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BRANCHING BROWNIAN MOTION IN A STRIP: SURVIVAL NEAR CRITICALITY

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We consider a branching Brownian motion with linear drift in which particles are killed on exiting the interval $(0, K)$ and study the evolution of the process on the event of survival as the width of the interval shrinks to the critical value at which survival is no longer possible. We combine spine techniques and a backbone decomposition to obtain exact asymptotics for the near-critical survival probability. This allows us to deduce the existence of a quasi-stationary limit result for the process conditioned on survival which reveals that the backbone thins down to a spine as we approach criticality.

This paper is motivated by recent work on survival of near critical branching Brownian motion with absorption at the origin by Aïdékon and Harris [Near-critical survival probability of branching Brownian motion with an absorbing barrier (2010) Unpublished manuscript] as well as the work of Berestycki et al. [*Ann. Probab.* **41** (2013) 527–618; *J. Stat. Phys.* **143** (2011) 833–854].

1. Introduction and main results.

1.1. *Introduction and main results.* We consider a branching diffusion in which each particle performs a Brownian motion with drift $-\mu$, for $\mu \geq 0$, and is killed on hitting 0 or K. All living particles undergo branching at constant rate β to be replaced by a random number of offspring particles, A, where A is an independent random variable with distribution $\{q_k; k = 0, 1, \dots\}$ and finite mean $m > 1$ and such that $E(A \log^+ A) < \infty$. Once born, offspring particles move off independently from their birth position, repeating the stochastic behaviour of their parent.

In other words, the motion of a single particle is governed by the infinitesimal generator

$$L = \frac{1}{2}\frac{d^2}{dx^2} - \mu\frac{d}{dx}, \qquad x \in (0, K), \tag{1.1}$$

defined for all functions $u \in C^2(0, K)$, the space of twice continuously differentiable functions on $(0, K)$, with $u(0+) = u(K-) = 0$. The branching activity is

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characterised by the branching mechanism

$$F(s) = \beta\big(G(s) - s\big), \qquad s \in [0, 1],$$

where $G(s) = \sum_{k=0}^{\infty} q_k s^k$ is the probability generating function of A.

Denote by N_t and $|N_t|$ the set of and the number of particles alive at time t, respectively. For a particle $u \in N_t$, we write $x_u(t)$ for its spatial position at time t. We define $X_t = \sum_{u \in N_t} \delta_{x_u(t)}$ to be the spatial configuration of particles alive at time t, and we set $X = (X_t, t \geq 0)$. Denote by P_ν^K the law of X with $X_0 = \nu$ where $\nu \in \mathcal{M}_a(0, K)$, the space of finite atomic measures on $(0, K)$ of the form $\sum_{i=1}^n \delta_{x_i}$ with $x_i \in (0, K)$ and $n \in \mathbb{N}$. If the process is initiated from a single particle at $x \in (0, K)$, then we simply write P_x^K (instead of $P_{\delta_x}^K$). We will sometimes neglect the dependence on the initial configuration and write P^K without a subscript. We call the process X a P^K-branching diffusion.

Further, $(\xi = (\xi_t, t \geq 0), \mathbb{P}_x^K)$ will henceforth denote a Brownian motion with drift $-\mu$ starting from $x \in (0, K)$ which is killed upon exiting the interval $(0, K)$. $\mathbb{P}^K$ is the law of the single particle motion under P^K.

For $x \in [0, K]$ we define the survival probability $p_K(x) = P_x^K(\zeta = \infty)$ where $\zeta = \inf\{t > 0 : |N_t| = 0\}$ is the time of extinction. As a first result we identify the critical width K_0 below at which survival is no longer possible.

PROPOSITION 1. *If* $\mu < \sqrt{2(m-1)\beta}$ *and* $K > K_0$ *where* $K_0 := \pi(\sqrt{2(m-1)\beta - \mu^2})^{-1}$, *then* $p_K(x) > 0$ *for all* $x \in (0, K)$; *otherwise* $p_K(x) = 0$ *for all* $x \in [0, K]$.

Proposition 1 is essentially not new as, in the case of binary branching, it is already implicit in Theorem 3 in Engländer and Kyprianou [16]; see also [15], Example 14. Nevertheless, we will give a short proof of Proposition 1 in Section 2 as the techniques therein will be important later. In particular, the proof uses a spine argument, decomposing X into a Brownian motion conditioned to stay in $(0, K)$ dressed with independent copies of (X, P^K) which "immigrate" along its path.

Our aim is to study the evolution of the P^K-branching diffusion on the event of survival. We will therefore develop a decomposition which identifies the particles with infinite genealogical lines of descent, that is, particles which produce a family of descendants which survives forever. To illustrate this, in a realisation of X, let us colour blue all particles with an infinite line of descent and colour red all remaining particles. Thus, on the event of survival, the resulting picture consists of a blue tree "dressed" with red trees whereas, on the event of extinction, we see a red tree only.

For the moment, let us consider the binary branching case only in which each particle splits into two. Suppose a particle dies and is replaced by two offspring at position y. For each of the offspring, the probability that it has an infinite genealogical line of descent is the survival probability $p_K(y)$, independent of the other offspring particle. Thus each offspring particle is blue with probability $p_K(y)$, and hence with probability $p_K(y)^2$ both offspring particles are blue.

Therefore, given the parent particle is blue, it branches into two blue particles at rate $\beta\frac{p_K(y)^2}{p_K(y)} = \beta p_K(y)$. Similarly, given the parent particle is red, it branches into two red particles at rate $\beta(1 - p_K(y))$.

Further, with probability $2p_K(y)(1 - p_K(y))$ one blue and one red particle are born. Then, given a particle is blue, it branches into one blue and one red particle at rate $2\beta(1 - p_K(y))$. We call such a branching event an *immigration*.

Intuitively speaking, as blue and red particles are so coloured as the result of statistically biased selection, we would expect their respective associated motions to be altered in a way that reflects this selection. The following two results, again in the setting of binary branching, make precise this heuristic. They show that a Doob h-transform of L using $h = p_K$ and $h = 1 - p_K$, for blue and red particles, respectively, describes the relevant motions *and* that the blue and red trees are, in fact, branching diffusions.

PROPOSITION 2 (The red tree). *Let $K > K_0$. In the case of binary branching, the red tree is a branching diffusion on $(0, K)$ with single particle motion described by the infinitesimal generator*

$$L^{R,K} = \frac{1}{2}\frac{d^2}{dy^2} - \left(\mu + \frac{p_K'(y)}{1 - p_K(y)}\right)\frac{d}{dy} \qquad \text{on } (0, K),$$

for $u \in C^2(0, K)$ with $u(0+) = u(K-) = 0$, and its particles branch at space-dependent rate $\beta(1 - p_K(y))$, $y \in (0, K)$.

THEOREM 3 (The dressed blue tree). *Let $K > K_0$. In the case of binary branching, the dressed blue tree is a branching diffusions on $(0, K)$ starting from an initial particle at $x \in (0, K)$, which evolves as follows*:

(i) *From x, we run a blue branching diffusion X^B, that is a branching diffusion with single particle movement has infinitesimal generator*

$$L^{B,K} = \frac{1}{2}\frac{d^2}{dy^2} - \left(\mu - \frac{p_K'(y)}{p_K(y)}\right)\frac{d}{dy} \qquad \text{on } (0, K),$$

defined for all $u \in C^2(0, K)$, and each particle branches at space-dependent rate $\beta p_K(y)$, $y \in (0, K)$.

(ii) *Conditionally on X^B, along the trajectory of each particle in X^B, an immigrant occurs at space-dependent rate $2\beta(1 - p_K(y))$, $y \in (0, K)$. Each immigrant initiates a red branching diffusion from the space–time position of its birth.*

In view of Theorem 3, we will sometimes refer to the blue branching diffusion X^B as the *backbone* and the theorem itself together with Proposition 2 as the *backbone decomposition*.

The corresponding results in the case of a general branching mechanism F are given as Proposition 11 and Theorem 12 in Section 3. In particular, we will see that a general branching mechanism induces a second type of immigration at the branching times of the backbone.

A significant convenience of these results is that the law of the P^K-branching diffusion conditioned on survival is the same as the law of the dressed blue branching diffusion. For example, instead of studying the quasi-stationary limit $\lim_{K\downarrow K_0} P_x^K(\cdot|\zeta=\infty)$ it suffices to study the evolution of the dressed blue branching diffusion as $K \downarrow K_0$.

To help understand the branching diffusion near criticality, we study the asymptotics of the survival probability p_K as $K \downarrow K_0$. As a first asymptotic result note that $u = 1 - p_K$ solves the differential equation $Lu + F(u) = 0$ on $(0, K)$ with boundary condition $u(0) = u(K) = 1$; cf. Remark 10. Near criticality we may assume that $p_K(x)$ is very small for a fixed x, and neglecting all terms of order $(p_K(x))^2$ and higher, we obtain the linearisation $Lp_K + (m-1)\beta p_K = 0$. This suggests $p_K(x) \sim C_K \sin(\pi x/K_0)e^{\mu x}$. In fact we have the following result.

THEOREM 4. *Define*

$$C_K := (K - K_0)\frac{(K_0^2\mu^2+\pi^2)(K_0^2\mu^2+9\pi^2)}{12(m-1)\beta\pi K_0^3(e^{\mu K_0}+1)}.$$

Then, as $K \downarrow K_0$, *we have* $C_K \downarrow 0$ *and*

$$p_K(x) \sim C_K \sin(\pi x/K_0)e^{\mu x}, \tag{1.2}$$

uniformly for all $x \in (0, K_0)$. *That is,* $p_K(x)/(C_K \sin(\pi x/K_0)e^{\mu x})$ *converges to* 1 *uniformly for all* $x \in (0, K_0)$, *as* $K \downarrow K_0$.

It is of particular note that we are able to determine C_K here. In Section 4.1 we will prove the first part of Theorem 4, that is equation (1.2) but without identifying C_K, in the fashion of [1] using spine techniques. In the sketch of the analytic argument above, we used that p_K asymptotically solves the linearisation $Lp_K + (m-1)\beta p_K = 0$. However, so does any multiple of p_K. Therefore, it is not possible to find the exact expression for C_K by studying this linearisation only. On the probabilistic side, using a spine approach is closely related to this linearised differential equation approach, and similarly does not assist in establishing an expression for C_K.

However, it turns out that the backbone decomposition in Theorem 3 captures enough information about the evolution of (X, P^K) on survival to derive the explicit expression for C_K. A heuristic argument and an outline of the proof using large deviation theory is given in Section 4.2.1, followed by a rigorous proof based on computations of the growth rate of the expected number of particles in the backbone in Section 4.2.2.

With Theorems 3 and 4 in hand we look for a quasi-stationary limit result for the law of the dressed blue branching diffusion, which agrees with the law of

(X, P^K) conditioned on survival, as we approach criticality. In the case of a binary branching mechanism as considered in Theorem 3, the blue branching rate βp_K drops down to 0 as $K \downarrow K_0$. At the same time the red branching rate $\beta(1 - p_K)$ increases to β and the rate of immigration $2\beta(1 - p_K)$ rises to 2β at criticality. It is therefore reasonable to believe that, over a fixed time interval $[0, T]$, the blue tree thins down to a single genealogical line of descent as $K \downarrow K_0$.

Let us formalise this idea by defining what we expect to be the limiting branching diffusion, now already for the case of a general branching mechanism, and giving the quasi-stationary limit thereafter.

DEFINITION 5. Let $x \in (0, K_0)$. Let $X^* = (X_t^*, t \geq 0)$ be a $\mathcal{M}_a(0, K_0)$-valued process which is constructed as follows.

X^* is initiated from a single particle at x performing a Brownian motion conditioned to stay in $(0, K_0)$, that is, a strong Markov process with infinitesimal generator

$$L^{K_0,*} = \frac{1}{2}\frac{d^2}{dy^2} + \frac{\pi/K_0}{\tan(\pi y/K_0)}\frac{d}{dy} \qquad \text{on } (0, K_0), \tag{1.3}$$

defined for all $u \in C^2(0, K_0)$. Along its path we immigrate $\tilde{A}$ independent copies of (X, P^{K_0}) at rate $m\beta$ where $\tilde{A}$ has the size-biased offspring distribution $(\tilde{q}_k, k = 0, 1, \ldots)$ with

$$\tilde{q}_k = q_{k+1}\frac{k+1}{m}, \qquad k \geq 0.$$

Denote the law of X^* by Q_x^*.

THEOREM 6. *Let $x \in (0, K_0)$. Then, for any fixed time $T > 0$, the law of $(X_t, 0 \leq t \leq T)$ under the measure $\lim_{K\downarrow K_0} P_x^K(\cdot|\zeta = \infty)$ is equal to $(X_t^*, 0 \leq t \leq T)$ under Q_x^*.*

To conclude this study, we demonstrate the robustness of our approach by applying the results for the P^K-branching diffusion to study the evolution of a supercritical super-Brownian motion with absorption at 0 and K near criticality. We outline a backbone decomposition analogous to Theorem 3 in which we will see that the backbone of the super-Brownian motion with absorption at 0 and K is the same as the backbone of an associated P^K-branching diffusion. This connection allows us to deduce asymptotic results for the survival rate of the super-Brownian motion with absorption on $(0, K)$ directly from the results on the survival probability of the associated P^K-branching diffusion. Further, we can find a quasi-stationary limit result for the super-Brownian motion equivalent to Theorem 6.

Our paper is organised as follows. In Section 2 we introduce some useful spine techniques along with the proof of Proposition 1. In Section 3 we establish the results corresponding to Proposition 2 and Theorem 3 for the case of a general branching mechanism; see Proposition 11 and Theorem 12. In doing so, we show that the red branching diffusion and the dressed blue branching diffusion arise from martingale changes of measure which condition (X, P^K) on extinction and survival, respectively. In Section 4, we give a heuristic large deviation argument for the constant C_K (see Section 4.2.1), and we prove the asymptotic results for the survival probability given in Theorem 4. The proof of the quasi-stationary limit result in Theorem 6 follows in Section 5. Section 6 sketches the analogous results for the super-Brownian motion on $(0, K)$.

1.2. *Literature overview.* Branching Brownian motion with an absorbing barrier was first studied by Kesten [25]. Our paper is particularly motivated by recent results on the asymptotics of the survival probability of branching Brownian motion with absorption found in Berestycki et al. [2] as well as Aïdékon and Harris [1]. A discussion of branching Brownian motion in the critical width strip can be found in Berestycki et al. [3].

Spine techniques of the type used in the proof of Proposition 1 were developed in Chauvin and Rouault [9], Lyons [32] and Lyons et al. [33] and are now a standard approach in the theory of branching processes. See, for example, Harris et al. [23] and Kyprianou [27] for related applications in the setting of branching Brownian motion with absorption at 0, respectively, absorption at a space–time barrier.

A backbone decomposition, similar in spirit to the one presented in Theorem 3 but for supercritical superprocesses, is given in Berestycki et al. [4]. It extends the earlier work of Evans and O'Connell [20], Fleischmann and Swart [21] and Engländer and Pinsky [17] as well as the corresponding decomposition for continuous-state branching processes in Duquesne and Winkel [11].

The results for superprocesses are complemented by the decomposition in Etheridge and Williams [18] which considers the $(1 + \beta)$-superprocess conditioned on survival. This work is of particular interest in the current context since it also presents the equivalent result for the approximating branching particle system. However, we should point out that in their case the immigrants are conditioned to become extinct up to a fixed time T whereas, in our setting, we condition on extinction in the strip $(0, K)$. Thus the underlying transformations in [18] are time-dependent in contrast to the space-dependent h-transforms we see in our setting.

Unlike all of the above mentioned backbone decompositions, the proof of the one we address in this paper is based on a new technique using entirely martingale changes of measure. This lends itself more favourably to the quasi-stationary limit theorem later in the paper.

The equivalent result to Theorem 4 in the setting of branching Brownian motion with absorption at the origin was shown in Berestycki et al. [2] and Aïdékon and Harris [1]. However, it has not been possible so far to give such an explicit expression for the constant which plays the analogous role to C_K for branching Brownian motion with absorption at the origin.

TABLE 1
Index of some notation

$(\xi, \mathbb{P}^K)$	Brownian motion with drift $-\mu$ and killing upon exiting $(0, K)$	
$(\xi, \mathbb{P}^{R,K})$	"Red" diffusion in $(0, K)$: drift $-(\mu + \frac{p_K'}{1-p_K})$ and killing upon exiting $(0, K)$	Equation (3.4)
$(\xi, \mathbb{P}^{B,K})$	"Blue" diffusion in $(0, K)$: drift $-(\mu - \frac{p_K'}{p_K})$	Equation (3.10)
(X, P^K)	BBM with killing upon exiting $(0, K)$	
$(X, \mathbf{P}^{R,K})$	Red branching diffusion in $(0, K)$	Proposition 11
$(X, \mathbf{P}^{B,K})$	Blue branching diffusion in $(0, K)$	Proposition 13
$(X, \mathbf{P}^{D,K})$	Dressed blue branching diffusion in $(0, K)$	Theorem 12
$(X, \mathbf{P}^K)$	Two-colour branching diffusion in $(0, K)$	Corollary 14
$(\xi, \mathbb{Q}^K)$	Brownian motion conditioned to stay in $(0, K)$	Equation (2.2)
(X, Q^K)	BBM with a spine conditioned to stay in $(0, K)$	Equation (2.3)
$(\mathcal{G}_t, t \geq 0)$	Natural filtration of ξ	
$(\mathcal{F}_t, t \geq 0)$	Natural filtration of X	

A similarly fashioned result to Theorem 6, albeit being temporal rather than spatial quasi-stationarity, was obtained in the aforementioned work by Etheridge and Williams [18]. Their result extends the Evans immortal particle representation for superprocesses in [19] which is the equivalent of the spine representation for branching processes.

1.3. *Table of notation.* Our results and proofs come with a number of changes of measure. For the benefit of the reader we include in Table 1 an index of some of the probability measures which will be used frequently throughout the paper.

2. Changes of measure and spine techniques: Proof of Proposition 1.

Let us begin this section by stating a general result on how martingale changes of measure affect the drift of a Brownian motion. Recall that we denote by $(\xi, \mathbb{P}_x^K)$ a Brownian motion with drift $-\mu$ initiated from $x \in (0, K)$ which is killed upon exiting $(0, K)$ and set $\mathcal{G}_t = \sigma(\xi_s : s \leq t)$.

We remind the reader of the following classical result, which is adapted from Revuz and Yor [34], Chapter VIII, Proposition 3.4 and the discussion preceding it, since we will make use of it several times.

LEMMA 7. *Let $x \in (0, K)$. Let $h \in C^2(0, K)$, and suppose that*

$$\frac{h(\xi_t)}{h(x)} \exp\left\{-\int_0^t \frac{Lh(\xi_s)}{h(\xi_s)}\, ds\right\}, \qquad t \geq 0, \tag{2.1}$$

is a $\mathbb{P}_x^K$-martingale. Define $\hat{\mathbb{P}}_x^K$ to be the probability measure which has martingale density (2.1) *with respect to $\mathbb{P}_x^K$ on $\mathcal{G}_t$.*

Under $\hat{\mathbb{P}}^K_x$, ξ *has infinitesimal generator* $L + \frac{h'(y)}{h(y)} dy$ *for all functions* $u \in C^2(0, K)$ *with* $u(0+) = u(K-) = 0$.

In this regard, a change of measure with a martingale of the form (2.1) is equivalent to a h-transform of the infinitesimal generator L.

The proof of Proposition 1 uses classical spine techniques developed in Chauvin and Rouault [9], Lyons et al. [33] and Lyons [32]; see, for example, Harris et al. [23] and Kyprianou [27] for related applications in the setting of branching Brownian motion with absorption at 0.

We will briefly recall the key steps in the spine construction. For a comprehensive account we refer the reader to Hardy and Harris [22].

Recall that we characterised the Brownian motion conditioned to stay in $(0, K)$ via its infinitesimal generator $L^{K,*}$ given in (1.3) in Definition 5 (where K_0 can be replaced by a general $K > 0$). In view of Lemma 7, it is not difficult to see that its law can be obtained from the law of $(\xi, \mathbb{P}^K)$ by a martingale change measure. In fact, note that the process

$$\Upsilon^K(t) = \sin(\pi\xi_t/K)e^{\mu\xi_t+(\mu^2/2+\pi^2/2K^2)t}, \qquad t \geq 0 \tag{2.2}$$

is a $\mathbb{P}^K$-martingale, and define $\mathbb{Q}^K_x$ to be the probability measure which has martingale density $\Upsilon^K(t)$ with respect to $\mathbb{P}^K_x$ on $\mathcal{G}_t$. Then, under $\mathbb{Q}^K_x$, ξ is a Brownian motion conditioned to stay in $(0, K)$. By Lemma 7 with $h(x) = \sin(\pi x/K)e^{\mu x}$, its infinitesimal generator is indeed given by $L^{K,*}$ as in (1.3).

This process was first introduced in Knight [26], Theorem 3.1 and referred to as the *taboo process*. Let us note that $(\xi, \mathbb{Q}^K_x)$ is positive recurrent and has invariant density $\frac{2}{K}\sin^2(\pi x/K)$, for $x \in (0, K)$.

Using ideas in [22], we can use Υ^K to construct a martingale with respect to $\mathcal{F}_t = \sigma(X_s, s \leq t)$, the filtration generated by the P^K-branching diffusion up to time t. For each $u \in N_t$, write $\Upsilon^K_u(t) = \sin(\pi x_u(t)/K)e^{\mu x_u(t)+(\mu^2/2+\pi^2/2K^2)t}$, $t \geq 0$. Define the process $Z^K = (Z^K(t), t \geq 0)$ as

$$Z^K(t) = \sum_{u\in N_t} e^{-(m-1)\beta t}\Upsilon_u(t) = \sum_{u\in N_t} e^{\mu x_u(t)-\lambda(K)t}\sin\bigl(\pi x_u(t)/K\bigr), \qquad t \geq 0,$$

where we set $\lambda(K) := (m-1)\beta - \mu^2/2 - \pi^2/2K^2$. Then Z is a nonnegative $(P^K_x, \mathcal{F}_t)$-martingale. For $x \in (0, K)$, we define a martingale change of measure on the probability space of the P^K-branching diffusion via

$$\left.\frac{dQ^K_x}{dP^K_x}\right|_{\mathcal{F}_t} = \frac{Z^K(t)}{Z^K(0)}. \tag{2.3}$$

This change of measure induces the following spine construction for the path of X under Q^K_x. From the initial position x, we run a $\mathbb{Q}^K_x$-diffusion, that is a Brownian motion conditioned to stay in $(0, K)$, and we call it a spine. At times of a Poisson process with rate $m\beta$ we immigrate $\tilde{A}$ independent copies of (X, P^K) rooted at the spatial position of the spine at this time. The number of immigrants $\tilde{A}$ has the size-biased offspring distribution

$$\tilde{q}_k = \frac{1+k}{m} q_{k+1}, \qquad k \geq 0.$$

From this description it is clear that the process (X, Q^K), survives a.s. since the spine survives. In light of the change of measure (2.3), survival of X under Q^K implies a positive probability of survival of X under P^K if the martingale Z^K is uniformly integrable. For this reason, we will now the study the large time behaviour of Z^K.

Since we assumed $E(A \log^+ A) < \infty$, the following proposition gives a necessary and sufficient condition for the $L^1(P_x^K)$-convergence of Z^K.

PROPOSITION 8. *Recall that* $\lambda(K) = (m-1)\beta - \mu^2/2 - \pi^2/2K^2$, *and let* $0 < x < K$.

(i) *If* $\lambda(K) > 0$, *then the martingale* Z^K *is* $L^1(P_x^K)$-*convergent and in particular uniformly integrable.*

(ii) *If* $\lambda(K) \leq 0$, *then* $\lim_{t\to\infty} Z^K(t) = 0$ P_x^K-*a.s.*

We refrain from giving the proof of Proposition 8 since it is a straightforward adaptation of the proof of Theorem 13 in Kyprianou [27] which presents the L^1-convergence result in the case of a branching Brownian motion with absorption at a space–time barrier; see also the proof of Theorem 1 therein, as well as the proof in [32] and the proof of Theorem A in [33].

We will now show that the martingale limit $Z^K(\infty)$ is zero if and only if (X, P^K) becomes extinct.

PROPOSITION 9. *For* $x \in (0, K)$, *the events* $\{Z^K(\infty) = 0\}$ *and* $\{\zeta < \infty\}$ *agree* P_x^K-*a.s.*

PROOF. Clearly $\{\zeta < \infty\} \subset \{Z^K(\infty) = 0\}$ and it remains to show that $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$ has zero probability. We consider the cases $\lambda(K) \leq 0$ and $\lambda(K) > 0$ separately.

Assume $\lambda(K) \leq 0$. Proposition 8 gives $Z^K(\infty) = 0$, P^K-a.s. As Z^K is the sum of the nonnegative terms $e^{-\lambda(K)t} \sin(\pi x_u(t)/K) e^{\mu x_u(t)}$, Z^K vanishes in the limit if and only if all its terms do. On extinction, this is certainly the case. On the event of survival, these terms can only vanish if all particles move arbitrarily close to the killing boundary as $\sin(\pi x/K)e^{\mu x} \approx 0$ for x close to 0 and K only. Let us show that this particle behaviour cannot occur.

We suppose for a contradiction that $Z^K(\infty) = 0$ on the event of survival. This assumption implies that, for any $\varepsilon > 0$, all particles leave the interval $(\varepsilon, K - \varepsilon)$

eventually, and thus we may assume without loss of generality that the process survives in the small strip $(0, \varepsilon)$. We will now lead this argument to a contradiction by showing that, for ε small enough, the P_x^ε-branching diffusion, $x \in (0, \varepsilon)$, will become extinct a.s.

Denote by $P_x^{(-\delta,\varepsilon+\delta)}$ the law under which X is our usual branching Brownian motion but with killing upon exiting the interval $(-\delta, \varepsilon + \delta)$, $\delta > 0$. For any $\delta > 0$, we can embed the P^ε-branching diffusion in a $P^{(-\delta,\varepsilon+\delta)}$-branching diffusion according to the following procedure. Let us write $v \le u$ if v is an ancestor of u (u is considered to be an ancestor of itself), in accordance with the classical Ulam–Harris notation; see, for instance, [22], page 290. Under $P_x^{(-\delta,\varepsilon+\delta)}$, we define

$$N_t|_{(0,\varepsilon)} = \{u \in N_t : \forall s \le t\ \forall v \in N_s \text{ s.t. } v \le u \text{ we have } x_v(s) \in (0, \varepsilon)\},$$

which is the set of particles $u \in N_t$ whose ancestors (not forgetting u itself) have not exited $(0, \varepsilon)$ up to time t. Now we can define the restriction of X to $(0, \varepsilon)$ under $P_x^{(-\delta,\varepsilon+\delta)}$ by

$$X_t|_{(0,\varepsilon)} = \sum_{u \in N_t|_{(0,\varepsilon)}} \delta_{x_u(t)}, \qquad t \ge 0.$$

Then we conclude immediately that, for an initial position in $(0, \varepsilon)$, the restricted process $X|_{(0,\varepsilon)} = (X_t|_{(0,\varepsilon)}, t \ge 0)$ under $P_x^{(-\delta,\varepsilon+\delta)}$ has the same law as (X, P_x^ε).

Now we choose δ and ε small enough such that $\lambda(\varepsilon + 2\delta) := (m-1)\beta - \mu^2/2 - \pi^2/2(\varepsilon + 2\delta)^2 < 0$. Then, under $P^{(-\delta,\varepsilon+\delta)}$, the process

$$\begin{aligned} Z^{(-\delta,\varepsilon+\delta)}&(t) \\ &:= \sum_{u \in N_t} \{e^{\mu(x_u(t)+\delta) - \lambda(\varepsilon+2\delta)t} \sin(\pi(x_u(t) + \delta)/(\varepsilon + 2\delta))\}, \qquad t \ge 0 \end{aligned}$$

is a martingale of the form in Proposition 8. Considering now the contribution coming from the particles in the set $N_t|_{(0,\varepsilon)}$ only, we first note that our assumption of survival of the P^ε-branching diffusion ensures that this set is nonempty for any time t. Further, for particles $u \in N_t|_{(0,\varepsilon)}$, the terms $e^{\mu(x_u(t)+\delta)} \sin(\pi(x_u(t) + \delta)/(\varepsilon + 2\delta))$ are uniformly bounded from below by a constant $c > 0$, and hence, under $P_x^{(-\delta,\varepsilon+\delta)}$, we get

$$Z^{(-\delta,\varepsilon+\delta)}(t) \ge c N_t|_{(0,\varepsilon)} e^{-\lambda(\varepsilon+2\delta)t}.$$

Since we have chosen δ and ε such that $\lambda(\varepsilon + 2\delta) < 0$, we now conclude that $Z^{(-\delta,\varepsilon+\delta)}(\infty) = \infty$, $P_x^{(-\delta,\varepsilon+\delta)}$-a.s. This is a contradiction since $Z^{(-\delta,\varepsilon+\delta)}$ is a positive martingale and therefore has a finite limit. Hence, for $\lambda(K) \le 0$, the martingale limit $Z^K(\infty)$ cannot be zero on survival.

Consider the case $\lambda(K) > 0$. Suppose for a contradiction that $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$ has positive probability. Let $z_K(x) = P_x^K(Z^K(\infty) = 0)$, for $x \in (0, K)$. Define $M_\infty := \mathbf{1}_{\{Z^K(\infty)=0\}}$, and set

$$M_t := E_x^K(M_\infty | \mathcal{F}_t) = \prod_{u \in N_t} z_K(x_u(t)), \tag{2.4}$$

where the second equality follows from the branching Markov property. Then the process $(M_t, t \geq 0)$ defined through (2.4) is a uniformly integrable P_x^K-martingale with limit $M_\infty = \mathbf{1}_{\{Z^K(\infty)=0\}}$. On the event $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$, we clearly have $M_\infty = 1$, P_x^K-a.s. This requires in turn that all particles $x_u(t), u \in N_t$ move toward 0 and K as $t \to \infty$, since we know from Proposition 8(i) that $z_K(x) < 1$ for x within $(0, K)$. The previous part of this proof already showed that this leads to a contradiction. Thus, for $\lambda(K) > 0$, the martingale limit cannot be zero on survival. This completes the proof. □

PROOF OF PROPOSITION 1. Note that $\lambda(K) \geq 0$ if and only if $\mu < \sqrt{2(m-1)\beta}$ and $K > K_0$. The result follows now immediately from Propositions 8 and 9. □

REMARK 10. In the proof of Proposition 1, we saw that the function $z_K(x) = P_x^K(Z^K(\infty) = 0)$ generates the product martingale $(M_t, t \geq 0)$ in (2.4). We can apply the same argument given there to show that

$$E_x^K(\mathbf{1}_{\{\zeta^K < \infty\}} | \mathcal{F}_t) = \prod_{u \in N_t} (1 - p_K(x_u(t))), \qquad t \geq 0$$

is a uniformly integrable product martingale. Followed by a classical Feynman–Kac argument (cf. Champneys et al. [7]), this gives that $z_K(x)$, respectively, $1 - p_K(x)$ solves

$$\begin{aligned} Lu + F(u) &= 0 \qquad \text{on } (0, K), \\ u(0) = u(K) &= 1. \end{aligned} \tag{2.5}$$

3. Backbone decomposition via martingale changes of measure.

In this section we decompose the P^K-branching diffusion into the blue and red branching diffusions corresponding to the blue and red trees described in our intuitive picture, for the binary branching case only, in Section 1.1. Recall that the blue tree consists of all genealogical lines of descent that will never become extinct while the red trees contain all remaining lines of descent. In Section 1.1, we only gave a characterisation of the red, blue, and dressed blue branching diffusion in the case of a binary branching mechanism (Proposition 2 and Theorem 3). For a general branching mechanism, the results will be presented in this section as Proposition 11 and Theorem 12.

Let us refer to the process corresponding to the coloured tree as the *two-colour branching diffusion*. The law $\mathbf{P}^K$ of the two-colour branching diffusion is defined by the law of X under P^K and a subsequent colouring of the particles. Let $c(u)$ denote the colour of a particle u. We say a particle u is blue if it has an infinite genealogical line of descent and we write $c(u) = b$; otherwise we say it is red and write $c(u) = r$. Let us remark that the natural filtration of $(X, \mathbf{P}^K)$ is

$\sigma(\mathcal{F}_t, c(u)_{u\in N_t})$, but this filtration will not play a role in the forthcoming analysis. Given $\mathcal{F}_\infty$, the colouring is deterministic.

Define $c(N_t) = \{(c_u)_{u\in N_t} : c_u \in \{b, r\}\}$ as the set of all possible colourings of N_t. Trivially, for all $t \geq 0$,

$$\frac{d\mathbf{P}_x^K}{dP_x^K}\bigg|_{\mathcal{F}_\infty} = \prod_{u\in N_t} (\mathbf{1}_{\{c(u)=b\}} + \mathbf{1}_{\{c(u)=r\}}) = 1$$

and thus

$$\begin{aligned}\frac{d\mathbf{P}_x^K}{dP_x^K}\bigg|_{\mathcal{F}_t} &= E_x^K\biggl(\prod_{u\in N_t} (\mathbf{1}_{\{c(u)=b\}} + \mathbf{1}_{\{c(u)=r\}})\Big|\mathcal{F}_t\biggr)\\ &= \sum_{\mathbf{c}\in c(N_t)} \prod_{u\in N_t} P_x^K\bigl(c(u) = c_u|\mathcal{F}_t\bigr)\\ &= \sum_{\mathbf{c}\in c(N_t)} \prod_{u\in N_t, c_u=b} p_K\bigl(x_u(t)\bigr) \prod_{u\in N_t, c_u=r} \bigl(1 - p_K\bigl(x_u(t)\bigr)\bigr) = 1,\end{aligned}$$

where the sum is taking over all possible colourings $\mathbf{c} = (c_u)_{u\in N_t}$ in $c(N_t)$. In particular, for $A \in \mathcal{F}_t$, we get

$$\begin{aligned}&\mathbf{P}_x^K\bigl(A; c(u) = c_u\ \forall u \in N_t|\mathcal{F}_t\bigr)\\ &\quad= \mathbf{1}_A \prod_{u\in N_t, c_u=b} p_K\bigl(x_u(t)\bigr) \prod_{u\in N_t, c_u=r} \bigl(1 - p_K\bigl(x_u(t)\bigr)\bigr).\end{aligned}$$

We can now derive the change of measure for the red branching diffusion. It is sufficient to consider one initial particle and we suppose that this particle is red. Let $A \in \mathcal{F}_t$, and write $c(\varnothing) = r$ for the event that the initial particle is red. Then

$$\begin{aligned}\mathbf{P}_x^{R,K}(A) := \mathbf{P}_x^K\bigl(A|c(\varnothing) = r\bigr) &= \frac{\mathbf{P}_x^K(A; c(u) = r\ \forall u \in N_t)}{\mathbf{P}_x^K(c(\varnothing) = r)}\\ &= \frac{E_x^K(\mathbf{1}_A \prod_{u\in N_t}(1 - p_K(x_u(t))))}{1 - p_K(x)}.\end{aligned}\tag{3.1}$$

Clearly, conditioning the initial particle to be red is the same as conditioning the process to become extinct, and therefore the law of X under $\mathbf{P}^{R,K}$ agrees with the law of X conditioned on extinction. The following proposition characterises X under $\mathbf{P}^{R,K}$ and generalises Proposition 2 in Section 1.1.

Throughout this section we will denote branching rates by β and offspring probabilities by q with superscripts indicating whether they belong to the red or blue branching diffusion or the immigration procedure.

PROPOSITION 11 (The red branching diffusion). *For $\nu \in \mathcal{M}_a(0, K)$, define $\mathbf{P}_\nu^{R,K}$ via (3.1). Then $(X, \mathbf{P}_\nu^{R,K})$ is a branching process with single particle motion* *characterised by the infinitesimal generator*

$$L^{R,K} = \frac{1}{2}\frac{d^2}{dy^2} - \left(\mu + \frac{p'_K(y)}{1 - p_K(y)}\right)\frac{d}{dy} \qquad \text{on } (0, K), \tag{3.2}$$

for $u \in C^2(0, K)$ *with* $u(0+) = u(K-) = 0$, *and the branching activity is governed by the space-dependent branching mechanism*

$$F^{R,K}(s, y) = \frac{1}{1 - p_K(y)}\bigl(F\bigl(s\bigl(1 - p_K(y)\bigr)\bigr) - sF\bigl(1 - p_K(y)\bigr)\bigr),$$

for $s \in [0, 1]$ *and* $y \in (0, K)$. *In particular,* $F^{R,K}$ *is of the form*

$$F^{R,K}(s, y) = \beta^R(y)\left(\sum_{k\geq 0} q_k^R(y)s^k - s\right),$$

where β^R *is a space-dependent branching rate and* $(q_k^R, k \geq 0)$ *a space-dependent offspring distribution* [*expressions for* β^R *and* $(q_k^R, k \geq 0)$ *are given in* (3.5) *and* (3.6) *below*].

PROOF. The change of measure in (3.1) preserves the branching property in the following sense. Let $\nu = \sum_{i=1}^n \delta_{x_i}$ be an initial configuration at time 0 in $(0, K)$ and $A \in \mathcal{F}_t$. Then

$$\begin{aligned}
\mathbf{P}_\nu^{R,K}(A) &= E_\nu^K\left(\mathbf{1}_A \frac{\prod_{u\in N_t}(1 - p_K(x_u(t)))}{\prod_{i=1}^n(1 - p_K(x_i))}\right) \\
&= \prod_{i=1}^n E_{x_i}^K\left(\mathbf{1}_A \frac{\prod_{u\in N_t^i}(1 - p_K(x_u(t)))}{1 - p_K(x_i)}\right) \\
&= \left(\bigotimes_{i=1}^n \mathbf{P}_{x_i}^{R,K}\right)(A),
\end{aligned}$$

where N_t^i is the set of descendants at time t of the ith initial particle.

The process $(X, \mathbf{P}^{R,K})$ is therefore completely characterised by its evolution up to the first branching time S. Let us denote by $\xi = \{\xi_t, 0 \leq t \leq S\}$ the path of the initial particle up to time S, noting that it is a Brownian motion with drift $-\mu$, killed upon exiting $(0, K)$ under P^K and $\mathbb{P}^K$. Let H be a positive bounded measurable functional of this path. We begin with considering the case $t < S$. Using the change of measure in (3.1) and the fact that S is exponentially distributed with parameter β, we have

$$\begin{aligned}
&\mathbf{E}_x^{R,K}\bigl(H(\xi_s, s \leq t); S > t\bigr) \\
&\quad = E_x^K\left(H(\xi_s, s \leq t)\frac{1 - p_K(\xi_t)}{1 - p_K(x)}; S > t\right) \qquad (3.3)
\end{aligned}$$

$$= e^{-\beta t}\mathbb{E}_x^K\biggl(H(\xi_s, s \le t)\frac{1-p_K(\xi_t)}{1-p_K(x)}\biggr)$$

$$= e^{-\beta t}\mathbb{E}_x^{R,K}\bigl(H(\xi_s, s \le t)e^{-\int_0^t F(1-p_K(\xi_s))/(1-p_K(\xi_s))\,ds}\bigr),$$

where $\mathbb{P}_x^{R,K}$ is defined by the change of measure

$$\frac{d\mathbb{P}_x^{R,K}}{d\mathbb{P}_x^K}\bigg|_{\mathcal{G}_t} = \frac{1-p_K(\xi_t)}{1-p_K(x)}e^{\int_0^t F(1-p_K(\xi_s))/(1-p_K(\xi_s))\,ds}, \qquad t \ge 0. \tag{3.4}$$

Thus the initial particle performs a $\mathbb{P}^{R,K}$-motion. It follows from Lemma 7 using $h = 1 - p_K$ and the fact that $L(1-p_K) + F(1-p_K) = 0$ (see Remark 10), that the motion under $\mathbb{P}^{R,K}$ is governed by the infinitesimal generator $L^{R,K}$ in (3.2). Note that $L^{R,K}$ depends on the branching mechanism F through p_K.

Taking $H = 1$ and differentiating in t at $t = 0$ in (3.3) above, we see that under $\mathbf{P}^{R,K}$ the branching rate changes to

$$\begin{aligned}\beta^R(y) &= \frac{F(1-p_K(y)) + \beta(1-p_K(y))}{1-p_K(y)}\\ &= \beta\sum_{k\ge 0} q_k\bigl(1-p_K(y)\bigr)^{k-1},\end{aligned} \tag{3.5}$$

for $y \in (0, K)$.

It remains to identify the offspring distribution, and we therefore study the process at its first branching time S. Using (3.1) in the first step, and then (3.4) together with the definition of β^R in (3.5) in the last, we get

$$\begin{aligned}
&\mathbf{E}_x^{R,K}\bigl(H(\xi_s, s\le S); S \in dt; N_S = k\bigr)\\
&\quad = E_x^K\biggl(\frac{(1-p_K(\xi_S))^{N_S}}{1-p_K(x)}H(\xi_s, s \le S); S \in dt; N_S = k\biggr)\\
&\quad = E_x^K\biggl(\frac{(1-p_K(\xi_t))^k}{1-p_K(x)}H(\xi_s, s \le t)\beta e^{-\beta t}q_k\biggr)\,dt\\
&\quad = \mathbb{E}_x^K\biggl(\frac{1-p_K(\xi_t)}{1-p_K(x)}e^{\int_0^t F(1-p_K(\xi_s))/(1-p_K(\xi_s))\,ds}H(\xi_s, s \le t)\\
&\qquad\qquad \times q_k\beta e^{-\beta t}e^{-\int_0^t F(1-p_K(\xi_s))/(1-p_K(\xi_s))\,ds}\bigl(1-p_K(\xi_t)\bigr)^{k-1}\biggr)\,dt\\
&\quad = \mathbb{E}_x^{R,K}\biggl(H(\xi_s, s \le t)\beta^R(\xi_t)e^{-\int_0^t \beta^R(\xi_s)\,ds}\frac{\beta}{\beta^R(\xi_t)}q_k\bigl(1-p_K(\xi_t)\bigr)^{k-1}\biggr)\,dt.
\end{aligned}$$

We see that, in addition to the change in the motion and the branching rate, the offspring distribution under $\mathbf{P}^{R,K}$ becomes $\{q_k^R, k \ge 0\}$ where, for $y \in (0, K)$,

$$q_k^R(y) = \beta\bigl(\beta^R(y)\bigr)^{-1}q_k\bigl(1-p_K(y)\bigr)^{k-1}, \qquad k \ge 0. \tag{3.6}$$

A simple computation shows that $F^{R,K}(s,y) = \beta^R(y)(\sum_{k\geq 0} q_k^R(y)s^k - s)$ takes the desired form. $\square$

The natural next step is to condition the initial particle to be blue and study the resulting law. Note that this will describe the evolution of a dressed blue branching diffusion, corresponding to a blue tree dressed with red trees, and from this process we will be able to recover the blue branching diffusion. We will give the change of measure for the blue branching diffusion in Proposition 13 following the next theorem.

Let us define the law of the dressed blue branching diffusion by

$$\begin{aligned}\mathbf{P}_x^{D,K}(A) &:= \mathbf{P}_x^K(A|c(\varnothing) = b)\\ &= \frac{\mathbf{P}_x^K(A; c(u) = b \text{ for at least one } u \in N_t)}{\mathbf{P}_x^K(c(\varnothing) = b)}\\ &= \frac{E_x^K(\mathbf{1}_A(1 - \prod_{u\in N_t}(1 - p_K(x_u(t)))))}{p_K(x)}.\end{aligned} \tag{3.7}$$

Then $(X, \mathbf{P}^{D,K})$ is the same as (X, P^K) conditioned on survival.

THEOREM 12 (The dressed blue branching diffusion). *Let $K > K_0$ and $x \in (0, K)$. The process $(X, \mathbf{P}_x^{D,K})$ evolves as follows.*

(i) *From x, we run a branching diffusion X^B with single particle movement according to the infinitesimal generator*

$$L^{B,K} = \frac{1}{2}\frac{d^2}{dy^2} - \left(\mu - \frac{p_K'(y)}{p_K(y)}\right)\frac{d}{dy} \qquad \text{on } (0, K), \tag{3.8}$$

defined for all $u \in C^2(0, K)$, and space-dependent branching mechanism $F^{B,K}$ of the form

$$F^{B,K}(s, y) = \beta^B(y)\left(\sum_{k\geq 0} q_k^B(y)s^k - s\right), \qquad s \in [0, 1], y \in (0, K),$$

where, for a fixed $y \in (0, K)$, the branching rate $\beta^B(y)$ and the offspring distribution $(q_k^B(y), k \geq 2)$ are given by

$$\beta^B(y) = \beta \sum_{k\geq 2}\sum_{n\geq k} q_n \binom{n}{k} p_K(y)^{k-1}(1 - p_K(y))^{n-k},$$

$$q_k^B(y) = \beta\beta^B(y)^{-1}\sum_{n\geq k} q_n \binom{n}{k} p_K(y)^{k-1}(1 - p_K(y))^{n-k}.$$

In particular, $F^{B,K}(s, y)$ can be written as

$$\frac{1}{p_K(y)}(F(sp_K(y) + (1 - p_K(y))) - (1 - s)F(1 - p_K(y))).$$

(ii) *Conditionally on the branching diffusion* X^B *in* (i), *we have the following*:

- (*Immigration along the trajectories.*) *Along the trajectories of each particle in* X^B, *an immigration with* $n \geq 1$ *immigrants occurs at rate*

$$\beta_n^{I,1}(y) = \beta q_{n+1}(n+1)\bigl(1 - p_K(y)\bigr)^n, \qquad y \in (0, K).$$

- (*Branch point immigration.*) *At a branch point of* X^B *at* $y \in (0, K)$ *with some fixed* $k \geq 2$ *offspring, the number of immigrants is distributed according to* $(q_{n,k}^{I,2}(y), n \geq 0)$, *in that we see an immigration of n immigrants with probability*

$$q_{n,k}^{I,2}(y) = \bigl(\kappa_k(y)\bigr)^{-1} q_{n+k} \binom{n+k}{k} p_K(y)^{k-1}\bigl(1 - p_K(y)\bigr)^n,$$

with normalising constant $\kappa_k(y) = q_k^B(y)\beta^{-1}\beta^B(y)$.

Each immigrant initiates an independent copy of $(X, \mathbf{P}^{R,K})$ *from the space–time position of its birth.*

PROOF. We use the same notation as in the proof of Proposition 11 and in addition let $T_{(0,K)}$ denote the first time the initial particle exits $(0, K)$. Consider the change of measure in (3.7) and note that, for any time $t < S$ and $A \in \mathcal{F}_t$, it becomes

$$\mathbf{P}_x^{D,K}(A) = E_x^K\biggl(\mathbf{1}_A \frac{p_K(\xi_t)}{p_K(x)}, T_{(0,K)} > t\biggr),$$

where the term $T_{(0,K)} > t$ appears since the product in the enumerator in (3.7) is empty if the initial particle gets killed before it reproduces. Then

$$\begin{aligned} &\mathbf{E}_x^{D,K}\bigl(H(\xi_s, s \leq t); S > t\bigr) \\ &\quad = e^{-\beta t}\mathbb{E}_x^K\biggl(H(\xi_s, s \leq t)\frac{p_K(\xi_t)}{p_K(x)}, T_{(0,K)} > t\biggr) \\ &\quad = e^{-\beta t}\mathbb{E}_x^{B,K}\bigl(H(\xi_s, s \leq t)e^{\int_0^t F(1-p_K(\xi_s))/(p_K(\xi_s)\,ds)}\bigr), \end{aligned} \tag{3.9}$$

where $\mathbb{P}_x^{B,K}$ is defined by the change of measure, for $t \geq 0$,

$$\frac{d\mathbb{P}_x^{B,K}}{d\mathbb{P}_x^K}\bigg|_{\mathcal{G}_t} = \frac{p_K(\xi_t)}{p_K(x)} \exp\biggl\{-\int_0^t \frac{F(1 - p_K(\xi_s))}{p_K(\xi_s)}\,ds\biggr\}\mathbf{1}_{\{T_{(0,K)}>t\}}. \tag{3.10}$$

By Lemma 7 using $h = p_K$ and $Lp_K - F(1-p_K) = 0$ (cf. Remark 10), the motion of ξ under $\mathbb{P}_x^{B,K}$ is governed by the infinitesimal generator $L^{B,K}$ as in (3.8). Note that $L^{B,K}$ depends on F through p_K. Then, setting

$$\begin{aligned} \beta^D(y) &= -\frac{F(1 - p_K(y)) - \beta p_K(y)}{p_K(y)} \\ &= \beta\frac{1 - \sum_{k=0}^{\infty}(1 - p_K(y))^k q_k}{p_K(y)} \qquad \text{for } y \in (0, K), \end{aligned} \tag{3.11}$$

we see that (3.9) simplifies to

$$\mathbf{E}_x^{D,K}\big(H(\xi_s, s \le t), S > t\big) = \mathbb{E}_x^{B,K}\big(H(\xi_s, s \le t)e^{-\int_0^t \beta^D(\xi_s)\,ds}\big).$$

We deduce from this that, under $\mathbf{P}^{D,K}$, the motion of the initial particle is given by the change of measure in (3.10), and it branches at space-dependent rate β^D as in (3.11).

It remains to specify the offspring distribution. We begin with the expression in (3.7) and then use (3.10) and the expression for β^D in (3.11) to get

$$\begin{aligned}
&\mathbf{E}_x^{D,K}\big(H(\xi_s, s \le S); S \in dt; N_S = k\big)\\
&\quad = E_x^K\left(H(\xi_s, s \le t)\frac{1-(1-p_K(\xi_t))^{N_S}}{p_K(x)}; S \in dt; N_S = k\right)\\
&\quad = \mathbb{E}_x^K\left(H(\xi_s, s \le t)\beta e^{-\beta t} q_k \frac{1-(1-p_K(\xi_t))^k}{p_K(x)}\,dt\right)\\
&\quad = \mathbb{E}_x^{B,K}\left(H(\xi_s, s \le t)\beta^D e^{-\int_0^t \beta^D(\xi_s)\,ds}\frac{\beta}{\beta^D(\xi_t)} q_k \frac{1-(1-p_K(\xi_t))^k}{p_K(\xi_t)}\,dt\right).
\end{aligned}$$

Again this reveals the evolution of the initial particle as described above, and we further see that the offspring distribution of the initial particle under $\mathbf{P}^{D,K}$ is given by $\{q_k^D, k \ge 0\}$ where

$$q_k^D(y) \propto q_k \frac{1-(1-p_K(y))^k}{p_K(y)} \qquad \text{for } y \in (0, K),$$

up to the normalising constant $\beta(\beta^D(y))^{-1}$. We note that $q_0(y) = 0$ for all $y \in (0, K)$ which we expected to see since $(X, \mathbf{P}^{D,K})$ is equal in law to (X, P^K) conditioned on survival. However, we have so far neglected the fact that the initial particle can give birth to particles of the same type, that is, blue particles (referred to as branching) and red particles which evolve as under $\mathbf{P}^{R,K}$ (referred to as immigration). We will split up the rate β^D and the offspring distribution q_k^D into terms corresponding to branching, respectively, immigration. Firstly, note that with the help of the binomial theorem we can decompose the rate β^D into

$$\begin{aligned}
\beta^D(y) &= \beta \frac{1-\sum_{k=0}^\infty (1-p_K(y))^k q_k}{p_K(y)}\\
&= \beta \sum_{k\ge 2}\sum_{n \ge k} q_n \binom{n}{k} p_K(y)^{k-1}(1-p_K(y))^{n-k}\\
&\quad + \beta \sum_{n\ge 1} q_n n (1-p_K(y))^{n-1} \qquad (3.12)\\
&=: \beta^B(y) + \sum_{n \ge 0} \beta_n^{I,1}(y).
\end{aligned}$$

Then $\beta_n^{I,1}$ is the rate at which the initial particle gives birth to one blue particle and n (red) immigrants while β^B is the rate at which the initial particle gives birth to at least two particles of the blue type and a random number of (red) immigrants. These rates agree with the immigration rates $\beta_n^{I,1}$ and β^B as stated in (ii), respectively, (i). Again using the binomial theorem, we can now rewrite the offspring distribution q_k^D, for each $k \geq 1$, as

$$
\begin{aligned}
q_k^D(y) &\propto q_k \frac{1-(1-p_K(y))^k}{p_K(y)} \\
&= q_k \sum_{i=2}^{k} \binom{k}{i} p_K(y)^{i-1}\bigl(1-p_K(y)\bigr)^{k-i} \qquad (3.13)\\
&\quad + q_k k\bigl(1-p_K(y)\bigr)^{k-1}, \qquad k \geq 1. \qquad (3.14)
\end{aligned}
$$

Then the term in (3.13) gives, up to normalisation, the sum of the probabilities that the initial particle branches into i blue particles, and at the same branching time, $k-i$ red particles immigrate. This gives the immigrant distribution at branching points, $(q_{n,k}^{I,2}(y), k \geq 2)$, as stated in (ii) as well as the offspring distribution of the blue branching diffusion in (i). The term in (3.14) is the probability that $k-1$ immigrants occur, again up to a normalising constant.

Note that $(X, \mathbf{P}^{D,K})$ inherits the branching Markov property from (X, P^K) by (3.7) in a similar spirit to the case of $(X, \mathbf{P}^{R,K})$; cf. the proof of Proposition 11. Thus the description of the initial particle also characterises the evolution of all particles of the blue type and together with the characterisation of the immigrating $\mathbf{P}^{R,K}$-branching diffusions in Proposition 11 we have completely characterised the evolution of X under $\mathbf{P}^{D,K}$. ☐

In light of Theorem 12, we call the blue branching diffusion X^B in step (i) the *backbone*. Let us give the change of measure under which X evolves like X^B. Using the classical Ulam–Harris notation (see, e.g., [22], page 290), we denote by τ_v and σ_v the birth, respectively, death time of a particle v, by $T_{(0,K)}^v$ its first exit time from $(0, K)$ and by A_v the random number of its offspring. Denote by $\mathcal{T}$ the set of all particles in a realisation of X. Let $\mathcal{T}_t$ be the set of all $v \in \mathcal{T}$ with $\tau_v < t$ and v is in $\mathcal{T}_{t-}$ if, in addition, $\sigma_v < t$.

PROPOSITION 13 (The backbone). *For $\nu \in \mathcal{M}_a(0, K)$ such that $\nu = \sum_{i=1}^n \delta_{x_i}$ with $x_i \in (0, K)$, $n \geq 1$, we define the measure $\mathbf{P}_\nu^{B,K}$ via the following change of measure. For $t \geq 0$,*

$$
\begin{aligned}
\frac{d\mathbf{P}_\nu^{B,K}}{dP_\nu^K}\bigg|_{\mathcal{F}_t} &= \prod_{v \in \mathcal{T}_t} \frac{p_K(x_v(\sigma_v \wedge t))}{p_K(x_v(\tau_v))} \mathbf{1}_{\{t < T_{(0,K)}^v\}} \\
&\quad \times \exp\left\{\int_{\tau_v}^{\sigma_v \wedge t} F'\bigl(1 - p_K\bigl(x_v(s)\bigr)\bigr) + \beta \, ds\right\} \\
&\quad \times \prod_{v \in \mathcal{T}_{t-}} \frac{q_{A_v}^B(x_v(\sigma_v))}{q_{A_v} \beta(\beta^B(x_v(\sigma_v)))^{-1}}.
\end{aligned}
$$

The branching diffusion $(X, \mathbf{P}_\nu^{B,K})$ *has single particle movement according to the infinitesimal generator* $L^{B,K}$ *and branching mechanism* $F^{B,K}$ *as given in step* (i) *of Theorem* 12.

PROOF. A simple computation, using (3.11) and (3.12), shows that

$$F'(1-p_K(y)) = -\frac{F(1-p_K(y))}{p_K(y)} - \beta^B(y), \qquad y \in (0, K).$$

The result then follows from rewriting the change of measure up to the first branching time S as

$$\begin{aligned}\frac{d\mathbf{P}_x^{B,K}}{dP_x^K}\bigg|_{\mathcal{F}_S} &= \frac{p_K(\xi_S)}{p_K(x)} \exp\left\{-\int_0^S \frac{F(1-p_K(\xi_s))}{p_K(\xi_s)}\,ds\right\}\mathbf{1}_{\{S<T_{(0,K)}\}} \\ &\quad \times \frac{1}{\beta}\beta^B(\xi_S)\exp\left\{-\int_0^S \beta^B(\xi_s)-\beta\,ds\right\} \times \frac{q_{N_S}^B(\xi_S)}{q_{N_S}},\end{aligned}$$

noting that the first line on the right-hand side accounts for the change of motion, the first term in the second line for the change in the branching rate and the last term in the second line for the change in the offspring distribution. ☐

COROLLARY 14 (The backbone decomposition). *Let* $K > K_0$ *and* $\nu \in \mathcal{M}_a(0, K)$ *such that* $\nu = \sum_{i=1}^n \delta_{x_i}$ *with* $x_i \in (0, K)$, $n \geq 1$.

Then $(X, \mathbf{P}_\nu^K)$ *has the same law as the process*

$$\sum_{i=1}^n (Y_i X_t^{D,i} + (1-Y_i)X_t^{R,i}), \qquad t \geq 0,$$

where $X^{R,i} = (X_t^{R,i}, t \geq 0)$ *are independent copies of* $(X, \mathbf{P}_{x_i}^{R,K})$, $X^{D,i} = (X_t^{D,i}, t \geq 0)$ *are independent copies of* $(X, \mathbf{P}_{x_i}^{D,K})$ *and the* Y_i *are independent Bernoulli random variables with respective parameters* $p_K(x_i)$.

Intuitively speaking, we can describe the evolution under $\mathbf{P}_\nu^K$ and thus also under P_ν^K as follows. Independently for each initial particle i with position x_i, we flip a coin with probability $p_K(x_i)$ of "heads". If it lands "heads", we initiate a copy of $(X, \mathbf{P}_{x_i}^{D,K})$, and otherwise we initiate a copy of $(X, \mathbf{P}_{x_i}^{R,K})$.

COROLLARY 15. *Given the number of particles of* (X, P_ν^K) *and their positions, say* $x_1, \dots, x_n$ *for some* $n \in \mathbb{N}$, *at a fixed time* t, *then the number of particles of* X_t^B *is the number of successes in a sequence of* n *independent Bernoulli trials, each with success probability* $p_K(x_1), \dots, p_K(x_n)$.

REMARK 16. With Theorem 12 in hand it can be shown that if the differential equation in (2.5) has a nontrivial, [0, 1]-valued solution, then it is unique. We sketch the argument here.

Assume that $g_K(x)$ is a nontrivial, $(0,1)$-valued solution to (2.5). By a Feynman–Kac argument (cf. Champneys et al. [7]), it follows that

$$M^K(t) = \prod_{u \in N_t} g_K\big(x_u(t)\big), \qquad t \geq 0,$$

is a P_x^K-product martingale. Since M^K is uniformly integrable, its limit $M^K(\infty)$ exits P_x^K-a.s. On the event of extinction, $M^K(\infty) = 1$. On the event of survival, it follows from Theorem 12 that

$$M^K(t) = \prod_{u \in N_t} g_K\big(x_u(t)\big) \leq \prod_{u \in N_t^B} g_K\big(x_u^B(t)\big), \tag{3.15}$$

where N_t^B is the set of particles in X_t^B.

Clearly, $|N_t^B| \to \infty$ as $t \to \infty$ since each particle in X^B is replaced by at least two offspring and there is no killing. Denote by $\xi^B = (\xi_t^B, t \geq 0)$ the path of an arbitrary line of descent of particles in X^B. Then ξ^B performs an ergodic motion in $(0, K)$ according to the infinitesimal generator $L^{B,K}$ in (3.2). By ergodicity, P^K-a.s., we have $\liminf_{t\to\infty} \xi_t^B = 0$ and $\limsup_{t\to\infty} \xi_t^B = K$ which implies

$$\liminf_{t\to\infty} g_K\big(\xi_t^B\big) = \inf_{y \in (0,K)} g_K(y) < 1, \tag{3.16}$$

since g_K is nontrivial and $(0,1)$-valued. At any time $t \geq 0$, we can choose $|N_t^B|$ lines of descent, each of them containing the path of one of the particles in N_t^B, and (3.16) holds true along these lines of descent. Loosely speaking, the right-hand side of (3.15) then tends to an infinite product of terms with lim inf strictly smaller than 1, and therefore it must converge to 0, that is,

$$\liminf_{t\to\infty} M^K(t) \leq \liminf_{t\to\infty} \prod_{u \in N_t^B} g_K\big(x_u^B(t)\big) = 0, \qquad P^K\text{-a.s.}$$

Since the limit $M^K(\infty)$ exists P^K-a.s., we get $M^K(\infty) = 0$, on the event of survival.

We conclude that $M^K(\infty) = \mathbf{1}_{\{\zeta < \infty\}}$. Taking expectations gives

$$g_K(x) = E_x^K\big(M^K(\infty)\big) = P_x^K(\zeta < \infty), \qquad x \in (0, K).$$

As this is true for any nontrivial, [0, 1]-valued solution to (2.5), we have established uniqueness of these solutions.

In Remark 10, we saw that the function $z_K(x) = P_x^K(Z^K(\infty) = 0)$ solves (2.5). With Proposition 8, this yields that (2.5) has a nontrivial solution if and only if $\mu < \sqrt{2(m-1)\beta}$ and $K > K_0$.

Again by Remark 10, $1 - p_K(x)$ is also a solution to (2.5). Thus we may derive again that the events $\{Z^K(\infty) = 0\}$ and $\{\zeta < \infty\}$ agree P_x^K-a.s.; cf. Proposition 9.

4. Proof of Theorem 4.

We break up Theorem 4 into two parts which will be proved in the subsequent sections.

PROPOSITION 17. *Uniformly for all* $x \in (0, K_0)$,

$$p_K(x) \sim c_K \sin(\pi x/K_0)e^{\mu x} \qquad \text{as } K \downarrow K_0,$$

where c_K *is independent of* x *and* $c_K \downarrow 0$ *as* $K \downarrow K_0$.

PROPOSITION 18. *The constant* c_K *in Proposition* 17 *satisfies*

$$c_K \sim (K - K_0)\frac{(K_0^2\mu^2 + \pi^2)(K_0^2\mu^2 + 9\pi^2)}{12(m-1)\beta\pi K_0^3(e^{\mu K_0} + 1)} \qquad \text{as } K \downarrow K_0. \tag{4.1}$$

Theorem 4 then follows by defining C_K to be the expression on the right-hand side in (4.1).

We will provide probabilistic proofs of the results above. We remark that, although it would take some effort to make rigorous, it is also possible to recover the asymptotics of p_K and the explicit constant C_K in an analytic approach using a careful asymptotic expansion of the nonlinear ODE $Lu + F(u) = 0$ with $u(0) = u(K) = 1$, as shown to us by Derrida.

4.1. *Proof of Proposition* 17.

We begin with a preliminary result which ensures that the survival probability p_K is right-continuous at K_0.

LEMMA 19. *Let* $x \in (0, K_0)$. *Then* $\lim_{K\downarrow K_0} p_K(x) = 0$.

PROOF. We fix $x \in (0, K_0)$ throughout the proof and consider $p_K(x)$ as a function in K. For $t > 0$, let us define the probability $p_K(x,t) := P_x^K[\text{survival in } (0, K) \text{ up to time } t]$. Since $p_K(x,t)$ is monotonically decreasing for $K \downarrow K_0$ and $t \to \infty$, we have

$$\lim_{K\downarrow K_0}\lim_{t\to\infty} p_K(x,t) = \lim_{t\to\infty}\lim_{K\downarrow K_0} p_K(x,t).$$

Further, by monotonicity of measures, we have, for any $K > 0$,

$$\lim_{t\to\infty} p_K(x,t) = p_K(x),$$

and, for $t \geq 0$,

$$\lim_{K\downarrow K_0} p_K(x,t) = P_x^K\big(\text{survival in } [0, K_0] \text{ up to time } t\big) = p_{K_0}(x,t),$$

where the last equality holds true as any particle that hits 0 or K will immediately pass below 0, respectively, above K. Putting the pieces together, we get

$$\begin{aligned}\lim_{K\downarrow K_0} p_K(x) &= \lim_{K\downarrow K_0}\lim_{t\to\infty} p_K(x,t) = \lim_{t\to\infty}\lim_{K\downarrow K_0} p_K(x,t)\\ &= \lim_{t\to\infty} p_{K_0}(x,t) = p_{K_0}(x).\end{aligned}$$

By Proposition 1, $p_{K_0}(x) = 0$, and so we have $\lim_{K\downarrow K_0} p_K(x) = 0$. □

Recall that we denoted by $\mathcal{T}$ the set of all particles in a realisation of X and $v < u$ means that v is a strict ancestor of u. For $y \in (0, K_0)$, let $\mathcal{L}_{(0,y)}$ be the set containing all particles which are the first ones in their genealogical line to exit the strip $(0, y)$, that is,

$$\mathcal{L}_{(0,y)} = \{u \in \mathcal{T} : \exists s \in [\tau_u, \sigma_u] \text{ s.t. } x_u(s) \notin (0, y) \\ \text{and } x_v(r) \in (0, y) \text{ for all } v < u, r \in [\tau_v, \sigma_v]\}. \tag{4.2}$$

The random set $\mathcal{L}_{(0,y)}$ is a stopping line in the sense of Biggins and Kyprianou [6]; see also Chauvin [8], which uses a slightly different definition though.

Let $|\mathcal{L}_{(0,y)}|$ be the number of particles which are the first ones in their line of descent to hit y (we do not count the ones exiting at 0), which can be written as

$$|\mathcal{L}_{(0,y)}| = \sum_{u \in \mathcal{L}_y} \mathbf{1}_{\{x_u(T^u_{(0,y)}) = y\}}, \tag{4.3}$$

recalling that we denoted by $T^u_{(0,y)}$ the first exit time of a particle u from $(0, y)$. Likewise we can define the stopping line $\mathcal{L}_{(y,K_0)}$ as the set containing all particles which are the first ones in their genealogical line to exit the strip (y, K_0) and $|\mathcal{L}_{(y,K_0)}|$ as the number of particles in $\mathcal{L}_{(y,K_0)}$ which have exited at y.

The quantity $|\mathcal{L}_{(0,y)}|$ will turn out to be the essential ingredient in the proof of Proposition 17. To begin with, let us show that $E^{K_0}_x(|\mathcal{L}_{(0,y)}|)$ is finite. In fact, we can compute this expectation explicitly.

LEMMA 20. *Let $x, y \in (0, K_0)$ with $x \leq y$. We have*

$$E^{K_0}_x(|\mathcal{L}_{(0,y)}|) = \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)} e^{\mu(x-y)}, \tag{4.4}$$

where $|\mathcal{L}_{(0,y)}|$ is defined in (4.3). For $x, y \in (0, K_0)$ with $x \geq y$, (4.4) holds true with $|\mathcal{L}_{(0,y)}|$ replaced by $|\mathcal{L}_{(y,K_0)}|$.

PROOF. To begin with, we note that a stopping line is called dissecting if there exists a P^K-a.s. finite time such that each particle alive at this time has descended from a particle in the stopping line; cf. [27]. Since we choose $y \in (0, K_0)$, the width of the strip $(0, y)$ is subcritical and hence, for any initial position $x \in (0, y)$, all particles will exit it eventually. This ensures that the stopping line $\mathcal{L}_{(0,y)}$ defined in (4.2) is a dissecting stopping line. Since $\mathcal{L}_{(0,y)}$ is dissecting it follows from Theorem 6 in [27] that we can apply the many-to-one lemma (see, e.g., [22], Theorem 8.5) for the stopping line $\mathcal{L}_{(0,y)}$. Let $T_{(0,y)}$ again be the first time ξ exists $(0, y)$, and recall the definition of $\mathbb{Q}^{K_0}_x$ via the martingale change of measure in (2.2). Then we get

$$E^{K_0}_x(|\mathcal{L}_{(0,y)}|) \\ = \mathbb{E}^{K_0}_x(e^{(m-1)\beta T_{(0,y)}} \mathbf{1}_{(\xi_{T_{(0,y)}} = y)})$$

$$= \mathbb{Q}_x^{K_0}\left(e^{(m-1)\beta T_{(0,y)}} \frac{\sin(\pi x/K_0) e^{\mu(x-\xi_{T_{(0,y)}})}}{\sin(\pi \xi_{T_{(0,y)}}/K_0) e^{(\mu^2/2+\pi^2/2K_0^2)T_{(0,y)}}} \mathbf{1}_{(\xi_{T_{(0,y)}}=y)}\right)$$

$$= \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)} e^{\mu(x-y)} \mathbb{Q}_x^{K_0}(\xi_{T_{(0,y)}} = y),$$

where we have used that $(m-1)\beta - \mu^2/2 - \pi^2/2K_0^2 = 0$ (and $\mathbb{Q}_x^{K_0}$ is used as an expectation operator). Under $\mathbb{Q}_x^{K_0}$, ξ will never hit 0 since it is conditioned to stay in $(0, K_0)$. However as ξ is positive recurrent it will eventually cross y and therefore $\mathbb{Q}_x^{K_0}(\xi_{T_{(0,y)}} = y) = 1$. This gives (4.4). The case $x \geq y$ follows in the same way. □

The following lemma is the essential part in the proof of Proposition 17.

LEMMA 21. *Let $x, y \in (0, K_0)$ with $x \leq y$. Let $|\mathcal{L}_{(0,y)}|$ be as defined in* (4.3). *Then we have*

$$\lim_{K \downarrow K_0} \frac{p_K(x)}{p_K(y)} = E_x^{K_0}(|\mathcal{L}_{(0,y)}|). \tag{4.5}$$

For $x, y \in (0, K_0)$ with $x \geq y$, (4.5) *holds true with $|\mathcal{L}_{(0,y)}|$ replaced by $|\mathcal{L}_{(y,K_0)}|$.*

PROOF. Fix $y \in (0, K_0)$. We begin with the case $0 < x \leq y$.

We recall from Remark 10 that $(\prod_{u \in N_t}(1 - p_K(x_u(t))), t \geq 0)$ is a P_x^K-martingale. Since $\mathcal{L}_{(0,y)}$ is dissecting, as noted in the proof of Lemma 20, it follows from [8] that we can stop the martingale at $\mathcal{L}_{(0,y)}$ and obtain, for $x \in (0, y)$,

$$\begin{aligned} 1 - p_K(x) &= E_x^K\Big(\prod_{u \in \mathcal{L}_{(0,y)}} 1 - p_K\big(x_u\big(T_{(0,y)}^u\big)\big)\Big) \\ &= E_x^K\big((1 - p_K(y))^{|\mathcal{L}_{(0,y)}|}\big), \end{aligned} \tag{4.6}$$

where we have used that the process started at zero becomes extinct immediately, that is, $p_K(0) = 0$. Further $|\mathcal{L}_{(0,y)}|$ has the same distribution under P_x^K and $P_x^{K_0}$ since we consider particles stopped at level y below K_0, and thus we can replace E_x^K by $E_x^{K_0}$ on the right-hand side above. Now, using first (4.6) and then the geometric sum $\sum_{j=0}^{n-1} a^j = \frac{1-a^n}{1-a}$, we get

$$\begin{aligned} \frac{p_K(x)}{p_K(y)} &= E_x^{K_0}\left(\frac{1 - (1 - p_K(y))^{|\mathcal{L}_{(0,y)}|}}{1 - (1 - p_K(y))}\right) \\ &= E_x^{K_0}\left(\sum_{j=0}^{|\mathcal{L}_{(0,y)}|-1} (1 - p_K(y))^j\right). \end{aligned} \tag{4.7}$$

The sum on the right-hand side is dominated by $|\mathcal{L}_{(0,y)}|$ which does not depend on K and has finite expectation; see Lemma 20. We can therefore apply the dominated convergence theorem to the right-hand side in (4.7), and we conclude that

$$\begin{aligned}\lim_{K\downarrow K_0} E_x^{K_0}\left(\sum_{j=0}^{|\mathcal{L}_{(0,y)}|-1}(1-p_K(y))^j\right)\\ = E_x^{K_0}\left(\sum_{j=0}^{|\mathcal{L}_{(0,y)}|-1}\lim_{K\downarrow K_0}(1-p_K(y))^j\right) = E_x^{K_0}(|\mathcal{L}_{(0,y)}|),\end{aligned} \tag{4.8}$$

where the convergence holds point-wise in $x \in (0, y)$. Combining (4.7) and (4.8) we get (4.5) for $x \in (0, y)$.

It remains to show that (4.5) also holds for $x \in (y, K_0)$. Instead of approaching criticality by taking the limit in K we can now fix a $K > K_0$ and consider a (supercritical) strip (z, K) and let $z \uparrow z_0$ where $z_0 := K - K_0$. Denote by $p_{(z,K)}(x + z)$ the probability of survival in the strip (z, K) when starting from $x + z$. We then have

$$\lim_{K\downarrow K_0}\frac{p_K(x)}{p_K(y)} = \lim_{z\uparrow z_0}\frac{p_{(z,K)}(x+z)}{p_{(z,K)}(y+z)}.$$

Hence (4.5) is equivalent to showing that

$$\lim_{z\uparrow z_0}\frac{p_{(z,K)}(x+z)}{p_{(z,K)}(y+z)} = E_{x+z_0}^{K}(|\mathcal{L}_{(y+z_0,K)}|) = E_x^{K_0}(|\mathcal{L}_{(y,K_0)}|).$$

Here $|\mathcal{L}_{(y+z_0,K)}|$ denotes the number of particles which are the first in their genealogical line to exit the strip $(y + z_0, K)$ at $y + z_0$. Noting that this has the same law under $P_{x+z}^{z,K}$ and $P_{x+z}^{z_0,K}$, we can then repeat the argument in the first part. □

The next step is to show that the convergence in Lemma 21 holds uniformly in x on $(0, K_0)$.

LEMMA 22. *Let $y \in (0, K_0)$. Then we have*

$$\lim_{K\downarrow K_0}\frac{p_K(x)}{p_K(y)} = \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)}e^{\mu(x-y)}, \tag{4.9}$$

uniformly for all $x \in (0, K_0)$.

PROOF. With Lemmas 21 and 20, it remains to show that, for fixed $y \in (0, K_0)$, the convergence in equation (4.5) of Lemma 21 holds uniformly for all $x \in (0, K_0)$. Taking a look back at the proof of Lemma 21, we see that it suffices to show that the convergence in (4.8) holds uniformly for all $x \in (0, K_0)$.

Let us fix a $y \in (0, K_0)$, and let $x \in (0, y)$. We set

$$\varphi(x, K) = E_x^{K_0}\left(\sum_{j=0}^{|\mathcal{L}_y|-1} (1 - p_K(y))^j \right) \qquad \text{for } x \in [0, y]$$

(with the convention that the P^K-branching diffusion becomes extinct immediately for the initial position $x = 0$, resp., stopped for $x = y$) and denote by $\varphi(x) = E_x^{K_0}(|\mathcal{L}_y|)$ its point-wise limit. Since $1 - p_K(y) \le 1 - p_{K'}(y)$, for $K \ge K'$, we have $\varphi(x, K) \le \varphi(x, K')$, and thus for any $x \in [0, y]$, the sequence $\varphi(x, K)$ is monotone increasing as $K \downarrow K_0$. Moreover the functions $\varphi(x, K)$ and $\varphi(x)$ are continuous in x, for any K. In conclusion, we have an increasing sequence of continuous functions on a compact set with a continuous point-wise limit, and therefore the convergence in (4.8) also holds uniformly in $x \in [0, y]$; see, for example, [36], Theorem 7.13. This implies now that, for fixed $y \in (0, K_0)$, (4.5), and thus (4.9) holds uniformly in $x \in (0, y)$.

As outlined in the proof of Lemma 21, we can adapt the argument to the case $x \in (y, K_0)$ to complete the proof. □

PROOF OF PROPOSITION 17. Choose a $y \in (0, K_0)$. Then an application of Lemma 22 gives, as $K \downarrow K_0$,

$$p_K(x) = p_K(y)\frac{p_K(x)}{p_K(y)} \sim p_K(y)\frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)}e^{\mu(x-y)} = c_K \sin(\pi x/K_0)e^{\mu x},$$

uniformly for all $x \in (0, K_0)$, where $c_K := \frac{p_K(y)}{\sin(\pi y/K_0)}e^{-\mu y}$. By Proposition 19, $c_K \downarrow 0$ as $K \downarrow K_0$ which completes the proof. □

4.2. *Proof of Proposition* 18. In this section we will present the proof of Proposition 18 which gives an explicit asymptotic expression for the constant c_K appearing in the asymptotics for the survival probability in Proposition 17 and Theorem 4. We begin with a heuristic that guides our proof.

4.2.1. *Heuristic argument.* The starting point for the proof of Proposition 18 is the following idea: By Corollary 14, at time t, given the spatial positions $x_u(t)$ of all particles $u \in N_t$, the number of blue particles is the number of successes in a sequence of Bernoulli trials with success probabilities $p_K(x_u(t))$. As this holds at any time t, we would expect that the proportion of blue particles, as a proportion of the whole population, roughly stays constant over time. This suggests that the blue particles (the backbone) grows at the same rate as the whole process on survival. Further, the immigrating red trees are conditioned to become extinct which suggests that they do not contribute to the survival of the process. Loosely speaking, we do not expect to lose too much information about the evolution of (X, P^K) on survival if we simply study the growth of the blue tree and ignore the contribution of the immigrating red trees.

We break up the heuristic argument into four steps.

Step (i) (*The growth rate of the backbone*). In this step, we derive an expression for the expected growth rate of the number of blue particles. The following heuristic argument is based on Donsker–Varadhan large deviation theory for occupation measures; for precise formulations, see, for instance, Deuschel and Stroock [10] or Chapter 5 in Stroock [37].

Consider a process $Y^B = (Y_t^B, t \geq 0)$ performing the single particle motion of the backbone, that is according to the infinitesimal generator $L^{B,K}$ which is given in (3.8) in Theorem 12 as

$$L^{B,K} = \frac{1}{2}\frac{d^2}{dy^2} - \left(\mu - \frac{p_K'}{p_K}\right)\frac{d}{dy} \qquad \text{on } (0, K),$$

with domain $C^2(0, K)$. Let $\Pi^{B,K}$ be the invariant density for $L^{B,K}$, that is, the positive solution of $\tilde{L}^{B,K}\Pi^{B,K} = 0$ where $\tilde{L}^{B,K}$ is the formal adjoint of $L^{B,K}$. Then we find

$$\Pi^{B,K}(y) \propto p_K(y)^2 e^{-2\mu y}, \qquad y \in (0, K).$$

For $t \geq 0$ and a set $A \subset [0, K]$, we define

$$\Gamma(t, A) = \int_0^t \mathbf{1}_{\{Y_s^B \in A\}}\, ds$$

to be the occupation time up to time t of Y^B in the set A. Then large deviation theory tells us that the probability that the occupation measure $t^{-1}\Gamma(t, \cdot)$ is "close" to the measure $\int_0^K \mathbf{1}_{\{\cdot\}}(y) f^2(y)\Pi^{B,K}(y)\, dy$ is roughly

$$\exp\left\{-t\int_0^K \frac{1}{2}(f'(y))^2 \Pi^{B,K}(y)\, dy\right\}. \tag{4.10}$$

Recall that each particle in the backbone moves according to $L^{B,K}$ and that the branching mechanism of the backbone is $F^{B,K}$ as defined in Theorem 12. For $y \in (0, K)$,

$$F^{B,K'}(1, y) := \frac{d}{ds}F^{B,K}(s, y)|_{s=1} = (m-1)\beta + \frac{F(1 - p_K(y))}{p_K(y)}$$

represents the *branching rate multiplied by the mean increment in population (the meath growth rate)* for particles at location y. Then from (4.10), we would *guess* that the expected number of particles with occupation densities "close to" $f^2\Pi^{B,K}$ at time t is very roughly

$$\exp\left\{t\int_0^K \left\{F^{B,K'}(1, y)f(y)^2 - \frac{1}{2}(f'(y))^2\right\}\Pi^{B,K}(y)\, dy\right\}.$$

By Laplace–Varadhan asymptotics, the expected growth rate of the blue tree should then be

$$\sup_f\left\{\int_0^K \left\{F^{B,K'}(1, y)f(y)^2 - \frac{1}{2}(f'(y))^2\right\}\Pi^{B,K}(y)\, dy\right\} \tag{4.11}$$

with the supremum taken over a suitable class of functions f with the normalisation $\int_0^K f^2(y)\Pi^{B,K}(y)\,dy=1$.

We assume henceforth that the supremum in (4.11) is taken over all functions f which satisfy, in addition the boundary condition,

$$\lim_{y\downarrow 0} f(y)f'(y)\Pi^{B,K}(y)=\lim_{y\uparrow K} f(y)f'(y)\Pi^{B,K}(y)=0. \tag{4.12}$$

Then an integration by parts shows that

$$\begin{aligned}\int_0^K &\{L^{B,K}f(y)\}f(y)\Pi^{B,K}(y)\,dy\\ &=-\frac{1}{2}\int_0^K (f'(y))^2\Pi^{B,K}(y)\,dy.\end{aligned} \tag{4.13}$$

Thus with (4.13), variational problem (4.11) can be written as

$$\sup_f\biggl\{\int_0^K \{[L^{B,K}+F^{B,K'}(1,y)]f(y)\}f(y)\Pi^{B,K}(y)\,dy\biggr\}. \tag{4.14}$$

Now set $h(y)=p_K(y)e^{-\mu y}f(y)$. Then h satisfies the normalisation $\int_0^K h(y)^2\,dy=1$ and $h(0)=0=h(K)$. An elementary computation shows that, instead of (4.14), we can consider the equivalent problem

$$\sup_h\biggl\{\int_0^K\biggl\{\frac{1}{2}h''(y)+\biggl((m-1)\beta-\frac{\mu^2}{2}\biggr)h(y)\biggr\}h(y)\,dy\biggr\}. \tag{4.15}$$

Equivalence means that the optimal solutions f^* and h^* of (4.14) and (4.15), respectively, satisfy $h^*(y)=p_K(y)e^{-\mu y}f^*(y)$. If we take the supremum in (4.15) over all functions $h\in L^2[0,K]$ with $h(0)=0=h(K)$ and $\int_0^K h(y)^2\,dy=1$, then (4.15) is a classical Sturm–Liouville eigenvalue problem. For this case, the optimal solution is $h^*(y)\propto\sin(\pi y/K)$, $y\in(0,K)$. Moreover, we get

$$f^*(y)=\frac{h^*(y)}{p_K(y)}e^{\mu y}\propto\frac{\sin(\pi y/K)}{p_K(y)}e^{\mu y},\qquad y\in(0,K), \tag{4.16}$$

up to a normalising constant. Further, f^* solves

$$[L^{B,K}+F^{B,K'}(1,y)]f^*(y)=\lambda(K)f^*(y)\qquad\text{in }(0,K), \tag{4.17}$$

where $\lambda(K)=(m-1)\beta-\mu^2/2-\pi^2/2K^2$.

In conclusion, under the assumption that f^* satisfies (4.12), we get from (4.17) and (4.13)

$$\begin{aligned}\lambda(K)&=\int_0^K\{[L^{B,K}+F^{B,K'}(1,y)]f^*(y)\}f^*(y)\Pi^{B,K}(y)\,dy\\ &=\int_0^K\biggl\{F^{B,K'}(1,y)f^*(y)^2-\frac{1}{2}(f^*(y)')^2\biggr\}\Pi^{B,K}(y)\,dy\\ &=\sup_f\biggl\{\int_0^K\biggl\{F^{B,K'}(1,y)f(y)^2-\frac{1}{2}(f'(y))^2\biggr\}\Pi^{B,K}(y)\,dy\biggr\}.\end{aligned} \tag{4.18}$$

Heuristically, this indicates that $\lambda(K)$ is the expected growth rate of the blue tree and this growth rate is attained by particles proportioning their time according to the optimal occupation density $(f^*)^2\Pi^{B,K}$, as opposed to the single particle occupation density $\Pi^{B,K}$.

Step (ii) [*Lower bound on* $\lambda(K)$]. Since f^* maximises the expression in (4.18), we get a lower bound on $\lambda(K)$ by taking $f = 1$, that is,

$$\int_0^K F^{B,K'}(1,y)\Pi^{B,K}(y)\,dy \le \lambda(K).$$

Step (iii) [*Upper bound on* $\lambda(K)$]. Let us define the "optimal" occupation density as

$$\Pi_*^{B,K}(y) := \big(f^*(y)\big)^2\Pi^{B,K}(y) = \frac{2}{K}\sin^2(\pi y/K), \qquad y \in (0,K).$$

Omitting the nonpositive term $-\frac{1}{2}(f^*(y)')^2$ in the second equality of (4.18) gives the upper bound

$$\lambda(K) \le \int_0^K F^{B,K'}(1,y)\Pi_*^{B,K}(y)\,dy.$$

Step (iv) (*Asymptotics*). By Theorem 4, $p_K(y) \sim c_K \sin(\pi y/K_0)e^{\mu y}$, as $K \downarrow K_0$, and we can easily deduce that

$$\Pi^{B,K}(y) \sim \Pi_{K_0}^{B,*}(y) \qquad \text{as } K \downarrow K_0.$$

We will make rigorous later that $F^{B,K'}(1,y) \sim (m-1)\beta c_K \sin(\pi y/K_0)e^{\mu y}$ as $K \downarrow K_0$. Our conjecture is therefore that

$$\lambda(K) \sim c_K \frac{2(m-1)\beta}{K_0}\int_0^{K_0} \sin^3(\pi y/K_0)e^{\mu y}\,dy \qquad \text{as } K \downarrow K_0.$$

Since we can calculate the integral explicitly this gives an exact asymptotic for c_K which agrees with the one given in Proposition 18 and Theorem 4. Intuitively, as we approach criticality, the single particle invariant measure, $\Pi^{B,K}$, becomes the optimal way for particles to proportion their time in order to maximise the growth of the blue tree, that is, $f^* \to 1$.

4.2.2. *Proof of Proposition* 18. We briefly recall some key quantities. Recall from equation (3.8) that the motion of the backbone particles is given by

$$L^{B,K} = \frac{1}{2}\frac{d^2}{dy^2} - \left(\mu - \frac{p_K'}{p_K}\right)\frac{d}{dy} \qquad \text{on } (0,K),$$

which has invariant density $\Pi^{B,K}$ satisfiying

$$\Pi^{B,K}(y) = \frac{p_K(y)^2 e^{-2\mu y}}{\int_0^K p_K(z)^2 e^{-2\mu z}\,dz}, \qquad y \in (0,K).$$

Recall from (4.11), the *mean growth rate* at position $y \in (0, K)$ is

$$F^{B,K\prime}(1, y) = (m-1)\beta + \frac{F(1 - p_K(y))}{p_K(y)}.$$

Throughout this section, let f^* be such that

$$f^*(y) \propto \frac{\sin(\pi y/K)}{p_K(y)} e^{\mu y}, \qquad y \in (0, K), \tag{4.19}$$

with the normalisation $\int_0^K f^*(y)^2 \Pi^{B,K}(y)\, dy = 1$. Finally we set $\Pi_*^{B,K}(y) := (f^*(y))^2 \Pi^{B,K}(y)$, so that

$$\Pi_*^{B,K}(y) = \frac{2}{K}\sin^2(\pi y/K), \qquad y \in (0, K).$$

Let us now come to the proof of Proposition 18. First, we want to confirm the conjecture that the expected number of particles of $(X, \mathbf{P}^{B,K})$ grows at rate $\lambda(K)$, which initiated the heuristic step (i).

PROPOSITION 23. *For $x \in (0, K)$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}_x^{B,K}(|N_t|) = \lambda(K).$$

PROOF. Let $x \in (0, K)$ and $t \geq 0$. We apply the many-to-one lemma (see, e.g., [23]), then the change of measure in (3.10) together with

$$F^{B,K\prime}(1, y) - \frac{F(1 - p_K(y))}{p_K(y)} = (m-1)\beta,$$

and finally the change of measure using (2.2), to get

$$\begin{aligned}
\mathbf{E}_x^{B,K}(|N_t|) &= \mathbb{E}_x^{B,K}\big(e^{\int_0^t F^{B,K\prime}(1,\xi_s)\,ds}\big)\\
&= e^{(m-1)\beta t}\mathbb{E}_x\left(\frac{p(\xi_t)}{p(x)}\mathbf{1}_{\{\xi_t < T_{(0,K)}\}}\right)\\
&= e^{\lambda(K)t}\mathbb{Q}_x^K\left(\frac{p(\xi_t)}{\sin(\pi\xi_t/K)}e^{-\mu\xi_t}\right)\frac{\sin(\pi x/K)}{p(x)}e^{\mu x}.
\end{aligned}\tag{4.20}$$

Since $(\xi, \mathbb{Q}_x^K)$ is an ergodic diffusion, whose transition density is explicitly known (cf. page 188 of [30]), with invariant distribution $\frac{2}{K}\sin^2(\pi x/K)\,dx$, it is easy to check that

$$\mathbb{Q}_x^K\left(\frac{p_K(\xi_t)}{\sin(\pi\xi_t/K)}e^{-\mu\xi_t}\right) \to \int_0^K p_K(y)e^{-\mu y}\frac{2}{\pi}\sin\left(\frac{\pi y}{K}\right)dy \qquad \text{as } t \to \infty.$$

Thus, after taking logarithms in (4.20), dividing by t and taking $t \to \infty$, the result follows. □

It is worth remarking at this point that, using ideas from [24], the following stronger version of the above proposition can be proved.

PROPOSITION 24. *For* $x \in (0, K)$,

$$\lim_{t\to\infty} \frac{1}{t} \log |N_t| = \lambda(K), \qquad \mathbf{P}_x^{B,K}\text{-}a.s.$$

The details are left to the reader.

We now give a short proof of the inequality in step (ii) of the heuristic by using a lower bound on the growth rate of the expected number of blue particles.

LEMMA 25. *For* $x \in (0, K)$, *we have*

$$\lambda(K) \geq \int_0^K F^{B,K'}(1, y)\Pi^{B,K}(y)\, dy.$$

PROOF. Using the many-to-one lemma (cf., e.g., [22]) and Jensen's inequality, we get for $x \in (0, K)$, $t \geq 0$,

$$\mathbf{E}_x^{B,K}(|N_t|) = \mathbb{E}_x^{B,K}\bigl(e^{\int_0^t F_K^{B'}(1,\xi_s)\, ds}\bigr) \geq \exp\biggl\{\mathbb{E}_x^{B,K}\biggl(\int_0^t F_K^{B'}(1, \xi_s)\, ds\biggr)\biggr\}.$$

Under $\mathbb{P}_x^{B,K}$, ξ has invariant distribution $\Pi^{B,K}(y)\, dy$. Therefore we can apply an ergodic theorem for diffusions [see, e.g., Rogers and Williams [35], Chapter V.53, Theorem (53.1) and Exercise (53.6)] which gives

$$\lim_{t\to\infty} \frac{1}{t}\biggl(\int_0^t F^{B,K'}(1, \xi_s)\, ds\biggr) = \int_0^K F_K^{B'}(1, y)\Pi^{B,K}(y)\, dy, \qquad \mathbb{P}_x^{B,K}\text{-a.s.}$$

Since $F^{B,K'}(1, y)$ is bounded for $y \in (0, K)$ [cf. the argument following (4.26) in the proof of Proposition 18], the bounded convergence theorem gives

$$\lim_{t\to\infty} \mathbb{E}_x^{B,K}\biggl(\frac{1}{t}\int_0^t F^{B,K'}(1, \xi_s)\, ds\biggr) = \int_0^K F_K^{B'}(1, y)\Pi^{B,K}(y)\, dy,$$

which, together with Proposition 23, gives the desired lower bound on $\lambda(K)$. □

Next we give the proof of the upper bound for $\lambda(K)$, again based on the discussion in the heuristic.

LEMMA 26. *For* $K > K_0$, *we have*

$$\lambda(K) \leq \int_0^K F^{B,K'}(1, y)\Pi_*^{B,K}(y)\, dy.$$

PROOF. The upper bound is described in part (iii) of the heuristic. The only part of the explanation there that is not rigorous is the need to verify condition (4.12), that is,

$$\lim_{y\downarrow 0}\bigl(f^*(y)\bigr)' f^*(y)\Pi^{B,K}(y) = \lim_{y\uparrow K}\bigl(f^*(y)\bigr)' f^*(y)\Pi^{B,K}(y) = 0, \tag{4.21}$$

in order to justify the integration by parts that leads to the second equality in (4.18).

To this end, we begin by showing that f^* is uniformly bounded in $(0, K)$. Recall from (4.19) that

$$f^*(y) \propto \frac{\sin(\pi y/K)}{p_K(y)} e^{\mu y}, \qquad y \in (0, K).$$

Since f^* is continuous in $(0, K)$ it is sufficient to show that $\limsup_{x\downarrow 0} f^*(x)$ and $\limsup_{x\uparrow K} f^*(x)$ are bounded.

An application of L'Hôpital's rule gives

$$\lim_{x\downarrow 0} \frac{\sin(\pi x/K)e^{\mu x}}{(\pi/(2K\mu))(1-e^{-2\mu x})} = 1. \tag{4.22}$$

To conclude that $\limsup_{x\downarrow 0} f^*(x) < \infty$, it therefore suffices to show that there exists a constant $c > 0$ such that

$$c(1-e^{2\mu x}) \le p_K(x) \qquad \text{for all } x \text{ sufficiently close to zero.}$$

By Remark 10, $(\prod_{u\in N_t}(1-p_K(x_u(t))), t \ge 0)$ is a P_x^K-martingale, and it follows then by a standard Feynman–Kac argument that $1 - p_K(x)$ satisfies

$$1 - p_K(x) = 1 + \mathbb{E}_x^K \int_0^{T_{(0,K)}} F(1 - p_K(\xi_s))\, ds, \qquad x \in (0, K),$$

where $T_{(0,K)}$ is the first time ξ exists the interval $(0, K)$. To compute the expectation above we use the potential density of ξ (see, e.g., Theorem 8.7 in [28]), and we get

$$\begin{aligned} -p_K(x) &= \mathbb{E}_x^K \int_0^{T_{(0,K)}} F(1-p_K(\xi_s))\, ds \\ &= \frac{1}{\mu}(1-e^{-2\mu x}) \int_0^K F(1-p_K(y)) \frac{(1-e^{-2\mu(K-y)})}{(1-e^{-2\mu K})}\, dy \\ &\quad - \frac{1}{\mu}\int_0^K F(1-p_K(y))(1-e^{-2\mu(x-y)})\, dy. \end{aligned} \tag{4.23}$$

Since $F(s) < 0$ for $0 < s < 1$, the first integral in the last equality on the right-hand side of (4.23) is strictly negative and bounded. Hence we can set

$$c := -\frac{1}{\mu}\int_0^K F(1-p_K(y))\frac{(1-e^{-2\mu(K-y)})}{(1-e^{-2\mu K})}\, dy > 0.$$

The second integral on the right-hand side of (4.23) is nonnegative, for x close to 0, since the term $1 - e^{-2\mu(x-y)}$ is nonpositive for $x \le y$. Therefore, we get

$$p_K(x) \ge c(1-e^{-2\mu x}) \qquad \text{for all } x \text{ sufficiently close to zero,}$$

which, together with (4.22), gives the desired result.

To establish boundedness as x approaches K, we observe that $p_K(x) = \bar{p}_K(K-x)$, where $\bar{p}_K$ denotes the survival probability for a branching diffusion which evolves as under P_x^K but with positive drift μ. Similarly to the previous argument we can then show that there exists a constant $c > 0$ such that $c\bar{p}_K(K-x) \geq \sin(\pi x/K)e^{\mu x}$, for x sufficiently close to K.

We can now show (4.21). Since f^* takes finite values at 0 and K, it suffices to show that $(f^*(y))'\Pi^{B,K}(y)$ evaluated at 0 and K is zero. Differentiating f^* and recalling that $\Pi^{B,K}(y) \propto p_K(y)^2e^{-2\mu y}$ gives

$$\begin{aligned}&(f^*(y))'\Pi^{B,K}(y)\\&\qquad\propto e^{-\mu y}\biggl(\biggl(\mu\sin(\pi y/K)+\frac{\pi}{K}\cos(\pi y/K)\biggr)p_K(y)-\sin(\pi y/K)p_K'(y)\biggr).\end{aligned}$$

Differentiating both sides of equation (4.23) with respect to x, it is easily seen that $p_K'(x)$ is bounded for all $x \in [0, K]$. Therefore $(f^*(y))'\Pi^{B,K}(y)$ is equal to 0 at 0 and K which completes the proof. □

We complete the proof of Proposition 18 by making step (iv) rigorous.

PROOF OF PROPOSITION 18. By Lemmas 25 and 26, we get the following bounds on $\lambda(K)$:

$$\int_0^K F^{B,K'}(1,y)\Pi^{B,K}(y)\,dy \leq \lambda(K) \leq \int_0^K F^{B,K'}(1,y)\Pi_*^{B,K}(y)\,dy. \tag{4.24}$$

Recall that $\Pi^{B,K}$ and $\Pi_*^{B,K}$ were defined as

$$\Pi^{B,K}(y) = \frac{p_K(y)^2e^{-2\mu y}}{\int_0^K p_K(z)^2e^{-2\mu z}\,dz} \quad\text{and}\quad \Pi_*^{B,K}(y) = \frac{2}{K}\sin^2(\pi y/K),$$

for $y \in (0, K)$. By Proposition 17, we have, as $K \downarrow K_0$,

$$\Pi^{B,K}(y) = \frac{p_K(y)^2e^{-2\mu y}}{\int_0^K p_K(z)^2e^{-2\mu z}\,dz} \sim \frac{2}{K_0}\sin^2(\pi y/K) = \Pi_{K_0}^{B,*}(y), \tag{4.25}$$

where we have used that the asymptotics in Proposition 17 hold uniformly to deal with the integral in the denominator. The uniformity in Proposition 17 also ensures that (4.25) holds uniformly for all $y \in (0, K_0)$. Further, we have

$$\lim_{s\uparrow 1}\frac{F(s)}{s(s-1)} = \lim_{s\uparrow 1}\frac{F'(s)}{2s-1} = (m-1)\beta,$$

where we applied L'Hôpital's rule in the first equality above. We apply this for $s = 1 - p_K(y)$ and $K \downarrow K_0$. Then, together with the definition of $F^{B,K'}(1,y)$ in

(4.11) and the asymptotics in Proposition 17, we obtain

$$\begin{aligned} F^{B,K'}(1,y) &= (m-1)\beta + \frac{F(1-p_K(y))}{p_K(y)} \\ &\sim (m-1)\beta - (m-1)\beta\big(1-p_K(y)\big) \qquad (4.26) \\ &\sim (m-1)\beta c_K \sin(\pi y/K_0)e^{\mu y} \qquad \text{as } K \downarrow K_0. \end{aligned}$$

Moreover, we note that for all $y \in (0,K)$,

$$\left|\frac{F(1-p_K(y))}{p_K(y)}\right| = \left|\frac{F(1)-F(1-p_K(y))}{1-(1-p_K(y))}\right| \le \max_{s\in[0,1]} F'(s).$$

Convexity of F yields that the maximum above is attained at either 0 or 1, and we know that $F'(0)$ and $F'(1)$ are both finite. Hence, by (4.26), $|F^{B,K'}(1,y)|$ is bounded in $(0,K)$, and we can therefore appeal to bounded convergence as we take the limit in (4.24). With (4.25) and (4.26) we get

$$\lambda(K) \sim c_K \frac{2(m-1)\beta}{K_0}\int_0^{K_0} \sin^3(\pi y/K_0)e^{\mu y}\,dy \qquad \text{as } K \downarrow K_0.$$

Evaluating the integral gives

$$\lambda(K) \sim c_K \frac{12(m-1)\beta\pi^3(e^{\mu K_0}+1)}{(K_0^2\mu^2+\pi^2)(K_0^2\mu^2+9\pi^2)} \qquad \text{as } K \downarrow K_0.$$

Finally, $\lambda(K) \sim \pi^2(K-K_0)K_0^{-3}$ as $K \downarrow K_0$ which follows from the linearisation

$$\begin{aligned} \lambda(K) &= (m-1)\beta - \frac{\mu^2}{2} - \frac{\pi^2}{2K^2} \\ &= \underbrace{(m-1)\beta - \frac{\mu^2}{2} - \frac{\pi^2}{2K_0^2}}_{=0} + \frac{\pi^2}{2K_0^2} - \frac{\pi^2}{2K^2} \\ &= \frac{\pi^2K^2}{2K_0^2K^2} - \frac{\pi^2}{2K^2} \\ &= \frac{\pi^2[(K-K_0)^2 + 2(K-K_0)K_0 + K_0^2]}{2K_0^2K^2} - \frac{\pi^2}{2K^2} \\ &= \frac{\pi^2(K-K_0)^2}{2K_0^2K^2} + \frac{\pi^2(K-K_0)}{K_0K^2} \end{aligned}$$

and noting that the second term in the last line is the leading order term as $K \downarrow K_0$. This completes the proof. $\square$

5. Proof of Theorem 6.

Recall that $(X, \mathbf{P}^{D,K})$ was defined as the process (X, P^K) conditioned on the event of survival and characterised via the change of measure in (3.7) and Theorem 12.

Fix a $K' > K_0$ and further denote by $N_t|_{(0,K)}$ the set of particles whose ancestors (including themselves) have not exited $(0, K)$ up to time t. Then, for $0 \le K \le K'$, and for $x \in (0, K_0)$ and $A \in \mathcal{F}_t$, we can write

$$\lim_{K \downarrow K_0} \mathbf{P}_x^{D,K}(A) = \lim_{K \downarrow K_0} E_x^{K'}\left(\mathbf{1}_A \frac{1 - \prod_{u \in N_t|_{(0,K)}}(1 - p_K(x_u(t)))}{p_K(x)}\right),$$

since $N_t|_{(0,K)}$ has the same law under P^K and $P^{K'}$. Suppose the particles in $N_t|_{(0,K)}$ are ordered, for instance, according to their spatial positions, and we write $u_1, \dots, u_{N_t|_{(0,K)}}$. We can now expand the term within the expectation on the right-hand side as

$$\begin{aligned} &\frac{1 - \prod_{u \in N_t|_{(0,K)}}(1 - p_K(x_u(t)))}{p_K(x)} \\ &\qquad = \sum_{i=1}^{|N_t|_{(0,K)}|} \frac{p_K(x_{u_i}(t))}{p_K(x)} \prod_{j<i} (1 - p(x_{u_j}(t))). \end{aligned} \tag{5.1}$$

By Lemma 22, for each u_i, we have

$$\lim_{K \downarrow K_0} \frac{p_K(x_{u_i}(t))}{p_K(x)} = \frac{\sin(\pi x_{u_i}(t)/K_0)}{\sin(\pi x/K_0)} e^{\mu(x_{u_i}(t) - x)} \mathbf{1}_{\{x_{u_i}(t) \in (0, K_0)\}}.$$

Further, $|N_t|_{(0,K)}|$ has finite expectation. Therefore, we can apply the dominated convergence theorem twice to get

$$\begin{aligned} \lim_{K \downarrow K_0} \mathbf{P}_x^{D,K}(A) &= E_x^{K'}\left(\mathbf{1}_A \lim_{K \downarrow K_0} \sum_{i=1}^{|N_t|_{(0,K)}|} \frac{p_K(x_{u_i}(t))}{p_K(x)} \prod_{j<i} (1 - p(x_{u_j}(t)))\right) \\ &= E_x^{K_0}\left(\mathbf{1}_A \sum_{i=1}^{|N_t|_{(0,K_0)}|} \frac{\sin(\pi x_{u_i}(t)/K_0) e^{\mu x_{u_i}(t)}}{\sin(\pi x/K_0) e^{\mu x}}\right) \\ &= E_x^{K_0}\left(\mathbf{1}_A \frac{Z^{K_0}(t)}{Z^{K_0}(0)}\right), \end{aligned}$$

where Z^{K_0} is the martingale used in the change of measure in (2.3) in Section 2. The evolution under this change of measure is described in the paragraph following (2.3) and agrees with that of (X^*, Q_x^*) as defined in Definition 5.

6. Super-Brownian motion in a strip.

Recall from (1.1) that the infinitesimal generator L is defined for all functions $u \in C^2(0, K)$ with $u(0+) = u(K-) = 0$. Change the domain to $u \in C^2(0, K)$ with $u''(0+) = u''(K-) = 0$. Then L corresponds to Brownian motion with absorption (instead of killing) at 0 and K. For technical reasons, we consider the absorption case from now on and denote by $\mathcal{P}^K = \{\mathcal{P}_t^K, t \geq 0\}$ the corresponding conservative diffusion semi-group. The results for branching Brownian motion with killing at 0 and K also hold in the absorption setting if we restrict the process with absorption to particles within $(0, K)$, in particular when defining N_t as the number of particles alive at time t who have not been absorbed.

Suppose $Y = (Y_t, t \geq 0)$ is a super-Brownian motion with associated semigroup $\mathcal{P}^K$ and branching mechanism ψ of the form

$$\psi(\lambda) = -a\lambda + b\lambda^2 + \int_0^\infty (e^{-\lambda y} - 1 + \lambda y)\Pi(dy), \qquad \lambda \geq 0,$$

where $a = -\psi'(0+) \in (0, \infty)$, $b > 0$, and Π is a measure concentrated on $(0, \infty)$ satisfying $\int_{(0,\infty)} (x \wedge x^2)\Pi(dx) < \infty$. For an initial configuration $\eta \in \mathcal{M}_f(0, K)$, the space of finite measures supported on $(0, K)$, we denote the law of Y by $\tilde{P}_\eta^K$. The existence of this class of superprocesses follows from [12].

Since $a = -\psi'(0+) > 0$, the function ψ is the branching mechanism of a supercritical continuous-state branching process (CSBP), say Z. We assume henceforth that ψ satisfies the nonexplosion condition $\int_{0+} |\psi(s)|^{-1}\, ds = \infty$ and further that $\psi(\infty) = \infty$. The last condition, together with $\psi'(0+) < 0$, ensures that ψ has a unique positive root λ^*. The parameter λ^* is the survival rate of Z in the sense that the probability of the event of becoming extinguished, namely $\{\lim_{t\to\infty} Z_t = 0\}$, given $Z_0 = x$ is $e^{-\lambda^* x}$, which is strictly positive. We further assume from now on that $\int^{+\infty} (\psi(s))^{-1}\, ds < \infty$, which guarantees that the event of becoming extinguished agrees with the event of extinction, that is, $\{\exists t > 0 : Z_t = 0\}$ a.s. This implies in turn that, for the super-Brownian motion Y, the event of becoming extinguished and the event of extinction agree $\tilde{P}^K$-a.s. We denote the event of extinction of Y by $\mathcal{E} = \{\exists t > 0 : Y_t(0, K) = 0\}$, where $Y_t(0, K)$ is the total mass within $(0, K)$ at time t.

We define the survival rate w_K of the $\tilde{P}^K$-superdiffusion as the function satisfying

$$-\log \tilde{P}_\eta^K(\mathcal{E}) = \langle w_K, \eta\rangle \qquad \text{for } \eta \in \mathcal{M}_f[0, K].$$

It can be shown (see, e.g., [13]) that w_K is a solution to

$$Lu - \psi(u) = 0 \qquad \text{with } u(0) = u(K) = 0. \tag{6.1}$$

Analogously to Proposition 1, and assuming henceforth in addition that the condition $\int_1^\infty x \log x\, \Pi(dx) < \infty$ is satisfied, it is possible to give a necessary and sufficient condition for a positive survival rate. This follows from a spine change of measure argument in the spirit of Section 2 and of Kyprianou et al. [29], now using the $\mathbf{P}_x^K$-martingale

$$\tilde{Z}^K(t) = \int_0^K \sin(\pi x/K) e^{\mu x - \lambda(K)t} Y_t(dx), \qquad t \geq 0, \tag{6.2}$$

where here $\lambda(K) = -\psi'(0+) - \mu^2/2 - \pi^2/2K^2$. One can then show that w_K is positive if $\tilde{Z}^K$ is an $L^1(\mathbf{P}_x^K)$-martingale and the latter holds if and only if $\lambda(K) > 0$ and $\int_1^\infty x \log x \Pi(dx) < \infty$.

Let us now establish the connection between the $\tilde{P}^K$-superdiffusion and a P^K-branching diffusion via the following relations. Set

$$F(s) = \frac{1}{\lambda^*}\psi\bigl(\lambda^*(1-s)\bigr), \qquad s \in (0,1), \tag{6.3}$$

$$\bar{w}_K(x) = \lambda^* p_K(x), \qquad x \in (0,K), \tag{6.4}$$

where p_K is the survival probability of the P^K-branching diffusion with branching mechanism F of (6.3). Bertoin et al. [5] show that (6.3) is the branching mechanism of a Galton–Watson process which they identify as the backbone of the CSBP with branching mechanism ψ.

THEOREM 27. (i) *If* $\mu < \sqrt{-2\psi'(0+)}$ *and* $K > K_0$ *where* $K_0 := \pi(\sqrt{-2\psi'(0+)})^{-1}$, *then* $w_K(x) > 0$ *for all* $x \in (0,K)$; *otherwise* $w_K(x) = 0$ *for all* $x \in [0,K]$.

(ii) *Uniformly for* $x \in (0,K_0)$, *as* $K \downarrow K_0$,

$$w_K(x) \sim \lambda^*(K-K_0)\frac{(K_0^2\mu^2+\pi^2)(K_0^2\mu^2+9\pi^2)}{12\psi'(0+)\pi K_0^3(e^{\mu K_0}+1)}\sin(\pi x/K_0)e^{\mu x}.$$

PROOF. The relation in (6.3) gives $(m-1)\beta = -\psi'(0+)$. Hence, the K_0 and the $\lambda(K)$ defined in this section are the same as the ones in Propositions 1 and 8.

Suppose $\mu < \sqrt{-2\psi'(0+)}$ and $K > K_0$. By Remark 16, p_K is the unique nontrivial solution to $Lu - F(1-u) = 0$ on $(0,K)$ with $u(0) = u(K) = 0$. Using (6.3) it follows then that $\bar{w}_K$ given by (6.4) solves (6.1). We can further deduce from this transformation that (6.1) has a unique nontrivial solution. On the other hand, we know that w_K solves (6.1), and by the spine argument we mentioned after (6.2), we know that w_K is positive within $(0,K)$. By uniqueness, we thus have $\bar{w}_K = w_K$.

Suppose $\mu \geq \sqrt{-2\psi'(0+)}$ or $K \leq K_0$. Then p_K is identically zero, and (2.5) does not have a nontrivial solution. By the transformation in (6.3), the same holds true for (6.1). Since w_K is always a solution to (6.1), it must therefore be equal to zero.

The result is now a consequence of Proposition 1 and Theorem 4. □

Let us outline the backbone decomposition for the $\tilde{P}_\eta^K$-superdiffusion. We begin by studying $(Y, \tilde{P}^K)$ conditioned on becoming extinct.

PROPOSITION 28. *For* $\eta \in \mathcal{M}_f[0,K]$ *and* $t \geq 0$, *we define*

$$\left.\frac{d\tilde{\mathbf{P}}_\eta^{R,K}}{d\tilde{P}_\eta^K}\right|_{\tilde{\mathcal{F}}_t} = \frac{e^{-\langle w_K, Y_t\rangle}}{e^{-\langle w_K, \eta\rangle}}, \tag{6.5}$$

where $(\tilde{\mathcal{F}}_t, t \geq 0)$ *is the natural filtration generated by* $(Y, \tilde{P}^K_\eta)$. *Then* $(Y, \tilde{\mathbf{P}}^{R,K}_\eta)$ *is equal in law to* $(Y, \tilde{P}^K_\eta(\cdot|\mathcal{E}))$. *Further* $(Y, \tilde{\mathbf{P}}^{R,K}_\eta)$ *has spatially dependent branching mechanism*

$$\psi^{R,K}(s,x) = \psi(s + w_K(x)) - \psi(w_K(x)), \qquad s \geq 0 \text{ and } x \in [0, K]$$

and diffusion semigroup $\mathcal{P}^K$.

The proof of Proposition 28 is just a straightforward adaptation of the proof of Lemma 2 in [4] and is thus omitted. We point out that the motion of the $\tilde{\mathbf{P}}^{R,K}$-superdiffusion remains unchanged, and it therefore differs from the motion of the analogous object for the P^K-branching diffusion, that is, the $\mathbf{P}^{R,K}$-branching diffusion in Proposition 11.

Let us introduce some notation before we proceed with the backbone decomposition. Associated to the laws $\{\tilde{\mathbf{P}}^{R,K}_{\delta_x}, x \in [0, K]\}$ is the family of the so-called excursion measures $\{\mathbb{N}^{R,K}_x, x \in [0, K]\}$, defined on the same measurable space, which satisfy

$$\mathbb{N}^{R,K}_x(1 - \exp\{-\langle f, Y_t\rangle\}) = -\log \tilde{\mathbf{E}}^{R,K}_{\delta_x}(\exp\{-\langle f, Y_t\rangle\}),$$

for any $f \in B_+[0, K]$ and $t \geq 0$. These measures are formally defined and studied in Dynkin and Kuznetov [14]. Further, we define

$$\rho_n(dy, x) = \frac{b w_K(x)^2 \delta_0(dy) \mathbf{1}_{\{n=2\}} + w_K(x)^n (y^n/n!) e^{w_K(x) y} \Pi(dy)}{q^{B,K}_n(x) w_K(x) \beta^{B,K}(x)},$$

for $n \geq 2$, $x \in (0, K)$.

DEFINITION 29. Let $K > K_0$ and $\nu \in \mathcal{M}_a(0, K)$. Let $X^B = (X^B_t, t \geq 0)$ be a $\mathbf{P}^{B,K}_\nu$-branching diffusion [which is the backbone of the P^K-branching diffusion with branching mechanism F given by (6.3)].

Dress the trajectories of X^B in such a way that a particle at space–time position $(x, t) \in \mathbb{R}^d \times [0, \infty)$ has an independent $\mathcal{M}_f(0, K)$-valued process grafted on with rate

$$2b\, dt \times d\mathbb{N}^R_x + \int_0^\infty y \exp\{-w_K(x) y\} \Pi(dy) \times dP^{R,K}_{y\delta_x}.$$

Moreover, when an individual in X^B gives birth to $n \geq 2$ offspring, then an additional independent copy of $(Y, \tilde{\mathbf{P}}^{R,K})$ with initial mass $y \geq 0$ is grafted on to the space–time branch point (x, t) with probability $\rho_n(dy, x)$.

For $t \geq 0$, let Y^D_t consists of the total dressed mass present at time t. We define the process $Y^D := (Y^D_t, t \geq 0)$ and denote its law by $\tilde{\mathbf{P}}^{D,K}_\nu$.

THEOREM 30 (Backbone decomposition). *Let* $K > K_0$ *and* $\eta \in \mathcal{M}_f[0, K]$. *Suppose that* ν *is a Poisson random measure on* $(0, K)$ *with intensity* $w_K(x)\eta(dx)$. *Let* $Y^R = (Y^R_t, t \geq 0)$ *be an independent copy of* $(Y, \tilde{\mathbf{P}}^{R,K}_\eta)$, *and let* $(Y^D, \tilde{\mathbf{P}}^{D,K}_\nu)$ *be the process constructed in Definition* 29. *Define the process* $\tilde{Y} = (\tilde{Y}_t, t \geq 0)$ *by*

$$\tilde{Y}_t = Y^R_t + Y^D_t, \qquad t \geq 0,$$

and denote its law by $\tilde{\mathbf{P}}^K_\eta$. *Then the process* $(\tilde{Y}, \tilde{\mathbf{P}}^K_\eta)$ *is Markovian and equal in law to* $(Y, \tilde{P}^K_\eta)$.

The proof of Theorem 30 is a simple adaptation of the proofs of Theorem 1 and 2 in [4] and is therefore omitted.

Conditioning $(Y, \tilde{P}^K_\eta)$ on nonextinction is the same as conditioning the Poisson random measure ν in Theorem 30 on having at least one atom from which a copy of $(Y^D, \tilde{\mathbf{P}}^{D,K})$ is then issued. In principle it should be possible to give a proof analogous to the ones presented in Section 3, using that $(Y, \tilde{P}^K_\eta)$ conditioned on nonextinction arises from a change of measure using the martingale

$$1 - e^{-\langle w_K, Y_t\rangle}, \qquad t \geq 0, \tag{6.6}$$

together with the martingale change of measure in (6.5) which conditions $(Y, \tilde{P}^K_\eta)$ on extinction.

The analogy between the P^K-branching diffusion and the $\tilde{P}^K$-super-diffusion indicates that there is a quasi-stationary limit result equivalent to Theorem 6.

We begin with constructing the limiting process. To this end, define the family of excursion-measures $\{\mathbb{N}^{K_0}_x, x \in [0, K_0]\}$, now associated with the laws $(\tilde{P}^{K_0}_{\delta_x}, x \in [0, K_0])$, satisfying

$$\mathbb{N}^{K_0}_x(1 - \exp\langle f, Y_t\rangle) = -\log \tilde{E}^{K_0}_{\delta_x}(e^{-\langle f, Y_t\rangle}) \qquad \text{for } f \in B^+(0, K), t \geq 0.$$

DEFINITION 31. Let $\eta \in \mathcal{M}_f(0, K)$. Suppose $\xi^* = (\xi^*_t, t \geq 0)$ is a Brownian motion conditioned to stay in $(0, K_0)$ with initial position x distributed according to

$$\frac{\sin(\pi x/K_0)e^{\mu x}}{\int_{(0,K_0)} \sin(\pi z/K_0)e^{\mu z}\eta(dz)}\eta(dx), \qquad x \in (0, K_0).$$

Along the space–time trajectory $\{(\xi^*_s, s) : s \geq 0\}$, we immigrate $M_f(0, K)$-valued processes at rate

$$2b\, ds \times d\mathbb{N}^{K_0}_{\xi^*_s} + \int_0^\infty y\Pi(dy) \times \tilde{P}^{K_0}_{y\delta_{\xi^*_s}}.$$

Then let $Y^* = (Y^*_t, t \geq 0)$ be such that Y^*_t consists of the total immigrated mass present at time t together with the mass present at time t of an independent copy of $(Y, \tilde{\mathbf{P}}^{K_0}_\nu)$ issued at time zero. We denote the law of Y^* by $\tilde{P}^*_\eta$.

The evolution of Y^* under $\tilde{P}^*$ can thus be seen as a path-wise description of Evans's immortal particle picture in [19] for the critical width K_0; for a similar construction of Evans's immortal particle picture see Kyprianou et al. [29].

Further, we note that $(Y^*, \tilde{P}_\eta^{K_0})$ has the same law as Y under the measure which has martingale density $\tilde{Z}^{K_0}(t)$ of (6.2) with respect to $\tilde{P}_\eta^{K_0}$; for similar results see, for instance, Engländer and Kyprianou [16], Kyprianou et al. [29] and Liu et al. [31].

THEOREM 32. *Let $K > K_0$ and $\eta \in \mathcal{M}_f[0, K_0]$. For a fixed time $t \geq 0$, the law of Y_t under the measure $\lim_{K \downarrow K_0} \tilde{P}_\eta^K(\cdot | \lim_{t\to\infty} \|Y_t\| > 0)$ is equal to Y_t^* under $\tilde{P}_\eta^*$.*

To prove Theorem 32 it suffices to show that the $\tilde{P}_\eta^K$-martingale in (6.6) converges to the martingale $\tilde{Z}^{K_0}$ in (6.2). This is a straightforward adaption of the proof of Theorem 6.

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