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REGULARITY OF THE ENTROPY FOR RANDOM WALKS ON HYPERBOLIC GROUPS[1]

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We consider nondegenerate, finitely supported random walks on a finitely generated Gromov hyperbolic group. We show that the entropy and the escape rate are Lipschitz functions of the probability if the support remains constant.

1. Introduction. This paper is an extension of [17] to finitely generated Gromov hyperbolic groups; see [9] and Section 2 below for the definition of hyperbolic groups. Let p be a finitely supported probability measure on an infinite group G, and define inductively, with $p^{(0)}$ being the Dirac measure at the identity e,

$$p^{(n)}(x) = [p^{(n-1)} \star p](x) = \sum_{y \in G} p^{(n-1)}(xy^{-1})p(y).$$

Define the entropy h_p and the escape rate ℓ_p^S by

$$h_p := \lim_n -\frac{1}{n} \sum_{x \in G} p^{(n)}(x) \ln p^{(n)}(x), \qquad \ell_p^S := \lim_n \frac{1}{n} \sum_{x \in G} |x| p^{(n)}(x),$$

where $|\cdot|$ is the word metric defined by some symmetric generating set S. The entropy h_p was introduced by Avez [2] and is related to bounded solutions of the equation on G $f(x) = \sum_{y \in G} f(xy)p(y)$; see, for example, [14]. Erschler and Kaimanovich have shown that, on Gromov hyperbolic groups, the entropy and the escape rate depend continuously on the probability p with finite first moment [8]. Here we are looking for a stronger regularity on a more restricted family of probability measures. We fix a finite set $F \subset G$ such that $\bigcup_n F^n = G$, and we consider probability measures in $\mathcal{P}(F)$, where $\mathcal{P}(F)$ is the set of probability measures p such that $p(x) > 0$ if, and only if, $x \in F$. The set $\mathcal{P}(F)$ is naturally identified with an open subset of the probabilities on F, which is a contractible open polygonal bounded convex domain in $\mathbb{R}^{|F|-1}$. We show:

THEOREM 1.1. *Assume G is a Gromov hyperbolic group, and F is a finite subset of G such that $\bigcup_n F^n = G$. Then, with the above notation, the functions $p \mapsto h_p$ and $p \mapsto \ell_p^S$ are Lipschitz continuous on $\mathcal{P}(F)$.*

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If the infinite hyperbolic group G is amenable, G is virtually cyclic, and the entropy is vanishing on $\mathcal{P}(F)$. Moreover, it follows from the formula in [15] that the escape rate is Lipschitz continuous in on $\mathcal{P}(F)$; see the remark after Formula (4) below. If G is a non-Abelian free group, and F a general finite generating set, then $p \mapsto h_p$ is real analytic ([17], Theorem 1.1) and $p \mapsto \ell_p^S$ as well [10]. This holds more generally for free products; see [10] and [11] for the precise conditions. A general nonamenable hyperbolic group has many common geometric features with non-Abelian free groups, and our proof follows the scheme of [17]. For Gromov hyperbolic groups, Ancona [1] proved that the Martin boundary of the random walk directed by the probability p is the Gromov geometric boundary. Let $K_\xi(x)$ be the Martin kernel associated to a point ξ of the geometric boundary. Our main technical result, Proposition 4.1, uses the description of the Martin kernel by Ancona (see also [12, 21]) to prove that $\ln K_\xi(x)$ is a Lipschitz continuous function of p as a Hölder continuous function on the geometric boundary. Then, like in [17], we can express h_p in terms of the exit measure p^∞ of the random walk on the geometric boundary ∂G and the Martin kernel. Unfortunately, it is not clear in that generality that the measure p^∞, seen as a linear functional on Hölder continuous functions on the geometric boundary, depends smoothly on the probability in $M(F)$. We use a symbolic representation of ∂G (see [6]) to express p^∞ as an combination of a finite number of symbolic measures. Each of these symbolic measures depends Lipschitz on p, and the entropy h_p is the maximum of a finite family of Lipschitz functions. The escape rate is expressed by an analogous formula: it is the maximum of the integrals of the Busemann kernel with respect of the stationary measures on the Busemann boundary. It turns out that the Busemann boundary can be described in terms of the same symbolic representation, and the Lipschitz regularity of the escape rate follows. It is likely that both entropy and rate of escape are more regular than what is obtained here, but this is what we can prove for the moment. Observe that for $G = \mathbb{Z}$, $S = \{\pm 1\}$, and F a finite generating subset, the function $p \mapsto \ell_p^S = |\sum_F i p_i|$ is Lipschitz continuous on $\mathcal{P}(F)$, but not C^1. For another example in the same spirit, we recall that Mairesse and Matheus [19] have shown that for the braid group $B_3 = \langle a, b | aba = bab\rangle$ and $F = \{a, a^{-1}, b, b^{-1}\}$, $p \mapsto \ell_p^F$ is Lipschitz, but not C^1 on $\mathcal{P}(F)$. The entropy is constant 0 in the case of $\mathbb{Z}$; the regularity of the entropy for the braid group is unknown.

In this note, the letter C stands for a real number independent of the other variables, but which may vary from line to line. The lower case c_0, c_1 will be constants which might depend only on $p \in \mathcal{P}(F)$. In the same way, the letter $\mathcal{O}_p$ stands for a neighborhood of p in $\mathcal{P}(F)$ which may vary from line to line.

2. Preliminaries.

2.1. *Hyperbolic groups.* We first recall basic facts about hyperbolic groups [9]. Let G be a finitely generated group with a symmetric finite set of generators S.

Let $d(x, y) = |x^{-1}y|$ be the word metric on G associated to S. For a subset $F \subset G$, we denote

$$N(F, R) := \{x \in G : d(x, F) \leq R\} \quad \text{and} \quad \partial F = \{x \in G : d(x, F) = 1\}.$$

For $x, y, z \in G$, the *Gromov product* $(x|y)_z$ is defined by the formula

$$(x|y)_z = \frac{1}{2}\big(d(x, z) + d(y, z) - d(x, y)\big).$$

We write $(x|y)$ for $(x|y)_e$, where e is the unit element. Let $\delta > 0$. The group G is said to be δ-hyperbolic if, for all $x, y, z, w \in G$,

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - \delta. \tag{1}$$

If G is δ-hyperbolic, then every geodesic triangle $\Delta = \{\alpha, \beta, \gamma\}$ in G is 4δ-slim, that is,

$$\alpha \subset N(\beta \cup \gamma, 4\delta), \qquad \beta \subset N(\gamma \cup \alpha, 4\delta), \qquad \gamma \subset N(\alpha \cup \beta, 4\delta).$$

A sequence $\{x_n\}_{n\geq 1}$ is said to converge to infinity if $\lim_{n,m\to\infty}(x_n|x_m) = \infty$. Two sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ converging to infinity are said to be equivalent if $\lim_{n\to\infty}(x_n|y_n) = \infty$. The geometric boundary ∂G is defined as the set of equivalence classes of sequences converging to infinity. The Gromov product extends to $G \cup \partial G$ by setting

$$(\xi|\eta) = \sup \liminf_{n,m\to\infty} (x_n|y_m),$$

where the sup runs over all sequences $\{x_n\}_{n\geq 1}$ converging to ξ and $\{y_m\}_{m\geq 1}$ converging to η. Recall that $G \cup \partial G$ is compact when equipped with the base $\{N(\{x\}, r)\} \cup \{V_r(\xi)\}$, where

$$V_r(\xi) := \{\eta \in G \cup \partial G : (\eta|\xi) > r\}.$$

One can introduce a metric ρ on ∂G such that, for some $a > 1$ and $C > 0$,

$$a^{-(\xi|\eta)-C} \leq \rho(\xi, \eta) \leq a^{-(\xi|\eta)+C}.$$

Another boundary is the Busemann boundary $\partial_B G$. Define, for $x \in G$, the function $\Psi_x(z)$ on G by

$$\Phi_x(z) = d(x, z) - d(x, e).$$

The assignment $x \mapsto \Psi_x$ is continuous, injective and takes values in a relatively compact set of functions for the topology of uniform convergence on compact subsets of G. The *Busemann compactification* $\overline{G}$ of G is the closure of G for that topology. The Busemann compactification $\overline{G}$ is a compact G-space. The *Busemann boundary* $\partial_B G := \overline{G} \setminus G$ is made of Lipschitz continuous functions h on G such that $h(e) = 0$, and such that the Lipschitz constant is at most 1. Moreover, they are horofunctions in the sense of [6]: they have the property that for all $\lambda \leq h(x)$, the distance of a point x to the set $h^{-1}(\lambda)$ is given by $h(x) - \lambda$; see Section 5.1 for more about horofunctions.

2.2. *Random walks*. Let Ξ be a compact space. Ξ is called a G-space if the group G acts by continuous transformations on Ξ. This action extends naturally to probability measures on Ξ. We say that the measure ν on Ξ is stationary if $\sum_{x\in G}(x_*\nu)p(x)=\nu$. The *entropy* of a stationary measure ν is defined by

$$h_p(\Xi,\nu)=-\sum_{x\in G}\left(\int_\Xi \ln\frac{dx_*^{-1}\nu}{d\nu}(\xi)\,d\nu(\xi)\right)p(x). \tag{2}$$

The entropy h_p and the escape rate ℓ_p are given by variational formulas over stationary measures (see [14], Section 3, for the entropy and [15], Theorem 18, for the escape rate)

$$h_p=\max\{h_p(\Xi,\nu);\ \Xi G\text{-space and }\nu\text{ stationary on }\Xi\}, \tag{3}$$

$$\ell_p^S=\max\left\{\sum_{x\in G}\left(\int_{\overline{G}} h(x^{-1})\,d\nu(h)\right)p(x);\ \nu\text{ stationary on }\overline{G}\right\}. \tag{4}$$

Moreover, the stationary measures in (4) are supported by $\partial_B G$. In particular, in the case when G is virtually cyclic, $\partial_B G$ is finite and not reduced to a point,[2] and ℓ_p^S is given by the maximum of a finite number of linear functions of p.

Let $\Omega=G^{\mathbb{N}}$ be the space of sequences of elements of G, M the product probability $p^{\mathbb{N}}$. The random walk is described by the probability $\mathbb{P}$ on the space of paths Ω, the image of M by the mapping

$$(\omega_n)_{n\in\mathbb{Z}}\mapsto(X_n)_{n\geq 0}\qquad\text{where }X_0=e\text{ and }X_n=X_{n-1}\omega_n\text{ for }n>0.$$

In particular, the distribution of X_n is the convolution $p^{(n)}$. We have:

THEOREM 2.1 ([1], Corollary 6.3, [13], Theorem 7.5). *There is a mapping $X_\infty:\Omega\to\partial G$ such that for M-a.e. ω,*

$$\lim_n X_n(\omega)=X_\infty(\omega).$$

The action of G over itself by left multiplications extends to ∂G and makes ∂G a G-space. The image measure $p^\infty:=(X_\infty)_*M$ is the only stationary probability measure on ∂G, and $(\partial G,\nu)$ achieves the maximum in (3) ([13], Theorem 7.6)

$$h_p=h_p(\partial G,p^\infty)=-\sum_{x\in F}\left(\int_{\partial G}\ln\frac{dx_*^{-1}p^\infty}{dp^\infty}(\xi)\,dp^\infty(\xi)\right)p(x). \tag{5}$$

The Green function $G(x)$ associated with (G,p) is defined by

$$G(x)=\sum_{n=0}^{\infty}p^{(n)}(x)$$

[2]The restriction of each limit function to a $\mathbb{Z}$ coset is of the form $\pm x+a$, where a can take a finite number of values, and there is at least one $\mathbb{Z}$ coset where both signs appear.

(see, e.g., Proposition 2.2 for the convergence of the series). For $y \in G$, the Martin kernel K_y is defined by

$$K_y(x) = \frac{G(x^{-1}y)}{G(y)}.$$

Ancona ([1], Théorème 6.2) showed that $y_n \to \xi \in \partial G$ if, and only if, the Martin kernels K_{y_n} converge toward a function K_ξ called the Martin kernel at ξ. We have

$$\frac{dx_* p^\infty}{dp^\infty}(\xi) = K_\xi(x). \tag{6}$$

2.3. *Differentiability.* We are going to use formula (3) and first show that the mapping $p \mapsto -\ln K_\xi(x)$ is Lipschitz continuous from a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ into a space of Hölder continuous functions on ∂G. The following properties are obtained exactly in the same way as in [17].

For $x, y \in G$, let $u(x, y)$ be the probability of eventually reaching y when starting from x. By left invariance, $u(x, y) = u(e, x^{-1}y)$. Moreover, by the strong Markov property, $G(x) = u(e, x)G(e)$ so that we have

$$K_y(x) = \frac{u(x, y)}{u(e, y)}. \tag{7}$$

By definition, we have $0 < u(x, y) \le 1$. The number $u(x, y)$ is given by the sum of the probabilities of the paths going from x to y which do not visit y before arriving at y. The next two results are classical:

PROPOSITION 2.2. *Let $p \in \mathcal{P}(F)$. There are numbers C and $\zeta, 0 < \zeta < 1$ and a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ such that for all $q \in \mathcal{O}_p$, all $x \in G$ and all $n \ge 0$,*

$$q^{(n)}(x) \le C\zeta^n.$$

PROOF. Let $q \in \mathcal{P}(F)$. Consider the convolution operator P_q in $\ell_2(G, \mathbb{R})$, defined by

$$P_q f(x) = \sum_{y \in F} f(xy^{-1})q(y).$$

Derriennic and Guivarc'h [7] showed that for $p \in \mathcal{P}(F)$, P_p has spectral radius smaller than one. In particular, there exists n_0 such that the operator norm of $P_p^{n_0}$ in $\ell_2(G)$ is smaller than one. Since F and F^{n_0} are finite, there is a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ such that for all $q \in \mathcal{O}_p$, $\|P_q^{n_0}\|_2 < \lambda$ for some $\lambda < 1$ and $\|P_q^k\|_2 \le C$ for $1 \le k \le n_0$. It follows that for all $q \in \mathcal{O}_p$, all $n \ge 0$,

$$\|P_q^n\|_2 \le C\lambda^{[n/n_0]}.$$

In particular, for all $x \in G$, $q^{(n)}(x) = [P_q^n \delta_e](x) \le |P_q^n \delta_e|_2 \le C\lambda^{[n/n_0]}|\delta_e|_2 \le C\lambda^{[n/n_0]}$. $\square$

COROLLARY 2.3 ([7]). *Let $p \in \mathcal{P}(F)$. There are numbers C and $\delta > 0$ such that for all $q \in \mathcal{O}_p$, all $x, y \in G$,*

$$G(x, y) \le Ce^{-\delta|x^{-1}y|}.$$

PROOF. We have $q^{(n)}(x^{-1}y) = 0$ for $n \le \frac{1}{r}|x^{-1}y|$; take $\delta = \frac{1}{r}\ln\frac{1}{\zeta}$. $\square$

Fix $p \in \mathcal{P}(F)$, and let Δ be a subset of G. We can define $G_\Delta(x, y), u_\Delta(x, y)$ by considering only the paths of the random walk which remain inside Δ. Clearly, $G_\Delta \le G, u_\Delta \le u$. For $x \in G$, V a subset of G and $v \in V$, let $\alpha_x^V(v)$ be the probability that the first visit in V of the random walk starting from x occurs at v ($\alpha_x^V(v) = u_{G\setminus V\cup\{v\}}(x, v)$). We have $0 \le \sum_{v\in V}\alpha_x^V(v) \le 1$ and the following:

PROPOSITION 2.4. *Fix x and V. For all $s > 1$, the mapping $p \mapsto \alpha_x^V(v)$ is a C^∞ function from $\mathcal{P}(F)$ into $\ell^s(V)$. Moreover, $\|\frac{\partial\alpha_x^V}{\partial p_i}\|_s$ is bounded independently of x and V.*

PROOF. By Proposition 2.2, there is a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ and numbers $C, \zeta, 0 < \zeta < 1$, such that for $q \in \mathcal{O}_p$ and for all $y \in G$,

$$q^{(n)}(y) \le C\zeta^n.$$

The number $\alpha_x^V(v)$ can be written as the sum of the probabilities $\alpha_x^{n,V}(v)$ of entering V at v in exactly n steps. We have

$$\alpha_x^{n,V}(v) \le q^{(n)}(x^{-1}v) \le C\zeta^n.$$

Moreover, the function $p \mapsto \alpha_x^{n,V}(v)$ is a homogeneous polynomial of degree n on $\mathcal{P}(F)$, since

$$\alpha_x^{n,V}(v) = \sum_{\mathcal{E}} q_{i_1}q_{i_2}\cdots q_{i_n},$$

where $\mathcal{E}$ is the set of paths $\{x, xi_1, xi_1i_2, \dots, xi_1i_2\cdots i_n = v\}$ of length n made of steps in F which start from x and enter V in v. It follows that for all $\alpha = \{n_1, n_2, \dots, n_{|B|}, n_i \in \mathbb{N}\cup\{0\}\}$, all $v \in V$,

$$\left|\frac{\partial^\alpha}{\partial p^\alpha}\alpha_x^{n,V}(v)\right| \le \frac{n^{|\alpha|}}{(\inf_{i\in F} p_i)^{|\alpha|}}\alpha_x^{n,V}(v) \le \frac{Cn^{|\alpha|}}{(\inf_{i\in F} p_i)^{|\alpha|}}\zeta^n,$$

where $|\alpha| = \sum_{i\in F} n_i$. Therefore,

$$\sum_{v\in V}\left|\frac{\partial^\alpha}{\partial p^\alpha}\alpha_x^{n,V}(v)\right|^s \le \frac{Cn^{s|\alpha|}}{(\inf_{i\in F} p_i)^{s|\alpha|}}\zeta^{(s-1)n}\sum_{v\in V}\alpha_x^{n,V}(v) \le \frac{Cn^{s|\alpha|}}{(\inf_{i\in F} p_i)^{s|\alpha|}}\zeta^{(s-1)n}.$$

Thus, $q \mapsto \frac{\partial^\alpha}{\partial p^\alpha}\alpha_x^V(v)$ is given locally by a uniformly converging series in $\ell^s(V)$ of derivatives. It follows that $q \mapsto \alpha_x^V(v)$ is a C^∞ function from $\mathcal{P}(F)$ into $\ell^s(V)$.

From the above computation, it follows that $\|\frac{\partial^\alpha}{\partial p^\alpha}\alpha_x^V(v)\|_s \le \sum_n \frac{Cn^{s|\alpha|}}{(\inf_{i\in F} p_i)^{s|\alpha|}} \times \zeta^{(s-1)n}$, independently of x, V. $\square$

PROPOSITION 2.5. *There exists T large enough that for $t > T$, for any y and V, the mapping $p \mapsto \alpha_v^{\{y\}}(y)$ is a C^∞ function from $\mathcal{P}(F)$ into $\ell^t(V)$. Moreover, $v \mapsto \frac{\partial \alpha_v^{\{y\}}(y)}{\partial p_i}$ is bounded in $\ell^t(V)$ independently of y and V.*

PROOF. It suffices to show that there is T such that the function $v \mapsto \alpha_v^{\{y\}}(y) \in \ell^T(V)$ and to apply the same arguments as in the proof of Proposition 2.4. Consider the probability $\check{p}$ with support F^{-1} defined by $\check{p}(x) = p(x^{-1})$, and define all quantities $(\check{p})^{(n)}, \check{G}(x), \check{u}(x, y)$. Observe that, since it is the sum of the same probabilities over the same set of paths, $G(v, y) = \check{G}(y, v)$. Therefore, we have, using Corollary 2.3 for the $\check{p}$ random walk,

$$\alpha_v^{\{y\}}(y) \le G(v, y) = \check{G}(y, v) \le e^{-\check{\delta}|y^{-1}v|}.$$

The group G has exponential growth: there is a v such that there are less than Ce^{vR} elements of G at distance less than R from y. It follows that for $T > v/\check{\delta}$, the function $v \mapsto \alpha_v^{\{y\}}(y) \in \ell^T(V)$. $\square$

2.4. *Projective contractions on cones.* In this subsection, we recall the Birkhoff theorem about linear maps preserving convex cones. Let $\mathcal{C}$ be a convex cone in a Banach space, and define on $\mathcal{C}$ the projective distance between half lines as

$$\vartheta(f, g) := \ln[\tau(f, g)\tau(g, f)],$$

where $\tau(f, g) := \inf\{s, s > 0, sf - g \in \mathcal{C}\}$. Let $\mathcal{D}$ be the space of directions in $\mathcal{C}$. Then, ϑ defines a distance on $\mathcal{D}$. Let A be an operator from $\mathcal{C}$ into $\mathcal{C}$, and let $T : \mathcal{D} \to \mathcal{D}$ be the projective action of A. Then, by [3],

$$\vartheta(Tf, Tg) \le \beta\vartheta(f, g) \qquad \text{where } \beta = \tanh\left(\tfrac{1}{4}\operatorname{Diam} T(\mathcal{D})\right). \tag{8}$$

In some cases, ϑ-diameters are easy to estimate: for example, in $\mathcal{C}^t = \{f \in \ell^t; f \ge 0\}$, the set $\mathcal{U}(g, c) := \{f : c^{-1}g \le f \le cg\}$, where $g \in \mathcal{C}$ and $c \ge 1$, has ϑ-diameter $4\ln c$. Moreover, the following observation is useful:

LEMMA 2.6 ([18], Lemma 1.3). *Let $f, g \in \mathcal{C}^t$, $\|f\|_t = \|g\|_t$. Then,*

$$\|f - g\|_t \le \left(e^{\vartheta(f,g)} - 1\right)\|f\|_t.$$

3. Obstacles. In this section, we show that the function Φ on ∂G defined by $\Phi(\xi) := -\ln K_\xi(x)$ is Hölder continuous for any fixed $x \in G$. This is not a new result [12]. Nevertheless, we present the construction and the proof in order to introduce the notation used in the next section to show that Φ is Lipschitz in p as a Hölder continuous function on ∂G. Like in [12], the proof is based on Ancona's Harnack inequality at infinity (see [1] and [12], Proposition 2.1, for the form used here): there exist a number R and a constant $c = c(p)$ such that if $[x, y]$ is a geodesic segment and $z \in [x, y]$, then for any $\Delta \subset G$, $N([x, y], R) \subset \Delta$, we have

$$c^{-1} u_\Delta(x, z) u_\Delta(z, y) \le u_\Delta(x, y) \le c u_\Delta(x, z) u_\Delta(z, y), \tag{9}$$

where $u_\Delta(v, w)$ is the probability of ever arriving at w starting from v before reaching $G \setminus \Delta$. Moreover, from the proof of (9) in [12] or [21], it follows that there exists a neighborhood $\mathcal{O}$ of p in $\mathcal{P}(F)$ and a constant C such that $c(p) \le C$ for $p \in \mathcal{O}$.

3.1. *Obstacles*. Without loss of generality, we may assume that F contains the set of generators, and δ is an integer. Set $r = \max\{|x|; x \in F, \delta\}$.

Fix M large. In particular, $M \geq R + 12r$, where R is given by (9). For a geodesic γ, we call an *obstacle* a family $U_0^- \subset U_0 \subset U_1^- \subset U_1$ of subsets of G such that

$$\begin{aligned} U_0^- &= \{x \in G : d(x, \gamma(-2M)) < d(x, \gamma(0))\}, \\ U_0 &= \{x \in G : d(x, \gamma(-2M)) < d(x, \gamma(4r))\}, \\ U_1^- &= \{x \in G : d(x, \gamma(0)) < d(x, \gamma(2M))\}, \\ U_1 &= \{x \in G : d(x, \gamma(0)) < d(x, \gamma(2M + 4r))\}. \end{aligned}$$

The subsets $U_i^\pm$ are connected and satisfy $U_0^- \subset U_0 \subset U_1^- \subset U_1$. More precisely, we have the two following elementary facts:

LEMMA 3.1. *If $x \in U_0^-$ and $[x, \gamma(-2M)]$ is a geodesic segment, then $[x, \gamma(-2M)] \subset U_0^-$.*

PROOF. Assume not. Then there is a $z \in [x, \gamma(-2M)]$ such that $d(z, \gamma(-2M)) \geq d(z, \gamma(0))$. Adding $d(z, x)$ to both sides of this inequality, we obtain

$$d(x, \gamma(-2M)) = d(x, z) + d(z, \gamma(-2M)) \geq d(x, z) + d(z, \gamma(0)) \geq d(x, \gamma(0)),$$

a contradiction to $x \in U_0^-$. $\square$

The statements and the proofs are the same for all $U_i^\pm$.

LEMMA 3.2. *If $x \in U_0^-$, then $B(x,r) \subset U_0$; if $x \in U_0$, then $B(x, M-3r) \subset U_1^-$.*

PROOF. Let $x \in U_0^-$ and $x' \in B(x,r)$. Writing (1) with $x = x'$, $y = \gamma(0)$, $z = \gamma(4r)$ and $w = \gamma(-2M)$, we get

$$\begin{aligned} &d(x', \gamma(-2M)) - d(x', \gamma(0)) + 2M \\ &\qquad \geq \min\{d(x', \gamma(-2M)) - d(x', \gamma(4r)) + 2M + 4r, 4M\} - 2\delta. \end{aligned}$$

Since $d(x', \gamma(-2M)) - d(x', \gamma(4r)) < d(x, \gamma(-2M)) + r - d(x, \gamma(0)) + 5r < 6r \leq 2M - 4r$, we get

$$\begin{aligned} d(x', \gamma(4r)) &\geq d(x', \gamma(0)) + 4r - 2\delta > d(x, \gamma(0)) + r \\ &> d(x, \gamma(-2M)) + r > d(x', \gamma(-2M)). \end{aligned}$$

Analogously, if $x \in U_0$ and $x' \in B(x, M-3r)$, we get, writing now (1) with $z = \gamma(2M)$,

$$\begin{aligned} &d(x', \gamma(-2M)) - d(x', \gamma(0)) + 2M \\ &\qquad \geq \min\{d(x', \gamma(-2M)) - d(x', \gamma(2M)) + 4M, 4M\} - 2\delta. \end{aligned}$$

Since the right-hand side is smaller than $4M - 2r$, it cannot exceed $4M - 2\delta$, and we get

$$d(x', \gamma(0)) \leq d(x', \gamma(2M)) - 2M + 2\delta < d(x', \gamma(2M)). \qquad \square$$

Lemma 3.2 implies that any trajectory of the random walk going from U_0^- to $G \setminus U_1$ has to cross successively $U_0 \setminus U_0^-$, $U_1^- \setminus U_0$ and $U_1 \setminus U_1^-$. For V_1, V_2 subsets of G, denote $A_{V_1}^{V_2}$ the (infinite) matrix such that the row vectors indexed by $v \in V_1$ are the $\alpha_v^{V_2}(w)$, $w \in V_2$. In particular, if $V_2 = \{y\}$, set $\omega_{V_1}^y$ for the (column) vector

$$\omega_{V_1}^y = A_{V_1}^{\{y\}} = (\alpha_v^{\{y\}}(y))_{v \in V_1} = (u(v,y))_{v \in V_1}.$$

Fix $t > T$. By Propositions 2.4 and 2.5, $\omega_{V_1}^y$ is a vector in $\ell^t(V_1)$ and $\alpha_x^{V_0} \in \ell^s(V_0)$, with $1/s + 1/t = 1$. With this notation, the strong Markov property yields, if $U_0^- \subset U_0 \subset U_1^- \subset U_1$ is an obstacle and $x \in U_0^-$, $y \notin U_1$,

$$u(x,y) = \sum_{v_0, v_1} \alpha_x^{V_0}(v_0) A_{V_0}^{V_1}(v_0, v_1) u(v_1, y) = \langle \alpha_x^{V_0}, A_{V_0}^{V_1} \omega_{V_1}^y \rangle$$

with the natural summation rules for matrices and for the (ℓ^s, ℓ^t) coupling. All series are bounded series with nonnegative terms, and we set $V_i = U_i \setminus U_i^-$.

Observe that an obstacle is completely determined by the directing geodesic segment $[\gamma(-2M), \ldots, \gamma(2M+4r)]$, so that there is a finite number of possible obstacles and therefore a finite number of spaces $\ell^t(V)$, of (infinite) matrices $A_{V_0}^{V_1}$, of vectors $\omega_{V_1}^z$ and $\alpha_x^{V_0}$ if the distances $d(z, \gamma(2M+4r+1))$ and $d(x, \gamma(-2M))$ are bounded.

3.2. *Properties of the matrix* $A_{V_0}^{V_1}$. Recall that the general entry of the matrix $A = A_{V_0}^{V_1}$ is $A(v_0, v_1)$, the probability that starting from $v_0 \in V_0$, the first visit in V_1 occurs at v_1. In particular, assume $A(v_0, v_1) = 0$. Then, all paths from v_0 to v_1 with steps in F have to enter V_1 elsewhere before reaching v_1. Since the support F of p contains the generators of the group, $A(v, v_1) = 0$ for all v's in the connected component of v_0 in U_1^-. By Lemma 3.1, all paths from $\gamma(0)$ to v_1 with steps in F have to enter V_1 before reaching v_1. Therefore this property depends neither on $v_0 \notin U_1^-$ nor on $p \in \mathcal{P}(F)$. We say that v_1 is active if $A(v_0, v_1) \neq 0$. In the sequel we will call V_1 the set of active elements of $U_1 \setminus U_1^-$. We have:

PROPOSITION 3.3. *Let γ be a geodesic, $U_0^- \subset U_0 \subset U_1^- \subset U_1$ an obstacle, $V_0 = U_0 \setminus U_0^-$, V_1 the active part of $U_1 \setminus U_1^-$. There exists a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ and a constant c_1 such that, for all $p \in \mathcal{O}_p$, all $v_0 \in V_0$, $v_1 \in V_1$,*

$$c_1^{-1} u_{G\setminus U_1^-}(v_0, \gamma(0)) \alpha_{\gamma(0)}^{V_1}(v_1) \le A(v_0, v_1) \le c_1 u_{G\setminus U_1^-}(v_0, \gamma(0)) \alpha_{\gamma(0)}^{V_1}(v_1). \tag{10}$$

PROOF. Introduce the set U_1^{--}, $U_1^{--} = \{x \in G : d(x, \gamma(0)) < d(x, \gamma(2M - 4r))\}$. By a variant of Lemma 3.2, we may write, for $v_0 \in V_0$, $v_1 \in V_1$,

$$A(v_0, v_1) = \sum_{w \in U_1^- \setminus U_1^{--}} u_{G\setminus U_1^-}(v_0, w) \alpha_w^{V_1}(v_1).$$

Using that $\alpha_{\gamma(0)}^{V_1}(v_1) = \sum_{w \in U_1^- \setminus U_1^{--}} u_{G\setminus U_1^-}(\gamma(0), w) \alpha_w^{V_1}(v_1)$, we see that it suffices to prove that, for all $p \in \mathcal{O}_p$, all $v_0 \in V_0$, $w \in U_1^{--} \setminus U_1^-$,

$$\begin{aligned} &c_1^{-1} u_{G\setminus U_1^-}(v_0, \gamma(0)) u_{G\setminus U_1^-}(\gamma(0), w) \\ &\qquad \le u_{G\setminus U_1^-}(v_0, w) \le c_1 u_{G\setminus U_1^-}(v_0, \gamma(0)) u_{G\setminus U_1^-}(\gamma(0), w). \end{aligned}$$

This will follow from a variant of (9) once we will have located the point $\gamma(0)$ with respect to the geodesic $[v_0, w]$.

Observe that if $v_0 \in U_0$, then $d(v_0, \gamma(0)) \ge M - 3r$. Indeed, writing that

$$\begin{aligned} &\big(\gamma(-2M), \gamma(4r)\big)_{v_0} \\ &\quad \ge \min\{\big(\gamma(-2M), \gamma(-M+2r)\big)_{v_0}, \big(\gamma(4r), \gamma(-M+2r)\big)_{v_0}\} - \delta \\ &\quad = \big(\gamma(-2M), \gamma(-M+2r)\big)_{v_0} - \delta, \end{aligned}$$

we get that $d(v_0, \gamma(4r)) \ge M + 2r - \delta \ge M - 3r$ and the claim follows. Since, by Lemma 3.1, the whole geodesic $[v_0, \gamma(-M+2r)]$ lies in U_0, we have $d(\gamma(0), [v_0, \gamma(-M+2r)]) \ge M - 3r$. But we know that $\gamma(0) \in N([v_0, \gamma(M)] \cup [v_0, \gamma(-M+2r)], 4\delta)$. It follows that there is a point $z_1 \in [v_0, \gamma(M)]$ with $d(\gamma(0), z_1) \le 4\delta$. In the same way, since $w \in G \setminus U_1^{--}$, $d(\gamma(0), [w, \gamma(M -$

$2r)]) \geq M - 3r$ and therefore $d(z_1, [w, \gamma(M-2r)]) \geq M - 3r - 4\delta$. It follows that there is a point $z \in [v_0, w]$ such that $d(z, \gamma(0)) \leq d(z, z_1) + d(z_1, \gamma(0)) \leq 8\delta$.

Let y_0 be the point in $[v_0, w]$ at distance R from w. Then $G \setminus U_1^-$ contains $N([v_0, y_0], R)$, and the point z belongs to $[v_0, y_0]$.[3] So we may apply (9) to the points v_0, z, y_0 and the domain $\Delta = G \setminus U_1^-$ to obtain, for all $p \in \mathcal{O}_p$, all $v_0 \in V_0$, $v_1 \in V_1$,

$$c_0^{-1} u_{G\setminus U_1^-}(v_0, z) u_{G\setminus U_1^-}(z, y_0) \leq u_{G\setminus U_1^-}(v_0, y_0) \leq c_0 u_{G\setminus U_1^-}(v_0, z) u_{G\setminus U_1^-}(z, y_0).$$

By changing the constant, we can replace y_0 by w [since $d(w, y_0) = R$] and z by $\gamma(0)$ [since $d(z, \gamma(0)) \leq 8\delta$]. We obtain the desired inequality. $\square$

For V a subset of G, $t > 0$, denote $\mathcal{C}_V^t$ the convex cone of nonnegative sequences in $\ell^t(V)$ and define on $\mathcal{C}_V^t$ the projective distance between half lines as

$$\vartheta(f, g) := \ln\bigl[\tau(f, g)\tau(g, f)\bigr],$$

where $\tau(f, g) := \inf\{s, s > 0, sf - g \in \mathcal{C}_V^t\}$. Represent the space of directions as the sector of the unit sphere $\mathcal{D}_V^t = \mathcal{C}_V^t \cap S_V^t$; then, ϑ defines a distance on $\mathcal{D}_V^t$ for which $\mathcal{D}_V^t$ is a complete space (Lemma 2.6). We fix $t > T$ such that the sequences $\alpha_v^{\{y\}}(y) \in \ell^t(V)$ and we consider the matrix $A_{V_0}^{V_1}$ as an operator from $\ell^t(V_1)$ into the space of sequences indexed on V_0. We have:

PROPOSITION 3.4. *Choose $t > T$ and s such that $1/s + 1/t = 1$. For any obstacle $U_0^- \subset U_0 \subset U_1^- \subset U_1$, all $p \in \mathcal{O}_p$, the operator $A_{V_0}^{V_1}$ sends $\mathcal{C}_{V_1}^t$ into $\mathcal{C}_{V_0}^t$, the adjoint operator $(A_{V_0}^{V_1})^*$ sends $\mathcal{C}_{V_0}^s$ into $\mathcal{C}_{V_1}^s$ and*

$$\operatorname{Diam}_{\mathcal{C}_{V_0}^t}\bigl(A_{V_0}^{V_1}(\mathcal{C}_{V_1}^t)\bigr) \leq 4\ln c_1, \qquad \operatorname{Diam}_{\mathcal{C}_{V_1}^s}\bigl((A_{V_0}^{V_1})^*(\mathcal{C}_{V_0}^s)\bigr) \leq 4\ln c_1,$$

where c_1 and $\mathcal{O}_p$ are the ones in (10).

PROOF. By definition, $\sum_{v_1 \in V_1} \alpha_{\gamma(0)}^{V_1}(v_1) \leq 1$ so that $\alpha_{\gamma(0)}^{V_1}(v_1) \in \ell^s(V_1)$. By (10), for any $v_0 \in V_0$, any $f \in \ell^t(V_1)$,

$$A_{V_0}^{V_1} f(v_0) \leq c_1 u_{G\setminus U_1}\bigl(v_0, \gamma(0)\bigr) \bigl\|\alpha_{\gamma(0)}^{V_1}(v_1)\bigr\|_s \|f\|_t.$$

By the same argument as in the proof of Proposition 2.5, we see that $v_0 \mapsto u_{G\setminus U_1^-}(v_0, \gamma(0)) \in \ell^t(V_0)$. It follows that for any $f \in \ell^t(V_1)$, $A_{V_0}^{V_1} f$ belongs to $\ell^t(V_0)$.

By (10), we know that for any $f \in \mathcal{C}_{V_1}^t$,

$$c_1^{-1} u_{G\setminus U_1}\bigl(v_0, \gamma(0)\bigr) \leq \frac{A_{V_0}^{V_1} f(v_0)}{\langle \alpha_{\gamma(0)}^{V_1}(\cdot), f(\cdot)\rangle} \leq c_1 u_{G\setminus U_1}\bigl(v_0, \gamma(0)\bigr). \tag{11}$$

[3] Since $w \notin U_1^{--}$, we have $d(w, z) \geq d(w, \gamma(0)) - 8\delta \geq M - 3r - 8\delta \geq M - 11r > R$.

It follows that $\mathrm{Diam}_{\mathcal{C}_{V_0^t}} A_{V_0}^{V_1}(\mathcal{C}_{V_1}^t) \le 4\ln c_1$. The same argument works for the adjoint operator $(A_{V_0}^{V_1})^*$, since we know that $v_0 \mapsto u_{G\setminus U_1}(v_0, \gamma(0)) \in \ell^t(V_0)$ and $v_1 \mapsto \alpha_{\gamma(0)}^{V_1}(v_1) \in \ell^s(V_1)$. $\square$

PROPOSITION 3.5. *Choose $t > T+1$, s such that $1/s+1/t=1$. The mapping $p \mapsto A_{V_0}^{V_1}$ [resp., $p \mapsto (A_{V_0}^{V_1})^*$] is C^∞ from $\mathcal{P}(F)$ into $\mathcal{L}(\ell^t(V_1), \ell^t(V_0))$ [resp., $\mathcal{L}(\ell^s(V_0), \ell^s(V_1))$].*

PROOF. We follow the scheme of the proofs of Propositions 2.4 and 2.5. By Proposition 2.2, there is a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ and numbers $C, \zeta, 0 < \zeta < 1$, such that for $q \in \mathcal{O}_p$ and for all $y \in G$,

$$q^{(n)}(y) \le C\zeta^n.$$

We write $\alpha_{v_0}^{V_1}(v_1)$ as the sum of the probabilities $\alpha_{v_0}^{n,V_1}(v_1)$ of entering V_1 at v_1 in exactly n steps. We have, for all $v_0 \in V_0, v_1 \in V_1$,

$$\alpha_{v_0}^{n,V_1}(v_1) \le q^{(n)}\big(v_0^{-1}v_1\big) \le C\zeta^n.$$

As before, the function $p \mapsto \alpha_{v_0}^{n,V_1}(v_1)$ is a homogeneous polynomial of degree n on $\mathcal{P}(F)$ and for all $\alpha = \{n_1, n_2, \dots, n_{|B|}, n_i \in \mathbb{N}\cup\{0\}\}$, all $v_0 \in V_0, v_1 \in V_1$,

$$\left|\frac{\partial^\alpha}{\partial p^\alpha}\alpha_{v_0}^{n,V_1}(v_1)\right| \le \frac{Cn^{|\alpha|}}{(\inf_{i\in F} p_i)^{|\alpha|}}\alpha_{v_0}^{n,V_1}(v_1) \le \frac{Cn^{|\alpha|}}{(\inf_{i\in F} p_i)^{|\alpha|}}\zeta^n.$$

Let $f \in \ell^t(V_1)$. Then,

$$\begin{aligned}\sum_{v_1}\left|\frac{\partial^\alpha}{\partial p^\alpha}\alpha_{v_0}^{n,V_1}(v_1)\right|\big|f(v_1)\big| &\le \left\|\frac{Cn^{|\alpha|}}{(\inf_{i\in F} p_i)^{|\alpha|}}\alpha_{v_0}^{n,V_1}(v_1)\right\|_s \|f\|_t \\ &\le Cn^{|\alpha|}\zeta^{(s-1)n/s}\big(u_{G\setminus U_1^-}\big(v_0,\gamma(0)\big)\big)^{1/s}\|f\|_t.\end{aligned}$$

To obtain the last inequality, we use that

$$\big\|\alpha_{v_0}^{n,V_1}(v_1)\big\|_s \le C\left(\zeta^{(s-1)n}\sum_{v_1}\alpha_{v_0}^{n,V_1}(v_1)\right)^{1/s},$$

(10) and $\sum_{v_1}\alpha_{\gamma(0)}^{V_1}(v_1) \le 1$. Therefore,

$$\begin{aligned}&\left\|\sum_{v_1}\left|\frac{\partial^\alpha}{\partial p^\alpha}\alpha_{v_0}^{n,V_1}(v_1)\right|\big|f(v_1)\big|\right\|_{\ell^t(V_0)} \\ &\qquad\le Cn^{|\alpha|}\zeta^{(s-1)n/s}\left(\sum_{v_0}\big(u_{G\setminus U_1}\big(v_0,\gamma(0)\big)\big)^{t/s}\right)^{1/t}\|f\|_t.\end{aligned}$$

Since $t > T+1$, $t/s > T$, the series $\sum_{v_0}(u_{G\setminus U_1}(v_0,\gamma(0)))^{t/s}$ converges, and the operator

$$f \mapsto \sum_{v_1} \frac{\partial^\alpha}{\partial p^\alpha}\alpha_{v_0}^{n,V_1}(v_1) f(v_1)$$

has norm smaller than $Cn^{|\alpha|}\zeta^{(s-1)n/s}$ in $\mathcal{L}(\ell^t(V_1),\ell^t(V_0))$. The series of operators which defines $\frac{\partial^\alpha}{\partial p^\alpha}A_{V_0}^{V_1}$ is converging.

The proof is the same for the adjoint operator. We estimate, for $g \in \ell^s(V_0)$,

$$\begin{aligned}\sum_{v_0}\left|\frac{\partial^\alpha}{\partial p^\alpha}\alpha_{v_0}^{n,V_1}(v_1)\right||g(v_0)| &\le \left\|\frac{Cn^{|\alpha|}}{(\inf_{i\in F} p_i)^{|\alpha|}}\alpha_{v_0}^{n,V_1}(v_1)\right\|_t \|g\|_s \\ &\le Cn^{|\alpha|}\zeta^{n/t}\big(\alpha_{\gamma(0)}^{V_1}(v_1)\big)^{(t-1)/t}\|g\|_s.\end{aligned}$$

As before, we find that the operator

$$g \mapsto \sum_{v_0}\frac{\partial^\alpha}{\partial p^\alpha}\alpha_{v_0}^{n,V_1}(v_1) g(v_0)$$

has norm smaller than

$$Cn^{|\alpha|}\zeta^{n/t}\sum_{v_1}\big(\alpha_{\gamma(0)}^{V_1}(v_1)\big)^{s(t-1)/t} \le Cn^{|\alpha|}\zeta^{n/t}$$

in $\mathcal{L}(\ell^s(V_0),\ell^s(V_1))$ (recall that $s\frac{t-1}{t}=1$). The series of operators which defines $\frac{\partial^\alpha}{\partial p^\alpha}(A_{V_0}^{V_1})^*$ is converging as well. $\square$

3.3. *Hölder regularity of the Martin kernel.* Fix $x \in G$ and a geodesic γ with $\gamma(0)=e$. Consider the family $U_0^- \subset U_0 \subset \cdots \subset U_n^- \subset U_n$ such that for all $j = 1,\dots,n-1$, $U_j^- \subset U_j \subset U_{j+1}^- \subset U_{j+1}$ is an obstacle for $\gamma\circ\sigma^{2jM+K}$. The integer K is chosen so that $x, e \in U_0^-$, for example, $K = 4M + |x|$. With that choice, $\gamma(n) \notin U_k$ as soon as $n > K + 2kM + 4r$. Iterating the strong Markov property, we get, for $z \notin U_k$,

$$\frac{u(x,z)}{u(e,z)} = \frac{\langle \alpha_x^{V_0}, A_{V_0}^{V_1}\cdots A_{V_{k-1}}^{V_k}\omega_{V_k}^z\rangle}{\langle \alpha_e^{V_0}, A_{V_0}^{V_1}\cdots A_{V_{k-1}}^{V_k}\omega_{V_k}^z\rangle}.$$

Choose $t > T+1$ and s such that $1/s + 1/t = 1$. Set $f_k(z) := \frac{\omega_{V_k}^z}{\|\omega_{V_k}^z\|_t}$, $\alpha := \alpha_e^{V_0}$, $\beta := \alpha_x^{V_0}$. For all $z \notin U_k$, $f_k(z) \in \mathcal{D}_{V_k}^t$ and $\alpha,\beta \in \mathcal{C}_{V_0}^s - \{0\}$. By Proposition 2.4, if $z, z' \notin U_k$, $\vartheta_{\mathcal{C}^t}(A_{V_{k-1}}^{V_k} f_k(z), A_{V_{k-1}}^{V_k} f_k(z')) \le 4\ln c_1$. Set $\tau = \frac{c_1^2-1}{c_1^2+1}$. By repeated application of (8), we have, as soon as $z, z' \notin U_k$,

$$
\begin{aligned}
&\vartheta_{\mathcal{C}^t}\big(A_{V_0}^{V_1}\cdots A_{V_{k-1}}^{V_k} f_k(z), A_{V_0}^{V_1}\cdots A_{V_{k-1}}^{V_k} f_k(z')\big)\\
&\qquad \le \tau^{k-1}\vartheta_{\mathcal{C}^t}\big(A_{V_{k-1}}^{V_k} f_k(z), A_{V_{k-1}}^{V_k} f_k(z')\big) \qquad (12)\\
&\qquad \le 4\ln c_1 \tau^{k-1}.
\end{aligned}
$$

We are interested in the function $\Phi : \partial G \to \mathbb{R}$,

$$\Phi(\xi) = -\ln K_x(\xi) = -\ln \lim_{x_n \to \xi} \frac{u(x, x_n)}{u(e, x_n)}.$$

If we choose the reference geodesic γ converging toward ξ, then setting $\Phi_n(\xi) = \frac{u(x,\gamma(n))}{u(e,\gamma(n))}$, we have $\lim_n \Phi_n(\xi) = \Phi(\xi)$. More precisely, as soon as $n, m > K + 2kM + 4r$, we may write

$$\Phi_n(\xi) - \Phi_m(\xi) = \ln \frac{\langle \alpha, T_{j_0}\cdots T_{j_{k-1}} f_k(\gamma(n))\rangle}{\langle \beta, T_{j_0}\cdots T_{j_{k-1}} f_k(\gamma(n))\rangle} \frac{\langle \beta, T_{j_0}\cdots T_{j_{k-1}} f_k(\gamma(m))\rangle}{\langle \alpha, T_{j_0}\cdots T_{j_{k-1}} f_k(\gamma(m))\rangle},$$

where T_{j_s} is the projective action of $A_{V_s}^{V_{s+1}}$. By (12) and Lemma 2.6, we have, as soon as $n, m > K + 2kM + 4r$, $|\Phi_n(\xi) - \Phi_m(\xi)| \le C\tau^k$. For the same reason, for any fixed family of $f_k \in \mathcal{D}_{V_k}^t$, the sequence $T_{j_0}\cdots T_{j_{k-1}} f_k$ converge in $\mathcal{D}_{V_0}^t$ toward some f_∞, independent of the choice of f_k and a priori depending on the geodesic γ converging toward ξ. In any case, we have

$$\|T_{j_0}\cdots T_{j_{k-1}} f_k - f_\infty\|_t \le C\tau^k \quad \text{and} \quad \Phi(\xi) = \ln \frac{\langle \alpha, f_\infty\rangle}{\langle \alpha_1, f_\infty\rangle}. \qquad (13)$$

Consider now two points $\xi, \eta \in \partial G$ such that $\rho(\xi, \eta) < a^{-n-C}$. Then there is a geodesic γ converging to ξ and a sequence $\{y_\ell\}_{\ell \ge 1}$ going to η such that for ℓ, m large enough, $(\gamma(m), y_\ell) > n$. For fixed x and $K = 4M + |x|$, consider the same family $U_0^- \subset U_0 \subset \cdots \subset U_k^- \subset U_k$ such that for all $j = 1, \dots, k-1$, $U_j^- \subset U_j \subset U_{j+1}^- \subset U_{j+1}$ is an obstacle for $\gamma \circ \sigma^{2jM+K}$. We have:

LEMMA 3.6. *Assume $2kM < n - K - 4r - 22\delta$ and ℓ large enough. Then, $y_\ell \notin U_k$.*

PROOF. Choose ℓ large enough that $\lim_{m\to\infty}(\gamma(m), y_\ell) > n$, and we choose a geodesic $[y_\ell, \xi]$ such that $(y_\ell, \xi) > n$. By definition of U_j, we have to show that $d(y_\ell, \gamma(2(j-1)M + K)) \ge d(y_\ell, \gamma(2jM + K + 4r))$ for $2jM + K + r + 22\delta < n$. By continuity, there is a point s_0 where the function $s \mapsto d(y_\ell, \gamma(s))$ attains its minimum. We are going to show that $s_0 \ge n - 12\delta$. By 8δ convexity of $s \mapsto d(y_\ell, \gamma(s))$ ([9], Proposition 25, page 45), this proves the claim.[4]

[4]Indeed, since $2jM + K + r + 22\delta < n$, $\gamma(2jM + K + 4r + 10\delta)$ lies between $\gamma(2(j-1)M + K)$ and s_0 and thus, by 8δ convexity of the distance, $d(y_\ell, \gamma(2jM + K + 4r + 10\delta)) \le d(y_\ell, \gamma(2(j -$

By continuity, there is a point s_1 such that $d(\gamma(s_1), [\gamma(0), y_\ell]) = d(\gamma(s_1), [y_\ell, \xi]) \leq 4\delta$. On the one hand,

$$s_1 \geq d\bigl(\gamma(0), [y_\ell, \xi]\bigr) - 4\delta \geq n - 4\delta$$

[recall that $(\xi, y_\ell) > n$]. On the other hand, we know that

$$d\bigl(y_\ell, \gamma(s_1)\bigr) \leq \bigl(\gamma(0), \xi\bigr)_{y_\ell} + 8\delta \leq d\bigl(y_\ell, \gamma(s_0)\bigr) + 8\delta$$

(see the proof of Lemma 22.4 in [21]). It follows that $s_0 \geq s_1 - 8\delta \geq n - 12\delta$. $\square$

We have that $\Phi(\xi) - \Phi(\eta) = \lim_{x_m \to \xi, y_\ell \to \eta} \ln(\frac{u(e,x_m)}{u(x,x_m)} \frac{u(x,y_\ell)}{u(e,y_\ell)})$. With the above notation, assume that k is such that $2kM < n - K - 4r - 22\delta$. If ℓ and m are large enough, $y_\ell, \gamma(m) \notin U_k$ and

$$\Phi(\xi) - \Phi(\eta) = \lim_{x_m \to \xi, y_\ell \to \eta} \ln \frac{\langle \beta, T_{j_0} \cdots T_{j_{k-1}} f_k(\gamma(m))\rangle}{\langle \alpha, T_{j_0} \cdots T_{j_{k-1}} f_k(\gamma(m))\rangle} \frac{\langle \alpha, T_{j_0} \cdots T_{j_{k-1}} f_k(y_\ell)\rangle}{\langle \beta, T_{j_0} \cdots T_{j_{k-1}} f_k(y_\ell)\rangle}.$$

Since as above, we have $\vartheta(T_{j_0} \cdots T_{j_{k-1}} f_k(\gamma(m)), T_{j_0} \cdots T_{j_{k-1}} f_k(y_\ell)) < C\tau^k$, and α, β take a finite number of values, we have

$$\bigl|\Phi(\xi) - \Phi(\eta)\bigr| \leq C\tau^k \leq C\rho_0^n$$

for a new constant C and $\rho_0 = \tau^{1/2M}$. This shows that for all $x \in G$, the function $\xi \mapsto -\ln K_x(\xi)$ is Hölder continuous on ∂G. Moreover, the Hölder exponent $|\ln \rho_0| / \ln a$ and the Hölder constant C are uniform on a neighborhood of p in $\mathcal{P}(F)$.

Let us choose $\kappa < 1, \kappa < -\frac{\ln \rho_0}{2 \ln a}$, and consider the space Γ_κ of functions ϕ on ∂G such that there is a constant C_κ with the property that $|\phi(\xi) - \phi(\eta)| \leq C_\kappa (d(\xi, \eta))^\kappa$. For $\phi \in \Gamma_\kappa$, denote $\|\phi\|_\kappa$ the best constant C_κ in this definition. The space Γ_κ is a Banach space for the norm $\|\phi\| := \|\phi\|_\kappa + \max_{\partial G} |\phi|$. In this subsection, we showed that for $p \in \mathcal{P}(F)$, $x \in G$, there exist $\kappa > 0$ and a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ such that for $p' \in \mathcal{O}_p$, the function $\Phi_{p'}(\xi) = -\ln K_\xi(x)$ belongs to Γ_κ and that the mapping $p' \mapsto \Phi_{p'}$ is bounded from $\mathcal{O}_p$ into Γ_κ.

4. The Martin kernel depends regularly on *p*.

PROPOSITION 4.1. *Fix $x \in G$. For all $p \in \mathcal{P}(F)$, there exist $\kappa > 0$ and a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ such that the mapping $p \mapsto \Phi(\xi) = -\ln K_\xi(x)$ is Lipschitz continuous from $\mathcal{O}_p$ into Γ_κ.*

$1)M + K)) + 8\delta$ [recall that s_0 achieves the minimum of $d(y_\ell, \gamma(s))$]. The inequality follows by writing the δ-hyperbolicity relation (1) with $x = y_\ell$, $y = \gamma(2jM + K + 4r)$, $z = \gamma(2jM + K + 4r + 10\delta)$ and $w = \gamma(2(j-1)M + K)$.

PROOF. Let $p \in \mathcal{P}(F)$ and choose $\kappa = \kappa(p)$ given by Section 3.3. We have to find a neighborhood $\mathcal{O}$ of p in $\mathcal{P}(F)$ and a constant C such that, for $p' \in \mathcal{O}$,

$$\|\Phi_p - \Phi_{p'}\| = \max_\xi \big|\Phi_p(\xi) - \Phi_{p'}(\xi)\big| + \|\Phi_p - \Phi_{p'}\|_\kappa \le C\vartheta\big(p, p'\big),$$

where, for convenience, we use on $\mathcal{P}(F)$ the already defined projective distance on $\mathbb{R}^F$. We treat the two terms separately.

CLAIM 1. $\max_\xi |\Phi_p(\xi) - \Phi_{p'}(\xi)| \le C\vartheta(p, p')$.

Choose the geodesic γ converging to ξ. Applying Section 3.3 and (13), there are vectors $f_\infty(p), f_\infty(p') \in \ell^t(V_0)$ such that

$$\big|\Phi_p(\xi) - \Phi_{p'}(\xi)\big| = \left|\ln \frac{\langle \alpha(p), f_\infty(p)\rangle}{\langle \alpha(p'), f_\infty(p')\rangle} \frac{\langle \beta(p'), f_\infty(p')\rangle}{\langle \beta(p), f_\infty(p)\rangle}\right|.$$

By Proposition 2.4, we make an error of order $C\vartheta(p, p')$ when replacing $\beta(p')$ by $\beta(p)$ and $\alpha(p')$ by $\alpha(p)$. The remaining term is

$$\lim_k \left|\ln \frac{\langle \alpha, T_{j_0} \cdots T_{j_{k-1}} f_k\rangle}{\langle \alpha, T'_{j_0} \cdots T'_{j_{k-1}} f_k\rangle} \frac{\langle \beta, T'_{j_0} \cdots T'_{j_{k-1}} f_k\rangle}{\langle \beta, T_{j_0} \cdots T_{j_{k-1}} f_k\rangle}\right|, \tag{14}$$

where T_{j_s} is the projective action of $A_{V_s}^{V_{s+1}}(p)$, T'_{j_s} the projective action of $A_{V_s}^{V_{s+1}}(p')$ and we have chosen once for all $f_k \in \ell^t(V_k)$, independent of $p \in \mathcal{O}$.

We have

$$\begin{aligned}
&\vartheta\big(T_{j_0} \cdots T_{j_{k-1}} f_k, T'_{j_0} \cdots T'_{j_{k-1}} f_k\big) \\
&\quad \le \sum_{i=1}^{k-1} \vartheta\big(T_{j_0} \cdots T_{j_{i-1}} T'_{j_i} \cdots T'_{j_{k-1}} f_k, T_{j_0} \cdots T_{j_i} T'_{j_{i+1}} \cdots T'_{j_{k-1}} f_k\big) \\
&\quad \le \sum_{i=1}^{k-1} \tau^{i-1} \vartheta\big(T'_{j_i} \cdots T'_{j_{k-1}} f_k, T_{j_i} T'_{j_{i+1}} \cdots T'_{j_{k-1}} f_k\big),
\end{aligned}$$

where we used (12) to write the last line. If the neighborhood $\mathcal{O}$ is relatively compact in $\mathcal{P}(F)$, all points $T'_{j_{i+1}} \cdots T'_{j_{k-1}} f_k$ are in a common bounded subset of $\mathcal{D}^t_{V_{j_{i+1}}}$. By Proposition 3.5, there is a constant C and a neighborhood $\mathcal{O}$ such that for $p' \in \mathcal{O}$, $i = 1, \dots, k-1$,

$$\vartheta\big(T'_{j_i} T'_{j_{i+1}} \cdots T'_{j_{k-1}} f_k, T_{j_i} T'_{j_{i+1}} \cdots T'_{j_{k-1}} f_k\big) \le C\vartheta\big(p, p'\big).$$

Finally, we get that for all k, $\vartheta(T_{j_0} \cdots T_{j_{k-1}} f_k, T'_{j_0} \cdots T'_{j_{k-1}} f_k) \le \frac{C}{1-\tau}\vartheta(p, p')$. Reporting in (14) proves Claim 1.

CLAIM 2. $\|\Phi_p - \Phi_{p'}\|_\kappa \le C\vartheta(p, p')$.

Let $\xi, \eta \in \partial G$ be such that $\rho(\xi, \eta) < a^{-n-C}$. We want to show that there is a constant C and a neighborhood $\mathcal{O}$, independent on n such that, for $p' \in \mathcal{O}$,

$$\left|\Phi_p(\xi) - \Phi_{p'}(\xi) - \Phi_p(\eta) + \Phi_{p'}(\eta)\right| \leq Ca^{-\kappa n}\vartheta(p, p').$$

Choose as before a geodesic γ converging to ξ and a sequence $\{y_\ell\}_{\ell \geq 1}$ going to η such that for ℓ, m large enough, $(\gamma(m), y_\ell) > n$. For fixed x and $K = 4M + |x|$, consider the same family $U_0^- \subset U_0 \subset \cdots \subset U_k^- \subset U_k$ such that for all $j = 1, \dots, k-1$, $U_j^- \subset U_j \subset U_{j+1}^- \subset U_{j+1}$ is an obstacle for $\gamma \circ \sigma^{2jM+K}$. By Lemma 3.6, for ℓ large enough, $y_\ell \notin U_k$, and we may write $\Phi_p(\xi) - \Phi_{p'}(\xi) - \Phi_p(\eta) + \Phi_{p'}(\eta)$ as

$$\begin{aligned}
&\lim_{x_m \to \xi, y_\ell \to \eta} \ln \frac{\langle \beta, T_{j_0} \cdots T_{j_{k-1}} g_k\rangle}{\langle \alpha, T_{j_0} \cdots T_{j_{k-1}} g_k\rangle} \frac{\langle \alpha', T'_{j_0} \cdots T'_{j_{k-1}} g'_k\rangle}{\langle \beta', T'_{j_0} \cdots T'_{j_{k-1}} g'_k\rangle} \\
&\qquad \times \frac{\langle \alpha, T_{j_0} \cdots T_{j_{k-1}} h_k\rangle}{\langle \beta, T_{j_0} \cdots T_{j_{k-1}} h_k\rangle} \frac{\langle \beta', T'_{j_0} \cdots T'_{j_{k-1}} h'_k\rangle}{\langle \alpha', T'_{j_0} \cdots T'_{j_{k-1}} h'_k\rangle},
\end{aligned} \tag{15}$$

where $\alpha = \alpha(p), \alpha' = \alpha(p'), \beta = \beta(p), \beta' = \beta(p')$, T_{j_s} is the projective action of $A_{V_s}^{V_{s+1}}(p)$, T'_{j_s} the projective action of $A_{V_s}^{V_{s+1}}(p')$ and g_k, g'_k are $f_k(\gamma(m))$ calculated with p and p', respectively, h_k, h'_k are $f_k(y_\ell)$ calculated with p and p'.

Recall that $g_k = f_k(\gamma(m))$ is the direction of $\omega_v^{\gamma(m)}$ in $\ell^t(V_k)$. It can obtained by a series of obstacles along γ between U_k and $\gamma(m)$. Let us show that we can choose m large enough (depending on p') such that we have $\vartheta(g_k, g'_k) \leq C\vartheta(p, p')$. Indeed,

$$\begin{aligned}
\vartheta(g_k, g'_k) &= \vartheta\big(f_k(\gamma(m)), f'_k(\gamma(m))\big) \\
&= \vartheta\big(T_{j_k} \cdots T_{j_{m-1}} f_m, T'_{j_k} \cdots T'_{j_{m-1}} f'_m\big).
\end{aligned}$$

We have $\vartheta(f_m, f'_m) < C$ and for m large enough,

$$\vartheta\big(T'_{j_k} \cdots T'_{j_{m-1}} f_m, T'_{j_k} \cdots T'_{j_{m-1}} f'_m\big) < \tau^{m-k} C \leq \vartheta(p, p').$$

By the same computation as in Claim 1, we then have

$$\vartheta\big(T_{j_k} \cdots T_{j_{m-1}} f_m, T'_{j_k} \cdots T'_{j_{m-1}} f_m\big) \leq C\vartheta(p, p').$$

Since $\vartheta(g_k, g'_k) \leq C\vartheta(p, p')$, using the contraction of the T_j, we can replace g'_k by g_k in (15) with an error less than $C\tau^k\vartheta(p, p') < C\rho_0^n\vartheta(p, p')$. In the same way, following obstacles along the geodesic between $\gamma(n)$ and y_ℓ, we have, for ℓ large enough, $\vartheta(h_k, h'_k) \leq \vartheta(p, p')$, and we can replace h'_k by h_k in (15) with an error less than $C\rho_0^n\vartheta(p, p')$.

Observe also that all terms $\dot\alpha = \alpha/\|\alpha\|_s$, $\dot\beta = \beta/\|\beta\|_s$ belong to $\mathcal{D}^s_{V_{j_0}}$. We may write, considering, for instance, $\langle \alpha', T'_{j_0}\cdots T'_{j_{k-1}} g'_k\rangle$,

$$\begin{aligned}\frac{\langle \alpha', T'_{j_0}\cdots T'_{j_{k-1}} g'_k\rangle}{\langle \alpha, T'_{j_0}\cdots T'_{j_{k-1}} g'_k\rangle} &= \frac{\langle \alpha', A'_{j_0}\cdots A'_{j_{k-1}} g'_k\rangle}{\langle \alpha, A'_{j_0}\cdots A'_{j_{k-1}} g'_k\rangle}\\ &= \frac{\langle (A'_{j_{k-1}})^*\cdots (A'_{j_0})^*\alpha', g'_k\rangle}{\langle (A'_{j_{k-1}})^*\cdots (A'_{j_0})^*\alpha, g'_k\rangle}\\ &= \frac{\|\alpha'\|_s}{\|\alpha\|_s}\frac{\langle (T'_{j_{k-1}})^*\cdots (T'_{j_0})^*\dot\alpha', g'_k\rangle}{\langle (T'_{j_{k-1}})^*\cdots (T'_{j_0})^*\dot\alpha, g'_k\rangle},\end{aligned}$$

where $(T'_j)^*$ denotes the projective action of $(A'_j)^*$ on $\mathcal{D}^s_{V_j}$. Observe that if we replace α' by α, β' by β in (15) and use the above equation and its analogs, the ratios $\frac{\|\alpha'\|_s}{\|\alpha\|_s}$, $\frac{\|\beta'\|_s}{\|\beta\|_s}$ cancel one another, and using the contraction of the $(T'_j)^*$, we make an other error of size at most $C\rho_0^n\vartheta(p,p')$.

We find that, up to an error of size at most $C\rho_0^n\vartheta(p,p')$, the difference $\Phi_p(\xi) - \Phi_{p'}(\xi) - \Phi_p(\eta) + \Phi_{p'}(\eta)$ is given by

$$\begin{aligned}\lim_{x_m\to\xi, y_\ell\to\eta} \ln &\frac{\langle \dot\beta, A_{j_0}\cdots A_{j_{k-1}} g_k\rangle}{\langle \dot\beta, A'_{j_0}\cdots A'_{j_{k-1}} g_k\rangle}\frac{\langle \dot\alpha, A'_{j_0}\cdots A'_{j_{k-1}} g_k\rangle}{\langle \dot\alpha, A_{j_0}\cdots A_{j_{k-1}} g_k\rangle}\\ &\times \frac{\langle \dot\alpha, A_{j_0}\cdots A_{j_{k-1}} h_k\rangle}{\langle \dot\alpha, A'_{j_0}\cdots A'_{j_{k-1}} h_k\rangle}\frac{\langle \dot\beta, A'_{j_0}\cdots A'_{j_{k-1}} h_k\rangle}{\langle \dot\beta, A_{j_0}\cdots A_{j_{k-1}} h_k\rangle},\end{aligned}$$

where we reordered the denominators to get a sum of four terms of the form

$$\pm\ln\frac{\langle \alpha, A_{j_0}\cdots A_{j_{k-1}} g\rangle}{\langle \alpha, A'_{j_0}\cdots A'_{j_{k-1}} g\rangle}$$

with $\alpha\in\mathcal{D}^s_{V_{j_0}}$, $g\in\mathcal{D}^t_{V_{j_k}}$. We can arrange each such term and write

$$\begin{aligned}&\frac{\langle \alpha, A_{j_0}\cdots A_{j_{k-1}} g\rangle}{\langle \alpha, A'_{j_0}\cdots A'_{j_{k-1}} g\rangle}\\ &\quad= \prod_{i=0}^{k-1}\frac{\langle \alpha, A_{j_0}\cdots A_{j_{i-1}}A_{j_i}A'_{j_{i+1}}\cdots A'_{j_{k-1}} g\rangle}{\langle \alpha, A_{j_0}\cdots A_{j_{i-1}}A'_{j_i}A'_{j_{i+1}}\cdots A'_{j_{k-1}} g\rangle}\\ &\quad= \prod_{i=0}^{[k/2]}\frac{\langle (A'_{j_{i-1}})^*\cdots (A'_{j_0})^*\alpha, A'_{j_i}g_i\rangle}{\langle (A'_{j_{i-1}})^*\cdots (A'_{j_0})^*\alpha, A_{j_i}g_i\rangle}\times\prod_{i=[k/2]+1}^{k-1}\frac{\langle (A_{j_i})^*\alpha_i, A'_{j_{i+1}}\cdots A'_{j_{k-1}} g\rangle}{\langle (A'_{j_i})^*\alpha_i, A'_{j_{i+1}}\cdots A'_{j_{k-1}} g\rangle}\\ &\quad= \prod_{i=0}^{[k/2]}\frac{\langle (A_{j_i})^*\alpha_i, g_i\rangle}{\langle (A'_{j_i})^*\alpha_i, g_i\rangle}\times\prod_{i=[k/2]+1}^{k-1}\frac{\langle \alpha_i, A_{j_i}g_i\rangle}{\langle \alpha_i, A'_{j_i}g_i\rangle},\end{aligned}$$

where $\alpha_i = (T_{j_{i-1}})^* \cdots (T_{j_0})^*\alpha$, $g_i = T'_{j_{i+1}} \cdots T'_{j_{k-1}} g$. Set $\beta_i = (T_{j_{i-1}})^* \cdots (T_{j_0})^*\beta$, $h_i = T'_{j_{i+1}} \cdots T'_{j_{k-1}} h$. We are reduced to estimate

$$\prod_{i=0}^{[k/2]} \frac{\langle (A_{j_i})^*\beta_i, g_i\rangle}{\langle (A'_{j_i})^*\beta_i, g_i\rangle} \frac{\langle (A'_{j_i})^*\alpha_i, g_i\rangle}{\langle (A_{j_i})^*\alpha_i, g_i\rangle} \frac{\langle (A_{j_i})^*\alpha_i, h_i\rangle}{\langle (A'_{j_i})^*\alpha_i, h_i\rangle} \frac{\langle (A'_{j_i})^*\beta_i, h_i\rangle}{\langle (A_{j_i})^*\beta_i, h_i\rangle}$$

$$\times \prod_{i=[k/2]+1}^{k-1} \frac{\langle \beta_i, A_{j_i} g_i\rangle}{\langle \beta_i, A'_{j_i} g_i\rangle} \frac{\langle \alpha_i, A'_{j_i} g_i\rangle}{\langle \alpha_i, A_{j_i} g_i\rangle} \frac{\langle \alpha_i, A_{j_i} h_i\rangle}{\langle \alpha_i, A'_{j_i} h_i\rangle} \frac{\langle \beta_i, A'_{j_i} h_i\rangle}{\langle \beta_i, A_{j_i} h_i\rangle}.$$

Since, g_i, h_i remain in a bounded part of the $\mathcal{D}_{V^t}$ and α_i, β_i in a bounded part of the $\mathcal{D}_{V^s}$, using Propositions 2.4 and 2.5, one gets a constant C such that $\vartheta(A_{j_i} g_i, A'_{j_i} g_i)$, $\vartheta(A_{j_i} h_i, A'_{j_i} h_i)$, $\vartheta((A_{j_i})^*\alpha_i, (A'_{j_i})^*\alpha_i)$ and $\vartheta((A_{j_i})^*\beta_i, (A'_{j_i})^*\beta_i)$ are all smaller than $C\vartheta(p, p')$. Furthermore, using the contraction of T_j and $(T_j)^*$ (Proposition 3.4) we see that

$$\vartheta(\alpha_i, \beta_i) \le C\tau^i, \qquad \vartheta(g_i, h_i) \le C\tau^{k-i}.$$

Moreover, all products in the formula are approximations of $\langle \alpha, f_\infty\rangle$ and thus are uniformly bounded away from 0. It follows that for $i \le k/2$,

$$\left|\ln \frac{\langle (A_{j_i})^*\beta_i, g_i\rangle \langle (A'_{j_i})^*\beta_i, h_i\rangle}{\langle (A'_{j_i})^*\beta_i, g_i\rangle \langle (A_{j_i})^*\beta_i, h_i\rangle}\right|,$$

$$\left|\ln \frac{\langle (A'_{j_i})^*\alpha_i, g_i\rangle \langle (A_{j_i})^*\alpha_i, h_i\rangle}{\langle (A_{j_i})^*\alpha_i, g_i\rangle \langle (A'_{j_i})^*\alpha_i, h_i\rangle}\right| \le C\tau^{k-i}\vartheta(p, p')$$

and for $i > k/2$,

$$\left|\ln \frac{\langle \beta_i, A_{j_i} g_i\rangle \langle \alpha_i, A'_{j_i} g_i\rangle}{\langle \beta_i, A'_{j_i} g_i\rangle \langle \alpha_i, A_{j_i} g_i\rangle}\right|, \left|\ln \frac{\langle \alpha_i, A_{j_i} h_i\rangle \langle \beta_i, A'_{j_i} h_i\rangle}{\langle \alpha_i, A'_{j_i} h_i\rangle \langle \beta_i, A_{j_i} h_i\rangle}\right| \le C\tau^i \vartheta(p, p'),$$

so that finally the main term of (15) is estimated by

$$\sum_{i=0}^{[k/2]} C\tau^{k-i}\vartheta(p, p') + \sum_{i=[k/2]+1}^{k-1} C\tau^i \vartheta(p, p') \le C\tau^{k/2}\vartheta(p, p') \le C\rho_0^{n/2}\vartheta(p, p').$$

Claim 2 is proven (recall that $\kappa < -\frac{\ln \rho_0}{2\ln a}$ so that $\rho^{1/2} < a^{-\kappa}$). $\square$

5. Markov coding and regularity of p^∞. In this section, we discuss the regularity of the mapping $p \mapsto p^\infty$ from $\mathcal{P}(F)$ into the space Γ_κ^* of continuous linear forms on Γ_κ. By Theorem 2.1, p^∞ is the only p-stationary measure for the action of ∂G, and thus depends continuously on p. In the case of the free group, p^∞ appears as the eigenform for an isolated maximal eigenvalue of an operator on Γ_κ (see [16], Chapter 4c) and therefore depends real analytically on p. This argument does not seem to work in all the generality of a hyperbolic group, and we are going to use the Markov representation of the boundary which was described by M. Coornaert and A. Papadopoulos in [6].

5.1. *Markov coding*. Following [6], we call *horofunctions* any integer valued function on G such that, for all $\lambda \leq h(x)$, the distance of a point x to the set $h^{-1}(\lambda)$ is given by $h(x) - \lambda$. Two horofunctions are said to be equivalent if they differ by a constant. Let Φ_0 be the set of classes of horofunctions. Equipped with the topology of uniform convergence on finite subsets of G, the space Φ_0 is a compact metric space. G acts naturally on Φ_0. The Busemann boundary $\partial_B G$ is a G-invariant subset of Φ_0. For each horofunction h, sequences $\{x_n\}_{n \geq 1}$ such that

$$d(x_n, x_{n+1}) = h(x_n) - h(x_{n+1}) = 1$$

converge to a common point in ∂G, the point at infinity of h. Two equivalent horofunctions have the same point at infinity. The mapping $\pi : \Phi_0 \to \partial G$ which associates to a class of horofunctions its point at infinity is continuous, surjective, G-equivariant and uniformly finite-to-one. Fix an arbitrary total order relation on the set of generators S. Define a map $\alpha : \Phi_0 \to \Phi_0$ by setting, for a class $\varphi = [h] \in \Phi_0$, $\alpha(\varphi) = a^{-1}\varphi$, where $a = a(\varphi)$ is the smallest element in S satisfying $h(e) - h(a) = 1$. In [6] is proven:

THEOREM 5.1 ([6]). *The dynamical system* (Φ_0, α) *is topologically conjugate to a subshift of finite type.*

We assume, as we may, that the number R_0 used in the construction of [6] satisfies $R_0 > r$. In order to fix notation, let (Σ, σ) be the subshift of finite type of Theorem 5.1. That is, there is a finite alphabet Z and a $Z \times Z$ matrix A with entries 0 or 1 such that Σ is the set of sequences $\underline{z} = \{z_n\}_{n \geq 0}$ such that for all n, $A_{z_n, z_{n+1}} = 1$ and σ is the left shift on Σ. We can decompose Σ into transitive components. Namely, there is a partition of the alphabet Z into the disjoint union of Z_j, $j = 0, \ldots, K$ in such a way that for $j = 1, \ldots, K$, $\Sigma_j := \{\underline{z}, z_0 \in Z_j\}$ is a σ-invariant transitive subshift of finite type and $\bigcup_{j=1}^{K} \Sigma_j$ is the ω-limit set of Σ. By construction, G-invariant closed subsets of Σ are unions of Σ_j for some $j \in \{1, \ldots, K\}$. We denote such G-invariant subsets by Σ_J, where J is the corresponding subset of $\{1, \ldots, K\}$. In particular, the supports of stationary measures on $\overline{G}$ are subsets of $\partial_B G$ which are identified with such Σ_J.

For $\chi > 0$ consider the space Γ_χ of functions ϕ on Σ such that there is a constant C_χ with the property that, if the points $\underline{z}$ and $\underline{z}'$ have the same first n coordinates, then $|\phi(\underline{z}) - \phi(\underline{z}')| < C_\chi \chi^n$. For $\phi \in \Gamma_\chi$, denote $\|\phi\|_\chi$ the best constant C_χ in this definition. The space Γ_χ is a Banach space for the norm $\|\phi\| := \|\phi\|_\chi + \max_\Sigma |\phi|$. Identifying Σ with Φ_0, we still write $\pi : \Sigma \to \partial G$ the mapping which associates to $\underline{z} \in \Sigma$ the point at infinity of the class of horofunctions represented by $\underline{z}$.

PROPOSITION 5.2. *The mapping* $\pi : \Sigma \to \partial G$ *is Hölder continuous.*

PROOF. Let $\underline{z}$ and $\underline{z}'$ be two elements of Σ such that $z_i = z_i'$ for $1 \le i \le n$. Denote h and h' the corresponding horofunctions with $h(e) = h'(e) = 0$. Let $\{x_n\}_{n\ge 0}$ be define inductively such that $x_0 = e$ and $(x_{n-1})^{-1}x_n$ is the smallest element a in S such that $h(x_{n-1}) - h(x_{n-1}a) = 1$. The sequence $\{x_n\}_{n\ge 0}$ is a geodesic and converges to $\pi(\underline{z})$. By [6], Lemma 6.5, h and h' coincide on $N(\{x_0, \dots, x_{n+L_0}\}, R_0)$, where L_0 and R_0 has been chosen as in [6], page 439. In particular, if one associates $\{x_n'\}_{n\ge 0}$ similarly to h', the sequence $\{x_n'\}_{n\ge 0}$ is a geodesic which converges to $\pi(\underline{z}')$, and we have $x_k = x_k'$ for $0 \le k \le n + L_0$. It follows that for all $m, m' > n + L_0$,

$$(x_m, x_{m'})_e = n + L_0 + (x_m, x_{m'})_{x_{n+L_0}} \ge n + L_0 \ge n.$$

Therefore

$$\bigl(\pi(\underline{z}), \pi(\underline{z}')\bigr)_e \ge \liminf_{m,m'} (x_m, x_{m'})_e \ge n \quad \text{and} \quad \rho\bigl(\pi(\underline{z}), \pi(\underline{z}')\bigr) \le e^{-an+c_1}. \qquad \square$$

In the same way, we have:

PROPOSITION 5.3. *Let x be fixed in G with $|x| < R_0$. Then the mapping $\underline{z} \mapsto h_{\underline{z}}(x)$ depends only on the first coordinate in Σ, where $h_{\underline{z}}$ is the horofunction representing $\underline{z}$ in Theorem 5.1.*

PROOF. As above, if $z_0 = z_0'$ and h, h' are the corresponding horofunctions with $h(e) = h'(e) = 0$, h and h' coincide on $N(e, R_0) \supset \{x\}$. $\square$

Let ν be a stationary probability measure on Φ_0. By equivariance of π, the measure $\pi_*\nu$ is stationary on ∂G and, by Theorem 2.1, we have $\pi_*\nu = p^\infty$. Actually, there is a more precise result:

PROPOSITION 5.4. *Let ν be a stationary measure on Φ_0. Then, for ν-a.e. $\varphi \in \Phi_0$, all x,*

$$\frac{dx_*\nu}{d\nu}(\varphi) = K_{\pi(\varphi)}(x). \tag{16}$$

PROOF. Since the mapping $\pi : \Phi_0 \to \partial F$ is G-equivariant and finite-to-one, the measure ν can be written as

$$\int \psi(\varphi)\, d\nu(\varphi) = \int \Bigl(\sum_{\varphi : \pi(\varphi) = \xi} \psi(\varphi) a(\varphi) \Bigr) dp^\infty(\xi),$$

where a is a nonnegative measurable function on Φ_0 such that $\sum_{\varphi : \pi(\varphi)=\xi} a(\varphi) = 1$ for p^∞-a.e. ξ. Moreover, since ∂G is a Poisson boundary for the random walk ([13], Theorem 7.6), the conditional measures $a(\varphi)$ has to satisfy $a(x\varphi) = a(\varphi)$ p^∞-a.s. ([14], Theorem 3.2). Formula (16) for the density then follows from formula (6). $\square$

Identifying Φ_0 with Σ, we see that, for $\underline{z} \in \Sigma$, $\sigma^{-1}\underline{z}$ is given by some $a\underline{z}$, where a is one of the generators. We can describe the restriction of a stationary measure to Σ_j. More precisely, we have:

PROPOSITION 5.5. *For each $j = 1, \dots, K$, there is a unique probability measure ν_j such that any p-stationary measure on Σ has the restriction to Σ_j proportional to ν_j. Moreover, for all $p \in \mathcal{P}(F)$, there exist $\chi > 0$ and a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ such that the mapping $p \mapsto \nu_j$ is Lipschitz continuous from $\mathcal{O}_p$ to $\Gamma_\chi^*(\Sigma_j)$.*

PROOF. Consider a p-stationary probability measure on Σ that has a nonzero restriction to Σ_j. Let ν_j be this (normalized) restriction. By (16), for all x such that $x^{-1}\Sigma_j = \Sigma_j$, we have $\frac{dx_*\nu_j}{d\nu_j}(\underline{z}) = K_{\pi(\underline{z})}(x)$. We shall show that there is a unique probability measure on Σ_j satisfying $\frac{d\sigma_*\nu_j}{d\nu_j}(\underline{z}) = K_{\pi(\underline{z})}(z_0)$ and that it depends Lipschitz continuously on p as an element of Γ_χ^* for some suitable χ.

We use thermodynamical formalism on the transitive subshift of finite type Σ_j. For $\chi < 1$ and $\phi \in \Gamma_\chi$ with real values, we define the transfer operator $\mathcal{L}_\phi$ on $\Gamma_\chi(\Sigma_j)$ by

$$\mathcal{L}_\phi \psi(\xi) := \sum_{\eta \in \sigma^{-1}\xi} e^{\phi(\eta)}\psi(\eta).$$

Then, $\mathcal{L}_\phi$ is a bounded operator in Γ_χ. Ruelle's transfer operator theorem (see [4], Theorem 1.7, and [20], Proposition 5.24) applies to $\mathcal{L}_\phi$, and there exists a number $P(\phi)$ and a linear functional N_ϕ on Γ_χ such that the operator $\mathcal{L}_\phi^*$ on $(\Gamma_\chi)^*$ satisfies $\mathcal{L}_\phi^* N_\phi = e^{P(\phi)} N_\phi$. The functional N_ϕ extends to a probability measure on Σ_j and is the only eigenvector of $\mathcal{L}_\phi^*$ with that property. Moreover, $\phi \mapsto \mathcal{L}_\phi$ is a real analytic map from Γ_χ to the space of linear operators on Γ_χ ([20], page 91). Consequently, the mapping $\phi \mapsto N_\phi$ is real analytic from Γ_χ into the dual space Γ_χ^*; see, for example, [5], Corollary 4.6. For $p \in \mathcal{P}(F)$, define $\phi_p(\underline{z}) = \ln K_{\pi(\underline{z})}(z_0)$. By Propositions 4.1 and 5.2, we can choose χ such that the mapping $p \mapsto \phi_p$ is Lipschitz continuous from a neighborhood $\mathcal{O}_p$ of p in $\mathcal{P}(F)$ into the space Γ_χ. It follows that the mapping $p \mapsto N_{\phi_p}$ is Lipschitz continuous from $\mathcal{O}_p$ into Γ_χ^*.

From the relation $\frac{d\sigma_*\nu_j}{d\nu_j}(\underline{z}) = K_{\pi(\underline{z})}(z_0)$, we know that ν_j is invariant under $\mathcal{L}_{\phi_p}^*$. This shows that ν_j is the only probability measure satisfying this relation, that $P(\phi_p) = 0$ and that ν_j extends N_{ϕ_p}. $\square$

Let Σ_J be a minimal, closed G-invariant subset of Σ. We know that Σ_J is a finite union of transitive subshifts of finite type. We have:

COROLLARY 5.6. *For $p \in \mathcal{P}(F)$, there is a unique p-stationary probability measure $\nu_J(p)$ on Σ_J. There is a χ and a neighborhood $\mathcal{O}$ of p such that the mapping $p \mapsto \nu_J(p)$ is Lipschitz continuous from $\mathcal{O}$ into $\Gamma_\chi^*(\Sigma_J)$.*

PROOF. Let ν_J be a p stationary measure on Σ_J. We know by Proposition 5.5 that the conditional measures on the transitive subsubshifts are unique and Lipschitz continuous from $\mathcal{O}$ into $\Gamma_\chi^*(\Sigma_k)$. We have to show that the $\nu_J(\Sigma_k)$ are well determined and Lipschitz continuous in p. Write again equation (16), but now for elements $x \in G$ that exchange the Σ_k within Σ_J and write that $\sum_k \nu_J(\Sigma_k) = 1$. We find that the $\nu_J(\Sigma_k)$ are given by a system of linear equations. By Propositions 4.1 and 5.5, we know that the coefficients of this linear system are Lipschitz continuous on $\mathcal{O}$. We know that there is a solution, and that it is unique, since otherwise there would be a whole line of solutions, in particular one which would give $\nu_J(\Sigma_k) = 0$ for some k and this is impossible. Then the unique solution is Lipschitz continuous. $\square$

6. Proof of Theorem 1.1.

Choose χ small enough and $\mathcal{O}$ a neighborhood of p in $\mathcal{P}(F)$ such that Proposition 4.1 and Corollary 5.6 apply: the mappings $p \mapsto \ln K_{\pi(\underline{z})}(x)$ and $p \mapsto \nu_J$ are Lipschitz continuous from $\mathcal{O}$ into, respectively, $\Gamma_\chi(\Sigma)$ and $\Gamma_\chi^*(\Sigma_J)$. Then, by definition (2), the function $p \mapsto h_p(\Sigma_J, \nu)$ is Lipschitz continuous on $\mathcal{O}$. By (3) and (5), the function h_p is the maximum of a finite number of Lipschitz continuous functions on $\mathcal{O}$; this proves the entropy part of Theorem 1.1.

For the escape rate part, recall that the Busemann boundary $\partial_B G$ is made of horofunctions so that it can be identified with a G-invariant subset of Σ. Stationary measures on $\partial_B G$ are therefore convex combinations of the $\nu_{J'}$, where J' are such that $\nu_{J'}(\partial_B G) = 1$. Formula (4) yields $\ell_p^S = \max_{J'}\{\sum_{x\in F}(\int_{\Sigma_{J'}} h(x^{-1})\, d\nu_{J'}(h)) \times p(x)\}$. By Proposition 5.3, for a fixed $x \in F$ the function $h(x)$ is in $\Gamma_\chi(\Sigma_j)$ for all χ. Therefore, Corollary 5.6 implies that each one of the functions $\int_{\Sigma_{J'}} h(x^{-1})\, d\nu_{J'}(h)$ is Lipschitz continuous on $\mathcal{O}$. This achieves the proof of Theorem 1.1 because the function $p \mapsto \ell_p^S$ is also written as the maximum of a finite number of Lipschitz continuous functions on $\mathcal{O}$.

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