{e}^{-\beta N \mathcal{H}^{\mathbf{y}}(\mathbf{x})}\, \mathrm{d}\mathbf{x}, \tag{6.10}$$

where

$$\mathcal{H}^{\mathbf{y}}(\mathbf{x}) = \sum_{i \in I} \frac{1}{2} V^{\mathbf{y}}(x_i) - \frac{1}{N} \sum_{\substack{i,j \in I \\ i<j}} \log|x_j - x_i|, \tag{6.11}$$

with

$$V^{\mathbf{y}}(x) = U(x) + \frac{x^2}{2} - \frac{2}{N} \sum_{i \notin I} \log|x - y_i| \tag{6.12}$$

an external potential and with $Z_U^{\mathbf{y}} \equiv Z_U^{\mathbf{y}}(\beta)$ a normalization. Following [31], we next introduce the notion of regular external potential:

DEFINITION 6.3. An external potential $V \equiv V^{\mathbf{y}}$ of a β-ensemble of K points in a configuration interval $\mathbf{I} = (a, b)$ is called K^{χ}-regular if the following bounds hold:

$$|\mathbf{I}| = \frac{\mathcal{K}}{N\rho(\bar{y})} + \mathcal{O}\Big(\frac{K^{\chi}}{N}\Big), \tag{6.13}$$

$$V'(x) = \rho(\bar{y}) \log \frac{d_+(x)}{d_-(x)} + \mathcal{O}\Big(\frac{K^{\chi}}{N d(x)}\Big), \tag{6.14}$$

$$V''(x) \ge 1 + \inf U''(x) + \frac{c}{d(x)}, \tag{6.15}$$

for $x \in \mathbf{I}$, with some $c > 0$ and for some small $\chi > 0$, where

$$d(x) := \min\{|x-a|, |x-b|\}$$

denotes the distance to the boundary of $\mathbf{I}$,

$$d_-(x) := d(x) + \rho(\bar{y})N^{-1}K^{\chi}$$

and

$$d_+(x) := \max\{|x-a|, |x-b|\} + \rho(\bar{y})N^{-1}K^{\chi}.$$

The main technical result we use in this section is Theorem 4.1 of [31]; see Theorem 6.5 below. It asserts that the local gap statistics of $\mu^{\mathbf{y}}$ are essentially independent of $\mathbf{y}$ and U, provided that $V^{\mathbf{y}}$ is K^{χ}-regular for some small $\chi > 0$.

6.3. *Comparison of local measures.* Fix small $\alpha, \sigma > 0$, and let K and L satisfy (6.4). Recall that we denote by $f_t\mu_G$ the distribution of the eigenvalues of the matrix in (5.2) and by $\widehat{\psi}_t\mu_G$ the reference β-ensemble defined in (5.10). Following the discussion in Section 6.2, we introduce the conditioned densities

$$f_t^{\mathbf{y}}\mu_G^{\mathbf{y}} = (f_t\mu_G)^{\mathbf{y}}, \qquad \widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}} = (\widehat{\psi}_t\mu_G)^{\mathbf{y}}. \tag{6.16}$$

Recall that we denote by $\widehat{\rho}_{\mathrm{fc}}(t)$ the equilibrium density of $\widehat{\psi}_t\mu_G$ and by $\widehat{\gamma}_k \equiv \widehat{\gamma}_k(t)$ the classical location of the kth particle with respect to $\widehat{\rho}_{\mathrm{fc}} \equiv \widehat{\rho}_{\mathrm{fc}}(t)$; cf. (3.14). Let $\varepsilon_0 > 0$ and define the set of "good" boundary conditions, $\mathcal{R}_{L,K} \equiv \mathcal{R}_{L,K}(\varepsilon_0, \alpha)$,

$$\begin{aligned}\mathcal{R}_{L,K} := \big\{\boldsymbol{\lambda} \in F^{(N)} : |\lambda_k - \widehat{\gamma}_k| \le N^{-1+\varepsilon_0}, \forall k \in [\![\alpha N, (1-\alpha)N]\!] \setminus I_{L,K}\big\} \\ \cap \big\{\boldsymbol{\lambda} \in F^{(N)} : |\lambda_k - \widehat{\gamma}_k| \le N^{-2/3+\varepsilon_0}, \forall k \in [\![1, N]\!]\big\}.\end{aligned} \tag{6.17}$$

The next result compares the local statistics of $f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$ and $\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$ for $\mathbf{y} \in \mathcal{R}_{L,K}$. Recall that $\mathfrak{a}$ stands for any number in $(0, 1/2)$.

PROPOSITION 6.4. *Fix small constants $\alpha, \sigma > 0$ [see (6.4)] and $\varepsilon_0 > 0$; see (6.17). Let K satisfy (6.4), and let O be an n-particle observable. Let $\varepsilon' > 0$, and choose τ satisfying $1 \gg \tau > N^{-2\mathfrak{a}}$. Then, for any $t \ge N^{\varepsilon'}\tau$ and any constant $\mathfrak{c} \in (0, 1)$, there is a set of configurations $\mathcal{G} \equiv \mathcal{G}_{L,K}(\varepsilon_0, \alpha) \subset \mathcal{R}_{L,K}(\varepsilon_0, \alpha)$, with*

$$\mathbb{P}^{f_t\mu_G}(\mathcal{G}) \ge 1 - \frac{N^{-\mathfrak{c}}}{2}, \tag{6.18}$$

such that

$$\left|\int O(\mathbf{x})\big(f_t^{\mathbf{y}}(\mathbf{x}) - \widehat{\psi}_t^{\mathbf{y}}(\mathbf{x})\big)\mu_G^{\mathbf{y}}(\mathrm{d}\mathbf{x})\right| \le C_O\sqrt{K}N^{\mathfrak{c}-\mathfrak{a}}\tau^{-1}, \tag{6.19}$$

$t \ge N^{\varepsilon'}\tau$, for N sufficiently large on Ω. The constant C_O, depends only on ε', α and O.

Moreover, there is $\upsilon > 0$, *such that*

$$\mathbb{P}^{f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\big(\{|x_k - \widehat{\gamma}_k(t)| < N^{-1+\varepsilon_0}, k \in I_{L,K}\}\big) \geq 1 - \mathrm{e}^{-\upsilon(\varphi_N)^{\xi}}, \tag{6.20}$$

$t \geq N^{\varepsilon'}\tau$, *for* N *sufficiently large on* Ω, *with* $\xi = A_0 \log\log N/2$; *see* (2.20).

PROOF. We follow closely the proof of Lemma 6.4 in [31]. Let τ satisfy $1 \gg \tau > N^{-2\mathfrak{a}}$, and choose $t \geq N^{\varepsilon'}\tau$. We estimate

$$\begin{aligned}\left|\int O(\mathbf{x})\big(f_t^{\mathbf{y}}(\mathbf{x}) - \widehat{\psi}_t^{\mathbf{y}}(\mathbf{x})\big)\mu_G^{\mathbf{y}}(\mathrm{d}\mathbf{x})\right| &\leq C_O \|f_t^{\mathbf{y}}\mu_G^{\mathbf{y}} - \widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}\|_1 \\ &\leq C_O\sqrt{S_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}(\widehat{g}_t^{\mathbf{y}})},\end{aligned} \tag{6.21}$$

where we use (5.9) and set $\widehat{g}_t := f_t/\widehat{\psi}_t$. For $\mathbf{y} \in \mathcal{R}_{L,K}$, we consider the locally constrained measure $\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$, explicitly given by

$$\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}(\mathrm{d}\mathbf{x}) = \frac{1}{Z^{\mathbf{y}}}\mathrm{e}^{-N\beta\widehat{\mathcal{H}}^{\mathbf{y}}(t,\mathbf{x})}\,\mathrm{d}\mathbf{x},$$

with

$$\begin{aligned}\widehat{\mathcal{H}}^{\mathbf{y}}(t,\mathbf{x}) = \sum_{k\in I}\left(\frac{\widehat{U}(t,x_k)}{2} + \frac{x_k^2}{4}\right) \\ - \frac{1}{N}\sum_{\substack{k,l\in I\\ k<l}}\log|x_k - x_l| - \frac{1}{N}\sum_{\substack{k\in I\\ l\notin I}}\log|x_k - y_l|.\end{aligned}$$

Here $I \equiv I_{L,K}$. From (5.20) of [31], we know that

$$\nabla_{\mathbf{x}}^2\widehat{\mathcal{H}}^{\mathbf{y}}(t,\mathbf{x}) \geq cN/K \qquad (\mathbf{y} \in \mathcal{R}_{L,K}), \tag{6.22}$$

for some $c > 0$ independent of t. Here, $\nabla_{\mathbf{x}}^2$ denotes the Hessian with respect the variables $\mathbf{x}$. Thus the Bakry–Émery criterion yields the logarithmic Sobolev inequality

$$S_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\big(\widehat{g}_t^{\mathbf{y}}\big) \leq \frac{CK}{N}D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big) \qquad (\mathbf{y} \in \mathcal{R}_{L,K}), \tag{6.23}$$

where the constant C can be chosen independent t.

For $k \in [\![1, N]\!]$, denote by $D_{\widehat{\psi}_t\mu_G,k}$ the Dirichlet form of the particle k, that is, $D_{\widehat{\psi}_t\mu_G,k}(f) := \frac{1}{2N}\int|\partial_k f|^2\widehat{\psi}_t\mu_G$, and by $D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}},k}$ its conditioned analogue (with $k \in I_{L,K}$). Using the notation of (6.5), we may write

$$\mathbb{E}^{f_t\mu_G}D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big) = \int D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big)f_t(\boldsymbol{\lambda})\mu_G(\mathrm{d}\boldsymbol{\lambda}),$$

and we can bound

$$\begin{aligned}\mathbb{E}^{f_t\mu_G} D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big) &= \mathbb{E}^{f_t\mu_G}\sum_{k\in I} D_{\widehat{\psi}_t^{\mathbf{y}}\mu_{G,k}^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big)\\ &\le D_{\widehat{\psi}_t\mu_G}(\sqrt{\widehat{g}_t})\\ &\le CN^{1-2\mathfrak{a}}\tau^{-2},\end{aligned} \tag{6.24}$$

for N sufficiently large, where we use Corollary 5.4 in the last line. Thus Markov's inequality implies, for $\mathfrak{c} > 0$, that there exists a set of configurations $\mathcal{G}^1 \subset \mathcal{R}$, with $\mathbb{P}^{f_t\mu_G}(\mathcal{G}^1) \ge 1 - N^{-\mathfrak{c}}$, such that, for $\mathbf{y} \in \mathcal{G}^1$,

$$D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big) \le CN^{2\mathfrak{c}}N^{1-2\mathfrak{a}}\tau^{-2} \tag{6.25}$$

holds for N sufficiently large on Ω. Substituting (6.25) into (6.23) and then into (6.21), we find that

$$\left|\int O(\mathbf{x})(f_t^{\mathbf{y}} - \widehat{\psi}_t^{\mathbf{y}})\mu_G^{\mathbf{y}}(\mathrm{d}\mathbf{x})\right| \le C_O\sqrt{K}N^{\mathfrak{c}}N^{-\mathfrak{a}}\tau^{-1},$$

on Ω for N sufficiently large. This proves (6.19).

To prove (6.20) note that the rigidity estimates of Lemma 5.6 imply

$$\begin{aligned}&\mathbb{E}^{f_t\mu_G}\big[\mathbb{P}^{f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\big(\{|x_k - \widehat{\gamma}_k(t)| > N^{-1+\varepsilon}, k \in I\}\big)\big]\\ &\quad= \mathbb{P}^{f_t\mu_G}\big(\{|x_k - \widehat{\gamma}_k(t)| > N^{-1+\varepsilon}, k \in I\}\big) \le \mathrm{e}^{-\upsilon(\varphi_N)^{\xi}},\end{aligned}$$

for some $\upsilon > 0$, where we have chosen $\xi = A_0 \log\log N/2$. By Markov's inequality we conclude that there is a set of configurations, $\mathcal{G}^2$, such that (6.20) holds with (ξ, υ)-high probability. Finally, set $\mathcal{G} := \mathcal{G}^1 \cap \mathcal{G}^2$, and note that $\mathcal{G}$ satisfies (6.18). □

6.4. *Gap universality for local measures.* In Section 6.3, we show that the local gap statistics of the measure $f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$ agree with those of $\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$ for boundary conditions $\mathbf{y}$ in the set $\mathcal{R}_{L,K}$. In this subsection, we are going to show that the local statistics of $\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$ are essentially independent of the precise form of $\mathbf{y}$, as is asserted by the main theorem of [31]. Recall the notion of external potential introduced in (6.12).

THEOREM 6.5 (Gap universality for local measures, Theorem 4.1 in [31]). *Let $L, \widetilde{L}$ and $\mathcal{K} = 2K + 1$ satisfy (6.4) with $\alpha, \sigma > 0$. Consider two boundary conditions $\mathbf{y}, \tilde{\mathbf{y}}$ such that the configuration intervals coincide, that is,*

$$\mathbf{I} = (y_{L-K-1}, y_{L+K+1}) = (\tilde{y}_{\widetilde{L}-K-1}, \tilde{y}_{\widetilde{L}+K+1}). \tag{6.26}$$

Consider two measures μ and $\tilde{\mu}$ in the form of (6.8), with possibly two different potentials U and $\widetilde{U}$, and consider the constrained measures $\mu^{\mathbf{y}}$ and $\tilde{\mu}^{\tilde{\mathbf{y}}}$. Let $\chi > 0$,

and assume that the external potentials $V^{\mathbf{y}}$ *and* $\tilde{V}^{\tilde{\mathbf{y}}}$ [*see* (6.12)] *are* K^{χ}*-regular*; *see Definition* 6.3. *In particular, assume that* $\mathbf{I}$ *satisfies*

$$|\mathbf{I}| = \frac{\mathcal{K}}{N\varrho_U(\bar{y})} + \mathcal{O}\Big(\frac{K^{\chi}}{N}\Big). \tag{6.27}$$

Assume further that

$$\max_{j\in I_{L,K}} \big|\mathbb{E}^{\mu^{\mathbf{y}}} x_j - \alpha_j\big| + \max_{j\in I_{\tilde{L},K}} \big|\mathbb{E}^{\tilde{\mu}^{\tilde{\mathbf{y}}}} x_j - \alpha_j\big| \le CN^{-1}K^{\chi}. \tag{6.28}$$

Let $p\in\mathbb{Z}$ *satisfy* $|p|\le K - K^{1-\chi'}$, *for some small* $\chi'>0$. *Fix* $n\in\mathbb{N}$. *Then there is a constant* χ_0, *such that if* $\chi,\chi'<\chi_0$, *then for any* n*-particle observable* O, *we have*

$$\begin{aligned}\big|&\mathbb{E}^{\mu^{\mathbf{y}}} O\big(N(x_{L+p+1}-x_{L+p}),\dots,N(x_{L+p+n}-x_{L+p})\big)\\ &- \mathbb{E}^{\tilde{\mu}^{\tilde{\mathbf{y}}}} O\big(N(x_{L+p+1}-x_{L+p}),\dots,N(x_{L+p+n}-x_{L+p})\big)\big| \le C_O K^{-\mathfrak{b}},\end{aligned}$$

for some constant $\mathfrak{b}>0$ *depending on* σ, α, *and for some constant* C_O *depending on* O. *This holds for* N *sufficiently large* [*depending on the* χ, χ', α *and* C *in* (6.28)].

Recall that the measure $\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$ can be written as the Gibbs measure

$$\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}(\mathrm{d}\mathbf{x}) = \frac{1}{Z^{\mathbf{y}}_{\widehat{\psi}_t}} \mathrm{e}^{-N\beta\mathcal{H}^{\mathbf{y}}(t,\mathbf{x})}\,\mathrm{d}\mathbf{x}, \tag{6.29}$$

where

$$\mathcal{H}^{\mathbf{y}}(t,\mathbf{x}) = \sum_{i\in I}\frac{1}{2}V^{\mathbf{y}}(t,x_i) - \frac{1}{N}\sum_{\substack{i,j\in I\\ i<j}}\log|x_j-x_i|, \tag{6.30}$$

with the external potential

$$V^{\mathbf{y}}(t,x) = \widehat{U}(t,x) + \frac{x^2}{2} - \frac{2}{N}\sum_{i\notin I}\log|x-y_i|. \tag{6.31}$$

Using Theorem 6.5 we first show that the local statistics of $\psi_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$ are virtually independent of $\mathbf{y}$; that is, we apply Theorem 6.5 with $\mu^{\mathbf{y}} = (\widehat{\psi}_t\mu_G)^{\mathbf{y}}$ and $\tilde{\mu}^{\tilde{\mathbf{y}}} = (\widehat{\psi}_t\mu_G)^{\tilde{\mathbf{y}}}$.

We first check the regularity assumption of the external potential $V^{\mathbf{y}}$. Recall the definition of K^{χ}-regular potential in Definition 6.3.

LEMMA 6.6. *Fix small constants* $\alpha,\sigma>0$; *see* (6.4). *Let* $\chi>0$, *and consider* $\mathbf{y}\in\mathcal{R}_{L,K}(\chi\sigma/2,\alpha/2)$. *Then, on the event* Ω, *the external potential* $V^{\mathbf{y}}(t,x)$ *in* (6.31) *is* K^{χ}*-regular on* $\mathbf{I} = (y_{L-K-1}, y_{L+K+1})$.

The proof of Lemma 6.6 follows almost verbatim the proof of Lemma 4.5 in the Appendix A of [31], and we therefore omit it here.

To check that assumption (6.28) of Theorem 6.5 holds, we use the following result. Recall the set of configurations $\mathcal{G}$ of Proposition 6.4.

LEMMA 6.7. *Under the assumptions of Proposition 6.4 the following holds. Let* $\mathbf{y} \in \mathcal{G}$. *Then, for all* $k \in I_{L,K}$,

$$\left|\mathbb{E}^{f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}} x_k - \mathbb{E}^{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}} x_k\right| \le C\frac{KN^{2\mathfrak{c}}}{N} N^{-\mathfrak{a}}\tau^{-1} \qquad \left(t \ge \tau N^{\varepsilon'}\right), \tag{6.32}$$

for N sufficiently large on Ω.

PROOF. We follow the proof of Lemma 6.5 of [31]. Fix $t \ge \tau N^{\varepsilon'}$, where $1 \gg \tau \ge N^{-2\mathfrak{a}}$. Let $\mathbf{y} \in \mathcal{G}$. Denote by $\mathcal{L}_t^{\mathbf{y}}$ the generator associated to the Dirichlet form $D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}$, that is,

$$\int f\mathcal{L}_t^{\mathbf{y}} g\widehat{\psi}_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}} = -\frac{1}{\beta N}\sum_{i\in I}\int \partial_i f\,\partial_i g\widehat{\psi}_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}} \qquad (I \equiv I_{L,K}).$$

Let q_s be the solution of the evolution equation $\partial_s q_s = \mathcal{L}_t^{\mathbf{y}} q_s$, $s \ge 0$, with initial condition $q_0 := \widehat{g}_t^{\mathbf{y}} = f_t^{\mathbf{y}}/\widehat{\psi}_t^{\mathbf{y}}$. Note that q_s is a density with respect to the reversible measure, $\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$, of this dynamics. Hence, we can write

$$\begin{aligned}\left|\mathbb{E}^{f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}} x_k - \mathbb{E}^{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}} x_k\right| &= \left|\int_0^\infty \mathrm{d}s \int x_k \mathcal{L}_t^{\mathbf{y}} q_s \widehat{\psi}_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}}\right| \\ &= \left|\frac{1}{\beta N}\int_0^\infty \mathrm{d}s \int \partial_k q_s \widehat{\psi}_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}}\right|.\end{aligned}$$

Recall that $\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}$ satisfies the logarithmic Sobolev inequality (6.23) with constant $\tau_K := CK/N$, provided that $\mathbf{y} \in \mathcal{R}_{L,K}$. Thus, upon using Cauchy–Schwarz and the exponential decay of the Dirichlet form $D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}(\sqrt{q_s})$, we obtain for some $\upsilon', c > 0$,

$$\left|\mathbb{E}^{f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}} x_k - \mathbb{E}^{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}} x_k\right| = \left|\frac{1}{\beta N}\int_0^{N^{\upsilon'}\tau_K} \mathrm{d}s \int \partial_k q_s \widehat{\psi}_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}}\right| + \mathcal{O}(\mathrm{e}^{-cN^{\upsilon'}}).$$

Using

$$|\partial_k q_s| = 2|\sqrt{q_s}\partial_k\sqrt{q_s}| \le R(\partial_k\sqrt{q_s})^2 + R^{-1}q_s,$$

where $R > 0$ is a free parameter, we obtain

$$\begin{aligned}&\left|\frac{1}{\beta N}\int_0^{N^{\upsilon'}\,\mathrm{d}s\tau_K} \int \partial_k q_s \widehat{\psi}_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}}\right| \\ &\qquad \le R\left[\int_0^{N^{\upsilon'}\tau_K} \mathrm{d}s\, D_{f_t^{\mathbf{y}}\mu_G}(\sqrt{q_s})\right] + \frac{1}{2}R^{-1}N^{-1+\upsilon'}\tau_K\end{aligned}$$

$$\leq RS_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}(\widehat{g}_t^{\mathbf{y}}) + \frac{1}{2}R^{-1}N^{-1+\upsilon'}\tau_K$$

$$\leq CR\tau_K D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big) + \frac{1}{2}R^{-1}N^{-1+\upsilon'}\tau_K,$$

where in the second line we use that the time integral of the Dirichlet form is bounded by the initial entropy (see, e.g., Theorem 2.3 in [30]) and in the final line we used the logarithmic Sobolev inequality (6.23). Optimizing over R, we get

$$\big|\mathbb{E}^{f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}x_k - \mathbb{E}^{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}x_k\big| \leq C\tau_K\Big(N^{-1+\upsilon'}D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big)\Big)^{1/2} + \mathcal{O}(\mathrm{e}^{-cN^{\upsilon'}})$$

$$\leq C\frac{KN^{\upsilon'/2}}{N^{3/2}}\Big(D_{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}\Big(\sqrt{\widehat{g}_t^{\mathbf{y}}}\Big)\Big)^{1/2} + \mathcal{O}(\mathrm{e}^{-cN^{\upsilon'}}),$$

where we used that $\tau_K = CK/N$. Using (6.25) we finally obtain

$$\big|\mathbb{E}^{f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}x_k - \mathbb{E}^{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}x_k\big| \leq C\frac{KN^{2\mathfrak{c}}}{N}N^{-\mathfrak{a}}\tau^{-1} + \mathcal{O}(\mathrm{e}^{-cN^{\upsilon'}}),$$

for N sufficiently large on Ω. □

LEMMA 6.8. *Fix small constants $\alpha, \sigma > 0$. Fix $\varepsilon' > 0$ and $t \geq \tau N^{\varepsilon'}$, where τ satisfies $1 \gg \tau \geq N^{-2\mathfrak{a}}$. Fix $n \in \mathbb{N}$, and consider an n-particle observable O. Let $\chi', \chi > 0$, with $\chi', \chi < \chi_0$, where χ_0 is the constant in Theorem 6.5. Then the following holds.*

Assume that $0 < \mathfrak{a} < 1/2$, $0 < \mathfrak{c} < 1$, $N^{-2\mathfrak{a}} \leq \tau \ll 1$ and $K \in [\![N^{\sigma}, N^{1/4}]\!]$ are chosen such that

$$\frac{KN^{2\mathfrak{c}}}{N}N^{-\mathfrak{a}}\tau^{-1} \leq \frac{K^{\chi}}{N}. \tag{6.33}$$

Let p be an integer satisfying $|p| \leq K - K^{1-\chi'}$. Let $\mathbf{y} \in \mathcal{G}_{L,K}(\frac{\chi^2\sigma}{2}, \frac{\alpha}{2})$. Then, for the observable G, as defined in (6.1), we have

$$\left|\int G_{L+p,n}(\mathbf{x})\big(f_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}} - \widehat{\psi}_t\,\mathrm{d}\mu_G\big)\right| \leq C_O K^{-\mathfrak{b}} + C_O\sqrt{K}N^{\mathfrak{c}}N^{-\mathfrak{a}}\tau^{-1}, \tag{6.34}$$

for N sufficiently large on Ω, where the constant C_O depends on O and ε', and the constant $\mathfrak{b} > 0$ depends on α and σ.

PROOF. We follow [31]. Fix $t \geq \tau N^{\varepsilon'}$ and $\chi > 0$. Let $\mathbf{y} \in \mathcal{G}_{L,K}(\frac{\chi^2\sigma}{2}, \alpha) \subset \mathcal{G}_{L,K}(\frac{\chi\sigma}{2}, \alpha)$. Then by Proposition 6.4 and the assumption in (6.33),

$$\big|\mathbb{E}^{f_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}x_k - \widehat{\gamma}_k(t)\big| \leq CK^{\chi}N^{-1},$$

for all $k \in I \equiv I_{L,K}$. Further, from Lemma 6.7 and the assumption in (6.33) we get

$$\big|\mathbb{E}^{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}}x_k - \widehat{\gamma}_k(t)\big| \leq CK^{\chi}N^{-1}, \tag{6.35}$$

for all $k \in I$. Recall from (6.7) that we denote by $\bar{y} := \frac{1}{2}(y_{L-K-1} + y_{L+K+1})$ the midpoint of the configuration interval $\mathbf{I}$ and that (α_k) denote $2K+1$ equidistant points in $\mathbf{I}$. As shown in Lemmas 4.5 and 5.2 of [31], we have

$$\left|\widehat{\gamma}_k(t) - \alpha_k\right| \le CK^{\chi}N^{-1},$$

for all $k \in I$, provided that $\mathbf{y} \in \mathcal{G}_{L,K}(\frac{\chi\sigma}{2}, \alpha)$. We hence obtain

$$\left|\mathbb{E}^{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}} x_k - \alpha_k\right| \le CK^{\chi}N^{-1}, \tag{6.36}$$

for N sufficiently large on Ω.

Proposition 6.4 implies that there is C_O such that

$$\left|\int G_{L+p,n}(\mathbf{x})\left(f_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}} - \widehat{\psi}_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}}\right)\right| \le C_O\sqrt{K}N^{\mathfrak{c}}N^{-\mathfrak{a}}\tau^{-1} \quad \left(t \ge N^{\varepsilon'}\tau\right), \tag{6.37}$$

for $\mathbf{y} \in \mathcal{G}_{L,K}(\frac{\chi\sigma}{2}, \alpha)$, N sufficiently large on Ω.

For $\alpha, \varepsilon_0, \varsigma_1 > 0$ and a β-ensemble μ on $I^{(N)}$, define a set of particle configurations $\mathcal{R}_\mu^* \equiv \mathcal{R}_\mu^*(\varepsilon_0, \alpha)$ by

$$\mathcal{R}_\mu^* := \left\{\mathbf{y} \in I^{(N-K)} : \mathbb{P}^{\mu^{\mathbf{y}}}\left(|x_k - \gamma_k| > N^{-1+\varepsilon_0}\right) \le \mathrm{e}^{-(1/2)N^{\varsigma_1}}, \forall k \in I_{L,K}\right\},$$

where γ_k denotes the classical location of the kth particle with respect to the equilibrium measure of μ.

As in the proof of Proposition 6.4, it follows from Markov's inequality and the rigidity bound for the β-ensemble $\widehat{\psi}_t\mu_G$ in Lemma 5.5 that we can choose $\mathcal{R}^*_{\widehat{\psi}_t\mu_G} \subset \mathcal{R}_{L,K}$ and that $\mathbb{P}^{\widehat{\psi}_t\mu_G}(\mathcal{R}^*_{\widehat{\psi}_t\mu_G}) \ge 1 - c\mathrm{e}^{-(1/2)N^{\varsigma_1}}$, for some $c > 0$, possibly after decreasing ς_1 by a small amount. For $\tilde{\mathbf{y}} \in \mathcal{R}^*_{\widehat{\psi}_t\mu_G}(\frac{\chi^2\sigma}{2}, \frac{\alpha}{2})$, Lemma 5.1 of [31] implies that

$$\left|\mathbb{E}^{\widehat{\psi}_t^{\tilde{\mathbf{y}}}\mu_G^{\tilde{\mathbf{y}}}} x_k - \alpha_k\right| \le CK^{\chi}N^{-1}, \tag{6.38}$$

for N sufficiently large on Ω. Thus together with (6.36), we have on Ω

$$\left|\mathbb{E}^{\widehat{\psi}_t^{\tilde{\mathbf{y}}}\mu_G^{\tilde{\mathbf{y}}}} x_k - \alpha_k\right| + \left|\mathbb{E}^{\widehat{\psi}_t^{\mathbf{y}}\mu_G^{\mathbf{y}}} x_k - \alpha_k\right| \le CK^{\chi}N^{-1}, \tag{6.39}$$

for N sufficiently large, for all $\mathbf{y} \in \mathcal{G}(\frac{\chi\sigma}{2}, \alpha)$ and all $\tilde{\mathbf{y}} \in \mathcal{R}^*_{\widehat{\psi}_t\mu_G}(\frac{\chi^2\sigma}{2}, \frac{\alpha}{2})$.

We now apply Theorem 6.5: let $\tilde{\mathbf{y}}$ and $\mathbf{y}$ be as above. By the scaling argument of Lemma 5.3 in [31], we can assume that the two configuration intervals $\tilde{\mathbf{I}}$ and $\mathbf{I}$ agree, so that assumption (6.26) of Theorem 6.5 holds. Moreover, by Lemma 6.6 we know that $V^{\mathbf{y}}$ and $V^{\tilde{\mathbf{y}}}$ are K^{χ}-regular external potentials. The assumption in (6.28) of Theorem 6.5 is satisfied by (6.39). Thus Theorem 6.5 implies that there is $\mathfrak{b} > 0$, depending on σ and α, such that

$$\left|\int G_{L+p,n}(\mathbf{x})\left(\widehat{\psi}_t^{\mathbf{y}}\,\mathrm{d}\mu_G^{\mathbf{y}} - \widehat{\psi}_t^{\tilde{\mathbf{y}}}\,\mathrm{d}\mu_G^{\tilde{\mathbf{y}}}\right)\right| \le C_O K^{-\mathfrak{b}}, \tag{6.40}$$

for N sufficiently large on Ω. Since estimate (6.40) holds for all $\tilde{\mathbf{y}} \in \mathcal{R}^*_{\widehat{\psi}_t \mu_G}$, and since $\mathbb{P}^{\widehat{\psi}_t \mu_G}(\mathcal{R}^*_{\widehat{\psi}_t \mu_G}) \geq 1 - \mathrm{e}^{-(1/2)N^{\varsigma_1}}$, we can integrate over $\tilde{\mathbf{y}}$ to find that

$$\left| \int G_{L+p,n}(\mathbf{x}) (\widehat{\psi}_t^{\mathbf{y}} \, \mathrm{d}\mu_G^{\mathbf{y}} - \widehat{\psi}_t \, \mathrm{d}\mu_G) \right| \leq C_O K^{-\mathfrak{b}},$$

for N sufficiently large on Ω. In combination with (6.37), this yields (6.34). □

6.5. *Proof of Theorems 6.1 and 6.2.* Lemma 6.8 compares the local statistics of the locally-constrained measure $f_t^{\mathbf{y}} \mu_G^{\mathbf{y}}$ with the β-ensemble $\widehat{\psi}_t \mu_G$. In order to compare with local statistics of the measure $f_t \mu_G$ with $\widehat{\psi}_t \mu_G$, we next integrate out the boundary conditions $\mathbf{y}$.

LEMMA 6.9. *Under the assumptions of Lemma 6.8 the following holds. Let $J \subset [\![\alpha N, (1-\alpha)N]\!]$ be an interval of consecutive integers in the bulk. Then*

$$\begin{aligned} &\left| \int \frac{1}{|J|} \sum_{j \in J} G_{j,n}(\mathbf{x}) (f_t \, \mathrm{d}\mu_G - \widehat{\psi}_t \, \mathrm{d}\mu_G) \right| \\ &\qquad \leq C_O \big(N^{-\mathfrak{c}} + K^{-\mathfrak{b}} + K^{-\chi'/2} \big) + C_O \sqrt{K} N^{\mathfrak{c}} N^{-\mathfrak{a}} \tau^{-1}, \end{aligned} \tag{6.41}$$

for N sufficiently large on Ω.

PROOF. For a small $\chi' > 0$ as in Lemma 6.8, set $\tilde{K} := K - K^{1-\chi'/2}$. We first assume that J is such that $|J| \leq 2\tilde{K} + 1$. We then choose L such that $J \subset I_{L,\tilde{K}} \subset I_{L,K}$. Recall the set of configurations $\mathcal{G}$ in Proposition 6.4. Using the conditioned measure $f_t^{\mathbf{y}} \mu_G^{\mathbf{y}}$ we estimate

$$\begin{aligned} &\mathbb{E}^{f_t \mu_G} \left[\frac{1}{|J|} \sum_{j \in J} G_{j,n} \right] \\ &\qquad = \mathbb{E}^{f_t \mu_G} \left[\frac{1}{|J|} \int \sum_{j \in J} G_{j,n}(\mathbf{x}) f_t^{\mathbf{y}} \, \mathrm{d}\mu_G^{\mathbf{y}} \mathbb{1}(\mathcal{G}) \right] + \mathcal{O}(N^{-\mathfrak{c}}), \end{aligned} \tag{6.42}$$

where we used (6.18). Next, using Lemma 6.8 we obtain on Ω

$$\begin{aligned} &\frac{1}{|J|} \int \sum_{j \in J} G_{j,n}(\mathbf{x}) f_t^{\mathbf{y}}(\mathbf{x}) \, \mathrm{d}\mu_G^{\mathbf{y}}(\mathbf{x}) \mathbb{1}(\mathcal{G}) \\ &\qquad = \frac{1}{|J|} \int \sum_{j \in J} G_{j,n} \widehat{\psi}_t \, \mathrm{d}\mu_G + \mathcal{O}(K^{-\mathfrak{b}}) + \mathcal{O}(\sqrt{K} N^{\mathfrak{c}} N^{-\mathfrak{a}} \tau^{-1}), \end{aligned}$$

on Ω. For the special case $|J| \leq 2\tilde{K} + 1$, this yields (6.41).

If $|J| \geq \tilde{K} + 1$, there are $L_a \in [\![\alpha N, (1-\alpha)N]\!]$, with $a \in [\![1, M_0]\!]$, such that the intervals $I_{L_a,K} = [\![L_a - K, L_a + K]\!]$ are nonintersecting with the properties

that $J \subset \bigcup_{a=1}^{M_0} I_{L_a,K}$ and $J \cap I_{L_a,K} \neq \varnothing$, for all $a \in [\![1, M_0]\!]$. Note that $M_0 \leq \frac{|J|}{K} + 2$. For simplicity of notation we abbreviate $I^{(a)} \equiv I_{L_a,K} = [\![L_a - K, L_a + K]\!]$ and $\tilde{I}^{(a)} \equiv [\![L_a - \tilde{K}, L_a + \tilde{K}]\!]$. We also label the interior and exterior points of a configuration $\boldsymbol{\lambda} \in F^{(N)}$ accordingly,

$$\mathbf{x}^{(a)} = (x_{L_a-K}, \ldots, x_{K+L_a}) \in F^{(\mathcal{K})},$$

respectively,

$$\mathbf{y}^{(a)} = (y_1, \ldots, y_{L_a-K-1}, y_{L_a+K+1}, \ldots, y_N) \in F^{(N-\mathcal{K})};$$

cf. (6.6). We let $\mathcal{G}^{(a)} \equiv \mathcal{G}_{L_a,K}(\varepsilon_0, \alpha) \subset \mathcal{R}_{L_a,K}(\varepsilon_0, \alpha)$ denote the set of configurations obtained in Proposition 6.4. Using this notation we can write

$$\begin{aligned} &\mathbb{E}^{f_t \mu_G}\Bigg[\frac{1}{|J|}\sum_{j\in J} G_{j,n}\Bigg] \\ &= \frac{1}{|J|}\sum_{a\,:\, I^{(a)}\cap J\neq\varnothing} \mathbb{E}^{f_t\mu_G}\Bigg[\int \sum_{j\in I^{(a)}\cap J} G_{j,n}\big(\mathbf{x}^{(a)}\big) f_t^{\mathbf{y}^{(a)}}\, \mathrm{d}\mu_G^{\mathbf{y}^{(a)}}\, \mathbb{1}\big(\mathcal{G}^{(a)}\big)\Bigg] \\ &\quad + \mathcal{O}\big(N^{-\mathfrak{c}}\big), \end{aligned} \tag{6.43}$$

on Ω, where the first summation on the right-hand side is over indices $a \in [\![1, M_0]\!]$ such that the intervals $(I^{(a)})$ satisfy $I^{(a)} \cap J \neq \varnothing$. Here, we also use the probability estimate on $\mathcal{G}^{(a)}$ in (6.18). In (6.43) we may further restrict, for each a, the summation over the index j from $I^{(a)}$ to $\tilde{I}^{(a)}$ at an expense of an error term of order $|I^{(a)} \setminus \tilde{I}^{(a)}| \leq K^{1-\chi'/2}$. Then summing over $a \in [\![1, M_0]\!]$, with $M_0 \sim |J|/K$, we get

$$\begin{aligned} &\mathbb{E}^{f_t \mu_G}\Bigg[\frac{1}{|J|}\sum_{j\in J} G_{j,n}\Bigg] \\ &= \frac{1}{|J|}\sum_{a\,:\, I^{(a)}\cap J\neq\varnothing} \mathbb{E}^{f_t\mu_G}\Bigg[\int \sum_{j\in \tilde{I}^{(a)}\cap J} G_{j,n}\big(\mathbf{x}^{(a)}\big) f_t^{\mathbf{y}^{(a)}}\, \mathrm{d}\mu_G^{\mathbf{y}^{(a)}}\, \mathbb{1}\big(\mathcal{G}^{(a)}\big)\Bigg] \\ &\quad + \mathcal{O}\big(N^{-\mathfrak{c}}\big) + \mathcal{O}\big(K^{-\chi'/2}\big), \end{aligned} \tag{6.44}$$

on Ω. Since for each choice of the index a the term in the expectation on the right-hand side of (6.44) can be dealt with as in the case $|J| \leq 2\tilde{K} + 1$ above, this completes the proof of (6.41) for general J. □

We can now give the proof of Theorem 6.1.

PROOF OF THEOREM 6.1. Let $\alpha > 0$. We first choose the constants $\mathfrak{a} \in (0, 1/2)$, $\mathfrak{c} \in (0, 1)$ and $\varepsilon' > 0$, and the parameter $K \in [\![N^{\sigma}, N^{1/4}]\!]$ appropriately: let $\delta > 0$ be a small constant. Then we set $\mathfrak{a} \equiv 1/2 - \delta$, $\mathfrak{c} \equiv \delta/4$, $K \equiv N^{\delta/4}$, $\varepsilon' \equiv \delta$,

$\sigma = \delta/8$. Note first that for this choice of K condition (6.4) is satisfied. Second, for sufficiently small $\delta > 0$, we observe that

$$KN^{2\mathfrak{c}}N^{-\mathfrak{a}}\tau^{-1} = N^{3\delta/4}N^{-\mathfrak{a}}\tau^{-1} \le K^{\chi},$$

holds, for example, for $\tau \ge N^{\delta}N^{-\mathfrak{a}}$ and $\chi > 0$ (with $\chi < \chi_0$). Thus (6.33) is satisfied with the above choices.

Hence, for $t \ge N^{2\delta}\tau$, Lemma 6.9 yields, for some $\mathfrak{b} > 0$,

$$\begin{aligned}\left| \int \frac{1}{|J|} \sum_{j \in J} G_{j,n}(\mathbf{x})(f_t \, \mathrm{d}\mu_G - \widehat{\psi}_t \, \mathrm{d}\mu_G) \right| \\ \le C_O K^{-\mathfrak{b}} + C_O N^{-\mathfrak{c}} + C_O K^{-\chi'/2} + C_O \sqrt{K} N^{\mathfrak{c}} N^{-\mathfrak{a}} \tau^{-1},\end{aligned}$$

for N sufficiently large on Ω. Thus, choosing $\tau \ge N^{\delta}N^{-\mathfrak{a}}$, there is a constant $\mathfrak{f} > 0$ such that (6.3) holds. This completes the proof of Theorem 6.1. □

Next, we sketch the proof of Theorem 6.2.

PROOF OF THEOREM 6.2. The proof of Theorem 6.2 is almost identical to the proof of Theorem 6.1. In fact, it suffices to establish Lemma 6.8 with μ_G replacing $\widehat{\psi}_t \mu_G$ on the left-hand side of (6.34). This can be accomplished by applying Theorem 6.5 with μ_G instead of $\widehat{\psi}_t \mu_G$: let $\tilde{\mathbf{y}} \in \mathcal{R}^*_{\mu_G}(\chi^2\sigma/2, \alpha/2)$, and let $\mathbf{y} \in \mathcal{G}(\chi^2\sigma/2, \alpha/2)$. Using the arguments of Proposition 5.2 in [31], we can rescale μ_G such that (6.26) and (6.27) are satisfied for $\mathbf{y}$ and $\tilde{\mathbf{y}}$. It is also straightforward to check that the external potentials leading to $\mu_G^{\tilde{\mathbf{y}}}$, $\tilde{\mathbf{y}} \in \mathcal{R}^*_{\mu_G}(\chi^2\sigma/2, \alpha/2)$, are K^{χ}-regular. By Lemma 5.1 of [31] we obtain

$$\left| \mathbb{E}^{\mu_G^{\tilde{\mathbf{y}}}} x_k - \alpha_k \right| \le C K^{\chi} N^{-1}.$$

Hence, using estimate (6.35), we conclude that assumption (6.28) is also satisfied. Thus Theorem 6.5 yields

$$\left| \int G_{L+p,n}(\mathbf{x}) \widehat{\psi}_t^{\mathbf{y}} \, \mathrm{d}\mu_G^{\mathbf{y}} - \int G_{L+p,n,\mathrm{sc}}(\mathbf{x}) \, \mathrm{d}\mu_G^{\tilde{\mathbf{y}}} \right| \le C_O K^{-\mathfrak{b}}, \tag{6.45}$$

for N sufficiently large on Ω. We refer to the proof of Proposition 5.2 in [31] for more details.

Since $\mathcal{R}^*_{\mu_G}(\chi^2\sigma/2, \alpha/2)$ has exponentially high probability under μ_G, we can integrate over $\tilde{\mathbf{y}}$ to find

$$\left| \int G_{L+p,n}(\mathbf{x}) \widehat{\psi}_t^{\mathbf{y}} \, \mathrm{d}\mu_G^{\mathbf{y}} - \int G_{L+p,n,\mathrm{sc}}(\mathbf{x}) \, \mathrm{d}\mu_G \right| \le C_O K^{-\mathfrak{b}},$$

for N sufficiently large on Ω.

The proof of Theorem 6.2 is now completed in the same way as the proof of Theorem 6.1. □

7. From gap statistics to correlation functions. In this section, we translate our results on the averaged local gap statistics into results on averaged correlation functions. Since this procedure is fairly standard (see, e.g., [29]), we refrain from stating all proofs in detail. We first need to slightly generalize the setup of Section 6.

Fix $n \in \mathbb{N}$, let O be an n-particle observable and consider an array of increasing positive integers,

$$\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathbb{N}^n. \tag{7.1}$$

Let $\alpha > 0$. We define for $j \in [\![\alpha N, (1-\alpha)N]\!]$ and $t \geq 0$ an observable $G_{j,\mathbf{m},t} \equiv G_{j,\mathbf{m}}$ by

$$\begin{aligned} G_{j,\mathbf{m}}(\mathbf{x}) := O\big(&N\rho_j(x_{j+m_1} - x_j), N\rho_j(x_{j+m_2} - x_j), \dots, \\ &N\rho_j(x_{j+m_n} - x_j)\big), \end{aligned} \tag{7.2}$$

where $\rho_j \equiv \widehat{\rho}_{\mathrm{fc}}(t, \widehat{\gamma}_j(t))$ denotes the density of the measure $\widehat{\rho}_{\mathrm{fc}}(t)$ at the classical location of the jth particle, $\widehat{\gamma}_j(t)$, with respect to the measure $\widehat{\rho}_{\mathrm{fc}}(t)$. We set $G_{j,\mathbf{m}} = 0$ if $j + m_n \geq (1-\alpha)N$. Similarly, we define $G_{j,\mathbf{m},\mathrm{sc}}$ by replacing ρ_j by the density of the standard semicircle law at the classical locations of the jth particle with respect to the semicircle law; cf. (6.2). The following theorem generalizes Theorem 6.2.

THEOREM 7.1. *Let $n \in \mathbb{N}$ be fixed, and let O be an n-particle observable. Fix small constants $\alpha, \delta > 0$, and consider an interval of consecutive integers $J \subset [\![\alpha N, (1-\alpha)N]\!]$ in the bulk. Then there are constants $\mathfrak{f}, \delta' > 0$ such that the following holds. Let $\mathbf{m} \in \mathbb{N}^n$ be an array of increasing integers [see (7.1)] such that $m_n \leq N^{\delta'}$, and consider the observable $G_{j,\mathbf{m}}$, respectively, $G_{j,\mathbf{m},\mathrm{sc}}$; see (7.2). Assume that $t \geq N^{-1/2+\delta}$, then*

$$\left| \int \frac{1}{|J|} \sum_{j \in J} G_{j,\mathbf{m}}(\mathbf{x}) f_t(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) - \int \frac{1}{|J|} \sum_{j \in J} G_{j,\mathbf{m},\mathrm{sc}}(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) \right| \leq C_O N^{-\mathfrak{f}},$$

for N sufficiently large on Ω. The constant C_O depends on α and O, and the constants $\mathfrak{f}$ and δ' depend on α and δ.

Theorem 7.1 is proven in the same way as Theorem 6.2. We remark that δ' is chosen such that $N^{\delta'} \ll K$; that is, m_n is much smaller than the size of the interval $I_{L,K}$.

For $n \geq 1$, define the n-point correlation function, $\varrho^N_{f_t,n}$, by

$$\varrho^N_{f_t,n}(x_1, \dots, x_n) := \int_{\mathbb{R}^{N-n}} (f_t \mu_G)^{\#} \, \mathrm{d}x_{n+1} \cdots \mathrm{d}x_N,$$

where $(f_t \mu_G)^{\#}$ denote the symmetrized versions of $f_t \mu_G$. Similarly, we denote by

$$\varrho^N_{G,n}(x_1, \dots, x_n) := \int_{\mathbb{R}^{N-n}} \mu_G^{\#} \, \mathrm{d}x_{n+1} \cdots \mathrm{d}x_N,$$

the n-point correlation functions of the Gaussian ensembles; see (4.13) with $U \equiv 0$.

Recall that we denote by $\widehat{L}_{\pm}(t)$, respectively, $L_{\pm}(t)$, the endpoints of the support of the measure $\widehat{\rho}_{\mathrm{fc}}(t)$, respectively, the measure $\rho_{\mathrm{fc}}(t)$. Recall that the two densities f_t and $\widehat{\psi}_t$ are both conditioned on V; that is, the entries (v_i) of V are considered fixed. We have the following result on the averaged correlation functions of $f_t\mu_G$ and $\widehat{\psi}_t\mu_G$.

THEOREM 7.2. *Fix $n \in \mathbb{N}$, and choose an n-particle observable O. Fix a small $\delta > 0$, and let $t \geq N^{-1/2+\delta}$. Let $\tilde{\alpha} > 0$ be a small constant, and consider two energies $E \in [L_-(t) + \tilde{\alpha}, L_+(t) - \tilde{\alpha}]$ and $E' \in [-2 + \tilde{\alpha}, 2 - \tilde{\alpha}]$. Then we have, for any $\varepsilon > 0$ and for $b \equiv b_N$ satisfying $\tilde{\alpha}/2 \geq b_N > 0$,*

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n O(\alpha_1, \ldots, \alpha_n) \right. \\
&\qquad \times \left[\int_{E-b}^{E+b} \frac{\mathrm{d}x}{2b} \frac{1}{[\rho_{\mathrm{fc}}(t,E)]^n} \varrho^N_{f_t,n}\left(x + \frac{\alpha_1}{N\rho_{\mathrm{fc}}(t,E)}, \ldots, x + \frac{\alpha_n}{N\rho_{\mathrm{fc}}(t,E)} \right) \right. \\
&\qquad \left. \left. - \int_{E'-b}^{E'+b} \frac{\mathrm{d}x}{2b} \frac{1}{[\rho_{\mathrm{sc}}(E')]^n} \varrho^N_{G,n}\left(x + \frac{\alpha_1}{N\rho_{\mathrm{sc}}(E')}, \ldots, x + \frac{\alpha_n}{N\rho_{\mathrm{sc}}(E')} \right) \right] \right| \\
&\qquad \leq C_O N^{2\varepsilon} \left(b^{-1} N^{-1+\varepsilon} + N^{-\mathfrak{f}} + N^{-c\alpha_0} \right),
\end{aligned} \tag{7.3}$$

for N sufficiently large on Ω. Here $\mathfrak{a}$ is the constant in the rigidity estimate (5.6), and $\mathfrak{f}$ is the constant in Theorem 7.1. Moreover, $\rho_{\mathrm{fc}}(t,E)$ stands for the density of the (N-independent) measure $\rho_{\mathrm{fc}}(t)$ at the energy E. The constant C_O depends on O and $\tilde{\alpha}$. Further, α_0 is the constant appearing in Assumption 2.2. The constant c depends on the measure ν.

Theorem 7.2 follows from Theorem 6.2. This is an application of Section 7 in [29]. The validity of Assumption IV in [29] is a direct consequence of the local law in Theorem 3.3. Further, we remark that the parameter b_N in Theorem 7.2 and the interval of consecutive integers J in Theorem 7.1 are related by $J = \{i : \widehat{\gamma}_i(t) \in [E - b_N, E + b_N]\}$, where $\widehat{\gamma}_i(t)$ are the classical locations with respect to the measure $\widehat{\rho}_{\mathrm{fc}}(t)$. This explains, up to minor technicalities, $b_N \gg N^{-1}$. Then Section 7 of [29] yields (7.3) formulated in terms of $\widehat{\rho}_{\mathrm{fc}}(t)$ instead of $\rho_{\mathrm{fc}}(t)$. Using (3.22) and the smoothness of O, we can replace $\widehat{\rho}_{\mathrm{fc}}$ by ρ_{fc} at the expense of an error of size $C_O N^{-c\alpha_0}$. This eventually gives (7.3) with $\rho_{\mathrm{fc}}(t)$.

8. Proofs of main results. Theorem 7.2 shows that the averaged local correlation functions of ensembles of the form

$$H_t = \mathrm{e}^{-(t-t_0)/2} V + \mathrm{e}^{-t/2} W + \left(1 - \mathrm{e}^{-t}\right)^{1/2} W',$$

with some small $t_0 \geq 0$, and with W' a GUE/GOE matrix independent of W and V, can be compared with the averaged local correlation functions of the GUE, respectively, GOE, for times satisfying $t \gg N^{-1/2}$. In this section, we explain how this can be used to prove the universality at time $t = 0$.

8.1. *Green function comparison theorem.* We start with a Green function comparison theorem. Assume that we are given two complex Hermitian or real symmetric Wigner matrices, X and Y, both satisfying the assumptions in Definition 2.1. Let V be a real random or deterministic diagonal matrix satisfying Assumptions 2.3 and 2.2. Consider the deformed Wigner matrices

$$H^X := V + X, \qquad H^Y := V + Y, \tag{8.1}$$

of size N. The main theorem of this subsection, Theorem 8.2, states that the correlation functions of the two matrices H^X and H^Y, when conditioned on V, are identical on scale $1/N$ provided that the first four moments of X and Y almost match. Theorem 8.2 is a direct consequence of the Green function comparison Theorem 8.1.

Denote the Green functions of H^X, H^Y, respectively, by

$$G^X(z) := \frac{1}{H^X - z}, \qquad G^Y(z) := \frac{1}{H^Y - z} \qquad (z \in \mathbb{C} \setminus \mathbb{R}),$$

and set $m_N^X(z) := N^{-1} \operatorname{Tr} G^X(z)$, $m_N^Y(z) := N^{-1} \operatorname{Tr} G^Y(z)$. From Theorem 3.3, we know that, for all $z \in \mathcal{D}_L$ [see (3.9)], with $L \geq 40\xi$,

$$\bigl|m_N^X(z) - \widehat{m}_{\mathrm{fc}}(z)\bigr| \leq (\varphi_N)^{c\xi} \frac{1}{N\eta} \tag{8.2}$$

and

$$\bigl|G_{ij}^X(z) - \delta_{ij}\widehat{g}_i(z)\bigr| \leq (\varphi_N)^{c\xi} \biggl(\sqrt{\frac{\operatorname{Im} \widehat{m}_{\mathrm{fc}}(z)}{N\eta}} + \frac{1}{N\eta} \biggr), \tag{8.3}$$

with (ξ, υ)-high probability on Ω for some $\upsilon > 0$ and $c > 0$, where

$$\widehat{g}_i(z) := \frac{1}{v_i - z - \widehat{m}_{\mathrm{fc}}(z)} \qquad (z \in \mathbb{C} \setminus \mathbb{R}).$$

Here, $\widehat{m}_{\mathrm{fc}}$, is the Stieltjes transform of the measure $\widehat{\rho}_{\mathrm{fc}}$, which agrees with $\widehat{\rho}_{\mathrm{fc}}^{\vartheta}$ for the choice $\vartheta = 1$ and with $\widehat{\rho}_{\mathrm{fc}}(t)$ for the choice $t = t_0$. The identical estimates hold true when X is replaced by Y.

Recall that we denote by $\widehat{L}_{\pm}$ the endpoints of the support of $\widehat{\rho}_{\mathrm{fc}}$, and that we denote by $\widehat{\kappa}_E \equiv \widehat{\kappa}$ the distance of $E \in [\widehat{L}_-, \widehat{L}_+]$ to the endpoints $\widehat{L}_{\pm}$. Adapting the Green function theorem of [32] we obtain the following theorem.

THEOREM 8.1 (Green function comparison theorem). *Assume that X and Y satisfy the assumptions in Definition 2.1, and let V satisfy Assumptions 2.2 and 2.3. Assume further that the first two moments of $X = (x_{ij})$ and $Y = (y_{ij})$ agree and that the third and forth moments satisfy*

$$|\mathbb{E}\bar{x}_{ij}^{p} x_{ij}^{3-p} - \mathbb{E}\bar{y}_{ij}^{p} y_{ij}^{3-p}| \le N^{-\delta-2} \qquad (p \in [\![0,3]\!]), \tag{8.4}$$

respectively,

$$|\mathbb{E}\bar{x}_{ij}^{q} x_{ij}^{4-q} - \mathbb{E}\bar{y}_{ij}^{q} y_{ij}^{4-q}| \le N^{-\delta} \qquad (q \in [\![0,4]\!]), \tag{8.5}$$

for some given $\delta > 0$.

Let $\varepsilon > 0$ be arbitrary, and let $N^{-1-\varepsilon} \le \eta \le N^{-1}$. Fix N-independent integers $k_1, \dots, k_n$ and energies $E_j^1, \dots, E_j^{k_j}$, $j = 1, \dots, n$, with $\widehat{\kappa} > \tilde{\alpha}$ for all E_j^k with some fixed $\tilde{\alpha} > 0$. Define $z_j^k := E_j^k \pm \mathrm{i}\eta$, with the sign arbitrarily chosen. Suppose that F is a smooth function such that for any multi-index $\sigma = (\sigma_1, \dots, \sigma_n)$, with $1 \le |\sigma| \le 5$, and any $\varepsilon' > 0$ sufficiently small, there is a $C_0 > 0$ such that

$$\max\left\{|\partial^{\sigma} F(x_1, \dots, x_n)| : \max_j |x_j| \le N^{\varepsilon'}\right\} \le N^{C_0\varepsilon'},$$

$$\max\left\{|\partial^{\sigma} F(x_1, \dots, x_n)| : \max_j |x_j| \le N^{2}\right\} \le N^{C_0},$$

for some C_0.

Then there exists a constant C_1, depending on $\sum_m k_m$, C_0 and the constants in (2.3), such that for any η with $N^{-1-\varepsilon} \le \eta \le N^{-1}$,

$$\begin{aligned} &\left|\mathbb{E}F\left(\frac{1}{N^{k_1}}\operatorname{Tr}\prod_{j=1}^{k_1} G^X(z_j^1), \dots, \frac{1}{N^{k_n}}\operatorname{Tr}\prod_{j=1}^{k_n} G^X(z_j^n)\right)\right. \\ &\qquad \left. - \mathbb{E}F\left(\frac{1}{N^{k_1}}\operatorname{Tr}\prod_{j=1}^{k_1} G^Y(z_j^1), \dots, \frac{1}{N^{k_n}}\operatorname{Tr}\prod_{j=1}^{k_n} G^Y(z_j^n)\right)\right| \\ &\qquad \le C_1 N^{-1/2+C_1\varepsilon} + C_1 N^{-1/2+\delta+C_1\varepsilon}, \end{aligned} \tag{8.6}$$

for N sufficiently large on Ω.

Theorem 8.1 is proven in the same way as Theorem 2.3 in [33] with the following modifications. Fix some labeling of $\{(i,j) : 1 \le i \le j \le N\}$ by $[\![1, \gamma(N)]\!]$, with $\gamma(N) := N(N+1)/2$, and write the γth element of this labeling as (i_γ, j_γ). Starting with $W^{(0)} \equiv X$, inductively define $W^{(\gamma)}$ by replacing the (i_γ, j_γ), (j_γ, i_γ) entries of $W^{(\gamma-1)}$ by the corresponding entries of Y. Moreover set $H^{(\gamma)} := V + W^{(\gamma)}$. Thus we have $H^{(0)} = H^X$, $H^{(\gamma(N))} = H^Y$, and $H^{(\gamma)} - H^{(\gamma-1)}$ is zero

in all but two entries for every γ. In short, we use a Lindeberg-type replacement strategy: we successively replace the entries of the matrix X by entries of the matrix Y. Note, however, that the entries of the matrix V are not changed.

The main technical input in the proof of Theorem 2.3 in [32] is estimate (2.21) in that publication. For the case at hand the corresponding estimate reads as follows: let ξ satisfy (2.20). Then, for all $\delta > 0$, and any $N^{-1/2} \gg y \geq N^{-1+\delta}$, we have

$$\mathbb{P}\Bigg(\max_{\gamma \leq \gamma(N)} \max_{k} \sup_{E\,:\,\widehat{\kappa}_E \geq \tilde{\alpha}} \left| \left(\frac{1}{H^{(\gamma)} - E - \mathrm{i}y} \right)_{kk} \right| \geq N^{2\delta} \Bigg) \leq \mathrm{e}^{-\upsilon(\varphi_N)^{\xi}}, \tag{8.7}$$

on Ω for N sufficiently large, where $\upsilon > 0$ depends only on $\tilde{\alpha}$, δ and the constants in (2.3). Estimate (8.7) follows easily from the local law in (8.3), the stability bound (3.21) and Lemma 3.6. The rest of the proof of Theorem 8.1 is identical to the proof in [32]. (The matching conditions in (8.4) are weaker than in [32], but the proof carries over without any changes.)

Lindeberg's replacement method was applied in random matrix theory in [15] to compare traces of Green functions. This idea was also used in [56] in the proof of the "four moment theorem" that compares individual eigenvalue distributions. The four-moment matching conditions (8.4) and (8.5) appeared first in [56] with $\delta = 0$. The "Green function comparison theorem" of [32] compares Green functions at fixed energies. Since the approach in [56] requires additional difficult estimates due to singularities from neighboring eigenvalues, we follow the method of [32], where difficulties stemming from such resonances are absent. For deformed Wigner matrices with deterministic potential the approach of [56] was recently followed in [45] where a "four moment theorem" was established. It allows one to compare local correlation functions of the matrices $V + W$ and $V + W'$ for fixed V, where W and W' are real symmetric or complex Hermitian Wigner matrices, provided that the moments of the off-diagonal entries of W and W' match to fourth order.

The Green function comparison theorem leads directly to the equivalence of local statistics for the matrices H^Y and H^X.

THEOREM 8.2. *Assume that X, Y are two complex Hermitian or two real symmetric Wigner matrices satisfying assumptions in Definition 2.1. Assume further that X and Y satisfy the matching conditions (8.4) and (8.5), for some $\delta > 0$. Let V be a deterministic real diagonal matrix satisfying the Assumptions 2.2 and 2.3. Denote by $\varrho^N_{H^X,n}$, $\varrho^N_{H^Y,n}$ the n-point correlation functions of the eigenvalues with respect to the probability laws of the matrices H^X, H^Y, respectively. Then, for any energy E in the interior of the support of ρ_{fc} and any n-particle*

observable O, we have

$$\lim_{N\to\infty}\int_{\mathbb{R}^k} \mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_n\, O(\alpha_1,\dots,\alpha_n) \times\left[\varrho^N_{H^X,n}\left(E+\frac{\alpha_1}{N},\dots,E+\frac{\alpha_n}{N}\right)-\varrho^N_{H^Y,n}\left(E+\frac{\alpha_1}{N},\dots,E+\frac{\alpha_n}{N}\right)\right] = 0,$$

for any fixed $n\in\mathbb{N}$.

Notice that this comparison theorem holds for any fixed energy E in the bulk. The proof of [32] applies almost verbatim. The only technical input in the proof is the local law for m^X_N, respectively, m^Y_N, on scales $\eta\sim N^{-1+\varepsilon}$, which we have established in Theorem 3.3; see also (8.3).

8.2. *Proof of Theorem 2.5.* In the remaining subsections, 8.2 and 8.3, we complete the proofs of our main results in Theorems 2.5 and 2.6. The proofs for deterministic and random V differ slightly. We start with the case of deterministic V in this subsection; the random case is treated in Section 8.3.

PROOF OF THEOREM 2.5. Assume that $W=(w_{ij})$ is a complex Hermitian or a real symmetric Wigner matrix satisfying the assumptions in Definition 2.1. Let $V=\operatorname{diag}(v_i)$ be a deterministic real diagonal matrix satisfying Assumptions 2.2 and 2.3. (Note that the event Ω then has full probability.) Set $H=(h_{ij})=V+W$. Let $E\in\mathbb{R}$ be inside the support of ρ_{fc}. Note that by Lemma 3.6, E is also contained in the support of $\widehat{\rho}_{\mathrm{fc}}$, for N sufficiently large. (Here we have $\widehat{\rho}_{\mathrm{fc}}=\widehat{\rho}^{\vartheta=1}_{\mathrm{fc}}$ and similarly for ρ_{fc}.) Fix $\delta'>0$, and set $t\equiv N^{-1/2+\delta'}$. We first claim that there exists an auxiliary complex Hermitian or real symmetric Wigner matrix, $U=(u_{ij})$, satisfying the assumptions in Definition 2.1 such that the following holds: set

$$Y := \mathrm{e}^{-t/2}U+\left(1-\mathrm{e}^{t/2}\right)W', \tag{8.8}$$

where W' is a GUE/GOE matrix independent of W. Then the moments of the entries of Y satisfy

$$\mathbb{E}\bar{y}^p_{ij}y^{3-p}_{ij}=\mathbb{E}\bar{w}^p_{ij}w^{3-p}_{ij},\qquad \left|\mathbb{E}\bar{y}^q_{ij}y^{4-q}_{ij}-\mathbb{E}\bar{w}^q_{ij}w^{4-q}_{ij}\right|\le Ct, \tag{8.9}$$

for $p\in[\![0,3]\!]$, $q\in[\![0,4]\!]$, where (w_{ij}) are the entries of the Wigner matrix W.

Assuming the existence of such a Wigner matrix U, we choose $t_0\equiv t$ and set

$$\begin{aligned} H_t &:= \mathrm{e}^{-(t-t_0)/2}V+\mathrm{e}^{-t/2}U+\left(1-\mathrm{e}^{-t}\right)^{1/2}W' \\ &= V+\mathrm{e}^{-t/2}U+\left(1-\mathrm{e}^{-t}\right)^{1/2}W'. \end{aligned}$$

Then the matrices H_t and $H=V+W$ satisfy the matching conditions (8.4) and (8.5) of Theorem 8.1 (with, say, $\delta=1/4-2\delta'$). This follows from (8.9). Thus

Theorem 8.2 implies that the correlation functions of H_t and H agree in the limit of large N, that is,

$$\begin{aligned}\lim_{N\to\infty}\int_{\mathbb{R}^n} &\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_n\, O(\alpha_1,\dots,\alpha_n)\\ &\times\Bigg[\frac{1}{2b}\int_{E-b}^{E+b}\frac{\mathrm{d}x}{[\rho_{\mathrm{fc}}(E)]^n}\varrho^N_{H_t,n}\Big(x+\frac{\alpha_1}{\rho_{\mathrm{fc}}(E)N},\dots,x+\frac{\alpha_n}{\rho_{\mathrm{fc}}(E)N}\Big)\\ &\quad-\frac{1}{2b}\int_{E-b}^{E+b}\frac{\mathrm{d}x}{[\rho_{\mathrm{fc}}(E)]^n}\varrho^N_{H,n}\Big(x+\frac{\alpha_1}{\rho_{\mathrm{fc}}(E)N},\dots,x+\frac{\alpha_n}{\rho_{\mathrm{fc}}(E)N}\Big)\Bigg]\\ &=0,\end{aligned}\tag{8.10}$$

where $(\varrho^N_{H,n})$ denote the correlation functions of $H=V+W$ and where $(\varrho^N_{H_t,n})$ denote the correlation functions of H_t. [In fact, (8.10) holds even without the averages in the energy around E.]

On the other hand, for small $\delta>0$, Theorem 7.2 assures that the local correlation functions of the matrix H_t agree with the correlation functions of the GUE (resp., GOE), when averaged over an interval of size b, with $1\gg b\geq N^{-\delta}$; that is, for any E' with $|E'|<2$,

$$\begin{aligned}\lim_{N\to\infty}\int_{\mathbb{R}^n} &\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_n\, O(\alpha_1,\dots,\alpha_n)\\ &\times\Bigg[\frac{1}{2b}\int_{E-b}^{E+b}\frac{\mathrm{d}x}{[\rho_{\mathrm{fc}}(E)]^n}\varrho^N_{H_t,n}\Big(x+\frac{\alpha_1}{\rho_{\mathrm{fc}}(E)N},\dots,x+\frac{\alpha_n}{\rho_{\mathrm{fc}}(E)N}\Big)\\ &\quad-\frac{1}{[\rho_{\mathrm{sc}}(E')]^n}\varrho^N_{G,n}\Big(E'+\frac{\alpha_1}{\rho_{\mathrm{sc}}(E')N},\dots,E'+\frac{\alpha_n}{\rho_{\mathrm{sc}}(E')N}\Big)\Bigg]\\ &=0,\end{aligned}\tag{8.11}$$

where $(\varrho^N_{G,n})$ denote the correlations functions of the GUE, respectively, GOE. Combining (8.10) and (8.11), we get (2.15).

Thus to complete the proof we need to show the existence of a Wigner matrix U with the properties described above. For a real random variables ζ, denote by $m_k(\zeta)=\mathbb{E}\zeta^k$, $k\in\mathbb{N}$, its moments.

LEMMA 8.3 (Lemma 6.5 in [32]). *Let m_3 and m_4 be two real numbers such that*

$$m_4-m_2^2-1\geq 0,\qquad m_4\leq C_1,$$

for some constant C_1. Let ζ_G be a Gaussian random variable with mean 0 and variance 1. Then for any sufficient small $\gamma>0$, depending on C_1, there exists a real random variable ζ_γ with subexponential decay and independent of ζ_G, such that the first three moments of

$$\zeta':=(1-\gamma)^{1/2}\zeta_\gamma+\gamma^{1/2}\zeta_G$$

are $m_1(\zeta') = 0$, $m_2(\zeta') = 1$, $m_3(\zeta') = m_3$, *and the forth moment* $m_4(\zeta')$ *satisfies*

$$|m_4(\zeta') - m_4| \le C\gamma,$$

for some C *depending on* C_1.

Since the real and imaginary parts of W are independent, it is sufficient to match them individually; that is, we apply Lemma 8.3 separately to the real and imaginary parts of (w_{ij}). This completes the proof of Theorem 2.5 for deterministic V. □

8.3. *Proof of Theorem 2.6.* Next, we prove Theorem 2.6. Assume that $W = (w_{ij})$ is a complex Hermitian or a real symmetric Wigner matrix satisfying the assumption in Definition 2.1. Let $V = \operatorname{diag}(v_i)$ be a random real diagonal matrix satisfying Assumptions 2.2 and 2.3. Denote by $\tilde{f}_t \mu_G$ the distribution of the eigenvalues of the matrix

$$H_t := V + \mathrm{e}^{-t/2} W + \left(1 - \mathrm{e}^{t}\right)^{1/2} W' \qquad (t \ge 0),$$

where W' is a GUE/GOE matrix independent of V and W. Let $\mathbb{E}^V$ stand for the expectation with respect to the law of the entries (v_i) of V. Recall the definition of the event Ω in Definition 3.3. Following the notation of Section 5, $f_t \mu_G \equiv f_t^V \mu_G$ denotes the density conditioned on V. For an n-particle observable O and for $G_{j,\mathbf{m}}$ as in (7.2), we may write

$$\begin{aligned}
&\int \frac{1}{|J|} \sum_{j \in J} G_{j,\mathbf{m}}(\mathbf{x}) \tilde{f}_t(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) \\
&\quad = \mathbb{E}^V \left[\left(\int \frac{1}{|J|} \sum_{j \in J} G_{j,\mathbf{m}}(\mathbf{x}) f_t^V(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) \right) \mathbb{1}(\Omega) \right] + \mathcal{O}\left(N^{-\mathfrak{t}}\right),
\end{aligned}$$

where $\mathfrak{t} > 0$ is the constant in (2.12) of the Assumptions in 2.3. Here we use the definition of Ω. Since (v_i) are i.i.d., (3.3) holds with exponentially high probability. Estimate (3.4) holds with probability large than $1 - N^{-\mathfrak{t}}$ by Assumption 2.3. Hence $\mathbb{P}^V(\Omega^c) \le cN^{-\mathfrak{t}}$, for some $c > 0$ and N sufficiently large.

Using Theorem 7.1, we find that

$$\begin{aligned}
&\int \frac{1}{|J|} \sum_{j \in J} G_{j,\mathbf{m}}(\mathbf{x}) \tilde{f}_t(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) \\
&\quad = \int \frac{1}{|J|} \sum_{j \in J} G_{j,\mathbf{m},\mathrm{sc}}(\mathbf{x}) \, \mathrm{d}\mu_G(\mathbf{x}) + \mathcal{O}\left(N^{-\mathfrak{f}}\right) + \mathcal{O}\left(N^{-\mathfrak{t}}\right),
\end{aligned} \tag{8.12}$$

where we use once more the estimates on the event Ω. Here $\mathfrak{f} > 0$ is the constant appearing in Theorem 7.1.

To establish the equivalent result to Theorem 7.2, we need a local deformed semicircle law for the setting when the entries (v_i) of V are not fixed. Recall that we denote by m_{fc} the Stieltjes transform of the deformed semicircle law $\rho_{\mathrm{fc}} = \rho_{\mathrm{fc}}^{\vartheta=1}$.

LEMMA 8.4 (Theorems 2.10 and 2.21 in [39]). *Let W be a complex Hermitian or a real symmetric Wigner matrix satisfying the assumptions in Definition 2.1. Let V be a random real diagonal matrix satisfying Assumptions 2.2 and 2.3. Set $H := V + W$, $G(z) := (H - z)^{-1}$ and $m_N(z) := N^{-1} \operatorname{Tr} G(z)$, $(z \in \mathbb{C}^+)$. Let $\xi = A_0 \log\log N/2$; see (2.20). Then there exists $\upsilon > 0$ and c [both depending on the constants in (2.3), the constants A_0, E_0 in (3.9) and the measure ν], such that for $L \geq 40\xi$, we have*

$$|m_N(z) - m_{\mathrm{fc}}(z)| \leq (\varphi_N)^{c\xi} \left(\min\left\{ \frac{1}{N^{1/4}}, \frac{1}{\sqrt{\kappa_E + \eta}} \frac{1}{\sqrt{N}} \right\} + \frac{1}{N\eta} \right), \tag{8.13}$$

and

$$\left|G_{ij}(z) - \delta_{ij} g_i(z)\right| \leq (\varphi_N)^{c\xi} \left(\sqrt{\frac{\operatorname{Im} m_{\mathrm{fc}}(z)}{N\eta}} + \frac{1}{N\eta} \right), \tag{8.14}$$

$i, j \in [\![1, N]\!]$, with (ξ, υ)-high probability, for all $z = E + \mathrm{i}\eta \in \mathcal{D}_L$; see (3.9). Here, we have set

$$g_i(z) := \frac{1}{v_i - z - m_{\mathrm{fc}}(z)} \qquad \left(z \in \mathbb{C}^+, i \in [\![1, N]\!]\right).$$

Moreover, fixing $\alpha > 0$, there is c_1 [depending on the constants in (2.3), the constants A_0, E_0 in (3.9), the measure ν and α], such that

$$|\lambda_i - \gamma_i| \leq (\varphi_N)^{c_1 \xi} \frac{1}{\sqrt{N}}, \tag{8.15}$$

with (ξ, υ)-high probability, for all $i \in [\![\alpha N, (1 - \alpha)N]\!]$. Here (λ_i) denote the eigenvalues of $H = V + W$, and (γ_i) are their classical locations with respect the deformed semicircle law ρ_{fc}.

Using the local law in Lemma 8.4, we obtain from (8.12) equivalent results to Theorem 7.2.

THEOREM 8.5. *Fix $n \in \mathbb{N}$, and consider an n-particle observable O. Fix $\delta > 0$, and let $t \geq N^{-1/4+\delta}$. Let $\tilde{\alpha} > 0$ be a small constant, and consider two energies $E \in [L_-(t) + \tilde{\alpha}, L_+(t) - \tilde{\alpha}]$ and $E' \in [-2 + \tilde{\alpha}, 2 - \tilde{\alpha}]$. Then, for any $\varepsilon > 0$ and for $b \equiv b_N$ satisfying $\tilde{\alpha}/2 \geq b_N > 0$, we have*

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n O(\alpha_1, \ldots, \alpha_n) \right. \\
&\quad \times \left[\int_{E-b}^{E+b} \frac{\mathrm{d}x}{2b} \frac{1}{[\rho_{\mathrm{fc}}(t, E)]^n} \varrho^N_{\tilde{f}_t, n} \left(x + \frac{\alpha_1}{N\rho_{\mathrm{fc}}(t, E)}, \ldots, x + \frac{\alpha_n}{N\rho_{\mathrm{fc}}(t, E)} \right) \right. \\
&\qquad \left. \left. - \int_{E'-b}^{E'+b} \frac{\mathrm{d}x}{2b} \frac{1}{[\rho_{\mathrm{sc}}(E')]^n} \varrho^N_{G, n} \left(x + \frac{\alpha_1}{N\rho_{\mathrm{sc}}(E')}, \ldots, x + \frac{\alpha_n}{N\rho_{\mathrm{sc}}(E')} \right) \right] \right| \\
&\quad \leq C_O N^{2\varepsilon} \left(b^{-1} N^{-1/2+\varepsilon} + N^{-\mathfrak{f}} + N^{-\mathfrak{t}} + N^{-1/4} \right),
\end{aligned}$$

for N sufficiently large. Here, $\mathfrak{f} > 0$ is the constant in Theorem 7.1. Moreover, $\rho_{\mathrm{fc}}(E)$ stands for the density of the (N-independent) measure ρ_{fc} at the energy E. The constant C_O depends on O, $\tilde{\alpha}$ and the measure ν. The constant $\mathfrak{f}$ depends on δ and $\tilde{\alpha}$.

The proof of Theorem 8.5 is an application of Section 7 in [29]. The validity of Assumption IV in [29] is a direct consequence of the local law in Lemma 8.4. Here and also below, we use that the local laws of Lemma 8.4 are only used on very small scales $\eta \sim N^{-1+\varepsilon}$ in the bulk. For such small η the first error term in (8.13) is negligible compared to the second error term. Also note that the first term on the right-hand side of the estimate in Theorem 8.5 is bigger than the corresponding term in (7.3). This is due to the weaker rigidity bounds in case V is random; see (8.15). We therefore have to impose that $b \gg N^{-1/2}$ in order to have a vanishing error term in the limit of large N. Finally, we mention that the error term $C_O N^{2\varepsilon} N^{-1/4}$ stems from replacing $\widehat{\rho}_{\mathrm{fc}}(t, E)$ by $\rho_{\mathrm{fc}}(t, E)$; see the comment below Theorem 7.2.

PROOF OF THEOREM 2.6. The proof Theorem 2.6 follows now along the lines of the proof of Theorem 2.5. First, we check that the Green function comparison Theorem 8.1 holds true for $H^X = V + X$, respectively, $H^Y = V + Y$ with random V. This is indeed the case, since the only input we used is estimate (8.7), which also holds for random V by the local laws in Lemma 8.4 and the stability estimate (3.21). Note that we are using that bound (8.7) is only required on scales $\eta \ll N^{-1/2}$. Similarly, we can establish Theorem 8.2 for random V using the Green function comparison theorem for random V, the local laws in Lemma 8.4 and the stability estimate (3.21). Finally, we note that the construction of the matrix U and Y [see (8.8)] and the moment matching in (8.9) do not involve V. We can thus complete the proof of Theorem 2.6 in the same way as the proof of Theorem 2.5. □

9. Edge universality for deformed Wigner matrices. In this section we prove Theorem 2.10. Its proof is a combination of Corollary 5.4 (bounds on the global Dirichlet form) and the method of [12]. In fact, the proof of the edge universality is very similar to the proof of the bulk universality: we first establish the edge universality for our model with a small Gaussian component (cf. Section 6 for the bulk), and then remove the small Gaussian component using Green function comparison and a moment matching; cf. Section 8 for the bulk.

9.1. *Edge universality with a small Gaussian component.* We mainly follow the exposition in Section 3 of [12]. We consider the local statistics at the lower edge; the upper edge is treated in exactly the same way.

9.1.1. *Preliminaries.* Recall the definition of the β-ensemble μ_U in (4.2) for a given potential U that is C^4 and "regular." To study the local statistics at the lower edge, we introduce two auxiliary measures, σ and $\check{\sigma}$, on $\mathcal{F}^{(N)}$ as follows. By a shift and a rescaling, we can assume that the equilibrium density, ϱ_U, of μ_U is supported on $[0, A_+]$, for some $A_+ > 0$. Fix a small $\varepsilon_0 > 0$, and set

$$\sigma(\mathrm{d}\boldsymbol{\lambda}) := \frac{1}{Z_\sigma} \mathrm{e}^{-\beta N \mathcal{H}_\sigma(\boldsymbol{\lambda})}\, \mathrm{d}\boldsymbol{\lambda}, \tag{9.1}$$

with

$$\begin{aligned} \mathcal{H}_\sigma(\boldsymbol{\lambda}) &= \mathcal{H}(\boldsymbol{\lambda}) + \frac{2}{N} \sum_{i=1}^{N} \Theta\big(N^{2/3-\varepsilon_0} \lambda_i\big), \\ \Theta(x) &:= (x+1)^2 \mathbb{1}(x < -1), \end{aligned} \tag{9.2}$$

where $\mathcal{H}$ is given in (4.3) and where $Z_\sigma \equiv Z_\sigma(\beta)$ is a normalization. Similarly, we introduce

$$\check{\sigma}(\mathrm{d}\boldsymbol{\lambda}) := \frac{1}{Z_{\check{\sigma}}} \mathrm{e}^{-\beta N \mathcal{H}_{\check{\sigma}}(\boldsymbol{\lambda})}\, \mathrm{d}\boldsymbol{\lambda}, \tag{9.3}$$

with

$$\mathcal{H}_{\check{\sigma}}(\boldsymbol{\lambda}) = \mathcal{H}(\boldsymbol{\lambda}) + \frac{1}{N} \sum_{i=1}^{N} \Theta\big(N^{2/3-\varepsilon_0} \lambda_i\big), \tag{9.4}$$

with $Z_{\check{\sigma}} \equiv Z_{\check{\sigma}}(\beta)$ a normalization. The potential Θ is added to avoid that the (x_i) deviate too far to the left, yet its influence on the local statistics at the edge is negligible; see Lemma 4.1 in [12]. Below, we choose $\beta = 1, 2$ depending on the symmetry class of our original matrix.

Following Section 3 of [12], we choose a small $\delta > \varepsilon_0$ and an integer K such that $K \in [\![N^\delta, N^{1-\delta}]\!]$. Denote by $I = [\![1, K]\!]$ the set of the first K indices. For $\boldsymbol{\lambda} \in \mathcal{F}^{(N)}$, we write

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) = (x_1, \ldots, x_K, y_{K+1}, \ldots, y_N), \tag{9.5}$$

and

$$\mathbf{x} = (x_1, \ldots, x_K) \in \mathcal{F}^{(K)}, \qquad \mathbf{y} = (y_{K+1}, \ldots, y_N) \in \mathcal{F}^{(N-K)}; \tag{9.6}$$

cf. (6.5) and (6.6). We further denote $\mathbf{I} := (-\infty, y_{K+1}]$. For fixed $\mathbf{y}$, we define the localized measures $\mu_U^{\mathbf{y}}$, $\sigma^{\mathbf{y}}$ and $\check{\sigma}^{\mathbf{y}}$ as in Section 6.2. (For simplicity of notation, we do not indicate the U and ε_0 dependences in the measures σ, $\check{\sigma}$.)

We introduce the set of "good" boundary conditions

$$\mathcal{R}(\varepsilon_0) \equiv \mathcal{R} := \big\{\mathbf{y} \in \mathcal{F}^{(N-K)} : |y_k - \gamma_k| \le N^{-2/3+\varepsilon_0} \check{k}, k \notin I\big\}, \tag{9.7}$$

with $\check{k} = \min\{k, N-k\}$, where (γ_k) denote the classical locations with respect to the equilibrium density. With our choices of δ and ε_0, we have $y_K - y_1 \sim (K/N)^{2/3}$.

9.1.2. *Comparison of the local measures at the edge.* Fix $t > 0$. Recall that we denote by $f_t \mu_G$ the distribution of $\boldsymbol{\lambda}(t)$ under the flow generated by (5.3). As in Section 6, we fix (v_i), and condition of the event Ω; see Definition 3.1. Also recall from (4.21) the definition of the time dependent reference β-ensemble $\widehat{\psi}_t \mu_G$, whose equilibrium density is $\widehat{\varrho}_{\mathrm{fc}}(t)$. By a simple shift and a scaling, we may assume, for fixed t, that $\operatorname{supp} \widehat{\varrho}_{\mathrm{fc}} = [0, \widehat{L}_+(t)]$ and that

$$\widehat{\varrho}_{\mathrm{fc}}(t, x) = \frac{1}{\pi} \sqrt{x} \big(1 + O(x)\big), \tag{9.8}$$

as $x \searrow 0$. This can easily be checked from the proofs of the Lemmas A.1 and 3.6 in the Appendix. For $\mathbf{y} \in \mathcal{R}$, we then introduce the localized measures $\widehat{\psi}_t^{\mathbf{y}} \mu_G^{\mathbf{y}}$ and $f_t^{\mathbf{y}} \mu_G^{\mathbf{y}}$ in the obvious way. For technical reasons, we also use the measures σ and $\check{\sigma}$, with the choice $U = \widehat{U}(t)$. (The Hamiltonians of the measures $\widehat{\psi}_t \mu_G$ and σ, $\check{\sigma}$, agree up to the confining potential Θ.)

In a first step, we compare the statistics of $\widehat{\psi}_t^{\mathbf{y}} \mu_G^{\mathbf{y}}$ and $\sigma^{\mathbf{y}}$. This is the analogue result to Proposition 6.4 above, respectively, to Lemma 5.4 in [12].

LEMMA 9.1. *Let $0 < \mathfrak{a} < 1/2$. Fix small constants $\delta > \varepsilon_0 > 0$. Let $K \in [\![N^\delta, N^{1-\delta}]\!]$, and let O be an n-particle observable. Let $\varepsilon' > 0$, and choose τ satisfying $1 \gg \tau > N^{-2\mathfrak{a}}$. Then, for any $t \geq N^{\varepsilon'} \tau$ and any constant $\mathfrak{c} \in (0, 1)$, there is a set of configurations $\mathcal{G}(\varepsilon_0) \equiv \mathcal{G} \subset \mathcal{R}$, with*

$$\mathbb{P}^{f_t \mu_G}(\mathcal{G}) \geq 1 - \frac{N^{-\mathfrak{c}}}{2}, \tag{9.9}$$

such that

$$\left| \int O(\mathbf{x}) \big(f_t^{\mathbf{y}}(\mathbf{x}) \mu_G^{\mathbf{y}}(\mathrm{d}\mathbf{x}) - \sigma^{\mathbf{y}}(\mathrm{d}\mathbf{x}) \big) \right| \leq C_O K^{1/6} N^{1/3} N^{\mathfrak{c}-\mathfrak{a}} \tau^{-1}, \tag{9.10}$$

$t \geq N^{\varepsilon'} \tau$, for N sufficiently large on Ω.

Moreover, there is $\upsilon > 0$, such that

$$\mathbb{P}^{f_t^{\mathbf{y}} \mu_G^{\mathbf{y}}} \big(\{ |x_k - \widehat{\gamma}_k(t)| < N^{-1+\varepsilon_0}, k \in I \} \big) \geq 1 - \mathrm{e}^{-\upsilon(\varphi_N)^\xi}, \tag{9.11}$$

$t \geq N^{\varepsilon'} \tau$, for N sufficiently large on Ω, with $\xi = A_0 \log\log N/2$; see (2.20).

PROOF. We follow the proof of Proposition 6.4 with some modifications. First, introduce the density q_t by demanding

$$q_t \sigma = f_t \mu_G.$$

Then we note that, at the lower edge,

$$\frac{1}{N} \sum_{k \in I^c} \frac{1}{(x - \widehat{\gamma}_k(t))^2} \geq c N^{1/3} / K^{1/3},$$

for $x \geq -N^{-2/3+\varepsilon_0}$ and $\mathbf{y} \in \mathcal{R}$. We thus have $\nabla^2_{\mathbf{x}} \mathcal{H}^{\mathbf{y}}_\sigma(\mathbf{x}) \geq cN^{1/3}/K^{1/3}$; cf. (6.10) for $\mathcal{H}^{\mathbf{y}}_\sigma$. Hence the logarithmic Sobolev inequality

$$S_{\sigma^{\mathbf{y}}}(q_t^{\mathbf{y}}) \leq C \frac{K^{1/3}}{N^{1/3}} D_{\sigma^{\mathbf{y}}}\Big(\sqrt{q_t^{\mathbf{y}}}\Big),$$

with the Dirichlet form $D_{\sigma^{\mathbf{y}}}(f) = \frac{1}{\beta N} \sum_{i \in I} \int |\partial_i f(\mathbf{x})|^2 \sigma^{\mathbf{y}}(\mathrm{d}\mathbf{x})$ holds. To bound the Dirichlet form, we proceed as in (6.24),

$$\begin{aligned}
\mathbb{E}^\sigma D_{\sigma^{\mathbf{y}}}\Big(\sqrt{q_t^{\mathbf{y}}}\Big) &\leq D_\sigma(\sqrt{q_t}) \\
&\leq 2D_{\widehat{\psi}_t \mu_G}(\sqrt{\widehat{g}_t}) + CN^{4/3} \sum_{i=1}^N \mathbb{E}^{\widehat{\psi}_t \mu_G} \big|\Theta'\big(N^{2/3-\varepsilon_0} x_i\big)\big|^2 \\
&\leq C \frac{N^{1-2\mathfrak{a}}}{\tau^2} + \mathrm{e}^{-N^c},
\end{aligned}$$

for some $c > 0$, with $\widehat{g}_t = f_t/\widehat{\psi}_t$, where we used the definitions of D_σ, $D_{\widehat{\psi}_t \mu_G}$ to get the second line. The third line follows from Corollary 5.4 and Lemma 5.5.

To complete the proof, we now follow mutatis mutandis the proof of Proposition 6.4. We leave the details aside. □

Eventually, we are going to apply Theorem 3.3 of [12], which shows that the statistics of $\sigma^{\mathbf{y}}$ are universal for most boundary conditions $\mathbf{y}$. In order to apply it, we need the analogue of Lemma 6.7 above.

LEMMA 9.2. *Under the assumptions of Proposition 9.1 the following holds. Let* $\mathbf{y} \in \mathcal{G}$. *Then, assuming that*

$$K^{1/3} N^{-1/3} N^{2\mathfrak{c}-\mathfrak{a}} \tau^{-1} \leq N^{\varepsilon_0} K^{-1/3} N^{-2/3}, \tag{9.12}$$

we get, for all $k \in I$,

$$\big|\mathbb{E}^{f_t^{\mathbf{y}} \mu_G^{\mathbf{y}}} x_k - \mathbb{E}^{\sigma^{\mathbf{y}}} x_k\big| \leq CN^{\varepsilon_0} K^{-1/3} N^{-2/3}, \tag{9.13}$$

for N sufficiently large on Ω.

PROOF. Replacing the constant $\tau_K = CK/N$ in the logarithmic Sobolev inequality 6.23 by $CK^{1/3}/N^{1/3}$, we can copy the proof of Lemma 6.7 (see also Lemma 5.5 in [12]) almost word by word. □

From (9.13), we immediately get, for $\mathbf{y} \in \mathcal{G}$, the estimate

$$\big|\mathbb{E}^{\sigma^{\mathbf{y}}}\big(x_k - \widehat{\gamma}_k(t)\big)\big| \leq CN^{\varepsilon_0} N^{-2/3} k^{-1/3}, \tag{9.14}$$

provided that (9.12) holds.

9.1.3. *Universality of the localized measures at the edge.* In this subsection, we establish the following result.

LEMMA 9.3. *Fix an integer $n > 0$. Then for any $1/4 > \varkappa$ the following holds on the event Ω. For any $\delta > 0$, there is a constant $\mathfrak{f} > 0$ such that, for $t \geq N^{-\delta}$ and for $\Lambda \subset [\![1, N^{\varkappa}]\!]$ with $|\Lambda| = n$,*

$$\begin{aligned} &\big|\mathbb{E}^{f_t \mu_G} O\big(\big(c_t N^{2/3} j^{1/3}(\lambda_j - \widehat{\gamma}_j(t))\big)_{j\in\Lambda}\big) \\ &\quad - \mathbb{E}^{\mu_G} O\big(\big(N^{2/3} j^{1/3}(\lambda_j - \gamma_j)\big)_{j\in\Lambda}\big)\big| \\ &\quad \leq C N^{-\mathfrak{f}}, \end{aligned} \tag{9.15}$$

where c_t depends only on $\widehat{\varrho}_{\mathrm{fc}}(t)$. Here, $(\widehat{\gamma}_j)$ denote the classical locations with respect the measure $\widehat{\varrho}_t$, and (γ_j) denote the classical locations with respect the semicircle law ϱ_{sc}.

PROOF. We follow the proof of Lemma 5.1 in [12]. We consider the case $n = 1$ only; the general case is proved in the same way. By a shift and a scaling, we may assume that $c_t = 1$ [see (9.8)], and we may replace $\widehat{\gamma}_j(t)$ by the γ_j. [Here, we implicitly use that we fixed (v_i) and conditioned on the event Ω.]

We will need two modifications of the set $\mathcal{R}(\varepsilon_0)$ of "good" boundary conditions. Let σ, $\check{\sigma}$ be given by (9.1), respectively (9.3) (with a generic potential U). Then set

$$\begin{aligned} \mathcal{R}^*(\varepsilon_0) := \big\{ \mathbf{y} \in \mathcal{R}(\varepsilon_0) : \forall k \in I, \big|\mathbb{E}^{\sigma^{\mathbf{y}}} x_k - \gamma_k\big| \leq N^{-2/3+\varepsilon_0} k^{-1/3}, \\ \mathbb{P}^{\check{\sigma}^{\mathbf{y}}}\big(x_1 \geq \gamma_1 - N^{-2/3+\varepsilon_0}\big) \geq 1/2 \big\}. \end{aligned} \tag{9.16}$$

We further need the set

$$\mathcal{R}^{\#}(\varepsilon_0) := \big\{ \mathbf{y} \in \mathcal{R}(\varepsilon_0/3) : |y_{K+1} - y_{K+2}| \geq N^{-2/3-\varepsilon_0} K^{-1/3} \big\}. \tag{9.17}$$

While the set $\mathcal{R}^*(\varepsilon_0)$ incorporates rigidity estimates in the sense that γ_k is a good approximation in expectation to x_k and that x_1 is not too much on the left, the set $\mathcal{R}^{\#}(\varepsilon_0)$ incorporates a level repulsion estimate. It has no counterpart in Section 6 above.

We now choose $\mathfrak{a} = 1/2 - \delta'$, $\mathfrak{c} = \delta'/2$ and $\tau = N^{-\delta'}$, for some small $1/12 > \delta' > 0$. With this choice, we have for $K \leq N^{1/4-6\delta'}$, that

$$K^{1/6} N^{1/3} N^{\mathfrak{c}-\mathfrak{a}} \tau^{-1} \leq N^{-\varepsilon_0}, \tag{9.18}$$

respectively,

$$K^{1/3} N^{-1/3} N^{2\mathfrak{c}-\mathfrak{a}} \tau^{-1} \leq N^{\varepsilon_0} K^{-1/3} N^{-2/3}, \tag{9.19}$$

for a small $\varepsilon_0 > 0$ (with $\delta' > \varepsilon_0$).

Then, from Lemma 9.1 we have, for $\mathbf{y} \in \mathcal{G}$,

$$\left| \int O\big(N^{2/3} j(x_j - \gamma_j)\big)\big(f_t^{\mathbf{y}} \, \mathrm{d}\mu_G^{\mathbf{y}} - \mathrm{d}\sigma^{\mathbf{y}}\big) \right| \leq C_O N^{-\chi} \qquad (j \in \Lambda), \tag{9.20}$$

for some $\chi > 0$. Here, the measure σ is given by (9.1) with the potential $\widehat{U}(t)$.

Let $\tilde{\sigma}$ denote the measure given by (9.1) with the potential $U \equiv 0$. For $\tilde{\mathbf{y}} \in \mathcal{R}(\varepsilon_0)$ (where the classical locations are taken with respect the semicircle law), we introduce the localized measure $\tilde{\sigma}^{\tilde{\mathbf{y}}}$. We now apply Theorem 3.3 of [12]: for $\mathbf{y} \in \mathcal{R}^{\#}(\varepsilon_0) \cap \mathcal{R}^{*}(\varepsilon_0)$, respectively, $\tilde{\mathbf{y}} \in \mathcal{R}^{\#}(\varepsilon_0) \cap \mathcal{R}^{*}(\varepsilon_0)$, we have

$$\left| \int O\big(N^{2/3} j(x_j - \gamma_j)\big)\big(\mathrm{d}\sigma^{\mathbf{y}} - \mathrm{d}\tilde{\sigma}^{\tilde{\mathbf{y}}}\big) \right| \leq C_O N^{-\chi}, \tag{9.21}$$

for sufficiently large N, by choosing $\chi > 0$ sufficiently small. From Lemma 4.1 of [12], we know that $\mathbb{P}^{\tilde{\sigma}}(\mathcal{R}^{\#}(\varepsilon_0) \cap \mathcal{R}^{*}(\varepsilon_0)) \geq 1 - N^{-c}$, for some $c > 0$. We further know from Lemma 4.1 of [12] that, for any bounded observable O, $|\mathbb{E}^{\tilde{\sigma}} O - \mathbb{E}^{\mu_G} O| \leq C_O \exp(-N^c)$, $c > 0$, where μ_G denotes the GUE/GOE. Thus, integrating out the boundary conditions $\tilde{\mathbf{y}}$ and replacing $\tilde{\sigma}$ with μ_G, we get from (9.20) and (9.21),

$$\left| \int O\big(N^{2/3} j(x_j - \gamma_j)\big)\big(f_t^{\mathbf{y}} \, \mathrm{d}\mu_G^{\mathbf{y}} - \mathrm{d}\mu_G\big) \right| \leq C_O N^{-\chi}, \tag{9.22}$$

for sufficiently small $\chi > 0$, where $\mathbf{y} \in \mathcal{G}(\varepsilon_0) \cap \mathcal{R}^{\#}(\varepsilon_0) \cap \mathcal{R}^{*}(\varepsilon_0)$. Once we have established that

$$\mathbb{P}^{f_t \mu_G}\big(\mathcal{G}(\varepsilon_0) \cap \mathcal{R}^{\#}(\varepsilon_0) \cap \mathcal{R}^{*}(\varepsilon_0)\big) \geq 1 - N^{-c}, \tag{9.23}$$

for some $c > 0$, we integrate out the boundary condition $\mathbf{y}$ in (9.22), and we get (9.15) for $n = 1$.

To prove (9.23) we follow the two steps of the proof of (5.23) in [12]. In a first step, one controls the probability of $\mathcal{R}^{\#}(\varepsilon_0)$ using the rigidity estimates for $f_t \mu_G$ (see Lemma 3.4), the level repulsion estimates for the measure $\sigma^{\mathbf{y}}$ in Theorem 3.2 of [12], Lemma 9.1 and the condition (9.19). In a second step, one shows that $\mathcal{G}(\varepsilon_0) \subset \mathcal{R}^{*}(\varepsilon_0)$. This follows from (9.2) and the arguments given in the proof of Lemma 5.1 of [12]. In this way (9.23) can be established; we leave the details to the interested reader. □

9.2. *Removal of the Gaussian component.* In this subsection we prove Theorem 2.10. We use the following version of the Green function comparison theorem at the edge. It is the counterpart to Theorem 8.1 above.

THEOREM 9.4. *Suppose we have two Wigner matrices X and Y satisfying the conditions in Definition 2.1. Set $H^X := V + X$, $H^Y := V + Y$; see (8.1). Denote by $\mathbb{P}^X$, $\mathbb{P}^Y$ the probability distributions of X,Y. Then on Ω the following holds true.*

For any $\varepsilon > 0$, there is $\delta > 0$ [depending on ε and the constants C_0, ϑ in (2.3)], such that

$$\mathbb{P}^X\big(N^{2/3}(\lambda_1 - \widehat{\gamma}_1) \le s - N^{-\varepsilon}\big) - N^{-\delta} \le \mathbb{P}^Y\big(N^{2/3}(\lambda_1 - \widehat{\gamma}_1) \le s\big)$$
$$\le \mathbb{P}^X\big(N^{2/3}(\lambda_1 - \widehat{\gamma}_1) \le s + N^{-\varepsilon}\big) + N^{-\delta}, \qquad s \in \mathbb{R},$$

for N sufficiently large, where $(\widehat{\gamma}_k)$ denote the classical locations of the measure $\widehat{\varrho}_{\mathrm{fc}} \equiv \widehat{\varrho}^{\theta}_{\mathrm{fc}}$, with $\theta = 1$. Analogous results hold for the joint distributions of the eigenvalues $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p}$, as long as $|i_p| \le N^{\varepsilon}$.

Theorem 9.4 is proven exactly in the same way as Theorem 2.4 of [33] for the Wigner case $V = 0$. It suffices to note that the entries of V are fixed in Theorem 9.4 and that the only input needed in the proof are the local laws for the Green functions of H^X and H^Y, which have been established in Theorem 3.3 above.

Given Theorem 9.4, we now complete the proof of Theorem 2.10. Following the arguments in Section 8.2, we construct an auxiliary Wigner matrix U such that the first two moments of the matrix

$$H_t = V + \mathrm{e}^{-t/2}U + \big(1 - \mathrm{e}^{-t}\big)^{1/2}W' \tag{9.24}$$

with $t = N^{-\delta}$ ($\delta > 0$ as in Lemma 9.3, and W' an independent GUE/GOE matrix) and the matrix $H = V + W$ match. By Lemma 9.3 the edge statistics of H_t are universal. By Theorem 9.4 the eigenvalue statistics of H_t and H at the edge agree for large N. The existence of such U is assured by Lemma 8.3. We have thus established that there is a small $\chi > 0$ such that

$$\begin{aligned}&\big|\mathbb{E}^{f_0\mu_G} O\big(\big(c_0 N^{2/3} j^{1/3}(\lambda_j - \widehat{\gamma}_j)\big)_{j\in\Lambda}\big)\\ &\quad - \mathbb{E}^{\mu_G} O\big(\big(N^{2/3} j^{1/3}(\lambda_j - \gamma_j)\big)_{j\in\Lambda}\big)\big| \le C N^{-\chi},\end{aligned} \tag{9.25}$$

for N sufficiently large on Ω, where μ_G is the GUE/GOE.

Finally, we use Assumptions 2.2 and 2.3 as well as a simple moment bound to average over $\widehat{\nu}$ (the empirical distribution of V) in (9.25). This completes the proof of Theorem 2.10.

APPENDIX

In this Appendix we prove the auxiliary results used in Sections 3 and 4: Lemmas 3.5, 3.6 and 4.2. We start with a more extended version of Lemma 3.5. Recall from (3.1) that we denote $\Theta_\varpi = [0, 1 + \varpi']$, $\varpi' = \varpi/10$. Also recall the definition of the domain $\mathcal{D}'$ of the spectral parameter z in (3.17).

LEMMA A.1. *Let ν satisfy Assumption 2.3 for some $\varpi > 0$. Then the following holds true for any $\vartheta \in \Theta_\varpi$. There are $L^\vartheta_-, L^\vartheta_+ \in \mathbb{R}$, with $L^\vartheta_- < 0 < L^\vartheta_+$, such that* $\operatorname{supp} \rho^\vartheta_{\mathrm{fc}} = [L^\vartheta_-, L^\vartheta_+]$, *and there is a constant $C > 1$ such that, for all $\vartheta \in \Theta_\varpi$,*

$$C^{-1}\sqrt{\kappa_E} \le \rho^\vartheta_{\mathrm{fc}}(E) \le C\sqrt{\kappa_E} \qquad \big(E \in [L^\vartheta_-, L^\vartheta_+]\big), \tag{A.1}$$

where κ_E denotes the distance of E to the endpoints of the support of $\rho_{\mathrm{fc}}^{\vartheta}$, that is,

$$\kappa_E := \min\{|E - L_-^{\vartheta}|, |E - L_+^{\vartheta}|\}. \tag{A.2}$$

The Stieltjes transform, $m_{\mathrm{fc}}^{\vartheta}$, of $\rho_{\mathrm{fc}}^{\vartheta}$ has the following properties:

(1) *for all $z = E + \mathrm{i}\eta \in \mathbb{C}^+$,*

$$\operatorname{Im} m_{\mathrm{fc}}^{\vartheta}(z) \sim \begin{cases} \sqrt{\kappa + \eta}, & E \in [L_-, L_+], \\ \dfrac{\eta}{\sqrt{\kappa + \eta}}, & E \in [L_-, L_+]^c; \end{cases} \tag{A.3}$$

(2) *there exists a constant $C > 1$ such that for all $z \in \mathcal{D}'$ and for all $x \in I_\nu$,*

$$C^{-1} \le \left|\vartheta x - z - m_{\mathrm{fc}}^{\vartheta}(z)\right| \le C; \tag{A.4}$$

(3) *there exists a constant $C > 1$ such that for all $z \in \mathcal{D}'$,*

$$C^{-1}\sqrt{\kappa + \eta} \le \left|1 - \int \frac{\mathrm{d}\nu(v)}{(\vartheta v - z - m_{\mathrm{fc}}^{\vartheta}(z))^2}\right| \le C\sqrt{\kappa + \eta}; \tag{A.5}$$

(4) *there are constants $C > 1$ and $c_0 > 0$ such for all $z = E + \mathrm{i}\eta \in \mathcal{D}'$ satisfying $\kappa_E + \eta \le c_0$,*

$$C^{-1} \le \left|\int \frac{\mathrm{d}\nu(v)}{(\vartheta v - z - m_{\mathrm{fc}}^{\vartheta}(z))^3}\right| \le C; \tag{A.6}$$

moreover, there is $C > 1$, such that for all $z \in \mathcal{D}'$,

$$\left|\int \frac{\mathrm{d}\nu(v)}{(\vartheta v - z - m_{\mathrm{fc}}^{\vartheta}(z))^3}\right| \le C. \tag{A.7}$$

The constants in statements (1)–(4) *can be chosen uniformly in $\vartheta \in \Theta_\varpi$.*

PROOF. We follow the proofs in [39, 51]. Let $\vartheta \in \Theta_\varpi$. Set $\zeta = z + m_{\mathrm{fc}}^{\vartheta}(z)$, and let

$$F(\zeta) := \zeta - \int \frac{\mathrm{d}\nu(v)}{\vartheta v - \zeta} \qquad (\zeta \in \mathbb{C}^+). \tag{A.8}$$

Then the functional equation (3.5) is equivalent to $z = F(\zeta)$. As is argued in [51], a point $E \in \mathbb{R}$ is inside the support of the measure $\rho_{\mathrm{fc}}^{\vartheta}$ if and only if $\zeta_E = E + m_{\mathrm{fc}}^{\vartheta}(E)$ satisfies $\operatorname{Im} F(\zeta_E) = 0$ and $\operatorname{Im}\zeta_E > 0$. Accordingly, the endpoints of the support are characterized as the solutions of

$$H(\zeta) := \int \frac{\mathrm{d}\nu(v)}{(\vartheta v - \zeta)^2} = 1 \qquad (\zeta \in \mathbb{R}). \tag{A.9}$$

Note that $H(\zeta)$ is a continuous function outside $\vartheta I_\nu \equiv \{x : x = \vartheta y,\, y \in I_\nu\}$ which is decreasing as $|\zeta|$ increases. Since $\vartheta \in \Theta_\varpi = [0, 1 + \varpi']$, with $\varpi' = \varpi/10$, we obtain from Assumption 2.3 that $H(\zeta) \ge 1 + \varpi/2$, for all $\zeta \in \vartheta I_\nu$. It thus follows

that there are only two solutions, $\zeta_\pm^\vartheta \in \mathbb{R} \setminus \vartheta I_\nu$, to $H(\zeta) = 1$, $\zeta \in \mathbb{R}$. In particular, $\zeta_-^\vartheta < 0$, $\zeta_+^\vartheta > 0$, and there is a constant $\mathfrak{g} > 0$, depending only on ν, such that

$$\inf_{\vartheta \in \Theta_\varpi} \operatorname{dist}\big(\{\zeta_\pm^\vartheta\}, \vartheta I_\nu\big) \geq \mathfrak{g}. \tag{A.10}$$

As argued in [39, 51], the set $\gamma := \{\zeta \in \mathbb{C}^+ : \operatorname{Im} F(\zeta) = 0, \operatorname{Im}\zeta > 0\}$ is, for each fixed $\vartheta \in \Theta_\varpi$, a finite curve in the upper half plane that is the graph of a continuous function which only connects to the real line at $\zeta_\pm^\vartheta$.

Since $\operatorname{dist}(\{\zeta_\pm^\vartheta\}, \vartheta I_\nu\} \geq \mathfrak{g} > 0$, $F(\zeta)$ is analytic in a neighborhood of $\zeta_\pm^\vartheta$. Thus for ζ in a neighborhood of ζ_+^ϑ, we may write

$$F(\zeta) = F\big(\zeta_+^\vartheta\big) + F'\big(\zeta_+^\vartheta\big)\big(\zeta - \zeta_+^\vartheta\big) + \frac{F''(\zeta_+^\vartheta)}{2}\big(\zeta - \zeta_+^\vartheta\big)^2 + \mathcal{O}\big(\big(\zeta - \zeta_+^\vartheta\big)^3\big).$$

Note that $F'(\zeta_+^\vartheta) = 0$ by the definition of ζ_+^ϑ. Moreover, we know that $\operatorname{Im} F(\zeta) = 0$, for ζ in a real neighborhood of ζ_+^ϑ, but we also have $\operatorname{Im} F(\zeta) = 0$, for $\zeta \in \gamma \cup \bar{\gamma}$. Thus $F''(\zeta_+^\vartheta) \neq 0$. We can therefore invert $F(\zeta) = z$ in a neighborhood of ζ_+ to obtain

$$\zeta(z) = F^{(-1)}(z) = \zeta_+^\vartheta + c_+^\vartheta \sqrt{z - L_+^\vartheta}\Big(1 + \mathcal{A}_+^\vartheta\Big(\sqrt{z - L_+^\vartheta}\Big)\Big) \tag{A.11}$$

[with the convention $\operatorname{Im} F^{(-1)}(z) \geq 0$], where L_+^ϑ is defined by $\zeta_+^\vartheta = L_+^\vartheta + m_{\mathrm{fc}}(L_+^\vartheta)$. Here, $c_+^\vartheta > 0$ is a real constant, and $\mathcal{A}_+^\vartheta$ is an analytic function that is real-valued on the real line and that satisfies $\mathcal{A}_+^\vartheta(0) = 0$. Recalling that $\zeta(z) = z + m_{\mathrm{fc}}^\vartheta(z)$ and taking the limit $\eta \to 0$ we obtain (A.1), for fixed ϑ. To achieve uniformity in ϑ, we use the (uniform) stability bound (A.10) and the (pointwise) positivity of $|F''(\zeta_+^\vartheta)|$: we differentiate (A.9) with respect to ϑ and observe that $\partial_\vartheta H(\zeta, \vartheta)|_{\zeta = \zeta_+^\vartheta} \neq 0$, for all $\vartheta \in \Theta_\varpi$, since $F''(\zeta_+^\vartheta) \neq 0$. Thus by the implicit function theorem, ζ_+^ϑ is a C^1 function of $\vartheta \in \Theta_\varpi$. Next, we observe that $F''(\zeta)$ is an analytic function of ζ, for ζ away from ϑI_ν. Thus, using once more (A.10), we can bound $|F''(\zeta_+^\vartheta)| \geq c$, for some $c > 0$, uniformly in $\vartheta \in \Theta_\varpi$. In fact, $F^{(n)}(\zeta_+^\vartheta)$, $n \in \mathbb{N}$ are all continuous functions of $\vartheta \in \Theta_\varpi$, and we can bound them uniformly in ϑ for each $n \in \mathbb{N}$. Repeating the same argument for ζ close to ζ_-^ϑ, we complete the proof of (A.1).

Statement (2) follows from (A.10) for z close to the edges. For z away from the edges, Assumption 2.3 assures that the curve γ stays away from the real line for all $\vartheta \in \Theta_\varpi$ as is readily checked. This implies the stability bound for that region.

For the proofs of the remaining statements, we refer to the Appendix of [39]. □

Next, we prove Lemma 3.6.

PROOF OF LEMMA 3.6. It follows from Assumption 2.3 that on Ω for all N sufficiently large,

$$\inf_{x\in I_{\widehat{\nu}}} \frac{1}{N}\sum_{i=1}^{N} \frac{1}{(\vartheta v_i - x)^2} \geq 1 + \varpi/2, \tag{A.12}$$

for all $\vartheta \in \Theta_\varpi = [0, 1+\varpi/10]$. The analogous statements of Lemma A.1, holding on Ω for N sufficiently large, follow in the same way as in the proof of that lemma. To get uniformity in N, it suffices to check that the analogous expression to (A.10) holds uniformly in N, for N sufficiency large: by (A.12) there are two real solutions $\widehat{\zeta}_\pm^\vartheta$ to $\widehat{H}(\zeta) := \frac{1}{N}\sum_{i=1}^{N} \frac{1}{(\vartheta v_i - \zeta)^2} = 1$ that both lie outside of the interval $\vartheta I_{\widehat{\nu}}$. Thus (3.3) and (3.4) imply that

$$\inf_{\vartheta\in\Theta_\varpi} \operatorname{dist}\big(\{\widehat{\zeta}_\pm^\vartheta\}, \vartheta I_{\widehat{\nu}}\big) \geq \mathfrak{g}/2, \tag{A.13}$$

on Ω for all N sufficiently large. Then we can bound

$$\widehat{F}''(\zeta) = -\frac{2}{N}\sum_{i=1}^{N} \frac{1}{(\vartheta v_i - \zeta)^3},$$

evaluated at $\widehat{\zeta}_\pm^\vartheta$, uniformly below in ϑ and N, for N sufficiently large, implying the uniformity in N of the constants in statements (1)–(4).

Next we prove (3.22). For simplicity we drop ϑ from the notation and work on Ω. As above, set $\zeta = z + m_{\mathrm{fc}}(z)$ and $\widehat{\zeta} = z + \widehat{m}_{\mathrm{fc}}(z)$. From the definitions of F, $\widehat{F}$ and equations (3.5), (3.6), we have $\widehat{F}(\widehat{\zeta}) = F(\zeta) = z$, for all $z \in \mathcal{D}'$. Using the stability bound (A.13) and equation (3.3) in the definition of Ω, we get, assuming that $|\widehat{\zeta} - \zeta| \ll 1$,

$$\begin{aligned}\big[F'(\zeta) + \mathcal{O}\big(N^{-\alpha_0}\big)\big](\widehat{\zeta} - \zeta) + \frac{F''(\zeta)}{2}(\widehat{\zeta} - \zeta)^2 \\ = o(1)(\widehat{\zeta} - \zeta)^2 + \mathcal{O}\big(N^{-\alpha_0}\big),\end{aligned} \tag{A.14}$$

uniformly in $\vartheta \in \Theta_\varpi$, for all $z \in \mathcal{D}'$. From Lemma A.1, we get $F'(\zeta) \sim \sqrt{\kappa + \eta}$ and $F''(\zeta) \leq C$, for all $z \in \mathcal{D}'$. We abbreviate $\Lambda := |\widehat{\zeta} - \zeta|$ in the following.

We first consider $z = E + \mathrm{i}\eta \in \mathcal{D}'$, such that $\kappa_E + \eta > N^{-\varepsilon}$, for some small $\varepsilon > 0$ (with $\varepsilon < \alpha_0$). Here κ_E is defined in (A.2). For such z we obtain from (A.14) that $\Lambda \leq CN^{\varepsilon}(\Lambda^2 + N^{-\alpha_0})$. Thus either $\Lambda \leq C_0 N^{\varepsilon} N^{-\alpha_0}$ or $C_0 N^{-\varepsilon} \leq \Lambda$, for some constant C_0. We now show that $|\Lambda| \leq C_0 N^{\varepsilon} N^{-\alpha_0}$, for all $z \in \mathcal{D}'$ such that $\kappa_E + \eta \geq N^{-\varepsilon}$. For $z \in \mathcal{D}'$ with $\eta = 2$, we have

$$\widehat{\zeta}(z) - \zeta(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{\widehat{\zeta}(z) - \zeta(z)}{(\vartheta v_i - \widehat{\zeta}(z))(\vartheta v_i - \zeta(z))} + \mathcal{O}\big(N^{-\alpha_0}\big),$$

where we use (3.3). Since $\eta = 2$ and $\operatorname{Im}\widehat{\zeta}, \operatorname{Im}\zeta \geq \eta$, we obtain $\Lambda \leq \frac{1}{4}\Lambda + \mathcal{O}(N^{-\alpha_0})$, that is, $\Lambda(z) \leq CN^{-\alpha_0}$, for $\eta = 2$. To extend the conclusion to all η,

we use the Lipschitz continuity of $\widehat{\zeta}(z)$, respectively, $\zeta(z)$. Differentiating $z = F(\zeta)$, with respect to z we obtain $\partial_z \zeta = (F'(\zeta))^{-1}$. Thus using property (2) of Lemma A.1, we infer that the Lipschitz constant of $\zeta(z)$ is, for $z \in \mathcal{D}'$ satisfying $\kappa_E + \eta > N^{-\varepsilon}$, bounded above by $N^{\varepsilon/2}$. The same conclusion also holds for $\widehat{\zeta}(z)$. Bootstrapping, we obtain

$$|\widehat{\zeta}(z) - \zeta(z)| \le CN^{\varepsilon} N^{-\alpha_0}, \tag{A.15}$$

on Ω for N sufficiently large, for all $z \in \mathcal{D}'$ satisfying $\kappa_E + \eta > N^{-\varepsilon}$.

In order to control $\widehat{\zeta}(z) - \zeta(z)$ for $z = E + \mathrm{i}\eta \in \mathcal{D}'$ with $\kappa_E + \eta \le N^{-\varepsilon}$, $\varepsilon > 0$, we first derive the estimate $|\widehat{L}_{\pm}^{\vartheta} - L_{\pm}^{\vartheta}| \le CN^{-\alpha_0}$, for some $c > 0$, on Ω. We recall that $\widehat{L}_{\pm}$, respectively, $L_{\pm}$, are obtained through the relations

$$\frac{1}{N}\sum_{i=1}^{N} \frac{1}{(\vartheta v_i - \widehat{\zeta}_{\pm})^2} = 1, \qquad \int \frac{\mathrm{d}\nu(v)}{(\vartheta v - \zeta_{\pm})^2} = 1.$$

Then a similar argument as given above shows that $|\widehat{\zeta}_{\pm} - \zeta_{\pm}| \le CN^{-\alpha_0}$ and $|\widehat{L}_{\pm} - L_{\pm}| \le CN^{-\alpha_0}$ on Ω, N sufficiently large. We refer to Section 4.3 in [40] for details.

Second, following the arguments in the proof of Lemma A.1, we may write, for $\widehat{\zeta}$ and ζ in a neighborhood of $\zeta_{\pm}$,

$$\begin{aligned} \widehat{\zeta}(z) - \widehat{\zeta}_{\pm} &= \widehat{c}_{\pm}\sqrt{z - \widehat{L}_{\pm}}\big(1 + \mathcal{O}(z - \widehat{L}_{\pm})\big), \\ \zeta(z) - \zeta_{\pm} &= c_{+}\sqrt{z - L_{\pm}}\big(1 + \mathcal{O}(z - L_{\pm})\big). \end{aligned} \tag{A.16}$$

We therefore get $|\widehat{\zeta}(z) - \zeta(z)| \le C\sqrt{\kappa_E + \eta} + CN^{-\alpha_0/2}$. Note that the constants can be chosen uniformly in $\vartheta \in \Theta_{\varpi}$. Choosing, for example, $\varepsilon = \alpha_0/4$, we get from (A.15) and (A.16) the desired inequality (3.22). □

We now move on to the construction of the potentials $\widehat{U}$ and U. We first record the following corollary of Lemma A.1. Set $\mathrm{B}_r(p) := \{z \in \mathbb{C} : |z - p| < r\}$. Recall the conventions in (2.8) and the definition of κ_E in (A.2).

COROLLARY A.2. *Under the assumptions of Lemma 3.5 there are constants c_+^{ϑ}, $r_+ > 0$, such that for any $E \in \mathrm{B}_{r_+}(L_+^{\vartheta}) \cap \mathbb{R}$,*

$$\operatorname{Im} m_{\mathrm{fc}}^{\vartheta}(E) = \begin{cases} \sqrt{\kappa_E}\big(c_+^{\vartheta} + \mathcal{B}_+^{\vartheta}(-\kappa_E)\big), & E \le L_+^{\vartheta}, \\ 0, & E \ge L_+^{\vartheta}, \end{cases} \tag{A.17}$$

and

$$\operatorname{Re} m_{\mathrm{fc}}^{\vartheta}(E) = \begin{cases} \mathcal{C}_+^{\vartheta}(-\kappa_E), & E \le L_+^{\vartheta}, \\ \sqrt{\kappa_E}\big(c_+^{\vartheta} + \mathcal{B}_+^{\vartheta}(\kappa_E)\big) + \mathcal{C}_+^{\vartheta}(\kappa_E), & E \ge L_+^{\vartheta}, \end{cases} \tag{A.18}$$

where $\mathcal{B}_+^\vartheta, \mathcal{C}_+^\vartheta$ are analytic functions on $\mathrm{B}_{r_+}(0)$ that are real-valued on $\mathbb{R}$ and that satisfy $\mathcal{B}_+^\vartheta(0) = 0$, $c_+^\vartheta + \mathcal{B}_+^\vartheta > 0$, respectively, $\mathcal{C}_+^\vartheta < 0$, on $\mathrm{B}_{r_+}(0) \cap \mathbb{R}$. Moreover, for all $z \in \mathrm{B}_{r_+}(L_+^\vartheta)$, the functions $\mathcal{B}_+^\vartheta, \mathcal{C}_+^\vartheta$, respectively, $\operatorname{Im} m_{\mathrm{fc}}^\vartheta$, $\operatorname{Re} m_{\mathrm{fc}}^\vartheta$, are continuous in $\vartheta \in \Theta_\varpi$.

Similar statements hold at the lower edge L_-^ϑ.

PROOF. Fix $\vartheta \in \Theta_\varpi$. As argued in the proof of Lemma A.1, the function $F(\zeta)$ can locally be inverted around $\zeta_\pm^\vartheta$; see (A.11) above. Thus for ζ in a neighborhood of ζ_+^ϑ, we may write

$$\begin{aligned} m_{\mathrm{fc}}^\vartheta(z) = F^{-1}(z) - z &= \zeta_+^\vartheta - z + c_+^\vartheta \sqrt{z - L_+^\vartheta}\Big(1 + \mathcal{A}_+^\vartheta\Big(\sqrt{z - L_+^\vartheta}\Big)\Big) \\ &= c_+^\vartheta \sqrt{z - L_+^\vartheta}\big(1 + \mathcal{B}_+^\vartheta(z - L_+^\vartheta)\big) + \mathcal{C}_+^\vartheta(z - L_+^\vartheta), \end{aligned}$$

for z in $\mathrm{B}_r(L_+^\vartheta)$, for some $r > 0$, where $\mathcal{B}_+^\vartheta$ and $\mathcal{C}_+^\vartheta$ are analytic in a neighborhood of zero and real-valued on the real line, since $\operatorname{Im} F^{-1}(E) = 0$, for $E \in [L_-^\vartheta, L_+^\vartheta]^c$. Equations (A.17) and (A.18) follow. From the proof of Lemma A.1, it is immediate that $c_+^\vartheta > 0$. Thus $c_+^\vartheta + \mathcal{B}_+^\vartheta > 0$ in a real neighborhood of zero. Since $x - L_+^\vartheta - m_{\mathrm{fc}}^\vartheta(L_+^\vartheta) < 0$, for all $x \in \vartheta I_\nu$, we must have $\mathcal{C}_+^\vartheta < 0$ in a real neighborhood of zero. Since $F(z)$ is analytic on $\mathrm{B}_r(L_+^\vartheta)$, for all $\vartheta \in \Theta_\varpi$, and since ζ_+^ϑ is a C^1 function of ϑ, the functions $\mathcal{B}_+^\vartheta$ and $\mathcal{C}_+^\vartheta$ are C^1 in $\vartheta \in \Theta_\varpi$. Then it is clear from (A.10) that we can choose $r > 0$ uniformly in $\vartheta \in \Theta_\varpi$. The same arguments apply for ζ close to ζ_-^ϑ. □

The analogous result to Corollary A.2 is stated next. Recall the notation $\widehat{\kappa}_E := \min\{|E - \widehat{L}_-^\vartheta|, |E - \widehat{L}_+^\vartheta|\}$.

COROLLARY A.3. *Under the assumptions of Lemma 3.6 the following holds on Ω, for N sufficiently large. There are constants $\widehat{c}_+^\vartheta$, r'_+, with $r_+ \geq r'_+ > 0$, such that for any $E \in \mathrm{B}_{r'_+}(L_+^\vartheta) \cap \mathbb{R}$,*

$$\operatorname{Im} \widehat{m}_{\mathrm{fc}}^\vartheta(E) = \begin{cases} \sqrt{\widehat{\kappa}_E}\big(\widehat{c}_+^\vartheta + \widehat{\mathcal{B}}_+^\vartheta(-\widehat{\kappa}_E)\big), & E \leq \widehat{L}_+^\vartheta, \\ 0, & E \geq \widehat{L}_+^\vartheta, \end{cases} \tag{A.19}$$

and

$$\operatorname{Re} \widehat{m}_{\mathrm{fc}}^\vartheta(E) = \begin{cases} \widehat{\mathcal{C}}_+^\vartheta(-\widehat{\kappa}_E), & E \leq \widehat{L}_+^\vartheta, \\ \sqrt{\widehat{\kappa}_E}\big(\widehat{c}_+^\vartheta + \widehat{\mathcal{B}}_+^\vartheta(\widehat{\kappa}_E)\big) + \widehat{\mathcal{C}}_+^\vartheta(\widehat{\kappa}_E), & E \geq \widehat{L}_+^\vartheta, \end{cases} \tag{A.20}$$

where $\widehat{\mathcal{B}}_+^\vartheta, \widehat{\mathcal{C}}_+^\vartheta$ are analytic functions on $\mathrm{B}_{r'_+}(0)$ that are real-valued on $\mathbb{R}$ and that satisfy $\widehat{\mathcal{B}}_+^\vartheta(0) = 0$ and $\widehat{c}_+^\vartheta + \widehat{\mathcal{B}}_+^\vartheta > 0$, respectively, $\widehat{\mathcal{C}}_+^\vartheta < 0$, on $\mathrm{B}_{r'_+}(0) \cap \mathbb{R}$. Moreover, the constant r'_+ can be chosen independent of $\vartheta \in \Theta_\varpi$ and N, for N sufficiently large.

Further, the functions $\widehat{\mathcal{B}}_+^\vartheta, \widehat{\mathcal{C}}_+^\vartheta$, *respectively,* $\operatorname{Im} \widehat{m}_{\mathrm{fc}}^\vartheta, \operatorname{Re} \widehat{m}_{\mathrm{fc}}^\vartheta$, *are continuous functions in* $\vartheta \in \Theta_\delta$, *for all* $z \in \mathrm{B}_{r'_+}(L_+^\vartheta)$. *There is* $c > 0$, *such that*

$$\text{(A.21)} \qquad \big|\widehat{\mathcal{B}}_+^\vartheta(z) - \mathcal{B}_+^\vartheta(z)\big| \le N^{-c\alpha_0/2}, \qquad \big|\widehat{\mathcal{C}}_+^\vartheta(z) - \mathcal{C}_+^\vartheta(z)\big| \le N^{-c\alpha_0/2},$$

for all $z \in \mathrm{B}_{r'_+}(L_+^\vartheta)$ *and all* $\vartheta \in \Theta_\varpi$, *on* Ω *for* N *sufficiently large.*

Similar statements hold at the lower edge $\widehat{L}_-^\vartheta$.

PROOF. Corollary A.3 is proven in the same way as Corollary A.2. The only things to be checked are that $r'_\pm > 0$ can be chosen uniformly in N, N sufficiently large, and the bounds in (A.21). The former statement is an immediate consequence of the stability bound (A.13). The latter follows from $z = F(\zeta) = \widehat{F}(\widehat{\zeta})$, with $\zeta = z + m_{\mathrm{fc}}^\vartheta(z)$ and $\widehat{\zeta} = z + \widehat{m}_{\mathrm{fc}}^\vartheta(z)$. Then using (3.3), the stability bound (A.13) and the uniform lower bound on $F''(\zeta_\pm^\vartheta)$, it is straightforward to derive estimate (A.21) from (3.22). □

Next we prove Lemma 4.2. Recall from (4.16) that we chose $\vartheta \equiv \vartheta(t) := \mathrm{e}^{-(t-t_0)/2}$.

PROOF OF LEMMA 4.2. For $c > 0$ and a measure ω on $\mathbb{R}$, we define $\operatorname{supp}_c \omega := \operatorname{supp} \omega + [-c, c]$. Recall the constants $r'_\pm > 0$ of Corollary A.3. Set $s := \min\{r'_-, r'_+\}/2$.

We specify the potentials $\widehat{U}$ and U through their spatial derivatives $\widehat{U}'$ and U'. For $t \ge 0$, we set

$$\widehat{U}'(t,x) + x := -2 \fint_{\mathbb{R}} \frac{\widehat{\rho}_{\mathrm{fc}}(t,y)}{y-x}\,\mathrm{d}y, \qquad U'(t,x) + x := -2 \fint_{\mathbb{R}} \frac{\rho_{\mathrm{fc}}(t,y)}{y-x}\,\mathrm{d}y,$$

for $x \in \operatorname{supp} \widehat{\rho}_{\mathrm{fc}}(t)$, respectively, $x \in \operatorname{supp} \rho_{\mathrm{fc}}(t)$.

For $x \in \mathbb{R}$ satisfying $|x - L_\pm(t)| \le s$, where $L_\pm(t)$ denote the endpoints of the support of the measure $\rho_{\mathrm{fc}}(t)$, we set

$$\text{(A.22)} \qquad \begin{aligned} \widehat{U}'(t,x) + x &:= -2\widehat{\mathcal{C}}_\pm^\vartheta(\widehat{k}_\pm), \qquad \widehat{k}_\pm \equiv x - \widehat{L}_\pm(t), \\ U'(t,x) + x &:= -2\mathcal{C}_\pm^\vartheta(k_\pm), \qquad k_\pm \equiv x - L_\pm(t), \end{aligned}$$

where $\widehat{\mathcal{C}}_\pm^\vartheta$ are the functions appearing in Corollary A.3 with $\vartheta \equiv \vartheta(t)$, and $\mathcal{C}_\pm^\vartheta$ are the functions appearing in Corollary A.2 with $\vartheta \equiv \vartheta(t)$. From Lemma A.1, Corollaries A.2 and A.3, we conclude that $\widehat{U}'(t,x)$, respectively, $U'(t,x)$ are well defined for $x \in \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$, $t \ge 0$, where $s = \min\{r'_-, r'_+\}/2$.

For $x \notin \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$, we define U' as a C^3 extension in x such that: (1) $U^{(n)}(t,x)$, $\partial_t U^{(n)}(t,x)$, $n \in [\![1,3]\!]$, are continuous in t; (2) for all $t \ge 0$ and for all $x \notin \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$, $|U'(t,x) + x| > |2 \operatorname{Re} m_{\mathrm{fc}}(t,x)|$ and $\widehat{U}''(t,x) \ge -C_U$, for some constant $C_U \ge 0$; (3) $U'(t,x) + x \sim x$ for all $t \ge 0$, as $|x| \to \infty$. Similarly, we define $\widehat{U}(t,x)$ as C^3 extensions such that: (1) $\widehat{U}^{(n)}(t,x)$, $\partial_t \widehat{U}^{(n)}(t,x)$,

$n \in [\![1,3]\!]$, are continuous in t; (2) there is $c > 0$ such that $\sup_{t\geq 0} |\widehat{U}^{(n)}(t,x) - U^{(n)}(t,x)| \leq N^{-c\alpha_0/2}$, $n \in [\![1,3]\!]$, for N sufficiently large on Ω.

We next show that the potential $U'(t,x)$ is a C^3 function in x. For simplicity, we often drop the t-dependence from the notation. Let $\zeta = z + m_{\mathrm{fc}}(z)$, and recall from the proof of Lemma A.1 that $\zeta(z)$ satisfies $\zeta(z) = F^{(-1)}(z)$, where $F(\zeta) = \zeta - \int \frac{\mathrm{d}\nu(v)}{(\vartheta v - \zeta)}$. Thus, to prove regularity of $U'(t,x)$ in x in the support of the measure $\rho_{\mathrm{fc}}(t)$, it suffices to show that $F'(\zeta) \neq 0$ on the curve $\gamma \cap \mathbb{C}^+$ where $\operatorname{Im} F = 0$. Recall that on γ we have

$$\tilde{H}(\zeta) := \int \frac{\mathrm{d}\nu(v)}{|\vartheta v - \zeta|^2} = 1, \tag{A.23}$$

where $\vartheta \equiv \vartheta(t)$. On the other hand, we have

$$\operatorname{Re} F'(\zeta) = 1 - \int \frac{(\vartheta v - \operatorname{Re}\zeta)^2 - (\operatorname{Im}\zeta)^2}{|\vartheta v - \zeta|^4}\,\mathrm{d}\nu(v).$$

Thus, on the curve γ,

$$\begin{aligned}\operatorname{Re} F'(\zeta) &= \int \frac{\mathrm{d}\nu(v)}{|\vartheta v - \zeta|^2} - \int \frac{(\vartheta v - \operatorname{Re}\zeta)^2 - (\operatorname{Im}\zeta)^2}{|\vartheta v - \zeta|^4}\,\mathrm{d}\nu(v)\\ &= \int \frac{2(\operatorname{Im}\zeta)^2}{|\vartheta v - \zeta|^4}\,\mathrm{d}\nu(v).\end{aligned}$$

From (2.10) we get

$$\int \frac{\mathrm{d}\nu(v)}{|\vartheta v - \zeta|^4} \geq \left(\int \frac{\mathrm{d}\nu(v)}{|\vartheta v - \zeta|^2}\right)^2 = 1, \tag{A.24}$$

on γ. Since $F' \neq 0$ on γ, the inverse function theorem implies that the real part of $m_{\mathrm{fc}}(t,x)$ is a smooth function in the interior of $\operatorname{supp}\rho_{\mathrm{fc}}(t)$, whose derivatives are continuous in t. For $x \in \mathrm{B}_s(L_\pm^\vartheta)$, we already showed in Lemma A.2 that $\mathcal{C}_\pm^\vartheta(x)$ is a smooth function, whose derivatives are continuous in t. Thus we have shown that $U'(t,x)$ is smooth in $\operatorname{supp}_s \rho_{\mathrm{fc}}(t)$. Outside $\operatorname{supp}_s \rho_{\mathrm{fc}}(t)$, $U'(t,x)$ is manifestly C^3 by definition: it is a C^3 extension of the functions $\mathcal{C}_\pm(t)$. Thus $\mathbb{R} \ni x \mapsto U'(t,x)$, $\partial_t U'(t,x)$ are C^3 functions for all $t \geq 0$.

Clearly, we can bound the derivatives $U^{(n)}(t,x)$, $\partial_t U^{(n)}(t,x)$, $n \in [\![1,3]\!]$, uniformly on compact sets. It is also immediate that $U^{(n)}(t,x)$ are continuous functions in $t \geq 0$. Thus we can bound $U^{(n)}$ uniformly in t and uniformly in x on compact sets, for $n \in [\![1,3]\!]$. For $x \in \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$, we have $U''(t,x) \geq -C$, for some $C \geq 0$. For $x \notin \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$, a similar bound holds true by construction. Thus $U'(t,x)$ satisfies (4.4) uniformly in $t \geq 0$. Further, since $U'(t,x) + x \sim x$, as $|x| \to \infty$, (4.5) also holds uniformly in $t \geq 0$.

On Ω, we can extend the reasoning above to $\widehat{U}'(t,x)$, $\partial_t \widehat{U}'(t,x)$, for N sufficiently large. For example, the arguments in (A.23)–(A.24) can be extended to the

finite N case by using (3.3) and Lemma 3.6. Let again $s \equiv \min\{r'_-, r'_+\}/2$. Then for $x \in \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$ we have by Lemma 3.6 that $|\widehat{m}_{\mathrm{fc}}(t, x+\mathrm{i}\eta) - m_{\mathrm{fc}}(t, x+\mathrm{i}\eta)| \leq N^{-c\alpha_0}$, for some $c > 0$, on Ω for all $\eta \geq 0$ and all $t \geq 0$. Together with (A.21) we can conclude that $|\widehat{U}'(t, x) - U'(t, x)| \leq N^{-c\alpha_0/2}$ on Ω, for $x \in \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$. We also have $|\partial_x \widehat{m}_{\mathrm{fc}}(t, x+\mathrm{i}\eta) - \partial_x m_{\mathrm{fc}}(t, x+\mathrm{i}\eta)| \leq C N^{-c\alpha_0}$, for x satisfying $\min\{|x - L_+|, |x - L_-|\} \geq s$, as can be checked as in the proof of Lemma 3.6. Hence, combining this last statement with the regularity of $\widehat{C}^{\vartheta}_{\pm}$ claimed in Lemma A.3, we have $|\widehat{U}''(t, x) - U''(t, x)| \leq N^{-c\alpha_0}$, for $x \in \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$, $t \geq 0$, on Ω for N sufficiently large. This conclusion can be extended to arbitrary $\widehat{U}^{(n)}$. Similarly, one checks that $U^{(n)}(t, x)$, $n \in [\![1, 3]\!]$ are continuous functions of $t \geq 0$. For $x \notin \operatorname{supp}_s \rho_{\mathrm{fc}}(t)$, these properties follow directly from the definition of $\widehat{U}'$ above. Thus $\widehat{U}'(t, x)$ satisfies (4.4) and (4.5) with uniform constants for all $t \geq 0$ and N sufficiently large on Ω.

Finally, the potentials $\widehat{U}(t)$ and $U(t)$ are "regular" as follows from Lemmas 3.5 and 3.6. □

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