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SUBLOGARITHMIC FLUCTUATIONS FOR INTERNAL DLA

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We consider internal diffusion limited aggregation in dimension larger than or equal to two. This is a random cluster growth model, where random walks start at the origin of the d-dimensional lattice, one at a time, and stop moving when reaching a site that is not occupied by previous walks. It is known that the asymptotic shape of the cluster is a sphere. When the dimension is two or more, we have shown in a previous paper that the inner (resp., outer) fluctuations of its radius is at most of order log(radius) [resp., $\log^2$(radius)]. Using the same approach, we improve the upper bound on the inner fluctuation to $\sqrt{\log(\text{radius})}$ when d is larger than or equal to three. The inner fluctuation is then used to obtain a similar upper bound on the outer fluctuation.

1. Introduction. This note is a companion to our paper [1]. There, we introduced a family of cluster growth models with a spherical asymptotic shape, but a wide diversity of shape fluctuations. Internal diffusion limited aggregation (internal DLA) was one member of this family. More precisely, the internal DLA cluster of volume N, say $A(N)$, is obtained inductively as follows. Initially, we assume that the explored region is empty, that is, $A(0) = \varnothing$. Then, consider N independent discrete-time random walks $S_1, \dots, S_N$ starting from 0. Assume $A(k-1)$ is obtained, and define

$$\tau_k = \inf\{t \geq 0 : S_k(t) \notin A(k-1)\} \quad \text{and} \quad A(k) = A(k-1) \cup \{S_k(\tau_k)\}. \tag{1.1}$$

We call explorers the random walks obeying the aggregation rule (1.1). We say that the kth explorer is *settled* on $S_k(\tau_k)$ after time τ_k, and is *unsettled* before time τ_k. The cluster $A(N)$ is interpreted as the positions of the N settled explorers.

In this paper we show how the tools developed in [1] lead in dimension $d \geq 3$ to sharper estimates on the fluctuations of $A(N)$ with respect to its spherical asymptotic shape. We keep the notation of [1], and recall the basic ones to make the paper as self-contained as possible. We denote with $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^d$. For any x in $\mathbb{R}^d$ and r in $\mathbb{R}$, set

$$B(x,r) = \{y \in \mathbb{R}^d : \|y - x\| < r\} \quad \text{and} \quad \mathbb{B}(x,r) = B(x,r) \cap \mathbb{Z}^d. \tag{1.2}$$

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For $\Lambda \subset \mathbb{Z}^d$, $|\Lambda|$ denotes the number of sites in Λ, and the boundary of Λ is $\partial\Lambda = \{z \notin \Lambda : \exists y \in \Lambda, \|y - z\| = 1\}$. For a simple random walk, let $H(\Lambda)$ denotes its first hitting time of Λ. The inner error $\delta_I(n)$ is such that

$$n - \delta_I(n) = \sup\{r \geq 0 : \mathbb{B}(0, r) \subset A(|\mathbb{B}(0, n)|)\}. \tag{1.3}$$

Also, the outer error $\delta_O(n)$ is such that

$$n + \delta_O(n) = \inf\{r \geq 0 : A(|\mathbb{B}(0, n)|) \subset \mathbb{B}(0, r)\}. \tag{1.4}$$

Our main result is as follows.

PROPOSITION 1.1. *There are constants $\{\alpha_d, \beta_d, d \geq 3\}$ such that in dimension $d \geq 3$, with probability* 1,

$$\limsup \frac{\delta_I(n)}{\sqrt{\log(n)}} \leq \alpha_d \quad \textit{and} \quad \limsup \frac{\delta_O(n)}{\sqrt{\log(n)}} \leq \beta_d. \tag{1.5}$$

REMARK 1.2. For $d = 2$ we show, with similar computations, that there are constants α_2, β_2 such that, with probability 1,

$$\limsup \frac{\delta_I(n)}{\log(n)} \leq \alpha_2 \quad \text{and} \quad \limsup \frac{\delta_O(n)}{\log(n)} \leq \beta_2. \tag{1.6}$$

The inner error bound in (1.6) was already obtained in all dimensions in [1]. Recently, Jerison, Levine and Sheffield [2] established, in dimension two and with a different method, the estimates (1.6). Also, they announced in [2] that the approach they followed could be adapted in dimension $d \geq 3$ to get (1.5).

Let us describe the main steps. The inner error is at the heart of the argument. It is based on a large deviation estimate which refines our previous estimates, with interest of its own. For a real x, let $\lfloor x \rfloor$ be the integer part of x.

LEMMA 1.3. *Choose R and A large enough. Assume that $\lfloor AR^d \rfloor$ explorers lie initially on $\mathbb{B}(0, R/2)$. We call η the initial configuration of these explorers and $A(\eta)$ the cluster they produce. There are positive constants $\{\kappa_d, d \geq 2\}$ independent of R and A, such that when $d \geq 3$,*

$$P(\mathbb{B}(0, R) \not\subset A(\eta)) \leq \exp(-\kappa_d AR^2), \tag{1.7}$$

and when $d = 2$, we have

$$P(\mathbb{B}(0, R) \not\subset A(\eta)) \leq \exp\left(-\kappa_2 \frac{AR^2}{\log(R)}\right). \tag{1.8}$$

REMARK 1.4. The reason behind the previous lemma, in $d \geq 3$, is that out of $\lfloor AR^d \rfloor$ explorers, only about AR^2 eventually hit a fixed site on the boundary of $\mathbb{B}(0, R)$, so that it is only these very explorers that need to be pushed away from this very site. The cost should be proportional to AR^2.

For the outer error, we use a large deviation estimate symmetrical to Lemma 1.3 as well as our coupling between internal DLA and the *flashing process* of [1]. The latter large deviation estimate was recently proved by Jerison, Levine and Sheffield in [2].

LEMMA 1.5 (Lemma A of Jerison, Levine and Sheffield [2]). *For β and R positive reals, assume that $\lfloor \beta R^d \rfloor$ explorers lie initially outside $\mathbb{B}(0, R)$. We call η the initial configuration of these explorers and $A(\eta)$ the cluster they produce. There are positive constants $\{\kappa'_d, d \geq 2\}$, such that for β small enough, we have when $d \geq 3$,*

$$P\big(0 \in A(\eta)\big) \leq \exp\big(-\kappa'_d R^2\big), \tag{1.9}$$

whereas when $d = 2$, we have

$$P\big(0 \in A(\eta)\big) \leq \exp\left(-\kappa'_2 \frac{R^2}{\log(R)}\right). \tag{1.10}$$

We give an alternative proof of this result, based on estimating the probability of crossing a shell, while avoiding traps.

LEMMA 1.6. *Consider $d \geq 2$. Fix a positive real R, and start a random walk on $z \in \partial\mathbb{B}(0, 2R)$. There are positive constants $\{\kappa_d, a_d\}$ such that for any V subset of the shell $\mathcal{S} = \mathbb{B}(0, 2R) \setminus \mathbb{B}(0, R)$, we have*

$$P_z\big(H\big(\mathbb{B}(0, R)\big) < H\big(V^c\big)\big) \leq \exp\left(a_d - \kappa_d \frac{R}{\rho}\right) \qquad \textit{where } \rho^{d-1} = \frac{|V|}{R}. \tag{1.11}$$

REMARK 1.7. $V^c = \mathcal{S} \setminus V$ is interpreted as traps. Note that ρ is proportional to the radius of a cylinder of height R and volume $|V|$. We can also read (1.11) in the following way:

$$P_z\big(H\big(\mathbb{B}(0, R)\big) < H\big(V^c\big)\big) \leq \exp\left(a_d - \kappa_d \left(\frac{R^d}{|V|}\right)^{1/(d-1)}\right). \tag{1.12}$$

This shows that for (1.12) to be an effective inequality, one needs that $|V|$ be smaller than R^d. The power $1/(d-1)$ on $R^d/|V|$ in (1.12) is not important in proving Lemma 1.5. If one were willing to accept the weaker power $1/d$, then one would have the following simple heuristics in dimension $d \geq 3$. Let t denote the time the walk spends in the annulus of height R. On one hand, the central limit scaling yields that this probability of such a stay is of order $\exp(-cR^2/t)$. On the other hand, all this time should be spent on sites of V, and it is well known that the probability is of order $\exp(-\kappa_d t/|V|^{2/d})$. Putting together these opposite requirements, and optimizing over t, we find a statement weaker than (1.12), but sufficient for our present purpose:

$$P_z\big(H\big(\mathbb{B}(0, R)\big) < H\big(V^c\big)\big) \leq \exp\left(a_d - \kappa_d \left(\frac{R^d}{|V|}\right)^{1/d}\right). \tag{1.13}$$

Even though it is not written in [1], inequality (1.13) was the motivation behind the introduction of flashing processes in [1], which were basically used to bypass this type of estimate. In this paper we show how the use of flashing explorers leads easily to Lemma 1.6.

The rest of the paper is organized as follows. In Section 2, we enounce some known results: we recall the approach of Lawler, Bramson and Griffeath [5] and useful large deviation estimates. Then the inner error estimate is proved in Section 3. In Section 4, we show how a flashing process permits a simple control on the outer error. Finally, we have gathered in an Appendix the proof of the large deviations Lemmas 1.3, 1.5 and 1.6.

2. Prerequisites.

2.1. *Notation.* We recall some notation of [1]. The state space of configurations is $\mathbb{N}^{\mathbb{Z}^d}$, its elements are denoted η and they represent starting conditions for a set of explorers, or random walks. Two types of initial configurations play an important role here: (i) the configuration $n\mathbf{1}_{z^*}$ formed by n trajectories starting on a given site z^* and (ii) for $\Lambda \subset \mathbb{Z}^d$, the configuration $\mathbf{1}_\Lambda$ that we simply identify with Λ. For any configuration $\eta \in \mathbb{N}^{\mathbb{Z}^d}$, we write

$$|\eta| = \sum_{z\in\mathbb{Z}^d} \eta(z). \tag{2.1}$$

DEFINITION 2.1. Let $R \in \mathbb{R}_+ \cup \{\infty\}$. For $z \in \mathbb{B}(0, R) \cup \partial\mathbb{B}(0, R)$, we denote by $M_R(\eta, z)$ [resp., $W_R(\eta, z)$] the number of simple random walks (resp., explorers) initially on η that hit z when or before exiting $\mathbb{B}(0, R)$. Thus, when $z \in \partial\mathbb{B}(0, R)$, $M_R(\eta, z)$ [resp., $W_R(\eta, z)$] is the number of simple random walks (resp., explorers) which exit $\mathbb{B}(0, R)$ exactly on z.

REMARK 2.2. Note that trajectories of walkers and explorers can be coupled to be the same up to the settling time of the explorer, the walker then proceeding along its simple random walk trajectories.

As in [4] (Section 3), it is useful to stop explorers as they reach $\partial B(0, R)$, for some $R > 0$, and then to define $A_R(\eta)$ as the set of positions of settled explorers.

DEFINITION 2.3. Consider $R \in \mathbb{R} \cup \{\infty\}$. We set

$$\forall z \in \mathbb{B}(0, R) \qquad \tilde{M}_R(\eta, z) = W_R(\eta, z) + M_R\big(A_R(\eta), z\big). \tag{2.2}$$

Finally, for any function $F : \mathbb{Z}^d \to \mathbb{R}$ and subset $\Lambda \subset \mathbb{Z}^d$, we denote

$$F(\Lambda) = \sum_{z\in\Lambda} F(z).$$

2.2. *On a classical approach.* We recall the approach of Lawler, Bramson and Griffeath in [5]. Send $N = |\mathbb{B}(0,n)|$ explorers from the origin. The approach of [5] is based on the following observations. (i) If explorers did not settle, they would just be independent random walks; (ii) exactly one explorer occupies each site of the cluster. Then, observations (i) and (ii) imply that for any integer n and $z \in \mathbb{B}(0,n)$,

$$\tilde{M}_n(N\mathbb{1}_0, z) := W_n(N\mathbb{1}_0, z) + M_n\big(A_n(N), z\big) \overset{\text{law}}{\geq} M_n(N\mathbb{1}_0, z). \tag{2.3}$$

When $z \in \partial\mathbb{B}(0,n)$, inequality (2.3) becomes an equality,

$$W_n(N\mathbb{1}_0, z) + M_n\big(A_n(N), z\big) \overset{\text{law}}{=} M_n(N\mathbb{1}_0, z). \tag{2.4}$$

Note that for any set $\Lambda \subset \mathbb{B}(0,n)$, $M_n(\Lambda, z)$ is a sum of independent Bernoulli variables. Note also that $A_n(N) \subset \mathbb{B}(0,n)$ so that for any $z \in \mathbb{B}(0,n) \cup \partial\mathbb{B}(0,n)$

$$W_n(N\mathbb{1}_0, z) + M_n\big(\mathbb{B}(0,n), z\big) \geq \tilde{M}_n(N\mathbb{1}_0, z). \tag{2.5}$$

However, Lawler et al. did not use that $W_n(N\mathbb{1}_0, z)$ and $M_n(\mathbb{B}(0,n), z)$ were independent. They could only obtain a rough estimate on the lower tail of $W_n(N\mathbb{1}_0, z)$. This in turn gave some estimates on the inner error, which was used to derive bounds on the outer error, by using that the cluster covers $\mathbb{B}(0, n - \delta_I(n))$. In other words, from (2.4), and the definition of $\delta_I(n)$, for $R > n$ and $z \in \partial\mathbb{B}(0,R)$,

$$W_R(N\mathbb{1}_0, z) + M_R\big(\mathbb{B}\big(0, n - \delta_I(n)\big), z\big) \leq \tilde{M}_R(N\mathbb{1}_0, z). \tag{2.6}$$

Therefore, if $\delta_I(n)$ is likely to be smaller than $r < n < R$, and $z \in \partial\mathbb{B}(0,R)$, we have

$$\mathbb{1}_{\{\delta_I(n) \leq r\}}\big(W_R(N\mathbb{1}_0, z) + M_R\big(\mathbb{B}(0, n-r), z\big) \leq \tilde{M}_R(N\mathbb{1}_0, z)\big). \tag{2.7}$$

We will also make use of the independence of the σ-fields generated by the events $\{\delta_I(n) \leq r\}$ and the random variables $W_R(N\mathbb{1}_0, z)$ on the one hand, and that generated by the random variable $M_R(\mathbb{B}(0, n-r), z)$ on the other.

2.3. *On sums of Bernoulli variables.* Let us now recall a simple tool of [1] in estimating deviations in view of (2.5) and (2.6). We first enounce the lower tail estimate.

LEMMA 2.4. *Suppose that a sequence of random variables* $\{W_n, M_n, L_n, \tilde{M}_n, n \in \mathbb{N}\}$, *and a sequence of real numbers* $\{c_n, n \in \mathbb{N}\}$, *satisfy for each* $n \in \mathbb{N}$,

$$W_n + L_n + c_n \geq \tilde{M}_n \quad \text{and} \quad \tilde{M}_n \overset{\text{law}}{=} M_n. \tag{2.8}$$

Assume that W_n *and* L_n *are independent, and that* L_n *and* M_n *both are sums of independent Bernoulli variables. Assume that the Bernoulli variables*

$\{Y_1^{(n)}, \dots, Y_{N_n}^{(n)}\}$ *whose sum is* L_n, *satisfy for some* $\kappa > 1$,

$$\text{(H1)} \quad \sup_n \sup_{i \le N_n} E\bigl[Y_i^{(n)}\bigr] < \frac{\kappa - 1}{\kappa},$$

$$\text{(H2)} \quad \mu_n := E[M_n] - E[L_n] \ge 0.$$

Then, for any n *in* $\mathbb{N}$ *and* ξ_n *in* $\mathbb{R}$, *we have for all* $\lambda \ge 0$,

$$P(W_n < \xi_n) \le \exp\left(-\lambda(\mu_n - \xi_n - c_n) + \frac{\lambda^2}{2}\left(\mu_n + \kappa \sum_{i=1}^{N_n} E\bigl[Y_i^{(n)}\bigr]^2\right)\right). \tag{2.9}$$

The upper tail estimate needs other assumptions.

LEMMA 2.5. *Assume for each* $n \in \mathbb{N}$, *and for an event* $\mathcal{A}_n$,

$$\mathbb{1}_{\mathcal{A}_n}(W_n + L_n) \le \tilde{M}_n \quad \textit{and} \quad \tilde{M}_n \overset{\text{law}}{=} M_n. \tag{2.10}$$

Assume that W_n *and* L_n *are independent*, $\mathbb{1}_{\mathcal{A}_n}$ *and* L_n *are independent and that* L_n *and* M_n *both are sums of independent Bernoulli variables such that* $\mu_n := E[M_n] - E[L_n] \ge 0$. *Then, for all* n *in* $\mathbb{N}$, ξ_n *in* $\mathbb{R}$ *and* $\lambda \in [0, \log 2]$,

$$P(W_n \ge \xi_n, \mathcal{A}_n) \le \exp\left(-\lambda(\xi_n - \mu_n) + \lambda^2\left(\mu_n + 4\sum_i E\bigl[Y_i^{(n)}\bigr]^2\right)\right). \tag{2.11}$$

REMARK 2.6. This lower (resp., upper) tail estimate turns out to be useful when $\xi_n + c_n$ is less than (resp., ξ_n is more than) $E[M_n] - E[L_n]$. By Lemmas 2.4 and 2.5 tail estimates reduce to a three-step strategy: (i) estimation of $E[M_n] - E[L_n]$; (ii) estimation of $\sum_i E^2[Y_i^{(n)}]$; (iii) optimization in λ. We emphasize that, in particular for the lower tail, this strategy does not require any control of the variance of W_n.

PROOF OF LEMMAS 2.4 AND 2.5. As in [1] this is an application of Lemma 2.3 of [1]. For the lower tail, using the exponential Chebyshev's inequality, the independence between W_n and L_n, formula (2.8) and centering the random variables, we get

$$P(W_n < \xi_n) \le \frac{E[e^{-\lambda(M_n - E[M_n])}]}{E[e^{-\lambda(L_n - E[L_n])}]} e^{-\lambda(E[M_n] - E[L_n] - \xi_n - c_n)}. \tag{2.12}$$

With, for all $t \in \mathbb{R}$, $f(t) = e^t - (1 + t)$ and $g(t) = (e^t - 1)^2$, by Lemma 2.3 of [1],

$$\begin{aligned} &\frac{E[e^{-\lambda(M_n - E[M_n])}]}{E[e^{-\lambda(L_n - E[L_n])}]} \\ &\quad \le \exp\left\{f(-\lambda)\bigl(E[M_n] - E[L_n]\bigr) + \frac{\kappa}{2} g(-\lambda) \sum_{i=1}^{N_n} E^2\bigl[Y_i^{(n)}\bigr]\right\}. \end{aligned} \tag{2.13}$$

We conclude by observing that for all $t \in \mathbb{R}$,

$$f(t) \le \frac{t^2}{2} e^{[t]_+} \quad \text{and} \quad g(t) \le t^2 e^{2[t]_+}, \tag{2.14}$$

where $[\cdot]_+$ stands for the positive part. The proof for the upper tail is similar. □

2.4. *On a discrete mean value property of Green's function.*

PROPOSITION 2.7. *Consider $d \ge 2$. There is a constant K_d such that, for any n and R with $n - n^{1/3} \le R \le n$ and z in $\mathbb{B}(0, R)$ with $n - \|z\| \le 1$,*

$$\left| |\mathbb{B}(0, R)| G_n(0, z) - \sum_{y \in \mathbb{B}(0,R)} G_n(y, z) \right| \le K_d. \tag{2.15}$$

PROOF. For $n - R$ large enough (larger than some constant that depends only on d) this is Theorem 5.2 of [1]. For $n = R$ this is a direct consequence of Lemmas 2 and 3 of [4]. For the remaining cases, one can use the same Lemmas in conjunction with Lemma 5 of [4]. □

REMARK 2.8. For the inner bound we will use Proposition 2.7 with $R = n$. For the outer bound we will use Proposition 2.7 with $n - R$ of order $\log n$ in dimension 2 and $\sqrt{\log n}$ in dimension $d \ge 3$.

3. Inner error.

3.1. *Exploration by waves.* We choose the following height sequence. For any positive integer n, $h(n) = \sqrt{\log(n)}$ in $d \ge 3$, and $h(n) = \log(n)$ in $d = 2$. We partition $\mathbb{Z}^d$ into concentric shells of heights $h(n)$. We define $\mathcal{S}_0 = \mathbb{B}(0, h(n))$, and for $k \ge 1$,

$$\mathcal{S}_k = \mathbb{B}(0, (k+1)h(n)) \setminus \mathbb{B}(0, kh(n)) \quad \text{and} \quad \Sigma_k = \partial\mathbb{B}(0, kh(n)). \tag{3.1}$$

We realize the internal DLA with $N = |\mathbb{B}(0, n)|$ explorers as an exploration wave process, where concentric shells are covered in turn; see Section 3 of [4].

We fix an integer k. For a site $z \in \Sigma_k$, we call *cell* centered on z, $\mathcal{C}(z) := \mathbb{B}(z, h(n)) \cap \mathcal{S}_k$, and we call *tile* centered on z, $\mathcal{T}(z) := \mathbb{B}(z, h(n)/2) \cap \Sigma_k$. A generic cell is denoted $\mathcal{C}$, and a generic tile is denoted $\mathcal{T}$. Note the obvious facts

$$\bigcup_{z \in \Sigma_k} \mathbb{B}(z, h(n)) \supset \mathcal{S}_k. \tag{3.2}$$

Before covering shell $\mathcal{S}_k$, one stops the unsettled explorers on Σ_k. Following [1], for $z \in \Sigma_k$, we prove that the $W_{kh(n)}(N\mathbb{1}_0, \mathcal{T})$ explorers stopped on $\mathcal{T} = \mathcal{T}(z)$ are likely to cover $\mathcal{C}(z)$, if $kh(n) \le n - Ah(n)$ for a large enough constant A. More precisely, we show that the probability of the event $\{\mathcal{S}_k \not\subset A(N)\}$ is smaller, for A large enough, than any given power of $1/n$. As first observed in [5],

$$W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) + M_{kh(n)}\big(\mathbb{B}\big(0, kh(n)\big), \mathcal{T}\big) \geq \tilde{M}_{kh(n)}(N\mathbb{1}_0, \mathcal{T}). \tag{3.3}$$

Since (3.3) corresponds to an inequality of type (2.8), we wish to use Lemma 2.4, but we need to ensure (H1) and (H2).

First, if $\tilde{\mathbb{B}}(r)$ denotes the sites of $\mathbb{B}(0, kh(n))$ at a distance less than r from $\mathcal{T}$, there is L and $\rho_d > 1$ (which depend only on the dimension), such that

$$\sup_{y \in \mathbb{B}(0,kh(n)) \setminus \tilde{\mathbb{B}}(Lh(n))} P_y\big(S\big(H(\Sigma_k)\big) \in \mathcal{T}\big) < \frac{\rho_d - 1}{\rho_d}; \tag{3.4}$$

(see Lemma 5.1 of [1]). Set $c_n = |\tilde{\mathbb{B}}(Lh(n))|$, and note that $c_n \leq c(Lh(n))^d$ for some constant c. From (3.3) we have

$$\begin{aligned} &W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) + M_{kh(n)}\big(\mathbb{B}\big(0, kh(n)\big) \setminus \tilde{\mathbb{B}}\big(Lh(n)\big), \mathcal{T}\big) \\ &\quad \geq \tilde{M}_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) - c_n. \end{aligned} \tag{3.5}$$

We will use Lemma 2.4 with $L_n = M_{kh(n)}(\mathbb{B}(0, kh(n)) \setminus \tilde{\mathbb{B}}(Lh(n)), \mathcal{T})$ and we note that (H1) is ensured by (3.4). Let us define

$$\mu(\mathcal{T}) = E\big[M_{kh(n)}(N\mathbb{1}_0, \mathcal{T})\big] - E\big[M_{kh(n)}\big(\mathbb{B}\big(0, kh(n)\big) \setminus \tilde{\mathbb{B}}_{\mathcal{T}}\big(Lh(n)\big), \mathcal{T}\big)\big]. \tag{3.6}$$

We consider the event that $\mathcal{S}_k$ is not covered, and use the bound

$$\begin{aligned} &P(\mathcal{S}_k \text{ not covered}) \\ &\quad \leq P\big(\exists \mathcal{T} \subset \Sigma_k : W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) < \tfrac{1}{3}\mu(\mathcal{T})\big) \\ &\qquad + P\big(\mathcal{S}_k \text{ not covered}, \forall \mathcal{T} \subset \Sigma_k : W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) \geq \tfrac{1}{3}\mu(\mathcal{T})\big). \end{aligned} \tag{3.7}$$

In the next sections, we compute $\mu(\mathcal{T})$, and estimate the probabilities of the two events on the right-hand side of (3.7).

3.1.1. *Mean number of explorers crossing a tile.* If $\mathcal{T}$ is a tile of a cell $\mathcal{C}$ which belongs to shell $\mathcal{S}_k \subset \mathbb{B}(0, n)$, at a distance $Ah(n)$ from $\mathbb{B}(0, n)$, then we show that for some positive constants $\{c_d, d \geq 2\}$,

$$\mu(\mathcal{T}) \geq c_d Ah(n)^d. \tag{3.8}$$

The inequality in (3.8) follows as in [1], Section 4.2, and relies on Proposition 2.7. Note that (3.8) ensures (H2).

3.1.2. *$W_{kh(n)}(N\mathbb{1}_0, \mathcal{T})$ is unlikely to be small.* Like in (4.17) and (4.18) of Section 4.2 of [1], there are constants C_d such that

$$\sum_{y \in \mathbb{B}(0,kh(n))} P_y^2\bigl(S\bigl(H(\Sigma_k)\bigr) \in \mathcal{T}\bigr) \leq \begin{cases} C_2 h^2(n)\log(n), & \text{for } d = 2, \\ C_d h^d(n), & \text{for } d \geq 3. \end{cases} \tag{3.9}$$

By Lemma 2.4, since for A large enough we have $\mu(\mathcal{T}) \geq 3cL^d h(n)^d \geq 3c_n$, there are positive constants $\{c'_d, d \geq 2\}$ such that

$$\begin{aligned} &P\bigl(W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) < \tfrac{1}{3}\mu(\mathcal{T})\bigr) \\ &\quad \leq \begin{cases} \exp\bigl(-\lambda\kappa_2 A h^2(n) + \lambda^2 c'_2 h^2(n)\log(n)\bigr), & \text{for } d = 2, \\ \exp\bigl(-\lambda\kappa_d A h^d(n) + \lambda^2 c'_d h^d(n)\bigr), & \text{for } d \geq 3. \end{cases} \end{aligned} \tag{3.10}$$

Thus, after optimizing over λ, we get

$$\begin{aligned} &P\biggl(\exists z \in \Sigma_k : W_{kh(n)}\bigl(N\mathbb{1}_0, \mathcal{T}(z)\bigr) < \frac{1}{3}\mu(\mathcal{T})\biggr) \\ &\quad \leq \begin{cases} n^2 \exp\biggl(-\dfrac{\kappa_2^2 A^2 h^2(n)}{4c'_2 \log(n)}\biggr), & \text{for } d = 2, \\ n^d \exp\biggl(-\dfrac{\kappa_d^2 A^2 h^d(n)}{4c'_d}\biggr), & \text{for } d \geq 3, \end{cases} \end{aligned} \tag{3.11}$$

and the event $\{W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) \leq \frac{1}{3}\mu(\mathcal{T})\}$ has a probability that decreases, for A large enough, faster than any given power of $1/n$.

3.1.3. *$\mathcal{C}$ is likely to be covered when $W_{kh(n)}(N\mathbb{1}_0, \mathcal{T})$ is large.* We consider here the event $\{\forall \mathcal{T} \subset \mathcal{S}, W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) \geq \frac{\kappa}{3} A h^d(n)\}$. Consider shell $\mathcal{S}_k$ at a distance $Ah(n)$ from $\partial\mathbb{B}(0, n)$. Since $\mathcal{S}_k$ is the union of $\mathbb{B}(z, h(n))$ when $z \in \Sigma_k$, Lemma 1.3 implies, when $d = 2$, that

$$\begin{aligned} &P\biggl(\mathcal{S}_k \notin A(N) \text{ and } W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) > \frac{1}{3}\mu(\mathcal{T}) \text{ for all } \mathcal{T}\biggr) \\ &\quad \leq |\mathcal{S}_k| \exp\biggl(-\kappa_2 \kappa A \frac{h^2(n)}{\log(n)}\biggr). \end{aligned} \tag{3.12}$$

We obtain a bound smaller than any power of $1/n$ when $h(n) = \log(n)$ and A is large enough. When $d \geq 3$, then we have

$$\begin{aligned} &P\bigl(\mathcal{S}_k \notin A(N) \text{ and } W_{kh(n)}(N\mathbb{1}_0, \mathcal{T}) > \tfrac{1}{3}\mu(\mathcal{T}) \text{ for all } \mathcal{T}\bigr) \\ &\quad \leq |\mathcal{S}_k| \exp\bigl(-\kappa_d \kappa A h^2(n)\bigr). \end{aligned} \tag{3.13}$$

For any given power of n, we obtain a negligible bound when $h^2(n) = \log(n)$ and A is large enough.

4. Outer error. In this section, we prove the outer error estimate (1.5). This is a consequence of our inner error estimates, of Lemma 1.5, combined with coupling with a flashing process of [1]. When dimension $d = 2$, and for A large to be chosen later, we decompose the event $\{\delta_O(n) \geq A\log(n)\}$, as

$$\{\delta_O(n) \geq A\log(n)\} = \bigcup_{i\geq 1}\{\delta_O(n) \in [A\log(n) + i - 1, A\log(n) + i[\}. \tag{4.1}$$

In dimension $d \geq 3$, $\sqrt{\log(n)}$ replaces $\log(n)$ in (4.1). Note that the index i is at most of order n^d. Now, we fix $i \geq 1$, and we set $3h(n) = A\sqrt{\log(n)} + i$ in $d \geq 3$, and $3h(n) = A\log(n) + i$ in $d = 2$. We now consider the event $\{\delta_O(n) \in [3h(n) - 1, 3h(n)[\}$. We also define

$$\Sigma = \mathbb{B}(0, n + 3h(n)) \setminus \mathbb{B}(0, n + 3h(n) - 1).$$

Note now that

$$\begin{aligned} &P(\delta_O(n) \in [3h(n) - 1, 3h(n)[) \\ &\quad \leq P\left(\bigcup_{z\in\Sigma}\{z \in A(N), \delta_O(n) = \|z\| - n\}\right). \end{aligned} \tag{4.2}$$

For $z \in \Sigma$, and in view of Lemma 1.5, we define

$$G(z) = \{z \in A(N), \delta_O(n) = \|z\| - n, |A(N) \cap \mathbb{B}(z, h(n))| > \beta h^d(n)\}. \tag{4.3}$$

To prove that $P(z \in A(N), \delta_O(n) = \|z\| - n)$ is smaller than any given power of $1/n$, we further split the event into two pieces:

$$\begin{aligned} &P(z \in A(N), \delta_O(n) = \|z\| - n) \\ &\quad \leq P(G(z)) + P(z \in A(N), |A(N) \cap \mathbb{B}(z, h(n))| \leq \beta h^d(n)). \end{aligned} \tag{4.4}$$

The second term on the right-hand side of (4.4) is dealt with using Lemma 1.5. We deal now with $G(z)$. Note that under $\{\delta_O(n) \in [3h(n) - 1, 3h(n)[\}$, no explorer escapes $\mathbb{B}(0, n + 3h(n))$. Thus, on $G(z)$, there are at least $\beta h^d(n)$ explorers which settle on $\mathbb{B}(z, h(n))$ before exiting $\mathbb{B}(0, n+3h(n))$. We now express the event $G(z)$ in term of *flashing explorers*, as introduced in [1].

4.1. *On a flashing process.* We refer the reader to Section 3.1 of [1] for a definition of flashing processes. Here, we partition $\mathbb{Z}^d$ into shells encaging $\mathbb{B}(0, n)$, with for $k \geq 0$,

$$\mathcal{S}_k = \mathbb{B}(0, n + 2(k + 1)h(n)) \setminus \mathbb{B}(0, n + 2kh(n)).$$

Also, for $k \geq 0$, let $\Sigma_k = \partial\mathbb{B}(0, n + (2k + 1)h(n))$. We now consider the flashing process. Explorers behave like internal DLA explorers, as long as they stay in $\mathbb{B}(0, n)$. After exiting $\mathbb{B}(0, n)$ they do not flash until their hitting of Σ_0, and behave like *flashing explorers* as defined in Section 3.1 of [1]. In shells $\{\mathcal{S}_k, k \geq 0\}$, cells and tiles have the meaning given in Section 4 of [1]. The key features the reader has to keep in mind are as follows:

- If a flashing explorer is unsettled up to time $H(\Sigma_k)$, then after time $H(\Sigma_k)$, it probes one site distributed almost uniformly over the *cell* centered at $S(H(\Sigma_k))$, and settles if the site is unoccupied.
- When an explorer leaves the cell centered on $S(H(\Sigma_k))$, it cannot afterward settle in $\mathcal{S}_k$, but perform a simple random walk, independent of other explorers, until it hits Σ_{k+1}. Thus, if we know that an explorer has reached at time t a site of $\mathbb{B}(0, n + (2k+1)h) \setminus \mathbb{B}(0, n + 2kh)$, then it performs after time t a simple random walk, independent of its surroundings, until it reaches Σ_k.
- We can build the internal DLA cluster, $A(N)$, and the flashing cluster $A^*(N)$ using the same trajectories $S_1, \dots, S_N$ such that

$$A(N) = \bigcup_{i=1}^{N} \{S_i(T(i))\} \quad \text{and} \quad A^*(N) = \bigcup_{i=1}^{N} \{S_i(T^*(i))\}, \tag{4.5}$$

  and for all $i = 1, \dots, N$, $T^*(i) \geq T(i)$. This last property, at the heart of our coupling argument between the flashing process and the original DLA, is fundamental. It implies that if a DLA explorer has crossed a site before settling, then the corresponding flashing explorer has also crossed the site before settling.

Before introducing more notation, let us explain the simple idea behind our estimate.

Heuristics. Using representation (4.5), event $G(z)$ for $z \in \Sigma$ implies that at least $\beta h^d(n)$ *flashing explorers* hit $\mathbb{B}(z, h(n))$ before exiting $\mathbb{B}(0, n + 3h(n))$. Consider these explorers after the moment they enter $\mathbb{B}(z, h(n)) \subset \mathcal{S}_1$ for the first time. They are behaving as independent random walks until they hit Σ_1. Now, a fraction must hit Σ_1 on $\mathbb{B}(z, 2h(n)) \cap \Sigma_1$. We show that this latter event has a probability we can estimate through the approach of [1].

For simplicity, let us call R_1 the radius of Σ_1, that is, $R_1 = n + 3h(n)$. Recall that for $\Lambda \subset \mathbb{B}(0, R_1) \cup \partial\mathbb{B}(0, R_1)$, we call $W_{R_1}(N\mathbb{1}_0, \Lambda)$ the number of flashing explorers which hit Λ before (or as) they hit Σ_1. In this section, the initial configuration is always $N\mathbb{1}_0$, and we omit this coordinate in W_{R_1} to simplify notation. Under our coupling (4.5), we have

$$G(z) \subset \{W_{R_1}(\mathbb{B}(z, h(n))) \geq \beta h^d(n)\}. \tag{4.6}$$

Let z' be the closest site of Σ_1 to the line $(0, z)$, and note that $\|z - z'\| \leq 1$. Note that a fraction of the $W_{R_1}(\mathbb{B}(z, h(n)))$ independent random walks in $\mathbb{B}(z, h(n)) \cap \mathbb{B}(0, n + 3h(n))$, must hit Σ_1 in a neighborhood of z'. Indeed, first note that since $z' \in \Sigma_1$, we have

$$|\partial\mathbb{B}(z', 2h(n)) \cap \mathbb{B}(0, n + 3h(n))| \geq \tfrac{1}{4}|\partial\mathbb{B}(z', 2h(n))|. \tag{4.7}$$

Now, for any $y \in \partial\mathbb{B}(z, h(n))$, a random walk starting on y, exits $\mathbb{B}(z', 2h(n))$ on any site of $\partial\mathbb{B}(z', 2h(n))$ with a probability proportional to $(2h(n))^{1-d}$. Thus, there is a positive constant ρ such that

$$\inf_{y \in \partial B(z, h(n))} P_y(S(H(\partial\mathbb{B}(z', 2h(n)))) \in \mathcal{S}_2) \geq \rho. \tag{4.8}$$

In other words, each flashing explorer stopped on $\partial\mathbb{B}(z,h(n))$ before hitting Σ_1 has a probability at least ρ to exit Σ_1 from $\Sigma_1\cap\mathbb{B}(z,2h(n))$. Thus, there is a positive constant I, such that for any large enough integer k,

$$P\Big(W_{R_1}\big(\mathbb{B}(z',2h(n))\cap\Sigma_1\big)<\frac{\rho}{2}k\Big|W_{R_1}\big(\mathbb{B}(z,h(n))\big)>k\Big)\le\exp(-Ik). \tag{4.9}$$

From (4.6), we have

$$\begin{aligned}\bigcup_{z\in\Sigma}G(z)\subset\bigcup_{z'\in\Sigma_1}&\Big\{W_{R_1}\big(\mathbb{B}(z',2h(n))\cap\Sigma_1\big)\ge\frac{\rho}{2}\beta h^d(n)\Big\}\\ \cup\bigcup_{z'\in\Sigma_1}&\Big\{W_{R_1}\big(\mathbb{B}(z',2h(n))\cap\Sigma_1\big)<\frac{\rho}{2}\beta h^d(n)\text{ and }\\ &\qquad W_{R_1}\big(\mathbb{B}(z,h(n))\big)\ge\beta h^d(n)\Big\}.\end{aligned} \tag{4.10}$$

Let us now define, for any $a>0$,

$$F(a)=\bigcup_{z\in\Sigma_1}\{W_{R_1}\big(\mathbb{B}(z,2h(n))\cap\Sigma_1\big)\ge ah^d(n)\}. \tag{4.11}$$

Thus, from (4.10) and (4.9), and for some constant $C>0$

$$\begin{aligned}P\Big(\bigcup_{z\in\Sigma}G(z)\Big)&\le P\Big(\bigcup_{z'\in\Sigma_1}\Big\{W_{R_1}\big(\mathbb{B}(z',2h(n))\cap\Sigma_1\big)\ge\frac{\rho}{2}\beta h^d(n)\Big\}\Big)\\ &\quad+|\Sigma_1|\sup_{z'\in\Sigma_1}P\Big(W_{R_1}\big(\mathbb{B}(z',2h(n))\cap\Sigma_1\big)<\frac{\rho}{2}\beta h^d(n)\Big|\\ &\qquad\qquad W_{R_1}\big(\mathbb{B}(z,h(n))\big)\ge\beta h^d(n)\Big)\\ &\le P\Big(F\Big(\frac{\rho}{2}\beta\Big)\Big)+Cn^{d-1}\exp\Big(-I\frac{\rho}{2}\beta h^d(n)\Big).\end{aligned} \tag{4.12}$$

It remains to show that for any fixed a, we can find A [defining $h(n)$] such that $P(F(a))$ is smaller than any given power of $1/n$.

4.2. *Estimating* $P(F(a))$. Note that by definition of $\delta_I(n)$, for $z\in\Sigma_1=\partial\mathbb{B}(0,n+3h(n))$ and $\mathcal{T}_z=\mathbb{B}(z,2h)\cap\Sigma_1$, $W_{R_1}(\mathcal{T}_z)$ satisfies the inequality

$$W_{R_1}(\mathcal{T}_z)+M_{R_1}\big(\mathbb{B}(0,n-\delta_I(n)),\mathcal{T}_z\big)\le\tilde{M}_{R_1}(N\mathbb{1}_0,\mathcal{T}_z). \tag{4.13}$$

Thus, for some large constant α_d to be chosen later, we have

$$\begin{aligned}&\mathbb{1}_{\delta_I(n)\le\alpha_d h(n)/(2A)}\Big(W_{R_1}(\mathcal{T}_z)+M_{R_1}\Big(\mathbb{B}\Big(0,n-\alpha_d\frac{h(n)}{2A}\Big)\Big)\Big)\\ &\quad\le\tilde{M}_{R_1}(N\mathbb{1}_0,\mathcal{T}_z).\end{aligned} \tag{4.14}$$

Inequality (4.13) puts us in the setting of Lemma 2.5. Thus, we first need to compute

$$\tilde{\mu}(\mathcal{T}_z) = E\big[M_{R_1}(N\mathbb{1}_0, \mathcal{T}_z)\big] - E\bigg[M_{R_1}\bigg(\mathbb{B}\bigg(0, n - \alpha_d \frac{h(n)}{2A}\bigg), \mathcal{T}_z\bigg)\bigg]. \tag{4.15}$$

Following the same computations as in Section 4.3 of [1], we have for some constant K

$$\tilde{\mu}(\mathcal{T}_z) \le K\bigg(\alpha_d \frac{h(n)}{A} n^{d-1}\bigg) \times \frac{h^{d-1}(n)}{n^{d-1}} \le \frac{K\alpha_d}{A} h^d(n). \tag{4.16}$$

Second, note that as in Section 4.3 of [1], we have that for constants $\{c_d, d \ge 2\}$,

$$\sum_{z \in \mathbb{B}(0,n)} P_z^2\big(S\big(H(\Sigma_1)\big) \in \mathcal{T}_z\big) \le \begin{cases} c_2 h^2(n)\log(n), & \text{if } d = 2, \\ c_d h^d(n), & \text{if } d \ge 3. \end{cases} \tag{4.17}$$

In optimizing over λ in (2.5), we find for (other) constants $\{c_d, d \ge 2\}$, if A is choosen large enough

$$\begin{aligned} &P\big(\exists z \in \Sigma_1 : W_{R_1}(\mathcal{T}_z) \ge a h^d(n)\big) \\ &\quad \le P\bigg(\delta_I(n) > \alpha_d \frac{h(n)}{2A}\bigg) + n^d \begin{cases} \exp\bigg(-c_2 \dfrac{h^2(n)}{\log(n)}\bigg), & \text{if } d = 2, \\ \exp\big(-c_d h^d(n)\big), & \text{if } d \ge 3. \end{cases} \end{aligned} \tag{4.18}$$

We conclude using the fact, for α_d large enough, the first term of the sum in the right-hand side of (4.18) is smaller than any given power of $1/n$. This was proved in Section 3 for the original internal DLA and the same proof can be adapted for the flashing process we consider here. The only difference is that we need a stronger version of Lemma 1.3 where $P(\mathbb{B}(0, R) \not\subset A(\eta))$ is replaced by $P(\mathbb{B}(0, R) \not\subset A_{\alpha R}(\eta))$ for some large α (this stronger version of the lemma is actually what we prove in the Appendix). Indeed, we can use Lemma 1.3 in the context of our flashing process by considering only explorers that do not exit $\mathbb{B}(0, n)$. Once α_d is fixed, we choose A large enough so that (4.18) holds.

APPENDIX A: PROOF OF LEMMA 1.3

We fix η, a configuration of AR^d explorers in $\mathbb{B}(0, R/2)$, and we choose $z \in \mathbb{B}(0, R)$. Then

$$\begin{aligned} P\big(\mathbb{B}(0, R) \not\subset A(\eta)\big) &\le P\big(\mathbb{B}(0, R) \not\subset A_{\alpha R}(\eta)\big) \\ &\le \sum_{z \in \mathbb{B}(0,R)} P\big(W_{\alpha R}(\eta, z) = 0\big) \end{aligned} \tag{A.1}$$

for any $\alpha > 1$ (in the sequel α will have to be taken large enough). Let L be a large positive real to be fixed later, and let ζ be the configuration with one explorer on each site of $\mathbb{B}(0, \alpha R) \setminus \mathbb{B}(z, L)$. We have

$$W_{\alpha R}(\eta, z) + M_{\alpha R}(\zeta, z) \ge \tilde{M}_{\alpha R}(\eta, z) - \big|\mathbb{B}(z, L)\big|. \tag{A.2}$$

Note that $W_{\alpha R}(\eta, z)$ and $M_{\alpha R}(\zeta, z)$ are independent: we are in the setting of Lemma 2.4. Assume for a moment that conditions (H1) and (H2) hold, and in addition,

$$E[M_{\alpha R}(\eta, z)] - E[M_{\alpha R}(\zeta, z)] \geq \max\Bigl(3|\mathbb{B}(z, L)|, \sum_{y \in \mathbb{B}(0,\alpha R)} P_y(H_z < H_{\alpha R})^2\Bigr). \tag{A.3}$$

Then, we have

$$P(W_{\alpha R}(\eta, z) = 0) \leq \exp(-C(E[M_{\alpha R}(\eta, z)] - E[M_{\alpha R}(\zeta, z)])). \tag{A.4}$$

We next consider separately the case $d \geq 3$ and the case $d = 2$, estimate the expectation of $\tilde{M}_{\alpha R}(\eta, z) - M_{\alpha R}(\zeta, z)$ and show (A.3).

A.1. The case $d \geq 3$. We show in this section that for some $\kappa_d > 0$, and A large enough,

$$E[\tilde{M}_{\alpha R}(\eta, z) - M_{\alpha R}(\zeta, z)] \geq \frac{\kappa_d}{2} A R^2 \gg 3|\mathbb{B}(z, L)|. \tag{A.5}$$

The proof is based on the following classical estimates. There are a_1, a_2 positive constants such that for any $y, z \in \mathbb{Z}^d$

$$\frac{a_1}{1 + \|y - z\|^{d-2}} \leq P_y(H_z < \infty) \leq \frac{a_2}{1 + \|y - z\|^{d-2}}. \tag{A.6}$$

Note first that when L is large enough, (H1) holds. Indeed,

$$\begin{aligned} \sup_{y:\|z-y\|>L} P_y(H_z < H_{\alpha R}) &\leq \sup_{y:\|z-y\|>L} P_y(H_z < \infty) \\ &\leq \frac{a_2}{1 + L^{d-2}} \leq \frac{\kappa - 1}{\kappa} \quad \text{with } \kappa > 1. \end{aligned} \tag{A.7}$$

We now estimate the mean number of explorers hitting z.

$$\begin{aligned} &E[M_{\alpha R}(\eta, z)] - E[M_{\alpha R}(\zeta, z)] \\ &\quad = \sum_{y \in \mathbb{B}(0,R/2)} \eta(y) P_y(H_z < H_{\alpha R}) - \sum_{y \in \mathbb{B}(0,\alpha R) \setminus \mathbb{B}(z,L)} P_y(H_z < H_{\alpha R}) \\ &\quad \geq \sum_{y \in \mathbb{B}(0,R/2)} \eta(y) P_y(H_z < H_{\alpha R}) - \sum_{y \in \mathbb{B}(0,\alpha R)} P_y(H_z < \infty). \end{aligned} \tag{A.8}$$

Note that for $y \in \mathbb{B}(0, R/2)$, we have

$$\begin{aligned} P_y(H_z < H_{\alpha R}) &= P_y(H_z < \infty) - E_y[\mathbb{1}_{H_{\alpha R} < H_z} P_{S(H_{\alpha R})}(H_z < \infty)] \\ &\geq \frac{a_1}{1 + \|y - z\|^{d-2}} - E_y\Bigl[\frac{a_2}{1 + \|S(H_{\alpha R}) - z\|^{d-2}}\Bigr] \\ &\geq \inf_{y \in \mathbb{B}(0,R/2)} \frac{a_1}{1 + \|y - z\|^{d-2}} - \sup_{y \in \partial\mathbb{B}(0,\alpha R)} \frac{a_2}{1 + \|y - z\|^{d-2}}. \end{aligned} \tag{A.9}$$

Now, for a constant α which depends only on a_1, a_2, there is $\kappa > 0$ such that

$$\inf_{y\in\mathbb{B}(0,R/2)} P_y(H_z < H_{\alpha R}) \geq \frac{\kappa}{R^{d-2}}. \tag{A.10}$$

Now, using (A.10) in (A.8), we have a constant c such that

$$\begin{aligned} E[\tilde{M}_{\alpha R}(\eta,z) - M_{\alpha R}(\zeta,z)] &\geq AR^d \frac{\kappa}{R^{d-2}} - \sum_{y:\|y-z\|<\alpha R} \frac{a_1}{1+\|y-z\|^{d-2}} \\ &\geq \kappa AR^2 - ca_2(\alpha R)^2. \end{aligned} \tag{A.11}$$

When A is chosen large enough, we obtain (A.5).

Finally, there are constants $\{C_d, d \geq 3\}$ such that for any $z \in \mathbb{B}(0, R)$

$$\sum_{y\in\mathbb{B}(0,\alpha R)} P_y(H_z < H_{\alpha R})^2 \leq \begin{cases} C_3\alpha R, & \text{for } d = 3, \\ C_4 \log(\alpha R), & \text{for } d = 4, \\ C_d, & \text{for } d \geq 5. \end{cases} \tag{A.12}$$

Thus, hypothesis (A.3) holds.

A.2. The case $d = 2$. We still have

$$\begin{aligned} P_y(H_z < H_{\alpha R}) &= \frac{G_{\alpha R}(y,z)}{G_{\alpha R}(z,z)} = \frac{G_{\alpha R}(z,y)}{G_{\alpha R}(z,z)} \quad \text{and} \\ G_{\alpha R}(z,y) &= E_z[a(S(H_{\alpha R}),y)] - a(z,y), \end{aligned} \tag{A.13}$$

where the *potential kernel* $a(.,.)$ replaces Green's function. Note that for $0 \leq \|z\| + R < \alpha R$, we have two positive constants K_2 and K_2' such that

$$K_2' \log(2\alpha R) \geq G_{B(z,2\alpha R)} \geq G_{\alpha R}(z,z) \geq G_{B(z,R)}(z,z) \geq K_2 \log(R), \tag{A.14}$$

by Proposition 1.6.6 of Lawler [3]. To estimate $G_{\alpha R}(z, y)$, we use Theorem 4.4.4 of [6] which establishes that for $z \neq 0$ (with γ the Euler constant),

$$\left| a(0,z) - \frac{2}{\pi}\log(\|z\|) - \frac{2\gamma + \log(8)}{\pi} \right| \leq \frac{K_g}{\|z\|^2}. \tag{A.15}$$

Thus, for $y \in \mathbb{B}(0, \alpha R)$, $0 \leq \|z\| \leq R$, and $y \neq z$

$$\left| G_{\alpha R}(z,y) - \frac{2}{\pi} E\left[\log\left(\frac{\|S(H_{\alpha R}) - z\|}{\|y - z\|}\right)\right]\right| \leq 2K_g. \tag{A.16}$$

When $y \in B(0, R/2)$, we get

$$G_{\alpha R}(z,y) \geq \frac{2}{\pi}\log(2(\alpha-1)/3) - 2K_g. \tag{A.17}$$

We choose α large enough so that for some constant C_1, we have, for all y in $B(0, R/2)$,

$$G_{\alpha R}(z,y) \geq C_1. \tag{A.18}$$

Formulas (A.13), (A.14) and (A.18) together imply that

$$\begin{aligned} E\big[M_{\alpha R}(\eta,z)\big] &= \sum_{y\in\mathbb{B}(0,R/2)} \eta(y)P_y(H_z<H_{\alpha R}) \\ &\geq \frac{C_1}{K_2'\log(2\alpha R)} \sum_{y\in\mathbb{B}(0,R/2)} \eta(y) \\ &= \frac{C_1 A R^2}{K_2'\log(2\alpha R)}. \end{aligned} \tag{A.19}$$

Using Lemma 3 of [4], we have, for some positive constant C_2,

$$E\big[M_{\alpha R}(\zeta,z)\big] \leq E\big[M_{\alpha R}\big(\mathbb{B}(0,\alpha R),z\big)\big] \leq \frac{C_2(\alpha R)^2}{\log(R)}. \tag{A.20}$$

We need now to choose L to have (H1) satisfied. Note that for $y \neq z$, (A.16) and (A.14) yields

$$P_y(H_z<H_{\alpha R}) \leq \frac{1}{K_2\log(R)} E\bigg[\frac{2}{\pi}\log\bigg(\frac{\|S(H_{\alpha R})-z\|}{\|y-z\|}\bigg)+2K_g\bigg]. \tag{A.21}$$

If $\|z-y\| > R/\log(R)$, we obtain, for some constant C_3,

$$P_y(H_z<H_{\alpha R}) \leq \frac{C_3\log((\alpha+1)\log(R))}{\log(R)}. \tag{A.22}$$

When R is large enough, we have that (H1) holds for $L = R/\log(R)$. Note that $|\mathbb{B}(0,L)|$ is of order $R^2/\log(R)^2$ and is much smaller than $R^2/\log(R)$.

Finally we need to control the sum of second moments. Simply note that, from (A.20),

$$\sum_{y\in\mathbb{B}(0,\alpha R)\setminus\mathbb{B}(0,L)} P_y^2(H_z<H_{\alpha R}) \leq E\big[M_{\alpha R}(\zeta,z)\big] \leq \frac{C_2\alpha^2R^2}{\log(R)}. \tag{A.23}$$

APPENDIX B: PROOF OF LEMMA 1.6

We will choose an h such that $R/2h$ is a positive integer. We divide $\mathcal{S} = B(0,2R)\setminus B(0,R)$ into $R/2h$ concentric shells of height $2h$. For $k=1,\dots,R/2h$, define

$$\begin{aligned} \mathcal{S}_k &= \mathbb{B}\big(0,2R-2(k-1)h\big)\setminus\mathbb{B}(0,2R-2kh) \quad \text{and} \\ \Sigma_k &:= \partial\mathbb{B}\big(0,2R-(2k-1)h\big). \end{aligned} \tag{B.1}$$

Also, we set $\mathcal{S}_0 = \mathbb{B}(0,2R)^c$. Then, we start on $z \in \partial\mathbb{B}(0,2R)$ a flashing explorer associated with this partition with an explored region V. The flashing setting is much simpler than the one introduced in Section 3.1 of [1]. There is an underlying simple random walk, say S^*, and each shell $\mathcal{S}_1,\mathcal{S}_2,\dots$ is associated with a flashing site. These flashing sites, say $\{Z_k, 0 \le k \le 2R/h\}$ are obtained as follows. We set $Z_0 = z$, and for $k \ge 1$ we draw a continuous random variable R_k on $[0, h]$ with density in $r \in [0, h] \mapsto dr^{d-1}/h^d$: the flashing site Z_k is the exit site from $\mathbb{B}(S^*(H(\Sigma_k)), R_k)$ after time $H(\Sigma_k)$. Then, the explorer settles on the first flashing site in $\mathcal{S} \setminus V$. The purpose of the flashing construction is that: (i) the flashing site is distributed almost uniformly inside the ball $\mathbb{B}(S^*(H(\Sigma_k)), h)$; and (ii) $P_z(H(\mathbb{B}(0, R)) < H(V^c))$ is bounded above by the probability that the explorer crosses $\mathcal{S}$.

For a small β to be chosen later, we say that $y \in \Sigma_k$ has a *dense neighborhood* if $|\mathbb{B}(y, h) \cap V| \ge \beta h^d$, and we call D_k their set. There is $\kappa > 0$ such that knowing that S^* has crossed $D_1, \dots, D_{k-1}$:

- if $S^*(H(\Sigma_k)) \notin D_k$, then the probability that S^* does not settle in $\mathcal{S}_k$ is smaller than $\kappa\beta$;
- the probability that $S^*(H(\Sigma_k)) \in D_k$ is smaller than $\kappa|D_k|/h^{d-1}$ (see Lemma 5 of [5]) uniformly over the position of the previous flashing site (in $\mathcal{S}_{k-1}$ or, exceptionally, on the border of S_{k-1}).

Now, the flashing explorer has crossed the annulus $\mathcal{S}$ if $Z_k \in V$ for all $k \ge 1$. In other words,

$$\{H(\mathbb{B}(0, R)) < H(V^c)\} \subset \bigcap_{k=1}^{R/2h} \{Z_k \in V\}. \tag{B.2}$$

By successive conditioning, we obtain

$$P_z\left(\bigcap_{k=1}^{R/2h} \{Z_k \in V\}\right) \le \prod_{k=1}^{R/2h} \left(\kappa\beta + \frac{\kappa|D_k|}{h_k^{d-1}}\right). \tag{B.3}$$

By the arithmetic–geometric inequality and (B.2), we obtain

$$P_z(H(\mathbb{B}(0, R)) < H(V^c)) \le \left(\kappa\beta + \frac{\kappa}{R/2h} \sum_{k=1}^{R/2h} \frac{|D_k|}{h^{d-1}}\right)^{R/2h}. \tag{B.4}$$

Note that each $y \in D_k$ satisfies $|\mathbb{B}(y, h) \cap V| \ge \beta h^d$, but each site in $\mathbb{B}(y, h) \cap V$ is in the neighborhood of at most h^{d-1} sites of D_k. Thus for some κ',

$$\sum_{k=1}^{R/2h} \frac{\beta|D_k|h^d}{h^{d-1}} \le \kappa'|V| \qquad \text{i.e.,} \quad \frac{1}{R/2h} \sum_{k=1}^{R/2h} \frac{|D_k|}{h^{d-1}} \le \frac{2\kappa'|V|}{\beta R h^{d-1}}. \tag{B.5}$$

We choose now β such that $4\kappa\beta < 1$, and we choose the smallest h such that $R/2h$ is a positive integer and

$$h \ge \max\left\{h_0, \left(\frac{2\kappa'|V|}{\beta^2 R}\right)^{1/(d-1)}\right\}. \tag{B.6}$$

This adds a constraint on $|V|$,

$$|V| \le \frac{\beta^2}{2^d \kappa'} R^d. \tag{B.7}$$

Instead of including (B.7) as a condition of our lemma, we find it more convenient to note that the probability we estimate is always less than 1, so that we deal with the case where (B.7) is violated with the constant a_d of (1.11).

APPENDIX C: PROOF OF LEMMA 1.5

Recall that a_d and κ_d are the constants appearing in Lemma 1.6. We define a positive constant

$$\gamma = \max\left(1, \left(\frac{2a_d}{\kappa_d}\right)^{d-1}\right). \tag{C.1}$$

Choose now $\beta > 0$ such that $4^d \beta\gamma \le 1$ and $h_0 = R/4 \ge 1$. Note that

$$\gamma|\eta| \le \gamma\beta R^d \le h_0^d. \tag{C.2}$$

We build now, by induction, a random subdivision of $\mathbb{B}(z, R)$ into shells of heights $h_0, h_1, \dots,$ in which, respectively, $N_0, N_1, \dots$ explorers of $A(\eta)$ have settled. We emphasize that the randomness comes from $A(\eta)$, and that the event $\{0 \in A(\eta)\}$ imposes to have $N_i \ge \lfloor h_i \rfloor$, for $i \ge 0$. Assume that $h_1, \dots, h_k$ have been defined such that

$$h_k \ge 1 \quad \text{and} \quad \sum_{i=1}^{k} h_i < \frac{R}{2}. \tag{C.3}$$

We define $h_{k+1}^d = \gamma N_k \le \gamma|\eta|$, and, by (C.2) we have $h_{k+1} \le h_0$. Note also that $h_{k+1} \ge 1$. Indeed, necessarily $N_k \ge \lfloor h_k \rfloor$, so that $h_{k+1}^d \ge \gamma \lfloor h_k \rfloor \ge \lfloor h_k \rfloor$. Since $\min(h_1, \dots, h_{k+1}) \ge 1$, the number of steps before we violate (C.3), say L, is finite. Obviously $L \le R$. Note that since $h_L \le h_0$,

$$\frac{R}{2} \le \sum_{i=1}^{L} h_i \le h_L + \sum_{i=1}^{L-1} h_i \le \frac{R}{4} + \frac{R}{2}. \tag{C.4}$$

Thus, we define

$$h_{L+1} = R - \left(\sum_{i=0}^{L} h_i\right) \ge 0. \tag{C.5}$$

For any choice of integers $l, n_0, \dots, n_l$, the event $\{L = l, N_0 = n_0, \dots, N_L = n_l\}$ implies that $n_1 + \cdots + n_l$ explorers have crossed a shell $B(z, R) \setminus B(z, R - h_0)$ by stepping on at most n_0 explorers settled in it, that $n_2 + \cdots + n_L$ explorers have crossed shell $B(z, R - h_0) \setminus B(z, R - h_0 - h_1)$ with n_1 explorers settled in it, and so on and so forth. Using Lemma 1.6, the fact that $n_i \le \beta R^d$, $l \le R$ and the notation $\delta = \frac{1}{d-1}$, we reach the following estimate:

$$
\begin{aligned}
&P(0 \in A(\eta)) \\
&\quad \le \sum_{\substack{l \le R, n_0, n_1, \dots, n_l \le |\eta| \\ \forall i, n_i \ge \lfloor h_i \rfloor}} P(L = l, N_0 = n_0, \dots, N_L = n_l) \\
&\quad \le R(\beta R^d)^{R+1} \sup_{\substack{l \le R, n_0, n_1, \dots, n_l \le |\eta| \\ \forall i, n_i \ge \lfloor h_i \rfloor}} e^{a_d \sum_{i=1}^{L} i n_i} \\
&\qquad \times \exp\left(-\kappa_d \sum_{i=1}^{L} n_i \left(\left(\frac{h_0^d}{n_0}\right)^{\delta} + \cdots + \left(\frac{h_{i-1}^d}{n_{i-1}}\right)^{\delta}\right)\right).
\end{aligned} \tag{C.6}
$$

Now, note that by the arithmetic–geometric inequality, for $1 \le i \le l$ (and using $h_i \le h_0$)

$$
\begin{aligned}
\frac{1}{i}\left(\left(\frac{h_0^d}{n_0}\right)^{\delta} + \cdots + \left(\frac{h_{i-1}^d}{n_{i-1}}\right)^{\delta}\right) &\ge \left(\frac{h_0^d}{n_0} \times \cdots \times \frac{h_{i-1}^d}{n_{i-1}}\right)^{\delta/i} \\
&= \left(\frac{h_0^d}{n_{i-1}} \gamma^{i-1}\right)^{\delta/i} = \left(\frac{h_0^d}{h_i^d} \gamma^{i}\right)^{\delta/i} \ge \frac{2a_d}{\kappa_d}.
\end{aligned} \tag{C.7}
$$

Thus, from (C.6) and (C.7), we have

$$
\begin{aligned}
P(0 \in A(\eta)) &\le R(\beta R^d)^{R+1} \max_{\substack{l \le R, n_0, n_1, \dots, n_l \le |\eta| \\ \forall i, n_i \ge \lfloor h_i \rfloor}} \exp\left(-a_d \sum_{i=1}^{L} i n_i\right) \\
&\le R(\beta R^d)^{R+1} \max_{\substack{l \le R, n_0, n_1, \dots, n_l \le |\eta| \\ \forall i, n_i \ge \lfloor h_i \rfloor}} \exp\left(-\frac{a_d}{\gamma} \sum_{i=1}^{L-1} i h_{i+1}^d\right).
\end{aligned} \tag{C.8}
$$

Since $h_1 \le R/4$, note that we have $h_2 + \cdots + h_L \ge R/4$ by (C.4). By Hölder's inequality, note that for constants $\{c_d, d \ge 2\}$,

$$
\begin{aligned}
\sum_{i=1}^{L-1} i h_{i+1}^d &\ge \frac{(\sum_{i=1}^{L-1} h_{i+1})^d}{(\sum_{i=1}^{L-1} 1/i^{1/(d-1)})^{d-1}} \\
&\ge \begin{cases} c_2 \dfrac{R^2}{\log(L)} \ge c_2 \dfrac{R^2}{\log(R)}, & \text{for } d = 2, \\[2ex] c_d \dfrac{R^d}{L^{d-2}} \ge c_d R^2, & \text{for } d \ge 3. \end{cases}
\end{aligned} \tag{C.9}
$$

This completes the proof.

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