LOSS OF REGULARITY FOR KOLMOGOROV EQUATIONS

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The celebrated Hörmander condition is a sufficient (and nearly necessary) condition for a second-order linear Kolmogorov partial differential equation (PDE) with smooth coefficients to be hypoelliptic. As a consequence, the solutions of Kolmogorov PDEs are smooth at all positive times if the coefficients of the PDE are smooth and satisfy Hörmander's condition even if the initial function is only continuous but not differentiable. First-order linear Kolmogorov PDEs with smooth coefficients do not have this smoothing effect but at least preserve regularity in the sense that solutions are smooth if their initial functions are smooth. In this article, we consider the intermediate regime of nonhypoelliptic second-order Kolmogorov PDEs with smooth coefficients. The main observation of this article is that there exist counterexamples to regularity preservation in that case. More precisely, we give an example of a second-order linear Kolmogorov PDE with globally bounded and smooth coefficients and a smooth initial function with compact support such that the unique globally bounded viscosity solution of the PDE is not even locally Hölder continuous. From the perspective of probability theory, the existence of this example PDE has the consequence that there exists a stochastic differential equation (SDE) with globally bounded and smooth coefficients and a smooth function with compact support which is mapped by the corresponding transition semigroup to a function which is not locally Hölder continuous. In other words, degenerate noise can have a roughening effect. A further implication of this loss of regularity phenomenon is that numerical approximations may converge without any arbitrarily small polynomial rate of convergence to the true solution of the SDE. More precisely, we prove for an example SDE with globally bounded and smooth coefficients that the standard Euler approximations converge to the exact solution of the SDE in the strong and numerically weak sense, but at a rate that is slower then any power law.

1. Introduction and main results. The key observation of this article is to reveal the phenomenon of *loss of regularity* in Kolmogorov partial differential equations (PDEs). This observation has a direct consequence on the literature on *regularity analysis of linear PDEs*, on the literature on *regularity analysis of stochastic differential equations* (SDEs) and on the literature on *numerical approximations of SDEs*. We will illustrate the implications for each field separately.

Regularity analysis of linear partial differential equations. For some $d, m \in \mathbb{N}$, let $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be smooth functions such that there exists a real number $\rho > 0$ such that $\langle x, \mu(x) \rangle \leq \rho(1 + \|x\|^2)$ and $\|\sigma(x)\|^2_{L(\mathbb{R}^m, \mathbb{R}^d)} \leq \rho(1 + \|x\|^2)$ for all $x \in \mathbb{R}^d$. (Here and below, we write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the Euclidean scalar product and norm on $\mathbb{R}^n$.) Let furthermore $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a globally bounded and continuous function and consider the second-order PDE

(1.1)
$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} \sigma_{i,k}(x) \cdot \sigma_{j,k}(x) \cdot \frac{\partial^2}{\partial x_i \, \partial x_j} u(t, x)$$
$$+ \sum_{i=1}^{d} \mu_i(x) \cdot \frac{\partial}{\partial x_i} u(t, x), \qquad u(0, x) = \varphi(x)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^d$. The PDE (1.1) is referred to as *Kolmogorov equation* in the literature (see, e.g., Cerrai [5], Da Prato [11], Röckner [64] and Röckner and Sobol [65]; it is also referred to as *Kolmogorov backward equation* or *Kolmogorov PDE*, see, e.g., Da Prato and Zabczyk [12], Øksendal [59]). It has a strong link to probability theory and appeared first (in a slightly different form; see display (125) in [44]) in Kolmogorov's celebrated paper [44]. Corollary 4.17 in Section 4 below implies that the PDE (1.1) admits a unique globally bounded viscosity solution. More precisely, Corollary 4.17 proves that there exists a unique globally bounded continuous function $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ such that $u|_{(0,\infty) \times \mathbb{R}^d}$ is a viscosity solution of (1.1) and such that $u(0, x) = \varphi(x)$ for all $x \in \mathbb{R}^d$. In this article, we are interested to know whether solutions u of the PDE (1.1) *preserve regularity* in the sense that $u|_{(0,\infty) \times \mathbb{R}^d}$ is smooth if the initial function $u(0, \cdot) = \varphi(\cdot)$ is smooth. In particular, we will answer the question whether smoothness and global boundedness of the initial function $\varphi : \mathbb{R}^d \to \mathbb{R}$ implies the existence of a *classical solution* of the PDE (1.1).

In the case of first-order Kolmogorov PDEs with smooth coefficients, that is, $\sigma \equiv 0$ in (1.1), regularity preservation of solutions of (1.1) is well known. More precisely, if $\sigma(x) = 0$ for all $x \in \mathbb{R}^d$ and if the initial function $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ in (1.1) is smooth, then it is well known that there exists a unique smooth classical solution of (1.1). In this sense, the PDE (1.1) is *regularity preserving* in the purely first-order case $\sigma \equiv 0$. In the second-order case $\sigma \not\equiv 0$, the situation may be even better in the sense that the PDE (1.1) often has a *smoothing effect*. More precisely, if the PDE (1.1) is *hypoelliptic*, then by definition solutions u of the PDE (1.1) are

smooth in the sense that $u|_{(0,\infty)\times\mathbb{R}^d}$ is infinitely often differentiable even if the initial function $u(0,\cdot)=\varphi(\cdot)$ is only continuous but not differentiable. In the seminal paper [31], Hörmander gave a sufficient (and also nearly necessary; see the discussion before Theorem 1.1 in [31] and Section 2 in Hairer [26]) condition for (1.1) to be hypoelliptic; see Theorem 1.1 in [31]. To formulate Hörmander's condition, set $\sigma_0(x)=\mu(x)-\frac{1}{2}\sum_{k=1}^m \sigma_k'(x)\sigma_k(x)$ for all $x\in\mathbb{R}^d$. Then the *Hörmander condition* is fulfilled if

$$(1.2) \quad \begin{aligned} \mathrm{span}\{\sigma_{i_0}(x), [\sigma_{i_0},\sigma_{i_1}](x), [[\sigma_{i_0},\sigma_{i_1}],\sigma_{i_2}](x), \ldots \in \mathbb{R}^d : \\ i_0, i_1, i_2, \ldots \in \{0, 1, \ldots, m\}, i_0 \neq 0\} = \mathbb{R}^d \end{aligned}$$

for all $x\in\mathbb{R}^d$ where $[f,g]$ denotes the Lie bracket of two smooth vector fields $f,g:\mathbb{R}^d\to\mathbb{R}^d$. Consequently, if Hörmander's condition (1.2) is satisfied, then the PDE (1.1) admits a unique globally bounded smooth classical solution even if the initial function $\varphi:\mathbb{R}^d\to\mathbb{R}$ is assumed to be continuous and globally bounded only. Clearly, there are many cases where the Hörmander condition (1.2) fails to be fulfilled and where (1.1) is not hypoelliptic, for example, if $\sigma\equiv 0$. Next, we point out that if all derivatives of the drift coefficient μ, of the diffusion coefficient σ and of the initial function φ are globally bounded (μ and σ are then, in particular, globally Lipschitz continuous), then smoothness of the solution of the PDE (1.1) is known even in the nonhypoelliptic case (see, e.g., Theorem 4.32 in Krylov [47] for twice differentiability of the solution; infinitely often differentiability of the solution follows analogously as in the proof of Theorem 4.32 in Krylov [47]). Obviously, there are many cases where μ and σ are not both globally Lipschitz continuous, for example, when μ is a polynomial with a degree greater or equal 2 (see, e.g., Section 4 in [34] for a list of examples). To the best of our knowledge, regularity of solutions of the PDE (1.1) is in general unknown in the nonhypoelliptic case if $\sigma\not\equiv 0$ and if μ and σ are not both globally Lipschitz continuous.

In this article, we address the question whether second-order linear PDEs with smooth coefficients of the form (1.1) at least preserve regularity in the nonhypoelliptic case. The following Theorem 1.1 answers this question to the negative. More precisely, the key observation of this article is to reveal the phenomenon of *loss of regularity* in the sense that the solution u of the PDE (1.1) starting with a smooth compactly supported function $u(0,\cdot)\in C_{\mathrm{cpt}}^\infty(\mathbb{R}^d,\mathbb{R})$ may turn into a nondifferentiable function $u(t,\cdot)\notin C^1(\mathbb{R}^d,\mathbb{R})$ for every positive time $t\in(0,\infty)$. In analogy to the well-known "smoothing effect" in the hypoelliptic case, we will say in the case of loss of regularity that the PDE (1.1) has a *roughening effect*. Here is a simple two-dimensional example with polynomial μ and linear σ which has this roughening effect. In the special case $d=2$, $m=1$ and $\mu(x)=(x_1\cdot x_2, -x_1^2)$ and $\sigma(x)=(0, x_2)$ for all $x=(x_1, x_2)\in\mathbb{R}^2$, the PDE (1.1) reads as

$$(1.3) \quad \frac{\partial}{\partial t}u(t,x) = \frac{x_2^2}{2}\frac{\partial^2}{\partial x_2^2}u(t,x) + x_1 x_2\frac{\partial}{\partial x_1}u(t,x) - x_1^2\frac{\partial}{\partial x_2}u(t,x)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^2$. Theorem 2.1 and Corollary 4.17 below imply that there exists an infinitely often differentiable function $\varphi \in C_{\text{cpt}}^\infty(\mathbb{R}^d, \mathbb{R})$ with compact support such that the unique globally bounded viscosity solution $u : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ to (1.3) with $u(0, \cdot) = \varphi(\cdot)$ has the property that $u|_{(0,\infty) \times \mathbb{R}^d}$ is not differentiable and not locally Lipschitz continuous. In particular, we thereby disprove the existence of a globally bounded classical solution of the PDE (1.3) with $u(0, \cdot) = \varphi(\cdot)$. Note that the drift coefficient μ of the PDE (1.3) grows superlinearly. One could wonder whether the roughening effect of example (1.3) is due to this superlinear growth of μ. To exclude this possibility, we prove for an example PDE with globally bounded and smooth coefficients that there exists a smooth initial function with compact support such that the solution u is not even locally Hölder continuous; see Theorem 1.1 below. In particular, Theorem 1.1 implies that, in general, the PDE (1.1) does not have a classical solution even if the coefficients and the initial function are *globally bounded* and *infinitely often differentiable*.

THEOREM 1.1 (Disprove of the existence of classical solutions of the Kolmogorov PDE with smooth and globally bounded coefficients and initial function). *There exists a natural number $d \in \mathbb{N}$, a globally bounded and infinitely often differentiable function $\mu : \mathbb{R}^d \to \mathbb{R}^d$, a symmetric nonnegative matrix $A = (A_{i,j})_{i,j \in \{1,2,\ldots,d\}} \in \mathbb{R}^{d \times d}$ and an infinitely often differentiable function $\varphi \in C_{\text{cpt}}^\infty(\mathbb{R}^d, \mathbb{R})$ with compact support such that there exists no globally bounded classical solution of the PDE*

$$\frac{\partial}{\partial t} u(t, x) = \sum_{i,j=1}^d A_{i,j} \cdot \frac{\partial^2}{\partial x_i \, \partial x_j} u(t, x) + \sum_{i=1}^d \mu_i(x) \cdot \frac{\partial}{\partial x_i} u(t, x),$$

(1.4)
$$u(0, x) = \varphi(x)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^d$. In addition, there exists a unique globally bounded viscosity solution $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ of (1.4) and this function fails to be locally Hölder continuous.

Theorem 1.1 follows immediately from Corollary 4.17 in Section 4 and from Theorem 3.1 in Section 3. More precisely, Corollary 4.17 and Theorem 3.1 imply that there exists an infinitely differentiable function $\varphi \in C_{\text{cpt}}^\infty(\mathbb{R}^3, \mathbb{R})$ with compact support such that the unique globally bounded viscosity solution $u : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ of the PDE

(1.5)
$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x_2^2} u(t, x) + \cos(x_3 \exp(x_2^3)) \cdot \frac{\partial}{\partial x_1} u(t, x)$$

with initial condition $u(0, x) = \varphi(x)$ for $(t, x) = (t, x_1, x_2, x_3) \in (0, \infty) \times \mathbb{R}^3$ is not locally Hölder continuous. In particular, the PDE (1.5) with $u(0, \cdot) = \varphi(\cdot)$ has no globally bounded classical solution. The PDE (1.5) has a globally bounded and

highly oscillating drift coefficient and a constant diffusion coefficient and serves as a counterexample to regularity preservation for Kolmogorov PDEs. An SDE with a globally bounded and highly oscillating diffusion coefficient and a vanishing drift coefficient has been presented in Li and Scheutzow [49] as a counterexample for strong completeness of SDEs. Another interesting observation is that the PDE (1.5) without the second-order term on the right-hand side of (1.5) preserves regularity and has a smooth classical solution and that the PDE (1.5) without the first-order term on the right-hand side of (1.5) also preserves regularity and has a smooth classical solution. Thus, the roughening effect of the PDE (1.5) is a consequence of the interplay between the first-order and the second-order term in (1.5). We add that Theorem 3.4 in Section 3 is a stronger version of Theorem 1.1 in which the roughening effect appears on every arbitrarily small open subset of the state space; see Section 3 and also Theorem 1.2 below for more details. Note that in both counterexamples to regularity preservation [PDE (1.5) and PDE (1.3)] it does not hold that all derivatives of μ and σ are globally bounded. Indeed, in both counterexamples the drift coefficient μ is not globally Lipschitz continuous. As observed above, regularity preservation is known if all derivatives of μ and σ are globally bounded. Moreover, note that the coefficients in our counterexample PDE (1.5) are analytic functions and that the initial function $\varphi : \mathbb{R}^d \to \mathbb{R}$ may be chosen to be analytic (see Theorem 3.1 for details). We emphasize that this does not contradict the classical Cauchy–Kovalevskaya theorem (e.g., Theorem 4.6.2 in Evans [18]) proving existence, uniqueness and analyticity of solutions of PDEs with analytic coefficients as the Cauchy–Kovalevskaya theorem applies to (1.4) in the case $A = 0$ only. Moreover, we would like to point out that Theorem 1.1 does not contradict to Theorems 7.1.3, 7.1.4 and 7.1.7 in Evans [18], which show the existence of a unique classical solution of (1.4) if A is strictly positive [note that A in (1.5) is nonnegative but not strictly positive].

Theorem 1.1 shows that a general existence theorem for globally bounded classical solutions of the PDE (1.1) cannot be established. However, it is possible to ensure the existence of a viscosity solution of the PDE (1.1) under rather general assumptions on the coefficients. More precisely, one of our main results, Theorem 4.16 below, establishes the existence of a within a certain class unique viscosity solution for every second-order linear Kolmogorov PDE whose coefficients are locally Lipschitz continuous and satisfy the Lyapunov-type inequality (4.74). To the best of our knowledge, this is the first result in the literature proving existence and uniqueness of solutions of the Kolmogorov PDE (1.1) in the above generality; see also the discussion after Theorem 4.16 for a short review of existence and uniqueness results for Kolmogorov PDEs. A crucial result on the route to Theorem 4.16 is the uniqueness result of Corollary 4.14 for viscosity solutions of degenerate parabolic second-order linear PDEs.

The roughening effect of the PDE (1.1) revealed in this first paragraph of this Introduction has a direct consequence on the literature on regularity analysis of SDEs. This is the subject of the next paragraph.

Regularity analysis of stochastic differential equations. For the rest of this Introduction, we use the following notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,\infty)}$ which supports a standard $(\mathcal{F}_t)_{t \in [0,\infty)}$-Brownian motion $W : [0,\infty) \times \Omega \to \mathbb{R}^m$ with continuous sample paths. It is a classical result that the above assumptions on μ and σ ensure the existence of a family $X^x = (X_1^x, \ldots, X_d^x) : [0,\infty) \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, of up to indistinguishability unique solution processes (see, e.g., Theorem 3.1.1 in [63]) with continuous sample paths of the SDE

$$(1.6) \qquad dX^x(t) = \mu\big(X^x(t)\big)\, dt + \sigma\big(X^x(t)\big)\, dW(t)$$

for $t \in (0,\infty)$ and $x \in \mathbb{R}^d$ and with $X^x(0) = x$ for all $x \in \mathbb{R}^d$ (see, e.g., Theorem 1 in Krylov [46]). Here, the function $\mu : \mathbb{R}^d \to \mathbb{R}^d$ is the infinitesimal mean and the function $\sigma \cdot \sigma^* : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is the infinitesimal covariance matrix of the SDE (1.6). It is also well known that the coercivity assumption on μ and the linear growth bound on σ additionally imply moment bounds $\sup_{x \in \{y \in \mathbb{R}^d : \|y\| \leq p\}} \mathbb{E}[\sup_{t \in [0,p]} \|X^x(t)\|^p] < \infty$ for all $p \in [0,\infty)$ for the solution processes of the SDE (1.6). The *transition semigroup* $P_t : C_b(\mathbb{R}^d, \mathbb{R}) \to C_b(\mathbb{R}^d, \mathbb{R})$, $t \in [0,\infty)$ of the SDE (1.6) is defined by $(P_t\varphi)(x) := \mathbb{E}[\varphi(X^x(t))]$ for all $t \in [0,\infty)$, $x \in \mathbb{R}^d$ and all $\varphi \in C_b(\mathbb{R}^d, \mathbb{R})$ where $C_b(\mathbb{R}^d, \mathbb{R})$ is as usual the space of globally bounded and continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$. Note for every $\varphi \in C_b(\mathbb{R}^d, \mathbb{R})$ that the function $\mathbb{R}^d \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ is continuous (see, e.g., Theorem 1.7 in Krylov [47]) and hence, the semigroup $(P_t)_{t \in [0,\infty)}$ is well defined. Observe also that the function $\mathbb{R}^d \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ is continuous for every $\varphi \in C_b(\mathbb{R}^d, \mathbb{R})$ although the SDE (1.6) is, in general, not strongly complete; see Li and Scheutzow [49] and see, for example, also Elworthy [15], Kunita [48] and Fang, Imkeller and Zhang [19] for further results on strong completeness of SDEs.

Theorem 1.1 in Hörmander [31] and Proposition 4.18 below imply that if the Hörmander condition (1.2) is fulfilled, then the semigroup is smoothing in the sense that $P_t(C_b(\mathbb{R}^d, \mathbb{R})) \subseteq C_b^\infty(\mathbb{R}^d, \mathbb{R})$ for all $t \in (0,\infty)$. To the best of our knowledge, it remained an open question in the nonhypoelliptic case whether SDEs with infinitely often differentiable coefficients such as (1.6) in general preserve regularity in the sense that $P_t(C_b^\infty(\mathbb{R}^d, \mathbb{R})) \subseteq C_b^\infty(\mathbb{R}^d, \mathbb{R})$ for all $t \in (0,\infty)$. This article answers this question to the negative. More precisely, the following theorem reveals that smooth functions with compact support may be mapped to nonsmooth functions by the transition semigroup of the SDE (1.6). In analogy to the well-known "smoothing effect" of many SDEs, we will say that the semigroup has a *roughening effect* in that case. Here is a simple two-dimensional example SDE with polynomial drift coefficient and linear diffusion coefficient which has this roughening effect. In the special case $d = 2$, $m = 1$ and $\mu(x) = (x_1 \cdot x_2, -x_1^2)$ and $\sigma(x) = (0, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, the SDE (1.6) reads as

$$(1.7) \qquad \begin{aligned} dX_1^x(t) &= X_1^x(t) \cdot X_2^x(t)\, dt, \\ dX_2^x(t) &= -X_1^x(t)^2\, dt + X_2^x(t)\, dW(t) \end{aligned}$$

for $t \in (0, \infty)$ and $x \in \mathbb{R}^2$. Observe that (1.3) is the Kolmogorov PDE of (1.7); see Corollary 4.17 for details. Moreover, note that $\langle x, \mu(x) \rangle = 0$ for all $x \in \mathbb{R}^2$ in this example. Thus, the solution process of the associated ordinary differential equation stays on the circle centered at $(0, 0) \in \mathbb{R}^2$ going through the starting point. Theorem 2.1 in Section 2 shows for the SDE (1.7) that there exists an infinitely often differentiable function $\varphi \in C^\infty_{\text{cpt}}(\mathbb{R}^d, \mathbb{R})$ with compact support such for every $t \in (0, \infty)$ the functions $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ and $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^2$ are continuous but not differentiable and not locally Lipschitz continuous. For every $t \in (0, \infty)$, we hence have the roughening effect $P_t(C^\infty_{\text{cpt}}(\mathbb{R}^d, \mathbb{R})) \not\subseteq C^1(\mathbb{R}^d, \mathbb{R})$ in the case of the SDE (1.7). The drift coefficient μ of the SDE (1.7) grows superlinearly. As above, the superlinear growth of μ is not necessary for the transition semigroup of the SDE to be roughening. This is subject of the next main result of this article.

THEOREM 1.2 (A counterexample to regularity preservation with degenerate additive noise). *There exists a natural number $d \in \mathbb{N}$, a globally bounded and infinitely often differentiable function $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and a constant function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, that is $\sigma(x) = \sigma(0)$ for all $x \in \mathbb{R}^d$, with the following properties. For every $t \in (0, \infty)$ the function $\mathbb{R}^d \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^d$ is continuous but nowhere locally Hölder continuous and for every nonempty open set $O \subset \mathbb{R}^d$ there exists an infinitely often differentiable function $\varphi \in C^\infty_{\text{cpt}}(\mathbb{R}^d, \mathbb{R})$ with compact support such that the function $O \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ is continuous but not locally Hölder continuous. In particular, for every $t \in (0, \infty)$ we have $P_t(C^\infty_{\text{cpt}}(\mathbb{R}^d, \mathbb{R})) \not\subseteq \bigcup_{\alpha \in (0, \infty)} C^\alpha(\mathbb{R}^d, \mathbb{R})$.*

Theorem 1.2 follows immediately from Theorem 3.4 in Section 3. The roughening effect of some SDEs with smooth coefficients revealed through example (1.7) and Theorem 1.2 above, has a direct consequence on the literature on numerical approximations of SDEs. This is the subject of the next paragraph.

Numerical approximations of stochastic differential equations. Starting with Maruyama's adaptation of Euler's method to SDEs in 1955 (see [51]), an extensive literature on the numerical approximation of solutions of SDEs has been published in the last six decades; see, for example, the books and overview articles [3, 23, 38, 41–43, 52, 53, 57] for extensive lists of references. A key objective in this field of research is to prove convergence of suitable numerical approximation processes to the solution process of the SDE and to establish a rate of convergence for the considered approximation scheme in the strong, in the almost sure or in the numerically weak sense.

Almost sure convergence rates of many numerical schemes such as the standard Euler method or the higher-order Milstein method are well known for the SDE (1.6) and even for a much larger class of nonlinear SDEs; see Gyöngy [22] and Jentzen, Kloeden and Neuenkirch [39]. Many applications, however, require

the numerical approximation of moments or other functionals of the solution process, for instance, the expected pay-off of an option in computational finance; see, for example, Glasserman [21] for details. For this reason, applications are particularly interested in strong and numerically weak convergence rates. The vast majority of research results establishing strong and numerically weak convergence rates assume that the coefficients of the SDE are globally Lipschitz continuous or at least that they satisfy the *global monotonicity condition* that there exists a real number $\rho \in \mathbb{R}$ such that $\langle x - y, \mu(x) - \mu(y) \rangle + \frac{1}{2} \sum_{k=1}^{m} \|\sigma_k(x) - \sigma_k(y)\|^2 \leq \rho \|x - y\|^2$ for all $x, y \in \mathbb{R}^d$ (see, e.g., Theorem 2.4 in Hu [33], Theorem 5.3 in Higham, Mao and Stuart [28], Schurz [67], Theorems 2 and 3 in Higham and Kloeden [27], Theorem 6.3 in Mao and Szpruch [50], Theorem 1.1 in Hutzenthaler, Jentzen and Kloeden [36], Theorem 3.2 in Wang and Gan [68]). Strong and numerically weak convergence rates without assuming global monotonicity are established in Gyöngy and Rásonyi [25] in the case of a class of scalar SDEs with globally Hölder continuous coefficients, in Dörsek [14] in the case of the two-dimensional stochastic Navier–Stokes equations and in Dereich, Neuenkirch and Szpruch [13], Alfonsi [1], Neuenkirch and Szpruch [58] in the case of a class of scalar SDEs (including, e.g., the Cox–Ingersoll–Ross process) that can be transformed in a suitable sense to SDEs that satisfy the global monotonicity assumption. The global monotonicity assumption is a serious restriction on the coefficients of the SDE and excludes many interesting SDEs in the literature (e.g., stochastic Lorenz equations, stochastic Duffing–van der Pol oscillators and the stochastic SIR model; see Section 4 in [34] for details and further examples). It remains an open problem to establish strong and numerically weak convergence rates in the general setting of the SDE (1.6).

In this article, we establish in the setting (1.6) the existence of an SDE with *globally bounded* and *infinitely often differentiable coefficients* for which the Euler approximations converge in the strong and in the numerically weak sense without any arbitrarily small polynomial rate of convergence. More precisely, our main result for the literature on the numerical approximation of SDEs is the following theorem.

THEOREM 1.3 (A counterexample to the rate of convergence in the numerical approximation of nonlinear SDEs with additive noise). *Let $T \in (0, \infty)$ and $x_0 \in \mathbb{R}^4$ be arbitrary. Then there exists a globally bounded and infinitely often differentiable function $\mu : \mathbb{R}^4 \to \mathbb{R}^4$ and a symmetric nonnegative matrix $B \in \mathbb{R}^{4 \times 4}$ such that the stochastic process $X : [0, T] \times \Omega \to \mathbb{R}^4$ with continuous sample paths satisfying $X(t) = x_0 + \int_0^t \mu(X(s)) \, ds + B W(t)$ for all $t \in [0, T]$ and its Euler–Maruyama approximations $Y^N : \{0, \frac{T}{N}, \frac{2T}{N}, \ldots, T\} \times \Omega \to \mathbb{R}^4$, $N \in \mathbb{N}$, satisfying $Y^N(0) = x_0$ and $Y^N(\frac{(n+1)T}{N}) = Y^N(\frac{nT}{N}) + \mu(Y^N(\frac{nT}{N}))\frac{T}{N} + B(W_{(n+1)T/N} -$*

$W_{nT/N})$ for all $n \in \{0, 1, \ldots, N - 1\}$, $N \in \mathbb{N}$, *fulfill that*

$$\lim_{N \to \infty} (N^{\alpha} \cdot \mathbb{E}[\|X(T) - Y^N(T)\|])$$

(1.8)
$$= \lim_{N \to \infty} (N^{\alpha} \cdot \|\mathbb{E}[X(T)] - \mathbb{E}[Y^N(T)]\|)$$

$$= \begin{cases} 0, & \alpha = 0, \\ \infty, & \alpha > 0, \end{cases}$$

for all $\alpha \in [0, \infty)$. In particular, for every $\alpha \in (0, \infty)$ there exists no real number $c_{\alpha} \in (0, \infty)$ such that $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^N(T)]\| \leq c_{\alpha} \cdot N^{-\alpha}$ for all $N \in \mathbb{N}$.

Theorem 1.3 follows immediately from Theorem 5.1 in Section 5. In the deterministic case $\sigma \equiv 0$, it is well known that the Euler approximations converge to the solution process of (1.6) with the rate 1. In the stochastic case $\sigma \not\equiv 0$, this rate of convergence can often not be achieved. In particular, Clark and Cameron [6] proved for an SDE in the setting of (1.6) that a class of Euler-type schemes cannot, in general, converge strongly with a higher-order than $\frac{1}{2}$. Since then, there have been many results on lower bounds of strong and numerically weak approximation errors for numerical approximation schemes of SDEs; see, for example, [4, 10, 29, 30, 35, 45, 55–57, 66] and the references therein. Now the observation of Theorem 1.3 is that there exist SDEs with smooth and globally bounded coefficients for which the standard Euler approximations converge in the strong and numerically weak sense *without any arbitrarily small polynomial rate of convergence*. To the best of our knowledge, Theorem 1.3 is the first result in the literature in which it has been established that Euler's method converges to the solution of an SDE with smooth coefficients in the strong and numerical weak sense without any arbitrarily small polynomial rate of convergence. Clearly, this lack of a rate of convergence is not a special property of the Euler scheme and holds for other schemes such as the Milstein scheme, too. It is based on the fact that our counterexample SDE for Theorem 1.3 [see (5.3)] suffers under the *roughening effect* revealed in Theorems 1.1 and 1.2 (see Corollary 5.2 and Theorem 5.1 in Section 5 for details).

Comparing Theorem 5.1 with Theorem 2.4 in Gyöngy [22] reveals the remarkable difference that the Euler approximations for some SDEs have almost sure convergence rate $\frac{1}{2}-$ but no strong and no numerically weak rate of convergence. More formally, Theorem 2.4 in [22] shows in the setting of Theorem 1.3 that there exist finite random variables $C_{\varepsilon} : \Omega \to [0, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, such that $\|X(T) - Y^N(T)\| \leq C_{\varepsilon} \cdot N^{(\varepsilon-1/2)}$, $\mathbb{P}$-a.s. for all $N \in \mathbb{N}$ and all $\varepsilon \in (0, \frac{1}{2})$. Taking expectation then results in $\mathbb{E}[\|X(T) - Y^N(T)\|] \leq \mathbb{E}[C_{\varepsilon}] \cdot N^{(\varepsilon-1/2)}$ for all $N \in \mathbb{N}$ and all $\varepsilon \in (0, \frac{1}{2})$ and from Theorem 1.3 we hence get that the error constants have infinite expectations, that is, $\mathbb{E}[C_{\varepsilon}] = \infty$ for all $\varepsilon \in (0, \frac{1}{2})$. In addition, we refer to Theorem 2.3 in Milstein and Tretyakov [54] for a weak convergence result restricted to certain subevents of the probability space. Finally, we emphasize that Monte Carlo simulations confirm the slow strong and numerically weak

convergence phenomenon of Euler's method revealed in Theorem 1.3. For details, the reader is referred to Figure 1 in Section 5 below.

2. Counterexamples to regularity preservation with linear multiplicative noise.

In this section, we establish the phenomenon of loss of regularity of the simple example SDE (1.7) with polynomial drift coefficient and linear diffusion coefficient. For this, we consider the following setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$, let $W : [0, \infty) \times \Omega \to \mathbb{R}$ be a one-dimensional standard $(\mathcal{F}_t)_{t \in [0, \infty)}$-Brownian motion with continuous sample paths and let $X^x = (X_1^x, X_2^x) : [0, \infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, be the up to indistinguishability unique solution processes with continuous sample paths of the SDE

$$
\begin{aligned}
(2.1) \qquad dX_1^x(t) &= X_1^x(t) \cdot X_2^x(t) \, dt, \\
dX_2^x(t) &= -\big(X_1^x(t)\big)^2 \, dt + X_2^x(t) \, dW(t)
\end{aligned}
$$

for $t \in (0, \infty)$ and $x \in \mathbb{R}^2$ satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^2$. Corollary 2.6 in Gyöngy and Krylov [24] ensures that the processes $X^x : [0, \infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, do indeed exist. The following Theorem 2.1 shows that the semigroup associated with the SDE (2.1) loses regularity in the sense that there exists an infinitely often differentiable function with compact support, which is mapped to a nonsmooth function by the semigroup.

THEOREM 2.1 (A counterexample to regularity preservation with linear multiplicative noise). *Let* $X^x : [0, \infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, *be solution processes of the SDE* (2.1) *with continuous sample paths and with* $X^x(0) = x$ *for all* $x \in \mathbb{R}^2$. *Then* $\sup_{x \in \{y \in \mathbb{R}^2 : \|y\| \leq p\}} \mathbb{E}[\sup_{t \in [0, p]} \|X^x(t)\|^p] < \infty$ *for all* $p \in [0, \infty)$ *and there exists an infinitely often differentiable function* $\varphi \in C_{\mathrm{cpt}}^\infty(\mathbb{R}^2, \mathbb{R})$ *with compact support such that for every* $t, p \in (0, \infty)$ *the mappings* $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^2$, $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ *and* $\mathbb{R}^2 \ni x \mapsto X^x(t) \in L^p(\Omega; \mathbb{R}^2)$ *are continuous but not locally Lipschitz continuous and not differentiable.*

The proof of Theorem 2.1 is deferred to the end of this section. The proof of Theorem 2.1 uses the following simple lemma.

LEMMA 2.2 (Restricted exponential integrals of a geometric Brownian motion). *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $W : [0, \infty) \times \Omega \to \mathbb{R}$ *be a one-dimensional standard Brownian motion with continuous sample paths. Then*

$$
(2.2) \qquad \mathbb{E}\bigg[\mathbb{1}_{\{a \leq e^{W(t)} \leq b\}} \exp\bigg(c \cdot \int_0^t e^{W(s)} \, ds \bigg) \bigg] = \infty
$$

for all $t, a, b, c \in (0, \infty)$ *with* $a < b$.

PROOF.   Independence of $W(t)$ from $(W(s) - \frac{s}{t}W(t))_{s \in [0,t]}$ for all $t \in (0, \infty)$ implies

$$\mathbb{E}\left[\mathbb{1}_{\{a \le e^{W(t)} \le b\}} \exp\left(c \cdot \int_0^t e^{W(s)}\, ds\right)\right]$$

$$\ge \mathbb{E}\left[\mathbb{1}_{\{a \le e^{W(t)} \le b\}} \exp\left(c \cdot \int_0^t e^{(W(s) - (s/t)W(t))} a^{s/t}\, ds\right)\right]$$

$$(2.3) \quad \ge \mathbb{P}[a \le e^{W(t)} \le b] \cdot \mathbb{E}\left[\exp\left(tc \cdot \min(a, 1) \cdot \frac{1}{t} \int_0^t e^{(W(s) - (s/t)W(t))}\, ds\right)\right]$$

$$\ge \mathbb{P}[a \le e^{W(t)} \le b]$$

$$\times \mathbb{E}\left[\exp\left(tc \cdot \min(a, 1) \cdot \exp\left(\frac{1}{t} \int_0^t W(s) - \frac{s}{t} W(t)\, ds\right)\right)\right]$$

for all $t, a, b, c \in (0, \infty)$ with $a < b$ where we used Jensen's inequality and convexity of the exponential function in the last step. The time integrated Brownian bridge $\int_0^t W(s) - \frac{s}{t} W(t)\, ds$ on the right-hand side of (2.3) is normally distributed with mean 0 and variance

$$\mathbb{E}\left[\left(\int_0^t W(s) - \frac{s}{t} W(t)\, ds\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t \int_0^t \left(W(s) - \frac{s}{t} W(t)\right)\left(W(r) - \frac{r}{t} W(t)\right) dr\, ds\right]$$

$$(2.4) \quad = \int_0^t \int_0^t \mathbb{E}\left[W(s)W(r) - \frac{r}{t} W(s)W(t) - \frac{s}{t} W(r)W(t) + \frac{sr}{t^2}(W(t))^2\right] dr\, ds$$

$$= \int_0^t \int_0^t \left(\min(r, s) - \frac{rs}{t} - \frac{sr}{t} + \frac{sr}{t}\right) dr\, ds$$

$$= 2 \int_0^t \int_0^s \left(r - \frac{rs}{t}\right) dr\, ds = \int_0^t \left(s^2 - \frac{s^3}{t}\right) ds = \frac{t^3}{12} \in (0, \infty)$$

for every $t \in (0, \infty)$. As the double exponential normal distribution has infinite mean, we conclude that the right-hand side of (2.3) is infinite for all $t, a, b, c \in (0, \infty)$. This finishes the proof Lemma 2.2. $\square$

The proof of the following Lemma 2.3 makes use of Lemma 2.2. Using Lemma 2.3, the proof of Theorem 2.1 is then completed at the end of this section.

LEMMA 2.3.   *Let* $X^x : [0, \infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, *be solution processes of the SDE* (2.1) *with continuous sample paths and with* $X^x(0) = x$ *for all* $x \in \mathbb{R}^2$. *Then*

$\sup_{x \in \{y \in \mathbb{R}^2 : \|y\| \le p\}} \mathbb{E}[\sup_{t \in [0,p]} \|X^x(t)\|^p] < \infty$ *for all* $p \in [0, \infty)$ *and*

$$\text{(2.5)} \quad \begin{aligned} \lim_{0 \ne x_1 \to 0} & \left(\frac{1}{x_1} \cdot \mathbb{E}[X_1^{(x_1, x_2)}(t) - X_1^{(0, x_2)}(t)] \right) \\ & = \infty = \lim_{0 \ne x_1 \to 0} \left(\frac{1}{|x_1|} \cdot \|X_1^{(x_1, x_2)}(t) - X_1^{(0, x_2)}(t)\|_{L^p(\Omega; \mathbb{R})} \right) \end{aligned}$$

for all $t, x_2, p \in (0, \infty)$ *and there exists an infinitely often differentiable function* $\varphi \in C_{\text{cpt}}^\infty(\mathbb{R}^2, \mathbb{R})$ *with compact support such that* $\lim_{0 \ne x_1 \to 0} (\frac{1}{x_1} \cdot \mathbb{E}[\varphi(X^{(x_1, x_2)}(t)) - \varphi(X^{(0, x_2)}(t))]) = \infty$ *for all* $t, x_2 \in (0, \infty)$.

PROOF. The global Lipschitz continuity of σ, the local Lipschitz continuity of μ and $\langle x, \mu(x) \rangle = 0$ for all $x \in \mathbb{R}^2$ imply that

$$\sup_{x \in \{y \in \mathbb{R}^2 : \|y\| \le p\}} \mathbb{E}\left[\sup_{t \in [0,p]} \|X^x(t)\|^p \right] < \infty$$

for all $p \in [0, \infty)$. Next, we disprove local Lipschitz continuity of the mapping $\mathbb{R}^2 \ni x \mapsto X_1^x(t) \in L^p(\Omega; \mathbb{R})$ for every $t, p \in (0, \infty)$. More precisely, aiming at a contradiction, we assume that the second equality in (2.5) is false. Then there exist positive real numbers $t, x_2, p \in (0, \infty)$ and a sequence of real numbers $h_n \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$, such that $\lim_{n \to \infty} h_n = 0$ and such that $\limsup_{n \to \infty} \frac{1}{|h_n|} \|X_1^{(h_n, x_2)}(t) - X_1^{(0, x_2)}(t)\|_{L^p(\Omega; \mathbb{R})} < \infty$. Theorem 1.7 in Krylov [47] (see also Proposition 3.2.1 in Prévôt and Röckner [63]) yields that $\sup_{s \in [0,t]} \|X^{(h_n, x_2)}(s) - X^{(0, x_2)}(s)\| \to 0$ in probability as $n \to \infty$. Hence, there exists a strictly increasing sequence $n_k \in \mathbb{N}$, $k \in \mathbb{N}$, of natural numbers such that $\lim_{k \to \infty} \sup_{s \in [0,t]} \|X^{(h_{n_k}, x_2)}(s) - X^{(0, x_2)}(s)\| = 0$, $\mathbb{P}$-a.s.; see, for example, Corollary 6.13 in Klenke [40]. Applying this, Fatou's lemma and Lemma 2.2 implies

$$\begin{aligned} \infty > & \limsup_{k \to \infty} \left(\frac{1}{|h_{n_k}|} \|X_1^{(h_{n_k}, x_2)}(t) - X_1^{(0, x_2)}(t)\|_{L^p(\Omega; \mathbb{R})} \right) \\ = & \limsup_{k \to \infty} \left(\frac{1}{|h_{n_k}|} \|X_1^{(h_{n_k}, x_2)}(t)\|_{L^p(\Omega; \mathbb{R})} \right) \\ = & \limsup_{k \to \infty} \left\| \exp\left(\int_0^t X_2^{(h_{n_k}, x_2)}(s) \, ds \right) \right\|_{L^p(\Omega; \mathbb{R})} \\ \ge & \left\| \liminf_{k \to \infty} \left\{ \exp\left(\int_0^t X_2^{(h_{n_k}, x_2)}(s) \, ds \right) \right\} \right\|_{L^p(\Omega; \mathbb{R})} \\ = & \left\| \exp\left(\int_0^t X_2^{(0, x_2)}(s) \, ds \right) \right\|_{L^p(\Omega; \mathbb{R})} \\ \text{(2.6)} \quad = & \left\| \exp\left(\int_0^t e^{(W(s) - s/2)} \, ds \cdot x_2 \right) \right\|_{L^p(\Omega; \mathbb{R})} \end{aligned}$$

$$\geq \left(\mathbb{E}\left[\exp\left(\int_0^t e^{W(s)}\, ds \cdot \frac{px_2}{e^{t/2}}\right) \cdot \mathbb{1}_{\{1 \leq e^{W(t)} \leq 2\}}\right]\right)^{1/p}$$
$$= \infty.$$

This contradiction implies that the second equality in (2.5) is true. The first equality in (2.5) follows from the second equality in (2.5) as $\frac{1}{x_1}(X_1^{(x_1,x_2)}(t) - X_1^{(0,x_2)}(t)) \in [0, \infty)$ for all $x_1 \in \mathbb{R} \setminus \{0\}$ and all $x_2 \in (0, \infty)$. In the next step, let $c \in (0, \infty)$ be an arbitrary fixed real number and let $\psi_1 \colon \mathbb{R} \to \mathbb{R}$ and $\psi_2 \colon \mathbb{R} \to [0, \infty)$ be two infinitely often differentiable functions with $x \cdot \psi_1(x) \geq 0$ for all $x \in \mathbb{R}$, with $\psi_1(x) = \psi_2(x) = 0$ for all $x \in \mathbb{R} \setminus [-c-1, c+1]$ and with $\psi_1(x) = x$ and $\psi_2(x) = 1$ for all $x \in [-c, c]$. Due to partition of unity, such functions indeed exist. Next, let $\varphi \colon \mathbb{R}^2 \to \mathbb{R}$ be given by $\varphi(x_1, x_2) = \psi_1(x_1) \cdot \psi_2(x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Note that $\varphi \in C_{\mathrm{cpt}}^\infty(\mathbb{R}^2, \mathbb{R})$ is an infinitely often differentiable function with compact support. We now show that $\lim_{0 \neq x_1 \to 0}(\frac{1}{x_1} \cdot \mathbb{E}[\varphi(X^{(x_1,x_2)}(t)) - \varphi(X^{(0,x_2)}(t))]) = \infty$ for all $t, x_2 \in (0, \infty)$. Aiming at a contradiction, assume that there exist positive real numbers $t, x_2 \in (0, \infty)$ and a sequence $h_n \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}$, such that $\lim_{n \to \infty} h_n = 0$ and such that

$$(2.7) \qquad \limsup_{n \to \infty}\left(\frac{1}{h_n} \cdot \mathbb{E}[\varphi(X_1^{(h_n,x_2)}(t)) - \varphi(X_1^{(0,x_2)}(t))]\right) < \infty.$$

Theorem 1.7 in Krylov [47] yields that $\sup_{s \in [0,t]} \|X^{(h_n,x_2)}(s) - X^{(0,x_2)}(s)\| \to 0$ in probability as $n \to \infty$. Hence, there exists a strictly increasing sequence $n_k \in \mathbb{N}$, $k \in \mathbb{N}$, of natural numbers such that $\lim_{k \to \infty} \sup_{s \in [0,t]} \|X^{(h_{n_k},x_2)}(s) - X^{(0,x_2)}(s)\| = 0$, $\mathbb{P}$-a.s.; see, for example, Corollary 6.13 in Klenke [40]. Applying this, the fact $\frac{1}{x_1}(\varphi(x_1, x_2) - \varphi(0, x_2)) \in [0, \infty)$ for all $x_1 \in \mathbb{R} \setminus \{0\}$ and all $x_2 \in (0, \infty)$, Fatou's lemma and Lemma 2.2 then results in

$$\infty > \limsup_{k \to \infty}\left(\frac{1}{h_{n_k}}\mathbb{E}[\varphi(X^{(h_{n_k},x_2)}(t)) - \varphi(X^{(0,x_2)}(t))]\right)$$
$$= \limsup_{k \to \infty} \mathbb{E}\left[\left|\frac{\varphi(X^{(h_{n_k},x_2)}(t)) - \varphi(X^{(0,x_2)}(t))}{h_{n_k}}\right|\right]$$
$$\geq \mathbb{E}\left[\liminf_{k \to \infty}\left|\frac{\varphi(X^{(h_{n_k},x_2)}(t)) - \varphi(X^{(0,x_2)}(t))}{h_{n_k}}\right|\right]$$
$$= \mathbb{E}\left[\liminf_{k \to \infty}\left(\frac{\varphi(X^{(h_{n_k},x_2)}(t)) - \varphi(X^{(0,x_2)}(t))}{h_{n_k}}\right)\right]$$
$$= \mathbb{E}\left[\psi_2(X_2^{(0,x_2)}(t))\left(\liminf_{k \to \infty} \frac{X_1^{(h_{n_k},x_2)}(t)}{h_{n_k}}\right)\right]$$
$$(2.8) \qquad = \mathbb{E}\left[\psi_2(X_2^{(0,x_2)}(t)) \cdot \exp\left(\int_0^t e^{(W(s)-s/2)}\, ds \cdot x_2\right)\right]$$

$$\geq \mathbb{E}\left[\mathbb{1}_{\{c/2 \leq x_2 \cdot \exp(W(t) - t/2) \leq c\}} \cdot \exp\left(\int_0^t e^{(W(s) - s/2)} \, ds \cdot x_2\right)\right]$$
$$= \infty.$$

This contradiction implies that $\lim_{0 \neq x_1 \to 0}(\frac{1}{x_1} \cdot \mathbb{E}[\varphi(X^{(x_1, x_2)}(t)) - \varphi(X^{(0, x_2)}(t))]) = \infty$ for all t, $x_2 \in (0, \infty)$. The proof of Lemma 2.3 is thus completed. $\square$

PROOF OF THEOREM 2.1.   Theorem 1.7 in Krylov [47] (see also Proposition 3.2.1 in Prévôt and Röckner [63]), in particular, shows for every $t \in [0, \infty)$ that the mapping

$$(2.9) \qquad\qquad \mathbb{R}^2 \ni x \mapsto X^x(t) \in L^0(\Omega; \mathbb{R}^2)$$

is continuous. This implies for every $\varphi \in C_{\text{cpt}}^\infty(\mathbb{R}^2, \mathbb{R})$ and every $t \in [0, \infty)$ that the mapping $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ is continuous. Moreover, Lemma 2.3 proves that $\sup_{x \in \{y \in \mathbb{R}^2 : \|y\| \leq p\}} \mathbb{E}[\sup_{t \in [0, p]} \|X^x(t)\|^p] < \infty$ for all $p \in [0, \infty)$. Combining this, (2.9), Corollary 6.21 in Klenke [40] and Theorem 6.25 in Klenke [40] shows for every $t, p \in [0, \infty)$ that the mappings $\mathbb{R}^2 \ni x \mapsto X^x(t) \in L^p(\Omega; \mathbb{R}^2)$ and $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^2$ are continuous. Furthermore, Lemma 2.3 implies that there exists an infinitely often differentiable function $\varphi \in C_{\text{cpt}}^\infty(\mathbb{R}^2, \mathbb{R})$ with compact support such that for every $t, p \in (0, \infty)$ the mappings $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^2$, $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ and $\mathbb{R}^2 \ni x \mapsto X^x(t) \in L^p(\Omega; \mathbb{R}^2)$ are not locally Lipschitz continuous and not differentiable. The proof of Theorem 2.1 is thus completed. $\square$

In the remainder of this section, we briefly consider slightly modified versions of the SDE (2.1). The generator of the SDE (2.1) is nowhere elliptic. We remark that the phenomenon of loss of regularity may also appear for an SDE whose generator is in many points of the state space elliptic. For example, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$, let $W = (W_1, W_2) : [0, \infty) \times \Omega \to \mathbb{R}^2$ be a two-dimensional standard $(\mathcal{F}_t)_{t \in [0, \infty)}$-Brownian motion and let $X^x = (X_1^x, X_2^x) : [0, \infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, be the up to indistinguishability unique solution processes with continuous sample paths of the SDE

$$(2.10) \qquad \begin{aligned} dX_1^x(t) &= X_1^x(t) \cdot X_2^x(t) \, dt + X_1^x(t) \, dW_1(t), \\ dX_2^x(t) &= -\left(X_1^x(t)\right)^2 dt + X_2^x(t) \, dW_2(t) \end{aligned}$$

for $t \in (0, \infty)$ and $x \in \mathbb{R}^2$ satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^2$. The generator of the SDE (2.10) is in every point $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 \cdot x_2 \neq 0$ elliptic but there exists a function $\varphi \in C_{\text{cpt}}^\infty(\mathbb{R}^d, \mathbb{R})$ such that for every $t \in (0, \infty)$ the functions $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^2$ and $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ are not locally Lipschitz continuous. The proof of this statement is completely analogous as in the case

of the SDE (2.1). Furthermore, the same statement holds if the two independent standard Brownian motion in (2.10) are replaced by one and the same standard Brownian motion. More precisely, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$ and if $W : [0, \infty) \times \Omega \to \mathbb{R}$ is a one-dimensional standard $(\mathcal{F}_t)_{t \in [0, \infty)}$-Brownian motion, then the up to indistinguishability unique solution processes $X^x = (X_1^x, X_2^x) : [0, \infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, of the SDE

$$(2.11) \qquad dX^x(t) = \begin{pmatrix} X_1^x(t) \cdot X_2^x(t) \\ -\left(X_1^x(t)\right)^2 \end{pmatrix} dt + X^x(t) \, dW(t)$$

for $t \in (0, \infty)$ and $x \in \mathbb{R}^2$ with continuous sample paths and with $X^x(0) = x$ for all $x \in \mathbb{R}^2$ fulfill that there exists a function $\varphi \in \mathrm{C}_{\mathrm{cpt}}^{\infty}(\mathbb{R}^2, \mathbb{R})$ such that for every $t \in (0, \infty)$ the functions $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^2$ and $\mathbb{R}^2 \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ are not locally Lipschitz continuous.

3. Counterexamples to regularity preservation with degenerate additive noise. In this section, we show the roughening effect for an example SDE with globally bounded and infinitely often differentiable coefficients. For this, it suffices to consider the following counterexample to regularity preservation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W : [0, \infty) \times \Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion and let $X^x = (X_1^x, X_2^x, X_3^x) : [0, \infty) \times \Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, be the up to indistinguishability unique solution processes with continuous sample paths of the SDE

$$\begin{aligned} dX_1^x(t) &= \cos\left(X_3^x(t) \cdot \exp(X_2^x(t)^3)\right) dt, \\ (3.1) \qquad dX_2^x(t) &= \sqrt{2} \, dW(t), \\ dX_3^x(t) &= 0 \, dt \end{aligned}$$

for $t \in [0, \infty)$ and $x \in \mathbb{R}^3$ satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^3$. Observe that

$$(3.2) \qquad X_1^x(t) = x_1 + \int_0^t \cos\left(x_3 \cdot \exp([x_2 + \sqrt{2}W(s)]^3)\right) ds,$$

$\mathbb{P}$-a.s. for all $t \in [0, \infty)$ and all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

THEOREM 3.1 (A counterexample to regularity preservation with degenerate additive noise). *Let $T \in (0, \infty)$ and let $X^x : [0, \infty) \times \Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, be solution processes of the SDE (3.1) satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^3$. Then there exists an infinitely often differentiable function $\varphi \in \mathrm{C}_{\mathrm{cpt}}^{\infty}(\mathbb{R}^3, \mathbb{R})$ with compact support such that for every $t \in (0, T]$ the functions $\mathbb{R}^3 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^3$ and $\mathbb{R}^3 \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ are continuous but not locally Hölder continuous.*

In the following, regularity properties of the solution processes $X^x = (X_1^x, X_2^x, X_3^x) : [0, \infty) \times \Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, of the SDE (3.1) are investigated in order to prove

Theorem 3.1. To do so, we first establish a few auxiliary results. We begin with a simple lemma on trigonometric integrals.

LEMMA 3.2.   *Let $a, b \in \mathbb{R}$ be real numbers with $a < b$, let $\psi : [a, b] \to [0, \infty)$ be a continuously differentiable function and let $\varphi : [a, b] \to \mathbb{R}$ be a twice continuously differentiable function with $e^{i \cdot \varphi(a)} = i$ and with $\varphi'(x) \geq 0$, $\varphi''(x) \geq 0$ and $\psi'(x) \leq 0$ for all $x \in [a, b]$. Then $\int_a^b \cos(\varphi(x)) \psi(x) \, dx \leq 0$.*

PROOF.   First, assume w.l.o.g. that $\varphi(b) \geq \varphi(a) + \pi$ (otherwise we have $\cos(\varphi(x)) \leq 0$ for all $x \in [a, b]$, and hence $\int_a^b \cos(\varphi(x)) \psi(x) \, dx \leq 0$). Moreover, assume w.l.o.g. that $\varphi'(x) > 0$ for all $x \in (a, b]$ (otherwise consider $\varphi|_{[\tilde{a}, b]} : [\tilde{a}, b] \to \mathbb{R}$ where $\tilde{a} := \inf(\{x \in [a, b] : \varphi'(x) > 0\} \cup \{b\})$ and observe that $\int_a^b \cos(\varphi(x)) \psi(x) \, dx = \int_{\tilde{a}}^b \cos(\varphi(x)) \psi(x) \, dx)$. In particular, $\varphi : [a, b] \to \mathbb{R}$ is strictly increasing and there exists a unique strictly increasing continuous function $\varphi^{-1} : [\varphi(a), \varphi(b)] \to [a, b]$ with $\varphi^{-1}(\varphi(x)) = x$ for all $x \in [a, b]$ and with $\varphi(\varphi^{-1}(x)) = x$ and $(\varphi^{-1})'(x) = \frac{1}{\varphi'(\varphi^{-1}(x))} > 0$ for all $x \in (\varphi(a), \varphi(b))$. Integration by substitution and integration by parts therefore imply

$$
\begin{aligned}
\int_a^b & \cos(\varphi(x)) \psi(x) \, dx \\
&= \int_{\varphi(a)}^{\varphi(b)} \cos(x) \cdot \psi(\varphi^{-1}(x)) \cdot (\varphi^{-1})'(x) \, dx \\
&= \int_{\varphi(a)}^{\varphi(b)} \frac{\cos(x) \cdot \psi(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} \, dx \\
&= \frac{[\sin(\varphi(b)) - 1]\psi(\varphi^{-1}(\varphi(b)))}{\varphi'(\varphi^{-1}(\varphi(b)))} \\
&\quad - \int_{\varphi(a)}^{\varphi(b)} [\sin(x) - 1] \left[\frac{\psi'(\varphi^{-1}(x))}{[\varphi'(\varphi^{-1}(x))]^2} - \frac{\psi(\varphi^{-1}(x))\varphi''(\varphi^{-1}(x))}{[\varphi'(\varphi^{-1}(x))]^3} \right] dx \\
&\leq 0.
\end{aligned}
$$

(3.3)

This completes the proof of Lemma 3.2.   □

The next lemma analyzes suitable regularity properties of the solution processes $X^x = (X_1^x, X_2^x, X_3^x) : [0, \infty) \times \Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, of the SDE (3.1). Its proof is based on Lemma 3.2.

LEMMA 3.3 (A lower bound).   *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W : [0, \infty) \times \Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion. Then*

$$
(3.4) \quad 1 - \mathbb{E}[\cos(h \cdot \exp([x + W(t)]^3))] \geq \exp\left(\frac{-8}{t} \left[\left| \ln\left(\frac{\pi}{2h} \right) \right|^{2/3} + x^2 \right] \right)
$$

for all $h \in (0, \frac{\pi}{2} \exp(-|[\sqrt{t} + x] \vee 0|^3)]$, $t \in (0, \infty)$ *and all* $x \in \mathbb{R}$ *and*

$$
\int_0^t \mathbb{E}\big[\mathbb{1}_{\{W(t) \in A\}}\big(1 - \cos(h \cdot e^{[x + W(s)]^3})\big)\big]\,ds
$$

(3.5)

$$
\geq \frac{t}{3} \cdot \mathbb{E}\big[\mathbb{1}_{\{W(t) \in A\}} e^{-64|W(t)|^2/t}\big] \cdot \exp\left(\frac{-64}{t}\Big[\Big|\ln\Big(\frac{\pi}{2h}\Big)\Big|^{2/3} + x^2\Big]\right)
$$

for all $h \in (0, \frac{\pi}{2} \exp(-[\sqrt{t} + |x| + \sup_{a \in A} |a|]^3)]$, $x \in \mathbb{R}$, $t \in (0, \infty)$ *and all bounded and Borel measurable sets* $A \subset \mathbb{R}$.

PROOF. First of all, define a family $\varphi_{t,x,h} : [\frac{[\ln(\pi/(2h))]^{1/3} - x}{\sqrt{t}}, \infty) \to \mathbb{R}$, $(t, x, h) \in \{(0, \infty) \times \mathbb{R} \times (0, \infty) : h \leq \frac{\pi}{2} \exp(-|x \vee 0|^3)\}$, of functions by

(3.6) $\varphi_{t,x,h}(y) := h \cdot \exp([x + \sqrt{t}y]^3)$

for all $y \in [\frac{[\ln(\pi/(2h))]^{1/3} - x}{\sqrt{t}}, \infty)$, $t \in (0, \infty)$, $h \in (0, \frac{\pi}{2} \exp(-|x \vee 0|^3)]$ and all $x \in \mathbb{R}$. Observe that

(3.7) $\varphi'_{t,x,h}(y) = 3\sqrt{t}[x + \sqrt{t}y]^2 \varphi_{t,x,h}(y) \geq 0$

and

(3.8) $\varphi''_{t,x,h}(y) = 6t[x + \sqrt{t}y]\varphi_{t,x,h}(y) + 9t[x + \sqrt{t}y]^4 \varphi_{t,x,h}(y) \geq 0$

for all $y \in [\frac{[\ln(\pi/(2h))]^{1/3} - x}{\sqrt{t}}, \infty)$, $t \in (0, \infty)$, $h \in (0, \frac{\pi}{2} \exp(-|x \vee 0|^3)]$ and all $x \in \mathbb{R}$. In addition, note that $\varphi_{t,x,h}(\frac{[\ln(\pi/(2h))]^{1/3} - x}{\sqrt{t}}) = \frac{\pi}{2}$ for all $t \in (0, \infty)$, $h \in (0, \frac{\pi}{2} \exp(-|x \vee 0|^3)]$ and all $x \in \mathbb{R}$. We can thus apply Lemma 3.2 to obtain that

(3.9) $\frac{1}{\sqrt{2\pi}} \int_{([\ln(\pi/(2h))]^{1/3} - x)/\sqrt{t}}^{\infty} \cos(h \cdot \exp([x + \sqrt{t}y]^3)) e^{-y^2/2}\,dy \leq 0$

for all $t \in (0, \infty)$, $h \in (0, \frac{\pi}{2} \exp(-|x \vee 0|^3)]$ and all $x \in \mathbb{R}$. This implies

$$
\mathbb{E}\big[\cos(h \cdot \exp([x + W(t)]^3))\big]
$$

$$
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(h \cdot \exp([x + \sqrt{t}y]^3)) e^{-y^2/2}\,dy
$$

(3.10) $\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{([\ln(\pi/(2h))]^{1/3} - x)/\sqrt{t}} \cos(h \cdot \exp([x + \sqrt{t}y]^3)) e^{-y^2/2}\,dy$

$$
\leq \mathbb{P}\left[W_1 \leq \frac{[\ln(\pi/(2h))]^{1/3} - x}{\sqrt{t}}\right]
$$

$$
= 1 - \mathbb{P}\left[W_1 > \frac{[\ln(\pi/(2h))]^{1/3} - x}{\sqrt{t}}\right]
$$

for all $t \in (0, \infty)$, $h \in (0, \frac{\pi}{2} \exp(-|x \vee 0|^3)]$ and all $x \in \mathbb{R}$. Moreover, Lemma 22.2 in Klenke [40] yields

$$(3.11) \qquad \mathbb{P}[W_1 > y] \geq \frac{e^{-y^2/2}}{y\sqrt{2\pi}(1 + y^{-2})} \geq \frac{e^{-y^2/2}}{y\sqrt{8\pi}} \geq e^{-4y^2}$$

for all $y \in [1, \infty)$. Combining this and inequality (3.10) then shows

$$
\begin{aligned}
1 - \mathbb{E}\big[\cos(h \cdot \exp([x + W(t)]^3))\big] &\geq \mathbb{P}\left[W_1 > \frac{[\ln(\pi/(2h))]^{1/3} - x}{\sqrt{t}}\right] \\
&\geq \exp\left(\frac{-4|[\ln(\pi/(2h))]^{1/3} - x|^2}{t}\right)
\end{aligned}
$$
(3.12)

for all $h \in (0, \frac{\pi}{2} \exp(-|[\sqrt{t} + x] \vee 0|^3)]$, $t \in (0, \infty)$ and all $x \in \mathbb{R}$ and the estimate $-|a + b|^2 \geq -2a^2 - 2b^2$ for all $a, b \in \mathbb{R}$ therefore results in the first inequality in (3.4). Next, the first inequality in (3.4) implies

$$
\begin{aligned}
\mathbb{E}\big[\mathbb{1}_{\{W(t)\in A\}} &|1 - \cos(h \cdot \exp([x + W(s)]^3))|\big] \\
&= \mathbb{E}\left[\mathbb{1}_{\{W(t)\in A\}}\mathbb{E}\left[1 - \cos\left(h \cdot \exp\left(\left[x + \frac{s}{t}W(t) + W(s)\right.\right.\right.\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\left.\left.\left.\left. - \frac{s}{t}W(t)\right]^3\right)\right)\bigg|W(t)\right]\right] \\
&\geq \mathbb{E}\left[\mathbb{1}_{\{W(t)\in A\}} \exp\left(\frac{-8t}{s(t-s)}\left[\left|\ln\left(\frac{\pi}{2h}\right)\right|^{2/3} + \left[x + \frac{s}{t}W(t)\right]^2\right]\right)\right]
\end{aligned}
$$
(3.13)

for all $h \in (0, \frac{\pi}{2} \exp(-[\sqrt{t} + |x| + \sup_{a\in A}|a|]^3)]$, $x \in \mathbb{R}$, $s, t \in (0, \infty)$ with $s < t$ and all bounded and Borel measurable sets $A \subset \mathbb{R}$. Hence, we get

$$
\begin{aligned}
\int_0^t &\mathbb{E}\big[\mathbb{1}_{\{W(t)\in A\}}|1 - \cos(h \cdot \exp([x + W(s)]^3))|\big]\,ds \\
&\geq \int_{t/3}^{2t/3} \mathbb{E}\big[\mathbb{1}_{\{W(t)\in A\}}|1 - \cos(h \cdot \exp([x + W(s)]^3))|\big]\,ds \\
&\geq \int_{t/3}^{2t/3} \mathbb{E}\left[\mathbb{1}_{\{W(t)\in A\}} \exp\left(\frac{-8t}{s(t-s)}\left[\left|\ln\left(\frac{\pi}{2h}\right)\right|^{2/3} + \left[x + \frac{s}{t}W(t)\right]^2\right]\right)\right]\,ds \\
&\geq \frac{t}{3} \cdot \mathbb{E}\left[\mathbb{1}_{\{W(t)\in A\}} \exp\left(\frac{-64}{t}\left[\left|\ln\left(\frac{\pi}{2h}\right)\right|^{2/3} + x^2 + |W(t)|^2\right]\right)\right]
\end{aligned}
$$
(3.14)

for all $h \in (0, \frac{\pi}{2} \exp(-[\sqrt{t} + |x| + \sup_{a\in A}|a|]^3)]$, $x \in \mathbb{R}$, $t \in (0, \infty)$ and all bounded and Borel measurable sets $A \subset \mathbb{R}$. This completes the proof of Lemma 3.3. $\square$

We are now ready to prove Theorem 3.1 stated at the beginning of this section. Its proof uses the lower bound established in Lemma 3.3 above.

PROOF OF THEOREM 3.1.   First of all, note that (3.2) and Lemma 3.3 imply that

$$\lim_{h \searrow 0}\left(\frac{\mathbb{E}[X_1^{(0,0,0)}(t) - X_1^{(0,0,h)}(t)]}{h^\varepsilon}\right)$$

$$= \lim_{h \searrow 0}\left(\frac{\mathbb{E}[\int_0^t 1 - \cos(h \cdot \exp([\sqrt{2}W(s)]^3))\,ds]}{h^\varepsilon}\right)$$

$$= \lim_{h \searrow 0}\left(\frac{\int_0^t 1 - \mathbb{E}[\cos(h \cdot \exp([W(2s)]^3))]\,ds}{h^\varepsilon}\right)$$

$$= \lim_{h \searrow 0}\left(\frac{\int_0^{2t} 1 - \mathbb{E}[\cos(h \cdot \exp([W(s)]^3))]\,ds}{2h^\varepsilon}\right)$$

(3.15)

$$\geq \lim_{h \searrow 0}\left(\frac{\int_t^{2t} 1 - \mathbb{E}[\cos(h \cdot \exp([W(s)]^3))]\,ds}{2h^\varepsilon}\right)$$

$$\geq \lim_{h \searrow 0}\left(\frac{\int_t^{2t} \exp((-8/t)|\ln(\pi/(2h))|^{2/3})\,ds}{2h^\varepsilon}\right)$$

$$= \lim_{h \searrow 0}\left(\frac{t}{2} \cdot \exp\left(\frac{-8}{t}\left|\ln\left(\frac{\pi}{2h}\right)\right|^{2/3} + \ln(h^{-\varepsilon})\right)\right)$$

$$= \frac{t}{2} \cdot \lim_{h \searrow 0}\left(\exp\left(\frac{-8}{t}\left|\ln\left(\frac{\pi}{2h}\right)\right|^{2/3} - \varepsilon \cdot \ln(h)\right)\right) = \infty$$

for all $\varepsilon, t \in (0, \infty)$. We hence get for every $t \in (0, \infty)$ that the function $\mathbb{R}^3 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^3$ is not locally Hölder continuous. Moreover, let $\psi : \mathbb{R} \to [0, 1]$ be an infinitely often differentiable function with compact support and with $\psi(x) = 1$ for all $x \in [-T, T]$ and let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a function given by $\varphi(x_1, x_2, x_3) = x_1 \psi(x_1)\psi(x_2)\psi(x_3)$ for all $x_1, x_2, x_3 \in \mathbb{R}$. Again (3.2) and Lemma 3.3 then show

$$\lim_{h \searrow 0}\left(h^{-\varepsilon} \cdot \mathbb{E}[\varphi(X^{(0,0,0)}(t)) - \varphi(X^{(0,0,h)}(t))]\right)$$

$$= \lim_{h \searrow 0}\left(h^{-\varepsilon} \cdot \mathbb{E}[(X_1^{(0,0,0)}(t) - X_1^{(0,0,h)}(t))\psi(\sqrt{2}W(t))]\right)$$

(3.16) $$\geq \lim_{h \searrow 0}\left(h^{-\varepsilon} \cdot \mathbb{E}[\mathbb{1}_{\{|\sqrt{2}W(t)| \leq T\}}(X_1^{(0,0,0)}(t) - X_1^{(0,0,h)}(t))]\right)$$

$$= \lim_{h \searrow 0}\left(h^{-\varepsilon} \cdot \mathbb{E}\left[\int_0^t \mathbb{1}_{\{|\sqrt{2}W(t)| \leq T\}}(1 - \cos(h \cdot \exp([\sqrt{2}W(s)]^3)))\,ds\right]\right)$$

$$= \lim_{h \searrow 0}\left(\frac{1}{2h^\varepsilon} \cdot \mathbb{E}\left[\int_0^{2t} \mathbb{1}_{\{|W(2t)| \leq T\}}(1 - \cos(h \cdot \exp([W(s)]^3)))\,ds\right]\right) = \infty$$

for all $t \in (0, T]$. The proof of Theorem 3.1 is thus completed. $\square$

In the remainder of this section, we briefly consider a slightly modified version of the SDE (3.1). More formally, let $(\mathbb{Z}_n)_{n \in \mathbb{N}_0}$ be a family of sets defined by $\mathbb{Z}_0 := \mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and by $\mathbb{Z}_n := \{z \in \mathbb{Z} : \frac{z}{2} \notin \mathbb{Z}\} = \{\ldots, -3, -1, 1, 3, \ldots\}$ for all $n \in \mathbb{N}$. Then let $\mu = (\mu_1, \mu_2, \mu_3) : \mathbb{R}^3 \to \mathbb{R}^3$ and $B \in \mathbb{R}^3$ be given by

$$\mu(x) = \begin{pmatrix} \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}_n} \frac{1}{4^{(n+|m|)}} \cos\left(\left(x_3 - \frac{m}{2^n}\right) \exp([x_2]^3)\right) \\ 0 \\ 0 \end{pmatrix} \quad \text{and}$$

(3.17)

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Note that $\mu : \mathbb{R}^3 \to \mathbb{R}^3$ is infinitely often differentiable and globally bounded by 2. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W : [0, \infty) \times \Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion and let $X^x : [0, \infty) \times \Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, be the up to indistinguishability unique solution processes with continuous sample paths of the SDE

$$(3.18) \qquad\qquad dX^x(t) = \mu(X^x(t)) \, dt + B \, dW(t)$$

for $t \in [0, \infty)$ and $x \in \mathbb{R}^3$ satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^3$. The following Theorem 3.4 establishes that the function $[0, \infty) \times \mathbb{R}^3 \to \mathbb{E}[X^x(t)] \in \mathbb{R}^3$ is nowhere locally Hölder continuous. Its proof is a straightforward consequence of Lemma 3.3 and, therefore, omitted.

THEOREM 3.4 (A further counterexample to regularity preservation with degenerate additive noise). *Let* $c, T \in (0, \infty)$ *and let* $X^x : [0, \infty) \times \Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, *be solution processes of the SDE* (3.18) *with continuous sample paths and with* $X^x(0) = x$ *for all* $x \in \mathbb{R}^3$. *Then for every* $t \in (0, \infty)$ *and every nonempty open set* $O \subset \mathbb{R}^3$, *the function* $O \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^3$ *is continuous but not locally Hölder continuous. Moreover, there exists an infinitely often differentiable function* $\varphi \in C_{\text{cpt}}^{\infty}(\mathbb{R}^3, \mathbb{R})$ *with compact support such that for every* $t \in (0, T]$ *and every nonempty open set* $O \subset (-c, c)^3$ *the function* $O \ni x \mapsto \mathbb{E}[\varphi(X^x(t))] \in \mathbb{R}$ *is continuous but not locally Hölder continuous.*

4. Solutions of Kolmogorov equations.

If the transition semigroup associated with an SDE is smooth, then it satisfies the Kolmogorov equation (which is a second-order linear PDE) corresponding to the SDE in the classical sense.

The transition semigroups in our counterexamples are, however, not locally Lipschitz continuous and are therefore no classical solutions of the Kolmogorov equations of the corresponding SDEs. The purpose of this section is to verify that the nonsmooth transition semigroup associated with such an SDE still satisfies the Kolmogorov equation but in a certain weak sense. More precisely, in Section 4.4, we show that the transition semigroups in our counterexamples are *viscosity solutions* of the associated Kolmogorov equations. Moreover, in Section 4.5, we show that the transition semigroups in our counterexamples are solutions of the associated Kolmogorov equations *in the distributional sense*. Throughout this section, the notation $\sup(\varnothing) := -\infty$ and $\inf(\varnothing) := \infty$ is used.

4.1. *Definition and basic properties of viscosity solutions.* Viscosity solutions were first introduced in Crandall and Lions [9] (see also [8, 16, 17]). The name *viscosity solution* is due to the method of vanishing viscosity; see the discussion in Section 10.1 in Evans [18]. For a review of the theory and for more references, we refer the reader to the well-known users's guide Crandall, Ishii and Lions [7].

For $d \in \mathbb{N}$, we denote by $\mathbb{S}_d = \{A \in \mathbb{R}^{d \times d} : A = A^*\}$ the set of all symmetric $d \times d$-matrices. Moreover, for $d \in \mathbb{N}$ and $A, B \in \mathbb{S}_d$ we write $A \leq B$ in the following if $\langle x, Ax \rangle \leq \langle x, Bx \rangle$ for all $x \in \mathbb{R}^d$. Furthermore, for $d \in \mathbb{N}$ and an open set $O \subset \mathbb{R}^d$ we call a function $F : O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ *degenerate elliptic* (see, e.g., (0.3) in Crandall, Ishii and Lions [7]) if $F(x, r, p, A) \leq F(x, r, p, B)$ for all $x \in O$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d$ and all $A, B \in \mathbb{S}_d$ with $A \geq B$. For convenience of the reader, we recall the definition of a viscosity solution (see, e.g., Section 2 in Crandall, Ishii and Lions [7] and also Definition 1.2 in Appendix C in Peng [61]).

DEFINITION 4.1 (Viscosity solution). Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set and let $F : O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function. A function $u : O \to \mathbb{R}$ is said to be a *viscosity subsolution* of $F = 0$ (or, equivalently, a viscosity solution of $F \leq 0$) if u is upper semicontinuous and if it holds for all $x \in O$ and all $\phi \in C^2(O, \mathbb{R})$ with $\phi \geq u$ and $\phi(x) = u(x)$ that

$$(4.1) \qquad F\big(x, \phi(x), (\nabla \phi)(x), (\operatorname{Hess} \phi)(x)\big) \leq 0.$$

Similarly, a function $u : O \to \mathbb{R}$ is said to be a *viscosity supersolution* of $F = 0$ (or, equivalently, a viscosity solution of $F \geq 0$) if u is lower semicontinuous and if it holds for all $x \in O$ and all $\phi \in C^2(O, \mathbb{R})$ with $\phi \leq u$ and $\phi(x) = u(x)$ that

$$(4.2) \qquad F\big(x, \phi(x), (\nabla \phi)(x), (\operatorname{Hess} \phi)(x)\big) \geq 0.$$

Finally, a function $u : O \to \mathbb{R}$ is said to be a *viscosity solution* of $F = 0$ if u is both a viscosity subsolution and a viscosity supersolution of $F = 0$.

In the proof of Corollary 4.11 below, the following elementary lemma (Lemma 4.2) is used. The proof of Lemma 4.2 is clear and, therefore, omitted.

LEMMA 4.2 (Sign changes of viscosity solutions). *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $F : O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function and let $u : O \to \mathbb{R}$ be a viscosity solution of $F \geq 0$. Then the function $\tilde{F} : O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ defined by $\tilde{F}(x, r, p, A) := -F(x, -r, -p, -A)$ for all $(x, r, p, A) \in O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$ is degenerate elliptic and the function $O \ni x \mapsto -u(x) \in \mathbb{R}$ is a viscosity solution of $\tilde{F} \leq 0$. The corresponding statement holds for viscosity solutions of $F \leq 0$ and $F = 0$, respectively.*

Above in Definition 4.1, the concept of viscosity solutions is presented via suitable test functions. An alternative instrument to characterize viscosity solutions are so-called *semijets* (see, e.g., Definition 2.2 in Crandall, Ishii and Lions [7]). They are recalled in the next definition.

DEFINITION 4.3 (Second-order semijets). Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set and let $u : O \to \mathbb{R}$ be a function. Then we define functions $J_+^2 u : O \to \mathcal{P}(\mathbb{R}^d \times \mathbb{S}_d)$, $J_-^2 u : O \to \mathcal{P}(\mathbb{R}^d \times \mathbb{S}_d)$, $\hat{J}_+^2 u : O \to \mathcal{P}(\mathbb{R}^d \times \mathbb{S}_d)$ and $\hat{J}_-^2 u : O \to \mathcal{P}(\mathbb{R}^d \times \mathbb{S}_d)$ by

$$(J_+^2 u)(x)$$

$$:= \left\{ (p, A) \in \mathbb{R}^d \times \mathbb{S}_d : \right.$$

$$\left. \limsup_{O \setminus \{x\} \ni y \to x} \left(\frac{u(y) - u(x) - \langle p, x - y \rangle - (1/2)\langle x - y, A(x - y) \rangle}{\|x - y\|^2} \right) \leq 0 \right\},$$

$$(\hat{J}_+^2 u)(x)$$

$$:= \left\{ (p, A) \in \mathbb{R}^d \times \mathbb{S}_d : \right.$$

$$\left. \left(\begin{matrix} \exists (x_n, p_n, A_n)_{n \in \mathbb{N}} \subset O \times \mathbb{R}^d \times \mathbb{S}_d : (\forall n \in \mathbb{N} : (p_n, A_n) \in (J_+^2 u)(x_n)) \\ \text{and } \lim_{n \to \infty} (x_n, u(x_n), p_n, A_n) = (x, u(x), p, A) \end{matrix} \right) \right\},$$

$$(J_-^2 u)(x)$$

$$:= \left\{ (p, A) \in \mathbb{R}^d \times \mathbb{S}_d : \right.$$

$$\left. \liminf_{O \setminus \{x\} \ni y \to x} \left(\frac{u(y) - u(x) - \langle p, x - y \rangle - (1/2)\langle x - y, A(x - y) \rangle}{\|x - y\|^2} \right) \geq 0 \right\}$$

and

$$(\hat{J}_-^2 u)(x)$$

$$:= \left\{ (p, A) \in \mathbb{R}^d \times \mathbb{S}_d : \right.$$

$$\left(\begin{array}{c} \exists (x_n, p_n, A_n)_{n \in \mathbb{N}} \subset O \times \mathbb{R}^d \times \mathbb{S}_d : (\forall n \in \mathbb{N} : (p_n, A_n) \in (J_-^2 u)(x_n)) \\ \text{and } \lim_{n \to \infty} (x_n, u(x_n), p_n, A_n) = (x, u(x), p, A) \end{array} \right) \right\}$$

for all $x \in O$.

The next lemma (Lemma 4.4), which is essentially one of the statements in Remark 2.3 in Crandall, Ishii and Lions [7], illustrates the relationship between semijets in the sense of Definition 4.3 and suitable test functions in the sense of Definition 4.1.

LEMMA 4.4 (Properties of semijets). *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set and let $u : O \to \mathbb{R}$ be a function. Then*

$$(J_+^2 u)(x) = \{ ((\nabla \phi)(x), (\mathrm{Hess}\, \phi)(x)) \in \mathbb{R}^d \times \mathbb{S}_d :$$

$$(\phi \in C^2(O, \mathbb{R}) \text{ with } u(x) = \phi(x) \text{ and } u \le \phi) \}$$

(4.3)

$$= \{ ((\nabla \phi)(x), (\mathrm{Hess}\, \phi)(x)) \in \mathbb{R}^d \times \mathbb{S}_d :$$

$$(\phi \in C^2(O, \mathbb{R}) \text{ and } u - \phi \text{ has a local maximum at } x) \}$$

and

$$(J_-^2 u)(x) = \{ ((\nabla \phi)(x), (\mathrm{Hess}\, \phi)(x)) \in \mathbb{R}^d \times \mathbb{S}_d :$$

$$(\phi \in C^2(O, \mathbb{R}) \text{ with } u(x) = \phi(x) \text{ and } u \ge \phi) \}$$

(4.4)

$$= \{ ((\nabla \phi)(x), (\mathrm{Hess}\, \phi)(x)) \in \mathbb{R}^d \times \mathbb{S}_d :$$

$$(\phi \in C^2(O, \mathbb{R}) \text{ and } u - \phi \text{ has a local minimum at } x) \}$$

for all $x \in O$.

The next corollary, which is essentially one of the statements in Remark 2.3 in Crandall, Ishii and Lions [7], is an immediate consequence of Lemma 4.4 above.

COROLLARY 4.5 (Characterizations of viscosity solutions). *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $F : O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function and let $u : O \to \mathbb{R}$ be an upper semicontinuous function. Then the following three assertions are equivalent:*

- *u is a viscosity subsolution of $F = 0$ (u is a viscosity solution of $F \le 0$),*

- *for every $x \in O$ and every $\phi \in \{\psi \in C^2(O, \mathbb{R}) : x$ is a local maximum of $(u - \psi) : O \to \mathbb{R}\}$ it holds that $F(x, u(x), (\nabla\phi)(x), (\text{Hess}\,\phi)(x)) \leq 0$,*
- *for every $x \in O$ and every $(p, A) \in (J^2_+ u)(x)$ it holds that $F(x, u(x), p, A) \leq 0$.*

The corresponding statement holds for viscosity supersolutions and viscosity solutions.

The next corollary, which is Remark 2.4 in Crandall, Ishii and Lions [7], illustrates a further characterization of viscosity solutions under the assumption that F is continuous. It follows immediately from Corollary 4.5 and from the semicontinuity of F.

COROLLARY 4.6 (Characterizations of viscosity solutions for semicontinuous left-hand sides). *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $F : O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic and lower semicontinuous function and let $u : O \to \mathbb{R}$ be an upper semicontinuous function. Then u is a viscosity solution of $F \leq 0$ if and only if it holds for every $x \in O$ and every $(p, A) \in (\hat{J}^2_+ u)(x)$ that $F(x, u(x), p, A) \leq 0$. The corresponding statement holds for viscosity solutions of $F \geq 0$ and $F = 0$, respectively.*

The next well-known remark (see, e.g., Section 2 in Crandall, Ishii and Lions [7]) illustrates that classical solutions are viscosity solutions. We will use it in the proof of Lemma 4.15 below.

REMARK 4.1 (Classical solutions are viscosity solutions). Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $F : O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function and let $u \in C^2(O, \mathbb{R})$ be a classical subsolution of $F = 0$, that is, suppose that

$$(4.5) \qquad F(x, u(x), (\nabla u)(x), (\text{Hess}\,u)(x)) \leq 0$$

for all $x \in O$. Then u is also a viscosity subsolution of $F = 0$. Indeed, for every $x \in O$ and every $\phi \in \{\psi \in C^2(O, \mathbb{R}) : x$ is a local maximum of $(u - \psi) : O \to \mathbb{R}\}$ it holds that $(\nabla(u - \phi))(x) = 0$ and $(\text{Hess}(u - \phi))(x) \leq 0$ and, therefore,

$$
\begin{aligned}
F(x, u(x), (\nabla\phi)(x), (\text{Hess}\,\phi)(x)) &= F(x, u(x), (\nabla u)(x), (\text{Hess}\,\phi)(x)) \\
(4.6) \qquad\qquad &\leq F(x, u(x), (\nabla u)(x), (\text{Hess}\,u)(x)) \\
&\leq 0
\end{aligned}
$$

due to (4.5) and due to the degenerate ellipticity assumption on F. The corresponding statement holds for classical supersolutions and classical solutions of $F = 0$.

For the convenience of the reader, we also state a special case of Theorem 3.2 in Crandall, Ishii and Lions [7] in the next lemma. It will be used in the proof of Lemma 4.10 below.

LEMMA 4.7 (Construction of suitable semijets).   *Let $d, k \in \mathbb{N}$, $\varepsilon \in (0, \infty)$, let $\mathcal{O} \subset \mathbb{R}^d$ be an open set, let $\Phi \in C^2(\mathcal{O}^k, \mathbb{R})$, let $u_i : \mathcal{O} \to \mathbb{R}$, $i \in \{1, \ldots, k\}$, be upper semicontinuous functions and let $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k) \in \mathcal{O}^k$ be a local maximum point of the function $\mathcal{O}^k \ni (x_1, \ldots, x_k) \mapsto (\sum_{i=1}^k u_i(x_i)) - \Phi(x_1, \ldots, x_k) \in \mathbb{R}$. Then there exist matrices $A_1 \in \mathbb{S}_d, \ldots, A_k \in \mathbb{S}_d$ such that for all $i \in \{1, \ldots, k\}$ it holds that $((\nabla_{x_i} \Phi)(\hat{x}), A_i) \in (\hat{J}_+^2 u_i)(\hat{x}_i)$ and such that*

$$
-\left(\frac{1}{\varepsilon} + \|(\mathrm{Hess}\,\Phi)(\hat{x})\|_{L(\mathbb{R}^{kd})}\right) I \le \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k \end{pmatrix}
$$

(4.7)

$$
\le (\mathrm{Hess}\,\Phi)(\hat{x}) + \varepsilon[(\mathrm{Hess}\,\Phi)(\hat{x})]^2.
$$

4.2. *An approximation result for viscosity solutions.*   The following approximation result for viscosity solutions is essentially well known (see Proposition 1.2 in Ishii [37] which refers to the first-order case in Theorem A.2 in Barles and Perthame [2]; see also Lemma 6.1 in Crandall, Ishii and Lions [7] and the remarks thereafter). For completeness, we give the proof here following the line of arguments for the first-order case in Theorem A.2 in Barles and Perthame [2]. In the remainder of this article, we use the notation $\mathrm{dist}(x, A) := \inf(\{\|x - y\| \in [0, \infty) : y \in A\} \cup \{\infty\}) \in [0, \infty]$ for all $x \in \mathbb{R}^d$, all $A \subset \mathbb{R}^d$ and all $d \in \mathbb{N}$.

LEMMA 4.8.   *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $u_n : O \to \mathbb{R}$, $n \in \mathbb{N}_0$, be functions and let $F_n : O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$, $n \in \mathbb{N}_0$, be degenerate elliptic functions such that F_0 is continuous. Moreover, assume that*

(4.8)

$$
\limsup_{n \to \infty} \sup_{(x,r,p,A) \in K} |F_n(x, r, p, A) - F_0(x, r, p, A)|
$$
$$
= 0 = \limsup_{n \to \infty} \sup_{x \in \bar{K}} |u_n(x) - u_0(x)|
$$

for all nonempty compact sets $K \subset O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$ and all nonempty compact sets $\bar{K} \subset O$ and assume for every $n \in \mathbb{N}$ that u_n is a viscosity solution of $F_n = 0$. Then u_0 is a viscosity solution of $F_0 = 0$.

PROOF.   The proof is divided into two steps.

Step 1: Let $x_0 \in O$ and let $\phi \in C^2(O, \mathbb{R})$ be a function such that x_0 is a strict maximum of $u_0 - \phi$, that is,

(4.9) $u_0(x) - \phi(x) < u_0(x_0) - \phi(x_0)$

for all $x \in O \setminus \{x_0\}$. Then we define $r := \min(1, \frac{1}{2} \mathrm{dist}(x_0, \mathbb{R}^d \setminus O)) \in [0, 1]$. Since $O \subset \mathbb{R}^d$ is an open set, we obtain that $r \in (0, 1]$. Furthermore, continuity of the function ϕ and of the functions u_n, $n \in \mathbb{N}$, together with compactness of the set $\{y \in \mathbb{R}^d : \|y - x_0\| \le r\} \subset O$ proves that there exists a sequence

$x_n \in \{y \in \mathbb{R}^d : \|y - x_0\| \leq r\} \subset O$, $n \in \mathbb{N}$, of vectors such that

$$(4.10) \qquad u_n(x) - \phi(x) \leq u_n(x_n) - \phi(x_n)$$

for all $x \in \{y \in \mathbb{R}^d : \|y - x_0\| \leq r\}$ and all $n \in \mathbb{N}$. We now prove that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to x_0. Aiming at a contraction, we assume that the sequence $(x_n)_{n \in \mathbb{N}}$ does not converge to x_0. Due to compactness of $\{y \in \mathbb{R}^d : \|y - x_0\| \leq r\}$, there exists a vector $\bar{x}_0 \in \{y \in \mathbb{R}^d : 0 < \|y - x_0\| \leq r\} \subset O$ and an increasing sequence $n_k \in \mathbb{N}$, $k \in \mathbb{N}$, such that $\lim_{k \to \infty} x_{n_k} = \bar{x}_0$. In particular, we obtain that the set $\{\bar{x}_0\} \cup (\bigcup_{k \in \mathbb{N}} \{x_{n_k}\})$ is compact. Assumption (4.8), inequality (4.10) and inequality (4.9) hence imply that

$$u_0(x_0) - \phi(x_0) = \lim_{k \to \infty} \left(u_{n_k}(x_0) - \phi(x_0) \right) \leq \limsup_{k \to \infty} \left(u_{n_k}(x_{n_k}) - \phi(x_{n_k}) \right)$$

$$= u_0(\bar{x}_0) - \phi(\bar{x}_0) < u_0(x_0) - \phi(x_0).$$

From this contradiction, we infer that $\lim_{n \to \infty} x_n = x_0$. Assumption (4.8) and continuity of $\nabla \phi : O \to \mathbb{R}^d$ and of $\operatorname{Hess} \phi : O \to \mathbb{S}_d$ hence imply that

$$(4.11) \qquad \begin{aligned} \lim_{n \to \infty} & \left(x_n, u_n(x_n), (\nabla \phi)(x_n), (\operatorname{Hess} \phi)(x_n) \right) \\ & = \left(x_0, u_0(x_0), (\nabla \phi)(x_0), (\operatorname{Hess} \phi)(x_0) \right). \end{aligned}$$

In addition, $\lim_{n \to \infty} x_n = x_0$ and (4.10) show that there exists a natural number $n_0 \in \mathbb{N}$ such that we have for all $n \in \{n_0, n_0 + 1, \ldots\}$ that $\|x_n - x_0\| < r$ and that $x_n \in O$ is a local maximum of the function $(u_n - \phi) : O \to \mathbb{R}$. Hence, Corollary 4.5 and the assumption that u_n is a viscosity solution of $F_n = 0$ show that

$$(4.12) \qquad F_n\left(x_n, u_n(x_n), (\nabla \phi)(x_n), (\operatorname{Hess} \phi)(x_n) \right) \leq 0$$

for all $n \in \{n_0, n_0 + 1, \ldots\}$. Continuity of F_0, equation (4.11), assumption (4.8), inequality (4.12) and compactness of the set $\bigcup_{n \in \mathbb{N}_0} \{(x_n, u_n(x_n), (\nabla \phi)(x_n), (\operatorname{Hess} \phi)(x_n))\}$ therefore yield that

$$(4.13) \qquad \begin{aligned} F_0 & \left(x_0, u_0(x_0), (\nabla \phi)(x_0), (\operatorname{Hess} \phi)(x_0) \right) \\ & = \lim_{n \to \infty} F_0\left(x_n, u_n(x_n), (\nabla \phi)(x_n), (\operatorname{Hess} \phi)(x_n) \right) \\ & = \lim_{n \to \infty} F_n\left(x_n, u_n(x_n), (\nabla \phi)(x_n), (\operatorname{Hess} \phi)(x_n) \right) \leq 0. \end{aligned}$$

We thus have proved that $F_0(x, u_0(x), (\nabla \phi)(x), (\operatorname{Hess} \phi)(x)) \leq 0$ for all $\phi \in \{\psi \in C^2(O, \mathbb{R}) : x$ is a strict maximum of $(u_0 - \psi) : O \to \mathbb{R}\}$ and all $x \in O$.

Step 2: Let $x_0 \in O$ and let $\phi \in C^2(O, \mathbb{R})$ be a function such that $\phi(x_0) = u_0(x_0)$ and $\phi \geq u_0$. Next define functions $\phi_\varepsilon : O \to \mathbb{R}$, $\varepsilon \in (0, 1)$, by $\phi_\varepsilon(x) = \phi(x) + \varepsilon \|x - x_0\|^2$ for all $x \in O$ and all $\varepsilon \in (0, 1)$. Note for every $\varepsilon \in (0, 1)$ that x_0 is a strict maximum of the function $(u_0 - \phi_\varepsilon) : O \to \mathbb{R}$. Step 1 can thus be applied to obtain

$$(4.14) \qquad F_0\left(x_0, u_0(x_0), (\nabla \phi_\varepsilon)(x_0), (\operatorname{Hess} \phi_\varepsilon)(x_0) \right) \leq 0$$

for all $\varepsilon \in (0, 1)$. Moreover, observe that $(\nabla \phi_\varepsilon)(x_0) = (\nabla \phi)(x_0)$ and that $(\operatorname{Hess} \phi_\varepsilon)(x_0) = (\operatorname{Hess} \phi)(x_0) + 2\varepsilon I_d$ for all $\varepsilon \in (0, 1)$ where $I_d \in \mathbb{S}^d$ is the $d \times d$-unit matrix. Consequently, we see that $\lim_{\varepsilon \searrow 0}(\nabla \phi_\varepsilon)(x_0) = (\nabla \phi)(x_0)$ and that $\lim_{\varepsilon \searrow 0}(\operatorname{Hess} \phi_\varepsilon)(x_0) = (\operatorname{Hess} \phi)(x_0)$. Continuity of F_0 and inequality (4.14) hence yield

$$
\begin{aligned}
(4.15) \quad & F_0\big(x_0, u_0(x_0), (\nabla \phi)(x_0), (\operatorname{Hess} \phi)(x_0)\big) \\
& = \lim_{\varepsilon \searrow 0} F_0\big(x_0, u_0(x_0), (\nabla \phi_\varepsilon)(x_0), (\operatorname{Hess} \phi_\varepsilon)(x_0)\big) \leq 0.
\end{aligned}
$$

We thus have proved that $F_0(x, u_0(x), (\nabla \phi)(x), (\operatorname{Hess} \phi)(x)) \leq 0$ for all $\phi \in C^2(O, \mathbb{R})$ with $\phi(x) = u_0(x)$ and $\phi \geq u_0$ and all $x \in O$. This shows that u_0 is a viscosity subsolution of $F_0 = 0$. In the same way, it can be shown that u_0 is a viscosity supersolution of $F_0 = 0$ and we thereby obtain that u_0 is a viscosity solution of $F_0 = 0$. The proof of Lemma 4.8 is thus completed. $\quad\square$

4.3. *Uniqueness of viscosity solutions of Kolmogorov equations.* A key result of this subsection (Corollary 4.14) establishes uniqueness of viscosity solutions of a second-order linear PDE within a certain class of functions and is apparently new. This uniqueness result is based on the well-known concept of superharmonic functions or—in the PDE language—on the idea of dominating supersolutions. More precisely, let $d \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$. For solution processes $X^x : [0, \infty) \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, of many SDEs, there exists a function $V \in C^2(\mathbb{R}^d, (0, \infty))$ [often $\mathbb{R}^d \ni x \mapsto 1 + \|x\|^2 \in (0, \infty)$] and a real number $\rho \in \mathbb{R}$ such that the stochastic processes $[0, \infty) \times \Omega \ni (t, \omega) \to e^{-\rho t} \cdot V(X^x(t)(\omega)) \in (0, \infty)$, $x \in \mathbb{R}^d$, are nonnegative supermartingales (so that $\mathbb{E}[V(X^x(t))] \leq e^{\rho t} \cdot V(x)$ for all $(t, x) \in [0, \infty) \times \mathbb{R}^d$); see, for example, the examples in Section 4 in [34]. For these stochastic processes to be supermartingales, it suffices that the Lyapunov function V satisfies

$$
(4.16) \qquad\qquad \mathcal{L} V(x) \leq \rho V(x)
$$

for all $x \in \mathbb{R}^d$, where $\mathcal{L}$ is the generator of the SDE under consideration. In other words, it suffices that the map $(0, \infty) \times \mathbb{R}^d \ni (t, x) \to e^{\rho t} \cdot V(x) \in (0, \infty)$ is a classical supersolution of the Kolmogorov equation. For $T \in (0, \infty)$, $d \in \mathbb{N}$ and an open set $O \subset \mathbb{R}^d$, a function $G : (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ is here called *degenerate elliptic* if $G(t, x, r, p, A) \leq G(t, x, r, p, B)$ for all $t \in (0, T)$, $x \in O$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d$ and all $A, B \in \mathbb{S}_d$ with $A \leq B$ (see, e.g., inequality (1.2) in Appendix C in Peng [61] and compare also with Section 4.1 above). To establish Corollary 4.14, we first state a few auxiliary results. For the convenience of the reader, we first state Proposition 3.7 from Crandall, Ishii and Lions [7] in the next lemma.

LEMMA 4.9. *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be a set, let $\eta : O \to \mathbb{R}$ be an upper semi-continuous function, let $\phi : O \to [0, \infty)$ be a lower semicontinuous function satisfying $\lim_{\alpha \to \infty} \sup_{y \in O} (\eta(y) - \alpha \cdot \phi(y)) \in \mathbb{R}$ and let $x : (0, \infty) \to O$ be a function satisfying*

$$(4.17) \qquad \lim_{\alpha \to \infty} \Big(\sup_{y \in O} \big(\eta(y) - \alpha \cdot \phi(y) \big) - \big(\eta(x(\alpha)) - \alpha \cdot \phi(x(\alpha)) \big) \Big) = 0.$$

Then $\lim_{\alpha \to \infty} \alpha \cdot \phi(x(\alpha)) = 0$ and for all $\alpha_n \in (0, \infty)$, $n \in \mathbb{N}$, with $\lim_{n \to \infty} \alpha_n = \infty$ and $\lim_{n \to \infty} x(\alpha_n) =: x_0 \in O$ it holds that $\phi(x_0) = 0$ and $\eta(x_0) = \lim_{\alpha \to \infty} \sup_{y \in O} (\eta(y) - \alpha \cdot \phi(y)) = \sup_{y \in \phi^{-1}(0)} \eta(y)$.

The next lemma essentially generalizes Theorem 2.2 in Appendix C in Peng [61] (which assumes the functions $G_1, \ldots, G_k$ to be uniformly continuous in the second argument uniformly in the last argument) and is a generalized analog of Theorem 8.2 in Crandall, Ishii and Lions [7] for unbounded domains. Given an open set $O \subset \mathbb{R}^d$, we define a sequence $O_n \subset O$, $n \in \mathbb{N}$, of compact sets by

$$(4.18) \qquad O_n := \left\{ x \in O : \mathrm{dist}(x, \mathbb{R}^d \setminus O) \geq \frac{1}{n} \text{ and } \|x\| \leq n \right\}$$

for all $n \in \mathbb{N}$. We also write $O_n^c := O \setminus O_n$ for the complement of O_n in O.

LEMMA 4.10 (A domination result for viscosity subsolutions). *Let $T \in (0, \infty)$, $d, k \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $G_1, \ldots, G_k : (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be degenerate elliptic and upper semicontinuous functions and let $u_1, \ldots, u_k : [0, T] \times O \to \mathbb{R}$ be upper semicontinuous functions such that for every $i \in \{1, \ldots, k\}$ it holds that $u_i|_{(0,T) \times O}$ is a viscosity subsolution of*

$$(4.19) \quad \frac{\partial}{\partial t} u_i(t, x) - G_i\big(t, x, u_i(t, x), (\nabla_x u_i)(t, x), (\mathrm{Hess}_x u_i)(t, x)\big) = 0$$

for $(t, x) \in (0, T) \times O$. Moreover, assume that

$$\limsup_{n \to \infty} \left[\sum_{i=1}^{k} G_i\big(t_i^{(n)}, x_i^{(n)}, r_i^{(n)}, \right.$$

$$(4.20) \qquad\qquad n\big(\mathbb{1}_{[2,k]}(i) \cdot [x_i^{(n)} - x_{i-1}^{(n)}] + \mathbb{1}_{[1,k-1]}(i) \cdot [x_i^{(n)} - x_{i+1}^{(n)}]\big), n A_i^{(n)}\big) \Bigg]$$

$$\leq 0$$

for all $(t_i^{(n)}, x_i^{(n)}, r_i^{(n)}, A_i^{(n)}) \in (0, T) \times O \times \mathbb{R} \times \mathbb{S}_d$, $n \in \mathbb{N}$, $i \in \{1, \ldots, k\}$, satisfying that $\lim_{n \to \infty} (t_1^{(n)}, x_1^{(n)}) \in (0, T) \times O$, that $\lim_{n \to \infty} (\sqrt{n} \sum_{i=2}^{k} \|(t_i^{(n)}, x_i^{(n)}) - (t_{i-1}^{(n)}, x_{i-1}^{(n)})\|) = 0$, that $\lim_{n \to \infty} \sum_{i=1}^{k} r_i^{(n)} > 0$, that $\sup_{n \in \mathbb{N}} \sum_{i=1}^{k} |r_i^{(n)}| < \infty$ and

that $\forall n \in \mathbb{N} : \forall z_1, \ldots, z_k \in \mathbb{R}^d : -5 \sum_{i=1}^{k} \|z_i\|^2 \le \sum_{i=1}^{k} \langle z_i, A_i^{(n)} z_i \rangle \le 5 \sum_{i=2}^{k} \|z_i - z_{i-1}\|^2$. Furthermore, assume that $\sum_{i=1}^{k} u_i(0, x) \le 0$ for all $x \in O$ and that

$$(4.21) \qquad \lim_{n \to \infty} \sup_{(t,x) \in (0,T) \times O_n^c} \sum_{i=1}^{k} u_i(t, x) \le 0.$$

Then $\sum_{i=1}^{k} u_i(t, x) \le 0$ for all $(t, x) \in [0, T) \times O$.

PROOF. If $O = \varnothing$, then the assertion is trivial. So for the rest of the proof, we assume that $O \ne \varnothing$. We will show that $\sum_{i=1}^{k} u_i(t, x) \le \frac{k\delta}{(T-t)}$ for all $(t, x) \in [0, T) \times O$ and all $\delta \in (0, 1]$. Letting $\delta \to 0$ will then yield that $\sum_{i=1}^{k} u_i(t, x) \le 0$ for all $(t, x) \in [0, T) \times O$. In the following, we thus fix $\delta \in (0, 1]$. In a first step of this proof, we modify the problem. More precisely, define functions $\tilde{u}_1, \ldots, \tilde{u}_k : [0, T) \times O \to [-\infty, \infty)$ by $\tilde{u}_i(t, x) := u_i(t, x) - \frac{\delta}{(T-t)}$ and functions $\tilde{G}_1, \ldots, \tilde{G}_k : (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ by

$$
\begin{aligned}
(4.22) \quad \tilde{G}_i(t, x, r, p, A) &:= G_i\left(t, x, r + \frac{\delta}{(T-t)}, p, A\right) - \frac{\partial}{\partial t}\left(\frac{\delta}{(T-t)}\right) \\
&= G_i\left(t, x, r + \frac{\delta}{(T-t)}, p, A\right) - \frac{\delta}{(T-t)^2}.
\end{aligned}
$$

Then it holds for every $i \in \{1, \ldots, k\}$ that $\tilde{u}_i|_{(0,T) \times O}$ is a viscosity subsolution of

$$(4.23) \quad \frac{\partial}{\partial t} \tilde{u}_i(t, x) - \tilde{G}_i\big(t, x, \tilde{u}_i(t, x), (\nabla_x \tilde{u}_i)(t, x), (\mathrm{Hess}_x \tilde{u}_i)(t, x)\big) = 0$$

for $(t, x) \in (0, T) \times O$. It remains to prove that $\sum_{i=1}^{k} \tilde{u}_i(z) \le 0$ for all $z \in [0, T) \times O$. Aiming at a contradiction, we assume that the extended real number $S_0 := \sup_{z \in [0,T) \times O} \sum_{i=1}^{k} \tilde{u}_i(z) \in (-\infty, \infty]$ satisfies that $S_0 \in (0, \infty]$. Assumption (4.21) then implies that there exists a natural number $n_0 \in \mathbb{N}$ such that $K := O_{n_0}$ is nonempty and such that $\sum_{i=1}^{k} \tilde{u}_i(z) \le \sum_{i=1}^{k} u_i(z) \le \min(1, \frac{S_0}{2})$ for all $z \in (0, T) \times K^c$. This, together with $\sum_{i=1}^{k} \tilde{u}_i(0, x) \le \sum_{i=1}^{k} u_i(0, x) \le 0$ and $\sum_{i=1}^{k} \tilde{u}_i(T, x) = -\infty$ for all $x \in O$ implies that

$$(4.24) \qquad \sup_{z \in [0,T] \times K^c} \sum_{i=1}^{k} \tilde{u}_i(z) \le \min\left(1, \frac{S_0}{2}\right) \le \frac{S_0}{2}.$$

Moreover, the function $\sum_{i=1}^{k} \tilde{u}_i : [0, T] \times O \to [-\infty, \infty)$ is upper semicontinuous and is hence bounded from above on the compact set $[0, T] \times K$. Combining this with (4.24) proves that $S_0 < \infty$ and we thus get $S_0 \in (0, \infty)$. In the next step, we define a function $\phi : ([0, T] \times O)^k \to [0, \infty)$ by $\phi(z_1, \ldots, z_k) = \frac{1}{2} \sum_{i=2}^{k} \|z_i - z_{i-1}\|^2$ for all $z_1, \ldots, z_k \in [0, T] \times O$. For several $n \in \mathbb{N}$, we will apply Lemma 4.7 with $\mathcal{O} = (0, T) \times O$, $\varepsilon = \frac{1}{n}$ and with $\Phi = n \cdot \phi|_{((0,T) \times O)^k}$ below. For this, we

now check the assumptions of Lemma 4.7. Define a function $\eta \colon ([0, T] \times K)^k \to [-\infty, \infty)$ by $\eta(z_1, \ldots, z_k) = \sum_{i=1}^{k} \tilde{u}_i(z_i)$ for all $z_1, \ldots, z_k \in [0, T] \times K$. Note for every $\alpha \in (0, \infty)$ that the function $([0, T] \times K)^k \ni \underline{z} \mapsto \eta(\underline{z}) - \alpha \cdot \phi(\underline{z}) \in [-\infty, \infty)$ is upper semicontinuous with a compact domain of definition and therefore, attains its maximum $S_\alpha := \sup_{\underline{z} \in ([0,T] \times K)^k} (\eta(\underline{z}) - \alpha \cdot \phi(\underline{z})) < \infty$ in a point $\underline{z}^{(\alpha)} = ((t_1^{(\alpha)}, x_1^{(\alpha)}), \ldots, (t_k^{(\alpha)}, x_k^{(\alpha)})) \in ([0, T] \times K)^k$. Next observe that

$$(4.25) \quad \infty > S_\alpha \geq \sup_{z \in [0,T) \times K} \eta(z, z, \ldots, z) = \sup_{z \in [0,T) \times K} \sum_{i=1}^{k} \tilde{u}_i(z) = S_0 > 0$$

for all $\alpha \in (0, \infty)$. This together with monotonicity of the function $(0, \infty) \ni \alpha \mapsto S_\alpha \in (0, \infty)$ implies that the limit $\lim_{\alpha \to \infty} S_\alpha$ exists in $(0, \infty)$, that is, it holds that $\lim_{\alpha \to \infty} S_\alpha \in (0, \infty)$. The set $\{\underline{z}^{(n)} : n \in \mathbb{N}\} \subset ([0, T] \times K)^k$ is relatively compact and, therefore, there exists a limit point $\underline{\hat{z}} = ((\hat{t}_1, \hat{x}_1), \ldots, (\hat{t}_k, \hat{x}_k)) \in ([0, T] \times K)^k$ of this set. Let $n_j \in \mathbb{N}$, $j \in \mathbb{N}$, be a strictly increasing sequence such that $\lim_{j \to \infty} \underline{z}^{(n_j)} = \underline{\hat{z}}$. Clearly, $\tilde{u}_i(T, x) = -\infty$ for all $x \in K$ and all $i \in \{1, \ldots, k\}$ implies that $t_1^{(\alpha)}, \ldots, t_k^{(\alpha)} \in [0, T)$ for all $\alpha \in (0, \infty)$. In addition, observe that if $(\hat{t}_1, \ldots, \hat{t}_k) \in [0, T]^k \setminus [0, T)^k$, then (4.25) implies that

$$(4.26) \quad \begin{aligned} 0 &< \lim_{j \to \infty} S_{n_j} = \lim_{j \to \infty} (\eta(\underline{z}^{(n_j)}) - n_j \cdot \phi(\underline{z}^{(n_j)})) \leq \lim_{j \to \infty} \eta(\underline{z}^{(n_j)}) \\ &\leq \left(\sum_{i=1}^{k} \left[\sup_{z \in [0,T] \times K} u_i(z) \right] \right) - \infty = -\infty \end{aligned}$$

and this contradiction shows that $(\hat{t}_1, \ldots, \hat{t}_k) \in [0, T)^k$. Next observe that

$$(4.27) \quad \begin{aligned} \lim_{\alpha \to \infty} &\Big[\sup_{z \in ([0,T) \times K)^k} (\eta(z) - \alpha \cdot \phi(z)) - (\eta(\underline{z}^{(\alpha)}) - \alpha \cdot \phi(\underline{z}^{(\alpha)})) \Big] \\ &= \lim_{\alpha \to \infty} [S_\alpha - S_\alpha] = 0. \end{aligned}$$

Hence, Lemma 4.9 applied to $\eta|_{([0,T) \times K)^k}$ and to $\phi|_{([0,T) \times K)^k}$ yields that

$$(4.28) \quad 0 = \lim_{\alpha \to \infty} [\alpha \cdot \phi(\underline{z}^{(\alpha)})] = \lim_{\alpha \to \infty} \left[\frac{\alpha}{2} \sum_{i=2}^{k} \| (t_i^{(\alpha)}, x_i^{(\alpha)}) - (t_{i-1}^{(\alpha)}, x_{i-1}^{(\alpha)}) \|^2 \right]$$

and that $\phi(\underline{\hat{z}}) = 0$. The definition of ϕ therefore ensures that $(\hat{t}_i, \hat{x}_i) = (\hat{t}_j, \hat{x}_j)$ for all $i, j \in \{1, \ldots, k\}$. Furthermore, observe that if $\hat{t}_1 = 0$, then (4.25) and the upper semicontinuity of η show that

$$(4.29) \quad \begin{aligned} 0 &< S_0 \leq \lim_{j \to \infty} S_{n_j} \leq \limsup_{j \to \infty} \eta(\underline{z}^{(n_j)}) \leq \eta(\underline{\hat{z}}) = \sum_{i=1}^{k} \tilde{u}_i(\hat{t}_1, \hat{x}_1) \\ &= \sum_{i=1}^{k} u_i(0, \hat{x}_1) - \frac{k\delta}{T} \leq 0 \end{aligned}$$

and this contradiction implies that $\hat{t}_1 = \hat{t}_2 = \cdots = \hat{t}_k \in (0, T)$. Consequently, there exists a natural number $j_0 \in \mathbb{N}$ such that for every $j \in \{j_0, j_0 + 1, \ldots\}$ it holds that $t_1^{(n_j)}, \ldots, t_k^{(n_j)} \in (0, T)$. Next, for every $n \in \mathcal{N} := \{m \in \mathbb{N} : t_1^{(m)}, \ldots, t_k^{(m)} \in (0, T)\}$, we apply Lemma 4.7 with $\mathcal{O} = (0, T) \times O$, with $\varepsilon = \frac{1}{n}$, with the functions $\tilde{u}_1|_{(0,T) \times O}, \ldots, \tilde{u}_k|_{(0,T) \times O}$ and $\Phi = n \cdot \phi|_{((0,T) \times O)^k}$ and with the local maximum point $\underline{z}^{(n)} \in ((0, T) \times O)^k$ to obtain the existence of matrices $(A_1^{(n)}, \ldots, A_k^{(n)}) = ((a_{i,j}^{n,1})_{i,j \in \{1,\ldots,d+1\}}, \ldots, (a_{i,j}^{n,k})_{i,j \in \{1,\ldots,d+1\}}) \in (\mathbb{S}_{d+1})^k$, $n \in \mathcal{N}$, such that for every $n \in \mathcal{N}$ and every $i \in \{1, \ldots, k\}$ it holds that

$$(4.30) \quad (n(\nabla_{(t_i,x_i)}\phi)((t_1^{(n)}, x_1^{(n)}), \ldots, (t_k^{(n)}, x_k^{(n)})), nA_i^{(n)}) \in (\hat{J}_+^2 \tilde{u}_i)(t_i^{(n)}, x_i^{(n)})$$

and

$$-[n + n\|(\mathrm{Hess}\,\phi)(\underline{z}^{(n)})\|_{L(\mathbb{R}^{(d+1)k})}]I \leq \begin{pmatrix} nA_1^{(n)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & nA_k^{(n)} \end{pmatrix}$$
$$\leq n(\mathrm{Hess}\,\phi)(\underline{z}^{(n)}) + \frac{1}{n}[n(\mathrm{Hess}\,\phi)(\underline{z}^{(n)})]^2.$$

Combining this with the identity $(\mathrm{Hess}\,\phi)(z) = (\mathrm{Hess}\,\phi)(0)$ for all $z \in ((0, T) \times O)^k$ then implies that

$$-[1 + \|(\mathrm{Hess}\,\phi)(0)\|_{L(\mathbb{R}^{(d+1)k})}]I \leq \begin{pmatrix} A_1^{(n)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k^{(n)} \end{pmatrix}$$

(4.31)
$$\leq (\mathrm{Hess}\,\phi)(0) + [(\mathrm{Hess}\,\phi)(0)]^2$$

for all $n \in \mathcal{N}$. To simplify the notation we define matrices $B_l^{(n)} \in \mathbb{S}_d$, $l \in \{1, \ldots, k\}$, $n \in \mathcal{N}$, by $B_l^{(n)} := (a_{i+1,j+1}^{n,l})_{i,j \in \{1,\ldots,d\}}$ for all $l \in \{1, \ldots, k\}$ and all $n \in \mathcal{N}$. Corollary 4.6 together with (4.30) and the fact that it holds for every $i \in \{1, \ldots, k\}$ that $\tilde{u}_i|_{(0,T) \times O}$ is a viscosity subsolution (4.23) then proves that

$$(4.32) \quad n\left(\frac{\partial}{\partial t_i}\phi\right)(\underline{z}^{(n)}) - \tilde{G}_i(t_i^{(n)}, x_i^{(n)}, \tilde{u}_i(t_i^{(n)}, x_i^{(n)}), n(\nabla_{x_i}\phi)(\underline{z}^{(n)}), nB_i^{(n)}) \leq 0$$

for all $i \in \{1, \ldots, k\}$ and all $n \in \mathcal{N}$. Summing over $i \in \{1, \ldots, k\}$ hence results in

(4.33)
$$n\sum_{i=1}^k \left(\frac{\partial}{\partial t_i}\phi\right)(\underline{z}^{(n)})$$
$$\leq \sum_{i=1}^k \tilde{G}_i(t_i^{(n)}, x_i^{(n)}, \tilde{u}_i(t_i^{(n)}, x_i^{(n)}), n(\nabla_{x_i}\phi)(\underline{z}^{(n)}), nB_i^{(n)})$$

for all $n \in \mathcal{N}$. Next note that the definition of ϕ ensures in the case $k \geq 2$ that

(4.34)
$$\left(\frac{\partial}{\partial t_i}\phi\right)((t_1, x_1), \ldots, (t_k, x_k)) = \frac{1}{2}\sum_{j=2}^{k}\frac{\partial}{\partial t_i}(t_j - t_{j-1})^2$$

$$= \begin{cases} t_1 - t_2, & i = 1, \\ 2t_i - t_{i-1} - t_{i+1}, & 1 < i < k, \\ t_k - t_{k-1}, & i = k, \end{cases}$$

for all $i \in \{1, \ldots, k\}$ and all $(t_1, x_1), \ldots, (t_k, x_k) \in (0, T) \times O$ and, therefore, we obtain that in the case $k \geq 2$ it holds that

$$\sum_{i=1}^{k}\left(\frac{\partial}{\partial t_i}\phi\right)((t_1, x_1), \ldots, (t_k, x_k))$$

$$= t_1 - t_2 + t_k - t_{k-1} + \sum_{i=2}^{k-1}(2t_i - t_{i-1} - t_{i+1})$$

(4.35)
$$= t_1 - t_2 + t_k - t_{k-1} + \left(\sum_{i=2}^{k-1}t_i - t_{i-1}\right) + \left(\sum_{i=2}^{k-1}t_i - t_{i+1}\right)$$

$$= \left(t_1 - t_{k-1} + \sum_{i=2}^{k-1}t_i - t_{i-1}\right) + \left(t_k - t_2 + \sum_{i=2}^{k-1}t_i - t_{i+1}\right)$$

$$= 0$$

for all $(t_1, x_1), \ldots, (t_k, x_k) \in (0, T) \times O$. Combining this with (4.33) results in

(4.36)
$$0 \leq \sum_{i=1}^{k}\tilde{G}_i\left(t_i^{(n)}, x_i^{(n)}, \tilde{u}_i(t_i^{(n)}, x_i^{(n)}), n(\nabla_{x_i}\phi)(\underline{z}^{(n)}), nB_i^{(n)}\right)$$

for all $n \in \mathcal{N}$. Therefore, we obtain from (4.22) and from $\hat{t}_1 = \cdots = \hat{t}_k \in (0, T)$ and $t_1^{(n_j)}, \ldots, t_k^{(n_j)} \in (0, T)$ for all $j \in \{j_0, j_0 + 1, \ldots\}$ that

(4.37)
$$\sum_{i=1}^{k}\frac{\delta}{(T - t_i^{(n_j)})^2}$$

$$\leq \sum_{i=1}^{k}G_i\left(t_i^{(n_j)}, x_i^{(n_j)}, \tilde{u}_i(t_i^{(n_j)}, x_i^{(n_j)})\right.$$

$$\left. + \frac{\delta}{(T - t_i^{(n_j)})}, n_j(\nabla_{x_i}\phi)(\underline{z}^{(n_j)}), n_j B_i^{(n_j)}\right)$$

for all $j \in \{j_0, j_0, \ldots\}$. In the next step, we define $(\mathbf{t}_i^{(n)}, \mathbf{x}_i^{(n)}, \mathbf{r}_i^{(n)}, \mathbf{A}_i^{(n)}) \in (0, T) \times O \times \mathbb{R} \times \mathbb{S}_d$, $i \in \{1, \ldots, k\}$, $n \in \mathbb{N}$, by

$$
\begin{aligned}
&(\mathbf{t}_i^{(n)}, \mathbf{x}_i^{(n)}, \mathbf{r}_i^{(n)}, \mathbf{A}_i^{(n)}) \\
&:= \begin{cases}
\left(t_i^{(n)}, x_i^{(n)}, \tilde{u}_i\left(t_i^{(n)}, x_i^{(n)}\right) + \dfrac{\delta}{(T - t_i^{(n)})}, B_i^{(n)} \right), \\
\qquad n \in \{n_j \in \mathbb{N} : j \in \{j_0, j_0 + 1, \ldots\}\}, \\
\left(\hat{t}_1, \hat{x}_1, \dfrac{\lim_{\alpha \to \infty} S_\alpha}{k} + \dfrac{\delta}{(T - \hat{t}_1)}, 0 \right), \\
\qquad \text{else,}
\end{cases}
\end{aligned}
\tag{4.38}
$$

for all $i \in \{1, \ldots, k\}$ and all $n \in \mathbb{N}$. Moreover, observe that in the case $k \geq 2$ it holds that

$$
\begin{aligned}
(\nabla_{x_i} \phi)\big((t_1, x_1), \ldots, (t_k, x_k)\big) &= \frac{1}{2} \sum_{j=2}^{k} \nabla_{x_i}\left(\|x_j - x_{j-1}\|^2\right) \\
&= \begin{cases}
x_1 - x_2, & i = 1, \\
2x_i - x_{i-1} - x_{i+1}, & 1 < i < k, \\
x_k - x_{k-1}, & i = k
\end{cases}
\end{aligned}
\tag{4.39}
$$

for all $i \in \{1, \ldots, k\}$ and all $(t_1, x_1), \ldots, (t_k, x_k) \in (0, T) \times O$. Then (4.37) ensures that

$$
\begin{aligned}
\frac{k\delta}{(T - \hat{t}_1)^2} \leq \limsup_{n \to \infty}\Bigg[&\sum_{i=1}^{k} G_i\big(\mathbf{t}_i^{(n)}, \mathbf{x}_i^{(n)}, \mathbf{r}_i^{(n)}, \\
&n\big(\mathbb{1}_{[2,k]}(i) \cdot \big[\mathbf{x}_i^{(n)} - \mathbf{x}_{i-1}^{(n)}\big] \\
&+ \mathbb{1}_{[1,k-1]}(i) \cdot \big[\mathbf{x}_i^{(n)} - \mathbf{x}_{i+1}^{(n)}\big]\big), n\mathbf{A}_i^{(n)}\big)\Bigg].
\end{aligned}
\tag{4.40}
$$

Next, we observe that the Taylor expansion $\phi(z) = \phi(0) + \langle(\nabla\phi)(0), z\rangle + \frac{1}{2}\langle z, (\operatorname{Hess}\phi)(0)z\rangle = \frac{1}{2}\langle z, (\operatorname{Hess}\phi)(0)z\rangle$ for all $z \in \mathbb{R}^{(d+1)k}$ implies that $(\nabla\phi)(z) = (\operatorname{Hess}\phi)(0)z$ for all $z \in \mathbb{R}^{(d+1)k}$. This together with (4.34), (4.39) and the estimate $(a + b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$ results in

$$
\begin{aligned}
\big\langle z, \big((\operatorname{Hess}\phi)(0)\big)^2 z\big\rangle &= \big\langle(\operatorname{Hess}\phi)(0)z, (\operatorname{Hess}\phi)(0)z\big\rangle = \big\|(\operatorname{Hess}\phi)(0)z\big\|^2 \\
&= \big\|(\nabla\phi)(z)\big\|^2 \\
&= \|z_1 - z_2\|^2 + \left[\sum_{i=2}^{k-1} \|2z_i - z_{i-1} - z_{i+1}\|^2\right] + \|z_k - z_{k-1}\|^2
\end{aligned}
\tag{4.41}
$$

$$\leq 2\|z_1 - z_2\|^2 + \left[\sum_{i=2}^{k-1} 2\left(\|z_i - z_{i-1}\|^2 + \|z_i - z_{i+1}\|^2\right)\right]$$

$$+ 2\|z_k - z_{k-1}\|^2$$

$$= 4\left[\sum_{i=2}^{k} \|z_i - z_{i-1}\|^2\right] \leq 8\left[\sum_{i=2}^{k} \|z_i\|^2\right] + 8\left[\sum_{i=2}^{k} \|z_{i-1}\|^2\right]$$

$$\leq 16\|z\|^2$$

for all $z = (z_1, \ldots, z_k) \in \mathbb{R}^{(d+1)k}$. Inequality (4.41) implies that $\|(\operatorname{Hess}\phi)(0)\|_{L(\mathbb{R}^{(d+1)\times k})} \leq 4$. Consequently, (4.31), (4.41) and $\langle z, (\operatorname{Hess}\phi)(0)z\rangle = 2\phi(z)$ for all $z \in \mathbb{R}^{(d+1)k}$ yield that

(4.42)
$$-5\|z\|^2 \leq \sum_{i=1}^{k}\langle z_i, A_i^{(n)}z_i\rangle \leq 2\phi(z) + \langle z, \left((\operatorname{Hess}\phi)(0)\right)^2 z\rangle$$

$$\leq 5\sum_{i=2}^{k}\|z_i - z_{i-1}\|^2$$

for all $z = (z_1, \ldots, z_k) \in \mathbb{R}^{(d+1)k}$ and all $n \in \mathcal{N}$. Inequality (4.42), in particular, implies $-5\|z\|^2 \leq \sum_{i=1}^{k}\langle z_i, B_i^{(n)}z_i\rangle = \sum_{i=1}^{k}\langle z_i, \mathbf{A}_i^{(n)}z_i\rangle \leq 5\sum_{i=2}^{k}\|z_i - z_{i-1}\|^2$ for all $z = (z_1, \ldots, z_k) \in \mathbb{R}^{dk}$ and all $n \in \mathbb{N}$. Combining this, the identities

(4.43)
$$\lim_{j\to\infty}\left[\sum_{i=1}^{k}\left(\tilde{u}_i(t_i^{(n_j)}, x_i^{(n_j)}) + \frac{\delta}{(T - t_i^{(n_j)})}\right)\right] = \left(\lim_{j\to\infty} S_{n_j}\right) + \frac{k\delta}{(T - \hat{t}_1)}$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{k}\mathbf{r}_i^{(n)}\right] > 0,$$

$\lim_{n\to\infty} n\sum_{i=2}^{k}\|(\mathbf{t}_i^{(n)}, \mathbf{x}_i^{(n)}) - (\mathbf{t}_{i-1}^{(n)}, \mathbf{x}_{i-1}^{(n)})\|^2 = 0$ [see (4.28)] and the estimate $\sup_{n\in\mathbb{N}}\max_{i\in\{1,\ldots,k\}}|\mathbf{r}_i^{(n)}| < \infty$ with assumption (4.20) and with (4.40) shows that

$$0 < \frac{k\delta}{(T - \hat{t}_1)^2}$$

$$\leq \limsup_{n\to\infty}\left[\sum_{i=1}^{k} G_i\left(\mathbf{t}_i^{(n)}, \mathbf{x}_i^{(n)}, \mathbf{r}_i^{(n)}, n\left(\mathbb{1}_{[2,\infty)}(i) \cdot \left[\mathbf{x}_i^{(n)} - \mathbf{x}_{i-1}^{(n)}\right]\right.\right.$$

$$\left.\left. + \mathbb{1}_{[0,k-1]}(i) \cdot \left[\mathbf{x}_i^{(n)} - \mathbf{x}_{i+1}^{(n)}\right]\right), n\mathbf{A}_i^{(n)}\right)\right]$$

$$\leq 0.$$

This contradiction implies that $S_0 \leq 0$. As $\delta \in (0, 1]$ was arbitrary, we conclude that $\sum_{i=1}^{k} u_i(t, x) \leq 0$ for all $(t, x) \in [0, T) \times O$. This finishes the proof of Lemma 4.10. $\square$

The next result, Corollary 4.11, establishes a comparison result for certain viscosity subsolutions and certain viscosity supersolutions of a PDE. It is a direct consequence of Lemma 4.10 above in the case $k = 2$. Corollary 4.11 essentially generalizes Theorem 2.4 in Appendix C in Peng [61] (which assumes the function G to be globally Lipschitz continuous in the third and last argument uniformly in the remaining arguments) and essentially generalizes Theorem 8.2 in Crandall, Ishii and Lions [7] (which assumes a bounded domain and a globally uniform estimate on the function G). Corollary 4.11 is an immediate consequence of Lemma 4.2 and Lemma 4.10 with $k = 2$. Its proof is therefore omitted.

COROLLARY 4.11 (A comparison result for viscosity subsolutions and viscosity supersolutions). *Let $T \in (0, \infty)$, $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $u_1, u_2 \in C([0, T] \times O, \mathbb{R})$, let $G : (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic and continuous function and assume that $u_1|_{(0,T) \times O}$ is a viscosity subsolution of*

$$(4.44) \qquad \frac{\partial}{\partial t} u(t, x) - G\big(t, x, u(t, x), (\nabla_x u)(t, x), (\mathrm{Hess}_x u)(t, x)\big) = 0$$

for $(t, x) \in (0, T) \times O$ and that $u_2|_{(0,T) \times O}$ is a viscosity supersolution of (4.44). Moreover, assume that

$$(4.45) \qquad \begin{aligned} \limsup_{n \to \infty} \big[&G\big(t_n, x_n, r_n, n(x_n - \hat{x}_n), n A_n\big) \\ &- G\big(\hat{t}_n, \hat{x}_n, \hat{r}_n, n(x_n - \hat{x}_n), n \hat{A}_n\big) \big] \leq 0 \end{aligned}$$

for all $(t_n, x_n, r_n, A_n), (\hat{t}_n, \hat{x}_n, \hat{r}_n, \hat{A}_n) \in (0, T) \times O \times \mathbb{R} \times \mathbb{S}_d$, $n \in \mathbb{N}$, satisfying that $\lim_{n \to \infty}(t_n, x_n) \in (0, T) \times O$, that $\lim_{n \to \infty}(\sqrt{n}\|(t_n, x_n) - (\hat{t}_n, \hat{x}_n)\|) = 0$, that $0 < \lim_{n \to \infty}(r_n - \hat{r}_n) \leq \sup_{n \in \mathbb{N}}(|r_n| + |\hat{r}_n|) < \infty$ and that $\forall n \in \mathbb{N}, z, \hat{z} \in \mathbb{R}^d : \langle z, A_n z \rangle - \langle \hat{z}, \hat{A}_n \hat{z} \rangle \leq 5\|z - \hat{z}\|^2$. Furthermore, assume that $u_1(0, x) \leq u_2(0, x)$ for all $x \in O$ and that

$$(4.46) \qquad \lim_{n \to \infty} \Big[\sup_{(t,x) \in (0,T) \times O_n^c} \big(u_1(t, x) - u_2(t, x)\big) \Big] \leq 0.$$

Then $u_1 \leq u_2$, that is, it holds that $u_1(t, x) \leq u_2(t, x)$ for all $(t, x) \in [0, T] \times O$.

Assumption (4.46) in Corollary 4.11 is in several cases difficult to verify. Lemma 4.13 below gives an extension of Corollary 4.11 which postulates a less restrictive condition than (4.46) by using a suitable Lyapunov type function [cf. (4.53) in Lemma 4.13 and (4.46) in Corollary 4.11]. In the proof of Lemma 4.13, the following elementary lemma is used.

LEMMA 4.12 (Scaling of viscosity subsolutions and viscosity supersolutions).
*Let $T \in (0, \infty)$, $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $V \in C^2((0, T) \times O, (0, \infty))$,
let $G : (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function, let
$u : (0, T) \times O \to \mathbb{R}$ be a viscosity subsolution (supersolution) of (4.44) and let
$\tilde{G} : (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a function defined by*

$$\tilde{G}(t, x, r, p, A)$$

$$:= \frac{1}{V(t, x)} G\big(t, x, rV(t, x), pV(t, x) + r(\nabla_x V)(t, x), AV(t, x)$$

(4.47)
$$+ p[(\nabla_x V)(t, x)]^* + (\nabla_x V)(t, x)p^*$$

$$+ r(\text{Hess}_x V)(t, x)\big)$$

$$- r\frac{(\partial/\partial t)V(t, x)}{V(t, x)}$$

*for all $(t, x, r, p, A) \in (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$. Then $\tilde{G}$ is degenerate elliptic
and the function $\tilde{u} : (0, T) \times O \to \mathbb{R}$ defined by $\tilde{u}(t, x) = \frac{u(t,x)}{V(t,x)}$ for all $(t, x) \in
(0, T) \times O$ is a viscosity subsolution (supersolution) of*

(4.48)
$$\frac{\partial}{\partial t}\tilde{u}(t, x) - \tilde{G}\big(t, x, \tilde{u}(t, x), (\nabla_x \tilde{u})(t, x), (\text{Hess}_x \tilde{u})(t, x)\big) = 0$$

for $(t, x) \in (0, T) \times O$.

PROOF. We proof Lemma 4.12 in the case where u is a viscosity subsolution
of (4.44). The case where u is a viscosity supersolution of (4.44) follows analo-
gously. We thus assume in the following that u is a viscosity subsolution of (4.44).
First, observe that $\tilde{u}$ is upper semicontinuous and that $\tilde{G}$ is degenerate elliptic. In
the next step assume that there exist a vector $(t, x) \in (0, T) \times O$ and a function
$\phi \in C^2((0, T) \times O, \mathbb{R})$ satisfying $\phi(t, x) = \tilde{u}(t, x)$ and $\phi \geq \tilde{u}$. Then the function
$(0, T) \times O \ni (s, y) \mapsto \phi(s, y)V(s, y) \in \mathbb{R}$ is in $C^2((0, T) \times O, \mathbb{R})$ and satisfies
$\phi(t, x)V(t, x) = \tilde{u}(t, x)V(t, x) = u(t, x)$ and $\phi \cdot V \geq \tilde{u} \cdot V = u$. As u is a viscos-
ity subsolution of (4.44), we get

(4.49)
$$V(t, x) \cdot \frac{\partial}{\partial t}\phi(t, x) + \phi(t, x) \cdot \frac{\partial}{\partial t}V(t, x)$$

$$\leq G\big(t, x, \phi(t, x)V(t, x), (\nabla_x(\phi V))(t, x), (\text{Hess}_x(\phi V))(t, x)\big).$$

Rearranging this inequality results in

$$\frac{\partial}{\partial t}\phi(t, x) \leq \frac{1}{V(t, x)} G\big(t, x, \phi(t, x)V(t, x), (\nabla_x(\phi V))(t, x),$$

$$(\text{Hess}_x(\phi V))(t, x)\big)$$

(4.50)
$$- \phi(t, x)\frac{(\partial/\partial t)V(t, x)}{V(t, x)}$$

$$= \frac{1}{V(t,x)} G\big(t, x, \phi(t,x)V(t,x), (\nabla_x \phi)(t,x)V(t,x)$$

$$+ \phi(t,x)(\nabla_x V)(t,x), (\mathrm{Hess}_x \phi)(t,x)V(t,x)$$

$$+ (\nabla_x \phi)(t,x)\big[(\nabla_x V)(t,x)\big]^*$$

$$+ (\nabla_x V)(t,x)\big[(\nabla_x \phi)(t,x)\big]^*$$

$$+ \phi(t,x)(\mathrm{Hess}_x V)(t,x)\big)$$

$$- \phi(t,x)\frac{(\partial/\partial t)V(t,x)}{V(t,x)}$$

$$= \tilde{G}\big(t, x, \phi(t,x), (\nabla_x \phi)(t,x), (\mathrm{Hess}_x \phi)(t,x)\big).$$

This proves inequality (4.50) for all $\phi \in \{\psi \in C^2((0,T) \times O, \mathbb{R}) : \psi(t,x) = \tilde{u}(t,x)$ and $\psi \geq \tilde{u}\}$ and all $(t,x) \in (0,T) \times O$. Therefore, $\tilde{u}$ is a viscosity sub-solution of (4.48) and the proof of Lemma 4.12 is completed. $\square$

LEMMA 4.13 (A further comparison result for viscosity subsolutions and viscosity supersolutions). *Let $T \in (0, \infty)$, $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $u_1, u_2 \in C([0,T] \times O, \mathbb{R})$, $V \in C([0,T] \times O, (0, \infty))$, let $G : (0,T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic and continuous function and assume that $u_1|_{(0,T) \times O}$ is a viscosity subsolution of*

$$(4.51) \qquad \frac{\partial}{\partial t}u(t,x) - G\big(t, x, u(t,x), (\nabla_x u)(t,x), (\mathrm{Hess}_x u)(t,x)\big) = 0$$

for $(t,x) \in (0,T) \times O$, that $u_2|_{(0,T) \times O}$ is a viscosity supersolution of (4.51) and that for every $r \in (0, \infty)$ it holds that $rV|_{(0,T) \times O} \in C^2((0,T) \times O, (0, \infty))$ is a classical supersolution of (4.51). Moreover, assume that

$$
\limsup_{n \to \infty} \bigg(\frac{G(t_n, x_n, r_n, p_n, A_n + nB_n V(t_n, x_n))}{V(t_n, x_n)}
$$

$$(4.52) \qquad\qquad - \frac{G(\hat{t}_n, \hat{x}_n, \hat{r}_n, \hat{p}_n, \hat{A}_n + n\hat{B}_n V(\hat{t}_n, \hat{x}_n))}{V(\hat{t}_n, \hat{x}_n)} \bigg)$$

$$\leq \frac{G(t_0, x_0, r_0, p_0, A_0)}{V(t_0, x_0)}$$

for all $(t_n, x_n, r_n, p_n, A_n, B_n), (\hat{t}_n, \hat{x}_n, \hat{r}_n, \hat{p}_n, \hat{A}_n, \hat{B}_n) \in (0,T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \times \mathbb{S}_d$, $n \in \mathbb{N}_0$, satisfying that $\lim_{n \to \infty}(t_n, x_n) = (t_0, x_0)$, that $\lim_{n \to \infty}(\sqrt{n}\|(t_n, x_n) - (\hat{t}_n, \hat{x}_n)\|) = 0$, that $0 < r_0 = \lim_{n \to \infty}(r_n - \hat{r}_n) \leq \sup_{n \in \mathbb{N}}(|r_n| + |\hat{r}_n|) < \infty$, that $\lim_{n \to \infty}(p_n - \hat{p}_n) = p_0$, that $\lim_{n \to \infty}(A_n - \hat{A}_n) = A_0$, that $\lim_{n \to \infty}(n^{-1/2}[\|\hat{p}_n\| + \|\hat{A}_n\|_{L(\mathbb{R}^d)}]) = 0$ and that $\forall n \in \mathbb{N}, z, \hat{z} \in \mathbb{R}^d : \langle z, B_n z \rangle - \langle \hat{z}, \hat{B}_n \hat{z} \rangle \leq 5\|z - \hat{z}\|^2$. Furthermore, assume that $u_1(0,x) \leq$

$u_2(0, x)$ *for all* $x \in O$ *and that*

$$(4.53) \qquad \lim_{n \to \infty} \left[\sup_{x \in O_n^c} \sup_{t \in (0,T)} \frac{(u_1(t, x) - u_2(t, x))}{V(t, x)} \right] \leq 0.$$

Then $u_1 \leq u_2$, *that is, it holds that* $u_1(t, x) \leq u_2(t, x)$ *for all* $(t, x) \in [0, T] \times O$.

PROOF. Define functions $\tilde{u}_1, \tilde{u}_2 : [0, T] \times O \to \mathbb{R}$ and $\tilde{G} : (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ by $\tilde{u}_1(t, x) = \frac{u_1(t,x)}{V(t,x)}$ and $\tilde{u}_2(t, x) = \frac{u_2(t,x)}{V(t,x)}$ for all $(t, x) \in [0, T] \times O$ and by

$$\begin{aligned} &\tilde{G}(t, x, r, p, A) \\ &:= \frac{1}{V(t, x)} G\big(t, x, rV(t, x), pV(t, x) + r(\nabla_x V)(t, x), AV(t, x) \\ &\qquad\qquad + p\big[(\nabla_x V)(t, x)\big]^* + (\nabla_x V)(t, x)p^* \\ &\qquad\qquad\qquad\qquad + r(\mathrm{Hess}_x V)(t, x)\big) \\ &\qquad - r\frac{(\partial/\partial t)V(t, x)}{V(t, x)} \end{aligned} \tag{4.54}$$

for all $(t, x, r, p, A) \in (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$. Lemma 4.12 then ensures that $\tilde{G}$ is degenerate elliptic, that $\tilde{u}_1|_{(0,T) \times O}$ is a viscosity subsolution of

$$(4.55) \qquad \frac{\partial}{\partial t} u(t, x) - \tilde{G}\big(t, x, u(t, x), (\nabla_x u)(t, x), (\mathrm{Hess}_x u)(t, x)\big) = 0$$

for $(t, x) \in (0, T) \times O$ and that $\tilde{u}_2|_{(0,T) \times O}$ is viscosity supersolution of (4.55). Below we will finish this proof by an application of Corollary 4.11 with $\tilde{u}_1, \tilde{u}_2$ and $\tilde{G}$. For this, we now check the assumptions of Corollary 4.11. First, observe that assumption (4.53) ensures that (4.46) is fulfilled. In addition, observe that the assumption $u_1(0, x) \leq u_2(0, x)$ for all $x \in O$ ensures that $\tilde{u}_1(0, x) \leq \tilde{u}_2(0, x)$ for all $x \in O$. In the next step, we verify (4.45). For this, let $(t_n, x_n, r_n, A_n), (\hat{t}_n, \hat{x}_n, \hat{r}_n, \hat{A}_n) \in (0, T) \times O \times \mathbb{R} \times \mathbb{S}_d$, $n \in \mathbb{N}_0$, be sequences satisfying that $\lim_{n \to \infty}(t_n, x_n) = (t_0, x_0) = (\hat{t}_0, \hat{x}_0) \in (0, T) \times O$, that $\lim_{n \to \infty}(\sqrt{n}\|(t_n, x_n) - (\hat{t}_n, \hat{x}_n)\|) = 0$, that $0 < r_0 = \hat{r}_0 = \lim_{n \to \infty}(r_n - \hat{r}_n) \leq \sup_{n \in \mathbb{N}}(|r_n| + |\hat{r}_n|) < \infty$ and that $\forall n \in \mathbb{N}, z, \hat{z} \in \mathbb{R}^d : \langle z, A_n z \rangle - \langle \hat{z}, \hat{A}_n \hat{z} \rangle \leq 5\|z - \hat{z}\|^2$. To verify (4.45), we will apply assumption (4.52). For this, we define $\tilde{V} : [0, T] \times O \to (0, \infty)$ and $(\mathbf{t}_n, \mathbf{x}_n, \mathbf{r}_n, \mathbf{p}_n, \mathbf{A}_n, \mathbf{B}_n), (\hat{\mathbf{t}}_n, \hat{\mathbf{x}}_n, \hat{\mathbf{r}}_n, \hat{\mathbf{p}}_n, \hat{\mathbf{A}}_n, \hat{\mathbf{B}}_n) \in (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d, n \in \mathbb{N}_0$, by $\tilde{V}(t, x) = r_0 \cdot V(t, x)$ for all $(t, x) \in [0, T] \times O$ and by $(\mathbf{t}_n, \mathbf{x}_n, \mathbf{r}_n) := (t_n, x_n, r_n V(t_n, x_n))$, $(\hat{\mathbf{t}}_n, \hat{\mathbf{x}}_n, \hat{\mathbf{r}}_n) := (\hat{t}_n, \hat{x}_n, \hat{r}_n V(\hat{t}_n, \hat{x}_n))$, $\mathbf{B}_n := A_n, \hat{\mathbf{B}}_n := \hat{A}_n$,

$$(4.56) \qquad \mathbf{p}_n := n(x_n - \hat{x}_n)V(t_n, x_n) + r_n(\nabla_x V)(t_n, x_n),$$

$$(4.57) \qquad \hat{\mathbf{p}}_n := n(x_n - \hat{x}_n)V(\hat{t}_n, \hat{x}_n) + \hat{r}_n(\nabla_x V)(\hat{t}_n, \hat{x}_n),$$

$$(4.58) \quad \mathbf{A}_n := n(x_n - \hat{x}_n)\big[(\nabla_x V)(t_n, x_n)\big]^* + (\nabla_x V)(t_n, x_n)n(x_n - \hat{x}_n)^*$$
$$+ r_n(\text{Hess}_x V)(t_n, x_n),$$

$$(4.59) \quad \hat{\mathbf{A}}_n := n(x_n - \hat{x}_n)\big[(\nabla_x V)(\hat{t}_n, \hat{x}_n)\big]^* + (\nabla_x V)(\hat{t}_n, \hat{x}_n)n(x_n - \hat{x}_n)^*$$
$$+ \hat{r}_n(\text{Hess}_x V)(\hat{t}_n, \hat{x}_n)$$

for all $n \in \mathbb{N}_0$. Continuity of V and $0 < r_0 = \lim_{n\to\infty}(r_n - \hat{r}_n) \le \sup_{n\in\mathbb{N}}(|r_n| + |\hat{r}_n|) < \infty$ then imply that

$$0 < \mathbf{r}_0 = r_0 V(t_0, x_0) = \lim_{n\to\infty}\big(r_n V(t_n, x_n) - \hat{r}_n V(\hat{t}_n, \hat{x}_n)\big)$$

$$(4.60) \qquad = \lim_{n\to\infty}(\mathbf{r}_n - \hat{\mathbf{r}}_n)$$

$$\le \sup_{n\in\mathbb{N}}\big(|\mathbf{r}_n| + |\hat{\mathbf{r}}_n|\big) < \infty.$$

Moreover, note that the local Lipschitz continuity of V and $\nabla_x V$ and the continuity of $\text{Hess}_x V$ together with the assumptions $\lim_{n\to\infty}(\sqrt{n}\|(t_n, x_n) - (\hat{t}_n, \hat{x}_n)\|) = \lim_{n\to\infty}(\sqrt{n}\|x_n - \hat{x}_n\|) = 0$, $\lim_{n\to\infty}(r_n - \hat{r}_n) = r_0$ and $\sup_{n\in\mathbb{N}}|\hat{r}_n| < \infty$ imply that

$$\lim_{n\to\infty}(\mathbf{p}_n - \hat{\mathbf{p}}_n) = \lim_{n\to\infty}\big[n(x_n - \hat{x}_n)\big(V(t_n, x_n) - V(\hat{t}_n, \hat{x}_n)\big)\big]$$

$$+ \lim_{n\to\infty}\big[(r_n - \hat{r}_n)(\nabla_x V)(t_n, x_n)\big]$$

$$(4.61) \qquad + \lim_{n\to\infty}\big[\hat{r}_n\big((\nabla_x V)(t_n, x_n) - (\nabla_x V)(\hat{t}_n, \hat{x}_n)\big)\big]$$

$$= r_0(\nabla_x V)(t_0, x_0) = \mathbf{p}_0,$$

$$\lim_{n\to\infty}(\mathbf{A}_n - \hat{\mathbf{A}}_n) = \lim_{n\to\infty}\big(n(x_n - \hat{x}_n)\big(\big[(\nabla_x V)(t_n, x_n)\big]^* - \big[(\nabla_x V)(\hat{t}_n, \hat{x}_n)\big]^*\big)\big)$$

$$+ \lim_{n\to\infty}\big(\big[(\nabla_x V)(t_n, x_n) - (\nabla_x V)(\hat{t}_n, \hat{x}_n)\big]n(x_n - \hat{x}_n)^*\big)$$

$$(4.62) \qquad + \lim_{n\to\infty}\big([r_n - \hat{r}_n](\text{Hess}_x V)(t_n, x_n)\big)$$

$$+ \lim_{n\to\infty}\big(\hat{r}_n\big[(\text{Hess}_x V)(t_n, x_n) - (\text{Hess}_x V)(\hat{t}_n, \hat{x}_n)\big]\big)$$

$$= r_0(\text{Hess}_x v)(t_0, x_0) = \mathbf{A}_0$$

and $\lim_{n\to\infty}(n^{-1/2}[\|\hat{\mathbf{p}}_n\| + \|\hat{\mathbf{A}}_n\|_{L(\mathbb{R}^d)}]) = 0$. Combining this and (4.60) with assumption (4.52) shows that

$$\limsup_{n\to\infty}\Bigg(\frac{G(\mathbf{t}_n, \mathbf{x}_n, \mathbf{r}_n, \mathbf{p}_n, \mathbf{A}_n + n\mathbf{B}_n V(\mathbf{t}_n, \mathbf{x}_n))}{V(\mathbf{t}_n, \mathbf{x}_n)}$$

$$(4.63) \qquad - \frac{G(\hat{\mathbf{t}}_n, \hat{\mathbf{x}}_n, \hat{\mathbf{r}}_n, \hat{\mathbf{p}}_n, \hat{\mathbf{A}}_n + n\hat{\mathbf{B}}_n V(\hat{\mathbf{t}}_n, \hat{\mathbf{x}}_n))}{V(\hat{\mathbf{t}}_n, \hat{\mathbf{x}}_n)}\Bigg)$$

$$\le \frac{G(\mathbf{t}_0, \mathbf{x}_0, \mathbf{r}_0, \mathbf{p}_0, \mathbf{A}_0)}{V(\mathbf{t}_0, \mathbf{x}_0)}.$$

The definition of $\tilde{G}$ hence implies that

$$
\begin{aligned}
\limsup_{n\to\infty} &\big(\tilde{G}(t_n, x_n, r_n, n(x_n - \hat{x}_n), nA_n) - \tilde{G}(\hat{t}_n, \hat{x}_n, \hat{r}_n, n(x_n - \hat{x}_n), n\hat{A}_n)\big) \\
&= \limsup_{n\to\infty} \Big(\frac{G(t_n, x_n, \mathbf{r}_n, \mathbf{p}_n, \mathbf{A}_n + n\mathbf{B}_n V(t_n, x_n)) - r_n(\partial/\partial t)V(t_n, x_n)}{V(t_n, x_n)} \\
&\qquad - \frac{G(\hat{t}_n, \hat{x}_n, \hat{\mathbf{r}}_n, \hat{\mathbf{p}}_n, \hat{\mathbf{A}}_n + n\hat{\mathbf{B}}_n V(\hat{t}_n, \hat{x}_n)) - \hat{r}_n(\partial/\partial t)V(\hat{t}_n, \hat{x}_n)}{V(\hat{t}_n, \hat{x}_n)} \Big) \\
&\leq \frac{G(t_0, x_0, \mathbf{r}_0, \mathbf{p}_0, \mathbf{A}_0)}{V(t_0, x_0)} - \frac{r_0(\partial/\partial t)V(t_0, x_0)}{V(t_0, x_0)} \\
&= (-[(\partial/\partial t)\tilde{V}(t_0, x_0) \\
&\qquad - G(t_0, x_0, \tilde{V}(t_0, x_0), (\nabla_x \tilde{V})(t_0, x_0), (\mathrm{Hess}_x \tilde{V})(t_0, x_0))]) \\
&\qquad /(V(t_0, x_0)) \\
&\leq 0
\end{aligned}
$$

(4.64)

as $\tilde{V}$ is by assumption a classical supersolution of (4.51). We can thus apply Corollary 4.11 to obtain that $\tilde{u}_1(t, x) = \frac{u_1(t,x)}{V(t,x)} \leq \frac{u_2(t,x)}{V(t,x)} = \tilde{u}_2(t, x)$ for all $(t, x) \in [0, T] \times O$. This finishes the proof of Lemma 4.13. $\quad\square$

The next result, Corollary 4.14, asserts uniqueness of the solution of a linear second-order PDE. We assume that the Lyapunov-type function $V : [0, T] \times O \to (0, \infty)$ in Lemma 4.13 is of the form $V(t, x) = e^{\rho t} \cdot \tilde{V}(x)$ for all $(t, x) \in [0, T] \times O$ where $\rho \in \mathbb{R}$ is a real number and where $\tilde{V} : O \to (0, \infty)$ is a twice continuously differentiable function.

COROLLARY 4.14 (Uniqueness of viscosity solutions of Kolmogorov type equations). *Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, $\rho \in \mathbb{R}$, let $O \subset \mathbb{R}^d$ be an open set, let $\varphi \in C(O, \mathbb{R})$, $v \in C((0, T) \times O, \mathbb{R})$, let $\mu : (0, T) \times O \to \mathbb{R}^d$ and $\sigma : (0, T) \times O \to \mathbb{R}^{d \times m}$ be locally Lipschitz continuous functions and let $V \in C^2(O, (0, \infty))$ satisfy*

$$
\begin{aligned}
v(t, x)V(x) &+ \langle \mu(t, x), (\nabla V)(x) \rangle + \mathrm{tr}\big(\sigma(t, x)[\sigma(t, x)]^*(\mathrm{Hess}\, V)(x)\big) \\
&\leq \rho \cdot V(x)
\end{aligned}
$$

(4.65)

for all $(t, x) \in (0, T) \times O$. Then there exists at most one continuous function $u : [0, T] \times O \to \mathbb{R}$ which fulfills $u(0, x) = \varphi(x)$ for all $x \in O$, which fulfills

$\lim_{n\to\infty} \sup_{(t,x)\in(0,T)\times O_n^c} \frac{|u(t,x)|}{V(x)} = 0$ *and which fulfills that* $u|_{(0,T)\times O}$ *is a viscosity solution of*

$$\frac{\partial}{\partial t}u(t,x) - v(t,x)u(t,x) - \langle \mu(t,x), (\nabla_x u)(t,x)\rangle$$

(4.66)
$$- \operatorname{tr}(\sigma(t,x)[\sigma(t,x)]^*(\operatorname{Hess}_x u)(t,x))$$

$$= 0$$

for $(t,x) \in (0,T) \times O$.

PROOF.   Let $u_1, u_2 : [0,T] \times O \to \mathbb{R}$ be two continuous functions such that $u_1(0,x) = \varphi(x) = u_2(0,x)$ for all $x \in O$, such that

$$\lim_{n\to\infty} \sup_{(t,x)\in(0,T)\times O_n^c} \frac{|u_1(t,x)| + |u_2(t,x)|}{V(x)} = 0$$

and such that $u_1|_{(0,T)\times O}$ and $u_2|_{(0,T)\times O}$ are viscosity solutions of (4.66). Then define a function $G : (0,T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ by $G(t,x,r,p,A) = v(t,x)r + \langle \mu(t,x), p\rangle + \operatorname{tr}(\sigma(t,x)[\sigma(t,x)]^*A)$. We show Corollary 4.14 by applying Lemma 4.13. To this end we now verify (4.52). For this, let $(t_n, x_n, r_n, p_n, A_n, B_n), (\hat{t}_n, \hat{x}_n, \hat{r}_n, \hat{p}_n, \hat{A}_n, \hat{B}_n) \in (0,T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \times \mathbb{S}_d, n \in \mathbb{N}_0$, satisfy that $\lim_{n\to\infty}(t_n, x_n) = (t_0, x_0)$, that $\lim_{n\to\infty}(\sqrt{n}\|(t_n, x_n) - (\hat{t}_n, \hat{x}_n)\|) = 0$, that $0 < r_0 = \lim_{n\to\infty}(r_n - \hat{r}_n) \le \sup_{n\in\mathbb{N}}(|r_n| + |\hat{r}_n|) < \infty$, that $\lim_{n\to\infty}(p_n - \hat{p}_n) = p_0$, that $\lim_{n\to\infty}(A_n - \hat{A}_n) = A_0$, that $\lim_{n\to\infty}(n^{-1/2}[\|\hat{p}_n\| + \|\hat{A}_n\|_{L(\mathbb{R}^d)}]) = 0$ and that $\forall n \in \mathbb{N}, z, \hat{z} \in \mathbb{R}^d : \langle z, B_n z\rangle - \langle \hat{z}, \hat{B}_n \hat{z}\rangle \le 5\|z - \hat{z}\|^2$. Then it holds that

$$\limsup_{n\to\infty}\left(\frac{1}{V(t_n, x_n)}G(t_n, x_n, r_n, p_n, A_n + nB_n V(t_n, x_n))\right.$$

$$\left. - \frac{1}{V(\hat{t}_n, \hat{x}_n)}G(\hat{t}_n, \hat{x}_n, \hat{r}_n, \hat{p}_n, \hat{A}_n + n\hat{B}_n V(\hat{t}_n, \hat{x}_n))\right)$$

$$\le \limsup_{n\to\infty}\left(\frac{v(t_n, x_n)r_n}{V(t_n, x_n)} - \frac{v(\hat{t}_n, \hat{x}_n)\hat{r}_n}{V(\hat{t}_n, \hat{x}_n)}\right)$$

$$+ \limsup_{n\to\infty}\left(\frac{\langle \mu(t_n, x_n), p_n\rangle}{V(t_n, x_n)} - \frac{\langle \mu(\hat{t}_n, \hat{x}_n), \hat{p}_n\rangle}{V(\hat{t}_n, \hat{x}_n)}\right)$$

$$+ \limsup_{n\to\infty}\left(\frac{\operatorname{tr}(\sigma(t_n, x_n)[\sigma(t_n, x_n)]^* A_n)}{V(t_n, x_n)}\right.$$

$$\left. - \frac{\operatorname{tr}(\sigma(\hat{t}_n, \hat{x}_n)[\sigma(\hat{t}_n, \hat{x}_n)]^*\hat{A}_n)}{V(\hat{t}_n, \hat{x}_n)}\right)$$

$$+ \limsup_{n\to\infty}(n[\operatorname{tr}([\sigma(t_n, x_n)]^* B_n \sigma(t_n, x_n))$$

(4.67)
$$- \operatorname{tr}([\sigma(\hat{t}_n, \hat{x}_n)]^* \hat{B}_n \sigma(\hat{t}_n, \hat{x}_n))])$$

$$\leq \limsup_{n\to\infty}\left(\frac{v(t_n, x_n)(r_n - \hat{r}_n)}{V(t_n, x_n)}\right)$$

$$+ \limsup_{n\to\infty}\left(\left[\frac{v(t_n, x_n)}{V(t_n, x_n)} - \frac{v(\hat{t}_n, \hat{x}_n)}{V(\hat{t}_n, \hat{x}_n)}\right]\hat{r}_n\right)$$

$$+ \limsup_{n\to\infty}\left(\frac{\langle\mu(t_n, x_n), p_n - \hat{p}_n\rangle}{V(t_n, x_n)}\right)$$

$$+ \limsup_{n\to\infty}\left(\left\langle\sqrt{n}\left[\frac{\mu(t_n, x_n)}{V(t_n, x_n)} - \frac{\mu(\hat{t}_n, \hat{x}_n)}{V(\hat{t}_n, \hat{x}_n)}\right], \frac{\hat{p}_n}{\sqrt{n}}\right\rangle\right)$$

$$+ \limsup_{n\to\infty}\left(\operatorname{tr}\left(\frac{\sigma(t_n, x_n)[\sigma(t_n, x_n)]^*}{V(t_n, x_n)}(A_n - \hat{A}_n)\right)\right)$$

$$+ \limsup_{n\to\infty}\left(\operatorname{tr}\left(\sqrt{n}\left[\frac{\sigma(t_n, x_n)[\sigma(t_n, x_n)]^*}{V(t_n, x_n)} - \frac{\sigma(\hat{t}_n, \hat{x}_n)[\sigma(\hat{t}_n, \hat{x}_n)]^*}{V(\hat{t}_n, \hat{x}_n)}\right]\frac{\hat{A}_n}{\sqrt{n}}\right)\right)$$

$$+ \limsup_{n\to\infty}\left(n\sum_{i=1}^{m}[\langle\sigma(t_n, x_n)e_i^{(m)}, B_n\sigma(t_n, x_n)e_i^{(m)}\rangle\right.$$

$$\left. - \langle\sigma(\hat{t}_n, \hat{x}_n)e_i^{(m)}, \hat{B}_n\sigma(\hat{t}_n, \hat{x}_n)e_i^{(m)}\rangle]\right).$$

Hence, the local Lipschitz continuity of the functions $\frac{\mu}{V}$ and $\frac{A}{V}$ together with the properties of $(t_n, x_n, r_n, p_n, A_n, B_n)$, $(\hat{t}_n, \hat{x}_n, \hat{r}_n, \hat{p}_n, \hat{A}_n, \hat{B}_n) \in (0, T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \times \mathbb{S}_d$, $n \in \mathbb{N}_0$, implies that

$$\limsup_{n\to\infty}\left(\frac{1}{V(t_n, x_n)}G(t_n, x_n, r_n, p_n, A_n + nB_nV(t_n, x_n))\right.$$

$$\left. - \frac{1}{V(\hat{t}_n, \hat{x}_n)}G(\hat{t}_n, \hat{x}_n, \hat{r}_n, \hat{p}_n, \hat{A}_n + n\hat{B}_nV(\hat{t}_n, \hat{x}_n))\right)$$

$$\leq \frac{v(t_0, x_0)r_0}{V(t_0, x_0)} + \frac{\langle\mu(t_0, x_0), p_0\rangle}{V(t_0, x_0)} + \operatorname{tr}\left(\frac{\sigma(t_0, x_0)[\sigma(t_0, x_0)]^*}{V(t_0, x_0)}A_0\right)$$

$$+ \limsup_{n\to\infty}\left(d\left[\sqrt{n}\left\|\frac{\sigma(t_n, x_n)[\sigma(t_n, x_n)]^*}{V(t_n, x_n)}\right.\right.\right.$$

$$\left.\left.\left. - \frac{\sigma(\hat{t}_n, \hat{x}_n)[\sigma(\hat{t}_n, \hat{x}_n)]^*}{V(\hat{t}_n, \hat{x}_n)}\right\|_{L(\mathbb{R}^d)}\right]\frac{\|\hat{A}_n\|_{L(\mathbb{R}^d)}}{\sqrt{n}}\right)$$

$$+ \limsup_{n\to\infty}\left(n\sum_{i=1}^{m}5\|\sigma(t_n, x_n)e_i^{(m)} - \sigma(\hat{t}_n, \hat{x}_n)e_i^{(m)}\|^2\right)$$

$$(4.68) \qquad = \frac{G(t_0, x_0, r_0, p_0, A_0)}{V(t_0, x_0)}$$

$$+ 5 \limsup_{n \to \infty} \left(n \left\| \sigma(t_n, x_n) - \sigma(\hat{t}_n, \hat{x}_n) \right\|^2_{HS(\mathbb{R}^m, \mathbb{R}^d)} \right)$$

$$= \frac{G(t_0, x_0, r_0, p_0, A_0)}{V(t_0, x_0)}.$$

This shows assumption (4.52). Moreover, by assumption, $u_1|_{(0,T) \times O}$ is a viscosity subsolution of (4.66) and $u_2|_{(0,T) \times O}$ is a viscosity supersolution of (4.66). Furthermore, (4.65) shows for every $r \in (0, \infty)$ that the function $(0, T) \times O \ni (t, x) \mapsto r \cdot e^{\rho t} \cdot V(x) \in (0, \infty)$ is a classical supersolution of (4.66). In addition, observe that (4.53) follows from $\lim_{n \to \infty} \sup_{(t,x) \in (0,T) \times O_n^c} \frac{|u_1(t,x)| + |u_2(t,x)|}{V(x)} = 0$. Consequently, Lemma 4.13 implies that $u_1 \leq u_2$. Repeating these arguments with u_1 and u_2 interchanged finally shows that $u_2 \leq u_1$ so that $u_1 = u_2$. This proves uniqueness and finishes the proof of Corollary 4.14. □

4.4. *Viscosity solutions of Kolmogorov equations.* The main result of this subsection, Theorem 4.16 below, establishes that the transition semigroup associated with a suitable SDE with locally Lipschitz continuous coefficients is within a certain class of functions the unique viscosity solution of the Kolmogorov equation of the SDE. To establish this result, we first prove an auxiliary result.

LEMMA 4.15 (Existence of viscosity solutions of Kolmogorov equations with globally Lipschitz continuous coefficients with compact support). *Let $d, m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,\infty)}$, let $W : [0, \infty) \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t \in [0,\infty)}$-Brownian motion, let $O \subset \mathbb{R}^d$ be an open set, let $\varphi : O \to \mathbb{R}$ be a continuous function and let $\mu : O \to \mathbb{R}^d$ and $\sigma : O \to \mathbb{R}^{d \times m}$ be locally Lipschitz continuous functions with compact support. Then there exists a family $X^x : [0, \infty) \times \Omega \to O$, $x \in O$, of up to indistinguishability unique adapted stochastic processes with continuous sample paths satisfying*

$$(4.69) \qquad X^x(t) = x + \int_0^t \mu(X^x(s)) \, ds + \int_0^t \sigma(X^x(s)) \, dW(s)$$

for all $t \in [0, \infty)$, $\mathbb{P}$-a.s. and all $x \in O$ and the function $u : (0, \infty) \times O \to \mathbb{R}$ given by $u(t, x) = \mathbb{E}[\varphi(X^x(t))]$ is a viscosity solution of

$$(4.70) \qquad \frac{\partial}{\partial t} u(t, x) - \langle (\nabla_x u)(t, x), \mu(x) \rangle - \frac{1}{2} \operatorname{tr}(\sigma(x)[\sigma(x)]^* (\operatorname{Hess}_x u)(t, x))$$
$$= 0$$

for $(t, x) \in (0, \infty) \times O$.

PROOF. First of all, observe that since μ and σ have compact supports, they are globally Lipschitz continuous, so that (4.69) has unique solutions. It thus remains to show that the function $u : (0, \infty) \times O \to \mathbb{R}$ introduced above is a viscosity solution of (4.70). Let $U \subset O$ be a relatively compact open set in O with

the property that $\text{supp}(\mu) \cup \text{supp}(\sigma) \subset U$. By assumption $\text{supp}(\mu)$ and $\text{supp}(\sigma)$ are compact sets, and hence such a set U does indeed exist. Next, let $\mu^{(n)} \in C_{\text{cpt}}^{\infty}(O, \mathbb{R}^d)$, $n \in \mathbb{N}$, and $\sigma^{(n)} \in C_{\text{cpt}}^{\infty}(O, \mathbb{R})$, $n \in \mathbb{N}$, be sequences of smooth functions satisfying $\lim_{n \to \infty} \sup_{x \in U} \|\mu(x) - \mu^{(n)}(x)\| = \lim_{n \to \infty} \sup_{x \in U} \|\sigma(x) - \sigma^{(n)}(x)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} = 0$ and $\text{supp}(\mu^{(n)}) \cup \text{supp}(\sigma^{(n)}) \subset U$ for all $n \in \mathbb{N}$ and denote by $X^{x,n} : [0, \infty) \times \Omega \to O$, $x \in O$, $n \in \mathbb{N}$, the solutions to the corresponding SDEs. Moreover, let $\varphi_k \in C^{\infty}(O, \mathbb{R})$, $k \in \mathbb{N}$, be a sequence of smooth functions satisfying $\sup_{x \in O_k} |\varphi(x) - \varphi_k(x)| < \frac{1}{k}$ for each $k \in \mathbb{N}$. Now we define functions $u^{n,k} : (0, \infty) \times O \to \mathbb{R}$, $n, k \in \mathbb{N}$, and $u^{(k)} : (0, \infty) \times O \to \mathbb{R}$, by $u^{n,k}(t, x) := \mathbb{E}[\varphi_k(X^{x,n}(t))]$ and $u^{(k)}(t, x) := \mathbb{E}[\varphi_k(X^x(t))]$. For any fixed n and k, the function $u^{n,k} : (0, \infty) \times O \to \mathbb{R}$, is smooth and globally Lipschitz continuous (see, e.g., Corollary 2.8.1 and Theorem 2.8.1 in [20]). Theorem 4.3 in [60] then shows that

$$(4.71) \qquad \begin{aligned} &\left(\frac{\partial}{\partial t} u^{n,k}\right)(t, x) - \langle (\nabla_x u^{n,k})(t, x), \mu^{(n)}(x) \rangle \\ &\qquad - \frac{1}{2} \text{tr}(\sigma^{(n)}(x)[\sigma^{(n)}(x)]^*(\text{Hess}_x u^{n,k})(t, x)) \\ &= 0 \end{aligned}$$

for all $(t, x) \in (0, \infty) \times O$, $n, k \in \mathbb{N}$. Remark 4.1 hence shows that the functions $u^{n,k}$, $n, k \in \mathbb{N}$, are also viscosity solutions to these equations. Furthermore, observe that the smoothness of the functions $\varphi_k \in C^{\infty}(O, \mathbb{R})$, $k \in \mathbb{N}$, and the global Lipschitz continuity of the functions $(\mu^{(n)})_{n \in \mathbb{N}}$, $(\sigma^{(n)})_{n \in \mathbb{N}}$, μ and σ imply that

$$(4.72) \qquad \begin{aligned} &\lim_{n \to \infty} \sup_{t \in (0,T]} \sup_{x \in O} |u^{(k)}(t, x) - u^{n,k}(t, x)| \\ &= \lim_{n \to \infty} \sup_{t \in (0,T]} \sup_{x \in \bar{U}} |\mathbb{E}[\varphi_k(X^{x,n}(t))] - \mathbb{E}[\varphi_k(X^x(t))]| \\ &\leq \lim_{n \to \infty} \sup_{t \in (0,T]} \sup_{x \in \bar{U}} \mathbb{E}[|\varphi_k(X^{x,n}(t)) - \varphi_k(X^x(t))|] \\ &\leq \left(\sup_{x \in \bar{U}} \|\varphi_k'(x)\|_{L(\mathbb{R}^d, \mathbb{R})}\right) \cdot \left(\lim_{n \to \infty} \sup_{t \in (0,T]} \sup_{x \in \bar{U}} \mathbb{E}[|X^{x,n}(t) - X^x(t)|]\right) = 0 \end{aligned}$$

for all $T \in (0, \infty)$ and all $k \in \mathbb{N}$. Combining this with Lemma 4.8 shows that for every $k \in \mathbb{N}$ it holds that $u^{(k)}$ is a viscosity solution of (4.70) with initial condition φ_k. In addition, note that

$$(4.73) \qquad \begin{aligned} &\lim_{k \to \infty} \sup_{(t,x) \in (0,\infty) \times K} |u(t, x) - u^{(k)}(t, x)| \\ &\leq \lim_{k \to \infty} \sup_{(t,x) \in (0,\infty) \times K} \mathbb{E}[|\varphi(X^x(t)) - \varphi_k(X^x(t))|] \\ &\leq \lim_{k \to \infty} \sup_{y \in \overline{U} \cup K} |\varphi(y) - \varphi_k(y)| = 0 \end{aligned}$$

for all compact sets $K \subset O$. Combining this with Lemma 4.8 eventually shows that u is indeed a viscosity solution of (4.70) as claimed. $\square$

The next result is a generalization and a consequence of Lemma 4.15 above and constitutes the main result of this section.

THEOREM 4.16 (Existence and uniqueness of viscosity solutions of Kolmogorov equations).   *Let* $d, m \in \mathbb{N}$, $\rho \in \mathbb{R}$, *let* $O \subset \mathbb{R}^d$ *be an open set, let* $\varphi : O \to \mathbb{R}$ *be a continuous function, let* $\mu : O \to \mathbb{R}^d$ *and* $\sigma : O \to \mathbb{R}^{d \times m}$ *be locally Lipschitz continuous functions and let* $V \in C^2(O, (0, \infty))$ *be such that* $\lim_{n \to \infty} \sup_{x \in O_n^c} \frac{|\varphi(x)|}{1 + V(x)} = 0$, *such that*

$$(4.74) \qquad \langle (\nabla V)(x), \mu(x) \rangle + \tfrac{1}{2} \operatorname{tr}(\sigma(x)[\sigma(x)]^*(\operatorname{Hess} V)(x)) \le \rho \cdot V(x)$$

for all $x \in O$ *and such that* $\lim_{n \to \infty} \inf\{V(x) : x \in O_n^c\} = \infty$. *Then there exists a unique continuous function* $u : [0, \infty) \times O \to \mathbb{R}$ *which fulfills* $u(0, x) = \varphi(x)$ *for all* $x \in O$, *which fulfills* $\lim_{n \to \infty} \sup_{(t,x) \in [0,T] \times O_n^c} \frac{|u(t,x)|}{V(x)} = 0$ *for all* $T \in (0, \infty)$ *and which is a viscosity solution of*

$$(4.75) \quad \begin{aligned} \frac{\partial}{\partial t} u(t, x) &- \langle (\nabla_x u)(t, x), \mu(x) \rangle - \frac{1}{2} \operatorname{tr}(\sigma(x)[\sigma(x)]^*(\operatorname{Hess}_x u)(t, x)) \\ &= 0 \end{aligned}$$

for $(t, x) \in (0, \infty) \times O$. *Moreover, if* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space with a normal filtration* $(\mathcal{F}_t)_{t \in [0,\infty)}$ *and if* $W : [0, \infty) \times \Omega \to \mathbb{R}^m$ *is a standard* $(\mathcal{F}_t)_{t \in [0,\infty)}$-*Brownian motion, then there exist up to indistinguishability unique global solutions* $X^x : [0, \infty) \times \Omega \to O$, $x \in O$, *to*

$$(4.76) \qquad X^x(t) = x + \int_0^t \mu(X^x(s)) \, ds + \int_0^t \sigma(X^x(s)) \, dW(s),$$

$\mathbb{P}$-*a.s. for all* $t \in [0, \infty)$ *and all* $x \in O$. *In that case,* u *has the probabilistic representation* $u(t, x) = \mathbb{E}[\varphi(X^x(t))]$ *for all* $(t, x) \in [0, \infty) \times O$.

PROOF.   W.l.o.g. we assume throughout this proof that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,\infty)}$ and that $W : [0, \infty) \times \Omega \to \mathbb{R}^m$ is a standard $(\mathcal{F}_t)_{t \in [0,\infty)}$-Brownian motion. Then, since V is a Lyapunov function, (4.76) does have global solutions which furthermore (assuming without loss of generality that $\rho \ge 0$) have the property that

$$(4.77) \qquad\qquad \mathbb{E}[V(X^x(t \wedge \tau))] \le e^{\rho t} V(x)$$

for any stopping time $\tau : \Omega \to [0, \infty)$. As a consequence, for every $(t, x) \in [0, \infty) \times O$ it holds that $\mathbb{E}[|\varphi(X^x(t))|]$ is finite so that we can *define* $u : [0, \infty) \times$

$O \to \mathbb{R}$ by $u(t, x) := \mathbb{E}[\varphi(X^x(t))]$ for all $(t, x) \in [0, \infty) \times O$. Note that as a consequence of our assumption on φ, for every $\delta \in (0, \infty)$ there exists a constant $C_\delta \in (0, \infty)$ such that

$$(4.78) \qquad |\varphi(x)| \leq C_\delta + \delta V(x)$$

holds for all $x \in O$. The bound (4.77) immediately implies a similar bound on $u(t, \cdot)$, so that u has the required behaviour at infinity. It therefore remains to show that u is indeed a viscosity solution of (4.75), as uniqueness of such a solution follows from Corollary 4.14. The proof for this goes again by approximation. Let $\mu^{(n)}$ and $\sigma^{(n)}$ for $n \in \mathbb{N}$ be any sequence of Lipschitz continuous functions such that for all $x \in O$ it holds that

$$(4.79) \qquad V(x) \leq n \quad \Rightarrow \quad \mu^{(n)}(x) = \mu(x), \qquad \sigma^{(n)}(x) = \sigma(x)$$

and

$$(4.80) \qquad V(x) \geq n + 1 \quad \Rightarrow \quad \mu^{(n)}(x) = 0, \qquad \sigma^{(n)}(x) = 0.$$

Denoting by $X^{x,n}$, $x \in O$, $n \in \mathbb{N}$, the solutions to the corresponding SDEs, we set $u_n(t, x) = \mathbb{E}[\varphi(X^{x,n}(t))]$ for all $(t, x) \in [0, \infty) \times O$, $n \in \mathbb{N}$. It then follows from Lemma 4.15 that $u_n|_{(0,\infty) \times O_n}$ is a viscosity solution to the equation analogous to (4.75). As a consequence of Lemma 4.8, it remains to show that $u_n \to u$, uniformly on compact subsets of $(0, \infty) \times O$. For this, we introduce the stopping times $\tau_n^x := \inf(\{t \in (0, \infty) : V(X^x(t)) \geq n\} \cup \{\infty\})$, $x \in O$, $n \in \mathbb{N}$. As a consequence of (4.78), the fact that $X^{x,n}$ and X^x coincide until time τ_n^x, and the fact that $V(X^{x,n}(t)) \leq n + 1$, $\mathbb{P}$-a.s. provided that $V(x) \leq n + 1$, we have for all $n \in \mathbb{N}$ and all $(t, x) \in [0, \infty) \times O$ with $V(x) \leq n + 1$ that

$$(4.81) \qquad \begin{aligned} &|u(t, x) - u_n(t, x)| \\ &\leq \mathbb{E}[\mathbb{1}_{\{\tau_n^x \leq t\}} |\varphi(X^x(t))|] + \mathbb{E}[\mathbb{1}_{\{\tau_n^x \leq t\}} |\varphi(X^{x,n}(t))|] \\ &\leq 2C_\delta \mathbb{P}[\tau_n^x \leq t] + \delta e^{\rho t} V(x) + \delta(n + 1) \mathbb{P}[\tau_n^x \leq t]. \end{aligned}$$

Using (4.77), we obtain from Chebychev's inequality that for all $(t, x) \in [0, \infty) \times O$ it holds that

$$(4.82) \quad \mathbb{P}[\tau_n^x \leq t] = \mathbb{P}[V(X^x(t \wedge \tau_n^x)) \geq n] \leq \frac{\mathbb{E}[V(X^x(t \wedge \tau_n^x))]}{n} \leq \frac{e^{\rho t} V(x)}{n}.$$

Inserting this into (4.81), the required locally uniform convergence follows at once. □

In the literature, there are many results proving an assertion similar to Theorem 4.16 and Corollary 4.14, respectively, under various assumptions on the functions μ and σ. Theorem 4.3 in Pardoux and Peng [60] implies that the transition semigroup associated with the SDE (4.76) is a viscosity solution of (4.75) if μ and σ are globally Lipschitz continuous; see also Peng [62]. Theorem C.2.4 in

Peng [61] can be applied if μ is locally Hölder continuous and if σ is constant and then proves uniqueness of an at most polynomially growing viscosity solution of (4.75). Uniqueness of the viscosity solution of (4.75) with given initial function follows from Theorem 8.2 in the User's guide Crandall, Ishii and Lions [7] if μ is globally one-sided Lipschitz continuous, that is, if there exists a constant $c \in \mathbb{R}$ such that $\langle x - y, \mu(x) - \mu(y) \rangle \leq c\|x - y\|^2$ for all $x, y \in \mathbb{R}^d$, and if σ is globally Lipschitz continuous. Moreover, Theorem 5.13 in Krylov [47] implies that the transition semigroup solves the Kolmogorov equation (4.75) in the sense of distributions if μ and σ are globally Lipschitz continuous. In addition, Theorems 7.1.3 and 7.1.4 in Evans [18] show that there exists a unique weak solution of the PDE (4.75) if the coefficients μ and σ are bounded and if the PDE (4.75) is uniformly parabolic.

In many situations, the open set $O \subset \mathbb{R}^d$ and the Lyapunov-type function $V : O \to \mathbb{R}$ in Theorem 4.16 satisfy $O = \mathbb{R}^d$ and $V(x) = (1 + \|x\|^2)^p$ for all $x \in \mathbb{R}^d$ where $p \in [1, \infty)$ is an arbitrary real number. This is subject of the following Corollary 4.17. It is a direct consequence of Theorem 4.16 and its proof is therefore omitted.

COROLLARY 4.17 (Existence and uniqueness of at most polynomially growing viscosity solutions of Kolmogorov equations). *Let $d, m \in \mathbb{N}$, let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a continuous and at most polynomially growing function, let $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be locally Lipschitz continuous functions with $\sup_{x \in \mathbb{R}^d} \frac{\langle x, \mu(x) \rangle}{(1 + \|x\|^2)} < \infty$ and $\sup_{x \in \mathbb{R}^d} \frac{\|\sigma(x)\|}{(1 + \|x\|)} < \infty$. Then there exists a unique continuous function $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ which fulfills $\limsup_{p \to \infty} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \frac{|u(t,x)|}{1 + \|x\|^p} < \infty$ for all $T \in (0, \infty)$, which fulfills $u(0, x) = \varphi(x)$ for all $x \in \mathbb{R}^d$, and which is a viscosity solution of*

$$
\begin{aligned}
(4.83) \quad &\frac{\partial}{\partial t} u(t, x) - \langle (\nabla_x u)(t, x), \mu(x) \rangle - \frac{1}{2} \operatorname{tr}\big(\sigma(x)[\sigma(x)]^*(\operatorname{Hess}_x u)(t, x)\big) \\
&= 0
\end{aligned}
$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^d$. Moreover, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$ and if $W : [0, \infty) \times \Omega \to \mathbb{R}^m$ is a standard $(\mathcal{F}_t)_{t \in [0, \infty)}$-Brownian motion, then u has the probabilistic representation $u(t, x) = \mathbb{E}[\varphi(X^x(t))]$ for all $(t, x) \in [0, \infty) \times \mathbb{R}^d$, where the stochastic processes $X^x : [0, \infty) \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, are as before.

Note that all examples in this article fulfill the assumptions of Corollary 4.17. In particular, observe that μ and σ from the SDE (2.1) in Section 2, μ and σ from the SDE (2.10) in Section 2, μ and σ from the SDE (2.11) in Section 2, μ and σ from the SDE (3.1) in Section 3, μ and σ from the SDE (3.18) in Section 3 as well as μ and σ from the SDE (5.3) in Section 5 all fulfill the assumptions of Corollary 4.17.

4.5. *Distributional solutions of Kolmogorov equations.* In this section, we formulate a slight extension to Theorem 5.13 in Krylov [47], which states that the semigroup associated to an SDE with smooth coefficients solves the corresponding Kolmogorov equation in the distributional sense, even if the coefficients are badly behaved near the boundary of the domain of definition O.

PROPOSITION 4.18. *Let $d, m \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $\mu = (\mu_1, \ldots, \mu_d) \in C^\infty(O, \mathbb{R}^d)$, $\sigma = (\sigma_{i,j})_{i \in \{1,\ldots,d\}, j \in \{1,\ldots,m\}} \in C^\infty(O, \mathbb{R}^{d \times m})$, let $\varphi \in C_b(O, \mathbb{R})$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filitration $(\mathcal{F}_t)_{t \in [0,\infty)}$, let $W : [0, \infty) \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t \in [0,\infty)}$-Brownian motion and let $X^x : [0, \infty) \times \Omega \to O$, $x \in O$, be solutions to*

$$(4.84) \qquad X^x(t) = x + \int_0^t \mu(X^x(s)) \, ds + \int_0^t \sigma(X^x(s)) \, dW(s),$$

$\mathbb{P}$-a.s. for all $(t, x) \in [0, \infty) \times \Omega$. Then the function $u : (0, \infty) \times O \to \mathbb{R}$ given by $u(t, x) = \mathbb{E}[\varphi(X^x(t))]$ for all $(t, x) \in [0, \infty) \times O$ solves the Kolmogorov equation

$$(4.85) \qquad \frac{\partial u}{\partial t} = \sum_{i=1}^d \mu_i \frac{\partial u}{\partial x_i} + \frac{1}{2} \sum_{l=1}^m \sum_{i,j=1}^d \sigma_{i,l} \sigma_{j,l} \frac{\partial^2 u}{\partial x_i \, \partial x_j}$$

in the distributional sense.

PROOF. Let O_n be as above, consider for every $n \in \mathbb{N}$ smooth and globally Lipschitz continuous functions $\mu^{(n)}$ and $\sigma^{(n)}$ which agree with μ and σ on O_n, and denote by $X^{x,n}$, $x \in O$, solutions of the corresponding SDE. Fix some final time $T \in (0, \infty)$, denote by P_x the law of X^x on $C([0, T], O)$ and for every $n \in \mathbb{N}$ by P_x^n the law of $X^{x,n}$ on $C([0, T], O)$. It then follows from the smoothness of the coefficients μ and σ that $O \ni x \mapsto P_x$ is weakly continuous; see Theorem 1.7 in Krylov [47]. In particular, this implies that u is continuous and that, for every compact $K \subset O$, the set $\{P_x : x \in K\}$ is tight. Let now $u_n(x, t) = \mathbb{E}[\varphi_n(X^{x,n}(t))]$ for all $(t, x) \in (0, \infty) \times O$, $n \in \mathbb{N}$, where $\varphi_n : O \to \mathbb{R}$, $n \in \mathbb{N}$, are smooth approximations of φ such that $\sup_{x \in O_n} |\varphi_n(x) - \varphi(x)| \leq 1/n$ and $\mathrm{supp}(\varphi_n) \subset O_{n+1}$ for all $n \in \mathbb{N}$ and such that $\sup_{n \in \mathbb{N}} \sup_{x \in O} |\varphi_n(x)| < \infty$. Note now that $P_x|_{\mathcal{B}(C([0,T],O_n))} = P_x^n|_{\mathcal{B}(C([0,T],O_n))}$ and that, locally uniformly in x, the P_x-measure of the set $C([0, T], O_n)$ converges to 1 as $n \to \infty$. In particular, there exists a real number $C \in [0, \infty)$ such that for all $(t, x) \in (0, T] \times O$ it holds that

$$(4.86) \qquad |u_n(x, t) - u(x, t)| \leq \frac{1}{n} + C[1 - P_x(C([0, T], O_n))].$$

As a consequence, one has $u_n \to u$, locally uniformly in x and t. The claim now follows at once from the fact that, by Theorem 5.13 in Krylov [47], each of the u_n solves the Kolmogorov equation with $\mu^{(n)}$ and $\sigma^{(n)}$. □

5. A counterexample to the rate of convergence of the Euler–Maruyama method. In this section, we use the results of Section 3 to establish the existence of an SDE with smooth and globally bounded coefficients for which the Euler–Maruyama method convergences *without any arbitrarily small polynomial rate of convergence*, thereby proving Theorem 1.3 of the Introduction. Denote by $\hat{C}$ the constant

$$(5.1) \qquad \hat{C} = \int_0^1 e^{-1/(1-u^2)} \, du,$$

and set

$$(5.2) \qquad \mu(x) = \begin{pmatrix} \mathbb{1}_{(1,\infty)}(x_4) \cdot \exp\left(-\dfrac{1}{x_4^2 - 1}\right) \cdot \cos((x_3 - \hat{C}) \cdot \exp(x_2^3)) \\ 0 \\ \mathbb{1}_{(-1,1)}(x_4) \cdot \exp\left(-\dfrac{1}{1 - x_4^2}\right) \\ 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for all $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The function $\mathbb{R} \ni x \mapsto \mathbb{1}_{(-1,1)}(x) \cdot \exp(-1/(1 - x^2)) \in [0, 1]$ that appears in μ has been used as a mollifier function in Lemma 1.2.3 in Hörmander [32]. Note that $\mu : \mathbb{R}^4 \to \mathbb{R}^4$ is infinitely often differentiable and globally bounded. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be any probability space supporting a four-dimensional standard Brownian motion $W : [0, \infty) \times \Omega \to \mathbb{R}^4$ with continuous sample paths. Then there exists a unique stochastic process $X : [0, \infty) \times \Omega \to \mathbb{R}^4$ with continuous sample paths which fulfills $X(t) = \int_0^t \mu(X(s)) \, ds + BW(t)$ for all $t \in [0, \infty)$. The stochastic process $X = (X_1, X_2, X_3, X_4) : [0, \infty) \times \Omega \to \mathbb{R}^4$ is thus a solution process of the SDE

$$
\begin{aligned}
dX_1(t) &= \mathbb{1}_{(1,\infty)}(X_4(t)) \cdot \exp\left(-\frac{1}{X_4(t)^2 - 1}\right) \\
&\qquad \times \cos((X_3(t) - \hat{C}) \cdot \exp(X_2(t)^3)) \, dt, \\
(5.3) \qquad dX_2(t) &= dW_2(t), \\
dX_3(t) &= \mathbb{1}_{(-1,1)}(X_4(t)) \cdot \exp\left(-\frac{1}{1 - X_4(t)^2}\right) dt, \\
dX_4(t) &= 1 \, dt
\end{aligned}
$$

for $t \in [0, \infty)$ satisfying $X(0) = 0$. In the next step, we define the Euler–Maruyama approximations for the SDE (5.3) using the following notation. Let

$\lfloor \cdot \rfloor_h : [0, \infty) \to [0, \infty)$, $h \in (0, \infty)$, be a family of mappings defined by

$$(5.4) \qquad \lfloor t \rfloor_h := \max\{s \in \{0, h, 2h, \ldots\} : s \le t\}$$

for all $t \in [0, \infty)$ and all $h \in (0, \infty)$. Then let $Y^h = (Y_1^h, Y_2^h, Y_3^h, Y_4^h) : [0, \infty) \times \Omega \to \mathbb{R}^4$, $h \in (0, \infty)$, be Euler–Maruyama approximation processes defined recursively by

$$Y^h(0) := 0 \quad \text{and}$$
$$(5.5)$$
$$Y^h(t) := Y^h(\lfloor t \rfloor_h) + \mu\big(Y^h(\lfloor t \rfloor_h)\big) \cdot (t - \lfloor t \rfloor_h) + B\big(W(t) - W(\lfloor t \rfloor_h)\big)$$

for all $t \in (nh, (n+1)h]$, $n \in \{0, 1, \ldots\}$ and all $h \in (0, \infty)$. Observe that this definition ensures that

$$Y_1^h(t) = \int_1^t \mathbb{1}_{(1,\infty)}(\lfloor s \rfloor_h) e^{-1/(\lfloor s \rfloor_h^2 - 1)}$$
$$(5.6)$$
$$\times \cos\left(\left(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u \rfloor_h) e^{-1/(1 - \lfloor u \rfloor_h^2)} \, du - \hat{C}\right) e^{W_2(\lfloor s \rfloor_h)^3}\right) ds$$

for all $t \in [1, \infty)$ and all $h \in (0, \infty)$. The following Theorem 5.1 proves that the Euler–Maruyama method (5.5) for the SDE (5.3) convergences *without any arbitrarily small polynomial rate of convergence*. Theorem 5.1 together with an elementary transformation argument [dealing with general $x_0 \in \mathbb{R}^4$ and general $T \in (0, \infty)$] then implies Theorem 1.3.

THEOREM 5.1 (A counterexample to the rate of convergence of the Euler–Maruyama method).   *Let* $X = (X_1, X_2, X_3, X_4) : [0, \infty) \times \Omega \to \mathbb{R}^4$ *be a solution process of the SDE* (5.3) *with continuous sample paths and with* $X(0) = 0$. *Then*

$$(5.7) \qquad \mathbb{E}[X_1(t)] - \mathbb{E}[Y_1^h(t)] \ge \exp\big(-14|\ln(h)|^{2/3}\big)$$

for all $h \in (0, \frac{1}{22}]$ *and all* $t \in [2, \infty)$ *and, therefore, we obtain*

$$(5.8) \qquad \lim_{h \searrow 0}\left(\frac{\mathbb{E}[\|X(t) - Y^h(t)\|]}{h^\alpha}\right) = \lim_{h \searrow 0}\left(\frac{\|\mathbb{E}[X(t)] - \mathbb{E}[Y^h(t)]\|}{h^\alpha}\right)$$
$$= \begin{cases} 0, & \alpha = 0, \\ \infty, & \alpha > 0, \end{cases}$$

for all $\alpha \in [0, \infty)$ *and all* $t \in [2, \infty)$. *In particular, for every* $t \in [2, \infty)$ *and every* $\alpha, C, h_0 \in (0, \infty)$ *there exists a real number* $h \in (0, h_0)$ *such that* $\|\mathbb{E}[X(t)] - \mathbb{E}[Y^h(t)]\| > C \cdot h^\alpha$.

The proof of Theorem 5.1 is deferred to the end of this section. To the best of our knowledge, the SDE (5.3) is the first SDE with smooth coefficients in the literature for which it has been established that the Euler–Maruyama scheme converges in the strong and numerical weak sense *without any arbitrarily small rate*

of convergence. Using the results of Section 3, one can show that the SDE (5.3) is not locally Hölder continuous with respect to the initial value. This is summarized in the next corollary of Lemma 3.3 in Section 3.

COROLLARY 5.2.   *Let $X^x : [0, \infty) \times \Omega \to \mathbb{R}^4$, $x \in \mathbb{R}^4$, be solution processes of the SDE* (5.3) *with continuous sample paths and with $X^x(0) = x$ for all $x \in \mathbb{R}^4$. Then for every $t \in (0, \infty)$ the function $\mathbb{R}^4 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^4$ is not locally Hölder continuous.*

PROOF.   Note that

$$\big\| \mathbb{E}\big[X^{(0,0,\hat{C},2)}(t)\big] - \mathbb{E}\big[X^{(0,0,h+\hat{C},2)}(t)\big]\big\|$$

(5.9)
$$\geq \big| \mathbb{E}\big[X_1^{(0,0,\hat{C},2)}(t) - X_1^{(0,0,h+\hat{C},2)}(t)\big]\big|$$

$$= \bigg| \int_0^t \exp\Big(\frac{-1}{((2+s)^2 - 1)} \Big) \mathbb{E}\big[1 - \cos(h \cdot \exp([W_2(s)]^3))\big] \, ds \bigg|$$

$$\geq \exp\Big(-\frac{1}{3}\Big) \int_0^t \big(1 - \mathbb{E}[\cos(h \cdot \exp([W_2(s)]^3))]\big) \, ds$$

for all $h, t \in (0, \infty)$. Combining this with Lemma 3.3 in Section 3 completes the proof of Corollary 5.2. $\square$

In the following, the size of the quantity $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^h(T)]\| \in [0, \infty)$ is analyzed for sufficiently small $h \in (0, \infty)$ and thereby Theorem 5.1 is established. To do so, we first establish a few auxiliary results. We begin with an elementary estimate for the numerical integration of concave functions.

LEMMA 5.3 (Numerical integration of concave functions).   *Let $\lfloor \cdot \rfloor_h : [0, \infty) \to [0, \infty)$, $h \in (0, \infty)$, be given by* (5.4), *let $b \in (0, \infty)$ be a real number and let $\psi : [0, b] \to \mathbb{R}$ be a continuously differentiable function with a nonincreasing derivative. Then*

(5.10)
$$\int_0^b \big(\psi(s) - \psi(\lfloor s \rfloor_h)\big) \, ds$$

$$\leq \frac{1}{2}\big[\psi'(0) \cdot h^2 + \big(\psi(\lfloor b \rfloor_h - h) - \psi(0)\big) \cdot h + \psi'(\lfloor b \rfloor_h) \cdot (b - \lfloor b \rfloor_h)^2\big]$$

for all $h \in (0, b]$.

PROOF.   The fundamental theorem of calculus and monotonicity of ψ' imply

$$\int_0^b \big(\psi(s) - \psi(\lfloor s \rfloor_h)\big) \, ds$$

(5.11)
$$= \int_0^b \int_{\lfloor s \rfloor_h}^s \psi'(u) \, du \, ds \leq \int_0^b \int_{\lfloor s \rfloor_h}^s \psi'(\lfloor s \rfloor_h) \, du \, ds$$

$$= \int_0^h \int_{\lfloor s \rfloor_h}^s \psi'(\lfloor s \rfloor_h) \, du \, ds + \int_h^{\lfloor b \rfloor_h} \int_{\lfloor s \rfloor_h}^s \psi'(\lfloor s \rfloor_h) \, du \, ds$$

$$+ \int_{\lfloor b \rfloor_h}^b \int_{\lfloor s \rfloor_h}^s \psi'(\lfloor s \rfloor_h) \, du \, ds$$

$$= \psi'(0) \cdot \frac{h^2}{2} + \frac{h^2}{2} \left(\sum_{n \in \mathbb{N}, nh < \lfloor b \rfloor_h} \psi'(nh) \right) + \psi'(\lfloor b \rfloor_h) \cdot \frac{(b - \lfloor b \rfloor_h)^2}{2}$$

$$\leq \psi'(0) \cdot \frac{h^2}{2} + \frac{h}{2} \left(\sum_{n \in \mathbb{N}, nh < \lfloor b \rfloor_h} \int_{(n-1)h}^{nh} \psi'(s) \, ds \right)$$

$$+ \psi'(\lfloor b \rfloor_h) \cdot \frac{(b - \lfloor b \rfloor_h)^2}{2}$$

$$= \psi'(0) \cdot \frac{h^2}{2} + \left(\psi(\lfloor b \rfloor_h - h) - \psi(0) \right) \cdot \frac{h}{2}$$

$$+ \psi'(\lfloor b \rfloor_h) \cdot \frac{(b - \lfloor b \rfloor_h)^2}{2}$$

for all $h \in (0, b]$. This finishes the proof of Lemma 5.3. $\quad\square$

Using Lemma 5.3, we establish in the next lemma a simple lower bound for the numerical integration of the function $\mathbb{1}_{(-1,1)}(x) \cdot \exp(-1/(1 - x^2))$, $x \in \mathbb{R}$, in the third component of $\mu : \mathbb{R}^4 \to \mathbb{R}^4$.

LEMMA 5.4 [Numerical integration of the function $\mathbb{1}_{(-1,1)}(x) \cdot \exp(-1/(1 - x^2))$, $x \in \mathbb{R}$]. *Let* $\lfloor \cdot \rfloor_h : [0, \infty) \to [0, \infty)$, $h \in (0, \infty)$, *be given by* (5.4). *Then*

$$(5.12) \qquad \frac{h}{20} \leq \int_0^\infty \mathbb{1}_{[0,1)}(\lfloor s \rfloor_h) \cdot \exp\left(-\frac{1}{1 - \lfloor s \rfloor_h^2} \right) ds - \hat{C} \leq 2h$$

for all $h \in (0, \frac{1}{8}]$.

PROOF. First of all, observe that

$$(5.13) \qquad \begin{aligned} \frac{d}{dx}\left(e^{-1/(1-x^2)}\right) &= \frac{-2x \cdot e^{-1/(1-x^2)}}{(1 - x^2)^2} \quad \text{and} \\ \frac{d^2}{dx^2}\left(e^{-1/(1-x^2)}\right) &= \frac{6 \cdot e^{-1/(1-x^2)}}{(1 - x^2)^4}\left(x^4 - \frac{1}{3} \right) \end{aligned}$$

for all $x \in (-1, 1)$. We hence obtain that the function $[0, 3^{-1/4}] \ni s \mapsto e^{-1/(1-s^2)} \in \mathbb{R}$ has a nonincreasing derivative. Applying Lemma 5.3 and using that the function

$[0, \infty) \ni s \mapsto \mathbb{1}_{[0,1)}(s) \cdot e^{-1/(1-s^2)} \in \mathbb{R}$ is nonincreasing therefore results in

$$\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor s \rfloor_h) \cdot \exp\left(\frac{-1}{(1 - |\lfloor s \rfloor_h|^2)}\right) ds - \int_0^1 \exp\left(\frac{-1}{(1 - s^2)}\right) ds$$

$$= \int_0^\infty \underbrace{\mathbb{1}_{[0,1)}(\lfloor s \rfloor_h) \cdot \exp\left(\frac{-1}{(1 - |\lfloor s \rfloor_h|^2)}\right) - \mathbb{1}_{[0,1)}(s) \cdot \exp\left(\frac{-1}{(1 - s^2)}\right)}_{\geq 0} ds$$

$$\geq \int_0^{3^{-1/4}} \exp\left(\frac{-1}{(1 - |\lfloor s \rfloor_h|^2)}\right) - \exp\left(\frac{-1}{(1 - s^2)}\right) ds$$

$$(5.14) \quad \geq \frac{h}{2} \cdot \left(\exp\left(\frac{-1}{(1 - 0^2)}\right) - \exp\left(\frac{-1}{(1 - |\lfloor 3^{-1/4} \rfloor_h - h|^2)}\right)\right)$$

$$+ \frac{2 \cdot \lfloor 3^{-1/4} \rfloor_h \cdot e^{-1/(1 - |\lfloor 3^{-1/4} \rfloor_h|^2)}}{[1 - |\lfloor 3^{-1/4} \rfloor_h|^2]^2} \cdot \frac{(3^{-1/4} - \lfloor 3^{-1/4} \rfloor_h)^2}{2}$$

$$\geq \frac{h}{2} \cdot \left(e^{-1} - \exp\left(\frac{-1}{(1 - [3^{-1/4} - 2h]^2)}\right)\right)$$

$$\geq \frac{h}{2} \cdot \left(e^{-1} - \exp\left(\frac{-1}{(1 - [1/2]^2)}\right)\right) = h \cdot \frac{(e^{-1} - e^{-4/3})}{2} > \frac{h}{20}$$

for all $h \in (0, \frac{1}{8}]$ where we used the estimate $3^{-1/4} - 2h \geq \frac{1}{3^{1/4}} - \frac{1}{4} \geq \frac{1}{2}$ for all $h \in (0, \frac{1}{8}]$ in the penultimate inequality in (5.14). Moreover, note that (5.13) implies that

$$\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor s \rfloor_h) \cdot \exp\left(\frac{-1}{(1 - |\lfloor s \rfloor_h|^2)}\right) ds$$

$$- \int_0^1 \exp\left(\frac{-1}{(1 - s^2)}\right) ds$$

$$\leq h + \int_0^1 \left|\exp\left(\frac{-1}{(1 - |\lfloor s \rfloor_h|^2)}\right) - \exp\left(\frac{-1}{(1 - s^2)}\right)\right| ds$$

$$(5.15) \quad \leq h + \sup_{x \in (0,1)} \left[\frac{2x \cdot e^{-1/(1-x^2)}}{(1 - x^2)^2}\right] \cdot h$$

$$= h + \left[\frac{2 \cdot 3^{-1/4} \cdot e^{-1/(1 - 3^{-1/2})}}{(1 - 3^{-1/2})^2}\right] \cdot h$$

$$= h + \left[\frac{6}{3^{1/4} \cdot (\sqrt{3} - 1)^2 \cdot e^{\sqrt{3}/(\sqrt{3}-1)}}\right] \cdot h \leq 2h$$

for all $h \in (0, \infty)$. Combining (5.14) and (5.15) completes the proof of Lemma 5.4. $\qquad \square$

We are now ready to prove Theorem 5.1. Its proof uses Lemma 5.4 as well as Lemma 3.3 in Section 3 above.

PROOF OF THEOREM 5.1. First of all, note that $X_1(t) = \int_1^t \exp(\frac{-1}{(s^2-1)}) \, ds$, $\mathbb{P}$-a.s. for all $t \in [1, \infty)$. Combining this with (5.6) then shows that

$$
\begin{aligned}
&\mathbb{E}[X_1(t)] - \mathbb{E}[Y_1^h(t)] \\
&= \underbrace{\int_1^t \exp\left(-\frac{1}{s^2-1}\right) - \mathbb{1}_{(1,\infty)}(\lfloor s \rfloor_h) \cdot \exp\left(-\frac{1}{\lfloor s \rfloor_h^2 - 1}\right) ds}_{\geq 0} \\
&\quad + \int_1^t \mathbb{1}_{(1,\infty)}(\lfloor s \rfloor_h) e^{-1/(\lfloor s \rfloor_h^2 - 1)} \\
&\qquad \times \mathbb{E}\left[1 - \cos\left(\left(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u \rfloor_h) e^{-1/(1-\lfloor u \rfloor_h^2)} \, du \right.\right.\right. \\
&\hspace{5cm} \left.\left.\left. - \int_0^1 e^{-1/(1-u^2)} \, du \right) e^{W_2(\lfloor s \rfloor_h)^3}\right)\right] ds \\
&\geq \int_{3/2}^t \mathbb{1}_{(1,\infty)}(\lfloor s \rfloor_h) e^{-1/(\lfloor s \rfloor_h^2 - 1)} \\
&\qquad \times \mathbb{E}\left[1 - \cos\left(\left(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u \rfloor_h) e^{-1/(1-\lfloor u \rfloor_h^2)} \, du \right.\right.\right. \\
&\hspace{5cm} \left.\left.\left. - \int_0^1 e^{-1/(1-u^2)} \, du \right) e^{W_2(\lfloor s \rfloor_h)^3}\right)\right] ds
\end{aligned}
$$

for all $t \in [\frac{3}{2}, \infty)$ and all $h \in (0, \infty)$. The estimate $\lfloor s \rfloor_h \geq \lfloor \frac{3}{2} \rfloor_h \geq \frac{3}{2} - h \geq \frac{11}{8}$ for all $s \in [\frac{3}{2}, \infty)$, $h \in (0, \frac{1}{8}]$ and Lemmas 5.4 and 3.3 therefore show that

$$
\begin{aligned}
&\mathbb{E}[X_1(t)] - \mathbb{E}[Y_1^h(t)] \\
&\quad \geq \exp\left(-\frac{1}{121/64 - 1}\right) \\
&\qquad \times \int_{3/2}^v \mathbb{E}\left[1 - \cos\left(\underbrace{\left(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u \rfloor_h) e^{-1/(1-|\lfloor u \rfloor_h|^2)} \, du - \int_0^1 e^{-1/(1-u^2)} \, du\right)}_{zh/20 \leq \cdots \leq 2h \text{ due to Lemma 5.4}}\right.\right. \\
&\hspace{8cm} \left.\left. \times e^{W_2(\lfloor s \rfloor_h)^3}\right)\right] ds \\
&\quad \geq e^{-64/57} \\
&\qquad \times \int_{3/2}^v \exp\left(\frac{-8}{\lfloor s \rfloor_h}\right)
\end{aligned}
$$

$$\times \left| \ln\left(\pi \Big/ \left(2 \left(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u \rfloor_h) \cdot e^{-1/(1-|\lfloor u \rfloor_h|^2)}\, du \right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \int_0^1 e^{-1/(1-u^2)}\, du \right) \right) \right) \right|^{2/3} \Bigg) ds$$

$$\geq \frac{(v-3/2)}{4} \cdot \exp\left(-\frac{64}{11} \left| \ln\left(\frac{10\pi}{h} \right) \right|^{2/3} \right)$$

for all $h \in (0, \min\{\frac{1}{8}, \frac{\pi}{4}\exp(-v^{3/2})\}]$, $t \in [v, \infty)$ and all $v \in [\frac{3}{2}, \infty)$. Hence, we finally obtain that

(5.16)
$$\mathbb{E}[X_1(t)] - \mathbb{E}[Y_1^h(t)]$$
$$\geq \exp\left(-\ln(8) - \frac{64}{11} |\ln(10\pi)|^{2/3} - \frac{64}{11} |\ln(h)|^{2/3} \right)$$

for all $h \in (0, \frac{1}{22}]$ and all $t \in [2, \infty)$. This completes the proof of Theorem 5.1. $\square$

In the next step, we illustrate the lower bound on the weak approximation error in Theorem 5.1 by a numerical simulation. More precisely, we ran Monte Carlo simulations and approximatively calculated the quantity $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^{T/N}(T)]\|$ for $T = 2$ and $N \in \{2^1, 2^2, \dots, 2^{29}, 2^{30}\}$. We approximated these differences of expectations with an average over 100,000 independent Monte Carlo realizations. Moreover, we discretized the integrals $X_1(2) = \int_1^2 \exp(\frac{-1}{(s^2-1)})\, ds$ and $X_3(2) = \int_0^1 \exp(\frac{-1}{(1-s^2)})\, ds$ in the exact solution with a uniform grid and mesh size $\frac{2}{2^{31}} = 2^{-30}$. Figure 1 depicts the resulting graph.

In addition to the weak approximation error $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^{T/N}(T)]\|$ for $T = 2$ and $N \in \{2^1, 2^2, \dots, 2^{29}, 2^{30}\}$, we also plotted the function

(5.17)
$$\{2^1, 2^2, \dots, 2^{30}\} \ni N$$
$$\mapsto \frac{1}{15 \cdot (\ln(N))^{1/3}} \exp\left(-\frac{1}{2T} \left(\ln(N) - \frac{1}{2T} (\ln(N))^{2/3} \right)^{2/3} \right) \in (0, 1]$$

(a function with order 0), the function $\{2^1, 2^2, \dots, 2^{30}\} \ni N \mapsto \frac{1}{15 \cdot \sqrt{N}} \in (0, 1]$ (order line $\frac{1}{2}$) and the function $\{2^1, 2^2, \dots, 2^{30}\} \ni N \mapsto \frac{1}{15 \cdot N} \in (0, 1]$ (order line 1) in Figure 1. In the standard literature in computational stochastics (see, e.g., Kloeden and Platen [42]) the Euler–Maruyama scheme is shown to converge in the numerically weak sense with order 1 if the coefficients of the SDE are smooth and globally Lipschitz continuous (see Chapter 8 in Kloeden and Platen [42] for the precise assumptions) and, therefore, the order line 1 is plotted in Figure 1. Moreover, the function with order 0 is included in Figure 1 so that one can compare the graph visually with a function which has convergence order 0. According to our simulations, the approximation error for the mean $\mathbb{E}[X(2)]$ does not drop far
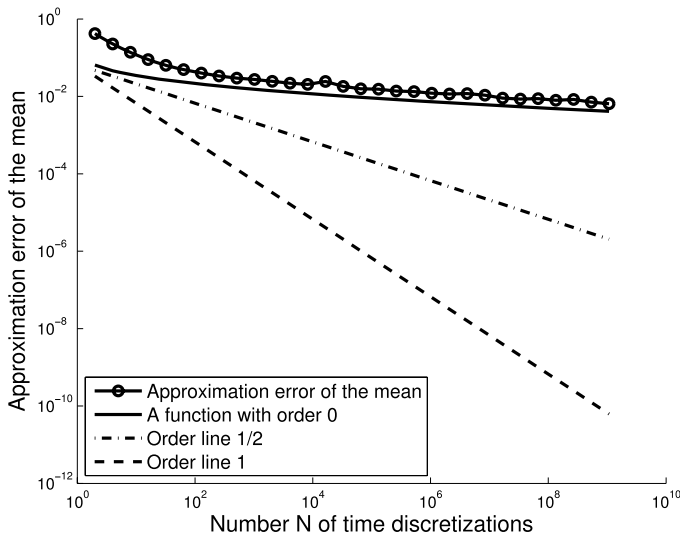
FIG. 1. *The norm* $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^{T/N}(T)]\|$ *of the difference between the mean of the solution of the SDE* (5.3) *and the mean of the Euler–Maruyama approximations* (5.5) *for* $T = 2$ *and* $N \in \{2^1, 2^2, \ldots, 2^{29}, 2^{30}\}$. *The function with convergence order* 0 *is given by* (5.17).

below $\frac{1}{100}$ even for $N = 2^{30} > 10^9$ time discretizations. This indicates that calculating the mean $\mathbb{E}[X(T)]$ with the Euler–Maruyama method up to a high precision requires a huge computational effort. In particular, this suggests for applications that an approximation cannot, in general, be assumed to be very close to the exact value even after a very high computational effort.

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