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SOBOLEV DIFFERENTIABLE STOCHASTIC FLOWS FOR SDES WITH SINGULAR COEFFICIENTS: APPLICATIONS TO THE TRANSPORT EQUATION

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In this paper, we establish the existence of a stochastic flow of Sobolev diffeomorphisms

$$\mathbb{R}^d \ni x \quad \longmapsto \quad \phi_{s,t}(x) \in \mathbb{R}^d, \qquad s,t \in \mathbb{R}$$

for a stochastic differential equation (SDE) of the form

$$dX_t = b(t, X_t)\,dt + dB_t, \qquad s,t \in \mathbb{R}, X_s = x \in \mathbb{R}^d.$$

The above SDE is driven by a *bounded measurable* drift coefficient $b:\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and a d-dimensional Brownian motion B. More specifically, we show that the stochastic flow $\phi_{s,t}(\cdot)$ of the SDE lives in the space $L^2(\Omega; W^{1,p}(\mathbb{R}^d, w))$ for all s,t and all $p \in (1,\infty)$, where $W^{1,p}(\mathbb{R}^d, w)$ denotes a weighted Sobolev space with weight w possessing a pth moment with respect to Lebesgue measure on $\mathbb{R}^d$. From the viewpoint of stochastic (and deterministic) dynamical systems, this is a striking result, since the dominant "culture" in these dynamical systems is that the flow "inherits" its spatial regularity from that of the driving vector fields.

The spatial regularity of the stochastic flow yields existence and uniqueness of a Sobolev differentiable weak solution of the (Stratonovich) stochastic transport equation

$$\begin{cases} d_t u(t,x) + \big(b(t,x) \cdot Du(t,x)\big)\,dt + \displaystyle\sum_{i=1}^d e_i \cdot Du(t,x) \circ dB_t^i = 0, \\ u(0,x) = u_0(x), \end{cases}$$

where b is *bounded and measurable*, u_0 is C_b^1 and $\{e_i\}_{i=1}^d$ a basis for $\mathbb{R}^d$. It is well known that the deterministic counterpart of the above equation does not in general have a solution.

1. Introduction.

An overview. This article offers the following novel contributions to the existing theory of stochastic differential equations (SDEs):

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- Well-posedness of the initial value problem for singular SDEs driven by *bounded measurable* drift vector fields and multidimensional Brownian motion. *No regularity or even continuity hypotheses are imposed on the drift vector fields.* Furthermore, under these hypotheses, we construct a unique stochastic flow of Sobolev diffeomorphisms for the singular SDE.
- The Sobolev flow of the singular SDE is employed as stochastic characteristics in order to generate a unique Sobolev differentiable solution to the stochastic transport equation with a *bounded measurable* drift coefficient. It is well known that the associated *deterministic* transport equation is in general ill-posed even with a differentiable drift; cf. [1, 7, 14].

From a dynamical systems perspective, the above result on singular SDEs is striking since the predominant intuition in the current literature on stochastic (and deterministic) dynamical systems is that the flow "inherits" its spatial regularity from the driving vector fields. Indeed, in the stochastic setting, the flow is in general even a *little rougher* in the space variable than the driving vector fields ([19, 23]). More specifically, it follows from work by Kunita ([19], pp. 178–179) that a SDE with $C^{k,\delta}$ coefficients ($\delta \in (0,1]$) generates a $C^{k,\varepsilon}$ stochastic flow with positive ε *strictly less* than δ. Here, the spatial $C^{k,\delta}$ regularity stands for k-times differentiability with the kth Fréchet derivative δ-Hölder continuous.

In contrast with its deterministic counterpart, the singular stochastic transport equation with multiplicative noise is well-posed due to the regularity of the stochastic characteristics and of their occupation measure. The latter properties have the effect of "smoothing out" the singularities of the drift coefficient. Needless to say, such an effect is not available in the singular deterministic setting.

The approach in the article is probabilistic, employing ideas from the Malliavin calculus coupled with new probabilistic estimates. In particular, the arguments are centered around a key relative compactness criteria for random variables developed by Nualart, Malliavin and Da Prato. See the Appendix. The authors are not aware of any other scenarios whereby the Malliavin calculus is employed to establish *almost sure spatial* regularity of stochastic flows for SDEs.

Background and statement of results. In this article, we analyze the spatial regularity in the initial condition $x \in \mathbb{R}^d$ for strong solutions $X_\cdot^x$ to the d-dimensional SDE

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x})\,du + B_t - B_s, \qquad s, t \in \mathbb{R}. \tag{1}$$

In the above SDE, the drift coefficient $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is only *Borel measurable and bounded*, and the equation is driven by standard Brownian motion $B_\cdot$ in $\mathbb{R}^d$.

More specifically, we construct a two-parameter pathwise Sobolev differentiable stochastic flow

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x) \quad \longmapsto \quad \phi_{s,t}(x) \in \mathbb{R}^d$$

for the SDE (1) such that each flow map

$$\mathbb{R}^d \ni x \quad \longmapsto \quad \phi_{s,t}(x) \in \mathbb{R}^d$$

is a Sobolev diffeomorphism in the sense that

$$\phi_{s,t}(\cdot) \text{ and } \phi_{s,t}^{-1}(\cdot) \in L^2\big(\Omega, W^{1,p}\big(\mathbb{R}^d; w\big)\big) \tag{2}$$

for all $s, t \in \mathbb{R}$ and all $p \in (1, \infty)$. In (2) above, $W^{1,p}(\mathbb{R}^d, w)$ denotes a weighted Sobolev space of mappings $\mathbb{R}^d \to \mathbb{R}^d$ with any measurable weight function $w : \mathbb{R}^d \to [0, \infty)$ satisfying the integrability requirement

$$\int_{\mathbb{R}^d} (1 + |x|^p) w(x)\, dx < \infty. \tag{3}$$

In particular, $\phi_{s,t}(\cdot)$ is locally α-Hölder continuous for all $\alpha < 1$. When the SDE (1) is autonomous, we show further that the stochastic flow corresponds to a Sobolev differentiable perfect cocycle on $\mathbb{R}^d$. For precise statements of the above results, see Theorem 3 and Corollary 5 in the next section.

A central objective of the article is to develop a new approach for constructing a Sobolev differentiable stochastic flow for the SDE (1). Our approach is based on Malliavin calculus ideas coupled with new probabilistic estimates on the spatial weak derivatives of solutions of the SDE. A unique (and striking) feature of these estimates is that they do not depend on the spatial regularity of the drift coefficient b.

The existence of a Sobolev differentiable stochastic flow for the SDE (1) is exploited (Section 3) to obtain a unique weak solution $u(t, x)$ of the (Stratonovich) stochastic transport equation

$$\begin{cases} d_t u(t,x) + \big(b(t,x) \cdot Du(t,x)\big)\, dt + \displaystyle\sum_{i=1}^{d} e_i \cdot Du(t,x) \circ dB_t^i = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{4}$$

when b is just bounded and measurable, $u_0 \in C_b^1(\mathbb{R}^d)$, and $\{e_i\}_{i=1}^d$ a basis for $\mathbb{R}^d$. This result is interesting in view of the fact that the corresponding deterministic transport equation is in general ill-posed; cf. [1, 7]. We also note that our result holds without the existence of the divergence of b, and furthermore, our solutions are spatially (and also Malliavin) Sobolev differentiable (cf. [14]).

SDEs with discontinuous coefficients and driven by Brownian motion (or more general noise) have been an important area of study in stochastic analysis and other related branches of mathematics. Important applications of this class of SDEs pertain to the modeling of the dynamics of interacting particles in statistical mechanics and the description of a variety of other random phenomena in areas such as biology or engineering. See, for example, [27] or [17] and the references therein.

Using estimates of solutions of parabolic PDEs and the Yamada–Watanabe principle, the existence of a global unique strong solution to the SDE (1) was first established by Zvonkin [32] in the one-dimensional case, when b is bounded and measurable. The latter work is a significant development in the theory of SDEs. Subsequently, the result was generalized by Veretennikov [30] to the multidimensional case. More recently, Krylov and Röckner employed local integrability criteria on the drift coefficient b to obtain unique strong solutions of the SDE (1) by using an argument of Portenko [27]. An alternative approach, which does not rely on a pathwise uniqueness argument and which also yields the Malliavin differentiability of solutions to (1) was recently developed in [22] and [21]. We also refer to the recent article [5] for an extension of the previous results to a Hilbert space setting. In [5], the authors employ techniques based on solutions of infinite-dimensional Kolmogorov equations.

Another important issue in the study of SDEs with (bounded) measurable coefficients is the regularity of their solutions with respect to the initial data and the existence of stochastic flows. See [19, 23] for more information on the existence and regularity of stochastic flows for SDEs, and [24, 25] in the case of stochastic differential systems with memory.

Using the method of stochastic characteristics, stochastic flows may be employed to prove uniqueness of solutions of stochastic transport equations under weak regularity hypotheses on the drift coefficient b. See, for example, [14], where the authors use estimates of solutions of backward Kolmogorov equations to show the existence of a stochastic flow of diffeomorphisms with α'-Hölder continuous derivatives for $\alpha' < \alpha$, where $b \in C([0,1]; C_b^\alpha(\mathbb{R}^d))$, and $C_b^\alpha(\mathbb{R}^d)$ is the space of bounded α-Hölder continuous functions. A similar result also holds true, when $b \in L^q([0,1]; L^p(\mathbb{R}^d))$ for p, q such that $p \geq 2, q > 2, \frac{d}{p} + \frac{2}{q} < 1$. See [12]. Here, the authors construct, for any $\alpha \in (0,1)$, a stochastic flow of α-Hölder continuous homeomorphisms for the SDE (1). Furthermore, it is shown in [12] that the solution map

$$\mathbb{R}^d \ni x \quad \longmapsto \quad X_\cdot^x \in L^p\big([0,1] \times \Omega; \mathbb{R}^d\big)$$

of the SDE (1) is differentiable in the $L^p(\Omega)$-sense for every $p \geq 2$.

The approach used in [12] is based on a Zvonkin-type transformation [32] and estimates of solutions of an associated backward parabolic PDE. We also mention the recent related works [9, 10] and [2]. For an overview of this topic, the reader may also consult the book [13].[2] In this connection, it should be noted that our method for constructing a stochastic flow for the SDE (1) is heavily dependent on Malliavin calculus ideas together with some difficult probabilistic estimates (cf. [21]).

Our paper is organized as follows: in Section 2 we introduce basic definitions and notations and provide some auxiliary results that are needed to prove the existence of a Sobolev differentiable stochastic flow for the SDE (1). See Theorem 3

[2] After completing the preparation of this article, personal communication with Flandoli indicated work in preparation with Fedrizzi [11] on similar issues regarding the regularity of stochastic flows for SDEs, using a different approach.

and Corollary 5 in Section 2. We also briefly discuss a specific extension of this result to SDEs with multiplicative noise. In Section 3, we give an application of our approach to the construction of a unique Sobolev differentiable solution to the (Stratonovich) stochastic transport equation (4). The Appendix specifies the relative compactness criterion of DaPrato, Malliavin and Nualart that is central to the construction of the Sobolev flow [6].

2. Existence of a Sobolev differentiable stochastic flow. Throughout this paper, we denote by $B_t = (B_t^{(1)}, \dots, B_t^{(d)})$, $t \in \mathbb{R}$, d-dimensional Brownian motion on the complete Wiener space $(\Omega, \mathcal{F}, \mu)$ where $\Omega := C(\mathbb{R}; \mathbb{R}^d)$ is given the compact open topology and $\mathcal{F}$ is its μ-completed Borel σ-field with respect to Wiener measure μ.

In order to describe the cocycle associated with the stochastic flow of our SDE, we define the μ-preserving (ergodic) Wiener shift $\theta(t, \cdot) : \Omega \to \Omega$ by

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \qquad \omega \in \Omega, t, s \in \mathbb{R}.$$

The Brownian motion is then a *perfect helix* with respect to θ: that is,

$$B_{t_1+t_2}(\omega) - B_{t_1}(\omega) = B_{t_2}\big(\theta(t_1, \omega)\big)$$

for all $t_1, t_2 \in \mathbb{R}$ and all $\omega \in \Omega$. The above helix property is a convenient pathwise expression of the fact that Brownian motion B has stationary ergodic increments.

Our main focus of study in this section is the d-dimensional SDE

$$X_t^{s,x} = x + \int_s^t b\big(u, X_u^{s,x}\big)\, du + B_t - B_s, \qquad s, t \in \mathbb{R}, x \in \mathbb{R}^d, \tag{5}$$

where the drift coefficient $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel-measurable function.

It is known that the above SDE has a unique strong global solution $X_\cdot^{s,x}$ for each $x \in \mathbb{R}^d$ ([30] or [21, 22]).

Here, we will establish the existence of a *Sobolev-differentiable* stochastic flow of diffeomorphisms for the SDE (5).

DEFINITION 1. A map $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x, \omega) \longmapsto \phi_{s,t}(x, \omega) \in \mathbb{R}^d$ is a *stochastic flow of homeomorphisms* for the SDE (5) if there exists a universal set $\Omega^* \in \mathcal{F}$ of full Wiener measure such that for all $\omega \in \Omega^*$, the following statements are true:

(i) For any $x \in \mathbb{R}^d$, the process $\phi_{s,t}(x, \omega)$, $s, t \in \mathbb{R}$, is a strong global solution to the SDE (5).

(ii) $\phi_{s,t}(x, \omega)$ is continuous in $(s, t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$.

(iii) $\phi_{s,t}(\cdot, \omega) = \phi_{u,t}(\cdot, \omega) \circ \phi_{s,u}(\cdot, \omega)$ for all $s, u, t \in \mathbb{R}$.

(iv) $\phi_{s,s}(x, \omega) = x$ for all $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$.

(v) $\phi_{s,t}(\cdot, \omega) : \mathbb{R}^d \to \mathbb{R}^d$ are homeomorphisms for all $s, t \in \mathbb{R}$.

A stochastic flow $\phi_{s,t}(\cdot,\omega)$ of homeomorphisms is said to be *Sobolev-differentiable* if for all $s,t\in\mathbb{R}$, the maps $\phi_{s,t}(\cdot,\omega)$ and $\phi_{s,t}^{-1}(\cdot,\omega)$ are Sobolev-differentiable in the sense described below.

From now on, we use $|\cdot|$ to denote the norm of a vector in $\mathbb{R}^d$ or a matrix in $\mathbb{R}^{d\times d}$.

In order to prove the existence of a Sobolev differentiable flow for the SDE (5), we need to introduce a suitable class of weighted Sobolev spaces. Fix $p\in(1,\infty)$ and let $w:\mathbb{R}^d\to(0,\infty)$ be a Borel-measurable function satisfying

$$\int_{\mathbb{R}^d}(1+|x|^p)w(x)\,dx<\infty. \tag{6}$$

Let $L^p(\mathbb{R}^d,w)$ denote the Banach space of all Borel-measurable functions $u=(u_1,\dots,u_d):\mathbb{R}^d\to\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d}|u(x)|^p w(x)\,dx<\infty \tag{7}$$

and equipped with the norm

$$\|u\|_{L^p(\mathbb{R}^d,w)}:=\left[\int_{\mathbb{R}^d}|u(x)|^p w(x)\,dx\right]^{1/p}.$$

Furthermore, denote by $W^{1,p}(\mathbb{R}^d,w)$ the linear space of functions $u\in L^p(\mathbb{R}^d,w)$ with weak partial derivatives $D_ju\in L^p(\mathbb{R}^d,w)$ for $j=1,\dots,d$. We equip this space with the complete norm

$$\|u\|_{1,p,w}:=\|u\|_{L^p(\mathbb{R}^d,w)}+\sum_{i,j=1}^{d}\|D_ju_i\|_{L^p(\mathbb{R}^d,w)}. \tag{8}$$

We will show that the strong solution $X_t^{s,\cdot}$ of the SDE (5) is in $L^2(\Omega,L^p(\mathbb{R}^d,w))$ when $p\in(1,\infty)$ (see Corollary 14). In fact, the SDE (5) implies the following estimate:

$$|X_t^{s,x}|^p\le c_p\big(|x|^p+|t-s|^p\|b\|_\infty^p+|B_t-B_s|^p\big)$$

for all $s,t\in\mathbb{R}$, $x\in\mathbb{R}^d$.

On the other hand, it is easy to see that the solutions $X_t^{s,\cdot}$ of SDE (5) are in general not in $L^p(\mathbb{R}^d,dx)$ with respect to Lebesgue measure dx on $\mathbb{R}^d$: just consider the special trivial case $b\equiv 0$. This implies that solutions of the SDE (5) (if they exist) may not belong to the Sobolev space $W^{1,p}(\mathbb{R}^d,dx)$, $p\in(1,\infty)$. However, we will show that such solutions do indeed belong to the weighted Sobolev spaces $W^{1,p}(\mathbb{R}^d,w)$ for $p\in(1,\infty)$.

REMARK 2. (i) Let $w:\mathbb{R}^d\to(0,\infty)$ be a weight function in Muckenhoupt's A_p-class ($1<p<\infty$), that is a locally (Lebesgue) integrable function on $\mathbb{R}^d$ such that

$$\sup\left(\frac{1}{\lambda_d(B)}\int_B w(x)\,dx\right)\left(\frac{1}{\lambda_d(B)}\int_B (w(x))^{1/(1-p)}\,dx\right)^{p-1}=:c_{w,p}<\infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^d$ and λ_d is Lebesgue measure on $\mathbb{R}^d$. For example, the function $w(x) = |x|^\gamma$ is an A_p-weight iff $-d < \gamma < d(p-1)$. Other examples of weights are given by positive superharmonic functions. See, for example, [16] and [18] and the references therein. Denote by $H^{1,p}(\mathbb{R}^d, w)$ the completion of $C^\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{1,p,w}$ in (8). If w is a A_p-weight, then we have

$$W^{1,p}(\mathbb{R}^d, w) = H^{1,p}(\mathbb{R}^d, w)$$

for all $1 < p < \infty$; see, for example, [16].

(ii) Let $p_0 = \inf\{q > 1 : w \text{ is a } A_q\text{-weight}\}$ and let $u \in W^{1,p}(\mathbb{R}^d, w)$. If $p_0 < p/d$, then u is locally Hölder continuous with any exponent α such that $0 < \alpha < 1 - dp_0/p$.

We now state our main result in this section which gives the existence of a Sobolev differentiable stochastic flow for the SDE (5).

THEOREM 3. *In the SDE* (5), *assume that the drift coefficient b is Borel-measurable and bounded. Then the SDE* (5) *has a Sobolev differentiable stochastic flow $\phi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$, $s, t \in \mathbb{R}$: that is,*

$$\phi_{s,t}(\cdot) \text{ and } \phi_{s,t}^{-1}(\cdot) \in L^2(\Omega, W^{1,p}(\mathbb{R}^d, w))$$

for all $s, t \in \mathbb{R}$ and all $p \in (1, \infty)$.

REMARK 4. If w is a A_p-weight, then it follows from Remark 2(ii) that a version of $\phi_{s,t}(\cdot)$ is locally Hölder continuous for all $0 < \alpha < 1$ and all s, t.

The following corollary is a consequence of Theorem 3 and the helix property of the Brownian motion.

COROLLARY 5. *Consider the autonomous SDE*

$$X_t^{s,x} = x + \int_s^t b(X_u^{s,x})\, du + B_t - B_s, \qquad s, t \in \mathbb{R} \tag{9}$$

with bounded Borel-measurable drift $b : \mathbb{R}^d \to \mathbb{R}^d$. Then the stochastic flow of the SDE (9) *has a version which generates a perfect Sobolev-differentiable cocycle $(\phi_{0,t}, \theta(t, \cdot))$ where $\theta(t, \cdot) : \Omega \to \Omega$ is the μ-preserving Wiener shift. More specifically, the following perfect cocycle property holds for all $\omega \in \Omega$ and all $t_1, t_2 \in \mathbb{R}$:*

$$\phi_{0,t_1+t_2}(\cdot, \omega) = \phi_{0,t_2}(\cdot, \theta(t_1, \omega)) \circ \phi_{0,t_1}(\cdot, \omega).$$

We will prove Theorem 3 through a sequence of lemmas and propositions. We begin by stating our main proposition.

PROPOSITION 6. *Let $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be bounded and measurable. Let U be an open and bounded subset of $\mathbb{R}^d$. For each $t \in \mathbb{R}$ and $p > 1$, we have*

$$X_t^{\cdot} \in L^2(\Omega; W^{1,p}(U)).$$

We will prove Proposition 6 using two steps. In the *first step*, we show that for a bounded smooth function $b : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ with compact support, it is possible to estimate the norm of $X_t^{\cdot}$ in $L^2(\Omega, W^{1,p}(U))$ independently of the size of the spatial Jacobian b' of b, with the estimate depending only on $\|b\|_\infty$.

In the *second step*, we will approximate our bounded measurable coefficient b by a sequence $\{b_n\}_{n=1}^{\infty}$ of smooth compactly supported functions as in step 1. We then show that the corresponding sequence $X_t^{n,\cdot}$ of solutions is relatively compact in $L^2(\Omega)$ when integrated against a test function on $\mathbb{R}^d$. By step 1, we use weak compactness of the above sequence in $L^2(\Omega, W^{1,p}(U))$ to conclude that the limit point $X_t^{\cdot}$ of the above sequence must also lie in this space.

We now turn to the first step of our procedure. Note that if b is a compactly supported smooth function, the corresponding solution of the SDE (1) is (strongly) differentiable with respect to x, and the first-order spatial Jacobian $\frac{\partial}{\partial x} X_t^x$ satisfies the linearized random ODE

$$\begin{cases} d \dfrac{\partial}{\partial x} X_t^x = b'(t, X_t^x) \dfrac{\partial}{\partial x} X_t^x \, dt, \\ \dfrac{\partial}{\partial x} X_0^x = \mathcal{I}_d. \end{cases} \tag{10}$$

In the above equation and throughout this section, $\mathcal{I}_d$ is the $d \times d$ identity matrix and $b'(t, x) := (\frac{\partial}{\partial x_i} b^{(j)}(t, x))_{1 \le i, j \le d}$ denotes the spatial Jacobian derivative of b.

A key estimate in the first step of the argument is provided by the following proposition.

PROPOSITION 7. *Assume that b is a smooth function with compact support. Then for any $p \in [1, \infty)$ and $t \in \mathbb{R}$, we have the following estimate for the solution of the linearized equation* (10):

$$\sup_{x \in \mathbb{R}^d} E\left[\left|\frac{\partial}{\partial x} X_t^x\right|^p\right] \le C_{d,p}(\|b\|_\infty),$$

where $C_{d,p}$ is an increasing continuous function depending only on d and p.

The proof of Proposition 7 relies on the following sequence of lemmas which provide estimates on expressions depending on the Gaussian distribution and its derivatives. To this end, we define $P(t, z) := (2\pi t)^{d/2} e^{-|z|^2/2t}$, $t > 0$, where $|z|$ is the Euclidean norm of a vector $z \in \mathbb{R}^d$.

LEMMA 8. *Let $\phi, h : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions such that $|\phi(s,z)| \le e^{-|z|^2/3s}$ and $\|h\|_\infty \le 1$. Also let $\alpha, \beta \in \{0,1\}^d$ be multiindices such that $|\alpha| = |\beta| = 1$. Then there exists a universal constant C (independent of ϕ, h, α and β) such that*

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(s,z) h(t,y) D^\alpha D^\beta P(t-s, y-z)\, dy\, dz\, ds\, dt \right| \le C.$$

Furthermore, there is a universal positive constant (also denoted by) C such that for measurable functions g and h bounded by 1, we have

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,z) P(s,z) h(t,y) D^\alpha D^\beta P(t-s, y-z)\, dy\, dz\, ds\, dt \right| \le C$$

and

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,z) D^\gamma P(s,z) h(t,y) D^\alpha D^\beta P(t-s, y-z)\, dy\, dz\, ds\, dt \right| \le C.$$

PROOF. We will only give a proof of the first estimate in the lemma. The proofs of the second and third estimates are left to the reader.

Denote the first integral in the lemma by I. Let $l, m \in \mathbb{Z}^d$ and define $[l, l+1) := [l^{(1)}, l^{(1)} + 1) \times \cdots \times [l^{(d)}, l^{(d)} + 1)$ and similarly for $[m, m+1)$. Truncate the functions ϕ, h by setting $\phi_l(s,z) := \phi(s,z) 1_{[l,l+1)}(z)$ and $h_m(t,y) := h(t,y) 1_{[m,m+1)}(y)$.

In the first integral, we replace ϕ, h by ϕ_l, h_m, respectively, and thus define

$$I_{l,m} := \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_l(s,z) h_m(t,y) D^\alpha D^\beta P(t-s, y-z)\, dy\, dz\, ds\, dt.$$

Therefore, we can write $I = \sum_{l,m \in \mathbb{Z}^d} I_{l,m}$. Below, we let C be a generic constant that may vary from line to line.

Assume $\|l - m\|_\infty := \max_i |l^{(i)} - m^{(i)}| \ge 2$. For $z \in [l, l+1)$ and $y \in [m, m+1)$ we have $|z - y| \ge \|l - m\|_\infty - 1$. If $\alpha \ne \beta$, we have that

$$D^\alpha D^\beta P(t-s, z-y) = \frac{(z^{(i)} - y^{(i)})(z^{(j)} - y^{(j)})}{(t-s)^2} P(t-s, y-z)$$

for a suitable choice of i, j. Then we can find C such that

$$\left| D^\alpha D^\beta P(t-s, z-y) \right| \le C e^{-(\|l-m\|_\infty - 2)^2/4}.$$

If $\alpha = \beta$, we have

$$(D^\alpha)^2 P(t-s, y-z) = \left(\frac{(y^{(i)} - z^{(i)})^2}{t-s} - 1 \right) \frac{P(t-s, y-z)}{t-s}$$

and similarly we find C such that

$$\left| (D^\alpha)^2 P(t-s, y-z) \right| \le C e^{-(\|l-m\|_\infty - 2)^2/4}.$$

In both cases we have $|I_{l,m}| \le Ce^{-|l|^2/8}e^{-(\|l-m\|_\infty-2)^2/4}$ and it follows that

$$\sum_{\|l-m\|_\infty \ge 2} |I_{l,m}| \le C.$$

Assume $\|l-m\|_\infty \le 1$ and let $\hat{\phi}_l(s,u)$ and $\hat{h}_m(t,u)$ be the Fourier transforms in the second variable, defined by

$$\hat{h}_m(t,u) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} h(t,x)e^{-i(u,x)}\,dx$$

and similarly for $\hat{\phi}_l(s,u)$. By the Plancherel theorem we have that

$$\int_{\mathbb{R}^d} \hat{\phi}_l(s,u)^2\,du = \int_{\mathbb{R}^d} \phi_l(s,z)^2\,dz \le Ce^{-|l|^2/6}$$

for all $s \in [0,1]$ and

$$\int_{\mathbb{R}^d} \hat{h}_m(t,u)^2\,du = \int_{\mathbb{R}^d} h_m(t,y)^2\,dy \le 1.$$

We can write

$$I_{l,m} = \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s,u)\hat{h}_m(t,-u)u^{(i)}u^{(j)}(t-s)e^{-(t-s)|u|^2/2}\,du\,ds\,dt. \tag{11}$$

To see this, start with the right-hand side. Then we have by Fubini's theorem

$$\begin{aligned}
&\int_{\mathbb{R}^d} \hat{h}_m(t,-u)\hat{\phi}_l(s,u)u^i u^j (t-s)e^{-(t-s)|u|^2/2}\,du \\
&\quad = (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_m(t,x)e^{i(u,x)}\phi_l(s,y)e^{-i(u,y)}u^i u^j (t-s) \\
&\qquad\qquad \times e^{-(t-s)|u|^2/2}\,du\,dx\,dy \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h_m(t,x)\phi_l(s,y)(t-s) \\
&\qquad\qquad \times \left[(2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(u,x-y)}u^i u^j e^{-(t-s)|u|^2/2}\,du\right]dx\,dy.
\end{aligned}$$

Now look at the expression in the square brackets. Substitute $v = \sqrt{t-s}u$ to get

$$\begin{aligned}
&(2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(u,x-y)}u^i u^j e^{-(t-s)|u|^2/2}\,du \\
&\quad = (2\pi)^{-d}(t-s)^{-d/2}\int_{\mathbb{R}^d} e^{i(v/\sqrt{t-s},x-y)}\frac{v^i}{\sqrt{t-s}}\frac{v^j}{\sqrt{t-s}}e^{-|v|^2/2}\,dv \\
&\quad = (2\pi)^{-d}(t-s)^{-d/2}(t-s)^{-1}\int_{\mathbb{R}^d} e^{i(v,(x-y)/\sqrt{t-s})}v^i v^j e^{-|v|^2/2}\,dv.
\end{aligned}$$

Now put $f(v) = e^{-|v|^2/2}$ and $p(v) = v^{(i)}v^{(j)}$. From properties of the Fourier transform, we know that $\widehat{pf} = D^\alpha D^\beta \hat{f}$ and $\hat{f} = f$. This gives that the above expression is equal to

$$(2\pi)^{-d/2}(t-s)^{-d/2}(t-s)^{-1}D^\alpha D^\beta f\left(\frac{x-y}{\sqrt{t-s}}\right)$$
$$= (t-s)^{-1}D^\alpha D^\beta P(t-s, x-y).$$

This proves equation (11).

Applying the inequality $ab \le \frac{1}{2}a^2c + \frac{1}{2}b^2c^{-1}$ to (13) with $a = \hat{\phi}_l(s,u)u^{(i)}$, $b = \hat{h}_m(t,-u)u^{(j)}$ and $c = e^{|l|^2/12}$ we get

$$\begin{aligned}|I_{l,m}| &\le \frac{1}{2}\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{\phi}_l(s,u)^2(u^{(i)})^2 e^{|l|^2/12}e^{-(t-s)|u|^2/2}\,du\,ds\,dt\\ &\quad+\frac{1}{2}\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{h}_m(t,-u)^2(u^{(j)})^2 e^{-|l|^2/12}e^{-(t-s)|u|^2/2}\,du\,ds\,dt\\ &\le \frac{1}{2}\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{\phi}_l(s,u)^2|u|^2 e^{|l|^2/12}e^{-(t-s)|u|^2/2}\,du\,ds\,dt\\ &\quad+\frac{1}{2}\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{h}_m(t,-u)^2|u|^2 e^{-|l|^2/12}e^{-(t-s)|u|^2/2}\,du\,ds\,dt.\end{aligned}$$

For the first term, integrate first with respect to t in order to get

$$\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{\phi}_l(s,u)^2|u|^2 e^{|l|^2/12}e^{-(t-s)|u|^2/2}\,du\,ds\,dt \le Ce^{-|l|^2/12}$$

and for the second term, integrate with respect to s first to get

$$\int_{1/2}^1\int_{t/2}^t\int_{\mathbb{R}^d}\hat{h}_m(t,-u)^2|u|^2 e^{-|l|^2/12}e^{-(t-s)|u|^2/2}\,du\,ds\,dt \le Ce^{-|l|^2/12},$$

which gives $|I_{l,m}| \le Ce^{-|l|^2/12}$, and hence

$$\sum_{\|l-m\|_\infty \le 1}|I_{l,m}| \le C.$$

□

Using the previous lemma we can show the following:

LEMMA 9. *Let $g, h : [0,1]\times\mathbb{R}^d \to \mathbb{R}$ be Borel-measurable and bounded by* 1 *and let $r \ge 0$. As before we let α, β, γ be multiindices with length* 1. *Then there exists a universal constant C such that*

$$\begin{aligned}&\left|\int_{t_0}^t\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(t_2,z)P(t_2-t_0,z)h(t_1,y)\right.\\ &\qquad\left.\times D^\alpha D^\beta P(t_1-t_2,y-z)(t-t_1)^r\,dy\,dz\,dt_2\,dt_1\right|\\ &\quad\le C(1+r)^{-1}(t-t_0)^{r+1}\end{aligned}$$

and

$$\begin{aligned}\bigg|\int_{t_0}^{t}\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & g(t_2,z)D^{\gamma}P(t_2-t_0,z)h(t_1,y)\\ & \times D^{\alpha}D^{\beta}P(t_1-t_2,y-z)(t-t_1)^r\,dy\,dz\,dt_2\,dt_1\bigg|\\ & \le C(1+r)^{-1/2}(t-t_0)^{r+1/2}.\end{aligned}$$

PROOF. We begin by proving the first estimate in the lemma for $t=1$, $t_0=0$. The following estimate holds for each integer $k\ge 0$:

$$\begin{aligned}\bigg|\int_{2^{-k-1}}^{2^{-k}}\int_{t/2}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & g(s,z)P(s,z)h(t,y)\\ & \times D^{\alpha}D^{\beta}P(t-s,y-z)(1-t)^r\,dy\,dz\,ds\,dt\bigg|\\ & \le C\big(1-2^{-k-1}\big)^r 2^{-k}.\end{aligned}$$

To see this, use the fact $P(at,z)=a^{-d/2}P(t,a^{-1/2}z)$ and make the following substitutions in the second estimate in Lemma 8: $t':=2^k t$ and $s':=2^k s$, $z':=2^{k/2}z$ and $y':=2^{k/2}y$, $\tilde h(t,y):=\frac{(1-t)^r}{(1-2^{-k-1})^r}h(t,y)$.

Summing the above inequalities over k gives

$$\begin{aligned}\bigg|\int_0^1\int_{t/2}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & g(s,z)P(s,z)h(t,y)D^{\alpha}D^{\beta}P(t-s,y-z)(1-t)^r\,dy\,dz\,ds\,dt\bigg|\\ & \le C(1+r)^{-1}.\end{aligned}$$

Moreover, it is easy to see that

$$\begin{aligned}\bigg|\int_0^1\int_0^{t/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & g(s,z)P(s,z)h(t,y)D^{\alpha}D^{\beta}P(t-s,y-z)(1-t)^r\,dy\,dz\,ds\,dt\bigg|\\ & \le C\int_0^1\int_0^{t/2}(t-s)^{-1}(1-t)^r\,ds\,dt\le C(1+r)^{-1}\end{aligned}$$

and combining these bounds gives the first assertion of the lemma for $t=1$, $t_0=0$. For general t and t_0 use the change of variables $t_1':=\frac{t_1-t_0}{t-t_0}$, $t_2':=\frac{t_2-t_0}{t-t_0}$, $y':=(t-t_0)^{-1/2}y$ and $z':=(t-t_0)^{-1/2}z$.

The second assertion of the lemma is proved similarly. $\square$

We now turn to the following key estimate (cf. [4], Proposition 2.2):

LEMMA 10. *Let B be a d-dimensional Brownian Motion starting from the origin and $b_1,\dots,b_n$ be compactly supported continuously differentiable functions $b_i:[0,1]\times\mathbb{R}^d\to\mathbb{R}$ for $i=1,2,\dots,n$. Let $\alpha_i\in\{0,1\}^d$ be a multiindex such that*

$|\alpha_i| = 1$ *for* $i = 1, 2, \ldots, n$. *Then there exists a universal constant* C (*independent of* $\{b_i\}_i$, n, *and* $\{\alpha_i\}_i$) *such that*

$$\left| E\left[\int_{t_0<t_1<\cdots<t_n<t} \left(\prod_{i=1}^{n} D^{\alpha_i} b_i(t_i, x + B_{t_i}) \right) dt_1 \cdots dt_n \right] \right| \le \frac{C^n \prod_{i=1}^{n} \|b_i\|_\infty (t - t_0)^{n/2}}{\Gamma((n/2) + 1)}, \tag{12}$$

where Γ *is the Gamma-function and* $x \in \mathbb{R}^d$. *Here*, D^{α_i} *denotes the partial derivative with respect to the* j'*th space variable*, *where* j *is the position of the* 1 *in* α_i.

PROOF. Without loss of generality, assume that $\|b_i\|_\infty \le 1$ for $i = 1, 2, \ldots, n$. Using the Gaussian density, we write the left-hand side of the estimate (12) in the form

$$\left| \int_{t_0<t_1<\cdots<t_n<t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^{n} D^{\alpha_i} b_i(t_i, x + z_i) \times P(t_i - t_{i-1}, z_i - z_{i-1})\, dz_1 \cdots dz_n\, dt_1 \cdots dt_n \right|.$$

Introduce the notation

$$J_n^\alpha(t_0, t, z_0) = \int_{t_0<t_1<\cdots<t_n<t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^{n} D^{\alpha_i} b_i(t_i, x + z_i) \times P(t_i - t_{i-1}, z_i - z_{i-1})\, dz_1 \cdots dz_n\, dt_1 \cdots dt_n,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^{nd}$. We shall show that $|J_n^\alpha(t_0, t, z_0)| \le C^n (t - t_0)^{n/2} / \Gamma(n/2 + 1)$, thus proving the proposition.

To do this, we will use integration by parts to shift the derivatives from the b_i's onto the Gaussian kernel. This will be done by introducing the alphabet

$$\mathcal{A}(\alpha) = \{P, D^{\alpha_1} P, \ldots, D^{\alpha_n} P, D^{\alpha_1} D^{\alpha_2} P, \ldots, D^{\alpha_{n-1}} D^{\alpha_n} P\},$$

where D^{α_i}, $D^{\alpha_i} D^{\alpha_{i+1}}$ denotes the derivatives in z of $P(t, z)$.

Take a string $S = S_1 \cdots S_n$ in $\mathcal{A}(\alpha)$ and define

$$I_S^\alpha(t_0, t, z_0) = \int_{t_0<\cdots<t_n<t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^{n} b_i(t_i, x + z_i) \times S_i(t_i - t_{i-1}, z_i - z_{i-1})\, dz_1 \cdots dz_n\, dt_1 \cdots dt_n.$$

We will need only a special type of strings: say that a string is *allowed* if, when all the $D^{\alpha_i}P$'s are removed from the string, a string of the form $P \cdot D^{\alpha_s} D^{\alpha_{s+1}} P \cdot P \cdot D^{\alpha_{s+1}} D^{\alpha_{s+2}} P \cdots P \cdot D^{\alpha_r} D^{\alpha_{r+1}} P$ for $s \geq 1$, $r \leq n-1$ remains. Also, we will require that the first derivatives $D^{\alpha_i}P$ are written in an increasing order with respect to i.

We now claim that

$$J_n^{\alpha}(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} \varepsilon_j I_{S^j}^{\alpha}(t_0, t, z_0),$$

where each ε_j is either -1 or 1 and each S^j is an allowed string in $\mathcal{A}(\alpha)$. To see this, we proceed by induction on $n \geq 1$.

The claim obviously holds for $n = 1$. Assume that it holds for $n \geq 1$, and let b_0 be another function satisfying the requirements of the lemma. Likewise with α_0. Then

$$\begin{aligned} &J_{n+1}^{(\alpha_0,\alpha)}(t_0, t, z_0) \\ &\quad = \int_{t_0}^{t} \int_{\mathbb{R}^d} D^{\alpha_0} b_0(t_1, x + z_1) P(t_1 - t_0, z_1 - z_0) J_n^{\alpha}(t_1, t, z_1)\, dz_1\, dt_1 \\ &\quad = -\int_{t_0}^{t} \int_{\mathbb{R}^d} b_0(t_1, x + z_1) D^{\alpha_0} P(t_1 - t_0, z_1 - z_0) J_n^{\alpha}(t_1, t, z_1)\, dz_1\, dt_1 \\ &\qquad - \int_{t_0}^{t} \int_{\mathbb{R}^d} b_0(t_1, x + z_1) P(t_1 - t_0, z_1 - z_0) D^{\alpha_0} J_n^{\alpha}(t_1, t, z_1)\, dz_1\, dt_1. \end{aligned}$$

Notice that

$$D^{\alpha_0} I_S^{\alpha}(t_1, t, z_1) = -I_{\widetilde{S}}^{(\alpha_0,\alpha)}(t_1, t, z_1),$$

where

$$\widetilde{S} = \begin{cases} D^{\alpha_0} P \cdot S_2 \cdots S_n & \text{if } S = P \cdot S_2 \cdots S_n, \\ D^{\alpha_0} D^{\alpha_1} P \cdot S_2 \cdots S_n & \text{if } S = D^{\alpha_1} P \cdot S_2 \cdots S_n. \end{cases}$$

Here, $\widetilde{S}$ is not an allowed string in $\mathcal{A}(\alpha)$. So from the induction hypothesis $D^{\alpha_0} J_n^{\alpha}(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} -\varepsilon_j I_{\widetilde{S}}^{(\alpha_0,\alpha)}(t_0, t, z_0)$. This gives

$$J_{n+1}^{(\alpha_0,\alpha)} = \sum_{j=1}^{2^{n-1}} -\varepsilon_j I_{D^{\alpha_0} P \cdot S^j}^{(\alpha_0,\alpha)} + \sum_{j=1}^{2^{n-1}} \varepsilon_j I_{P \cdot \widetilde{S}^j}.$$

It is easily checked that when S^j is an allowed string in $\mathcal{A}(\alpha)$, both $D^{\alpha_0} P \cdot S^j$ and $P \cdot \widetilde{S}^j$ are allowed strings in $\mathcal{A}(\alpha_0, \alpha)$.

This proves the claim.

For the rest of the proof of Lemma 10 we will bound I_S^α when S is an allowed string; that is, we will show that there is a positive constant M such that

$$I_S^\alpha(t_0,t,z_0) \le \frac{M^n(t-t_0)^{n/2}}{\Gamma((n/2)+1)}$$

for all integers $n \ge 1$ and for each allowed string S in the alphabet $\mathcal{A}(\alpha)$.

We proceed by induction on $n \ge 0$: the case $n = 0$ is immediate, so assume $n > 0$ and that this holds for all allowed strings of length less than n. We consider the three cases:

(1) $S = D^{\alpha_1}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_2,\dots,\alpha_n)$;
(2) $S = P \cdot D^{\alpha_1}D^{\alpha_2}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_3,\dots,\alpha_n)$;
(3) $S = P \cdot D^{\alpha_1}P \cdots D^{\alpha_m}P \cdot D^{\alpha_{m+1}}D^{\alpha_{m+2}}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_{m+3},\dots,\alpha_n)$.

In all the above cases, S' is an allowed string in the alphabet.

(1) We use the inductive hypothesis to bound $I_{S'}^{\alpha'}(t_1,t,z_1)$ and the bound

$$\int_{\mathbb{R}^d} \left|D^\alpha P(t,z)\right| dz \le Ct^{-1/2} \tag{13}$$

to get

$$\begin{aligned}
&\left|I_S^\alpha(t_0,t,z_0)\right| \\
&\quad= \left|\int_{t_0}^t \int_{\mathbb{R}^d} b_1(t_1,z_1)D^{\alpha_1}P(t_1-t_0,z_1-z_0)I_{S'}^{\alpha'}(t_1,t,z_1)\,dz_1\,dt_1\right| \\
&\quad\le \frac{M^{n-1}}{\Gamma((n+1)/2)} \int_{t_0}^t (t-t_1)^{(n-1)/2} \int_{\mathbb{R}^d} \left|D^{\alpha_1}P(t_1-t_0,z_1-z_0)\right| dz_1\,dt_1 \\
&\quad\le \frac{M^{n-1}C}{\Gamma((n+1)/2)} \int_{t_0}^t (t-t_1)^{(n-1)/2}(t_1-t_0)^{-1/2}\,dt_1 \\
&\quad= \frac{M^{n-1}C\sqrt{\pi}(t-t_0)^{k/2}}{\Gamma((n/2)+1)}.
\end{aligned}$$

The result follows if $M \ge \max\{C\sqrt{\pi},1\}$.

(2) For this case, we can write

$$\begin{aligned}
&I_S^\alpha(t_0,t,z_0) \\
&\quad= \int_{t_0}^t \int_{t_1}^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b_1(t_1,z_1)b_2(t_2,z_2)P(t_1-t_0,z_1-z_0) \\
&\qquad\qquad \times D^{\alpha_1}D^{\alpha_2}P(t_2-t_1,z_2-z_1)I_{S'}^{\alpha'}(t_2,t,z_2)\,dz_1\,dz_2\,dt_2\,dt_1.
\end{aligned}$$

We set $h(t_2,z_2) := b_2(t_2,z_2)I_{S'}^{\alpha'}(t_2,z_2)(t-t_2)^{1-n/2}$ so that by the inductive hypothesis we have

$$\|h\|_\infty \le M^{n-2}/\Gamma(n/2).$$

Use the above estimate in the first assertion of Lemma 9 with $g = b_1$ and integrate with respect to t_2 first, to get

$$\left|I_S^\alpha(t_0, t, z_0)\right| \le \frac{CM^{n-2}(t-t_0)^{n/2}}{n\Gamma(n/2)}$$

and the result follows if $M \ge \max\{C, 1\}$.

(3) We have

$$\begin{aligned}
I_S^\alpha(t_0, t, z_0) &= \int_{t_0<\cdots t_{m+2}<t} \int_{\mathbb{R}^{(m+2)d}} P(t_1 - t_0, z_1 - z_0) \\
&\qquad \times \prod_{j=1}^{m+2} b_j(t_j, z_j) \\
&\qquad \times \prod_{j=2}^{m} D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) \\
&\qquad \quad \times D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) \\
&\qquad \quad \times I_{S'}^{\alpha'}(t_{m+2}, t, z_{m+2})\, dz_1 \cdots dz_{m+2}\, dt_1 \cdots dt_{m+2}.
\end{aligned}$$

Set $h(t_{m+2}, z_{m+2}) := b_{m+2}(t_{m+2}, z_{m+2}) I_{S'}^{\alpha'}(t_{m+2}, t, z)(t - t_{m+2})^{(2+m-n)/2}$. Then from the inductive hypothesis we have $\|h\|_\infty \le M^{n-m-2}/\Gamma((n-m)/2)$. Define

$$\begin{aligned}
A(t_m, z_m) &:= \int_{t_m}^{t} \int_{t_{m+1}}^{t} \int_{\mathbb{R}^{2d}} b_{m+1}(t_{m+1}, z_{m+1}) h(t_{m+2}, z_{m+2})(t - t_{m+2})^{(n-m-2)/2} \\
&\qquad \times D^{\alpha_m} P(t_{m+1} - t_m, z_{m+1} - z) D^{\alpha_{m+1}} D^{\alpha_{m+2}} \\
&\qquad \times P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1})\, dz_{m+1}\, dz_{m+2}\, dt_{m+1}\, dt_{m+2}.
\end{aligned}$$

Then Lemma 9 implies that

$$\left|A(t_m, z_m)\right| \le \frac{C(n-m)^{-1/2} M^{n-m-2}(t-t_m)^{(n-m-1)/2}}{\Gamma((n-m)/2)}.$$

Using this in

$$\begin{aligned}
I_S^\alpha(t_0, t, z_0) = \int_{t_0<\cdots t_{m+2}<t} \int_{\mathbb{R}^{(m+2)d}} & P(t_1 - t_0, z_1 - z_0) \\
& \times \prod_{j=1}^{m} b_j(t_j, z_j) \\
& \times \prod_{j=1}^{m-1} D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) \\
& \quad \times \Omega(t_m, z_m)\, dz_1 \cdots dz_m\, dt_1 \cdots dt_m
\end{aligned}$$

and using the bound (13) several times gives

$$
\begin{aligned}
&\left|I_S^\alpha(t_0,t,z_0)\right| \\
&\quad\le C^{m+1}(n-m)^{-1/2}\frac{M^{n-m-2}}{\Gamma((n-m)/2)} \\
&\qquad\times\int_{t_0<\cdots t_m<t}(t_2-t_1)^{-1/2}\cdots(t_m-t_{m-1})^{-1/2}(t-t_m)^{(n-m-1)/2}\,dt_1\cdots dt_m \\
&\quad= C^{m+1}(n-m)^{-1/2}\frac{M^{n-m-2}\pi^{(m-1)/2}\Gamma((n-m+1)/2)}{\Gamma((n-m)/2)\Gamma((n/2)+1)}(t-t_0)^{n/2}.
\end{aligned}
$$

We can choose M so large that the result holds. This completes the induction argument. □

REMARK 11. Assume $\psi\in L^\infty([0,1]\times\Omega;\mathbb{R}^d)$ is adapted to the filtration generated by the Brownian motion. Then we can bound the Doleans–Dade exponential $\mathcal{E}(\int_0^1\psi(u)\,dB_u)$ in $L^p(\Omega)$ by an increasing continuous function of $\|\psi\|_{L^\infty([0,1]\times\Omega)}$.

To see this, notice that $M_t:=\mathcal{E}(\int_0^t\psi(u)\,dB_u)$ is the unique solution to the linear SDE

$$
dM_t=M(t)\psi(t)\,dB_t,\qquad M_0=1.
$$

By Itô's formula, we get

$$
\begin{aligned}
E[M_t^p]&=1+\frac{p(p-1)}{2}\int_0^t E[M_u^p|\psi(u)|^2]\,du \\
&\le 1+\frac{p(p-1)}{2}\|\psi\|_{L^\infty([0,1]\times\Omega)}\int_0^t E[M_u^p]\,du;
\end{aligned}
$$

and

$$
E[M_t^p]\le\exp\left\{\frac{tp(p-1)\|\psi\|_{L^\infty([0,1]\times\Omega)}}{2}\right\},
$$

where we have used Grönwall's lemma in the last inequality.

We are now ready to complete the proof of Proposition 7.

PROOF OF PROPOSITION 7. Let $t\in[0,1]$. Iterating the linearized equation (10), we obtain

$$
\frac{\partial}{\partial x}X_t^x=\mathcal{I}_d+\sum_{n=1}^\infty\int_{0<s_1<\cdots s_n<t}b'(s_1,X_{s_1}^x)\cdots b'(s_n,X_{s_n}^x)\,ds_1\cdots ds_n,
$$

where, as before, b' stands for the spatial Jacobian matrix of b. Let $p \in [1, \infty)$ and choose $r, s \in [1, \infty)$ such that $sp = 2^q$ for some integer q and $\frac{1}{r} + \frac{1}{s} = 1$. Then by Girsanov's theorem and Hölder's inequality

$$\begin{aligned}
E\left[\left|\frac{\partial}{\partial x} X_t^x\right|^p\right]
&= E\Biggl[\left|\mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0<s_1<\cdots s_n<t} b'(s_1, x + B_{s_1}) \cdots b'(s_n, x + B_{s_n})\, ds_1 \cdots ds_n\right|^p \\
&\qquad\qquad \times \mathcal{E}\left(\int_0^1 b(u, x + B_u)\, dB_u\right)\Biggr] \\
&\le C_1(\|b\|_\infty) \biggl\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0<s_1<\cdots s_n<t} b'(s_1, x + B_{s_1}) \cdots \\
&\qquad\qquad \times b'(s_n, x + B_{s_n})\, ds_1 \cdots ds_n \biggr\|_{L^{ps}(\Omega, \mathbb{R}^{d\times d})}^p,
\end{aligned}$$

where C_1 is a continuous increasing function as in Remark 11.

Then we obtain

$$\begin{aligned}
E\left|\frac{\partial}{\partial x} X_t^x\right|^p
&\le C_1(\|b\|_\infty) \\
&\quad \times \biggl\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0<s_1<\cdots s_n<t} b'(s_1, x + B_{s_1}) \cdots \\
&\qquad\qquad \times b'(s_n, x + B_{s_n})\, ds_1 \cdots ds_n \biggr\|_{L^{sp}(\Omega, \mathbb{R}^{d\times d})}^p \\
&\le C_1(\|b\|_\infty) \\
&\quad \times \Biggl(1 + \sum_{n=1}^{\infty} \sum_{i,j=1}^{d} \sum_{l_1,\ldots,l_{n-1}=1}^{d} \biggl\| \int_{0<s_1<\cdots<s_n<t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \\
&\qquad\qquad \times \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x + B_{s_2}) \cdots \\
&\qquad\qquad \times \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, x + B_{s_n}) \\
&\qquad\qquad \times ds_1 \cdots ds_n \biggr\|_{L^{ps}(\Omega, \mathbb{R})} \Biggr)^p.
\end{aligned}$$

Now consider the expression

$$A := \int_{0<s_1<\cdots<s_n<t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x + B_{s_2}) \cdots \times \frac{\partial}{\partial x_{l_n}} b^{(l_n)}(s_n, x + B_{s_n})\, ds_1 \cdots ds_n.$$

Then, using (deterministic) integration by parts, repeatedly, it is easy to see that A^2 can be written as a sum of at most 2^{2n} terms of the form

$$\int_{0<s_1<\cdots<s_{2n}<t} g_1(s_1) \cdots g_{2n}(s_{2n})\, ds_1 \cdots ds_{2n}, \tag{14}$$

where $g_l \in \{\frac{\partial}{\partial x_j} b^{(i)}(\cdot, x + B_\cdot) : 1 \le i, j \le d\}$, $l = 1, 2, \dots, 2n$. Similarly, by induction it follows that A^{2^q} is the sum of at most $2^{q2^q n}$ terms of the form

$$\int_{0<s_1<\cdots<s_{2n}<t} g_1(s_1) \cdots g_{2^q n}(s_{2^q n})\, ds_1 \cdots ds_{2^q n}. \tag{15}$$

Combining this with Lemma 10, we obtain the following estimate:

$$\begin{aligned}&\biggl\| \int_{0<s_1<\cdots<s_n<t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x + B_{s_2}) \cdots \\ &\qquad\qquad \times \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, x + B_{s_n})\, ds_1 \cdots ds_n \biggr\|_{L^{2^q}(\Omega,\mathbb{R})} \\ &\quad \le \left(\frac{2^{q2^q n} C^{2^q n} \|b\|_\infty^{2^q n} t^{2^{q-1} n}}{\Gamma(2^{q-1} n + 1)} \right)^{2^{-q}} \le \frac{2^{qn} C^n \|b\|_\infty^n}{((2^{q-1} n)!)^{2^{-q}}}.\end{aligned}$$

Then it follows that

$$E\left[\left| \frac{\partial}{\partial x} X_t^x \right|^p \right] \le C_1(\|b\|_\infty) \left(1 + \sum_{n=1}^\infty \frac{d^{n+2} 2^{qn} C^n \|b\|_\infty^n}{((2^{q-1} n)!)^{2^{-q}}} \right)^p = C_{d,p}(\|b\|_\infty).$$

The right-hand side of this inequality is independent of $x \in \mathbb{R}^d$, and the result follows. □

For the rest of the paper, we will fix a bounded and measurable $b : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$. It is proved in [30] (and [21]) that the corresponding SDE (5) has a unique strong solution, denoted by $X_\cdot^{s,x}$. Suppose $b_n : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ is a sequence of compactly supported smooth functions such that $b_n(t, x) \to b(t, x)\, dt \times dx$-a.e. and for some positive constant M, $|b_n(t, x)| \le M < \infty$ for all n, t, x. Denote by $X_\cdot^{n,s,x}$ the solution of (5) when b is replaced by b_n, $n \ge 1$. We then have the following.

LEMMA 12. *Fix $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Then the sequence $X_t^{n,s,x}$ converges weakly in $L^2(\Omega; \mathbb{R}^d)$ to $X_t^{s,x}$.*

PROOF. For simplicity, consider $d = 1$ and $s = 0$. We start by noting that the set

$$\left\{\mathcal{E}\left(\int_0^1 h(u)\,dB_u\right) : h \in C_b^1(\mathbb{R})\right\}$$

spans a dense subspace of $L^2(\Omega;\mathbb{R})$. So, it suffices to prove the convergence $E[X_t^{n,x}\mathcal{E}(\int_0^1 h(u)\,dB_u)] \to E[X_t^x\mathcal{E}(\int_0^1 h(u)\,dB_u)]$.

By the Cameron–Martin theorem, we have

$$E\left[X_t^x\mathcal{E}\left(\int_0^1 h(u)\,dB_u\right)\right] = \int_\Omega X_t^x(\omega + h)\,d\mu(\omega).$$

The function $(u,x) \mapsto b(u,x) + h'(u)$ is still bounded, and so $X_t^x(\cdot + h)$ must coincide with the solution to (5) when b is replaced by $b + h'$. Hence, by uniqueness in law of (5), we may write

$$E\left[X_t^x\mathcal{E}\left(\int_0^1 h(u)\,dB_u\right)\right] = E\left[(x + B_t)\mathcal{E}\left(\int_0^1 [b(u, x + B_u) + h'(u)]\,dB_u\right)\right]$$

and similarly for $X_t^{n,x}$. We thus get

$$\begin{aligned}
&E\left[X_t^{n,x}\mathcal{E}\left(\int_0^1 h(u)\,dB_u\right)\right] - E\left[X_t^x\mathcal{E}\left(\int_0^1 h(u)\,dB_u\right)\right]\\
&\quad = E\left[(x + B_t)\left(\mathcal{E}\left(\int_0^1 [b_n(u, x + B_u) + h'(u)]\,dB_u\right)\right.\right.\\
&\qquad\qquad \left.\left. - \mathcal{E}\left(\int_0^1 [b(u, x + B_u) + h'(u)]\,dB_u\right)\right)\right].
\end{aligned}$$

Using the inequality $|e^a - e^b| \le |e^a + e^b||a - b|$, Hölder's inequality and Burkholder–Davis–Gundy inequality we find a constant C such that the above is bounded by

$$\begin{aligned}
&C\left(E\left[\left(\mathcal{E}\left(\int_0^1 [b_n(u, x + B_u) + h'(u)]\,dB_u\right)\right.\right.\right.\\
&\qquad \left.\left.\left. + \mathcal{E}\left(\int_0^1 [b(u, x + B_u) + h'(u)]\,dB_u\right)\right)^4\right]\right)^{1/4}\\
&\quad \times \left(E\left[\left(\int_0^1 (b_n(u, x + B_u) - b(u, x + B_u))^2\,du\right)^2\right.\right.\\
&\qquad + \left(\int_0^1 (b(u, x + B_u) + h'(u))^2\right.\\
&\qquad\qquad \left.\left.\left. - (b_n(u, x + B_u) + h'(u))^2\,du\right)^4\right]\right)^{1/4}.
\end{aligned}$$

From Remark 11, since b_n is uniformly bounded we get that $\{\mathcal{E}(\int_0^1 [b_n(u, x + B_u) + h'(u)]\, dB_u)\}_n$ is bounded in $L^4(\Omega)$, so that the first factor above is uniformly bounded. The second factor converges to zero by bounded convergence. $\square$

We can actually strengthen the above lemma to get the following theorem.

THEOREM 13. *For any fixed $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, the sequence $\{X_t^{n,s,x}\}_{n=1}^{\infty}$ converges strongly in $L^2(\Omega; \mathbb{R}^d)$ to $X_t^{s,x}$.*

PROOF. For simplicity, consider the special case $s = 0$. We first give a sketch of the proof that $\{X_t^{n,x}\}_{n=1}^{\infty}$ is relatively compact in $L^2(\Omega; \mathbb{R}^d)$. We notice that by Corollary 27, it is enough to find a constant $C > 0$ such that

$$\sup_n E\big[\big|D_\theta X_t^{n,x} - D_{\theta'} X_t^{n,x}\big|^2\big] \le C|\theta - \theta'| \tag{16}$$

for $\theta, \theta' \in [0, t]$ and

$$\sup_n \sup_{\theta\in[0,t]} E\big[\big|D_\theta X_t^{n,x}\big|^2\big] \le C. \tag{17}$$

We begin by noticing that the Malliavin derivative satisfies the linearized equation

$$D_\theta X_t^{n,x} = \mathcal{I}_d + \int_\theta^t b'\big[\big(u, X_u^{n,x}\big) D_\theta X_u^{n,x}\big]\, du,$$

which is the same equation as for $\frac{\partial}{\partial x} X_t^{n,x}$ when we let $\theta = 0$. The above inequalities can then be obtained in a similar manner as for the bounds developed in Proposition 7. Indeed, we may iterate the above linearized equation to obtain

$$D_\theta X_t^{n,x} = \mathcal{I}_d + \sum_{k=1}^{\infty} \int_{\theta<u_1<\cdots u_k<t} b_n'\big(u_1, X_{u_1}^{n,x}\big) \cdots b_n'\big(u_k, X_{u_k}^{n,x}\big)\, du_1 \cdots du_k.$$

As in the proof of Proposition 7 with $p = 2$, we get the bound

$$E\big[\big|D_\theta X_t^{n,x}\big|^2\big] \le C_{d,2}\big(\|b\|_\infty\big),$$

where the right-hand side is independent of n, θ, t and x. This proves (17).

Suppose now that $\theta < \theta'$, and write

$$\begin{aligned}
&D_\theta X_t^{n,x} - D_{\theta'} X_t^{n,x} \\
&\quad = \int_\theta^t b_n'\big(u, X_u^{n,x}\big) D_\theta X_u^{n,x}\, du - \int_{\theta'}^t b_n'\big(u, X_u^{n,x}\big) D_{\theta'} X_u^{n,x}\, du \\
&\quad = \int_\theta^{\theta'} b_n'\big(u, X_u^{n,x}\big) D_\theta X_u^{n,x}\, du + \int_{\theta'}^t b_n'\big(u, X_u^{n,x}\big)\big(D_\theta X_u^{n,x} - D_{\theta'} X_u^{n,x}\big)\, du \\
&\quad = D_\theta X_{\theta'}^{n,x} - \mathcal{I}_d + \int_{\theta'}^t b_n'\big(u, X_u^{n,x}\big)\big(D_\theta X_u^{n,x} - D_{\theta'} X_u^{n,x}\big)\, du.
\end{aligned}$$

Iterating the above linear equation, we get

$$\begin{aligned} D_\theta X_t^{n,x} &- D_{\theta'} X_t^{n,x} \\ &= \left(\mathcal{I}_d + \sum_{k=1}^{\infty} \int_{\theta'<u_1<\cdots<u_k<t} b_n'(u_1, X_{u_1}^{n,x}) \cdots b_n'(u_k, X_{u_k}^{n,x})\, du_1 \cdots du_k\right) \\ &\quad \times \left(D_\theta X_{\theta'}^{n,x} - \mathcal{I}_d\right). \end{aligned}$$

On the other hand, note that

$$D_\theta X_{\theta'}^{n,x} - \mathcal{I}_d = \sum_{k=1}^{\infty} \int_{\theta<u_1<\cdots<u_k<\theta'} b_n'(u_1, X_{u_1}^{n,x}) \cdots b_n'(u_k, X_{u_k}^{n,x})\, du_1 \cdots du_k$$

and so

$$\begin{aligned} E\big[\big|D_\theta X_t^{n,x} &- D_{\theta'} X_t^{n,x}\big|^2\big] \\ &\le E\Bigg[\bigg|\mathcal{I}_d + \sum_{k=1}^{\infty} \int_{\theta'<u_1<\cdots<u_k<t} b_n'(u_1, x + B_{u_1}) \cdots \\ &\qquad\qquad \times b_n'(u_k, x + B_{u_k})\, du_1 \cdots du_k\bigg|^2 \\ &\qquad \times \bigg|\sum_{k=1}^{\infty} \int_{\theta<u_1<\cdots<u_k<\theta'} b_n'(u_1, x + B_{u_1}) \cdots \\ &\qquad\qquad \times b_n'(u_k, x + B_{u_k})\, du_1 \cdots du_k\bigg|^2 \\ &\qquad\qquad \times \mathcal{E}\left(\sum_{j=1}^{d} \int_0^1 b_n^{(j)}(u, B_u)\, dB_u\right)\Bigg]. \end{aligned}$$

By a similar argument as in the proof of Proposition 7, we get

$$E\big[\big|D_\theta X_t^{n,x} - D_{\theta'} X_t^{n,x}\big|^2\big] \le C_{d,2}(\|b\|_\infty)|\theta' - \theta|,$$

which proves (16).

Let $\{X_t^{n_k,s,x}\}_{k\ge1}$ be a subsequence of $\{X_t^{n,s,x}\}_{n\ge1}$. Applying the above compactness criterion to this subsequence, we have that this subsequence is relatively compact in $L^2(\Omega, \mathbb{R}^d)$. Thus, we can extract a further subsequence which by Lemma 12 must converge strongly to the limit $X_t^{s,x}$. Since $L^2(\Omega; \mathbb{R}^d)$ is a Banach space, the full sequence must converge strongly to $X_t^{s,x}$. □

As a consequence of Proposition 7 and the above discussion, we obtain the following result.

COROLLARY 14. *Let $X^{s,x}_\cdot$ be the unique strong solution to the SDE* (5) *and $q > 1$ an integer. Then there exists a constant $C = C(d, \|b\|_\infty, q) < \infty$ such that*

$$E\big[\big|X_{t_1}^{s_1,x_1} - X_{t_2}^{s_2,x_2}\big|^q\big] \le C\big(|s_1 - s_2|^{q/2} + |t_1 - t_2|^{q/2} + |x_1 - x_2|^q\big)$$

for all $s_1, s_2, t_1, t_2, x_1, x_2$.

In particular, there exists a continuous version of the random field $(s,t,x) \longmapsto X_t^{s,x}$ with Hölder continuous trajectories of Hölder constant $\alpha < \frac{1}{2}$ in s,t and $\alpha < 1$ in x, locally (*see* [19]).

PROOF. Retain the above notation. Without loss of generality, let $0 \le s_1 < s_2 < t_1 < t_2$. Then

$$\begin{aligned}
&X_{t_1}^{n,s_1,x_1} - X_{t_2}^{n,s_2,x_2}\\
&\quad= x_1 - x_2 + \int_{s_1}^{t_1} b_n\big(u, X_u^{n,s_1,x_1}\big)\,du - \int_{s_2}^{t_2} b_n\big(u, X_u^{n,s_2,x_2}\big)\,du\\
&\qquad + (B_{t_1} - B_{s_1}) - (B_{t_2} - B_{s_2})\\
&\quad= x_1 - x_2 + \int_{s_1}^{s_2} b_n\big(u, X_u^{n,s_1,x_1}\big)\,du + \int_{s_2}^{t_1} b_n\big(u, X_u^{n,s_1,x_1}\big)\,du\\
&\qquad - \int_{s_2}^{t_1} b_n\big(u, X_u^{n,s_2,x_2}\big)\,du - \int_{t_1}^{t_2} b_n\big(u, X_u^{n,s_2,x_2}\big)\,du\\
&\qquad + (B_{t_1} - B_{t_2}) + (B_{s_2} - B_{s_1})\\
&\quad= x_1 - x_2 + \int_{s_1}^{s_2} b_n\big(u, X_u^{n,s_1,x_1}\big)\,du - \int_{t_1}^{t_2} b_n\big(u, X_u^{n,s_2,x_2}\big)\,du\\
&\qquad + \int_{s_2}^{t_1} \big(b_n\big(u, X_u^{n,s_1,x_1}\big) - b_n\big(u, X_u^{n,s_1,x_2}\big)\big)\,du\\
&\qquad + \int_{s_2}^{t_1} \big(b_n\big(u, X_u^{n,s_1,x_2}\big) - b_n\big(u, X_u^{n,s_2,x_2}\big)\big)\,du\\
&\qquad + (B_{t_1} - B_{t_2}) + (B_{s_2} - B_{s_1}).
\end{aligned}$$

So, due to the uniform boundedness of $b_n, n \ge 1$, we get

$$\begin{aligned}
&E\big[\big|X_{t_1}^{n,s_1,x_1} - X_{t_2}^{n,s_2,x_2}\big|^q\big]\\
&\quad\le C_q\bigg(|x_1 - x_2|^q + |s_1 - s_2|^{q/2} + |t_1 - t_2|^{q/2}\\
&\qquad\qquad + E\bigg[\bigg|\int_{s_2}^{t_1} \big(b_n\big(u, X_u^{n,s_1,x_1}\big) - b_n\big(u, X_u^{n,s_1,x_2}\big)\big)\,du\bigg|^q\bigg]\\
&\qquad\qquad + E\bigg[\bigg|\int_{s_2}^{t_1} \big(b_n\big(u, X_u^{n,s_1,x_2}\big) - b_n\big(u, X_u^{n,s_2,x_2}\big)\big)\,du\bigg|^q\bigg]\bigg).
\end{aligned} \tag{18}$$

Using the fact that $X_t^{n,\cdot,s}$ is a stochastic flow of diffeomorphisms (see, e.g., [19]), the mean value theorem and Proposition 7, we get

$$
\begin{aligned}
E\Big[\Big|\int_{s_2}^{t_1} \big(b_n\big(u, X_u^{n,s_1,x_1}\big) - b_n\big(u, X_u^{n,s_1,x_2}\big)\big)\,du\Big|^q\Big] \\
&= |x_1 - x_2|^q \\
&\quad \times E\Big[\Big|\int_{s_2}^{t_1}\int_0^1 \Big(b_n'\big(u, X_u^{n,s_1,x_1+\tau(x_2-x_1)}\big)\frac{\partial}{\partial x} X_u^{n,s_1,x_1+\tau(x_2-x_1)}\Big)\,d\tau\,du\Big|^q\Big] \\
&\le |x_1 - x_2|^q \\
&\quad \times \int_0^1 E\Big[\Big|\int_{s_2}^{t_1}\Big(b_n'\big(u, X_u^{n,s_1,x_1+\tau(x_2-x_1)}\big)\frac{\partial}{\partial x} X_u^{n,s_1,x_1+\tau(x_2-x_1)}\Big)\,du\Big|^q\Big]\,d\tau \qquad (19) \\
&= |x_1 - x_2|^q \\
&\quad \times \int_0^1 E\Big[\Big|\frac{\partial}{\partial x} X_{t_1}^{n,s_1,x_1+\tau(x_2-x_1)} - \frac{\partial}{\partial x} X_{s_2}^{n,s_1,x_1+\tau(x_2-x_1)}\Big|^q\Big]\,d\tau \\
&\le C_q|x_1 - x_2|^q \sup_{t\in[s_1,1],x\in\mathbb{R}^d} E\Big[\Big|\frac{\partial}{\partial x} X_{t_1}^{n,s_1,x}\Big|^q\Big] \\
&\le C_{d,q}\big(\|b\|_\infty\big)|x_1 - x_2|^q.
\end{aligned}
$$

Finally, we observe that estimation of the last term of the right-hand side of (18) can be reduced to the previous case (19) by applying the Markov property, since

$$
\begin{aligned}
E\Big[\Big|\int_{s_2}^{t_1}\big(b_n\big(u, X_u^{n,s_1,x_2}\big) - b_n\big(u, X_u^{n,s_2,x_2}\big)\big)\,du\Big|^q\Big] \\
&\le \int_{s_2}^{t_1} E\big[\big|b_n\big(u, X_u^{n,s_1,x_2}\big) - b_n\big(u, X_u^{n,s_2,x_2}\big)\big|^q\big]\,du \\
&= \int_{s_2}^{t_1} E\big[E\big[\big|b_n\big(u, X_u^{n,s_2,y}\big) - b_n\big(u, X_u^{n,s_2,x_2}\big)\big|^q\big]\big|_{y=X_{s_2}^{n,s_1,x_2}}\big]\,du \\
&\le C E\big[\big|X_{s_2}^{n,s_1,x_2} - x_2\big|^q\big] = C E\big[\big|X_{s_2}^{n,s_1,x_2} - X_{s_1}^{n,s_1,x_2}\big|^q\big] \\
&\le M_q|s_2 - s_1|^{q/2}
\end{aligned}
$$

for a positive constant $M_q < \infty$.

Therefore, we have

$$
E\big[\big|X_{t_1}^{n,s_1,x_1} - X_{t_2}^{n,s_2,x_2}\big|^q\big] \le C_q\big(|s_1 - s_2|^{q/2} + |t_1 - t_2|^{q/2} + |x_1 - x_2|^q\big)
$$

for a constant C_q independent of n.

To complete the proof of the corollary, we use the fact that $X_{t_1}^{n,s_1,x_1} \to X_{t_1}^{s_1,x_1}$ and $X_{t_2}^{n,s_2,x_2} \to X_{t_2}^{s_2,x_2}$ strongly in $L^2(\Omega;\mathbb{R}^d)$ as $n \to \infty$ (Theorem 13), together with Fatou's lemma applied to a.e. convergent subsequences of $\{X_{t_1}^{n,s_1,x_1}\}_{n=1}^\infty$ and $\{X_{t_2}^{n,s_2,x_2}\}_{n=1}^\infty$. $\square$

This concludes step one of our program. We next proceed to step 2.

For simplicity, we consider $s = 0$, that is, we look at the sequence $\{X_t^{n,x}\}_{n\geq 1} := \{X_t^{n,0,x}\}_{n\geq 1}$ and $X_t^x := X_t^{0,x}$. The following lemma establishes convergence of the above sequence.

LEMMA 15. *For any $\varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and $t \in [0, 1]$, the sequence*

$$\langle X_t^n, \varphi\rangle = \int_{\mathbb{R}^d} \langle X_t^{n,x}, \varphi(x)\rangle_{\mathbb{R}^d}\, dx$$

converges to $\langle X_t, \varphi\rangle$ in $L^2(\Omega, \mathbb{R})$.

PROOF. Denote by D_s the Malliavin derivative (see the Appendix) and by U the compact support of φ. By noting the inequalities

$$E\big[\big|D_s\langle X_t^n, \varphi\rangle\big|^2\big] = E\big[\big|\langle D_s X_t^n, \varphi\rangle\big|^2\big] \leq \|\varphi\|_{L^2(\mathbb{R}^d)}^2 |U| \sup_{x\in U} E\big[\big|D_s X_t^{n,x}\big|^2\big]$$

and

$$\begin{aligned} E\big[\big|D_s\langle X_t^n, \varphi\rangle_{L^2(\mathbb{R}^d)} - D_{s'}\langle X_t^n, \varphi\rangle\big|^2\big] &\\ = E\big[\big|\langle D_s X_t^n - D_{s'} X_t^n, \varphi\rangle\big|^2\big] &\\ \leq \|\varphi\|_{L^2(\mathbb{R}^d)}^2 |U| \sup_{x\in U} E\big[\big|D_s X_t^{n,x} - D_{s'} X_t^{n,x}\big|^2\big] & \end{aligned}$$

we can invoke Corollary 27 in the Appendix to obtain a subsequence $\langle X_t^{n(k)}, \varphi\rangle$ converging in $L^2(\Omega, \mathbb{R})$ as $k \to \infty$. Denote the limit by $Y(\varphi)$.

Similar to the proof of Lemma 12 one can show that $E[\langle X_t^n, \varphi\rangle \mathcal{E}(\int_0^1 h(u)\, dB_u)]$ converges to $E[\langle X_t, \varphi\rangle \mathcal{E}(\int_0^1 h(u)\, dB_u)]$ for all $h \in C_b^1(\mathbb{R}; \mathbb{R}^d)$. We then get that $\langle X_t^n, \varphi\rangle$ converges weakly to $\langle X_t, \varphi\rangle$, and so, by uniqueness of the limits, we can conclude that

$$Y(\varphi) = \langle X_t, \varphi\rangle.$$

To see that the full sequence converges, we assume that there exist an $\varepsilon > 0$ and a subsequence $\langle X_t^{n(k)}, \varphi\rangle$ such that

$$\|\langle X_t^{n(k)}, \varphi\rangle - \langle X_t, \varphi\rangle\| \geq \varepsilon$$

for every k. Applying the above procedure to $\langle X_t^{n(k)}, \varphi\rangle$ gives a further subsequence converging to $\langle X_t, \varphi\rangle$ thus giving a contradiction. □

We are now able to finalize the proof of Proposition 6.

PROOF OF PROPOSITION 6. Using Proposition 7, we have

$$\sup_n \sup_{x\in\mathbb{R}^d} E\left[\left|\frac{\partial}{\partial x} X_t^{n,x}\right|^p\right] < \infty.$$

Hence, there exists a subsequence of $\frac{\partial}{\partial x} X_t^{n(k),x}$ converging in the weak topology of $L^2(\Omega, L^p(U))$ to an element Y. Then we have for any $A \in \mathcal{F}$ and $\varphi \in C_0^\infty(U; \mathbb{R}^d)$

$$\begin{aligned} E[1_A\langle X_t, \varphi' \rangle] &= \lim_{k\to\infty} E[1_A\langle X_t^{n(k)}, \varphi'\rangle] \\ &= -\lim_{k\to\infty} E\left[1_A\left\langle \frac{\partial}{\partial x} X_t^{n(k)}, \varphi\right\rangle\right] = -E[1_A\langle Y, \varphi\rangle]. \end{aligned}$$

Hence, we have for $\varphi \in C_0^\infty$:

$$\langle X_t, \varphi' \rangle = -\langle Y, \varphi \rangle \tag{20}$$

P-a.s. Finally, we need to show that there exists a measurable set $\Omega_0 \subset \Omega$ with full measure such that $X_t^\cdot$ has a weak derivative on this subset. To this end, choose a sequence $\{\varphi_n\}$ in $C^\infty(U; \mathbb{R}^d)$ dense in $W_0^{1,2}(U; \mathbb{R}^d)$. Choose a measurable subset Ω_n of Ω with full measure such that (20) holds on Ω_n with φ replaced by φ_n. Then $\Omega_0 := \bigcap_{n\geq 1} \Omega_n$ satisfies the desired property. □

We now return to the weighted Sobolev spaces. Using the same techniques as in the above lemma, we prove the following.

LEMMA 16. *For all* $p \in (1, \infty)$, *we have*

$$X_t^\cdot \in L^2(\Omega, W^{1,p}(\mathbb{R}^d, w)).$$

PROOF. For simplicity, we consider the case $d = 1$. It suffices to show that $E[(\int |\frac{\partial}{\partial x} X_t^x|^p w(x)\, dx)^{2/p}] < \infty$. To this end, let $X_t^{n,x}$ denote the sequence approximating X_t^x as in the previous lemma. Assume first that $p \geq 2$. Then by Hölder's inequality w.r.t. the Wiener measure μ, we have

$$\begin{aligned} E&\left[\left(\int \left|\frac{\partial}{\partial x} X_t^{n,x}\right|^p w(x)\, dx\right)^{2/p}\right] \\ &\leq \left(E \int \left|\frac{\partial}{\partial x} X_t^{n,x}\right|^p w(x)\, dx\right)^{2/p} \\ &\leq \left(\int w(x)\, dx\right)^{p/2} \left(\sup_{x\in\mathbb{R}} E\left|\frac{\partial}{\partial x} X_t^{n,x}\right|^p\right)^{2/p}. \end{aligned}$$

For $1 < p \leq 2$, by Hölder's inequality w.r.t. $w(x)\, dx$, we have

$$E\left[\left(\int \left|\frac{\partial}{\partial x} X_t^{n,x}\right|^p w(x)\, dx\right)^{2/p}\right] \leq \left(\int w(x)\, dx\right)^{(4-p)/2} \sup_{x\in\mathbb{R}^d} E\left[\left|\frac{\partial}{\partial x} X_t^{n,x}\right|^2\right].$$

In both cases, we can find a subsequence of $\frac{\partial}{\partial x} X_t^{n,x}$ converging to an element $Y \in L^2(\Omega, L^p(\mathbb{R}^d, w))$ in the weak topology. In particular for every $A \in \mathcal{F}$ and

$f \in L^q(\mathbb{R}^d, w)$ (q is the Sobolev conjugate of p) we have

$$\lim_{k\to\infty} E\left[1_A \int \frac{\partial}{\partial x} X_t^{n(k),x} f(x)w(x)\,dx\right] = E\left[1_A \int Y(x) f(x) w(x)\,dx\right]$$

by choosing f such that $fw \in L^q(\mathbb{R}, dx)$ [e.g., put $f(x) = e^{-w(x)}\varphi(x)$ for $\varphi \in C_0^\infty(\mathbb{R})$]. It follows that Y must coincide with the weak derivative of X_t^x. This proves the lemma. □

We now complete the proof of our main theorem in this section (Theorem 3) and its corollary.

PROOF OF THEOREM 3. Denote by $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x) \longmapsto \phi_{s,t}(x) \in \mathbb{R}^d$ the continuous version of the solution map $(s, t, x) \longmapsto X_t^{s,x}$ provided by Corollary 14. Let Ω^* be the set of all $\omega \in \Omega$ such that the SDE (5) has a unique spatially Sobolev differentiable family of solutions. Then by completeness of the probability space $(\Omega, \mathcal{F}, \mu)$, it follows that $\Omega^* \in \mathcal{F}$ and $\mu(\Omega^*) = 1$. Furthermore, by uniqueness of solutions of the SDE (5), it is easy to check that the following two-parameter group property

$$\phi_{s,t}(\cdot, \omega) = \phi_{u,t}(\cdot, \omega) \circ \phi_{s,u}(\cdot, \omega), \qquad \phi_{s,s}(x, \omega) = x, \tag{21}$$

holds for all $s, u, t \in \mathbb{R}$, all $x \in \mathbb{R}^d$ and all $\omega \in \Omega^*$. Finally, we apply Lemma 16 and use the relation $\phi_{s,t}(\cdot, \omega) = \phi_{t,s}^{-1}(\cdot, \omega)$, to complete the proof of the theorem. □

PROOF OF COROLLARY 5. Let Ω^* denote the set of full Wiener measure introduced in the above proof of Theorem 3. We claim that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$. To see this, let $\omega \in \Omega^*$ and fix an arbitrary $t_1 \in \mathbb{R}$. Then from the autonomous SDE (9) it follows that

$$X_{t+t_1}^{t_1,x}(\omega) = x + \int_{t_1}^{t+t_1} b\bigl(X_u^{t_1,x}(\omega)\bigr)\,du + B_{t+t_1}(\omega) - B_{t_1}(\omega), \qquad t_1, t \in \mathbb{R}. \tag{22}$$

By the helix property of B and a simple change of variable the above relation implies

$$X_{t+t_1}^{t_1,x}(\omega) = x + \int_0^t b\bigl(X_{u+t_1}^{t_1,x}(\omega)\bigr)\,du + B_t\bigl(\theta\bigl(t_1(\omega)\bigr)\bigr), \qquad t \in \mathbb{R}. \tag{23}$$

The above relation implies that the SDE (9) admits a Sobolev differentiable family of solutions when ω is replaced by $\theta(t_1, \omega)$. Hence, $\theta(t_1, \omega) \in \Omega^*$. Thus $\theta(t_1, \cdot)(\Omega^*) \subseteq \Omega^*$, and since $t_1 \in \mathbb{R}$ is arbitrary, this proves our claim. Furthermore, using uniqueness in the integral equation (22) it follows that

$$X_{t_2+t_1}^{t_1,x}(\omega) = X_{t_2}^{0,x}\bigl(\theta(t_1, \omega)\bigr) \tag{24}$$

for all $t_1, t_2 \in \mathbb{R}$, all $x \in \mathbb{R}^d$ and $\omega \in \Omega^*$. To prove the following cocycle property for all $\omega \in \Omega^*$:

$$\phi_{0,t_1+t_2}(\cdot, \omega) = \phi_{0,t_2}(\cdot, \theta(t_1, \omega)) \circ \phi_{0,t_1}(\cdot, \omega)$$

we rewrite the identity (24) in the form

$$\phi_{t_1,t_1+t_2}(x, \omega) = \phi_{0,t_2}(x, \theta(t_1, \omega)), \qquad t_1, t_2 \in \mathbb{R}, x \in \mathbb{R}^d, \omega \in \Omega^*, \tag{25}$$

replace x by $\phi_{0,t_1}(x, \omega)$ in the above identity and invoke the two-parameter flow property (21). This completes the proof of Corollary 5. □

Finally, we give an extension of Theorem 3 to a class of nondegenerate d-dimensional Itô-diffusions.

THEOREM 17. *Consider the time-homogeneous $\mathbb{R}^d$-valued SDE*

$$dX_t^x = b(X_t^x)\,dt + \sigma(X_t^x)\,dB_t, \qquad X_0^x = x \in \mathbb{R}^d, 0 \le t \le 1, \tag{26}$$

where the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable. Suppose that $\sigma(x)$ has an inverse $\sigma^{-1}(x)$ for all $x \in \mathbb{R}^d$. Further assume that $\sigma^{-1} : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is continuously differentiable such that

$$\frac{\partial}{\partial x_k}\sigma_{lj}^{-1} = \frac{\partial}{\partial x_j}\sigma_{lk}^{-1}$$

for all $l, k, j = 1, \dots, d$. In addition, require that the function $\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\Lambda(x) := \int_0^1 \sigma^{-1}(tx) \cdot x\,dt$$

possesses a Lipschitz continuous inverse $\Lambda^{-1} : \mathbb{R}^d \to \mathbb{R}^d$. Let $D\Lambda : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$ and $D^2\Lambda : \mathbb{R}^d \to L(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ be the existing corresponding derivatives of Λ.

Assume that the function $b^* : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\begin{aligned} b^*(x) &:= D\Lambda(\Lambda^{-1}(x))[b(\Lambda^{-1}(x))] \\ &\quad + \frac{1}{2}D^2\Lambda(\Lambda^{-1}(x))\left[\sum_{i=1}^d \sigma(\Lambda^{-1}(x))[e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(x))[e_i]\right] \end{aligned}$$

is bounded and Borel measurable, where e_i, $i = 1, \dots, d$, is a basis of $\mathbb{R}^d$.

Then there exists a stochastic flow $(s, t, x) \longmapsto \phi_{s,t}(x)$ of the SDE (26) such that

$$\phi_{s,t}(\cdot) \in L^2(\Omega, W^p(\mathbb{R}^d, w))$$

for all $0 \le s \le t \le 1$ and all $p > 1$.

PROOF. Because of our assumptions, we see that Λ^{-1} is twice continuously differentiable and that

$$D\Lambda(y)\sigma(y) = \mathcal{I}_d$$

for all $y \in \mathbb{R}^d$.

Then Itô's lemma applied to (5) implies that

$$\begin{aligned} dY_t^x &= D\Lambda(\Lambda^{-1}(Y_t^x))[b(\Lambda^{-1}(Y_t^x))] \\ &\quad + \frac{1}{2} D^2\Lambda(\Lambda^{-1}(Y_t^x)) \\ &\quad \times \left[\sum_{i=1}^d \sigma(\Lambda^{-1}(Y_t^x))[e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(Y_t^x))[e_i]\right] dt + dB_t, \\ Y_0^x &= \Lambda(x), \qquad 0 \le t \le 1, \end{aligned}$$

where $Y_t^x = \Lambda(X_t^x)$. Because of Theorem 3 and a chain rule for functions in Sobolev spaces (see, e.g., [31]) there exists a stochastic flow $(s, t, x) \longmapsto \phi_{s,t}(x)$ of the SDE (26) such that $\phi_{s,t}(\cdot) \in L^2(\Omega, W^p(\mathbb{R}^d, w))$ for all $0 \le s \le t \le 1$ and all $p > 1$. □

3. Application to the stochastic transport equation. In this section, we will study the stochastic transport equation

$$\begin{cases} d_t u(t,x) + \big(b(t,x) \cdot Du(t,x)\big)\,dt + \sum_{i=1}^d e_i \cdot Du(t,x) \circ dB_t^i = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{27}$$

where $e_1, \dots, e_d$ is the canonical basis of $\mathbb{R}^d$, $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ is a given bounded measurable vector field and $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a given initial data. The stochastic integration is understood in the Stratonovich sense.

In [19], it is proved that for smooth data and a sufficiently regular vector field b, (27) has an explicit solution $u(t,x) = u_0(\phi_t^{-1}(x))$ where $\phi_t(x)$ is the flow map generated by the strong solutions $(X_t^x)_{t \ge 0}$ of the SDE (5). In fact, this solution of the transport equation is strong in the sense that $u(t, \cdot)$ is differentiable everywhere in x almost surely for all t, and it satisfies the integral equation

$$u(t,x) + \int_0^t Du(s,x) \cdot b(s,x)\,ds + \sum_{i=1}^d \int_0^t e_i \cdot Du(s,x) \circ dB_s^i = u_0(x)$$

almost surely, for every t.

We shall use the following notion of weak solution (cf. Definition 12 in [14]).

DEFINITION 18. Let b be bounded and measurable and $u_0 \in L^\infty(\mathbb{R}^d)$. A *weak solution* of the transport equation (27) is a stochastic process $u \in L^\infty(\Omega \times$

$[0,1]\times\mathbb{R}^d$) such that, for every t, the function $u(t,\cdot)$ is weakly differentiable a.s. with $\sup_{0\le s\le 1, x\in\mathbb{R}^d} E[|Du(s,x)|^4]<\infty$ and for every test function $\theta\in C_0^\infty(\mathbb{R}^d)$, the process $\int_{\mathbb{R}^d}\theta(x)u(t,x)\,dx$ has a continuous modification which is an $\mathcal{F}_t$-semimartingale satisfying

$$\begin{aligned}\int_{\mathbb{R}^d}\theta(x)u(t,x)\,dx &= \int_{\mathbb{R}^d}\theta(x)u_0(x)\,dx\\ &\quad -\int_0^t\int_{\mathbb{R}^d} Du(s,x)\cdot b(s,x)\theta(x)\,dx\,ds\\ &\quad +\sum_{i=1}^d\int_0^t\left(\int_{\mathbb{R}^d} u(s,x)D_i\theta(x)\,dx\right)\circ dB_s^i,\end{aligned}\tag{28}$$

where $Du(t,x)$ is the weak derivative of $u(t,x)$ in the following space-variable.

Our definition of a weak solution for (27) differs slightly from that in [14] due to the fact that we do not require any regularity on the coefficient b except Borel measurability and boundedness. To compensate for it, the expression depends on the weak derivative of $u(t,x)$.

It is easy to see that equation (28) can be written in the equivalent Itô form.

LEMMA 19. *A process $u\in L^\infty(\Omega\times[0,1]\times\mathbb{R}^d)$ is a weak solution of the transport equation (27) if and only if, for every t, the function $u(t,\cdot)$ is weakly differentiable a.s. with $\sup_{0\le s\le 1,x\in\mathbb{R}^d} E[|Du(s,x)|^4]<\infty$, and for every test function $\theta\in C_0^\infty(\mathbb{R}^d)$, the process $\int_{\mathbb{R}^d}\theta(x)u(t,x)\,dx$ has a continuous $\mathcal{F}_t$-adapted modification satisfying the following equation a.s.:*

$$\begin{aligned}\int_{\mathbb{R}^d}\theta(x)u(t,x)\,dx &= \int_{\mathbb{R}^d}\theta(x)u_0(x)\,dx\\ &\quad -\int_0^t\int_{\mathbb{R}^d} Du(s,x)\cdot b(s,x)\theta(x)\,dx\,ds\\ &\quad +\sum_{i=1}^d\int_0^t\left(\int_{\mathbb{R}^d} u(s,x)D_i\theta(x)\,dx\right)dB_s^i\\ &\quad +\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(s,x)\Delta\theta(x)\,dx\,ds.\end{aligned}$$

The main result of this section is the following existence and uniqueness theorem for solutions of the stochastic transport equation (27).

THEOREM 20. *Let b be bounded and Borel measurable. Suppose $u_0\in C_b^1(\mathbb{R}^d)$. Then there exists a unique weak solution $u(t,x)$ to the stochastic transport equation (27). For each $t>0$ and all $p\in(1,\infty)$, the weak solution $u(t,\cdot)$ belongs a.s. to $W^{1,p}(\mathbb{R}^d,w)$, the weighted Sobolev space introduced in Section 1. Moreover, for fixed t and x, $u(t,\cdot,x)$ is Malliavin-differentiable.*

REMARK 21. As noted in [14], the deterministic transport equation is generally ill-posed under the conditions of Theorem 20. It is remarkable that Brownian forcing on the transport equation induces uniqueness and regularity of the solution.

We shall prove Theorem 20 using a sequence $b_n : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ of uniformly bounded smooth functions with compact support converging almost everywhere to b. We then study the corresponding sequence of solutions of the transport equation (27) when b is replaced by b_n.

For the rest of this section, we denote by ϕ_t the flow of the SDE (5) driven by the vector field b, and by $\phi_{n,t}$ the flow of the SDE (5) with b_n in place of b.

We begin with the following lemma.

LEMMA 22. *Let $u_0 \in C_b^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Then the sequence*

$$\left(\int_{\mathbb{R}^d} u_0\big(\phi_{n,s}^{-1}(x)\big) f(x)\, dx\right)_{n\geq 1}$$

converges to $\int_{\mathbb{R}^d} u_0(\phi_s^{-1}(x)) f(x)\, dx$ in $L^2(\Omega)$ for every $s \in [0, 1]$.

PROOF. Consider

$$\left\| \int_{\mathbb{R}^d} u_0\big(\phi_{n,s}^{-1}(x)\big) f(x)\, dx - \int_{\mathbb{R}^d} u_0\big(\phi_s^{-1}(x)\big) f(x)\, dx \right\|_{L^2(\Omega)}$$
$$\leq \int_{\mathbb{R}^d} \big\| u_0\big(\phi_{n,s}^{-1}(x)\big) - u_0\big(\phi_s^{-1}(x)\big)\big\|_{L^2(\Omega)} |f(x)|\, dx.$$

We have $\|u_0(\phi_{n,s}^{-1}(x)) - u_0(\phi_s^{-1}(x))\|_{L^2(\Omega)} \leq \|Du_0\|_\infty \|\phi_{n,s}^{-1}(x) - \phi_s^{-1}(x)\|_{L^2(\Omega)}$ which goes to zero for every s and x. Now

$$\big\| u_0\big(\phi_{n,s}^{-1}\big) - u_0\big(\phi_s^{-1}\big)\big\|_{L^2(\Omega)} |f| \leq 2\|u_0\|_\infty |f| \in L^1\big(\mathbb{R}^d\big)$$

and the result follows by dominated convergence. □

We also need the following result (see Theorem 2 in [15] and also [28, 29]).

THEOREM 23. *Let $\mathcal{U}$ be open subset of $\mathbb{R}^d$ and $f \in W^{1,d}(\mathcal{U})$ be a homeomorphism. Then f satisfies the Lusin's condition, that is,*

$$E \subset \mathcal{U}, \qquad |E| = 0 \quad \Longrightarrow \quad \big|f(E)\big| = 0.$$

Here, $|A|$ stands for the Lebesgue measure of a set A.

Moreover, for every measurable function $g : \mathcal{U} \longrightarrow [0, \infty)$ and a measurable set $E \subset \mathcal{U}$ the following change of variable formula is valid:

$$\int_E (g \circ f) |\det Jf|\, dx = \int_{f(E)} g(y)\, dy,$$

where $\det Jf$ *is the determinant of the Jacobian of* f.

REMARK 24. The random diffeomorphisms $\phi_t(\cdot), \phi_t^{-1}(\cdot) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. and satisfy the conditions of Theorem 23 on each bounded and open subset $\mathcal{U}$ of $\mathbb{R}^d$.

We are now ready to prove Theorem 20:

PROOF OF THEOREM 20. (1) *Existence of a weak solution*. We consider the approximation $\{b_n\}$ of b as described prior to Lemma 12. Then we know that there exists a unique strong solution to the transport equation (27) when b is replaced by b_n, which is given by $u_n(t,x) = u_0(\phi_{n,t}^{-1}(x)), n \geq 1$. In particular, u_n is a differentiable, weak L^∞-solution, such that for every $\theta \in C^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} \theta(x) u_n(t,x)\,dx = {} & \int_{\mathbb{R}^d} \theta(x) u_0(x)\,dx \\ & - \int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x) \theta(x)\,dx\,ds \\ & + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u_n(s,x) D_i \theta(x)\,dx \right) dB_s^i \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u_n(s,x) \Delta\theta(x)\,dx\,ds.\end{aligned} \tag{29}$$

Let us now define $u(t,x) := u_0(\phi_t^{-1}(x))$ so that $u \in L^\infty(\Omega \times [0,1] \times \mathbb{R}^d)$, and $u(t,\cdot)$ is weakly differentiable, a.s. We now let n go to infinity to get that $u(t,x)$ is a solution of the transport equation.

The following two limits exist in $L^2(\Omega)$ by Lemma 22 and dominated convergence:

$$\int_{\mathbb{R}^d} \theta(x) u_n(t,x)\,dx \to \int_{\mathbb{R}^d} \theta(x) u(t,x)\,dx,$$

$$\int_0^t \int_{\mathbb{R}^d} u_n(s,x) \Delta\theta(x)\,dx\,ds \to \int_0^t \int_{\mathbb{R}^d} u(s,x) \Delta\theta(x)\,dx\,ds.$$

By the Itô isometry, we have

$$\sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u_n(s,x) D_i\theta(x)\,dx \right) dB_s^i \to \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x) D_i\theta(x)\,dx \right) dB_s^i$$

in $L^2(\Omega)$. Finally, we claim that

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x) \theta(x)\,dx\,ds \\ &\qquad \to \int_0^t \int_{\mathbb{R}^d} Du(s,x) \cdot b(s,x) \theta(x)\,dx\,ds\end{aligned}$$

in $L^2(\Omega)$. To see this, observe that

$$\left(\int_0^t \int_{\mathbb{R}^d} Du_n(s,x)\cdot b_n(s,x)\theta(x)\,dx\,ds\right)_n$$

is convergent in $L^2(\Omega)$ because of the convergence of the other terms in equality (29). Then the claim is proved once we show that $\int_0^t \int_{\mathbb{R}^d} Du_n(s,x)\cdot b_n(s,x)\theta(x)\,dx\,ds$ converges *weakly* to $\int_0^t \int_{\mathbb{R}^d} Du(s,x)\cdot b(s,x)\theta(x)\,dx\,ds$. Then the strong and weak limit must coincide.

To prove weak convergence, we write the difference in three parts, namely:

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} Du_n(s,x)\cdot b_n(s,x)\theta(x)\,dx\,ds - \int_0^t \int_{\mathbb{R}^d} Du(s,x)\cdot b(s,x)\theta(x)\,dx\,ds\\
&\quad= \int_0^t \int_{\mathbb{R}^d} Du_n(s,x)\cdot b_n(s,x)\theta(x)\,dx\,ds\\
&\qquad - \int_0^t \int_{\mathbb{R}^d} Du_n(s,x)\cdot b(s,x)\theta(x)\,dx\,ds\\
&\qquad + \int_0^t \int_{\mathbb{R}^d} Du_0\big(\phi_{n,s}^{-1}(x)\big)D\phi_{n,s}^{-1}(x)\cdot b(s,x)\theta(x)\,dx\,ds\\
&\qquad - \int_0^t \int_{\mathbb{R}^d} Du_0\big(\phi_s^{-1}(x)\big)D\phi_{n,s}^{-1}(x)\cdot b(s,x)\theta(x)\,dx\,ds\\
&\qquad + \int_0^t \int_{\mathbb{R}^d} Du_0\big(\phi_s^{-1}(x)\big)D\phi_{n,s}^{-1}(x)\cdot b(s,x)\theta(x)\,dx\,ds\\
&\qquad - \int_0^t \int_{\mathbb{R}^d} Du_0\big(\phi_s^{-1}(x)\big)D\phi_s^{-1}(x)\cdot b(s,x)\theta(x)\,dx\,ds\\
&\quad= (i)_n + (ii)_n + (iii)_n.
\end{aligned}$$

We shall deal with these terms separately.

(α): the first term $(i)_n$ converges to 0 strongly in $L^2(\Omega)$ as $n\to\infty$, since by Hölder's inequality and Fubini's theorem

$$\begin{aligned}
E\big[(i)_n^2\big] &= E\left[\left(\int_0^t \int_{\mathbb{R}^d} Du_n(s,x)\cdot\big(b_n(s,x)-b(s,x)\big)\theta(x)\,dx\,ds\right)^2\right]\\
&\le \int_0^t \int_{\mathbb{R}^d} E\big[\big|Du_n(s,x)\big|^2\big]\big|b_n(s,x)-b(s,x)\big|^2\big|\theta(x)\big|\,dx\,\|\theta\|_{L^1(\mathbb{R})}.
\end{aligned}$$

We have that

$$E\big[\big|Du_n(s,x)\big|^2\big] \le \|Du_0\|_\infty^2 E\big[\big|D\phi_{n,s}^{-1}(x)\big|^2\big],$$

which is uniformly bounded in n, s and x by Proposition 7. Then, using dominated convergence, we obtain $\lim_{n\to\infty}(i)_n = 0$.

(β): the second term converges strongly to 0 in $L^2(\Omega)$, because of the following estimates:

$$
\begin{aligned}
E\big[(ii)_n^2\big] &\le \|b\|_\infty^2 \\
&\quad \times E\bigg(\int_0^t \int_{\mathbb{R}^d} \big|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))\big|\big|D\phi_{n,s}^{-1}(x)\big|\big|\theta(x)\big|\,dx\,ds\bigg)^2 \\
&\le \|b\|_\infty^2 t \|\theta\|_{L^1(\mathbb{R}^d)} \\
&\quad \times \int_0^t \int_{\mathbb{R}^d} E\big[\big|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))\big|^2 \big|D\phi_{n,s}^{-1}(x)\big|^2\big]\big|\theta(x)\big|\,dx\,ds \\
&\le \|b\|_\infty^2 t \|\theta\|_{L^1(\mathbb{R}^d)} \\
&\quad \times \int_0^t \int_{\mathbb{R}^d} \big(E\big[\big|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))\big|^4\big]\big)^{1/2} \\
&\qquad\qquad \times \big(E\big[\big|D\phi_{n,s}^{-1}(x)\big|^4\big]\big)^{1/2}\big|\theta(x)\big|\,dx\,ds \\
&\le \|b\|_\infty^2 t \|\theta\|_{L^1(\mathbb{R}^d)} \sup_{k,r,y}\big(E\big[\big|D\phi_{k,r}^{-1}(y)\big|^4\big]\big)^{1/2} \\
&\quad \times \int_0^t \int_{\mathbb{R}^d} \big(E\big[\big|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))\big|^4\big]\big)^{1/2}\big|\theta(x)\big|\,dx\,ds.
\end{aligned}
$$

The above estimates are consequences of Hölder's inequality. Since Du_0 is bounded and continuous, the right-hand side of the above inequality converges to 0 by dominated convergence.

(γ): for the last term, let $X \in L^2(\Omega)$ and consider

$$
\begin{aligned}
&E\big[(iii)_n X\big] \\
&\quad = \int_0^t E\bigg[\int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x))\big(D\phi_{n,s}^{-1}(x) - D\phi_s^{-1}(x)\big)\cdot b(s,x)\theta(x) X\,dx\bigg]ds.
\end{aligned}
$$

Now, for each s, since Du_0, b and θ are bounded and $D\phi_s^{-1}$ is the weak limit of $D\phi_{n,s}^{-1}$, this expression tends to 0 as $n \to \infty$.

(2) *Uniqueness of weak solutions*. Let us assume that u is a weak solution to the stochastic transport equation (28) (with $\sup_{0\le s\le 1, x\in\mathbb{R}^d} E[|Du(s,x)|^4] < \infty$). We will show that

$$
u(t,x) = u_0(\phi_t^{-1}(x)) \qquad \text{a.e.}
$$

This will guarantee uniqueness of the weak solution to the transport equation. So, let V be a bounded and open subset of $\mathbb{R}^d$ and consider for the locally integrable function $u(t,\cdot)$ on $\mathbb{R}^d$ its mollification

$$
u_\varepsilon(t,x) = (u * \eta_\varepsilon)(x) = \int_{\mathbb{R}^d} u(t,y)\eta_\varepsilon(x-y)\,dy
$$

with respect to the standard mollifier η.

We observe that u_ε satisfies the equation

$$u_\varepsilon(t,x) = u_{0,\varepsilon}(x) - \int_0^t (b\cdot Du)_\varepsilon(s,x)\,ds - \int_0^t (Du)_\varepsilon(s,x)\circ dB_s.$$

Then using the Itô–Ventzell formula applied to u_ε and $\phi_t(x)$ (see [19]) gives

$$\begin{aligned} &u_\varepsilon\big(t,\phi_t(x)\big) \\ &\quad = u_{0,\varepsilon}(x) + \int_0^t \big((Du)_\varepsilon\big(s,\phi_s(x)\big)\cdot b\big(s,\phi_s(x)\big) - (b\cdot Du)_\varepsilon\big(s,\phi_s(x)\big)\big)\,ds. \end{aligned} \tag{30}$$

Now let $\tau \in L^\infty(\Omega)$ and θ be a smooth function with compact support in $V \subseteq \mathbb{R}^d$. Denote by χ_V the indicator function of V. Then it follows from (30) that

$$E\bigg[\tau\int_V \theta(x)u_\varepsilon\big(t,\phi_t(x)\big)\,dx\bigg]$$

$$= E\bigg[\tau\int_V \theta(x)u_{0,\varepsilon}(x)\,dx\bigg] \tag{31}$$

$$\begin{aligned} &+ E\bigg[\tau\int_0^t\int_V \theta(x)\big((Du)_\varepsilon\big(s,\phi_s(x)\big)\cdot b\big(s,\phi_s(x)\big) \\ &\qquad - (b\cdot Du)_\varepsilon\big(s,\phi_s(x)\big)\big)\,dx\,ds\bigg]. \end{aligned} \tag{32}$$

Using Theorem 23 applied to $\phi_t^{-1}(\cdot)$, we obtain

$$\begin{aligned} &E\bigg[\tau\int_0^t\int_V \theta(x)\big((Du)_\varepsilon\big(s,\phi_s(x)\big)\cdot b\big(s,\phi_s(x)\big) - (b\cdot Du)_\varepsilon\big(s,\phi_s(x)\big)\big)\,dx\,ds\bigg] \\ &\quad = E\bigg[\tau\int_0^t\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta\big(\phi_s^{-1}(x)\big) \\ &\qquad\qquad \times \big((Du)_\varepsilon(s,x)\cdot b(s,x) - (b\cdot Du)_\varepsilon(s,x)\big) \\ &\qquad\qquad\qquad \times \big|\det\big(J\phi_s^{-1}(x)\big)\big|\,dx\,ds\bigg] \\ &\quad = I_1 + I_2, \end{aligned} \tag{33}$$

where

$$\begin{aligned} I_1 := E\bigg[\tau\int_0^t\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta\big(\phi_s^{-1}(x)\big)\big((Du)_\varepsilon(s,x)\cdot b(s,x)\big) \\ \times \big|\det\big(J\phi_s^{-1}(x)\big)\big|\,dx\,ds\bigg] \end{aligned} \tag{34}$$

and

$$\begin{aligned} I_2 := -E\bigg[\tau\int_0^t\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta\big(\phi_s^{-1}(x)\big)(b\cdot Du)_\varepsilon(s,x) \\ \times \big|\det\big(J\phi_s^{-1}(x)\big)\big|\,dx\,ds\bigg]. \end{aligned} \tag{35}$$

Since V is bounded, there exists a $n \in \mathbb{N}$ such that $V \subset \overline{V} \subset W := (-n, n)^d$. Then we get

$$\begin{aligned} \|(Du)_\varepsilon\|_{L^2(\phi_s(V))} &\le \|Du\|_{L^2(\phi_s(W))}, \\ \|(b \cdot Du)_\varepsilon\|_{L^2(\phi_s(V))} &\le \|b \cdot Du\|_{L^2(\phi_s(W))} \\ &\le \|b\|_\infty \|Du\|_{L^2(\phi_s(W))}. \end{aligned} \tag{36}$$

Using (36), Hölder's inequality, Fubini's theorem and Theorem 23, we obtain

$$\begin{aligned} I_1 &\le C E\bigg[\int_0^t \bigg(\int_{\mathbb{R}^d} \big(\chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))b(s,x)\big|\det\big(J\phi_s^{-1}(x)\big)\big|\big)^2\,dx\bigg)^{1/2} \\ &\qquad\qquad \times \bigg(\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x)\big|Du(s,x)\big|^2\,dx\bigg)^{1/2} ds\bigg] \\ &\le C\int_0^t E\bigg[\int_{\mathbb{R}^d} \big(\chi_{\phi_s(V)}(x)\theta(\phi_s^{-1}(x))b(s,x)\big|\det\big(J\phi_s^{-1}(x)\big)\big|\big)^2\,dx\bigg]^{1/2} \\ &\qquad \times E\bigg[\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x)\big|Du(s,x)\big|^2\,dx\bigg]^{1/2} ds \\ &\le C\int_0^t E\bigg[\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\big|\det\big(J\phi_s^{-1}(x)\big)\big|^2\,dx\bigg]^{1/2} \\ &\qquad \times E\bigg[\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x)\big|Du(s,x)\big|^2\,dx\bigg]^{1/2} ds \\ &\le C\int_0^t \bigg(\int_{\mathbb{R}^d} E\big[\chi_{\phi_s(V)}(x)\big]^{1/2} E\big[\big|\det\big(J\phi_s^{-1}(x)\big)\big|^4\big]^{1/2}\,dx\bigg)^{1/2} \\ &\qquad \times \bigg(\int_{\mathbb{R}^d} E\big[\chi_{\phi_s(W)}(x)\big]^{1/2} E\big[\big|Du(s,x)\big|^4\big]^{1/2}\,dx\bigg)^{1/2} ds \\ &\le C \sup_{0\le s\le 1, x\in\mathbb{R}^d} E\big[\big|\det\big(J\phi_s^{-1}(x)\big)\big|^4\big]^{1/2} \sup_{0\le s\le 1, x\in\mathbb{R}^d} E\big[\big|Du(s,x)\big|^4\big]^{1/2} \\ &\qquad \times \int_0^t \bigg(\int_{\mathbb{R}^d} E\big[\chi_{\phi_s(V)}(x)\big]^{1/2}\,dx\bigg) ds \\ &\le C\int_0^t \bigg(\int_{\mathbb{R}^d} E\big[\chi_{\phi_s(V)}(x)\big]^{1/2}\,dx\bigg) ds \end{aligned} \tag{37}$$

for a constant C depending on the sizes of V, θ and b, since

$$\sup_{0\le s\le 1, x\in\mathbb{R}^d} E\big[\big|\det\big(J\phi_s^{-1}(x)\big)\big|^4\big] \le M < \infty$$

because of Proposition 7 applied to $\phi_s^{-1}(x)$.

Further, it follows from Girsanov's theorem, Hölder's inequality and the symmetry of the distribution of the Brownian motion that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} E\big[\chi_{\phi_s(W)}(x)\big]^{1/2}\,dx\,ds \\
&\qquad = \int_0^t \int_{\mathbb{R}^d} \big(\mu\big(\phi_s^{-1}(x) \in W\big)\big)^{1/2}\,dx\,ds \\
&\qquad \le C \int_0^t \int_{\mathbb{R}^d} \big(\mu(B_s + x \in W)\big)^{1/4}\,dx\,ds \\
&\qquad = C \int_0^t \int_{\mathbb{R}^d} \big(\mu\big(B_s + x \in (-n,n)^d\big)\big)^{1/4}\,dx\,ds \\
&\qquad \le C \int_0^t \left(2 \int_0^\infty \left(1 - \Phi\left(\frac{-n+y}{\sqrt{s}}\right)\right)^{1/4} dy\right)^d ds,
\end{aligned}
\tag{38}
$$

where Φ is the standard normal distribution function.

On the other hand, we know that

$$
1 - \Phi(x) \le \frac{1}{2\pi x}\exp(-x^2/2)
$$

for all $x > 0$ (see [3]).

So,

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} E\big[\chi_{\phi_s(W)}(x)\big]^{1/2}\,dx\,ds \\
&\quad \le C \int_0^t \Bigg(2 \int_0^n \left(1 - \Phi\left(\frac{-n+y}{\sqrt{s}}\right)\right)^{1/4} dy \\
&\qquad\qquad + 2 \int_n^\infty \left(1 - \Phi\left(\frac{-n+y}{\sqrt{s}}\right)\right)^{1/4} dy\Bigg)^d ds \\
&\quad \le K \int_0^t \Bigg(\left(\int_0^n \left(1 - \Phi\left(\frac{-n+y}{\sqrt{s}}\right)\right)^{1/4} dy\right)^d \\
&\qquad\qquad + \left(\int_n^\infty \left(1 - \Phi\left(\frac{-n+y}{\sqrt{s}}\right)\right)^{1/4} dy\right)^d\Bigg) ds \\
&\quad \le M\left(1 + \int_0^t \left(\int_n^\infty \left(\frac{\sqrt{s}}{2\pi(y-n)}\exp\big(-(y-n)^2/2s\big)\right)^{1/4} dy\right)^d ds\right) \\
&\quad = M\left(1 + \int_0^t \left(\int_0^\infty \left(\frac{\sqrt{s}}{2\pi y}\exp(-y^2/2s)\right)^{1/4} dy\right)^d ds\right) \\
&\quad = M\left(1 + \int_0^t \left(\int_0^\infty \sqrt{s}\left(\frac{1}{2\pi y}\exp(-y^2/2)\right)^{1/4} dy\right)^d ds\right) \le L < \infty.
\end{aligned}
\tag{39}
$$

Furthermore, since

$$(Du)_\varepsilon \longrightarrow Du \qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^d)$$

for all $p > 1$ and since

$$\int_{\mathbb{R}^d} \bigl(\chi_{\phi_s(V)}(x)\theta\bigl(\phi_s^{-1}(x)\bigr)b(s,x)\bigl|\det\bigl(J\phi_s^{-1}(x)\bigr)\bigr|\bigr)^2\,dx < \infty \qquad \text{a.e.}$$

because of the above estimates, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta\bigl(\phi_s^{-1}(x)\bigr)\bigl((Du)_\varepsilon(s,x)\cdot b(s,x)\bigr)\bigl|\det\bigl(J\phi_s^{-1}(x)\bigr)\bigr|\,dx\\ &\qquad \longrightarrow \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta\bigl(\phi_s^{-1}(x)\bigr)\bigl((Du)(s,x)\cdot b(s,x)\bigr)\bigl|\det\bigl(J\phi_s^{-1}(x)\bigr)\bigr|\,dx\end{aligned}$$

for $\varepsilon \searrow 0 \mu \times ds$-a.e.

On the other hand, the latter expression w.r.t. ε is dominated by the integrable term

$$\begin{aligned}&\biggl(\int_{\mathbb{R}^d} \bigl(\chi_{\phi_s(V)}(x)\theta\bigl(\phi_s^{-1}(x)\bigr)b(s,x)\bigl|\det\bigl(J\phi_s^{-1}(x)\bigr)\bigr|\bigr)^2\,dx\biggr)^{1/2}\\ &\qquad \times \biggl(\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x)\bigl|Du(s,x)\bigr|^2\,dx\biggr)^{1/2}.\end{aligned}$$

So, using dominated convergence it follows from (37) and (39) that

$$\begin{aligned}I_1 &= I_1(\varepsilon)\\ &\longrightarrow E\biggl[\tau\int_0^t\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta\bigl(\phi_s^{-1}(x)\bigr)\bigl((Du)(s,x)\cdot b(s,x)\bigr)\\ &\qquad\qquad \times \bigl|\det\bigl(J\phi_s^{-1}(x)\bigr)\bigr|\,dx\,ds\biggr]\\ &\qquad\qquad\qquad \text{as } \varepsilon \searrow 0.\end{aligned} \tag{40}$$

Similarly, we also get

$$\begin{aligned}I_2 &= I_2(\varepsilon)\\ &\longrightarrow -E\biggl[\tau\int_0^t\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x)\theta\bigl(\phi_s^{-1}(x)\bigr)(b\cdot Du)(s,x)\\ &\qquad\qquad \times \bigl|\det\bigl(J\phi_s^{-1}(x)\bigr)\bigr|\,dx\,ds\biggr] \qquad \text{as } \varepsilon \searrow 0\end{aligned} \tag{41}$$

and

$$E\biggl[\tau\int_V \theta(x)u_\varepsilon\bigl(t,\phi_t(x)\bigr)\,dx\biggr] \longrightarrow E\biggl[\tau\int_V \theta(x)u\bigl(t,\phi_t(x)\bigr)\,dx\biggr] \tag{42}$$

as $\varepsilon \searrow 0$.

In addition, because of the assumptions on u_0 it is clear that

$$E\left[\tau \int_V \theta(x) u_{0,\varepsilon}(x)\, dx\right] \longrightarrow E\left[\tau \int_V \theta(x) u_0(x)\, dx\right]$$

as $\varepsilon \searrow 0$.

Altogether we can conclude that

$$E\left[\tau \int_{\mathbb{R}^d} \theta(x) u\bigl(t, \phi_t(x)\bigr)\, dx\right] = E\left[\tau \int_{\mathbb{R}^d} \theta(x) u_0(x)\, dx\right]$$

for all $\tau \in L^\infty(\Omega)$ and compactly supported smooth functions θ. Hence,

$$u\bigl(t, \phi_t(x)\bigr) = u_0(x)$$

$\mu \times dx$-a.e.

Since $\phi_t^{-1}(\cdot)$ satisfies the Lusin condition in Theorem 23 on bounded open subsets, we can find a Ω^* with $\mu(\Omega^*) = 1$ such that for all $\omega \in \Omega^*$

$$u(t, x) = u_0\bigl(\phi_t^{-1}(x)\bigr)\, dx\text{-a.e.}$$

Due to the continuity of u with respect to time, the latter relation also holds uniformly in t.

Finally, the Malliavin differentiability of (a version) of $u(t, x)$ is a consequence of the fact that $\phi_t^{-1}(x)$ is Malliavin differentiable (see [21]) and of the chain rule for Malliavin derivatives (see [26]). □

APPENDIX

The following result which is due to [6] provides a compactness criterion for subsets of $L^2(\Omega; \mathbb{R}^d)$ using Malliavin calculus. See, for example, [20, 26] or [8] for more information about Malliavin calculus.

THEOREM 25. *Let $\{(\Omega, \mathcal{A}, P); H\}$ be a Gaussian probability space, that is $(\Omega, \mathcal{A}, P)$ is a probability space and H a separable closed subspace of Gaussian random variables of $L^2(\Omega)$, which generate the σ-field $\mathcal{A}$. Denote by $\mathbf{D}$ the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}\bigl(f(h_1, \dots, h_n)\bigr) = \sum_{i=1}^n \partial_i f(h_1, \dots, h_n) h_i, \qquad h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further, let $\mathbf{D}_{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega; H)}.$$

Assume that C is a self-adjoint compact operator on H with dense image. Then for any $c > 0$, the set

$$\mathcal{G} = \{G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1}\mathbf{D}G\|_{L^2(\Omega; H)} \leq c\}$$

is relatively compact in $L^2(\Omega)$.

In order to formulate compactness criteria useful for our purposes, we need the following technical result which also can be found in [6].

LEMMA 26. *Let $v_s, s \geq 0$ be the Haar basis of $L^2([0, 1])$. For any $0 < \alpha < 1/2$ define the operator A_α on $L^2([0, 1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s \qquad \text{if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all β with $\alpha < \beta < (1/2)$, there exists a constant c_1 such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} \, dt \, dt' \right)^{1/2} \right\}.$$

A direct consequence of Theorem 25 and Lemma 26 is now the following compactness criterion which is essential for the proof of Theorem 13 and Lemma 15.

COROLLARY 27. *Let $X_n \in \mathbb{D}_{1,2}$, $n = 1, 2, \dots,$ be a sequence of $\mathcal{F}_1$-measurable random variables such that there are constants $\alpha > 0$ and $C > 0$ with*

$$\sup_n E[\|X_n\|^2] \leq C,$$

$$\sup_n E[\|D_t X_n - D_{t'} X_n\|^2] \leq C|t - t'|^\alpha$$

for $0 \leq t' \leq t \leq 1$ and

$$\sup_n \sup_{0 \leq t \leq 1} E[\|D_t X_n\|^2] \leq C,$$

where D_t denotes Malliavin differentiation. Then the sequence X_n, $n = 1, 2, \dots,$ is relatively compact in $L^2(\Omega; \mathbb{R}^d)$ (D_t stands for the Malliavin derivative).

REFERENCES

[1] AMBROSIO, L. (2004). Transport equation and Cauchy problem for BV vector fields. *Invent. Math.* **158** 227–260. MR2096794

[2] ATTANASIO, S. (2010). Stochastic flows of diffeomorphisms for one-dimensional SDE with discontinuous drift. *Electron. Commun. Probab.* **15** 213–226. MR2653726

[3] BORODIN, A. N. and SALMINEN, P. (2002). *Handbook of Brownian Motion—Facts and Formulae*, 2nd ed. Birkhäuser, Basel. MR1912205

[4] DAVIE, A. M. (2007). Uniqueness of solutions of stochastic differential equations. *Int. Math. Res. Not. IMRN* **24** Art. ID rnm124, 26. MR2377011

[5] DA PRATO, G., FLANDOLI, F., PRIOLA, E. and RÖCKNER, M. (2013). Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.* **41** 3306–3344. MR3127884

[6] DA PRATO, G., MALLIAVIN, P. and NUALART, D. (1992). Compact families of Wiener functionals. *C. R. Acad. Sci. Paris Sér. I Math.* **315** 1287–1291. MR1194537

[7] DIPERNA, R. J. and LIONS, P.-L. (1989). Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** 511–547. MR1022305

[8] DI NUNNO, G., ØKSENDAL, B. and PROSKE, F. (2009). *Malliavin Calculus for Lévy Processes with Applications to Finance*. Springer, Berlin. MR2460554

[9] FEDRIZZI, E. (2009). Uniqueness and flow theorems for solutions of SDE's with low regularity of the drift. Tesi di Laurea in Matematica, Univ. di Pisa.

[10] FEDRIZZI, E. and FLANDOLI, F. (2011). Pathwise uniqueness and continuous dependence of SDEs with non-regular drift. *Stochastics* **83** 241–257. MR2810591

[11] FEDRIZZI, E. and FLANDOLI, F. (2013). Noise prevents singularities in linear transport equations. *J. Funct. Anal.* **264** 1329–1354. MR3017266

[12] FEDRIZZI, E. and FLANDOLI, F. (2013). Hölder flow and differentiability for SDEs with nonregular drift. *Stoch. Anal. Appl.* **31** 708–736. MR3175794

[13] FLANDOLI, F. (2011). *Random Perturbation of PDEs and Fluid Dynamic Models*. *Lecture Notes in Math.* **2015**. Springer, Heidelberg. MR2796837

[14] FLANDOLI, F., GUBINELLI, M. and PRIOLA, E. (2010). Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** 1–53. MR2593276

[15] HAJŁASZ, P. (1993). Change of variables formula under minimal assumptions. *Colloq. Math.* **64** 93–101. MR1201446

[16] HEINONEN, J., KILPELÄINEN, T. and MARTIO, O. (1993). *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Oxford Univ. Press, New York. MR1207810

[17] KRYLOV, N. V. and RÖCKNER, M. (2005). Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** 154–196. MR2117951

[18] KUFNER, A. (1985). *Weighted Sobolev Spaces*. Wiley, New York. MR0802206

[19] KUNITA, H. (1990). *Stochastic Flows and Stochastic Differential Equations*. Cambridge Univ. Press, Cambridge. MR1070361

[20] MALLIAVIN, P. (1997). *Stochastic Analysis*. *Grundlehren der Mathematischen Wissenschaften* **313**. Springer, Berlin. MR1450093

[21] MENOUKEU-PAMEN, O., MEYER-BRANDIS, T., NILSSEN, T., PROSKE, F. and ZHANG, T. (2013). A variational approach to the construction and Malliavin differentiability of strong solutions of SDE's. *Math. Ann.* **357** 761–799. MR3096525

[22] MEYER-BRANDIS, T. and PROSKE, F. (2010). Construction of strong solutions of SDE's via Malliavin calculus. *J. Funct. Anal.* **258** 3922–3953. MR2606880

[23] MOHAMMED, S.-E. A. and SCHEUTZOW, M. K. R. (1998). Spatial estimates for stochastic flows in Euclidean space. *Ann. Probab.* **26** 56–77. MR1617041

[24] MOHAMMED, S.-E. A. and SCHEUTZOW, M. K. R. (2003). The stable manifold theorem for non-linear stochastic systems with memory. I. Existence of the semiflow. *J. Funct. Anal.* **205** 271–305. MR2017689

[25] MOHAMMED, S.-E. A. and SCHEUTZOW, M. K. R. (2004). The stable manifold theorem for non-linear stochastic systems with memory. II. The local stable manifold theorem. *J. Funct. Anal.* **206** 253–306. MR2021848

[26] NUALART, D. (1995). *The Malliavin Calculus and Related Topics*. Springer, New York. MR1344217

[27] PORTENKO, N. I. (1990). *Generalized Diffusion Processes*. *Translations of Mathematical Monographs* **83**. Amer. Math. Soc., Providence, RI. MR1104660

[28] RESHETNYAK, Y. G. (1966). Some geometrical properties of functions and mappings with generalized derivatives. *Sibirsk*. *Mat*. *Zh*. **7** 886–919.

[29] RESHETNYAK, Y. G. (1987). On the condition N for mappings of class $W^1_{n,\mathrm{loc}}$. *Sibirsk*. *Mat*. *Zh*. **28** 149–153.

[30] VERETENNIKOV, A. Y. (1979). On the strong solutions of stochastic differential equations. *Theory Probab. Appl*. **24** 354–366.

[31] ZIEMER, W. P. (1989). *Weakly Differentiable Functions*. Springer, New York. MR1014685

[32] ZVONKIN, A. K. (1974). A transformation of the phase space of a diffusion process that will remove the drift. *Mat*. *Sb*. **93** 129–149, 152. MR0336813

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