The Annals of Probability
2016, Vol. 44, No. 1, 1–41
DOI: 10.1214/14-AOP965

QUANTITATIVE STABLE LIMIT THEOREMS ON THE WIENER SPACE

BY IVAN NOURDIN[1], DAVID NUALART[2] AND GIOVANNI PECCATI[3]

Université de Lorraine, Kansas University and Université du Luxembourg

We use Malliavin operators in order to prove quantitative stable limit theorems on the Wiener space, where the target distribution is given by a possibly multidimensional mixture of Gaussian distributions. Our findings refine and generalize previous works by Nourdin and Nualart [*J. Theoret. Probab.* **23** (2010) 39–64] and Harnett and Nualart [*Stochastic Process. Appl.* **122** (2012) 3460–3505], and provide a substantial contribution to a recent line of research, focussing on limit theorems on the Wiener space, obtained by means of the Malliavin calculus of variations. Applications are given to quadratic functionals and weighted quadratic variations of a fractional Brownian motion.

1. Introduction and overview. Originally introduced by Rényi in the landmark paper [33], the notion of *stable convergence* for random variables (see Definition 2.2 below) is an intermediate concept, bridging convergence in distribution (which is a weaker notion) and convergence in probability (which is stronger). One crucial feature of stably converging sequences is that they can be naturally paired with sequences converging in probability (see, e.g., the statement of Lemma 2.3 below), thus yielding a vast array of noncentral limit results—most notably convergence toward *mixtures* of Gaussian distributions. This last feature makes indeed stable convergence extremely useful for applications, in particular to the asymptotic analysis of functionals of semimartingales, such as power variations, empirical covariances, and other objects of statistical relevance. See the classical reference [11], Chapter VIII.5, as well as the recent survey [31], for a discussion of stable convergence results in a semimartingale context.

Outside the (semi)martingale setting, the problem of characterizing stably converging sequences is for the time being much more delicate. Within the framework of limit theorems for functionals of general Gaussian fields, a step in this direction appears in the paper [28], by Peccati and Tudor, where it is shown that central limit theorems (CLTs) involving sequences of multiple Wiener–Itô integrals of order ≥ 2 are always stable. Such a result is indeed an immediate consequence of a general multidimensional CLT for chaotic random variables, and of

Received May 2013; revised August 2014.
[1]Supported in part by the French ANR Grant ANR-10-BLAN-0121.
[2]Supported in part by NSF Grant DMS-12-08625.
[3]Supported in part by Grant F1R-MTH-PUL-12PAMP (PAMPAS), from Luxembourg University.
MSC2010 subject classifications. 60F05, 60H07, 60G15.
Key words and phrases. Stable convergence, Malliavin calculus, fractional Brownian motion.

the well-known fact that the first Wiener chaos of a Gaussian field coincides with the L^2-closed Gaussian space generated by the field itself (see [20], Chapter 6, for a general discussion of multidimensional CLTs on the Wiener space). Some distinguished applications of the results in [28] appear, for example, in the two papers [1, 4], respectively, by Corcuera et al. and by Barndorff-Nielsen et al., where the authors establish stable limit theorems (toward a Gaussian mixture) for the power variations of pathwise stochastic integrals with respect to a Gaussian process with stationary increments. See [19] for applications to the weighted variations of an iterated Brownian motion. See [2] for some quantitative analogues of the findings of [28] for functionals of a Poisson measure.

Albeit useful for many applications, the results proved in [28] do not provide any intrinsic criterion for stable convergence toward Gaussian mixtures. In particular, the applications developed in [1, 4, 19] basically require that one is able to represent a given sequence of functionals as the combination of three components—one converging in probability to some nontrivial random element, one living in a finite sum of Wiener chaoses and one vanishing in the limit—so that the results from [28] can be directly applied. This is in general a highly nontrivial task, and such a strategy is technically too demanding to be put into practice in several situations (e.g., when the chaotic decomposition of a given functional cannot be easily computed or assessed).

The problem of finding effective intrinsic criteria for stable convergence on the Wiener space toward mixtures of Gaussian distributions—without resorting to chaotic decompositions—was eventually tackled by Nourdin and Nualart in [17], where one can find general sufficient conditions ensuring that a sequence of *multiple Skorohod integrals* stably converges to a mixture of Gaussian distributions. Multiple Skorohod integrals are a generalization of multiple Wiener–Itô integrals (in particular, they allow for random integrands), and are formally defined in Section 2.1 below. It is interesting to note that the main results of [17] are proved by using a generalization of a characteristic function method, originally applied by Nualart and Ortiz-Latorre in [25] to provide a Malliavin calculus proof of the CLTs established in [26, 28]. In particular, when specialized to multiple Wiener–Itô integrals, the results of [17] allow to recover the "fourth moment theorem" by Nualart and Peccati [26]. A first application of these stable limit theorems appears in [17], Section 5, where one can find stable mixed Gaussian limit theorems for the weighted quadratic variations of the fractional Brownian motion (fBm), complementing some previous findings from [18]. Another class of remarkable applications of the results of [17] are the so-called *Itô formulae in law*; see [8, 9, 22, 23]. Reference [9] also contains some multidimensional extensions of the abstract results proved in [17] (with a proof again based on the characteristic function method). Further applications of these techniques can be found in [34]. An alternative approach to stable convergence on the Wiener space, based on decoupling techniques, has been developed by Peccati and Taqqu in [27].

One evident limitation of the abstract results of [9, 17] is that they do not provide any information about rates of convergence. The aim of this paper is to prove several *quantitative versions* of the abstract results proved in [9, 17], that is, statements allowing one to explicitly assess quantities of the type

$$\big|E\big[\varphi\big(\delta^{q_1}(u_1),\dots,\delta^{q_d}(u_d)\big)\big]-E\big[\varphi(F)\big]\big|,$$

where φ is an appropriate test function on $\mathbb{R}^d$, each $\delta^{q_i}(u_i)$ is a multiple Skorohod integral of order $q_i \geq 1$, and F is a d-dimensional mixture of Gaussian distributions. Most importantly, we shall show that our bounds also yield natural sufficient conditions for stable convergence toward F. To do this, we must overcome a number of technical difficulties, in particular:

- We will work in a general framework and without any underlying semimartingale structure, in such a way that the powerful theory of stable convergence for semimartingales (see again [11]) cannot be applied.
- Although there are many versions of Stein's method allowing one to deal with general continuous non-Gaussian targets (see, e.g., [3, 5–7, 12, 13, 32]), it seems that none of them can be reasonably applied to the limit theorems that are studied in this paper. Indeed, the above quoted contributions fall mainly in two categories: either those requiring that the density of the target distribution is explicitly known (and in this case the so-called "density approach" can be applied—see, e.g., [3, 5–7]), or those requiring that the target distribution is the invariant measure of some diffusion process (so that the "generator approach" can be used—see, e.g., [12, 13, 32]). In both instances, a detailed analytical description of the target distribution must be available. In contrast, in the present paper we consider limit distributions given by the law of random elements of the type $S\cdot\eta = (S_1\eta_1,\dots,S_d\eta_d)$, where $\eta = (\eta_1,\dots,\eta_d)$ is a Gaussian vector, and $S = (S_1,\dots,S_d)$ is an independent random element that is suitably regular in the sense of Malliavin calculus. In particular, in our framework *no a priori knowledge* of the distribution of S (and therefore of $S\cdot\eta$) is required. One should note that in [3] one can find an application of Stein's method to the law of random objects with the form $S\eta$, where η is a one-dimensional Gaussian random variable and S has a law with a two-point support (of course, in this case the density of $S\eta$ can be directly computed by elementary arguments).

Our techniques rely on an interpolation procedure and on the use of Malliavin operators. To our knowledge, the main bounds proved in this paper, that is, the ones appearing in Proposition 3.1, Theorems 3.4 and 5.1, are first ever explicit upper bounds for mixed normal approximations in a nonsemimartingale setting.

Note that, in our discussion, we shall separate the case of one-dimensional Skorohod integrals of order 1 (discussed in Section 3) from the general case (discussed in Section 5), since in the former setting one can exploit some useful simplifications, as well as obtain some effective bounds in the Wasserstein and Kolmogorov distances. As discussed below, our results can be seen as abstract versions of classic limit theorems for Brownian martingales, such as the ones discussed in [35], Chapter VIII.

Although our results deal only with Skorohod integrals, they can be applied in the context of Stratonovich integrals. In fact, the Stratonovich integral can be expressed as a Skorohod integral plus a complementary term and in many problems this complementary term does not contribute to the limit. Examples of this situation are the Itô formulas in law for different types of Stratonovich integrals obtained by Harnett and Nualart in [8, 9] and the weak convergence of weighted variations established by Nourdin and Nualart in [17].

To illustrate our findings, we provide applications to quadratic functionals of a fractional Brownian motion (Section 3.3) and to weighted quadratic variations (Section 6). The results of Section 3.3 generalize some previous findings by Peccati and Yor [29, 30], whereas those of Section 6 complement some findings by Nourdin, Nualart and Tudor [18].

The paper is organized as follows. Section 2 contains some preliminaries on Gaussian analysis and stable convergence. In Section 3, we first derive estimates for the distance between the laws of a Skorohod integral of order 1 and of a mixture of Gaussian distributions (see Proposition 3.1). As a corollary, we deduce the stable limit theorem for a sequence of multiple Skorohod integrals of order 1 obtained in [9], and we obtain rates of convergence in the Wasserstein and Kolmogorov distances. We apply these results to a sequence of quadratic functionals of the fractional Brownian motion. Section 4 contains some additional notation and a technical lemma that are used in Section 5 to establish bounds in the multidimensional case for Skorohod integrals of general orders. Finally, in Section 6 we present the applications of these results to the case of weighted quadratic variations of the fractional Brownian motion. The Appendix contains some technical lemmas needed in Section 6.

2. Gaussian analysis and stable convergence.

In the next two subsections, we discuss some basic notions of Gaussian analysis and Malliavin calculus. The reader is referred to the monographs [24] and [20] for any unexplained definition or result.

2.1. *Elements of Gaussian analysis.*

Let $\mathfrak{H}$ be a real separable infinite-dimensional Hilbert space. For any integer $q \geq 1$, we denote by $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$, respectively, the qth tensor product and the qth symmetric tensor product of $\mathfrak{H}$. In what follows, we write $X = \{X(h) : h \in \mathfrak{H}\}$ to indicate an isonormal Gaussian process over $\mathfrak{H}$. This means that X is a centered Gaussian family, defined on some probability space $(\Omega, \mathcal{F}, P)$, with a covariance structure given by

$$E\big[X(h)X(g)\big] = \langle h, g\rangle_{\mathfrak{H}}, \qquad h, g \in \mathfrak{H}. \tag{2.1}$$

From now on, we assume that $\mathcal{F}$ is the P-completion of the σ-field generated by X. For every integer $q \geq 1$, we let $\mathcal{H}_q$ be the qth *Wiener chaos* of X, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial defined by

$$H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q}(e^{-x^2/2}).$$

We denote by $\mathcal{H}_0$ the space of constant random variables. For any $q \geq 1$, the mapping $I_q(h^{\otimes q}) = q! H_q(X(h))$ provides a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$ [equipped with the $L^2(\Omega)$ norm]. For $q = 0$, we set by convention $\mathcal{H}_0 = \mathbb{R}$ and I_0 equal to the identity map.

It is well known (Wiener chaos expansion) that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$, that is: any square integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.2}$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geq 1$, are uniquely determined by F. For every $q \geq 0$, we denote by J_q the orthogonal projection operator on the qth Wiener chaos. In particular, if $F \in L^2(\Omega)$ is as in (2.2), then $J_q F = I_q(f_q)$ for every $q \geq 0$.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $r \in \{0, \dots, p \wedge q\}$, the rth *contraction* of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\dots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.3}$$

Notice that $f \otimes_r g$ is not necessarily symmetric. We denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$. Contraction operators are useful for dealing with products of multiple Wiener–Itô integrals.

In the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, with $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite and nonatomic measure, one has that $\mathfrak{H}^{\odot q} = L_s^2(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on A^q. Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the multiple Wiener–Itô integral of order q of f with respect to X (as defined, e.g., in [24], Section 1.1.2) and (2.3) can be written as

$$\begin{aligned}
&(f \otimes_r g)(t_1, \dots, t_{p+q-2r}) \\
&\quad = \int_{A^r} f(t_1, \dots, t_{p-r}, s_1, \dots, s_r) \\
&\qquad \times g(t_{p-r+1}, \dots, t_{p+q-2r}, s_1, \dots, s_r)\, d\mu(s_1) \cdots d\mu(s_r).
\end{aligned}$$

2.2. *Malliavin calculus*. Let us now introduce some elements of the Malliavin calculus of variations with respect to the isonormal Gaussian process X. Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form

$$F = g\big(X(\phi_1), \dots, X(\phi_n)\big), \tag{2.4}$$

where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a infinitely differentiable function with compact support, and $\phi_i \in \mathfrak{H}$. The *Malliavin derivative* of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\big(X(\phi_1), \dots, X(\phi_n)\big)\phi_i.$$

By iteration, one can define the qth derivative $D^q F$ for every $q \geq 2$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot q})$.

For $q \geq 1$ and $p \geq 1$, $\mathbb{D}^{q,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,p}}$, defined by the relation

$$\|F\|_{\mathbb{D}^{q,p}}^p = E\big[|F|^p\big] + \sum_{i=1}^{q} E\big(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\big).$$

The Malliavin derivative D verifies the following chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \dots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) DF_i.$$

We denote by δ the adjoint of the operator D, also called the *divergence operator* or *Skorohod integral* (see, e.g., [24], Section 1.3.2, for an explanation of this terminology). A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ, noted $\operatorname{Dom} \delta$, if and only if it verifies

$$\big|E\big(\langle DF, u\rangle_{\mathfrak{H}}\big)\big| \leq c_u \sqrt{E(F^2)}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \operatorname{Dom} \delta$, then the random variable $\delta(u)$ is defined by the duality relationship (called "integration by parts formula"):

$$E\big(F\delta(u)\big) = E\big(\langle DF, u\rangle_{\mathfrak{H}}\big), \tag{2.5}$$

which holds for every $F \in \mathbb{D}^{1,2}$. The formula (2.5) extends to the multiple Skorohod integral δ^q, and we have

$$E\big(F\delta^q(u)\big) = E\big(\langle D^q F, u\rangle_{\mathfrak{H}^{\otimes q}}\big), \tag{2.6}$$

for any element u in the domain of δ^q and any random variable $F \in \mathbb{D}^{q,2}$. Moreover, $\delta^q(h) = I_q(h)$ for any $h \in \mathfrak{H}^{\odot q}$.

The following statement will be used in the paper, and is proved in [17].

LEMMA 2.1. *Let $q \geq 1$ be an integer. Suppose that $F \in \mathbb{D}^{q,2}$, and let u be a symmetric element in* $\mathrm{Dom}\,\delta^q$. *Assume that, for any $0 \leq r + j \leq q$, $\langle D^r F, \delta^j(u)\rangle_{\mathfrak{H}^{\otimes r}} \in L^2(\Omega, \mathfrak{H}^{\otimes q-r-j})$. Then, for any $r = 0, \dots, q-1$, $\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}$ belongs to the domain of δ^{q-r} and we have*

$$F\delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}\big(\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}\big). \tag{2.7}$$

[*With the convention that $\delta^0(v) = v$, $v \in L^2(\Omega)$ and $D^0 F = F$, $F \in L^2(\Omega)$.*]

For any Hilbert space V, we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of V-valued random variables (see [24], page 31). The operator δ^q is continuous from $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ to $\mathbb{D}^{k-q,p}$, for any $p > 1$ and any integers $k \geq q \geq 1$, that is, we have

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})}, \tag{2.8}$$

for all $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$, and some constant $c_{k,p} > 0$. These estimates are consequences of Meyer inequalities (see [24], Proposition 1.5.7). In particular, these estimates imply that $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q}) \subset \mathrm{Dom}\,\delta^q$ for any integer $q \geq 1$.

The following commutation relationship between the Malliavin derivative and the Skorohod integral (see [24], Proposition 1.3.2) is also useful:

$$D\delta(u) = u + \delta(Du), \tag{2.9}$$

for any $u \in \mathbb{D}^{2,2}(\mathfrak{H})$. By induction, we can show the following formula for any symmetric element u in $\mathbb{D}^{j+k,2}(\mathfrak{H}^{\otimes j})$

$$D^k\delta^j(u) = \sum_{i=0}^{j\wedge k} \binom{k}{i}\binom{j}{i} i!\,\delta^{j-i}(D^{k-i}u). \tag{2.10}$$

Also, we will make sometimes use of the following formula for the variance of a multiple Skorohod integral. Let $u, v \in \mathbb{D}^{2q,2}(\mathfrak{H}^{\otimes q}) \subset \mathrm{Dom}\,\delta^q$ be two symmetric functions. Then

$$\begin{aligned} E\big(\delta^q(u)\delta^q(v)\big) &= E\big(\langle u, D^q(\delta^q(v))\rangle_{\mathfrak{H}^{\otimes q}}\big) \\ &= \sum_{i=0}^{q} \binom{q}{i}^2 i!\,E\big(\langle u, \delta^{q-i}(D^{q-i}v)\rangle_{\mathfrak{H}^{\otimes q}}\big) \\ &= \sum_{i=0}^{q} \binom{q}{i}^2 i!\,E\big(D^{q-i}u \,\widehat{\otimes}_{2q-i}\, D^{q-i}v\big), \end{aligned} \tag{2.11}$$

with the notation

$$\begin{aligned} &D^{q-i}u \,\widehat{\otimes}_{2q-i}\, D^{q-i}v \\ &\quad = \sum_{j,k,\ell=1}^{\infty} \langle D^{q-i}\langle u, \xi_j \otimes \eta_\ell\rangle_{\mathfrak{H}^{\otimes q}}, \xi_k\rangle_{\mathfrak{H}^{\otimes q-i}} \langle D^{q-i}\langle v, \xi_k \otimes \eta_\ell\rangle_{\mathfrak{H}^{\otimes q}}, \xi_j\rangle_{\mathfrak{H}^{\otimes q-i}}, \end{aligned}$$

where $\{\xi_j, j \geq 1\}$ and $\{\eta_\ell, \ell \geq 1\}$ are complete orthonormal systems in $\mathfrak{H}^{\otimes q-i}$ and $\mathfrak{H}^{\otimes i}$, respectively.

The operator L is defined on the Wiener chaos expansion as $L = \sum_{q=0}^{\infty} -q J_q$, and is called the *infinitesimal generator of the Ornstein–Uhlenbeck semigroup*. The domain of this operator in $L^2(\Omega)$ is the set

$$\operatorname{Dom} L = \left\{ F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \|J_q F\|_{L^2(\Omega)}^2 < \infty \right\} = \mathbb{D}^{2,2}.$$

There is an important relationship between the operators D, δ and L (see [24], Proposition 1.4.3). A random variable F belongs to the domain of L if and only if $F \in \operatorname{Dom}(\delta D)$ (i.e., $F \in \mathbb{D}^{1,2}$ and $DF \in \operatorname{Dom} \delta$), and in this case

$$\delta D F = -LF. \tag{2.12}$$

Note also that a random variable F as in (2.2) is in $\mathbb{D}^{1,2}$ if and only if $\sum_{q=1}^{\infty} q q! \|f_q\|_{\mathfrak{H}^{\otimes q}}^2 < \infty$, and, in this case, $E(\|DF\|_{\mathfrak{H}}^2) = \sum_{q \geq 1} q q! \|f_q\|_{\mathfrak{H}^{\otimes q}}^2$. If $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ nonatomic), then the derivative of a random variable F as in (2.2) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a F = \sum_{q=1}^{\infty} q I_{q-1}\big(f_q(\cdot, a)\big), \qquad a \in A. \tag{2.13}$$

2.3. *Stable convergence.* The notion of stable convergence used in this paper is provided in the next definition. Recall that the probability space $(\Omega, \mathcal{F}, P)$ is such that $\mathcal{F}$ is the P-completion of the σ-field generated by the isonormal process X.

DEFINITION 2.2 (Stable convergence). Fix $d \geq 1$. Let $\{F_n\}$ be a sequence of random variables with values in $\mathbb{R}^d$, all defined on the probability space $(\Omega, \mathcal{F}, P)$. Let F be a $\mathbb{R}^d$-valued random variable defined on some extended probability space $(\Omega', \mathcal{F}', P')$. We say that F_n *converges stably* to F, written $F_n \stackrel{\text{st}}{\to} F$, if

$$\lim_{n \to \infty} E\big[Z e^{i\langle \lambda, F_n \rangle_{\mathbb{R}^d}}\big] = E'\big[Z e^{i\langle \lambda, F \rangle_{\mathbb{R}^d}}\big], \tag{2.14}$$

for every $\lambda \in \mathbb{R}^d$ and every bounded $\mathcal{F}$-measurable random variable Z.

Choosing $Z = 1$ in (2.14), we see that stable convergence implies convergence in distribution. For future reference, we now list some useful properties of stable convergence. The reader is referred, for example, to [11], Chapter 4, for proofs. From now on, we will use the symbol $\stackrel{P}{\to}$ to indicate convergence in probability with respect to P.

LEMMA 2.3. *Let $d \geq 1$, and let $\{F_n\}$ be a sequence of random variables with values in $\mathbb{R}^d$.*

1. $F_n \stackrel{\text{st}}{\to} F$ *if and only if* $(F_n, Z) \stackrel{\text{law}}{\to} (F, Z)$, *for every* $\mathcal{F}$*-measurable random variable* Z.

2. $F_n \stackrel{\text{st}}{\to} F$ *if and only if* $(F_n, Z) \stackrel{\text{law}}{\to} (F, Z)$, *for every random variable* Z *belonging to some set* $\mathscr{Z} = \{Z_\alpha : \alpha \in A\}$ *such that the* P*-completion of* $\sigma(\mathscr{Z})$ *coincides with* $\mathcal{F}$.

3. *If* $F_n \stackrel{\text{st}}{\to} F$ *and* F *is* $\mathcal{F}$*-measurable, then necessarily* $F_n \stackrel{P}{\to} F$.

4. *If* $F_n \stackrel{\text{st}}{\to} F$ *and* $\{Y_n\}$ *is another sequence of random elements, defined on* $(\Omega, \mathcal{F}, P)$ *and such that* $Y_n \stackrel{P}{\to} Y$, *then* $(F_n, Y_n) \stackrel{\text{st}}{\to} (F, Y)$.

The following statement (to which we will compare many results of the present paper) contains criteria for the stable convergence of vectors of multiple Skorohod integrals of the same order. The case $d = 1$ was proved in [17], Corollary 3.3, whereas the case of a general d is dealt with in [9], Theorem 3.2. Given $d \geq 1$, $\boldsymbol{\mu} \in \mathbb{R}^d$ and a nonnegative definite $d \times d$ matrix C, we shall denote by $\mathcal{N}_d(\boldsymbol{\mu}, C)$ the law of a d-dimensional Gaussian vector with mean $\boldsymbol{\mu}$ and covariance matrix C.

THEOREM 2.4. *Let* $q, d \geq 1$ *be integers, and suppose that* F_n *is a sequence of random variables in* $\mathbb{R}^d$ *of the form* $F_n = \delta^q(u_n) = (\delta^q(u_n^1), \ldots, \delta^q(u_n^d))$, *for a sequence of* $\mathbb{R}^d$*-valued symmetric functions* u_n *in* $\mathbb{D}^{2q,2q}(\mathfrak{H}^{\otimes q})$. *Suppose that the sequence* F_n *is bounded in* $L^1(\Omega)$ *and that*:

1. $\langle u_n^j, \bigotimes_{\ell=1}^m (D^{a_\ell} F_n^{j_\ell}) \otimes h\rangle_{\mathfrak{H}^{\otimes q}}$ *converges to zero in* $L^1(\Omega)$ *for all integers* $1 \leq j, j_\ell \leq d$, *all integers* $1 \leq a_1, \ldots, a_m, r \leq q-1$ *such that* $a_1 + \cdots + a_m + r = q$, *and all* $h \in \mathfrak{H}^{\otimes r}$.
2. *For each* $1 \leq i, j \leq d$, $\langle u_n^i, D^q F_n^j\rangle_{\mathfrak{H}^{\otimes q}}$ *converges in* $L^1(\Omega)$ *to a random variable* s_{ij}, *such that the random matrix* $\Sigma := (s_{ij})_{d\times d}$ *is nonnegative definite.*

Then $F_n \stackrel{\text{st}}{\to} F$, *where* F *is a random variable with values in* $\mathbb{R}^d$ *and with conditional Gaussian distribution* $\mathcal{N}_d(0, \Sigma)$ *given* X.

2.4. *Distances.* For future reference, we recall the definition of some useful distances between the laws of two real-valued random variables F, G.

- The *Wasserstein distance* between the laws of F and G is defined by

$$d_{\mathrm{W}}(F, G) = \sup_{\varphi \in \mathrm{Lip}(1)} \big|E[\varphi(F)] - E[\varphi(G)]\big|,$$

  where Lip(1) indicates the collection of all Lipschitz functions φ with Lipschitz constant less than or equal to 1.
- The *Kolmogorov distance* is

$$d_{\mathrm{Kol}}(F, G) = \sup_{x \in \mathbb{R}} \big|P(F \leq x) - P(G \leq x)\big|.$$

- The *total variation distance* is

$$d_{\mathrm{TV}}(F,G)=\sup_{A\in\mathscr{B}(\mathbb{R})}\big|P(F\in A)-P(G\in A)\big|.$$

- The *Fortet–Mourier distance* is

$$d_{\mathrm{FM}}(F,G)=\sup_{\varphi\in\mathrm{Lip}(1),\,\|\varphi\|_\infty\le 1}\big|E[\varphi(F)]-E[\varphi(G)]\big|.$$

Plainly, $d_{\mathrm{W}}\ge d_{\mathrm{FM}}$ and $d_{\mathrm{TV}}\ge d_{\mathrm{Kol}}$. We recall that the topologies induced by d_{W}, d_{Kol} and d_{TV}, over the class of probability measures on the real line, are strictly stronger than the topology of convergence in distribution, whereas d_{FM} metrizes convergence in distribution (see, e.g., [20], Appendix C, for a review of these facts).

3. Quantitative stable convergence in dimension one. We start by focussing on stable limits for one-dimensional Skorohod integrals of order one, that is, random variables having the form $F=\delta(u)$, where $u\in\mathbb{D}^{1,2}(\mathfrak{H})$. As already discussed, this framework permits some interesting simplifications that are not available for higher order integrals and higher dimensions. Notice that any random variable F such that $E[F]=0$ and $E[F^2]<\infty$ can be written as $F=\delta(u)$ for some $u\in\operatorname{Dom}\delta$. For example, we can take $u=-DL^{-1}F$, or in the context of the standard Brownian motion, we can take u an adapted and square integrable process.

3.1. *Explicit estimates for smooth distances and stable CLTs.* The following estimate measures the distance between a Skorohod integral of order 1, and a (suitably regular) mixture of Gaussian distributions. In order to deduce a stable convergence result in the subsequent Corollary 3.2, we also consider an element $I_1(h)$ in the first chaos of the isonormal process X.

PROPOSITION 3.1. *Let $F\in\mathbb{D}^{1,2}$ be such that $E[F]=0$. Assume $F=\delta(u)$ for some $u\in\mathbb{D}^{1,2}(\mathfrak{H})$. Let $S\ge 0$ be such that $S^2\in\mathbb{D}^{1,2}$, and let $\eta\sim\mathcal{N}(0,1)$ indicate a standard Gaussian random variable independent of the underlying isonormal Gaussian process X. Let $h\in\mathfrak{H}$. Assume that $\varphi:\mathbb{R}\to\mathbb{R}$ is C^3 with $\|\varphi''\|_\infty,\|\varphi'''\|_\infty<\infty$. Then*

$$\begin{aligned}
&\big|E[\varphi(F+I_1(h))]-E[\varphi(S\eta+I_1(h))]\big|\\
&\quad\le \tfrac12\|\varphi''\|_\infty E[2|\langle u,h\rangle_{\mathfrak{H}}|+|\langle u,DF\rangle_{\mathfrak{H}}-S^2|]\\
&\qquad+\tfrac13\|\varphi'''\|_\infty E[|\langle u,DS^2\rangle_{\mathfrak{H}}|].
\end{aligned}\tag{3.1}$$

PROOF. We proceed by interpolation. Fix $\varepsilon>0$ and set $S_\varepsilon=\sqrt{S^2+\varepsilon}$. Clearly, $S_\varepsilon\in\mathbb{D}^{1,2}$. Let $g(t)=E[\varphi(I_1(h)+\sqrt{t}F+\sqrt{1-t}S_\varepsilon\eta)]$, $t\in[0,1]$, and observe that $E[\varphi(F+I_1(h))]-E[\varphi(S_\varepsilon\eta+I_1(h))]=g(1)-g(0)=\int_0^1 g'(t)\,dt$. For $t\in(0,1)$, integrating by parts yields

$$\begin{aligned}
g'(t)&=\frac{1}{2}E\bigg[\varphi'\big(I_1(h)+\sqrt{t}F+\sqrt{1-t}S_\varepsilon\eta\big)\bigg(\frac{F}{\sqrt{t}}-\frac{S_\varepsilon\eta}{\sqrt{1-t}}\bigg)\bigg]\\
&=\frac{1}{2}E\bigg[\varphi'\big(I_1(h)+\sqrt{t}F+\sqrt{1-t}S_\varepsilon\eta\big)\bigg(\frac{\delta(u)}{\sqrt{t}}-\frac{S_\varepsilon\eta}{\sqrt{1-t}}\bigg)\bigg]\\
&=\frac{1}{2}E\bigg[\varphi''\big(I_1(h)+\sqrt{t}F+\sqrt{1-t}S_\varepsilon\eta\big)\\
&\qquad\times\bigg(\frac{1}{\sqrt{t}}\langle u,h\rangle_{\mathfrak{H}}+\langle u,DF\rangle_{\mathfrak{H}}+\frac{\sqrt{1-t}}{\sqrt{t}}\eta\langle u,DS_\varepsilon\rangle_{\mathfrak{H}}-S_\varepsilon^2\bigg)\bigg].
\end{aligned}$$

Integrating again by parts with respect to the law of η yields

$$\begin{aligned}
g'(t)&=\frac{1}{2}E\big[\varphi''\big(I_1(h)+\sqrt{t}F+\sqrt{1-t}S_\varepsilon\eta\big)\big(t^{-1/2}\langle u,h\rangle_{\mathfrak{H}}+\langle u,DF\rangle_{\mathfrak{H}}-S_\varepsilon^2\big)\big]\\
&\quad+\frac{1-t}{4\sqrt{t}}E\big[\varphi'''\big(I_1(h)+\sqrt{t}F+\sqrt{1-t}S_\varepsilon\eta\big)\langle u,DS^2\rangle_{\mathfrak{H}}\big],
\end{aligned}$$

where we have used the fact that $S_\varepsilon DS_\varepsilon=\frac{1}{2}DS_\varepsilon^2=\frac{1}{2}DS^2$. Therefore,

$$\begin{aligned}
&\big|E\big[\varphi\big(I_1(h)+F\big)\big]-E\big[\varphi\big(I_1(h)+S_\varepsilon\eta\big)\big]\big|\\
&\qquad\le\frac{1}{2}\|\varphi''\|_\infty E\big[2|\langle u,h\rangle_{\mathfrak{H}}|+|\langle u,DF\rangle_{\mathfrak{H}}-S^2-\varepsilon|\big]\\
&\qquad\quad+\|\varphi'''\|_\infty E\big[|\langle u,DS^2\rangle_{\mathfrak{H}}|\big]\int_0^1\frac{1-t}{4\sqrt{t}}\,dt,
\end{aligned}$$

and the conclusion follows letting ε go to zero, because $\int_0^1\frac{1-t}{4\sqrt{t}}\,dt=\frac{1}{3}$.

The following statement provides a stable limit theorem based on Proposition 3.1.

COROLLARY 3.2. *Let S and η be as in the statement of Proposition 3.1. Let $\{F_n\}$ be a sequence of random variables such that $E[F_n]=0$ and $F_n=\delta(u_n)$, where $u_n\in\mathbb{D}^{1,2}(\mathfrak{H})$. Assume that the following conditions hold as $n\to\infty$:*

1. *$\langle u_n,DF_n\rangle_{\mathfrak{H}}\to S^2$ in $L^1(\Omega)$;*
2. *$\langle u_n,h\rangle_{\mathfrak{H}}\to 0$ in $L^1(\Omega)$, for every $h\in\mathfrak{H}$;*
3. *$\langle u_n,DS^2\rangle_{\mathfrak{H}}\to 0$ in $L^1(\Omega)$.*

Then $F_n\overset{\text{st}}{\to}S\eta$, and selecting $h=0$ in (3.1) provides an upper bound for the rate of convergence of the difference $|E[\varphi(F_n)]-E[\varphi(S\eta)]|$, for every φ of class C^3 with bounded second and third derivatives.

PROOF. Relation (3.1) implies that, if conditions 1–3 in the statement hold true, then $|E[\varphi(F_n + I_1(h))] - E[\varphi(S\eta + I_1(h))]| \to 0$ for every $h \in \mathfrak{H}$ and every smooth test function φ. Selecting φ to be a complex exponential and using point 2 of Lemma 2.3 yields the desired conclusion. □

REMARK 3.3. (a) Corollary 3.2 should be compared with Theorem 2.4 in the case $d = q = 1$ (which exactly corresponds to [17], Corollary 3.3). This result states that, if (i) $u_n \in \mathbb{D}^{2,2}(\mathfrak{H})$ and (ii) $\{F_n\}$ is bounded in $L^1(\Omega)$, then it is sufficient to check conditions 1–2 in the statement of Corollary 3.2 for some S^2 in $L^1(\Omega)$ in order to deduce the stable convergence of F_n to $S\eta$. The fact that Corollary 3.2 requires more regularity on S^2, as well as the additional condition 3, is compensated by the less stringent assumptions on u_n, as well as by the fact that we obtain explicit rates of convergence for a large class of smooth functions.

(b) The statement of [17], Corollary 3.3, allows one also to recover a modification of the so-called *asymptotic Knight Theorem* for Brownian martingales, as stated in [35], Theorem VIII.2.3. To see this, assume that X is the isonormal Gaussian process associated with a standard Brownian motion $B = \{B_t : t \geq 0\}$ [corresponding to the case $\mathfrak{H} = L^2(\mathbb{R}_+, ds)$] and also that the sequence $\{u_n : n \geq 1\}$ is composed of square-integrable processes adapted to the natural filtration of B. Then, $F_n = \delta(u_n) = \int_0^\infty u_n(s)\,dB_s$, where the stochastic integral is in the Itô sense, and the aforementioned asymptotic Knight theorem yields that the stable convergence of F_n to $S\eta$ is implied by the following: (A) $\int_0^t u_n(s)\,ds \xrightarrow{P} 0$, uniformly in t in compact sets and (B) $\int_0^\infty u_n(s)^2\,ds \to S^2$ in $L^1(\Omega)$.

3.2. *Wasserstein and Kolmogorov distances*. The following statement provides a way to deduce rates of convergence in the Wasserstein and Kolmogorov distance from the previous results.

THEOREM 3.4. *Let $F \in \mathbb{D}^{1,2}$ be such that $E[F] = 0$. Write $F = \delta(u)$ for some $u \in \mathbb{D}^{1,2}(\mathfrak{H})$. Let $S \in \mathbb{D}^{1,4}$, and let $\eta \sim \mathcal{N}(0,1)$ indicate a standard Gaussian random variable independent of the isonormal process X. Set*

$$\Delta = 3\left(\frac{1}{\sqrt{2\pi}} E[|\langle u, DF\rangle_{\mathfrak{H}} - S^2|] + \frac{\sqrt{2}}{3} E[|\langle u, DS^2\rangle_{\mathfrak{H}}|]\right)^{1/3}$$

$$\times \max\left\{\frac{1}{\sqrt{2\pi}} E[|\langle u, DF\rangle_{\mathfrak{H}} - S^2|] + \frac{\sqrt{2}}{3} E[|\langle u, DS^2\rangle_{\mathfrak{H}}|],\right. \quad (3.2)$$

$$\left.\sqrt{\frac{2}{\pi}}(2 + E[S]) + E[|F|]\right\}^{2/3}.$$

Then $d_{\mathrm{W}}(F, S\eta) \leq \Delta$. Moreover, if there exists $\alpha \in (0,1]$ such that $E[|S|^{-\alpha}] < \infty$, then

$$d_{\mathrm{Kol}}(F, S\eta) \leq \Delta^{\alpha/(\alpha+1)}(1 + E[|S|^{-\alpha}]). \quad (3.3)$$

REMARK 3.5. Theorem 3.4 is specifically relevant whenever one deals with sequences of random variables living in a finite sum of Wiener chaoses. Indeed, in [21], Theorem 3.1, the following fact is proved: let $\{F_n : n \geq 1\}$ be a sequence of random variables living in the subspace $\bigoplus_{k=0}^{p} \mathcal{H}_k$, and assume that F_n converges in distribution to a nonzero random variable F_∞; then, there exists a finite constant $c > 0$ (independent of n) such that

$$d_{\mathrm{TV}}(F_n, F_\infty) \leq c d_{\mathrm{FM}}(F_n, F_\infty)^{1/(1+2p)} \leq c d_{\mathrm{W}}(F_n, F_\infty)^{1/(1+2p)}, \quad n \geq 1. \tag{3.4}$$

Exploiting this estimate, and in the framework of random variables with a finite chaotic expansion, the bounds in the Wasserstein distance obtained in Theorem 3.4 can be used to deduce rates of convergence in total variation toward mixtures of Gaussian distributions. The forthcoming Section 3.3 provides an explicit demonstration of this strategy, as applied to quadratic functionals of a (fractional) Brownian motion.

PROOF OF THEOREM 3.4. It is divided into two steps.

Step 1: *Wasserstein distance*. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class C^3 which is bounded together with all its first three derivatives. For any $t \in (0, 1)$, define

$$\varphi_t(x) = \int_{\mathbb{R}} \varphi(\sqrt{t} y + \sqrt{1-t} x) \, d\gamma(y),$$

where $d\gamma(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$ denotes the standard Gaussian measure. Then, we may differentiate and integrate by parts to get

$$\begin{aligned} \varphi_t''(x) &= \frac{1-t}{\sqrt{t}} \int_{\mathbb{R}} y \varphi'(\sqrt{t} y + \sqrt{1-t} x) \, d\gamma(y) \\ &= \frac{1-t}{t} \int_{\mathbb{R}} (y^2 - 1) \varphi(\sqrt{t} y + \sqrt{1-t} x) \, d\gamma(y) \end{aligned}$$

and

$$\varphi_t'''(x) = \frac{(1-t)^{3/2}}{t} \int_{\mathbb{R}} (y^2 - 1) \varphi'(\sqrt{t} y + \sqrt{1-t} x) \, d\gamma(y).$$

Hence, for $0 < t < 1$ we may bound

$$\|\varphi_t''\|_\infty \leq \frac{1-t}{\sqrt{t}} \|\varphi'\|_\infty \int_{\mathbb{R}} |y| \, d\gamma(y) \leq \sqrt{\frac{2}{\pi}} \frac{\|\varphi'\|_\infty}{t} \tag{3.5}$$

and

$$\begin{aligned} \|\varphi_t'''\|_\infty &\leq \frac{(1-t)^{3/2}}{t} \|\varphi'\|_\infty \int_{\mathbb{R}} |y^2 - 1| \, d\gamma(y) \\ &\leq \frac{\|\varphi'\|_\infty}{t} \sqrt{\int_{\mathbb{R}} (y^2 - 1)^2 \, d\gamma(y)} = \frac{\sqrt{2} \|\varphi'\|_\infty}{t}. \end{aligned} \tag{3.6}$$

Taylor expansion gives that

$$
\begin{aligned}
\bigl|E[\varphi(F)] - E[\varphi_t(F)]\bigr| &\le \int_{\mathbb{R}} E\bigl[\bigl|\varphi(\sqrt{t}y + \sqrt{1-t}F) - \varphi(\sqrt{1-t}F)\bigr|\bigr]\,d\gamma(y) \\
&\quad + E\bigl[\bigl|\varphi(\sqrt{1-t}F) - \varphi(F)\bigr|\bigr] \\
&\le \|\varphi'\|_\infty \sqrt{t}\int_{\mathbb{R}} |y|\,d\gamma(y) + \|\varphi'\|_\infty |\sqrt{1-t} - 1| E\bigl[|F|\bigr] \\
&\le \sqrt{t}\|\varphi'\|_\infty \biggl\{\sqrt{\frac{2}{\pi}} + E\bigl[|F|\bigr]\biggr\}.
\end{aligned}
$$

Here, we used that $|\sqrt{1-t} - 1| = t/(\sqrt{1-t}+1) \le \sqrt{t}$. Similarly,

$$
\begin{aligned}
\bigl|E[\varphi(S\eta)] - E[\varphi_t(S\eta)]\bigr| &\le \sqrt{t}\|\varphi'\|_\infty \biggl\{\sqrt{\frac{2}{\pi}} + E\bigl[|S\eta|\bigr]\biggr\} \\
&= \sqrt{\frac{2}{\pi}}\sqrt{t}\|\varphi'\|_\infty \{1 + E[S]\}.
\end{aligned}
$$

Using (3.1) with (3.5)–(3.6) together with the triangle inequality and the previous inequalities, we have

$$
\begin{aligned}
&\bigl|E[\varphi(F)] - E[\varphi(S\eta)]\bigr| \\
&\quad\le \sqrt{t}\|\varphi'\|_\infty \biggl(\sqrt{\frac{2}{\pi}}\{2 + E[S]\} + E\bigl[|F|\bigr]\biggr) \qquad (3.7)\\
&\qquad + \frac{\|\varphi'\|_\infty}{t}\biggl\{\frac{1}{\sqrt{2\pi}} E\bigl[|\langle u, DF\rangle_{\mathfrak{H}} - S^2|\bigr] + \frac{\sqrt{2}}{3} E\bigl[|\langle u, DS^2\rangle_{\mathfrak{H}}|\bigr]\biggr\}.
\end{aligned}
$$

Set

$$
\Phi_1 = \sqrt{\frac{2}{\pi}}\{2 + E[S]\} + E\bigl[|F|\bigr]
$$

and

$$
\Phi_2 = \frac{1}{\sqrt{2\pi}} E\bigl[|\langle u, DF\rangle_{\mathfrak{H}} - S^2|\bigr] + \frac{\sqrt{2}}{3} E\bigl[|\langle u, DS^2\rangle_{\mathfrak{H}}|\bigr].
$$

The function $t \mapsto \sqrt{t}\Phi_1 + \frac{1}{t}\Phi_2$ attains its minimum at $t_0 = (\frac{2\Phi_2}{\Phi_1})^{2/3}$. Then, if $t_0 \le 1$ we choose $t = t_0$ and if $t_0 > 1$ we choose $t = 1$. With these choices we obtain

$$
\begin{aligned}
&\bigl|E[\varphi(F)] - E[\varphi(S\eta)]\bigr| \qquad (3.8)\\
&\quad\le \|\varphi'\|_\infty \Phi_2^{1/3} \max\bigl((2^{-2/3} + 2^{1/3})\Phi_1^{2/3}, 3\Phi_2^{2/3}\bigr) \le \|\varphi'\|_\infty \Delta.
\end{aligned}
$$

This inequality can be extended to all Lispchitz functions φ, and this immediately yields that $d_{\mathrm{W}}(F, S\eta) \le \Delta$.

Step 2: *Kolmogorov distance*. Fix $z \in \mathbb{R}$ and $h > 0$. Consider the function $\varphi_h : \mathbb{R} \to [0, 1]$ defined by

$$\varphi_h(x) = \begin{cases} 1, & \text{if } x \leq z, \\ 0, & \text{if } x \geq z+h, \\ \text{linear}, & \text{if } z \leq x \leq z+h, \end{cases}$$

and observe that φ_h is Lipschitz with $\|\varphi_h'\|_\infty = 1/h$. Using that $\mathbf{1}_{(-\infty,z]} \leq \varphi_h \leq \mathbf{1}_{(-\infty,z+h]}$ as well as (3.8), we get

$$\begin{aligned} P[F \leq z] &- P[S\eta \leq z] \\ &\leq E\big[\varphi_h(F)\big] - E\big[\mathbf{1}_{(-\infty,z]}(S\eta)\big] \\ &= E\big[\varphi_h(F)\big] - E\big[\varphi_h(S\eta)\big] + E\big[\varphi_h(S\eta)\big] - E\big[\mathbf{1}_{(-\infty,z]}(S\eta)\big] \\ &\leq \frac{\Delta}{h} + P[z \leq S\eta \leq z+h]. \end{aligned}$$

On the other hand, we can write

$$\begin{aligned} P[z &\leq S\eta \leq z+h] \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{-x^2/2} \mathbf{1}_{[z,z+h]}(sx)\, dP_S(s)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \bigg(\int_{\mathbb{R}_+} dP_S(s) \int_{z/s}^{(z+h)/s} e^{-x^2/2}\, dx + \int_{\mathbb{R}_-} dP_S(s) \int_{(z+h)/s}^{z/s} e^{-x^2/2}\, dx \bigg) \\ &\leq \frac{|h|^\alpha}{\sqrt{2\pi}} \int_{\mathbb{R}} |s|^{-\alpha}\, dP_S(s) \bigg(\int_{\mathbb{R}} e^{-x^2/(2(1-\alpha))}\, dx \bigg)^{1-\alpha} \\ &\leq |h|^\alpha E\big[|S|^{-\alpha}\big], \end{aligned}$$

because $(\int_{\mathbb{R}} e^{-x^2/(2(1-\alpha))}\, dx)^{1-\alpha} = (\sqrt{1-\alpha} \int_{\mathbb{R}} e^{-y^2/2}\, dy)^{1-\alpha} \leq \sqrt{2\pi}$, so that

$$P[F \leq z] - P[S\eta \leq z] \leq \frac{\Delta}{h} + |h|^\alpha E\big[|S|^{-\alpha}\big].$$

Hence, by choosing $h = \Delta^{1/(\alpha+1)}$, we get that

$$P[F \leq z] - P[S\eta \leq z] \leq \Delta^{\alpha/(\alpha+1)}\big(1 + E\big[|S|^{-\alpha}\big]\big).$$

We prove similarly that

$$P[F \leq z] - P[S\eta \leq z] \geq -\Delta^{\alpha/(\alpha+1)}\big(1 + E\big[|S|^{-\alpha}\big]\big),$$

so the proof of (3.3) is done. □

3.3. *Quadratic functionals of Brownian motion and fractional Brownian motion.* We will now apply the results of the previous sections to some nonlinear functionals of a fractional Brownian motion with Hurst parameter $H \geq \frac{1}{2}$. Recall that a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B = \{B_t : t \geq 0\}$ with covariance function

$$E(B_s B_t) = \tfrac{1}{2}\big(t^{2H} + s^{2H} - |t - s|^{2H}\big).$$

Notice that for $H = \frac{1}{2}$ the process B is a standard Brownian motion. We denote by $\mathcal{E}$ the set of step functions on $[0, \infty)$. Let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathfrak{H}} = E(B_s B_t).$$

The mapping $\mathbf{1}_{[0,t]} \to B_t$ can be extended to a linear isometry between the Hilbert space $\mathfrak{H}$ and the Gaussian space spanned by B. We denote this isometry by $\phi \to B(\phi)$. In this way, $\{B(\phi) : \phi \in \mathfrak{H}\}$ is an isonormal Gaussian process. In the case, $H > \frac{1}{2}$, the space $\mathfrak{H}$ contains all measurable functions $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\int_0^\infty \int_0^\infty |\varphi(s)||\varphi(t)||t - s|^{2H-2}\, ds\, dt < \infty,$$

and in this case if φ and ϕ are functions satisfying this integrability condition,

$$\langle \varphi, \phi\rangle_{\mathfrak{H}} = H(2H - 1) \int_0^\infty \int_0^\infty \varphi(s)\phi(t)|t - s|^{2H-2}\, ds\, dt. \tag{3.9}$$

Furthermore, $L^{1/H}([0, \infty))$ is continuously embedded into $\mathfrak{H}$. In what follows, we shall write

$$c_H = \sqrt{H(2H - 1)\Gamma(2H - 1)}, \qquad H > 1/2, \tag{3.10}$$

and also $c_{1/2} := \lim_{H \downarrow 1/2} c_H = \frac{1}{\sqrt{2}}$.

The following statement contains explicit estimates in total variation for sequences of quadratic Brownian functionals converging to a mixture of Gaussian distributions. It represents a significant refinement of [29], Proposition 2.1 and [27], Proposition 18.

THEOREM 3.6. *Let $\{B_t : t \geq 0\}$ be a fBm of Hurst index $H \geq \frac{1}{2}$. For every $n \geq 1$, define*

$$A_n := \frac{n^{1+H}}{2} \int_0^1 t^{n-1}\big(B_1^2 - B_t^2\big)\, dt.$$

As $n \to \infty$, the sequence A_n converges stably to $S\eta$, where η is a random variable independent of B with law $\mathcal{N}(0, 1)$ and $S = c_H|B_1|$. Moreover, there exists a constant k (independent of n) such that

$$d_{\mathrm{TV}}(A_n, S\eta) \leq k n^{-(1-H)/15}, \qquad n \geq 1.$$

The proof of Theorem 3.6 is based on the forthcoming Proposition 3.7 and Proposition 3.8, dealing with the stable convergence of some auxiliary stochastic integrals, respectively in the cases $H = 1/2$ and $H > 1/2$. Notice that, since $\lim_{H\downarrow 1/2} c_H = c_{1/2} = \frac{1}{\sqrt{2}}$, the statement of Proposition 3.7 can be regarded as the limit of the statement of Proposition 3.8, as $H \downarrow \frac{1}{2}$.

PROPOSITION 3.7. *Let $B = \{B_t : t \geq 0\}$ be a standard Brownian motion. Consider the sequence of Itô integrals*

$$F_n = \sqrt{n}\int_0^1 t^n B_t \, dB_t, \qquad n \geq 1.$$

Then the sequence F_n converges stably to $S\eta$ as $n \to \infty$, where η is a random variable independent of B with law $\mathcal{N}(0,1)$ and $S = \frac{|B_1|}{\sqrt{2}}$. Furthermore, we have the following bounds for the Wasserstein and Kolmogorov distances

$$d_{\mathrm{Kol}}(F_n, S\eta) \leq C_\gamma n^{-\gamma},$$

for any $\gamma < \frac{1}{12}$, where C_γ is a constant depending on γ, and

$$d_{\mathrm{W}}(F_n, S\eta) \leq C n^{-1/6},$$

where C is a finite constant independent of n.

PROOF. Taking into account that the Skorohod integral coincides with the Itô integral, we can write $F_n = \delta(u_n)$, where $u_n(t) = \sqrt{n} t^n B_t \mathbf{1}_{[0,1]}(t)$. In order to apply Theorem 3.4, we need to estimate the quantities $E(|\langle u_n, DF_n\rangle_{\mathfrak{H}} - S^2|)$ and $E(|\langle u_n, DS^2\rangle_{\mathfrak{H}}|)$. We recall that $\mathfrak{H} = L^2(\mathbb{R}_+, ds)$. For $s \in [0,1]$, we can write

$$D_s F_n = \sqrt{n} s^n B_s + \sqrt{n}\int_s^1 t^n \, dB_t.$$

As a consequence,

$$\langle u_n, DF_n\rangle_{\mathfrak{H}} = n\int_0^1 s^{2n} B_s^2 \, ds + n\int_0^1 s^n B_s \left(\int_s^1 t^n \, dB_t\right) ds.$$

From the estimates,

$$\begin{aligned} E\left(\left|n\int_0^1 s^{2n} B_s^2 \, ds - \frac{B_1^2}{2}\right|\right) &\leq n\int_0^1 s^{2n} E\left(|B_s^2 - B_1^2|\right) ds + \left|\frac{n}{2n+1} - \frac{1}{2}\right| \\ &\leq 2n\int_0^1 s^{2n}\sqrt{1-s}\, ds + \frac{1}{2(2n+1)} \\ &\leq \frac{2n}{\sqrt{2n+1}}\sqrt{\int_0^1 s^{2n}(1-s)\, ds} + \frac{1}{2(2n+1)} \\ &\leq \frac{1}{\sqrt{2n}} + \frac{1}{4n} \end{aligned}$$

and

$$nE\left(\left|\int_0^1 s^n B_s\left(\int_s^1 t^n\,dB_t\right)ds\right|\right) \le \frac{n}{\sqrt{2n+1}}\int_0^1 s^{n+1/2}\sqrt{1-s^{2n+1}}\,ds$$
$$\le \frac{n}{(n+3/2)\sqrt{2n+1}} \le \frac{1}{\sqrt{2n}},$$

we obtain

$$E\left(\left|\langle u_n, DF_n\rangle_{\mathfrak{H}} - S^2\right|\right) \le \frac{\sqrt{2}}{\sqrt{n}} + \frac{1}{4n}. \tag{3.11}$$

On the other hand,

$$\left|\langle u_n, DS^2\rangle_{\mathfrak{H}}\right| = \sqrt{n}E\left(\left|B_1\int_0^1 s^n B_s\,ds\right|\right) \le \frac{\sqrt{n}}{n+3/2} \le \frac{1}{\sqrt{n}}. \tag{3.12}$$

Notice that

$$E(|F_n|) \le \frac{\sqrt{n}}{\sqrt{2n+2}} \le \frac{1}{\sqrt{2}}. \tag{3.13}$$

Therefore, using (3.11), (3.12) and (3.13) and with the notation of Theorem 3.4, for any constant $C < C_0$, where

$$C_0 = 3\left(\frac{1}{\sqrt{2\pi}}\left(\sqrt{2}+\frac{1}{4}\right)+\frac{\sqrt{2}}{3}\right)^{1/3}\left(\sqrt{\frac{2}{\pi}}\left(2+\frac{1}{\sqrt{\pi}}+\frac{1}{\sqrt{2}}\right)\right)^{2/3},$$

there exists n_0 such that for all $n \ge n_0$ we have $\Delta \le Cn^{-1/6}$. Therefore, $d_{\mathrm{W}}(F_n, S\eta) \le Cn^{-1/6}$ for $n \ge n_0$. Moreover, $E[|S|^{-\alpha}] < \infty$ for any $\alpha < 1$, which implies that

$$d_{\mathrm{Kol}}(F_n, S\eta) \le C_\gamma n^{-\gamma},$$

for any $\gamma < \frac{1}{12}$. This completes the proof of the proposition. □

As announced, the next result is an extension of Proposition 3.7 to the case of the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

PROPOSITION 3.8. *Let $B = \{B_t : t \ge 0\}$ be fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Consider the sequence of random variables $F_n = \delta(u_n)$, $n \ge 1$, where*

$$u_n(t) = n^H t^n B_t \mathbf{1}_{[0,1]}(t).$$

Then, the sequence F_n converges stably to $S\eta$ as $n \to \infty$, where η is a random variable independent of B with law $\mathcal{N}(0,1)$ and $S = c_H|B_1|$. Furthermore, we have the following bounds for the Wasserstein and Kolmogorov distances

$$d_{\mathrm{Kol}}(F_n, S\eta) \le C_{\gamma,H} n^{-\gamma},$$

for any $\gamma < \frac{1-H}{6}$, *where* $C_{\gamma,H}$ *is a constant depending on* γ *and* H, *and*

$$d_{\mathrm{W}}(F_n, S\eta) \le C_H n^{-(1-H)/3},$$

where C_H *is a constant depending on* H.

PROOF. Let us compute

$$D_s F_n = n^H s^n B_s + n^H \int_s^1 t^n \, dB_t.$$

As a consequence,

$$\langle u_n, DF_n \rangle_{\mathfrak{H}} = \|u_n\|_{\mathfrak{H}}^2 + n^H \left\langle u_n, \int_\cdot^1 t^n \, dB_t \right\rangle_{\mathfrak{H}}.$$

As in the proof of Proposition 3.7, we need to estimate the following quantities:

$$\varepsilon_n = E\big(\big|\|u_n\|_{\mathfrak{H}}^2 - S^2\big|\big)$$

and

$$\delta_n = E\left(\left| n^H \left\langle u_n, \int_\cdot^1 t^n \, dB_t \right\rangle_{\mathfrak{H}} \right|\right).$$

We have, using (3.9),

$$\begin{aligned}
\varepsilon_n &\le H(2H-1) E\left(\left| 2n^{2H} \int_0^1 \int_0^t s^n t^n B_s B_t (t-s)^{2H-2} \, ds \, dt - \Gamma(2H-1) B_1^2 \right| \right) \\
&\le H(2H-1) n^{2H} E\left(\left| 2 \int_0^1 \int_0^t s^n t^n [B_s B_t - B_1^2] (t-s)^{2H-2} \, ds \, dt \right| \right) \\
&\quad + H(2H-1) \left| 2n^{2H} \int_0^1 \int_0^t s^n t^n (t-s)^{2H-2} \, ds \, dt - \Gamma(2H-1) \right| \\
&= a_n + b_n.
\end{aligned}$$

We can write for any $s \le t$

$$\begin{aligned}
E\big(\big|B_s B_t - B_1^2\big|\big) &= E\big(\big|B_s B_t - B_s B_1 + B_s B_1 - B_1^2\big|\big) \\
&\le (1-t)^H + (1-s)^H \le 2(1-s)^H.
\end{aligned}$$

Using this estimate, we get

$$a_n \le 4H(2H-1) n^{2H} \int_0^1 \int_0^t s^n t^n (1-s)^H (t-s)^{2H-2} \, ds \, dt.$$

For any positive integers n, m set

$$\rho_{n,m} = \int_0^1 \int_0^t s^n t^m (t-s)^{2H-2} \, ds \, dt = \frac{\Gamma(n+1)\Gamma(2H-1)}{\Gamma(n+2H)(n+m+2H)}. \tag{3.14}$$

Then, by Hölder's inequality,

$$a_n \le 4H(2H-1)n^{2H}\rho_{n,n}^{1-H}\left(\int_0^1\int_0^t s^n t^n (1-s)(t-s)^{2H-2}\,ds\,dt\right)^H$$
$$= 4H(2H-1)n^{2H}\rho_{n,n}^{1-H}(\rho_{n,n}-\rho_{n+1,n})^H.$$

Taking into account that

$$\rho_{n,n}-\rho_{n+1,n} = \frac{\Gamma(n+1)(n(2H+1)+4H^2)}{\Gamma(n+2H)(2n+H)(n+2H)(2n+1+2H)},$$

and using Stirling's formula, we obtain that $\rho_{n,n}$ is less than or equal to a constant times n^{-2H} and $\rho_{n,n}-\rho_{n+1,n}$ is less than or equal to a constant times n^{-2H-1}. This implies that $a_n \le C_H n^{-H}$, for some constant C_H depending on H.

For the term b_n, using (3.14) we can write

$$b_n = H(2H-1)\Gamma(2H-1)\left|\frac{2n^{2H}\Gamma(n+1)}{\Gamma(n+2H)(2n+2H)} - 1\right|,$$

which converges to zero, by Stirling's formula, at the rate n^{-1}.

On the other hand,

$$\text{(3.15)}\qquad \begin{aligned}\delta_n &= H(2H-1)n^{2H}E\left(\left|\int_0^1\int_0^1 s^n B_s\left(\int_t^1 r^n\,dB_r\right)|t-s|^{2H-2}\,ds\,dt\right|\right)\\ &\le H(2H-1)n^{2H}\int_0^1\int_0^1 s^{n+H}\left[E\left(\left|\int_t^1 r^n\,dB_r\right|^2\right)\right]^{1/2}|t-s|^{2H-2}\,ds\,dt.\end{aligned}$$

We can write, using the fact that $L^{1/H}([0,\infty))$ is continuously embedded into $\mathfrak{H}$,

$$\text{(3.16)}\qquad E\left(\left|\int_t^1 r^n\,dB_r\right|^2\right) \le C_H\left(\int_t^1 r^{n/H}\,dr\right)^{2H} \le \frac{C_H}{(n/H+1)^{2H}}.$$

Substituting (6.13) into (6.14) we obtain $\delta_n \le C_H n^{H-1}$, for some constant C_H, depending on H. Thus,

$$E(|\langle u_n, DF_n\rangle_{\mathfrak{H}} - S^2|) \le C_H n^{H-1}.$$

Finally,

$$E(|\langle u_n, DS^2\rangle_{\mathfrak{H}}|) = n^H E\left(\left|\int_0^1\int_0^1 s^n B_s|t-s|^{2H-2}\,ds\,dt\right|\right)$$
$$\le n^H\left|\int_0^1\int_0^1 s^{n+H}|t-s|^{2H-2}\,ds\,dt\right| \le C_H n^{H-1}.$$

Notice that in this case $E(|\langle u_n, DF_n\rangle_{\mathfrak{H}} - S^2|)$ converges to zero faster than $E(|\langle u_n, DS^2\rangle_{\mathfrak{H}}|)$. As a consequence, $\Delta \le C_H n^{(H-1)/3}$, for some constant C_H and we conclude the proof using Theorem 3.4. □

PROOF OF THEOREM 3.6. Using Itô's formula (in its classical form for $H = \frac{1}{2}$, and in the form discussed, e.g., in [24], pages 293–294, for the case $H > \frac{1}{2}$) yields that

$$\tfrac{1}{2}(B_1^2 - B_t^2) = \delta(B.\mathbf{1}_{[t,1]}(\cdot)) + \tfrac{1}{2}(1 - t^{2H})$$

[note that $\delta(B.\mathbf{1}_{[t,1]}(\cdot))$ is a classical Itô integral in the case $H = \frac{1}{2}$]. Interchanging deterministic and stochastic integration by means of a stochastic Fubini theorem yields therefore that

$$A_n = F_n + H\frac{n^H}{2H+n}.$$

In view of Propositions 3.7 and 3.8, this implies that A_n converges in distribution to $S\eta$. The crucial point is now that each random variable A_n belongs to the direct sum $\mathcal{H}_0 \oplus \mathcal{H}_2$: it follows that one can exploit the estimate (3.4) in the case $p = 2$ to deduce that there exists a constant c such that

$$d_{\mathrm{TV}}(A_n, S\eta) \le c d_{\mathrm{W}}(A_n, S\eta)^{1/5} \le c\big(d_{\mathrm{W}}(F_n, S\eta) + d_{\mathrm{W}}(A_n, F_n)\big)^{1/5},$$

where we have applied the triangle inequality. Since (trivially) $d_{\mathrm{W}}(A_n, F_n) \le H\frac{n^H}{2H+n} < n^{H-1}$, we deduce the desired conclusion by applying the estimates in the Wasserstein distance stated in Propositions 3.7 and 3.8. □

4. Further notation and a technical lemma.

4.1. *A technical lemma.* The following technical lemma is needed in the subsequent sections.

LEMMA 4.1. *Let $\eta_1, \dots, \eta_d$ be a collection of i.i.d. $\mathcal{N}(0,1)$ random variables. Fix $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ and integers $k_1, \dots, k_d \ge 0$. Then, for every $f : \mathbb{R}^d \to \mathbb{R}$ of class $C^{(k,\dots,k)}$ (where $k = k_1 + \dots + k_d$) such that f and all its partial derivatives have polynomial growth,*

$$\begin{aligned} &E[f(\alpha_1\eta_1, \dots, \alpha_d\eta_d)\eta_1^{k_1}\cdots\eta_d^{k_d}] \\ &\quad = \sum_{j_1=0}^{\lfloor k_1/2\rfloor}\cdots\sum_{j_d=0}^{\lfloor k_d/2\rfloor}\prod_{l=1}^{d}\left\{\frac{k_l!}{2^{j_l}(k-2j_l)!j!}\alpha^{k_l-2j_l}\right\} \\ &\qquad\qquad \times E\left[\frac{\partial^{k_1+\dots+k_d-2(j_1+\dots+j_d)}}{\partial x_1^{k_1-2j_1}\cdots\partial x_d^{k_d-2j_d}}f(\alpha_1\eta_1,\dots,\alpha_d\eta_d)\right]. \end{aligned}$$

PROOF. By independence and conditioning, it suffices to prove the claim for $d = 1$, and in this case we write $\eta_1 = \eta$, $k_1 = k$, and so on. The decomposition of the random variable η^k in terms of Hermite polynomials is given by

$$\eta^k = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{2^j (k-2j)! j!} H_{k-2j}(\eta),$$

where $H_{k-2j}(x)$ is the $(k-2j)$th Hermite polynomial. Using the relation $E[f(\alpha\eta)H_{k-2j}(\eta)] = \alpha^{k-2j} E[f^{(k-2j)}(\alpha\eta)]$, we deduce the desired conclusion. □

4.2. *Notation*. The following notation is needed in order to state our next results. For the rest of this section, we fix integers $m \geq 0$ and $d \geq 1$.

(i) In what follows, we shall consider smooth functions

$$\psi : \mathbb{R}^{m \times d} \to \mathbb{R} : (y_1, \ldots, y_m; x_1, \ldots, x_d) \mapsto \psi(y_1, \ldots, y_m; x_1, \ldots, x_d). \tag{4.1}$$

Here, the implicit convention is that, if $m = 0$, then ψ does not depend on $(y_1, \ldots, y_m)$. We also write

$$\psi_{x_k} = \frac{\partial}{\partial x_k} \psi, \qquad k = 1, \ldots, d.$$

(ii) For every integer $q \geq 1$, we write $\mathscr{A}(q) = \mathscr{A}(q; m, d)$ (the dependence on m, d is dropped whenever there is no risk of confusion) to indicate the collection of all $(m + q(1+d))$-dimensional vectors with nonnegative integer entries of the type

$$\alpha^{(q)} = (k_1, \ldots, k_q; a_1, \ldots, a_m; b_{ij}, i = 1, \ldots, q, j = 1, \ldots, d), \tag{4.2}$$

verifying the set of Diophantine equations

$$\begin{aligned} k_1 + 2k_2 + \cdots + qk_q &= q, \\ a_1 + \cdots + a_m + b_{11} + \cdots + b_{1d} &= k_1, \\ b_{21} + \cdots + b_{2d} &= k_2, \\ &\cdots \\ b_{q1} + \cdots + b_{qd} &= k_q. \end{aligned} \tag{4.3}$$

(iii) Given $q \geq 1$ and $\alpha^{(q)}$ as in (4.2), we define

$$C(\alpha^{(q)}) := \frac{q!}{\prod_{i=1}^q i!^{k_i} \prod_{l=1}^m a_l! \prod_{i=1}^q \prod_{j=1}^d b_{ij}!}. \tag{4.4}$$

(iv) Given a smooth function ψ as in (4.1) and a vector $\alpha^{(q)} \in \mathscr{A}(q)$ as in (4.2), we set

$$\partial^{\alpha^{(q)}} \psi := \frac{\partial^{k_1 + \cdots + k_d}}{\partial y_1^{a_1} \cdots \partial y_m^{a_m} \partial x_1^{b_{11} + \cdots + b_{q1}} \cdots \partial x_d^{b_{1d} + \cdots + b_{qd}}} \psi. \tag{4.5}$$

The coefficients $C(\alpha^{(q)})$ and the differential operators $\partial^{\alpha^{(q)}}$, defined respectively in (4.4) and (4.5), enter the generalized Faa di Bruno formula (as proved, e.g., in [14]) that we will use in the proof of our main results.

(v) For every integer $q \geq 1$, the symbol $\mathscr{B}(q) = \mathscr{B}(q; m, d)$ indicates the class of all $(m + q(1+2d))$-dimensional vectors with nonnegative integer entries of the type

$$\beta^{(q)} = \big(k_1, \dots, k_q; a_1, \dots, a_m; b'_{ij}, b''_{ij}, i = 1, \dots, q, j = 1, \dots, d\big), \tag{4.6}$$

such that

$$\alpha\big(\beta^{(q)}\big) := \big(k_1, \dots, k_q; a_1, \dots, a_m; b'_{ij} + b''_{ij}, i = 1, \dots, q, j = 1, \dots, d\big), \tag{4.7}$$

is an element of $\mathscr{A}(q)$, as defined at point (ii). Given $\beta^{(q)}$ as in (4.6), we also adopt the notation

$$\begin{aligned} |b'| &:= \sum_{i=1}^{q} \sum_{j=1}^{d} b'_{ij}, \qquad |b''| := \sum_{i=1}^{q} \sum_{j=1}^{d} b''_{ij}, \\ |b''_{\bullet j}| &:= \sum_{i=1}^{q} b''_{ij}, \qquad j = 1, \dots, d. \end{aligned} \tag{4.8}$$

(vi) For every $\beta^{(q)} \in \mathscr{B}(q)$ as in (4.6) and every $(l_1, \dots, l_d)$ such that $l_s \in \{0, \dots, \lfloor |b''_{\bullet s}|/2 \rfloor\}$, $s = 1, \dots, d$, we set

$$\begin{aligned} &W\big(\beta^{(q)}; l_1, \dots, l_d\big) \\ &\quad := C\big(\alpha\big(\beta^{(q)}\big)\big) \prod_{i=1}^{q} \prod_{j=1}^{d} \binom{b'_{ij} + b''_{ij}}{b'_{ij}} \prod_{s=1}^{d} \frac{|b''_{\bullet s}|!}{2^{l_s} (|b''_{\bullet s}| - 2l_s)! l_s!}, \end{aligned} \tag{4.9}$$

where $C(\alpha(\beta^{(q)}))$ is defined in (4.4), and

$$\partial_{\star}^{(\beta^{(q)}; l_1, \dots, l_d)} := \partial^{\alpha(\beta^{(q)})} \frac{\partial^{|b''| - 2(l_1 + \cdots + l_d)}}{\partial x_1^{|b''_{\bullet 1}| - 2l_1} \cdots \partial x_d^{|b''_{\bullet d}| - 2l_d}}, \tag{4.10}$$

where $\alpha(\beta^{(q)})$ is given in (4.7), and $\partial^{\alpha(\beta^{(q)})}$ is defined according to (4.5).

(vii) The Beta function $B(u, v)$ is defined as

$$B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} \, dt, \qquad u, v > 0.$$

5. Bounds for general orders and dimensions.

5.1. *A general statement.* The following statement contains a general upper bound, yielding stable limit theorems and associated explicit rates of convergence on the Wiener space.

THEOREM 5.1. *Fix integers $m \geq 0$, $d \geq 1$ and $q_j \geq 1$, $j = 1, \dots, d$. Let $\eta = (\eta_1, \dots, \eta_d)$ be a vector of i.i.d. $\mathscr{N}(0,1)$ random variables independent of the isonormal Gaussian process X. Define $\hat{q} = \max_{j=1,\dots,d} q_j$. For every $j = 1, \dots, d$, consider a symmetric random element $u_j \in \mathbb{D}^{2\hat{q},4\hat{q}}(\mathfrak{H}^{2q_j})$, and introduce the following notation*:

- $F_j := \delta^{q_j}(u_j)$ *and* $F := (F_1, \dots, F_d)$;
- $(S_1, \dots, S_d)$ *is a vector of real-valued elements of* $\mathbb{D}^{\hat{q},4\hat{q}}$, *and*

$$S \cdot \eta := (S_1\eta_1, \dots, S_d\eta_d).$$

Assume that the function $\varphi : \mathbb{R}^{m\times d} \to \mathbb{R}$ admits continuous and bounded partial derivatives up to the order $2\hat{q} + 1$. Then, for every $h_1, \dots, h_m \in \mathfrak{H}$,

$$\big|E\big[\varphi\big(X(h_1), \dots, X(h_m); F\big)\big] - E\big[\varphi\big(X(h_1), \dots, X(h_m); S\cdot\eta\big)\big]\big|$$

$$\leq \frac{1}{2}\sum_{k,j=1}^{d}\left\|\frac{\partial^2}{\partial x_k\,\partial x_j}\varphi\right\|_\infty E\big[\big|\langle D^{q_k}F_j, u_k\rangle_{\mathfrak{H}^{\otimes q_k}} - \mathbf{1}_{j=k}S_j^2\big|\big] \tag{5.1}$$

$$\begin{aligned} &+ \frac{1}{2}\sum_{k=1}^{d}\sum_{\beta^{(q_k)}\in\mathscr{B}_0(q_k)}\sum_{l_1=0}^{\lfloor|b''_{\bullet 1}|/2\rfloor}\cdots\sum_{l_d=0}^{\lfloor|b''_{\bullet d}|/2\rfloor}\widehat{W}\big(\beta^{(q_k)}; l_1, \dots, l_d\big) \\ &\times \big\|\partial_\star^{(\beta^{(q_k)};l_1,\dots,l_d)}\varphi_{x_k}\big\|_\infty \\ &\times E\Bigg[\prod_{s=1}^{d} S^{|b''_{\bullet s}|-2l_s} \\ &\times \Bigg|\Bigg\langle u_k, h_1^{\otimes a_1}\otimes\cdots\otimes h_m^{\otimes a_m}\bigotimes_{i=1}^{q_k}\bigotimes_{j=1}^{d}\big\{\big(D^iF_j\big)^{\otimes b'_{ij}}\otimes\big(D^iS_j\big)^{\otimes b''_{ij}}\big\}\Bigg\rangle_{\mathfrak{H}^{\otimes q_k}}\Bigg|\Bigg], \end{aligned} \tag{5.2}$$

where we have adopted the same notation as in Section 4.2, with the following additional conventions: (a) *$\mathscr{B}_0(q)$ is the subset of $\mathscr{B}(q)$ composed of those $\beta(q_k)$ as in (4.6) such that $b'_{qj} = 0$ for $j = 1, \dots, d$,* (b) *$\widehat{W}(\beta^{(q_k)}; l_1, \dots, l_d) := W(\beta^{(q_k)}; l_1, \dots, l_d) \times B(|b'|/2 + 1/2; |b''|/2 + 1)$, where B is the Beta function.*

5.2. *Case $m=0$, $d=1$.* Specializing Theorem 5.1 to the choice of parameters $m=0$, $d=1$ and $q\geq 1$ yields the following estimate on the distance between the laws of a (multiple) Skorohod integral and of a mixture of Gaussian distributions.

PROPOSITION 5.2. *Suppose that $u\in\mathbb{D}^{2q,4q}(\mathfrak{H}^{2q})$ is symmetric. Let $F=\delta^q(u)$. Let $S\in\mathbb{D}^{q,4q}$, and let $\eta\sim\mathcal{N}(0,1)$ indicate a standard Gaussian random variable, independent of the underlying isonormal process X. Assume that $\varphi:\mathbb{R}\to\mathbb{R}$ is C^{2q+1} with $\|\varphi^{(k)}\|_\infty<\infty$ for any $k=0,\dots,2q+1$. Then*

$$\begin{aligned}
&\bigl|E\bigl[\varphi(F)\bigr]-E\bigl[\varphi(S\eta)\bigr]\bigr|\\
&\quad\leq\frac{1}{2}\|\varphi''\|_\infty E\bigl[\bigl|\langle u,D^qF\rangle_{\mathfrak{H}^{\otimes q}}-S^2\bigr|\bigr]\\
&\qquad+\sum_{(b',b'')\in\mathcal{Q},b'_q=0}\ \sum_{j=0}^{\lfloor|b''|/2\rfloor}c_{q,b',b'',j}\bigl\|\varphi^{(1+|b'|+2|b''|-2j)}\bigr\|_\infty\\
&\qquad\qquad\times E\bigl[S^{|b''|-2j}\\
&\qquad\qquad\qquad\times\bigl|\bigl\langle u,(DF)^{\otimes b'_1}\otimes\cdots\otimes\bigl(D^{q-1}F\bigr)^{\otimes b'_{q-1}}\\
&\qquad\qquad\qquad\qquad\otimes(DS)^{\otimes b''_1}\otimes\cdots\otimes\bigl(D^qS\bigr)^{\otimes b''_q}\bigr\rangle_{\mathfrak{H}^{\otimes q}}\bigr|\bigr],
\end{aligned}$$

where $\mathcal{Q}$ is the set of all pairs of q-ples $b'=(b'_1,b'_2,\dots,b'_q)$ and $b''=(b''_1,\dots,b''_q)$ of nonnegative integers satisfying the constraint $b'_1+2b'_2+\cdots+qb'_q+b''_1+2b''_2+\cdots+qb''_q=q$. The constants $c_{q,b',b'',j}$ are given by

$$\begin{aligned}
c_{q,b',b'',j}&=\frac{1}{2}B\bigl(|b'|/2+1/2,|b''|/2+1\bigr)\\
&\quad\times\prod_{i=1}^q\binom{b_i}{b'_i}\times\frac{|b''|!}{2^j(|b''|-2j)!j!}\times\frac{q!}{\prod_{i=1}^q i!^{b_i}b_i!},
\end{aligned}$$

where $b=b'+b''$.

In the particular case $q=2$ we obtain the following result.

PROPOSITION 5.3. *Suppose that $u\in\mathbb{D}^{4,8}(\mathfrak{H}^4)$ is symmetric. Let $F=\delta^2(u)$. Let $S\in\mathbb{D}^{2,8}$, and let $\eta\sim\mathcal{N}(0,1)$ indicate a standard Gaussian random variable, independent of the underlying isonormal process X. Assume that $\varphi:\mathbb{R}\to\mathbb{R}$ is C^5 with $\|\varphi^{(k)}\|_\infty<\infty$ for any $k=0,\dots,5$. Then*

$$\begin{aligned}
&\bigl|E\bigl[\varphi(F)\bigr]-E\bigl[\varphi(S\eta)\bigr]\bigr|\\
&\quad\leq\frac{1}{2}\|\varphi''\|_\infty E\bigl[\bigl|\langle u,D^2F\rangle_{\mathfrak{H}^{\otimes 2}}-S^2\bigr|\bigr]
\end{aligned}$$

$$+ C_0 \max_{3\le i\le 5} \|\varphi^{(i)}\|_\infty \bigl(E\bigl[|\langle u, (DF)^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}}|\bigr] + E\bigl[S|\langle u, DF\otimes DS\rangle_{\mathfrak{H}^{\otimes 2}}|\bigr] + E\bigl[(S^2+1)|\langle u, (DS)^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}}|\bigr] + E\bigl[S|\langle u, D^2S\rangle_{\mathfrak{H}^{\otimes 2}}|\bigr]\bigr),$$

where $C_0 = \frac{1}{2}B(\frac{1}{2},\frac{3}{2}) + \frac{3}{2}B(\frac{3}{2},1) + B(\frac{1}{2},2)$.

Taking into account that $DS^2 = 2SDS$ and $D^2S^2 = 2DS\otimes DS + 2SD^2S$, we can write the above estimate in terms of the derivatives of S^2, which is helpful in the applications. In this way, we obtain

$$\begin{aligned} &\bigl|E[\varphi(F)] - E[\varphi(S\eta)]\bigr| \\ &\quad\le \frac{1}{2}\|\varphi''\|_\infty E\bigl[|\langle u, D^2F\rangle_{\mathfrak{H}^{\otimes 2}} - S^2|\bigr] \\ &\qquad + C_0 \max_{3\le i\le 5}\|\varphi^{(i)}\|_\infty \bigl(E\bigl[|\langle u, (DF)^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}}|\bigr] + E\bigl[|\langle u, DF\otimes DS^2\rangle_{\mathfrak{H}^{\otimes 2}}|\bigr] \\ &\qquad + E\bigl[(S^{-2}+1)|\langle u, (DS^2)^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}}|\bigr] + E\bigl[|\langle u, D^2S^2\rangle_{\mathfrak{H}^{\otimes 2}}|\bigr]\bigr). \end{aligned} \tag{5.3}$$

Notice that a factor S^{-2} appears in the right-hand side of the above inequality.

5.3. *Case $m > 0$, $d = 1$.* Fix $q \ge 1$. In the case $m > 0$, $d = 1$, the class $\mathscr{B}(q)$ is the collection of all vectors with nonnegative integer entries of the type $\beta^{(q)} = (a_1,\dots,a_m; b'_1, b''_1,\dots,b'_q,b''_q)$ verifying

$$a_1 + \cdots + a_m + (b'_1 + b''_1) + \cdots + q(b'_q + b''_q) = q,$$

whereas $\mathscr{B}_0(q)$ is the subset of $\mathscr{B}(q)$ verifying $b'_q = 0$. Specializing Theorem 5.1 yields upper bounds for one-dimensional $\sigma(X)$-stable convergence.

PROPOSITION 5.4. *Suppose that $u \in \mathbb{D}^{2q,4q}(\mathfrak{H}^{2q})$ is symmetric, select $h_1,\dots,h_m \in \mathfrak{H}$, and write $\mathbf{X} = (X(h_1),\dots,X(h_m))$. Let $F = \delta^q(u)$. Let $S \in \mathbb{D}^{q,4q}$, and let $\eta \sim \mathscr{N}(0,1)$ indicate a standard Gaussian random variable, independent of the underlying Gaussian field X. Assume that*

$$\varphi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R} : (y_1,\dots,y_m,x) \mapsto \varphi(y_1,\dots,y_m,x)$$

admits continuous and bounded partial derivatives up to the order $2q+1$. Then

$$\begin{aligned} &\bigl|E[\varphi(\mathbf{X},F)] - E[\varphi(\mathbf{X},S\eta)]\bigr| \\ &\quad\le \frac{1}{2}\left\|\frac{\partial^2}{\partial x^2}\varphi\right\|_\infty E\bigl[|\langle u, D^qF\rangle_{\mathfrak{H}^{\otimes q}} - S^2|\bigr] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{\beta^{(q)}\in\mathscr{B}_0(q)}\sum_{j=0}^{\lfloor |b''|/2\rfloor}\widehat{W}(\beta^{(q)},j)\left\|\frac{\partial^{|a|}}{\partial y_1^{a_1}\cdots\partial y_m^{a_m}}\frac{\partial^{1+|b'|+2|b''|-2j}}{\partial x^{1+|b'|+2|b''|-2j}}\varphi\right\|_\infty\\
&\quad\times E\biggl[S^{|b''|-2j}\biggl|\biggl\langle u,h_1^{\otimes a_1}\otimes\cdots\\
&\qquad\qquad\otimes h_m^{\otimes a_m}\bigotimes_{i=1}^{q}\{(D^iF)^{\otimes b_i'}\otimes(D^iS)^{\otimes b_i''}\}\biggr\rangle_{\mathfrak{H}^{\otimes q}}\biggr|\biggr],
\end{aligned}
$$

where $|a|=a_1+\cdots+a_m$.

5.4. *Proof of Theorem* 5.1. The proof is based on the use of an interpolation argument. Write $\mathbf{X}=(X(h_1),\dots,X(h_m))$ and $g(t)=E[\varphi(\mathbf{X};\sqrt{t}F+\sqrt{1-t}S\cdot\eta)]$, $t\in[0,1]$, and observe that $E[\varphi(\mathbf{X};F)]-E[\varphi(\mathbf{X};S\eta)]=g(1)-g(0)=\int_0^1 g'(t)\,dt$. For $t\in(0,1)$, by integrating by parts with respect either to F or to η, we get

$$
\begin{aligned}
g'(t)&=\frac{1}{2}\sum_{k=1}^{d}E\biggl[\varphi_{x_k}(\mathbf{X};\sqrt{t}F+\sqrt{1-t}S\cdot\eta)\biggl(\frac{F_k}{\sqrt{t}}-\frac{S_k\eta_k}{\sqrt{1-t}}\biggr)\biggr]\\
&=\frac{1}{2}\sum_{k=1}^{d}E\biggl[\varphi_{x_k}(\mathbf{X};\sqrt{t}F+\sqrt{1-t}S\cdot\eta)\biggl(\frac{\delta^{q_k}(u_k)}{\sqrt{t}}-\frac{S_k\eta_k}{\sqrt{1-t}}\biggr)\biggr]\\
&=\frac{1}{2\sqrt{t}}\sum_{k=1}^{d}E[\langle D^{q_k}\varphi_{x_k}(\mathbf{X};\sqrt{t}F+\sqrt{1-t}S\cdot\eta),u_k\rangle_{\mathfrak{H}^{\otimes q_k}}]\\
&\quad-\frac{1}{2}\sum_{k=1}^{d}E\biggl[\frac{\partial^2}{\partial x_k^2}\varphi(\mathbf{X};\sqrt{t}F+\sqrt{1-t}S\cdot\eta)S_k^2\biggr].
\end{aligned}
$$

Using the Faa di Bruno formula for the iterated derivative of the composition of a function with a vector of functions (see [14], Theorem 2.1), we infer that, for every $k=1,\dots,d$,

$$
\begin{aligned}
&\langle D^{q_k}\varphi_{x_k}(\mathbf{X};\sqrt{t}F+\sqrt{1-t}S\cdot\eta),u_k\rangle_{\mathfrak{H}^{\otimes q_k}}\\
&\quad=\sum_{\alpha^{(q_k)}\in\mathscr{A}(q_k)}C(\alpha^{(q_k)})\partial^{(\alpha^{(q_k)})}\varphi_{x_k}(\mathbf{X};\sqrt{t}F+\sqrt{1-t}S\cdot\eta)\\
&\qquad\times\biggl\langle h_1^{\otimes a_1}\otimes\cdots\otimes h_m^{\otimes a_m}\bigotimes_{i=1}^{q_k}\bigotimes_{j=1}^{d}(D^i(\sqrt{t}F_j+\sqrt{1-t}S_j\eta_j))^{\otimes b_{ij}},u_k\biggr\rangle_{\mathfrak{H}^{\otimes q_k}}.
\end{aligned}
\tag{5.4}
$$

For every $i = 1, \dots, q_k$, every $j = 1, \dots, d$ and every symmetric $v \in \mathfrak{H}^{\otimes b_{ij}}$, we have

$$\begin{aligned}
&\langle (D^i(\sqrt{t}F_j + \sqrt{1-t}S_j\eta_j))^{\otimes b_{ij}}, v\rangle_{\mathfrak{H}^{\otimes b_{ij}}} \\
&\quad = \sum_{u=0}^{b_{ij}} \binom{b_{ij}}{u} t^{u/2}(1-t)^{(b_{ij}-u)/2}\eta^{(b_{ij}-u)} \\
&\qquad \times \langle (D^i F_j)^{\otimes u} \otimes (D^i S_j)^{\otimes(b_{ij}-u)}, v\rangle_{\mathfrak{H}^{\otimes b_{ij}}}.
\end{aligned} \tag{5.5}$$

Substituting (5.5) into (5.4), and taking into account the symmetry of u_k, yields

$$\begin{aligned}
&E[\langle D^{q_k}\varphi_{x_k}(\mathbf{X}; \sqrt{t}F + \sqrt{1-t}S\cdot\eta), u_k\rangle_{\mathfrak{H}^{\otimes q_k}}] \\
&\quad = \sum_{\beta^{(q_k)}\in\mathscr{B}(q_k)} C(\alpha^{(q_k)}) t^{|b'|/2}(1-t)^{|b''|/2} \prod_{i=1}^{q_k}\prod_{j=1}^{d} \binom{b'_{ij}+b''_{ij}}{b'_{ij}} \\
&\qquad \times E\Bigg[\partial^{\alpha(\beta^{(q_k)})}\varphi_{x_k}(\mathbf{X}; \sqrt{t}F + \sqrt{1-t}S\cdot\eta) \prod_{j=1}^{d} \eta_j^{|b''_{\bullet j}|} \\
&\qquad \times \Bigg\langle u_k, h_1^{\otimes a_1}\otimes\cdots\otimes h_m^{\otimes a_m} \bigotimes_{i=1}^{q_k}\bigotimes_{j=1}^{d}\{(D^i F_j)^{\otimes b'_{ij}} \otimes (D^i S_j)^{\otimes b''_{ij}}\}\Bigg\rangle_{\mathfrak{H}^{\otimes q_k}}\Bigg].
\end{aligned}$$

Notice that if $\beta^{(q_k)}$ does not belong to $\mathscr{B}_0(q_k)$, then $b'_{q_k l} \geq 1$ for some index $l = 1, \dots, d$. Taking into account the relations (4.3) this implies that $b'_{q_k l} = 1$, $b'_{q_k j} = 0$ for all $j \neq l$, $k_{q_k} = 1$ and all the other entries of $\beta^{(q_k)}$ must be equal to zero. In this way, the above sum can be decomposed as follows:

$$\begin{aligned}
&\sum_{\beta^{(q_k)}\in\mathscr{B}_0(q_k)} C(\alpha^{(q_k)}) t^{|b'|/2}(1-t)^{|b''|/2} \prod_{i=1}^{q_k}\prod_{j=1}^{d} \binom{b'_{ij}+b''_{ij}}{b'_{ij}} \\
&\quad \times E\Bigg[\partial^{\alpha(\beta^{(q_k)})}\varphi_{x_k}(\mathbf{X}; \sqrt{t}F + \sqrt{1-t}S\cdot\eta) \prod_{j=1}^{d} \eta_j^{|b''_{\bullet j}|} \\
&\quad \times \Bigg\langle u_k, h_1^{\otimes a_1}\otimes\cdots\otimes h_m^{\otimes a_m} \bigotimes_{i=1}^{q_k}\bigotimes_{j=1}^{d}\{(D^i F_j)^{\otimes b'_{ij}} \otimes (D^i S_j)^{\otimes b''_{ij}}\}\Bigg\rangle_{\mathfrak{H}^{\otimes q_k}}\Bigg] \\
&\quad + \sum_{l=1}^{d}\sqrt{t}E\bigg[\frac{\partial^2}{\partial x_k\,\partial x_l}\varphi(\mathbf{X}; \sqrt{t}F + \sqrt{1-t}S\cdot\eta)\langle D^{q_k}F_l, u_k\rangle_{\mathfrak{H}^{\otimes q_k}}\bigg] \\
&\quad := D(k,t) + F(k,t).
\end{aligned}$$

Since

$$\left|\frac{1}{2\sqrt{t}}\sum_{k=1}^{d} F(k,t) - \frac{1}{2}\sum_{k=1}^{d} E\left[\frac{\partial^2}{\partial x_k^2}\varphi(\mathbf{X}; \sqrt{t}F + \sqrt{1-t}S\cdot\eta)S_k^2\right]\right| \leq (5.1),$$

the theorem is proved once we show that

$$\sum_{k=1}^{d}\int_0^1 \frac{1}{2\sqrt{t}}\left|D(k,t)\right| dt$$

is less than the sum in (5.2). Using the independence of η and X, conditioning with respect to X and applying Lemma 4.1 yields

$$\begin{aligned}
&E\Bigg[\partial^{\alpha(\beta^{(q_k)})}\varphi_{x_k}(\mathbf{X}; \sqrt{t}F + \sqrt{1-t}S\cdot\eta)\prod_{j=1}^{d}\eta_j^{|b''_{\bullet j}|} \\
&\quad\times \left\langle u_k, h_1^{\otimes a_1}\otimes\cdots\otimes h_m^{\otimes a_m}\bigotimes_{i=1}^{q_k}\bigotimes_{j=1}^{d}\{(D^iF_j)^{\otimes b'_{ij}}\otimes (D^iS_j)^{\otimes b''_{ij}}\}\right\rangle_{\mathfrak{H}^{\otimes q_k}}\Bigg] \\
&= \sum_{l_1=0}^{\lfloor |b''_{\bullet 1}|/2\rfloor}\cdots\sum_{l_d=0}^{\lfloor |b''_{\bullet d}|/2\rfloor}\prod_{s=1}^{d}\frac{|b''_{\bullet s}|!}{2^{l_s}(|b''_{\bullet s}|-2l_s)!l_s!} \\
&\quad\times E\Bigg[\left\langle u_k, h_1^{\otimes a_1}\otimes\cdots\otimes h_m^{\otimes a_m}\bigotimes_{i=1}^{q_k}\bigotimes_{j=1}^{d}\{(D^iF_j)^{\otimes b'_{ij}}\otimes (D^iS_j)^{\otimes b''_{ij}}\}\right\rangle_{\mathfrak{H}^{\otimes q_k}} \\
&\qquad\times \prod_{s=1}^{d} S^{|b''_{\bullet s}|-2l_s}\partial_\star^{(\beta^{(q_k)};l_1,\ldots,l_d)}\varphi_{x_k}(\mathbf{X}; \sqrt{t}F + \sqrt{1-t}S\cdot\eta)\Bigg].
\end{aligned}$$

Then, estimating the term $\partial_\star^{(\beta^{(q_k)};l_1,\ldots,l_d)}\varphi_{x_k}(\mathbf{X}; \sqrt{t}F + \sqrt{1-t}S\cdot\eta)$ by $\|\partial_\star^{(\beta^{(q_k)};l_1,\ldots,l_d)}\varphi_{x_k}\|_\infty$, which does not depend on t, and using the equation

$$\int_0^1 \frac{1}{\sqrt{t}}t^{|b'|/2}(1-t)^{|b''|/2}\,dt = B\big(|b'|/2+1/2, |b''|/2+1\big),$$

we obtain the desired estimate. □

6. Application to weighted quadratic variations. In this section, we apply the previous results to the case of weighted quadratic variations of fractional Brownian motion. Let us introduce first some notation.

Given a measurable function $f:\mathbb{R}\to\mathbb{R}$, an integer $N\geq 0$ and a real number $p\geq 1$ we define the seminorm

$$\|f\|_{N,p} = \sum_{i=0}^{N}\sup_{0\leq t\leq 1}\|f^{(i)}\|_{L^p(\mathbb{R},\gamma_t)}, \tag{6.1}$$

where γ_t is the normal distribution $N(0,t)$.

We say that a function $f:\mathbb{R}\to\mathbb{R}$ has *moderate growth* if there exist positive constants A, B and $\alpha<2$ such that for all $x\in\mathbb{R}$, $|f(x)|\le A\exp(B|x|^{\alpha})$. Notice that the seminorm (6.1) is finite if f and all its derivatives up to the order N have moderate growth.

Consider a fractional Brownian motion $B=\{B_t:t\in[0,1]\}$ with Hurst parameter $H\in(0,1)$. That is, B is a zero mean Gaussian process with covariance $E(B_tB_s)=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$. The process B can be extended to an isonormal Gaussian process indexed by the Hilbert space $\mathfrak{H}$, which is the closure of the set of simple functions on $[0,1]$ with respect to the inner product $\langle \mathbf{1}_{[0,t]},\mathbf{1}_{[0,s]}\rangle_{\mathfrak{H}}=E(B_tB_s)$. We refer the reader to the basic references [16, 24] for a detailed account on this process. We denote by

$$\rho_H(k)=\tfrac{1}{2}\big(|k+1|^{2H}+|k-1|^{2H}-2|k|^{2H}\big),\qquad k\in\mathbb{Z},\tag{6.2}$$

the covariance function of the stationary sequence $\{B(k+1)-B(k):k\ge 0\}$.

We consider the uniform partition of the interval $[0,1]$, and for any $n\ge 1$ and $k=0,\dots,n-1$ we denote $\Delta B_{k/n}=B_{(k+1)/n}-B_{k/n}$, $\delta_{k/n}=\mathbf{1}_{[k/n,(k+1)/n]}$ and $\varepsilon_{k,n}=\mathbf{1}_{[0,k/n]}$. We will also make use of the notation $\beta_{j,k}=\langle\delta_{j/n},\delta_{k/n}\rangle_{\mathfrak{H}}$ and $\alpha_{j,t}=\langle\delta_{j/n},\mathbf{1}_{[0,t]}\rangle_{\mathfrak{H}}$, for any $t\in[0,1]$ and $j,k=0,\dots,n-1$.

Given a function $f:\mathbb{R}\to\mathbb{R}$, we define

$$u_n=n^{2H-1/2}\sum_{k=0}^{n-1}f(B_{k/n})\delta_{k/n}^{\otimes 2}.$$

We are interested in the asymptotic behavior of the weighted quadratic functionals

$$\begin{aligned}F_n&=n^{2H-1/2}\sum_{k=0}^{n-1}f(B_{k/n})\big[(\Delta B_{k/n})^2-n^{-2H}\big]\\&=n^{2H-1/2}\sum_{k=0}^{n-1}f(B_{k/n})I_2\big(\delta_{k/n}^{\otimes 2}\big).\end{aligned}\tag{6.3}$$

It is known (see, e.g., [15, 17, 18]) that for $\frac14<H<\frac34$, F_n converges in law to a mixture of Gaussian distributions. When the Hurst parameter H is not in this range, a different phenomenon occurs, as it was observed by Nourdin in [15]. More precisely, for $H<\frac14$, $n^{2H-1/2}F_n$ converges in $L^2(\Omega)$ to $\frac14\int_0^1 f''(B_s)\,ds$, whereas for $H>\frac34$, $n^{3/2-2H}F_n$ converges in $L^2(\Omega)$ to $\int_0^1 f(B_s)\,dZ_s$, where Z is the Rosenblatt process (see [15, 18]). In the critical case $H=\frac14$, there is convergence in law to a linear combination of the limits in the cases $H<\frac14$ and $\frac14<H<\frac34$, and in the critical case $H=\frac34$ there is convergence in law with an additional logarithmic factor (see [15, 18]).

In view of these results, we will focus on the case $\frac14<H<\frac34$, although our result could easily be extended to the limit case $H=\frac34$. Outside the interval $[\frac14,\frac34]$

the convergence is in $L^2(\Omega)$ and our methodology does not seem to be well suited to study the rate of convergence. Applying the general approach developed in previous sections, we are able to show the following rate of convergence in the asymptotic behavior of F_n, in the case $H \in (\frac{1}{4}, \frac{3}{4})$. This represents a quantitative version of the convergence in law proved in [18].

PROPOSITION 6.1. *Assume that the Hurst index H of B belongs to $(\frac{1}{4}, \frac{3}{4})$. Consider a function $f:\mathbb{R} \to \mathbb{R}$ of class C^4 such that f and its first 4 derivatives have moderate growth. Suppose in addition that $E[(\int_0^1 f^2(B_s)\,ds)^{-\alpha}] < \infty$ for some $\alpha > 1$. Consider the sequence of random variables F_n defined by (6.3). Set $S = \sqrt{\sigma_H \int_0^1 f^2(B_s)\,ds}$, with $\sigma_H^2 = \sum_{k=-\infty}^{\infty} \rho_H(k)^2$, where ρ_H is defined in (6.2). Then, for any function $\varphi:\mathbb{R} \to \mathbb{R}$ of class C^5 with $\|\varphi^{(k)}\|_\infty < \infty$ for any $k = 0, \dots, 5$ we have*

$$\big|E[\varphi(F_n)] - E[\varphi(S\eta)]\big| \le C_{f,H} \max_{1\le i\le 5} \|\varphi^{(i)}\|_\infty n^{-(|2H-1/2|\wedge|2H-3/2|)}, \tag{6.4}$$

where η is a standard normal variable independent of B. The constant $C_{f,H}$ has the form $C_{f,H} = C_H \max(1, \|f\|_{4,4}^4, (1 + |E[S^{-2\alpha}]|^{1/\alpha} \|f\|_{1,5\beta}^5))$, where C_H depends on H and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

PROOF. Along the proof C will denote a generic constant that might depend on H.

Notice first that the random variable F_n does not coincide with $\delta^2(u_n)$, except in the case $H = \frac{1}{2}$. For this reason, we define $G_n = \delta^2(u_n)$, and show the following estimate for the difference $F_n - G_n$:

$$E\big[|F_n - G_n|\big] \le C\|f\|_{3,2} n^{-(|2H-1/2|\wedge|2H-3/2|)}. \tag{6.5}$$

To show (6.5), we first apply Lemma 2.1 and we obtain

$$F_n - G_n = n^{2H-1/2} \sum_{k=0}^{n-1} 2\delta\big(f'(B_{k/n})\delta_{k/n}\big)\alpha_{k,k/n} + n^{2H-1/2} \sum_{k=0}^{n-1} f''(B_{k/n})\alpha_{k,k/n}^2.$$

Using the equality $\delta(f'(B_{k/n})\delta_{k/n}) = f'(B_{k/n})I_1(\delta_{k/n}) - f''(B_{k/n})\alpha_{k,k/n}$, yields

$$\begin{aligned} F_n - G_n &= 2n^{2H-1/2} \sum_{k=0}^{n-1} f'(B_{k/n})I_1(\delta_{k/n})\alpha_{k,k/n} - n^{2H-1/2} \sum_{k=0}^{n-1} f''(B_{k/n})\alpha_{k,k/n}^2 \\ &:= 2M_n - R_n. \end{aligned}$$

Point (a) of Lemma A.1 implies $|\alpha_{k,k/n}| \le n^{-(2H)\wedge 1}$ and we can write

$$E\big[|R_n|\big] \le \|f\|_{2,1} n^{1/2+2H-(4H\wedge 2)}. \tag{6.6}$$

On the other hand,

$$E\big[M_n^2\big] = n^{4H-1} \sum_{j,k=0}^{n-1} E\big[f'(B_{j/n}) f'(B_{k/n}) I_1(\delta_{j/n}) I_1(\delta_{k/n})\big] \alpha_{j,j/n} \alpha_{k,k/n},$$

and using the relation

$$I_1(\delta_{j/n}) I_1(\delta_{k/n}) = I_2(\delta_{j/n} \widetilde{\otimes} \delta_{k/n}) + \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}}$$

the duality relationship (2.5) yields

$$E\big[M_n^2\big] \le \|f\|_{3,2}^2 n^{4H-1} \sum_{j,k=0}^{n-1} \big[|\beta_{j,k}| + |\alpha_{j,j/n}\alpha_{k,k/n}| + |\alpha_{j,k/n}\alpha_{k,j/n}|\big] \times |\alpha_{j,j/n}\alpha_{k,k/n}|.$$

Finally, applying points (a) and (c) of Lemma A.1, we obtain,

$$E\big[M_n^2\big] \le C \|f\|_{3,2}^2 n^{4H-1} \big(n^{(1-2H)\vee 0} + n^{2-(4H\wedge 2)}\big) n^{-(4H\wedge 2)}. \tag{6.7}$$

If $H < \frac{1}{2}$, we obtain a rate of the form n^{1-4H} and if $H \ge \frac{1}{2}$ we obtain the bound n^{4H-3}. Then the estimates (6.6) and (6.7) imply (6.5).

Taking into account the estimate (6.5), the estimate (6.4) will follow from (5.3), provided we show the following inequalities for some constant C depending on H and for any $\beta > 1$:

$$E\big(\big|\langle u_n, D^2 G_n\rangle_{\mathfrak{H}^{\otimes 2}} - S^2\big|\big) \le C \|f\|_{4,2}^2 n^{-(|2H-1/2|\wedge|2H-3/2|)}, \tag{6.8}$$

$$E\big(\big|\langle u_n, D G_n^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}}\big|\big) \le C \|f\|_{3,3}^3 n^{-(|2H-1/2|\wedge|2H-3/2|)}, \tag{6.9}$$

$$\big\|\langle u_n, D(S^2)^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}}\big\|_{L^\beta(\Omega)} \le C \|f\|_{1,5\beta}^5 n^{-(|2H-1/2|\wedge|2H-3/2|)}, \tag{6.10}$$

$$E\big(\big|\langle u_n, D^2(S^2)\rangle_{\mathfrak{H}^{\otimes 2}}\big|\big) \le C \|f\|_{2,3}^3 n^{-(|2H-1/2|\wedge|2H-3/2|)}, \tag{6.11}$$

$$E\big(\big|\langle u_n, D G_n \otimes D(S^2)\rangle_{\mathfrak{H}^{\otimes 2}}\big|\big) \le C \|f\|_{3,4}^4 n^{-(|2H-1/2|\wedge|2H-3/2|)}. \tag{6.12}$$

The derivatives S^2 are given by the following expressions:

$$D(S^2) = 2\sigma_H \int_0^1 (ff')(B_s) \mathbf{1}_{[0,s]}\, ds,$$
$$D^2(S^2) = 2\sigma_H \int_0^1 (f'^2 + ff'')(B_s) \mathbf{1}_{[0,s]^2}\, ds.$$

On the other hand, applying formula (2.10) we obtain the following expressions for the derivatives of G_n

$$DG_n = \delta(u_n) + \delta^2(Du_n),$$
$$D^2 G_n = u_n + 2\delta(Du_n) + \delta^2(D^2 u_n).$$

We are now ready to prove (6.8)–(6.12). The proof will be based on the estimates obtained in Lemma A.2 of the Appendix. □

PROOF OF (6.8). We have

$$\begin{aligned}
&|\langle u_n, D^2 G_n\rangle_{\mathfrak{H}^{\otimes 2}} - S^2| \\
&\quad \le \big|\|u_n\|^2_{\mathfrak{H}^{\otimes 2}} - S^2\big| + 2|\langle u_n, \delta(Du_n)\rangle_{\mathfrak{H}^{\otimes 2}}| + |\langle u_n, \delta^2(D^2 u_n)\rangle_{\mathfrak{H}^{\otimes 2}}| \\
&\quad =: |A_n| + 2|B_n| + |C_n|.
\end{aligned}$$

To estimate $E[|A_n|]$, we write

$$\begin{aligned}
\|u_n\|^2_{\mathfrak{H}^{\otimes 2}} &= n^{4H-1}\sum_{j,k=0}^{n-1} f(B_{j/n}) f(B_{k/n})\beta^2_{j,k} \\
&= \frac{1}{n}\sum_{j,k=0}^{n-1} f(B_{j/n}) f(B_{k/n})\rho_H(k-j)^2 \\
&= \frac{1}{n}\sum_{p=-n+1}^{n-1}\sum_{j=0\vee -p}^{(n-1)\wedge(n-1-p)} f(B_{j/n}) f(B_{(j+p)/n})\rho_H(p)^2.
\end{aligned}$$

If we replace $f(B_{(j+p)/n})$ by $f(B_{j/n})$ we make an error in expectation of $(p/n)^H$, so this produces a total error of n^{-H}. On the other hand, the sequence $\sum_{|p|>n}\rho_H(p)^2$ converges to zero at the rate n^{4H-3}. As a consequence,

$$\begin{aligned}
E\big[|A_n|\big] &\le C\big(\|f\|^2_{1,2} n^{-H} + \|f\|^2_{0,2} n^{4H-3}\big) \\
&\quad + \sigma_H^2 E\left[\left|\frac{1}{n}\sum_{k=0}^{n-1} f^2(B_{k/n}) - \int_0^1 f^2(B_s)\,ds\right|\right].
\end{aligned}$$

It remains to estimate

$$\frac{1}{n}\sum_{k=0}^{n-1} f^2(B_{k/n}) - \int_0^1 f^2(B_s)\,ds = \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}\big[f^2(B_{k/n}) - f^2(B_s)\big]\,ds.$$

Using that $E[|f^2(B_{k/n}) - f^2(B_s)|] \le C\|f\|^2_{1,2} n^{-H}$ for $s\in[k/n,(k+1)/n]$, we obtain:

$$E\big[|A_n|\big] \le C\big(\|f\|^2_{1,2} n^{-H} + \|f\|^2_{0,2} n^{4H-3}\big). \tag{6.13}$$

For the term B_n we can write, using (A.2) and Meyer's inequalities:

$$\begin{aligned}
E\big[|B_n|\big] &\le n^{2H-1/2}\sum_{k=0}^{n-1} E\big[\big|f(B_{k/n})\delta\big(D_{k/n}(u_n\otimes_1 \delta_{k/n})\big)\big|\big] \\
&\le C\|f\|^2_{2,2} n^{2H-(3H\wedge 3/2)}.
\end{aligned} \tag{6.14}$$

The term C_n is handled in the same way, by using Meyer's inequalities and point (d) of Lemma A.1:

$$\begin{aligned}
E\big[|C_n|\big] &\le n^{2H-1/2}\sum_{k=0}^{n-1} E\big[\big|f(B_{k/n})\delta^2\big(D^2_{k/n,k/n}u_n\big)\big|\big] \\
&\le Cn^{2H-1/2}\|f\|^2_{4,2}\sum_{j,k=0}^{n-1}\beta^2_{j,k} \le Cn^{1/2-2H}\|f\|^2_{4,2}.
\end{aligned} \tag{6.15}$$

Then (6.8) follows from (6.13), (6.14) and (6.15). □

PROOF OF (6.9). We have

$$\begin{aligned}
\langle u_n, DG_n^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}} &= \langle u_n, \delta(u_n)\otimes\delta(u_n)\rangle_{\mathfrak{H}^{\otimes 2}} + 2\langle u_n, \delta(u_n)\otimes\delta^2(Du_n)\rangle_{\mathfrak{H}^{\otimes 2}} \\
&\quad + \langle u_n, \delta^2(Du_n)\otimes\delta^2(Du_n)\rangle_{\mathfrak{H}^{\otimes 2}} \\
&=: A_n + 2B_n + C_n.
\end{aligned}$$

For the term A_n we have, applying Hölder's and Meyer's inequalities and the estimate (A.1),

$$\begin{aligned}
E\big[|A_n|\big] &\le n^{2H-1/2}\sum_{k=0}^{n-1} E\big[\big|f(B_{k/n})\big(\delta(u_n\otimes_1\delta_{k/n})\big)^2\big|\big] \\
&\le C\|f\|^3_{1,3}n^{2H-1/2-(2H\wedge 1)}.
\end{aligned}$$

Similarly, using Hölder's and Meyer's inequalities and the estimates (A.1) and (A.3) yields

$$\begin{aligned}
E\big[|B_n|\big] &\le n^{2H-1/2}\sum_{k=0}^{n-1} E\big[\big|f(B_{k/n})\delta(u_n\otimes_1\delta_{k/n})\delta^2(D_{k/n}u_n)\big|\big] \\
&\le C\|f\|^3_{3,3}n^{2H-(3H\wedge 3/2)}.
\end{aligned}$$

Finally, using again Hölder's and Meyer's inequalities and the estimate (A.1) yields

$$\begin{aligned}
E\big[|C_n|\big] &\le n^{2H-1/2}\sum_{k=0}^{n-1} E\big[\big|f(B_{k/n})\big(\delta^2(D_{k/n}u_n)\big)^2\big|\big] \\
&\le C\|f\|^3_{3,3}n^{2H-1/2-(2H\wedge 1)}.
\end{aligned}$$ □

PROOF OF (6.10). We have

$$\begin{aligned}
&\langle u_n, D(S^2)^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}} \\
&\quad = 16n^{2H-1/2}\sum_{k=0}^{n-1} f(B_{k/n})\int_0^1\int_0^1 (ff')(B_s)(ff')(B_t)\alpha_{k,t}\alpha_{k,s}\,ds\,dt.
\end{aligned}$$

Then, we can write, using points (a) and (b) of Lemma A.1,

$$
\begin{aligned}
E\big[|\langle u_n, D(S^2)^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}}|\big] &\le C\|f\|_{1,5\beta}^5 n^{2H-1/2} \sup_{s,t\in[0,1]} \sum_{k=0}^{n-1} |\alpha_{k,t}\alpha_{k,s}| \\
&\le C\|f\|_{1,5\beta}^5 n^{2H-1/2-(2H\wedge 1)},
\end{aligned}
$$

for any $\beta \ge 1$. □

PROOF OF (6.11). We have

$$
\langle u_n, D^2(S^2)\rangle_{\mathfrak{H}^{\otimes 2}} = 4n^{2H-1/2} \sum_{k=0}^{n-1} f(B_{k/n}) \int_0^1 (f'^2 + ff'')(B_s)\alpha_{k,t}^2\, ds.
$$

As a consequence, applying points (a) and (b) of Lemma A.1 yields

$$
\begin{aligned}
E\big[|\langle u_n, D^2(S^2)\rangle_{\mathfrak{H}^{\otimes 2}}|\big] &\le C\|f\|_{2,3}^3 n^{2H-1/2} \sup_{s\in[0,1]} \sum_{k=0}^{n-1} \alpha_{k,s}^2 \\
&\le C\|f\|_{2,3}^3 n^{2H-1/2-(2H\wedge 1)}.
\end{aligned}
$$

□

PROOF OF (6.12). We have

$$
\begin{aligned}
\langle u_n, DG_n \otimes D(S^2)\rangle_{\mathfrak{H}^{\otimes 2}} &= \langle u_n, \delta(u_n) \otimes D(S^2)\rangle_{\mathfrak{H}^{\otimes 2}} + \langle u_n, \delta^2(Du_n) \otimes D(S^2)\rangle_{\mathfrak{H}^{\otimes 2}} \\
&=: A_n + B_n.
\end{aligned}
$$

For the term A_n we can write, applying Hölder's and Meyer's inequalities and the estimate (A.1),

$$
\begin{aligned}
E\big[|A_n|\big] &\le n^{2H-1/2} \sum_{k=0}^{n-1} E\big[|f(B_{k/n})\delta(u_n \otimes_1 \delta_{k/n})D_{k/n}(S^2)|\big] \\
&\le C\|f\|_{1,4}^4 n^{2H-1/2-(2H\wedge 1)}.
\end{aligned}
$$

For the term A_n we can write, applying Hölder's and Meyer's inequalities and the estimate (A.1),

$$
\begin{aligned}
E\big[|A_n|\big] &\le n^{2H-1/2} \sum_{k=0}^{n-1} E\big[|f(B_{k/n})\delta^2(D_{k/n}u_n)D_{k/n}(S^2)|\big] \\
&\le C\|f\|_{3,4}^4 n^{2H+1/2-(4H\wedge 2)}.
\end{aligned}
$$

This completes the proof of Proposition 6.1. □

REMARK 6.2. Note that the exponent in the rate $\delta = -(|2H - \frac{1}{2}| \wedge |2H - \frac{3}{2}|)$ is minimum when $H = \frac{1}{2}$ with $\delta = -\frac{1}{2}$. On the other hand, it becomes worst when H goes away from $\frac{1}{2}$ either from below or from above, and it converges to zero as H tends to $\frac{1}{4}$ or $\frac{3}{4}$. This is natural in view of the limit results for the weighted quadratic variations obtained in [15, 18]. This phenomenon has not been observed in other asymptotic problems, such as the rate of convergence for Euler-type numerical approximations of stochastic differential equations, where the rate $-(2H-\frac{1}{2})$ improves when H increases from $\frac{1}{2}$ up to $\frac{3}{4}$ (see [10]).

REMARK 6.3. In the case $H=\frac{1}{2}$, the process B is a Brownian motion, and it has independent increments. As consequence $\beta_{j,k}=0$ for $j\neq k$. Moreover, $F_n=G_n$. Therefore, the estimate (6.4) can be replaced by

$$\bigl|E[\varphi(F_n)]-E[\varphi(S\eta)]\bigr|\le C_f\max_{2\le i\le 5}\|\varphi^{(i)}\|_\infty n^{-1/2},$$

where $S^2=2\int_0^1 f(B_s)^2\,ds$.

REMARK 6.4. The extension to weighted power variations of any order or to Euler numerical schemes for stochastic differential equations driven by a fractional Brownian motion seems more involved. In the case of Euler numerical schemes, the results that could be obtained applying the methodology developed in this paper would lead to a precise analysis of the rate of convergence of the error to a particular distribution, which is usually a mixture of Gaussian laws. That is, we would be able to establish how close is the error to a limit distribution in terms of a distance between probabilities defined by means of regular functions.

APPENDIX

In this section, we will show two technical lemmas that play a fundamental role in the analysis of the asymptotic quadratic variation of the fractional Brownian motion. The notation in both lemmas is taken from Section 6.

LEMMA A.1. *Let $0<H<1$ and $n\ge 1$. We have, for some constant C_H:*

(a) $|\alpha_{k,t}|\le n^{-(2H\wedge 1)}$ *for any* $t\in[0,1]$ *and* $k=0,\dots,n-1$.
(b) $\sup_{t\in[0,1]}\sum_{k=0}^{n-1}|\alpha_{k,t}|\le C_H$.
(c) $\sum_{k,j=0}^{n-1}|\beta_{j,k}|\le C_H n^{(1-2H)\vee 0}$.
(d) *If* $H<\frac{3}{4}$, *then* $\sum_{k,j=0}^{n-1}\beta_{j,k}^2\le C_H n^{1-4H}$.
(e) $\sum_{k,j=0}^{n-1}|\beta_{k,l}\beta_{j,l}|\le C_H n^{-(4H\wedge 2)}$ *for any* $l=0,\dots,n-1$.
(f) *If* $H<\frac{3}{4}$, *then* $\sum_{k,j=0}^{n-1}|\beta_{k,l},\beta_{j,l}\beta_{j,k}|\le C_H n^{-4H-(2H\wedge 1)}$ *for any* $l=0,\dots,n-1$.

PROOF. Parts (a), (c) and (d) are contained in Lemmas 5 and 6 of [18]. Part (b) has been proved in Lemma 5.1 of [17] in the case $H<\frac{1}{2}$ and the proof actually works for any $H \in (0, 1)$. Part (e) follows easily from

$$\sum_{k,j=0}^{n-1} |\beta_{k,l}\beta_{j,l}| = \frac{1}{4} n^{-4H} \sum_{k,j=0}^{n-1} |\rho_H(k-l)\rho_H(j-l)|,$$

and the fact that the series $\sum_{p\in\mathbb{Z}} |\rho_H(p)|$ is convergent if $0 < H \le \frac{1}{2}$ and it diverges at the rate n^{2H-1} if $H > \frac{1}{2}$. Finally, to prove (f) we write, using Young's inequality,

$$\begin{aligned}\sum_{k,j=0}^{n-1} |\beta_{k,l}\beta_{j,l}\beta_{j,k}| &= \frac{1}{8} n^{-6H} \sum_{k,j=0}^{n-1} |\rho_H(k-l)\rho_H(j-l)\rho_H(j-k)| \\ &\le \frac{1}{8} n^{-6H} \Big(\sum_{p\in\mathbb{Z}} \rho_H(p)^2\Big)\Big(\sum_{p=-n}^{n} |\rho_H(p)|\Big) \\ &\le C_H n^{-4H-(2H\wedge 1)},\end{aligned}$$

where we have exploited the fact that $\sum_{p\in\mathbb{Z}} \rho_H(p)^2$ is convergent (because $H < \frac{3}{4}$), together with the asymptotic behavior of the mapping $n \mapsto \sum_{p=-n}^{n} |\rho_H(p)|$. □

The next lemma provides some technical estimates.

LEMMA A.2. *For any integer $M \ge 0$ and any real number $p > 1$, there exists a constant C depending on M, p and the Hurst parameter H such that*:

$$\|u_n \otimes_1 \delta_{k/n}\|_{M,p} \le C\|f\|_{M,p} n^{-1/2-(H\wedge 1/2)}, \tag{A.1}$$

$$\|D_{k/n}(u_n \otimes_1 \delta_{k/n})\|_{M,p} \le C\|f\|_{M+1,p} n^{-1/2-(3H\wedge 3/2)}, \tag{A.2}$$

$$\|D_{k/n} u_n\|_{M,p} \le C\|f\|_{M+1,p} n^{-(2H\wedge 1)}, \tag{A.3}$$

where $D_{k/n}F$ means $\langle DF, \delta_{k/n}\rangle_{\mathfrak{H}}$, for a given random variable F.

PROOF. In order to show the first estimate, we can write, for any integer $0 \le m \le M$,

$$D^m(u_n \otimes_1 \delta_{k/n}) = n^{2H-1/2} \sum_{j=0}^{n-1} f^{(m)}(B_{j/n})\beta_{k,j}\delta_{j/n} \widetilde{\otimes} \varepsilon_{j/n}^{\otimes m}.$$

Then, using points (a), (e) and (f) of Lemma A.1 we obtain

$$\begin{aligned}&\big(E\big[\|D^m(u_n \otimes_1 \delta_{k/n})\|_{\mathfrak{H}^{\otimes(m+1)}}^p\big]\big)^{1/p} \\ &\quad \le C n^{2H-1/2} \|f\|_{m,p}\end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{j,j'=0}^{n-1}\left|\beta_{k,j}\beta_{k,j'}\langle\delta_{j/n}\widetilde{\otimes}\varepsilon_{j/n}^{\otimes m},\delta_{j'/n}\widetilde{\otimes}\varepsilon_{j'/n}^{\otimes m}\rangle_{\mathfrak{H}^{\otimes(m+1)}}\right|\right)^{1/2}\\
&\le C\|f\|_{m,p}n^{2H-1/2}\left(\sum_{j,j'=0}^{n-1}|\beta_{k,j}\beta_{k,j'}|\big(|\beta_{j,j'}|+|\alpha_{j,j'/n}\alpha_{j',j/n}|\big)\right)^{1/2}\\
&\le C\|f\|_{m,p}n^{2H-1/2}\big(n^{-2H-(H\wedge 1/2)}+n^{-(4H\wedge 2)}\big)\\
&\le C\|f\|_{m,p}n^{-1/2-(H\wedge 1/2)},
\end{aligned}
$$

which shows (A.1).

To show the second estimate, we can write, for any integer $0\le m\le M$,

$$
D^m D_{k/n}(u_n\otimes_1\delta_{k/n})=n^{2H-1/2}\sum_{j=0}^{n-1}f^{(m+1)}(B_{j/n})\beta_{k,j}\alpha_{k,j/n}\delta_{j/n}\widetilde{\otimes}\varepsilon_{j/n}^{\otimes m}.
$$

Then, using points (a), (e) and (f) of Lemma A.1 we obtain

$$
\begin{aligned}
&\big(E\big[\|D^m D_{k/n}(u_n\otimes_1\delta_{k/n})\|_{\mathfrak{H}^{\otimes(m+1)}}^p\big]\big)^{1/p}\\
&\quad\le Cn^{2H-1/2}\|f\|_{m+1,p}\\
&\qquad\times\left(\sum_{j,j'=0}^{n-1}\left|\beta_{k,j}\alpha_{k,j/n}\beta_{k,j'}\alpha_{k,j'/n}\langle\delta_{j/n}\widetilde{\otimes}\varepsilon_{j/n}^{\otimes m},\delta_{j'/n}\widetilde{\otimes}\varepsilon_{j'/n}^{\otimes m}\rangle_{\mathfrak{H}^{\otimes(m+1)}}\right|\right)^{1/2}\\
&\quad\le Cn^{2H-1/2-(2H\wedge 1)}\|f\|_{m+1,p}\\
&\qquad\times\left(\sum_{j,j'=0}^{n-1}|\beta_{k,j}\beta_{k,j'}|\big(|\beta_{j,j'}|+|\alpha_{j,j'/n}\alpha_{j',j/n}|\big)\right)^{1/2}\\
&\quad\le C\|f\|_{m+1,p}n^{2H-1/2-(2H\wedge 1)}\big(n^{-2H-(H\wedge 1/2)}+n^{-(4H\wedge 2)}\big)\\
&\quad\le C\|f\|_{m+1,p}n^{-1/2-(3H\wedge 3/2)},
\end{aligned}
$$

and (A.2) follows.

Finally, for the estimate (A.3) we can write

$$
D^m D_{k/n}u_n=n^{2H-1/2}\sum_{j=0}^{n-1}f^{(m+1)}(B_{j/n})\alpha_{k,j/n}\delta_{j/n}^{\otimes 2}\widetilde{\otimes}\varepsilon_{j/n}^{\otimes m},
$$

which implies, using points (a), (c) and (d) of Lemma A.1,

$$
\begin{aligned}
&\big(E\big[\|D^m D_{k/n}u_n\|_{\mathfrak{H}^{\otimes(m+2)}}^p\big]\big)^{1/p}\\
&\quad\le Cn^{2H-1/2}\|f\|_{m+1,p}
\end{aligned}
$$

$$\times \left(\sum_{j,j'=0}^{n-1} \left| \alpha_{k,j/n}\alpha_{k,j'/n} \langle \delta_{j/n}^{\otimes 2} \widetilde{\otimes} \varepsilon_{j/n}^{\otimes m}, \delta_{j'/n}^{\otimes 2} \widetilde{\otimes} \varepsilon_{j'/n}^{\otimes m} \rangle_{\mathfrak{H}^{\otimes(m+2)}} \right| \right)^{1/2}$$

$$\le C n^{2H-1/2-(2H\wedge 1)} \| f \|_{m+1,p}$$

$$\times \left(\sum_{j,j'=0}^{n-1} (\beta_{j,j'}^2 + |\beta_{j,j'}\alpha_{j,j'/n}\alpha_{j',j/n}| + \alpha_{j,j'/n}^2 \alpha_{j',j/n}^2) \right)^{1/2}$$

$$\le C \| f \|_{m+1,p} n^{2H-1/2-(2H\wedge 1)}$$

$$\times \left(n^{1/2-2H} + n^{[(1/2-H)\vee 0]-(2H\wedge 1)} + n^{-(4H\wedge 2)} \right).$$

This shows (A.3) and the proof of the lemma is complete. □

Acknowledgments. We are grateful to two anonymous referees for a thorough reading and a number of helpful suggestions.

REFERENCES

[1] BARNDORFF-NIELSEN, O. E., CORCUERA, J. M. and PODOLSKIJ, M. (2009). Power variation for Gaussian processes with stationary increments. *Stochastic Process. Appl.* **119** 1845–1865. MR2519347

[2] BOURGUIN, S. and PECCATI, G. (2014). Portmanteau inequalities on the Poisson space: Mixed regimes and multidimensional clustering. *Electron. J. Probab.* **19** no. 66, 1–42. MR3248195

[3] CHATTERJEE, S. (2010). Spin glasses and Stein's method. *Probab. Theory Related Fields* **148** 567–600. MR2678899

[4] CORCUERA, J. M., NUALART, D. and WOERNER, J. H. C. (2006). Power variation of some integral fractional processes. *Bernoulli* **12** 713–735. MR2248234

[5] DÖBLER, C. (2013). Stein's method of exchangeable pairs for absolutely continuous, univariate distributions with applications to the Polya urn model. Preprint.

[6] EDEN, R. and VIENS, F. (2013). General upper and lower tail estimates using Malliavin calculus and Stein's equations. In *Seminar on Stochastic Analysis, Random Fields and Applications VII* (R. C. Dalang, M. Dozzi and F. Russo, eds.). *Progress in Probability* **67**. Birkhäuser, Basel.

[7] EDEN, R. and VÍQUEZ, J. (2012). Nourdin–Peccati analysis on Wiener and Wiener–Poisson space for general distributions. *Stochastic Process. Appl.* 125 182–216. MR3274696

[8] HARNETT, D. and NUALART, D. (2012). Weak convergence of the Stratonovich integral with respect to a class of Gaussian processes. *Stochastic Process. Appl.* **122** 3460–3505. MR2956113

[9] HARNETT, D. and NUALART, D. (2013). Central limit theorem for a Stratonovich integral with Malliavin calculus. *Ann. Probab.* **41** 2820–2879. MR3112933

[10] HU, Y., LIU, Y. and NUALART, D. (2013). Modified Euler approximation scheme for stochastic differential equations driven by fractional Brownian motions. Preprint.

[11] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR0959133

[12] KUSUOKA, S. and TUDOR, C. A. (2012). Stein's method for invariant measures of diffusions via Malliavin calculus. *Stochastic Process. Appl.* **122** 1627–1651. MR2914766

[13] KUSUOKA, S. and TUDOR, C. A. (2013). Extension of the Fourth Moment Theorem to invariant measures of diffusions. Preprint.

[14] MISHKOV, R. L. (2000). Generalization of the formula of Faa di Bruno for a composite function with a vector argument. *Int. J. Math. Math. Sci.* **24** 481–491. MR1781515

[15] NOURDIN, I. (2008). Asymptotic behavior of weighted quadratic and cubic variations of fractional Brownian motion. *Ann. Probab.* **36** 2159–2175. MR2478679

[16] NOURDIN, I. (2012). *Selected Aspects of Fractional Brownian Motion. Bocconi & Springer Series* **4**. Springer, Milan. MR3076266

[17] NOURDIN, I. and NUALART, D. (2010). Central limit theorems for multiple Skorokhod integrals. *J. Theoret. Probab.* **23** 39–64. MR2591903

[18] NOURDIN, I., NUALART, D. and TUDOR, C. A. (2010). Central and non-central limit theorems for weighted power variations of fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 1055–1079. MR2744886

[19] NOURDIN, I. and PECCATI, G. (2008). Weighted power variations of iterated Brownian motion. *Electron. J. Probab.* **13** 1229–1256. MR2430706

[20] NOURDIN, I. and PECCATI, G. (2012). *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality. Cambridge Tracts in Mathematics* **192**. Cambridge Univ. Press, Cambridge. MR2962301

[21] NOURDIN, I. and POLY, G. (2013). Convergence in total variation on Wiener chaos. *Stochastic Process. Appl.* **123** 651–674. MR3003367

[22] NOURDIN, I. and RÉVEILLAC, A. (2009). Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: The critical case $H = 1/4$. *Ann. Probab.* **37** 2200–2230. MR2573556

[23] NOURDIN, I., RÉVEILLAC, A. and SWANSON, J. (2010). The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter $1/6$. *Electron. J. Probab.* **15** 2117–2162. MR2745728

[24] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications (New York)*. Springer, Berlin. MR2200233

[25] NUALART, D. and ORTIZ-LATORRE, S. (2008). Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Process. Appl.* **118** 614–628. MR2394845

[26] NUALART, D. and PECCATI, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** 177–193. MR2118863

[27] PECCATI, G. and TAQQU, M. S. (2008). Stable convergence of multiple Wiener–Itô integrals. *J. Theoret. Probab.* **21** 527–570. MR2425357

[28] PECCATI, G. and TUDOR, C. A. (2005). Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII. Lecture Notes in Math.* **1857** 247–262. Springer, Berlin. MR2126978

[29] PECCATI, G. and YOR, M. (2004). Four limit theorems for quadratic functionals of Brownian motion and Brownian bridge. In *Asymptotic Methods in Stochastics. Fields Inst. Commun.* **44** 75–87. Amer. Math. Soc., Providence, RI. MR2106849

[30] PECCATI, G. and YOR, M. (2004). Hardy's inequality in $L^2([0,1])$ and principal values of Brownian local times. In *Asymptotic Methods in Stochastics. Fields Inst. Commun.* **44** 49–74. Amer. Math. Soc., Providence, RI. MR2106848

[31] PODOLSKIJ, M. and VETTER, M. (2010). Understanding limit theorems for semimartingales: A short survey. *Stat. Neerl.* **64** 329–351. MR2683464

[32] REINERT, G. (2005). Three general approaches to Stein's method. In *An Introduction to Stein's Method. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* **4** 183–221. Singapore Univ. Press, Singapore. MR2235451

[33] RÉNYI, A. (1963). On stable sequences of events. *Sankhya, Ser. A* **25** 293–302. MR0170385

[34] RÉVEILLAC, A. (2009). Convergence of finite-dimensional laws of the weighted quadratic variations process for some fractional Brownian sheets. *Stoch. Anal. Appl.* **27** 51–73. MR2473140

[35] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357

I. NOURDIN
G. PECCATI
FSTC—UR EN MATHÉMATIQUES
UNIVERSITÉ DU LUXEMBOURG
6 RUE RICHARD COUDENHOVE-KALERGI
LUXEMBOURG CITY 1359
LUXEMBOURG
E-MAIL: ivan.nourdin@uni.lu
giovanni.peccati@gmail.com

D. NUALART
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
1460 JAYHAWK BOULEVARD
LAWRENCE, KANSAS 66045
USA
E-MAIL: nualart@math.ku.edu