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SUPREMA OF LÉVY PROCESSES[1]

BY MATEUSZ KWAŚNICKI[2], JACEK MAŁECKI AND MICHAŁ RYZNAR

Wrocław University of Technology and Polish Academy of Sciences, Wrocław University of Technology and Université d'Angers, and Wrocław University of Technology

In this paper we study the supremum functional $M_t = \sup_{0 \le s \le t} X_s$, where X_t, $t \ge 0$, is a one-dimensional Lévy process. Under very mild assumptions we provide a simple, uniform estimate of the cumulative distribution function of M_t. In the symmetric case we find an integral representation of the Laplace transform of the distribution of M_t if the Lévy–Khintchin exponent of the process increases on $(0, \infty)$.

1. Introduction. By a classical reflection argument, the supremum functional $M_t = \sup_{0 \le s \le t} X_s$ of the Brownian motion X_t has truncated normal distribution, $\mathbf{P}(M_t \ge x) = 2\mathbf{P}(X_t \ge x)$ $(x \ge 0)$. A similar question for symmetric α-stable processes was first studied by Darling [11], and the case of general Lévy processes X_t was addressed by Baxter and Donsker [3]. Theorem 1 therein gives a formula for the double Laplace transform of the distribution of M_t, which for a *symmetric* Lévy process X_t with Lévy–Khintchin exponent $\Psi(\xi)$ reads

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-\xi x - zt} \mathbf{P}(M_t \in dx)\, dt \\ = \frac{1}{\sqrt{z}} \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\xi \log(z + \Psi(\zeta))}{\xi^2 + \zeta^2}\, d\zeta\right).\end{aligned} \tag{1.1}$$

Inversion of the double Laplace transform is typically a very difficult task. Apart from the Brownian motion case, an explicit formula for the distribution of M_t was found for the Cauchy process (the symmetric 1-stable process) by Darling [11], for a compound Poisson process with $\Psi(\xi) = 1 - \cos \xi$ by Baxter and Donsker [3] and for the Poisson process with drift by Pyke [32].

The development of the fluctuation theory for Lévy processes resulted in many new identities involving the supremum functional M_t; see, for example, [5, 13, 31, 33]. There are numerous other representations for the distribution of M_t, at least in the stable case; see [4, 7, 11, 12, 15, 16, 19, 20, 27, 28, 30, 36]. The main goal

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of this article is to give a more explicit formula for $\mathbf{P}(M_t < x)$ and simple sharp bounds for $\mathbf{P}(M_t < x)$ in terms of the Lévy–Khintchin exponent $\Psi(\xi)$ for a class of Lévy processes. Most estimates of the cumulative distribution function of M_t are proved for very general Lévy processes, without symmetry assumptions.

Let τ_x denote the first passage time through a barrier at the level x for the process X_t,

$$\tau_x = \inf\{t \geq 0 : X_t \geq x\}, \qquad x \geq 0,$$

with the infimum understood to be infinity when the set is empty. We always assume that $X_0 = 0$. Since $\mathbf{P}(M_t < x) = \mathbf{P}(\tau_x > t)$, the problems of finding the cumulative distribution functions of M_t and τ_x are the same. The supremum functional and first passage time statistics are important in various areas of applied probability [1, 2], as well as in mathematical physics [21, 26]. The recent progress in the potential theory of Lévy processes is, in part, due to the application of fluctuation theory; see [9, 10, 18, 22–25].

The paper is organized as follows. Section 2 contains some preliminary material related to Bernstein functions, Stieltjes functions and estimates for the Laplace transform. In Section 3 (Theorem 3.1 and Corollary 3.2) we prove, under mild assumptions, the estimate

$$\mathbf{P}(M_t < x) \approx \min\bigl(1, \kappa(1/t, 0) V(x)\bigr), \qquad t, x > 0,$$

where $V(x)$ and $\kappa(z, 0)$ are the renewal function for the ascending ladder-height process, and the Laplace exponent of the the ascending ladder-time process corresponding to X_t, respectively. Here $f(x) \approx g(x)$ means that there are constants $c_1, c_2 > 0$ such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$. In Section 4 we show that in the symmetric case, given some regularity of $\Psi(\xi)$, we have

$$V(x) \approx \frac{1}{\sqrt{\Psi(1/x)}}, \qquad x > 0;$$

see Theorem 4.4. Therefore the estimate of the above cumulative distribution function of M_t takes a very explicit form,

$$\mathbf{P}(M_t < x) \approx \min\left(1, \frac{1}{\sqrt{t\Psi(1/x)}}\right), \qquad t, x > 0.$$

The other main result of Section 4 is an explicit formula for the (single, in the space variable) Laplace transform of the distribution of M_t (Theorem 4.1), under the assumption that X_t is symmetric and $\Psi(\xi)$ is increasing on $[0, \infty)$.

When $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function $\psi(\xi)$, the above results can be significantly improved. Following the approach of [30], a (rather complicated) explicit formula for $\mathbf{P}(M_t < x)$ can be given, and estimates and asymptotic formulae for $\mathbf{P}(M_t < x)$ extend to $(d/dt)^n \mathbf{P}(M_t < x)$ when x is small or t is large. These results will be covered in a forthcoming paper.

NOTATION. We denote by C, C_1, C_2, etc. constants in theorems, and by c, c_1, c_2, etc. temporary constants in proofs. Any dependence of a constant on some parameters is always indicated by writing, for example, $c(n, \varepsilon)$. We write $f(x) \sim g(x)$ when $f(x)/g(x) \to 1$. We use the terms *increasing*, *decreasing*, *concave*, *convex function*, etc. in the weak sense.

2. Preliminaries.

2.1. *Complete Bernstein and Stieltjes functions.* A function $\psi(\xi)$ is said to be a *complete Bernstein function* (CBF) if

$$\psi(\xi) = c_1 + c_2\xi + \frac{1}{\pi}\int_{0+}^{\infty} \frac{\xi}{\xi + \zeta} \frac{\mu(d\zeta)}{\zeta}, \qquad \xi \in \mathbf{C} \setminus (-\infty, 0), \tag{2.1}$$

where $c_1, c_2 \geq 0$, and μ is a measure on $(0, \infty)$ such that the integral $\int_0^\infty \min(\zeta^{-1}, \zeta^{-2})\mu(d\zeta)$ is finite. A function $\tilde{\psi}(\xi)$ is said to be a *Stieltjes functions* if

$$\tilde{\psi}(\xi) = \frac{\tilde{c}_1}{\xi} + \tilde{c}_2 + \frac{1}{\pi}\int_{0+}^{\infty} \frac{1}{\xi + \zeta}\tilde{\mu}(d\zeta), \qquad \xi \in \mathbf{C} \setminus (-\infty, 0], \tag{2.2}$$

for some $\tilde{c}_1, \tilde{c}_2 \geq 0$ and some measure $\tilde{\mu}$ on $(0, \infty)$ such that the integral $\int_0^\infty \min(1, \zeta^{-1})\tilde{\mu}(d\zeta)$ is finite. See [34] for a general account on complete Bernstein functions, Stieltjes functions and related notions.

It is known that $\psi(\xi)$ is a CBF if and only if $\psi(\xi)$ is nonnegative and increasing on $(0, \infty)$, holomorphic in $\mathbf{C} \setminus (-\infty, 0]$, and $\operatorname{Im} \psi(\xi) > 0$ when $\operatorname{Im} \xi > 0$. Furthermore, if $\psi(\xi)$ is a CBF, then $\xi/\psi(\xi)$ is a CBF, and $1/\psi(\xi)$ and $\psi(\xi)/\xi$ are Stieltjes functions

The function $\tilde{\psi}(\xi)$ given by (2.2) is the Laplace transform of $\tilde{c}_2\delta_0(dx) + (\tilde{c}_1 + \mathcal{L}\tilde{\mu}(x))\,dx$ ([34], Theorem 2.2). Furthermore, $\pi\tilde{c}_1\delta_0(d\zeta) + \tilde{\mu}(d\zeta)$ is the limit of measures $-\operatorname{Im}(\tilde{\psi}(-\zeta + i\varepsilon))\,d\zeta$ as $\varepsilon \to 0^+$ ([34], Corollary 6.3 and Comments 6.12), so, in a sense, it is the boundary value of $\tilde{\psi}$. Therefore, we use a shorthand notation $-\operatorname{Im}(\tilde{\psi}^+(-\zeta))\,d\zeta$ for $\tilde{\mu}(d\zeta)$. Furthermore, we have $\tilde{c}_1 = \lim_{\xi \to 0}(\xi\tilde{\psi}(\xi))$ and $\tilde{c}_2 = \lim_{\xi \to \infty} \tilde{\psi}(\xi)$.

Following [30], we define

$$\psi^\dagger(\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log \psi(\zeta^2)}{\xi^2 + \zeta^2}\,d\zeta\right), \qquad \operatorname{Re} \xi > 0, \tag{2.3}$$

for any function $\psi(\xi)$ such that $\min(1, \zeta^{-2}) \log \psi(\zeta^2)$ is integrable in $\zeta > 0$. By a simple substitution,

$$\psi^\dagger(\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log \psi(\xi^2\zeta^2)}{1 + \zeta^2}\,d\zeta\right), \qquad \xi > 0. \tag{2.4}$$

By [30], Lemma 4, if $\psi(\xi)$ is a CBF, then also $\psi^\dagger(\xi)$ is a CBF (this was independently proved in [24], Proposition 2.4), and

$$\psi^\dagger(\xi)\psi^\dagger(-\xi) = \psi(-\xi^2), \qquad \xi \in \mathbf{C} \setminus \mathbf{R}. \tag{2.5}$$

PROPOSITION 2.1. *If $\psi(\xi)$ is nonnegative on $(0,\infty)$, and both $\psi(\xi)$ and $\xi/\psi(\xi)$ are increasing on $(0,\infty)$, then*

$$e^{-2\mathcal{C}/\pi}\sqrt{\psi(\xi^2)} \le \psi^\dagger(\xi) \le e^{2\mathcal{C}/\pi}\sqrt{\psi(\xi^2)}, \tag{2.6}$$

where $\mathcal{C} \approx 0.916$ is the Catalan constant. Note that $e^{2\mathcal{C}/\pi} \le 2$.

If, in addition, $\psi(\xi)$ is regularly varying at ∞, then

$$\psi^\dagger(\xi) \sim \sqrt{\psi(\xi^2)}, \qquad \xi \to \infty. \tag{2.7}$$

An analogous statement for $\xi \to 0$ holds for $\psi(\xi)$ regularly varying at 0.

In particular, (2.6) holds for any CBF. Likewise, (2.7) holds for any regularly varying CBF.

A result similar to (2.6) was obtained independently in [25], Proposition 3.7, while (2.7) for CBFs was derived in [22], Proposition 2.2.

PROOF. By the assumptions, we have

$$\psi(\xi^2)\min(1,\zeta^2) \le \psi(\xi^2\zeta^2) \le \psi(\xi^2)\max(1,\zeta^2), \qquad \xi,\zeta > 0. \tag{2.8}$$

It follows that

$$\begin{aligned}\psi^\dagger(\xi) &= \exp\biggl(\frac{1}{\pi}\int_0^\infty \frac{\log\psi(\xi^2\zeta^2)}{1+\zeta^2}\,d\zeta\biggr)\\ &\le \sqrt{\psi(\xi^2)}\exp\biggl(\frac{1}{\pi}\int_1^\infty \frac{\log\zeta^2}{1+\zeta^2}\,d\zeta\biggr) = e^{2\mathcal{C}/\pi}\sqrt{\psi(\xi^2)}.\end{aligned}$$

The lower bound is obtained in a similar manner.

The second statement of the proposition is proved in a very similar manner to Lemma 15 in [30]. Define an auxiliary function $h(\xi,\zeta) = \psi(\xi^2\zeta^2)/\psi(\xi^2)$. By (2.8) we have $|\log h(\xi,\zeta)| \le 2|\log\zeta|$, $\xi,\zeta > 0$. Since ψ is regularly varying at infinity, for some α, $\lim_{\xi\to\infty} h(\xi,\zeta) = \zeta^{2\alpha}$ for each $\zeta > 0$. Hence, by dominated convergence,

$$\lim_{\xi\to\infty}\int_0^\infty \frac{\log h(\xi,\zeta)}{1+\zeta^2}\,d\zeta = \int_0^\infty \frac{\log\zeta^{2\alpha}}{1+\zeta^2}\,d\zeta = 0.$$

It follows that

$$\lim_{\xi\to\infty}\biggl(\int_0^\infty \frac{\log\psi(\xi^2\zeta^2)}{1+\zeta^2}\,d\zeta - \frac{\pi}{2}\log\psi(\xi^2)\biggr) = 0,$$

and so finally $\lim_{\xi\to\infty}\psi^\dagger(\xi)/\sqrt{\psi(\xi^2)} = 1$, as desired. Regular variation at 0 is proved in a similar way. □

As in [30], for differentiable functions $\psi(\xi)$ with positive derivative, we define

$$\psi_\lambda(\xi) = \frac{1-\xi/\lambda^2}{1-\psi(\xi)/\psi(\lambda^2)}, \qquad \lambda > 0, \xi \in \mathbf{C} \setminus (-\infty, 0). \tag{2.9}$$

This definition is extended continuously by $\psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2\psi'(\lambda^2))$. Note that if $\psi(0) = 0$, then $\psi_\lambda(0) = 1$. For simplicity, we denote $\psi_\lambda^\dagger(\xi) = (\psi_\lambda)^\dagger(\xi)$. By [30], Lemma 2, if $\psi(\xi)$ is a CBF, then $\psi_\lambda(\xi)$ is a CBF for any $\lambda > 0$.

2.2. *Estimates for the Laplace transform.* This short section contains some rather standard estimates for the inverse Laplace transform.

PROPOSITION 2.2. *Let* $a > 0$, $c \geq 1$. *If* f *is nonnegative and* $f(x) \leq cf(a)\max(1, x/a)$ $(x > 0)$, *then for any* $\xi > 0$,

$$f(a) \geq \frac{\xi \mathcal{L} f(\xi)}{c(1 + (a\xi)^{-1}e^{-a\xi})}.$$

PROOF. We have

$$\begin{aligned}
\xi \mathcal{L} f(\xi) &= \int_0^a \xi e^{-\xi x} f(x)\,dx + \int_a^\infty \xi e^{-\xi x} f(x)\,dx \\
&\leq cf(a)\int_0^a \xi e^{-\xi x}\,dx + \frac{cf(a)}{a}\int_a^\infty \xi x e^{-\xi x}\,dx \\
&= cf(a)(1 - e^{-a\xi}) + \frac{cf(a)}{a\xi}(1 + a\xi)e^{-a\xi} = cf(a)\bigl(1 + (a\xi)^{-1}e^{-a\xi}\bigr),
\end{aligned}$$

as desired. □

PROPOSITION 2.3. *If* f *is nonnegative and increasing*, *then for* $a, \xi > 0$,

$$f(a) \leq e^{a\xi}\xi \mathcal{L} f(\xi).$$

PROOF. As before,

$$\begin{aligned}
\xi \mathcal{L} f(\xi) &= \int_0^a \xi e^{-\xi x} f(x)\,dx + \int_a^\infty \xi e^{-\xi x} f(x)\,dx \\
&\geq f(a)\int_a^\infty \xi e^{-\xi x}\,dx = f(a)e^{-a\xi},
\end{aligned}$$

as claimed. □

PROPOSITION 2.4. *If* f *is nonnegative and decreasing*, *then for* $a, \xi > 0$,

$$f(a) \leq \frac{\xi \mathcal{L} f(\xi)}{1 - e^{-a\xi}}.$$

PROOF. Again,

$$\xi \mathcal{L} f(\xi) = \int_0^a \xi e^{-\xi x} f(x)\,dx + \int_a^\infty \xi e^{-\xi x} f(x)\,dx$$
$$\geq f(a) \int_0^a \xi e^{-\xi x}\,dx = f(a)(1 - e^{-a\xi}),$$

as claimed. □

3. Suprema of general Lévy processes. We briefly recall the basic notions of the fluctuation theory for Lévy processes. Let L_t be the local time of the process X_t reflected at its supremum M_t, and denote by L_s^{-1} the right-continuous inverse of L_t, the ascending ladder-time process for X_t. This is a (possibly killed) subordinator, and $H_s = X(L_s^{-1}) = M(L_s^{-1})$ is another (possibly killed) subordinator, called the ascending ladder-height process. The Laplace exponent of the ascending ladder process, that is, the (possibly killed) bivariate subordinator (L_s^{-1}, H_s) $(s < L(\infty))$, is denoted by $\kappa(z, \xi)$. By [5], Corollary VI.10,

$$\kappa(z,\xi) = c \exp\biggl(\int_0^\infty \int_{[0,\infty)} (e^{-t} - e^{-zt-\xi x}) t^{-1} \mathbf{P}(X_t \in dx)\,dt\biggr), \tag{3.1}$$

where c is a normalization constant of the local time. Since our results are not affected by the choice of c, we assume that $c = 1$. We note that $\kappa(z, 0)$ is a Bernstein function of z, and also $z/\kappa(z, 0)$ is a Bernstein function (this follows from (3.1) by Frullani's integral; see [5], formula (VI.3) for the case when X_t is not a compound Poisson process). For a more in-depth account of the fluctuation theory, we refer the reader to [5, 13, 31]. In general, there is no closed-form formula for $\kappa(z, \xi)$. For a list of special cases, see [29] and the references therein. For a symmetric process which is not a compound Poisson process, we have $\kappa(z, 0) = \sqrt{z}$.

As usual, τ_x denotes the first passage time through a barrier at $x \geq 0$ for X_t (or for M_t). Following [5], for $x, z \geq 0$, we define

$$V^z(x) = \mathbf{E}\biggl(\int_0^\infty \exp(-z L_s^{-1}) \mathbf{1}_{[0,x)}(H_s)\,ds\biggr) = \mathbf{E}\biggl(\int_0^\infty e^{-zt} \mathbf{1}_{[0,x)}(M_t)\,dL_t\biggr).$$

For $z = 0$, we simply have $V^0(x) = \int_0^\infty \mathbf{P}(H_s < x)\,ds$, so that $V^0(x) = V(x)$ is the renewal function of the process H_s, studied in more detail for symmetric Lévy processes in Section 4. By [5], formula (VI.8),

$$\int_0^\infty e^{-zt} \mathbf{P}(M_t < x)\,dt = \frac{\kappa(z,0) V^z(x)}{z}, \qquad x, z \geq 0. \tag{3.2}$$

(Note that in [5], a weak inequality $M_t \leq x$ is used in the definition of $V^z(x)$.) Hence, for a symmetric process X_t which is not a compound Poisson process, we have

$$\int_0^\infty e^{-zt} \mathbf{P}(M_t < x)\,dt = \frac{V^z(x)}{\sqrt{z}}, \qquad x, z \geq 0. \tag{3.3}$$

This is a partial inverse of the double Laplace transform in (1.1); however, there is no known explicit formula for $V^z(x)$. For a different and, in a sense, more explicit partial inverse, see (4.2) below.

By [5], Section VI.4, the Laplace transform of $V^z(x)$ is $1/(\xi\kappa(z,\xi))$. Hence, when X_t is symmetric and it is not a compound Poisson process, the right-hand side of the Baxter–Donsker formula (1.1) can be written as $\sqrt{z}/(z\kappa(z,\xi))$; see [14], Corollary 9.7.

THEOREM 3.1. *Let X_t be a Lévy process, $M_t = \sup_{0\le s\le t} X_s$ and let $\kappa(z,\xi)$ be the bivariate Laplace exponent of its ascending ladder process. Suppose that*

$$K(s) = \int_s^\infty \frac{\kappa(z,0)}{z^2}\,dz < \infty, \qquad s > 0 \tag{3.4}$$

and that $\kappa(z,0)/z$ is unbounded (near 0). For $t, x > 0$, we have

$$\begin{aligned} \min\bigl(C_1, C_2(\kappa,t)\kappa(1/t,0)V(x)\bigr) &\le \mathbf{P}(M_t < x) \\ &\le \min\left(1, \frac{e}{e-1}\kappa(1/t,0)V(x)\right). \end{aligned} \tag{3.5}$$

Here

$$C_1 = \frac{e-1}{8e^2} \quad \textit{and} \quad C_2(\kappa,t) = \frac{zt}{2e},$$

where $z \in (0, 1/t)$ solves

$$\frac{\kappa(z,0)}{z} = \frac{4e^2}{e-1}K(1/t).$$

PROOF. The upper bound in (3.5) is a direct consequence of (3.2) and Proposition 2.4 with $\xi = 1/t$.

Following [5], Lemma VI.21, we find a lower bound for $V^z(x)$. We have

$$\begin{aligned} V(x) &= \mathbf{E}\left(\int_0^\infty \mathbf{1}_{[0,x)}(M_t)\,dL_t\right) \\ &\le e\mathbf{E}\left(\int_0^{1/z} e^{-zt}\mathbf{1}_{[0,x)}(M_t)\,dL_t\right) + \mathbf{E}\left(\int_{1/z}^\infty \mathbf{1}_{[0,x)}(M_t)\,dL_t\right), \end{aligned}$$

which implies

$$eV^z(x) \ge V(x) - \mathbf{E}\left(\int_{1/z}^\infty \mathbf{1}_{[0,x)}(M_t)\,dL_t\right). \tag{3.6}$$

Let $\sigma_z = \inf\{t \ge 1/z : X_t = M_t\} = L^{-1}(L_{1/z})$; σ_z is a stopping time. Since the support of the measure dL_t is contained in the set $\{t : X_t = M_t\}$ of zeros of the reflected process, we have

$$\mathbf{E}\Big(\int_{1/z}^{\infty} \mathbf{1}_{[0,x)}(M_t)\,dL_t\Big) = \mathbf{E}\Big(\int_{\sigma_z}^{\infty} \mathbf{1}_{[0,x)}(M_t)\,dL_t; M_{1/z} < x\Big)$$
$$\le \mathbf{E}\Big(\int_{\sigma_z}^{\infty} \mathbf{1}_{[0,x)}(M_t - M_{\sigma_z})\,dL_t; M_{1/z} < x\Big).$$

Next, observe that $M_{\sigma_z} = X_{\sigma_z}$, so that

$$M_t - M_{\sigma_z} = \sup_{s \le t-\sigma_z} (X_{\sigma_z+s} - X_{\sigma_z}), \qquad t \ge \sigma_z.$$

Hence,

$$\mathbf{E}\Big(\int_{1/z}^{\infty} \mathbf{1}_{[0,x)}(M_t)\,dL_t\Big)$$
$$\le \mathbf{E}\Big(\int_{\sigma_z}^{\infty} \mathbf{1}_{[0,x)}\Big(\sup_{s \le t-\sigma_z} (X_{\sigma_z+s} - X_{\sigma_z})\Big)\,dL_t; M_{1/z} < x\Big)$$
$$= \mathbf{E}\Big(\int_{0}^{\infty} \mathbf{1}_{[0,x)}\Big(\sup_{s \le u} (X_{\sigma_z+s} - X_{\sigma_z})\Big)\,d(L_{\sigma_z+u} - L_{\sigma_z}); M_{1/z} < x\Big).$$

Since $\sigma_z \ge 1/z$, by the strong Markov property,

$$\mathbf{E}\Big(\int_{1/z}^{\infty} \mathbf{1}_{[0,x)}(M_t)\,dL_t\Big) \le \mathbf{P}(M_{1/z} < x)\mathbf{E}\Big(\int_{0}^{\infty} \mathbf{1}_{[0,x)}(M_u)\,dL_u\Big)$$
$$= \mathbf{P}(M_{1/z} < x)V(x),$$

which, by (3.6), yields

$$V^z(x) \ge \frac{(1 - \mathbf{P}(M_{1/z} < x))V(x)}{e} = \frac{\mathbf{P}(M_{1/z} \ge x)V(x)}{e}.$$

Let $k > 0$. By (3.2) and the already proved upper bound of (3.5),

$$V^z(x)\kappa(z,0) = z\int_0^{k/z} e^{-zt}\mathbf{P}(M_t < x)\,dt + z\int_{k/z}^{\infty} e^{-zt}\mathbf{P}(M_t < x)\,dt$$
$$\le \frac{e}{e-1} V(x) z\int_0^{k/z} e^{-zt}\kappa(1/t,0)\,dt + \mathbf{P}(M_{k/z} < x).$$

The last two estimates give

$$\mathbf{P}(M_{k/z} < x) \ge \frac{\kappa(z,0)\mathbf{P}(M_{1/z} \ge x)V(x)}{e} - \frac{e}{e-1} V(x) z\int_0^{k/z} \kappa(1/t,0)\,dt$$
$$= \frac{V(x)\kappa(z,0)}{e}\Big(\mathbf{P}(M_{1/z} \ge x) - \frac{e^2}{e-1}\frac{zK(z/k)}{\kappa(z,0)}\Big). \tag{3.7}$$

Fix $\varepsilon \in (0,1)$ (later we choose $\varepsilon = 1/4$). Note that the function $\kappa(z,0)/z$ is continuous, decreasing and unbounded. Hence, it maps the interval $(0, 1/t]$ onto the interval $[t\kappa(1/t,0),\infty)$. Furthermore, $\kappa(z,0)$ is increasing, so that $K(z) \geq \kappa(z,0)/z$. In particular, $\frac{e^2}{\varepsilon(e-1)}K(1/t) > K(1/t) \geq t\kappa(1/t,0)$. It follows that we can choose $z = z(t) < 1/t$ such that

$$\frac{\kappa(z,0)}{z} = \frac{e^2}{\varepsilon(e-1)}K(1/t).$$

Setting $k = zt < 1$, the above equality can be rewritten as

$$\frac{e^2}{e-1}\frac{zK(z/k)}{\kappa(z,0)} = \varepsilon. \tag{3.8}$$

Suppose now that $V(x)\kappa(z,0) \leq \varepsilon(e-1)/e$. Then, by the upper bound of (3.5), we have $\mathbf{P}(M_{1/z} \geq x) = 1 - \mathbf{P}(M_{1/z} < x) \geq 1 - \varepsilon$. This, (3.7) and (3.8) give

$$\mathbf{P}(M_t < x) = \mathbf{P}(M_{k/z} < x) \geq \frac{V(x)\kappa(z,0)}{e}(1-2\varepsilon).$$

This estimate holds for $t \geq t_0$, where $V(x)\kappa(z(t_0),0) = \varepsilon(e-1)/e$ [here we use continuity of $\kappa(z(t),0)$ as a function of t]. Hence, by monotonicity of $\mathbf{P}(M_t < x)$ in t,

$$\mathbf{P}(M_t < x) \geq \min\left(\frac{\varepsilon(1-2\varepsilon)(e-1)}{e^2}, \frac{(1-2\varepsilon)V(x)\kappa(z,0)}{e}\right).$$

The lower bound in (3.5) follows by taking $\varepsilon = 1/4$ and using the inequality $\kappa(z,0) = \kappa(k/t,0) \geq k\kappa(1/t,0)$. □

To formulate the next result we define the following *upper scaling conditions*:

$$\text{for some } \varrho \in (0,1) \text{ and } c > 0, \qquad \frac{\kappa(z_2,0)}{\kappa(z_1,0)} \leq c\frac{z_2^\varrho}{z_1^\varrho} \quad \text{when } 0 < z_1 < z_2 < 1, \tag{3.9}$$

$$\text{for some } \varrho \in (0,1) \text{ and } c > 0, \qquad \frac{\kappa(z_2,0)}{\kappa(z_1,0)} \leq c\frac{z_2^\varrho}{z_1^\varrho} \quad \text{when } 1 < z_1 < z_2. \tag{3.10}$$

Observe that condition (3.10) implies that for any $z^* > 0$, there is c^* such that

$$\frac{\kappa(z_2,0)}{\kappa(z_1,0)} \leq c^*\frac{z_2^\varrho}{z_1^\varrho} \qquad \text{when } z^* < z_1 < z_2. \tag{3.11}$$

COROLLARY 3.2. *Let X_t be a Lévy process, $M_t = \sup_{0\leq s\leq t} X_s$ and let $\kappa(z,\xi)$ be the bivariate Laplace exponent of its ascending ladder process. If $\kappa(z,0)$ satisfies condition* (3.9) *with $0 < \varrho < 1$ and the integral $\int_1^\infty \kappa(z,0)z^{-2}\,dz$ is finite, then*

$$C(\kappa)\min\bigl(1, \kappa(1/t,0)V(x)\bigr) \leq \mathbf{P}(M_t < x) \leq \min\bigl(1, 2\kappa(1/t,0)V(x)\bigr), \tag{3.12}$$

for every $x > 0$ *and* $t \geq 1$. *If* $\kappa(z, 0)$ *satisfies* (3.10) *with* $0 < \varrho < 1$ *and* $\lim_{z \to 0} z/\kappa(z, 0) = 0$, *then* (3.12) *holds for* $x > 0$ *and* $t \leq 1$.

In particular, if $\kappa(z, 0)$ *satisfies both* (3.9) *and* (3.10), *that is, there are* $c > 0$ *and* $\varrho \in (0, 1)$ *such that* $\kappa(\lambda z, 0) \leq c\lambda^{\varrho}\kappa(z, 0)$ *for* $\lambda \geq 1$ *and* $z > 0$, *then* (3.12) *is true for every* $x > 0$ *and* $t > 0$.

PROOF. We begin with the first part of the statement. By condition (3.9),

$$\kappa(z, 0) \leq c_1(\kappa)\left(\frac{z}{s}\right)^{\varrho}\kappa(s, 0), \qquad s \leq z \leq 1.$$

In particular, $\kappa(s, 0)/s$ is unbounded. Furthermore, using also finiteness of the integral $\int_1^{\infty} \kappa(z, 0)z^{-2}\, dz$, we obtain

$$K(s) \leq c_2(\kappa)\frac{\kappa(s, 0)}{s}, \qquad s \leq 1. \tag{3.13}$$

This implies that the assumptions of Theorem 3.1 are satisfied.

Let $t \geq 1$ and define $z = z(t) \in (0, 1/t)$ as in Theorem 3.1. By condition (3.9) we have

$$\frac{\kappa(1/t, 0)}{\kappa(z, 0)} \leq \frac{c_3(\kappa)}{(zt)^{\varrho}}.$$

By definition of z and (3.13) (with $s = 1/t$), we have

$$\frac{1}{z} = \frac{4e^2}{e-1}\frac{K(1/t)}{\kappa(z, 0)} \leq \frac{4e^2 c_2(\kappa)c_3(\kappa)}{e-1}\frac{t}{(tz)^{\varrho}},$$

which gives $zt \geq c_4(\kappa)$. Hence, the constant C_2 in Theorem 3.1 satisfies $C_2 = zt/(2e) \geq c_4(\kappa)/(2e)$. This ends the proof of the first part.

The second part can be justified in a similar way, since condition (3.10) implies that

$$K(s) \leq c_5(\kappa)\frac{\kappa(s, 0)}{s}, \qquad s \geq 1.$$

Moreover, for $t < 1$ and $z = z(t)$ selected according to Theorem 3.1 we have $z(1) \leq z(t) < 1/t$. Applying (3.10) [with $z^* = z(1)$], we obtain

$$\frac{\kappa(1/t, 0)}{\kappa(z, 0)} \leq \frac{c_6(\kappa)}{(zt)^{\varrho}}, \qquad z \leq \frac{1}{t}.$$

Finally, the last statement is a direct consequence of the previous ones. □

REMARK 3.3. Due to Potter's theorem ([8], Theorem 1.5.6) condition (3.9) is implied by regular variation of $\kappa(z, 0)$ at zero with index $0 < \varrho^* < 1$. Likewise, condition (3.10) is implied by regular variation of $\kappa(z,0)$ at ∞ with index $0 < \varrho^* < 1$.

In the second part of the above corollary the assumption $\lim_{z\to 0} z/\kappa(z,0) = 0$ can be removed at the expence that the lower bound holds for $t \le t_0$, where $t_0 = t_0(\kappa)$ is sufficiently small. This is due to the fact that since $\lim_{t\searrow 0} K(1/t) = 0$, $z = z(t)$ in Theorem 3.1 is well defined for t small enough.

By the results of [5], Theorem VI.14 and [6], the regular variation of order $\varrho \in (0,1)$ of $\kappa(z,0)$ at 0 or at ∞ is equivalent to the existence of the limit of $\mathbf{P}(X_t \ge 0)$ as $t \to \infty$ or $t \to 0^+$, respectively. Hence, Corollary 3.2 implies the following result.

COROLLARY 3.4. *Let X_t be a Lévy process and $M_t = \sup_{0\le s\le t} X_s$. If*

$$\lim_{t\to\infty} \mathbf{P}(X_t \ge 0) \in (0,1) \quad \text{and} \quad \limsup_{t\to 0^+} \mathbf{P}(X_t \ge 0) < 1,$$

then (3.12) holds for $x > 0$ and $t \ge 1$. If

$$\lim_{t\to 0^+} \mathbf{P}(X_t \ge 0) \in (0,1) \quad \text{and} \quad \limsup_{t\to \infty} \mathbf{P}(X_t \ge 0) < 1,$$

then (3.12) is true for $x > 0$ and $t \le 1$. Finally, if

$$\lim_{t\to\infty} \mathbf{P}(X_t \ge 0) \in (0,1) \quad \text{and} \quad \lim_{t\to 0^+} \mathbf{P}(X_t \ge 0) \in (0,1),$$

then (3.12) holds for every $x > 0$ and $t > 0$.

PROOF. We only need to verify that $\kappa(z,0)/z^2$ is integrable at infinity, and that $\lim_{z\to 0^+}(z/\kappa(z,0)) = 0$. In each of the cases, there is $\varepsilon > 0$ such that $\mathbf{P}(X_t \ge 0) \le 1-\varepsilon$ for all $t > 0$. Therefore, by (3.1) and the Frullani integral, $\kappa(z,0) \le z^{1-\varepsilon}$ for $z \ge 1$, and $\kappa(z,0) \ge z^{1-\varepsilon}$ when $0 < z < 1$. The result follows. □

REMARK 3.5. The uniform estimates of Corollary 3.4 complement the existing results from [17] about the asymptotic behavior of $\mathbf{P}(M_t < x)$, where it was shown that

$$\lim_{t\to\infty} \frac{\sqrt{\pi}}{\kappa(1/t,0)} \mathbf{P}(M_t < x) = V(x),$$

under the assumption that $\kappa(z,0)$ is regularly varying at zero with index $\varrho \in (0,1)$.

4. Suprema of symmetric Lévy processes. In this section we assume that X_t is a *symmetric* Lévy process with Lévy–Khintchin exponent $\Psi(\xi)$. In a rather general setting, we can invert the Laplace transform in time variable in (1.1).

THEOREM 4.1. *Suppose that X_t is a symmetric Lévy process with Lévy–Khintchin exponent $\Psi(\xi)$. Suppose that $\Psi(\xi)$ is increasing in $\xi > 0$. If $M_t = \sup_{0 \le s \le t} X_s$, then*

$$\begin{aligned}\mathbf{E}e^{-\xi M_t} &= \frac{1}{\pi}\int_0^\infty \frac{\xi \Psi'(\lambda)}{(\lambda^2+\xi^2)\sqrt{\Psi(\lambda)}} \\ &\qquad \times \exp\biggl(\frac{1}{\pi}\int_0^\infty \frac{\xi \log(\lambda^2-\zeta^2)/(\Psi(\lambda)-\Psi(\zeta))}{\xi^2+\zeta^2}\,d\zeta\biggr) e^{-t\Psi(\lambda)}\,d\lambda.\end{aligned} \tag{4.1}$$

Since $\mathbf{P}(M_t < x) = \mathbf{P}(\tau_x > t)$, the following integrated form of (4.1) is sometimes more convenient.

COROLLARY 4.2. *With the notation and assumptions of Theorem 4.1,*

$$\begin{aligned}&\int_0^\infty e^{-\xi x}\mathbf{P}(\tau_x > t)\,dx \\ &\quad = \frac{\mathbf{E}e^{-\xi M_t}}{\xi} \\ &\quad = \frac{1}{\pi}\int_0^\infty \frac{\Psi'(\lambda)}{(\lambda^2+\xi^2)\sqrt{\Psi(\lambda)}} \\ &\qquad \times \exp\biggl(\frac{1}{\pi}\int_0^\infty \frac{\xi \log((\lambda^2-\zeta^2)/(\Psi(\lambda)-\Psi(\zeta)))}{\xi^2+\zeta^2}\,d\zeta\biggr) e^{-t\Psi(\lambda)}\,d\lambda.\end{aligned} \tag{4.2}$$

PROOF OF THEOREM 4.1. Let $\psi(\xi) = \Psi(\sqrt{\xi})$ for $\xi > 0$. For any $z \in \mathbf{C} \setminus (-\infty, 0]$ and $\xi > 0$, we define [see (1.1) and (2.3)]

$$\begin{aligned}\varphi(\xi, z) &= \sqrt{z}\exp\biggl(-\frac{1}{\pi}\int_0^\infty \frac{\xi \log(z+\Psi(\zeta))}{\xi^2+\zeta^2}\,d\zeta\biggr) \\ &= \exp\biggl(-\frac{1}{\pi}\int_0^\infty \frac{\xi \log(1+\psi(\zeta^2)/z)}{\xi^2+\zeta^2}\,d\zeta\biggr).\end{aligned}$$

For any $\xi > 0$, the function $\varphi(\xi, z)$ is positive and increasing in $z \in (0, \infty)$. As $z \to 0$ or $z \to \infty$, $\varphi(\xi, z)$ converges to 0 and 1, respectively. Furthermore, if $\operatorname{Im} z > 0$, then $\arg(1 + \psi(\zeta^2)/z) \in (-\pi, 0)$ for all $\zeta > 0$, and therefore

$$\arg \varphi(\xi, z) = -\frac{1}{\pi}\int_0^\infty \frac{\xi \arg(1+\psi(\zeta^2)/z)}{\xi^2+\zeta^2}\,d\zeta \in (0, \pi/2).$$

Hence, for any $\xi > 0$, $\varphi(\xi, z)$ [and even $(\varphi(\xi, z))^2$] is a complete Bernstein function of z. Note that the continuous boundary limit $\varphi^+(\xi, -z)$ exists for $z > 0$: if

$z = \psi(\lambda^2)$, or $\lambda = \sqrt{\psi^{-1}(z)}$, then

$$\begin{aligned}\varphi^+(\xi, -z) &= \exp\biggl(-\frac{1}{\pi}\int_0^\infty \frac{\xi \log^-(1 - \psi(\zeta^2)/\psi(\lambda^2))}{\xi^2 + \zeta^2}\,d\zeta\biggr) \\ &= \exp\biggl(-\frac{1}{\pi}\int_0^\infty \frac{\xi \log|1 - \psi(\zeta^2)/\psi(\lambda^2)|}{\xi^2 + \zeta^2}\,d\zeta + i\int_\lambda^\infty \frac{\xi}{\xi^2 + \zeta^2}\,d\zeta\biggr) \\ &= \exp\biggl(\frac{1}{\pi}\int_0^\infty \frac{\xi(\log \psi_\lambda(\zeta^2) - \log|1 - \zeta^2/\lambda^2|)}{\xi^2 + \zeta^2}\,d\zeta + i \arctan\frac{\xi}{\lambda}\biggr);\end{aligned}$$

see (2.9) for the notation. Here $\log^-$ denotes the boundary limit on $(-\infty, 0)$ approached from below, $\log^-(-\zeta) = -i\pi/2 + \log\zeta$ for $\zeta > 0$. The function $\log|1 - \zeta^2/\lambda^2|$ is harmonic in the upper half-plane $\operatorname{Im}\zeta > 0$, so that

$$\frac{1}{\pi}\int_0^\infty \frac{\xi \log|1 - \zeta^2/\lambda^2|}{\xi^2 + \zeta^2}\,d\zeta = \frac{1}{2}\log\biggl(1 + \frac{\xi^2}{\lambda^2}\biggr).$$

Furthermore, $\exp(i \arctan(\xi/\lambda)) = (\lambda + i\xi)/\sqrt{\lambda^2 + \xi^2}$. Therefore, with $z = \psi(\lambda^2)$,

$$\begin{aligned}\varphi^+(\xi, -z) &= \frac{\lambda(\lambda + i\xi)}{\lambda^2 + \xi^2}\exp\biggl(\frac{1}{\pi}\int_0^\infty \frac{\xi \log \psi_\lambda(\zeta^2)}{\xi^2 + \zeta^2}\,d\zeta\biggr) \\ &= \frac{\lambda(\lambda + i\xi)\psi_\lambda^\dagger(\xi)}{\lambda^2 + \xi^2};\end{aligned} \tag{4.3}$$

see (2.3) for the notation. Note that if $\psi(\xi)$ is bounded on $(0, \infty)$ and $z \ge \sup_{\xi > 0}\psi(\xi)$, then $\varphi^+(\xi, -z)$ is real.

By (1.1), $\varphi(\xi, z)/z$ is the double Laplace transform of the distribution of M_t. But for all $\xi > 0$, $\varphi(\xi, z)/z$ is a Stieltjes function of z. Therefore, by (2.2),

$$\begin{aligned}\frac{\varphi(\xi, z)}{z} &= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{\varphi^+(\xi, -\zeta)}{\zeta}\frac{1}{z + \zeta}\,d\zeta \\ &= \frac{1}{\pi}\int_0^\infty 2\lambda\psi'(\lambda^2)\operatorname{Im}\frac{\varphi^+(\xi, -\psi(\lambda^2))}{\psi(\lambda^2)}\frac{1}{z + \psi(\lambda^2)}\,d\lambda \\ &= \frac{2}{\pi}\int_0^\infty \frac{\lambda\psi'(\lambda^2)}{\psi(\lambda^2)}\frac{\lambda\xi\psi_\lambda^\dagger(\xi)}{\lambda^2 + \xi^2}\frac{1}{z + \psi(\lambda^2)}\,d\lambda.\end{aligned}$$

Note that the second equality holds true also when $\psi(\xi)$ is bounded. Since $1/(z + \psi(\lambda^2)) = \int_0^\infty e^{-t\psi(\lambda^2)}e^{-zt}\,dt$, we have

$$\frac{\varphi(\xi, z)}{z} = \int_0^\infty \biggl(\frac{2}{\pi}\int_0^\infty \frac{\lambda\psi'(\lambda^2)}{\psi(\lambda^2)}\frac{\lambda\xi\psi_\lambda^\dagger(\xi)}{\lambda^2 + \xi^2}e^{-t\psi(\lambda^2)}\,d\lambda\biggr)e^{-zt}\,dt.$$

The theorem follows by the uniqueness of the Laplace transform. □

Let $V(x) = V^0(x)$ be the renewal function for the ascending ladder-height process H_s corresponding to X_t; see Section 3 for the definition. When X_t satisfies the *absolute continuity condition* [e.g., if $1/(1 + \Psi(\xi))$ is integrable in ξ], then $V(x)$ is the (unique up to a multiplicative constant) increasing harmonic function for X_t on $(0, \infty)$, and $V'(x)$ is the decreasing harmonic function for X_t on $(0, \infty)$; cf. [35]. It is known ([5], formula (VI.6)) that for $\xi > 0$,

$$\mathcal{L}V(\xi) = \frac{1}{\xi\kappa(0, \xi)},$$

Moreover, if X_t is not a compound Poisson process, then by [14], Corollary 9.7,

$$\kappa(0, \xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log \Psi(\zeta)}{\xi^2 + \zeta^2}\, d\zeta\right) = \psi^\dagger(\xi),$$

where $\Psi(\xi) = \psi(\xi^2)$; see (2.3) for the notation. Clearly, we have $\mathcal{L}V'(\xi) = \xi\mathcal{L}V(\xi) = 1/\psi^\dagger(\xi)$; here V' is the distributional derivative of V on $[0, \infty)$. We remark that when X_t is a compound Poisson process, then, also by [14], Corollary 9.7,

$$\kappa(0, \xi) = c\psi^\dagger(\xi) \qquad \text{with } c = \exp\left(-\frac{1}{2}\int_0^\infty \frac{1 - e^{-t}}{t}\mathbf{P}(X_t = 0)\, dt\right). \tag{4.4}$$

For simplicity, we state the next three results only for the case when X_t is not a compound Poisson process. However, extensions for compound Poisson processes are straightforward due to (4.4).

As an immediate consequence of Proposition 2.1 and Karamata's Tauberian theorem ([8], Theorem 1.7.1), we obtain the following result, which in the case of complete Bernstein functions was derived in Proposition 2.7 of [22].

PROPOSITION 4.3. *Let $\Psi(\xi)$ be the Lévy–Khintchin exponent of a symmetric Lévy process X_t, which is not a compound Poisson process, and suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$. If $\Psi(\xi)$ is regularly varying at ∞, then V is regularly varying at 0 and $\Gamma(1+\alpha)V(x) \sim 1/\sqrt{\Psi(1/x)}$ as $x \to 0$. Similarly, if $\Psi(\xi)$ is regularly varying at 0, then $\Gamma(1+\alpha)V(x) \sim 1/\sqrt{\Psi(1/x)}$ as $x \to \infty$.*

Another consequence of Proposition 2.1 is a uniform estimate of the renewal function; see also Proposition 3.9 of [25].

THEOREM 4.4. *Let $\Psi(\xi)$ be the Lévy–Khintchin exponent of a symmetric Lévy process X_t, which is not a compound Poisson process, and suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$. Then*

$$\frac{1}{5}\frac{1}{\sqrt{\Psi(1/x)}} \le V(x) \le 5\frac{1}{\sqrt{\Psi(1/x)}}, \qquad x > 0.$$

PROOF. Let $\psi(\xi) = \Psi(\sqrt{\xi})$ for $\xi > 0$. By Proposition 2.1, we obtain $e^{-2\mathcal{C}/\pi}/\sqrt{\xi^2\psi(\xi^2)} \le \mathcal{L}V(\xi) \le e^{2\mathcal{C}/\pi}/\sqrt{\xi^2\psi(\xi^2)}$, $\xi > 0$. Since V is increasing, Proposition 2.3 gives

$$V(x) \le \frac{e\mathcal{L}V(1/x)}{x} \le \frac{e^{1+2\mathcal{C}/\pi}}{\sqrt{\psi(1/x^2)}} \le \frac{5}{\sqrt{\psi(1/x^2)}}.$$

Furthermore, using subadditivity and monotonicity of V (see [5], Section III.1), for $x = ka + r$ ($k \ge 0$, $r \in [0, a)$) we obtain $V(x) \le kV(a) + V(r) \le (k+1)V(a)$. It follows that $V(x) \le 2V(a)\max(1, x/a)$ for all $a, x > 0$, and so, by Proposition 2.2,

$$V(x) \ge \frac{\mathcal{L}V(1/x)}{2x(1+e^{-1})} \ge \frac{1}{2(1+e^{-1})e^{2\mathcal{C}/\pi}\sqrt{\psi(1/x^2)}} \ge \frac{1}{5\sqrt{\psi(1/x^2)}},$$

as desired. □

We remark that when V is a concave function on $(0,\infty)$ (e.g., when ψ is a complete Bernstein function, see below), then clearly $V(x) \le \max(1, x/a)V(a)$, so that the lower bound in Theorem 4.4 holds with constant $2/5$ instead of $1/5$.

If $\psi(\xi)$ is a complete Bernstein function [CBF, see (2.1)], then $\psi^\dagger(\xi)$ and $\xi/\psi^\dagger(\xi)$ are CBFs, and hence $1/\psi^\dagger(\xi)$ is a Stieltjes function; see (2.2). Therefore, $V'(x)$ is a completely monotone function on $(0,\infty)$, and $V(x)$ is a Bernstein function; see [34] for the relation between completely monotone, Bernstein, complete Bernstein and Stieltjes functions. More precisely, we have the following result.

PROPOSITION 4.5. *Let $\Psi(\xi)$ be the Lévy–Khintchin exponent of a symmetric Lévy process X_t, which is not a compound Poisson process, and suppose that $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function ψ. Then V is a Bernstein function, and*

$$V(x) = bx + \frac{1}{\pi}\int_{0^+}^\infty \operatorname{Im}\left(-\frac{1}{\psi^+(-\xi^2)}\right)\frac{\psi^\dagger(\xi)}{\xi}(1 - e^{-x\xi})\,d\xi, \qquad x > 0, \tag{4.5}$$

$$V'(x) = b + \frac{1}{\pi}\int_{0^+}^\infty \operatorname{Im}\left(-\frac{1}{\psi^+(-\xi^2)}\right)\psi^\dagger(\xi)e^{-x\xi}\,d\xi, \qquad x > 0, \tag{4.6}$$

where $b = \lim_{\xi\to 0^+}(\xi/\sqrt{\psi(\xi^2)})$.

As explained after formula (2.2), the expression $\operatorname{Im}(-1/\psi^+(-\xi^2))\,d\xi$ in (4.5) and (4.6) should be understood in the distributional sense, as a weak limit of measures $\operatorname{Im}(-1/\psi(-\xi^2 + i\varepsilon))\,d\xi$ on $(0,\infty)$ as $\varepsilon \to 0^+$. The measure $\operatorname{Im}(-1/\psi^+(-\xi))\,d\xi$ has an atom of mass πb at 0, and this atom is not included in the integrals from 0^+ to ∞ in (4.5) and (4.6).

PROOF. Since $1/\psi^\dagger(\xi)$ is a Stieltjes function, it has the form (2.2),

$$\mathcal{L}V'(\xi) = \frac{1}{\psi^\dagger(\xi)} = a + \frac{b}{\xi} + \frac{1}{\pi} \int_{0+}^{\infty} \frac{1}{\xi + \zeta} \tilde{\mu}(d\zeta), \qquad \xi \in \mathbf{C} \setminus (-\infty, 0],$$

where, using (2.5),

$$\tilde{\mu}(d\xi) = -\operatorname{Im}\left(\frac{1}{(\psi^\dagger)^+(-\xi)}\right) d\xi = -\operatorname{Im}\left(\frac{\psi^\dagger(\xi)}{\psi^+(-\xi^2)}\right) d\xi$$

and

$$a = \lim_{\xi \to \infty} \frac{1}{\psi^\dagger(\xi)}, \qquad b = \lim_{\xi \to 0^+} \frac{\xi}{\psi^\dagger(\xi)}.$$

Using Proposition 2.1, we can express a and b in terms of ψ. Since ψ is unbounded, also $\psi^\dagger$ is unbounded [by (2.6)], and so in fact $a = 0$. In a similar way, if $\xi/\psi(\xi)$ converges to 0 as $\xi \to 0^+$, then (2.6) gives $\xi/\psi^\dagger(\xi) \to 0$, so that $b = 0$. When the limit of $\xi/\psi(\xi)$ is positive [since $\xi/\psi(\xi)$ is a CBF, the limit always exists], then ψ is regularly varying at 0, and so $b = \lim_{\xi \to 0^+}(\xi/\sqrt{\psi(\xi^2)})$, as desired. By the uniqueness of the Laplace transform,

$$V'(x) = b + \frac{1}{\pi} \int_{0+}^{\infty} e^{-x\xi} \tilde{\mu}(d\xi), \qquad x > 0.$$

The result follows by integration in x. □

Note that for a compound Poisson process, we have $a > 0$, so there is an extra positive constant in (4.5).

As a combination of Theorem 3.1 and Theorem 4.4, we obtain the following result.

THEOREM 4.6. *Let $\Psi(\xi)$ be the Lévy–Khintchin exponent of a symmetric Lévy process X_t. Suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$. If $M_t = \sup_{0 \le s \le t} X_s$, then for all $t, x > 0$,*

$$\frac{1}{100} \min\left(1, \frac{1}{200\sqrt{t\Psi(1/x)}}\right) \le \mathbf{P}(M_t < x) \le \min\left(1, \frac{10}{\sqrt{t\Psi(1/x)}}\right).$$

PROOF. When X_t is not a compound Poisson process, then the result follows from Theorems 3.1 and 4.4, and from $\kappa(z, 0) = \sqrt{z}$. Suppose that X_t is a compound Poisson process. For $\varepsilon > 0$ consider $X_t^\varepsilon = \varepsilon B_t + X_t$, where the Brownian motion B_t is independent of X_t. Then the Lévy–Khintchin exponent of X_t^ε equals to $\Psi_\varepsilon(\xi) = (\varepsilon\xi)^2 + \Psi(\xi)$. It is easy to check that $\xi^2/\Psi_\varepsilon(\xi)$ is increasing. Moreover, M_t^ε converges in distribution to M_t as $\varepsilon \to 0$. The result follows by the continuity of $\Psi(\xi)$. □

REMARK 4.7. Clearly, the condition "$\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$," in Theorem 4.4, Proposition 4.3 and Theorem 4.6, can be replaced with

$$0 < \Psi'(\xi) < \frac{2\Psi(\xi)}{\xi}, \qquad \xi > 0. \tag{4.7}$$

If $\Psi(\xi) = \psi(\xi^2)$, then (4.7) reads

$$0 < \psi'(\xi) < \frac{\psi(\xi)}{\xi}, \qquad \xi > 0. \tag{4.8}$$

Using the standard representation of Bernstein functions, it is easy to check that any Bernstein function $\psi(\xi)$ (not necessarily a complete one) satisfies (4.8). Hence, Theorem 4.6 applies to any *subordinate Brownian motion*: a process $X_t = B_{\eta_t}$, where $B(s)$ is the standard Brownian motion [with $\mathbf{E}(B_s) = 0$ and $\mathrm{Var}(B_s) = 2s$], η_t is a subordinator [with $\mathbf{E}(e^{-\xi\eta_t}) = e^{-t\psi(\xi)}$], and B_s and η_t are independent processes.

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M. KWAŚNICKI
INSTITUTE OF MATHEMATICS
AND COMPUTER SCIENCE
WROCŁAW UNIVERSITY OF TECHNOLOGY
UL. WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW
POLAND
AND
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
UL. ŚNIADECKICH 8
00-976 WARSZAWA
POLAND
E-MAIL: mateusz.kwasnicki@pwr.wroc.pl

J. MAŁECKI
INSTITUTE OF MATHEMATICS
AND COMPUTER SCIENCE
WROCŁAW UNIVERSITY OF TECHNOLOGY
UL. WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW
POLAND
AND
LAREMA
UNIVERSITÉ D'ANGERS
2 BD LAVOISIER
49045 ANGERS CEDEX 1
FRANCE
E-MAIL: jacek.malecki@pwr.wroc.pl

M. RYZNAR
INSTITUTE OF MATHEMATICS
AND COMPUTER SCIENCE
WROCŁAW UNIVERSITY OF TECHNOLOGY
UL. WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW
POLAND
E-MAIL: michal.ryznar@pwr.wroc.pl