

THE GENEALOGY OF BRANCHING BROWNIAN MOTION WITH ABSORPTION

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We consider a system of particles which perform branching Brownian motion with negative drift and are killed upon reaching zero, in the near-critical regime where the total population stays roughly constant with approximately N particles. We show that the characteristic time scale for the evolution of this population is of order $(\log N)^3$, in the sense that when time is measured in these units, the scaled number of particles converges to a variant of Neveu’s continuous-state branching process. Furthermore, the genealogy of the particles is then governed by a coalescent process known as the Bolthausen–Sznitman coalescent. This validates the nonrigorous predictions by Brunet, Derrida, Muller and Munier for a closely related model.

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1. Introduction. Branching Brownian motion is a stochastic process in which, at time zero, there is a single particle at the origin. Each particle moves according to a standard Brownian motion for an exponentially distributed time with mean one, at which point it splits into two particles. Early work on branching Brownian motion, going back to McKean [54], focused on the position $M(t)$ of the

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right-most particle. Bramson [16, 17] obtained asymptotics for the median of the distribution of $M(t)$, and Lalley and Sellke [47] found the asymptotic distribution of $M(t)$.

In 1978, Kesten [43] introduced branching Brownian motion with absorption. This process follows the same dynamics as branching Brownian motion except that the initial particle is located at $x > 0$, the Brownian particles have a drift of $-\mu$, where $\mu > 0$, and particles are killed when they reach the origin. Kesten showed that there exists a critical value $\mu_c = \sqrt{2}$ such that if $\mu \geq \mu_c$, then the process dies out almost surely, while if $\mu < \mu_c$, the process survives with positive probability. More recent work on this process can be found in [38] and [39].

Our interest in branching Brownian motion with absorption comes from its possible interpretation as a model of a population undergoing selection. To see this connection, imagine that each individual in a population is represented by a position on the real line, which measures her fitness. The fitness of an individual evolves according to Brownian motion due to mutations, and initially the fitness of a child is identical to the fitness of the parent. Selection progressively eliminates all individuals whose fitness becomes too low; we effectively imagine selection as a moving wall with constant speed μ . Every individual whose fitness falls beyond the current threshold is instantly removed from the population.

To obtain asymptotic results as the population size tends to infinity, we consider a sequence of branching Brownian motions with absorption. For each positive integer N , we have a branching Brownian motion with absorption $(X_N(t), t \geq 0)$. We consider the near-critical case, where the drift μ depends on N and for $N \geq 2$,

$$(1) \quad \mu = \sqrt{2 - \frac{2\pi^2}{(\log N + 3 \log \log N)^2}}.$$

We also start the process with many particles, rather than just one, at time zero, and we make some rather technical assumptions on the initial conditions, which are given later in Proposition 1. While (1) and the initial conditions may seem unnatural, they are necessary to ensure that the number of particles in the system stays of order N on the time scale of interest, so that the process can be viewed as a model of a population of size approximately N .

We focus on understanding the genealogy of a sample from the population after a large time. We show that the time to the most recent common ancestor of a sample behaves like $(\log N)^3$. Moreover we identify the limiting geometry of the coalescence tree of a sample, which we show is governed by a coalescent process $(\Pi(t), t \geq 0)$ known as the Bolthausen–Sznitman coalescent. The Bolthausen–Sznitman coalescent, which is defined precisely in Section 1.3, is a coalescent process that allows many ancestral lines to merge at once. This result is in sharp contrast with the standard case of the Moran model or the Wright–Fisher model, where random genetic drift leads to a characteristic genealogical time of N generations and a genealogical tree given by Kingman’s coalescent, which permits only pairwise mergers of ancestral lines.

The main result of this paper can thus be stated as follows. Fix $t > 0$. Choose n particles uniformly at random from the population at time $(\log N)^3 t$, and label these particles at random by the integers $1, \dots, n$. For $0 \leq s \leq 2\pi t$, define $\Pi_N(s)$ to be the partition of $\{1, \dots, n\}$ such that i and j are in the same block of $\Pi_N(s)$ if and only if the particles labeled i and j are descended from the same ancestor at time $(t - s/2\pi)(\log N)^3$. This is the standard “ancestral partition” of the sample. Then, with our initial conditions, we have the following result, which is stated precisely later as Theorem 3.

MAIN RESULT. *The sequence of processes $(\Pi_N(s), 0 \leq s \leq 2\pi t)$ converges in the sense of finite-dimensional distributions as $N \rightarrow \infty$ to the Bolthausen–Sznitman coalescent $(\Pi(s), 0 \leq s \leq 2\pi t)$.*

The reason for the multiple mergers is that when a particle gets very far to the right [in fact, at position $\frac{1}{\sqrt{2}}(\log N + 3 \log \log N + O(1))$], many descendants of this particle survive for a long time, as they are able to avoid being killed at zero. They quickly generate a positive fraction of the population. As a result, when a sample of particles is taken far into the future, many of their ancestral lines get traced back to this particle and coalesce at nearly the same time. Our result is accompanied by Theorem 2, which gives the evolution of the total number of particles $M_N(t)$ in the system. Under the same assumptions, $M_N((\log N)^3 t)/(2\pi N)$ converges in the sense of finite-dimensional distributions toward a continuous-state branching process with branching mechanism $\Psi(u) = au + 2\pi^2 u \log u$ for some constant $a \in \mathbb{R}$.

1.1. *Related models and conjectures.* Our inspiration for this model comes from the work of Brunet et al. [21, 22] concerning the effect of natural selection on the genealogy of a population. They considered a model of a population with fixed size N in which each individual has a fitness. They assumed that each individual has $k \geq 2$ offspring in the next generation, and that the fitness of each offspring is the parent’s fitness plus an independent random variable with some distribution μ . Of the kN offspring, the N with the highest fitness survive to form the next generation. This process repeats itself in each generation. Brunet et al. [21, 22] gave a detailed and intricate, but not mathematically rigorous, analysis of this model and arrived at the following three conjectures:

(1) If L_m is the maximum of the fitnesses of the N individuals in generation m , then L_m/m converges almost surely to some limiting velocity v_N . Furthermore, the limit $v_\infty = \lim_{N \rightarrow \infty} v_N$ exists, and there is a constant C such that

$$(2) \quad v_\infty - v_N \sim \frac{C}{(\log N)^2}.$$

(2) If two individuals are sampled from the population at random in some generation, then the number of generations that we need to look back to find their most recent common ancestor is of order $(\log N)^3$.

(3) If n individuals are sampled from the population at random in some generation, and their ancestral lines are traced backwards in time, the coalescence of these lineages can be described by the Bolthausen–Sznitman coalescent.

This model is similar to a branching random walk in which the positions of the particles correspond to the fitnesses of the individuals. Indeed, this model would be precisely a branching random walk if all individuals were permitted to survive. The limiting velocity v_∞ that appears in the first conjecture is the limiting velocity of the right-most particle in branching random walk, which was studied in the 1970s by Kingman [44], Hammersley [37] and Biggins [10]. Interest in variations of the branching random walk in which the number of particles stays fixed is more recent. Bérard and Gouéré [3] recently proved the first conjecture in the form stated above, in the case $k = 2$, under suitable regularity conditions on μ . Their proof builds on previous work of Gantert, Hu and Shi [34] and Pemantle [58]. See also the work of Durrett and Mayberry [29], who considered a model very similar to this one while studying predator-prey systems, and Durrett and Remenik [30].

The analysis of Brunet et al. involves studying solutions $u(x, t)$ to the noisy FKPP equation

$$(3) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2 + \sqrt{\frac{u(1-u)}{N}} W(x, t),$$

where $W(x, t)$ is space–time white noise. If the noise term were removed, this partial differential equation would be the well-known FKPP equation, which was introduced in 1937 by Fisher [33] and by Kolmogorov, Petrovskii and Piscunov [46] and is one of the simplest nonlinear partial differential equations that admits traveling wave solutions. The link between the FKPP equation and branching Brownian motion has been known since the work of McKean [54], who showed that if $M(t)$ denotes the position of the right-most particle at time t for branching Brownian motion with variance parameter 2 and $u(t, x) = P(M(t) > x)$, then u is the unique solution to the FKPP equation with the initial condition $u(0, x) = \mathbf{1}_{\{x < 0\}}$. In [39], Harris, Harris and Kyprianou use branching Brownian motion with absorption to give a probabilistic analysis of solutions to the FKPP equation.

The first conjecture above can also be expressed as a conjecture about the velocity of solutions to equations such as (3). This form of the conjecture goes back to the work [19] of Brunet and Derrida, who refined their analysis and simulations in [18, 20]. In this form, the conjecture states that the velocity of traveling wave solutions to the original FKPP equation exceeds the velocity of solutions to equation (3) with the noise term by a quantity that is of the order $1/(\log N)^2$. Recently Mueller, Mytnik and Quastel [55] proved this result.

With the first conjecture having been largely settled, the purpose of the present paper is to provide rigorous versions of the second and third conjectures. As explained above, the model that we work with is not exactly the model studied in [21, 22]. Instead, to simplify the analysis, we replace branching random walk by branching Brownian motion, and rather than keeping the population size exactly fixed, we control the population size by killing particles that drift too far to the left. Note in particular that with our choice (1),

$$(4) \quad \mu_c - \mu \sim \frac{\pi^2}{\sqrt{2}(\log N)^2}$$

as $N \rightarrow \infty$, which matches precisely (2) in Conjecture 1 above. Models with non-constant population size were already discussed by Derrida and Simon in [25, 65] using nonrigorous methods. Although they do not study genealogies, their analysis strongly suggests that a result similar to ours may be expected.

Another question of interest related to these nearly critical branching particle systems (in the sense that the drift μ of particles is slightly above the critical value $\mu_c = \sqrt{2}$), concerns asymptotics for the survival probability. This is a topic that has attracted a considerable amount of attention in recent years; see, for example, [1, 3, 4, 32, 34, 39, 41]. In [5], we use the techniques developed here to derive fairly sharp estimates for the survival probability of nearly critical branching Brownian motion.

We also emphasize that the Bolthausen–Sznitman coalescent describes precisely the ultrametric structure that is expected to emerge in the low-temperature regime of mean-field spin glass models such as the well-known Sherrington–Kirkpatrick model. This is perhaps not a coincidence, as the model which we study here may be seen as a degenerate form of spin glass models, with the position of the particles being approximately given by a Gaussian field with a covariance structure which is closely related to their genealogy.

1.2. *Continuous-state branching processes.* A continuous-state branching process is a $[0, \infty]$ -valued Markov process $(Z(t), t \geq 0)$ whose transition functions $p_t(x, \cdot)$ satisfy

$$p_t(x + y, \cdot) = p_t(x, \cdot) * p_t(y, \cdot) \quad \text{for all } x, y \geq 0.$$

That is, the sum of independent copies of the process started from x and y has the same law as the process started from $x + y$. Continuous-state branching processes were introduced by Jirina [42]. Lamperti [48] showed that continuous-state branching processes are precisely the processes that can be obtained by taking scaling limits of Galton–Watson processes. Lamperti [49] and Silverstein [64] observed a one-to-one correspondence between continuous-state branching processes and Lévy processes with no negative jumps, by showing that it is possible to obtain

any continuous-state branching process through a time change of the corresponding Lévy process; see also [23] for a very readable account of this theory and proofs.

If we exclude processes that can make an instantaneous jump to infinity, continuous-state branching processes can be characterized by a function $\Psi : [0, \infty) \rightarrow \mathbb{R}$ of the form

$$\Psi(u) = \alpha u + \beta u^2 + \int_0^\infty (e^{-ux} - 1 + ux \mathbf{1}_{\{x \leq 1\}}) \nu(dx),$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and ν is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x^2) \nu(dx) < \infty$. The function Ψ is called the branching mechanism. If $(Z(t), t \geq 0)$ is a continuous-state branching process with branching mechanism Ψ , then for $\lambda \geq 0$,

$$(5) \quad E[e^{-\lambda Z(t)} | Z(0) = a] = e^{-au_t(\lambda)},$$

where the function $t \mapsto u_t(\lambda)$ is a solution to the differential equation

$$(6) \quad \frac{\partial}{\partial t} u_t(\lambda) = -\Psi(u_t(\lambda)), \quad u_0(\lambda) = \lambda.$$

Neveu [57] studied the continuous-state branching process with $\Psi(u) = u \log u$. We will be interested, more generally, in a continuous-state branching process $(Z(t), t \geq 0)$ whose branching mechanism is of the form $\Psi(u) = au + bu \log u$, where $a \in \mathbb{R}$ and $b > 0$. In this case (see, e.g., page 256 of [7]), there exists a real number c such that

$$\Psi(u) = -cu + b \int_0^\infty (e^{-ux} - 1 + ux \mathbf{1}_{\{x \leq 1\}}) x^{-2} dx.$$

Also, it is not difficult to solve (6) to obtain

$$(7) \quad u_t(\lambda) = \lambda e^{-bt} e^{a(e^{-bt}-1)/b}.$$

Because $\int_0^\delta 1/\Psi(u) du = \infty$ for all $\delta > 0$, the process does not explode. That is, almost surely $Z(t) < \infty$ for all t . Because $\int_\delta^\infty 1/\Psi(u) du = \infty$ for all $\delta > 0$, the process does not go extinct, that is, almost surely $Z(t) > 0$ for all t . Proofs of these facts can be found in [36].

1.3. *The Bolthausen–Sznitman coalescent.* In mathematical population genetics, it is standard to represent the ancestral relationships among a sample of n individuals using a coalescent process $(\Pi(t), t \geq 0)$, which is a continuous-time Markov process taking its values in the set of partitions of $\{1, \dots, n\}$. Here $\Pi(0)$ is the partition of $\{1, \dots, n\}$ into n singletons, and blocks of the partition merge over time. The merging of blocks of the partition corresponds to the merging of ancestral lines when the ancestral lines of the n sampled individuals are traced backwards in time. The standard coalescent model is Kingman’s coalescent. Kingman’s coalescent was introduced in [45] and is now the basis for much work in

mathematical population genetics. Kingman’s coalescent has the property that only two blocks of the partition ever merge at a time, and each transition that involves two blocks merging into one happens at rate one.

Within the last decade, alternative models of coalescence, allowing for multiple ancestral lines to merge at once, have been studied in some depth. These coalescent processes, known as coalescents with multiple mergers or Λ -coalescents, were introduced by Pitman [59] and Sagitov [62]. If Λ is a finite measure on $[0, 1]$, then the Λ -coalescent has the property that whenever there are b blocks, each transition that involves merging k blocks of the partition into one happens at rate

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k} \Lambda(dx).$$

Kingman’s coalescent is the special case of the Λ -coalescent in which Λ is the unit mass at zero.

If Λ is the uniform distribution on $[0, 1]$, then the Λ -coalescent is known as the Bolthausen–Sznitman coalescent. The Bolthausen–Sznitman coalescent was introduced in [14] in the context of Ruelle’s probability cascades. The Bolthausen–Sznitman coalescent has been studied extensively, and has been found to be related to stable subordinators [9] and random recursive trees [35]. It also shows up in Derrida’s generalized random energy model [15]. Properties of the Bolthausen–Sznitman coalescent have been worked out, for example, in [2, 27, 59].

Bertoin and Le Gall [7] showed how to define precisely the notion of the genealogy of a continuous-state branching process. They found that the genealogy of Neveu’s continuous-state branching process is given by the Bolthausen–Sznitman coalescent. These results were extended in [13], where it was shown that the genealogy of any continuous-state branching process whose branching mechanism is of the form $\Psi(u) = au + bu \log u$ can still be described by the Bolthausen–Sznitman coalescent. This connection between the Bolthausen–Sznitman coalescent and Neveu’s continuous-state branching process played a central role in Bovier and Kurkova’s analysis of Derrida’s generalized random energy model [15]. A survey of this material can be found in [6].

1.4. *Main results.* Recall that for each positive integer N , we have a branching Brownian motion $(X_N(t), t \geq 0)$. We denote by $M_N(t)$ the number of particles at time t , and we denote the positions of these particles by $X_{1,N}(t) \geq X_{2,N}(t) \geq \dots \geq X_{M_N(t),N}(t)$. We further define the process $(Z_N(t), t \geq 0)$ by setting

$$(8) \quad L = \frac{1}{\sqrt{2}}(\log N + 3 \log \log N)$$

and then letting

$$(9) \quad Z_N(t) = \sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)} \sin\left(\frac{\pi X_{i,N}(t)}{L}\right) \mathbf{1}_{\{X_{i,N}(t) \leq L\}}.$$

Note that only particles to the left of L contribute to $Z_N(t)$, and the level L depends on N . As we will see later, $Z_N(t)$ is a good measure of the “size” of the process at time t , in the sense that it predicts the number of particles shortly after time t . Also let

$$(10) \quad Y_N(t) = \sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)}.$$

We will see that as the branching Brownian motion evolves, most particles stay well to the left of L , and as long as this is the case, the number of particles changes little. However, occasionally a small number of particles get very far to the right. Because the descendants of these particles are able to avoid the barrier at zero, the number of particles increases rapidly. Indeed, the increase in the number of particles is so rapid that when we take the scaling limit as $N \rightarrow \infty$, we get a process with jumps. The proposition below shows that this limiting process is a continuous-state branching process.

PROPOSITION 1. *For all positive integers N , define the process $(V_N(t), t \geq 0)$ by*

$$(11) \quad V_N(t) = \frac{1}{N(\log N)^2} Z_N((\log N)^3 t).$$

Suppose as $N \rightarrow \infty$, the distribution of $V_N(0)$ converges to ν , where ν is a probability distribution on $[0, \infty)$. Suppose also that $Y_N(0)/N(\log N)^3$ converges to zero in probability as $N \rightarrow \infty$. Then there exists a constant $a \in \mathbb{R}$ such that as $N \rightarrow \infty$, the finite-dimensional distributions of the process $(V_N(t), t \geq 0)$ converge to the finite-dimensional distributions of the continuous-state branching process with branching mechanism $\Psi(u) = au + 2\pi^2 u \log u$ started with distribution ν at time zero.

The condition on $V_N(0)$ ensures that the number of particles in the system is of order N , as shown below with the scaling in Theorem 2. The condition on $Y_N(0)$ ensures that no single particle at time 0 is likely to have descendants that constitute a large fraction of the population a short time later. If we begin with N particles in what is a relatively “stable” configuration, then the initial conditions will hold. Furthermore, as shown in Proposition 3 of [5], if there is initially a single particle near L , then these conditions will be satisfied after a time of order L^2 .

Note that because the processes $(V_N(t), t \geq 0)$ for fixed N can increase very rapidly in a short time but do not have large jumps, the sequence of processes $(V_N, N \geq 1)$ is not tight, and convergence in the Skorohod topology does not hold.

The theorem below converts this result about the scaling limit of $(Z_N(t), t \geq 0)$ to a result about the number of particles. This convergence result holds only for $t > 0$. The hypothesis at time $t = 0$ still involves the processes $(V_N(t), t \geq 0)$,

which may not imply convergence of the number of particles at time zero. The result needs to be stated in this way because it is the value of $Z_N(t)$ rather than $M_N(t)$ that predicts the number of particles that will be alive a short time later.

THEOREM 2. *Assume the hypotheses of Proposition 1 hold. Then as $N \rightarrow \infty$, the finite-dimensional distributions of the process*

$$\left(\frac{1}{2\pi N} M_N((\log N)^3 t), t > 0 \right)$$

converge to the finite-dimensional distributions of the continuous-state branching process with branching mechanism $\Psi(u) = au + 2\pi^2 u \log u$ started with distribution ν at time zero, where a is the constant from Proposition 1.

The next result shows that if we pick n particles at random from branching Brownian motion with absorption at some time and trace back their ancestral lines, the resulting process, properly scaled, converges to the Bolthausen–Sznitman coalescent. This is a precise formulation of the result stated in the [Introduction](#). Choose n particles uniformly at random from the $M_N((\log N)^3 t)$ particles at time $(\log N)^3 t$, and label these particles at random by the integers $1, \dots, n$. Fix $t > 0$. For $0 \leq s \leq 2\pi t$, define $\Pi_N(s)$ to be the partition of $\{1, \dots, n\}$ such that i and j are in the same block of $\Pi_N(s)$ if and only if the particles labeled i and j are descended from the same ancestor at time $(t - s/2\pi)(\log N)^3$. Let $(\Pi(s), 0 \leq s \leq 2\pi t)$ be the Bolthausen–Sznitman coalescent run for time $2\pi t$ and restricted to $\{1, \dots, n\}$.

THEOREM 3. *Assume the hypotheses of Proposition 1 hold, and assume that $\nu(\{0\}) = 0$. Then as $N \rightarrow \infty$, the finite-dimensional distributions of $(\Pi_N(s), 0 \leq s \leq 2\pi t)$ converge to those of $(\Pi(s), 0 \leq s \leq 2\pi t)$.*

As discussed earlier, this result is, of course, the analog for this model of the third conjecture of Brunet et al. [21, 22] stated above. The $(\log N)^3$ time scaling that appears here, as well as in Proposition 1 and Theorem 2, matches the second conjecture stated above. If two particles are chosen at random, the time back to their most recent common ancestor is of the order $(\log N)^3$.

1.5. Overview of the proofs. Because the proofs of Proposition 1 and Theorems 2 and 3 are rather long, we outline the basic strategy here. The key idea is to treat separately the particles that reach approximately the level L . These are the particles that will produce a large number of descendants within a short time, leading to jumps in the population size when we look forward in time, and multiple mergers of ancestral lines going backwards in time.

The first step, carried out in Section 2, is to collect some results that we need pertaining to branching Brownian motion in a strip, which are important both for

the proofs and for understanding the heuristics behind our choices of parameters. Most importantly, we observe that if a branching Brownian motion is started with a single particle at x , and particles are killed upon reaching 0 or L , then the expected number of particles in a set B at a sufficiently large time t is approximately $\int_B p_t(x, y) dy$, where

$$(12) \quad p_t(x, y) = \frac{2}{L} e^{(1-\mu^2/2-\pi^2/2L^2)t} \cdot e^{\mu x} \sin\left(\frac{\pi x}{L}\right) \cdot e^{-\mu y} \sin\left(\frac{\pi y}{L}\right).$$

From this formula, we can make several observations concerning the behavior of the branching Brownian motion. First, note that the time parameter t appears in the formula only in the first exponential factor, so the population size should be roughly constant over time provided that $1 - \mu^2/2 - \pi^2/2L^2 = 0$. Indeed, we have chosen the parameters μ and L above [see (1) and (8)] to satisfy this equation, as this is the drift needed to stabilize the population size. Second, notice that the formula is proportional to $e^{\mu x} \sin(\pi x/L)$, which will equal $Z_N(t)$ if we sum over the positions of all particles at time t . Thus, it is $Z_N(t)$ that predicts the number of particles that will be in a given set at a later time, which is why $Z_N(t)$ provides a useful measure of the “size” of the process. Third, notice that the formula is proportional to $e^{-\mu y} \sin(\pi y/L)$. Consequently, regardless of the starting configuration, once t is large enough for the approximation to be valid, the particles will have settled into a “stable” configuration in which the “density” of particles at position y is proportional to $e^{-\mu y} \sin(\pi y/L)$. We will see in Lemma 5 that this approximation becomes accurate when t gets to be larger than $(\log N)^2$.

If we begin at time zero with N particles that are approximately in the stable configuration, so that their “density” is $CL e^{-\mu y} \sin(\pi y/L)$, where CL is a normalizing constant, then the value of $Z_N(0)$ should be approximately

$$N \int_0^L e^{\mu y} \sin\left(\frac{\pi y}{L}\right) \cdot CL e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy,$$

which is of the order NL^2 . On the other hand, if we begin instead with a single particle at L , then one can show typically the right-most descendant of this particle will reach a level that exceeds L by only a constant. This is essentially true because critical branching Brownian motion dies out, and can be seen from Proposition 16 below which shows that particles reach L at a much faster rate than they reach any level that is much greater than L . Consequently, we can estimate the typical contribution of the descendants of this particle at time t by using (12) with L in place of x and $L + \alpha$ in place of L , where $\alpha > 0$ is a constant. This means that the value of $Z_N(t)$ should be of the same order as

$$\int_0^L e^{\mu y} \sin\left(\frac{\pi y}{L}\right) \cdot \frac{2}{L + \alpha} e^{\mu L} \sin\left(\frac{\pi L}{L + \alpha}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L + \alpha}\right) dy,$$

which is of the order $L^{-1} e^{\mu L}$. We have chosen L so that particles that reach L produce substantial increases in the population size. Indeed, note that $L^{-1} e^{\mu L}$ and

NL^2 are of the same order precisely when L is within a constant of the value in (8).

In Section 3, we therefore define

$$(13) \quad L_A = \frac{1}{\sqrt{2}}(\log N + 3 \log \log N - A),$$

where $A \in \mathbb{R}$, and study the particles that stay to the left of L_A . That is, we consider branching Brownian motion with particles killed at 0 and at L_A . Using (12), it is possible to estimate first and second moments of various quantities. In Section 3, we apply these results to calculate the first and second moments of $Z_N(t)$, conditional on the process a time $\theta(\log N)^3$ earlier, where θ is a small constant. The first moment calculation is Lemma 11, while the variance bound appears in Lemma 12. The variance bound is sufficient to establish that when A is large, there is a law of large numbers, with the value of $Z_N(t)$ being close to its expectation. A similar variance bound for the number of particles is given in Lemma 14. Such results would not be possible without the truncation at L_A , because without truncation the expected number of particles is dominated by rare events in which one particle moves far to the right and produces a large number of surviving offspring. The analysis in Section 3 is motivated by some of the arguments based on moment bounds in [43].

In Section 3.2 we tackle the question of how many particles reach the level L_A . An estimate of the expected number is given in Proposition 16. From this result, one can deduce that if we start with N particles that are in approximately the “stable” configuration described above, then the time that it will take before a particle reaches L_A is of the order $(\log N)^3$, which explains the $(\log N)^3$ time scaling in our main results. To see heuristically why this scaling occurs, note that if $\beta > 0$ is a constant, then the number of particles between $L - \beta$ and L at time t is of the order

$$N \int_{L-\beta}^L CLe^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy,$$

which is of the order $1/(\log N)^3$. Such particles have a positive probability of reaching L between times t and $t + 1$, but the calculation in Proposition 16 shows that particles that are more than a constant distance from L at time t are unlikely to hit L by time $t + 1$. Thus, $O(1/(\log N)^3)$ particles hit L per unit time.

Since branching by particles close to L_A may enable several particles to hit L_A at nearly the same time, we also require the second moment estimate in Proposition 18 to establish that the expected number of particles that reach L_A within a time interval of length $\theta(\log N)^3$, conditional on at least one particle reaching L_A , is bounded by a constant. Then in Section 3.3, we show in Proposition 23 that a “good” event on which the bounds in Sections 3.1 and 3.2 are valid occurs with high probability.

In Section 4, we begin to consider the contribution from particles after they reach the level L_A . The key to this analysis is Proposition 24, which comes from [56]. This result states that if a particle starts at L_A , and y is a large constant, then the number of descendants of the particle that reach $L_A - y$ is approximately $y^{-1}e^{\sqrt{2}y}W$, where W is a random variable. Some analysis that involves a Tauberian theorem leads to Proposition 27, which says that for large x , we have $P(W > x) \sim B/x$. Conceptually, this result is the reason why the genealogy of the population is described by the Bolthausen–Sznitman coalescent. The contribution to the population of the particle at L_A will be approximately proportional to the number of descendants that hit y , if y is sufficiently large. The fact that a jump of size greater than x results from a particle at L_A with probability proportional to $1/x$ implies that the Lévy measure of the limiting continuous-state branching process will have a density proportional to x^{-2} , which in turn leads to the duality with the Bolthausen–Sznitman coalescent.

In Section 5, we show how to combine all of the previous estimates to get sharp results for the behavior of the process $(Z_N(t), t \geq 0)$. The key results are Proposition 39, which bounds the expected change in Z_N over a time interval of length $\theta(\log N)^3$ when there is no large jump, and Proposition 41, which estimates the probability that Z_N increases by at least $rN(\log N)^2$ over a time interval of length $\theta(\log N)^3$. These estimates on how the process behaves over a short time interval can be matched with the infinitesimal generator of the continuous-state branching process. This work is done in Section 6 and leads to a proof of Proposition 1. Once Proposition 1 is established, we are able to prove Theorem 2 by arguing that the value of $Z_N(t)$ can be used to predict accurately the number of particles shortly after time t .

The proof that the genealogy of the process converges to the Bolthausen–Sznitman coalescent is completed in Section 7. We represent the genealogy of the branching Brownian motion using a “flow of bridges,” a tool introduced by Bertoin and Le Gall in [8]. Using Proposition 1 and Theorem 2, we establish convergence to the flow of bridges associated with the continuous-state branching process, which is known to correspond to the Bolthausen–Sznitman coalescent.

1.6. *Notational conventions and index of notation.* For the benefit of the reader, we include in Table 1 an index of some of the notation.

Some constraints on the constants ε , A and θ are introduced at the beginning of Section 3; see equations (32)–(35). Further constraints on these constants, as well as the choices of the constants δ , η , y and ζ , are set out in Section 5.1; see equations (95)–(106).

Throughout the rest of the paper, C will denote a positive finite constant whose value may change from line to line. The constant C may depend on u and s , but may not depend on N or on the seven constants ε , A , θ , δ , η , y and ζ . We say a sequence of random variables $(R_N)_{N=1}^\infty$ is $o(f(N))$ if for any choices of the constants u , s , ε , A , θ , δ , η , y and ζ satisfying the constraints mentioned above, there

TABLE 1

Index of some of the notation that is used throughout the paper

A	used to control the level at which particles are killed; see the definition of L_A .
$G_{N,k}$	event that $Z_N(t_j)$ and $Y_N(t_j)$ are sufficiently small for $j \leq k$.
$G_N(\varepsilon)$	event that $G_{N,k}$ occurs for all k .
$h(N)$	slowly increasing function used to upper bound Y_N .
L	level, given by (8), such that descendants of a particle that get near this level will likely constitute a significant fraction of the population in the future.
L_A	level at which particles are killed, defined in (13).
$M_N(t)$	number of particles at time t .
R_k	number of particles killed at L_A between t_{k-1} and t_k .
s	the process Z_N is often studied between times $u(\log N)^3$ and $(u+s)(\log N)^3$.
t_k	the process Z_N is frequently studied at the times t_k .
u	the process Z_N is often studied between times $u(\log N)^3$ and $(u+s)(\log N)^3$.
V_N	normalization of the process Z_N , defined in (11).
$X_N(t)$	the branching Brownian motion at time t .
$X_{i,N}(t)$	position of the i th particle from the right at time t .
y	large constant; the number of descendants of a particle at L_A that reach $L_A - y$ plays a central role in the paper.
$Y_N(t)$	weighted sum of particle positions at time t , defined in (10), such that a particle at x contributes $e^{\mu x}$ to the sum.
$Z_N(t)$	measure of the “size” of the process at time t , defined in (9), such that a particle at $x \leq L$ contributes $e^{\mu x} \sin(\pi x/L)$.
$Z_{N,1}$	similar to Z_N , but with particles killed at L_A , defined in (36).
$Z'_{N,1}$	similar to $Z_{N,1}$, with L_A used in place of L in the sine function; see (37).
Z_y	number of descendants of a particle at zero that reach $-y$.
δ	small constant used to bound the error in an estimate of a branching process limit; see (96).
ε	small constant used to bound Z_N above by $\varepsilon^{-1/2} N(\log N)^2$.
μ	drift of the branching Brownian motion, given by (1).
η	small constant used to bound the difference between Z_y and its limit.
θ	small constant such that t_k and t_{k+1} are $\theta s(\log N)^3$ apart.
ζ	large constant chosen so that with high probability, descendants of a particle at zero will have reached $-y$ by time ζ .

is a deterministic sequence $(b_N)_{N=1}^\infty$ tending to zero such that $|R_N| \leq b_N f(N)$ for all N . Note in particular that throughout this paper, the bounds implicit in the notation $o(1)$ or $o(f(N))$ are nonrandom and depend solely on the choices of parameters.

Also, if g is a function of some of the constants $\varepsilon, A, \theta, \delta, \eta, y, \zeta$ and N , we will occasionally use the notation $O(g(\varepsilon, A, \theta, \delta, \eta, y, \zeta, N))$ to denote an expression whose absolute value is bounded by $Cg(\varepsilon, A, \theta, \delta, \eta, y, \zeta, N)$, where C is defined as above.

2. Branching Brownian motion in a strip. Suppose $(B_t)_{t \geq 0}$ is Brownian motion started at x , with $0 < x < K$, and assume the process is killed when it hits

0 or K . Then (see, e.g., page 188 of [50]) the density of the process at time t , restricted to $(0, K)$, is

$$\begin{aligned}
 (14) \quad v_t(x, y) &= \frac{\pi}{K} u_{\pi^2 t / K^2}(\pi x / K, \pi y / K) \\
 &= \frac{2}{K} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t / 2K^2} \sin\left(\frac{n\pi x}{K}\right) \sin\left(\frac{n\pi y}{K}\right).
 \end{aligned}$$

Consider now branching Brownian motion in a strip in which each particle gives birth at rate one, drifts to the left at rate $\mu > 0$, and is killed upon reaching 0 or K . We will need to estimate the expected number of particles at time t when t is large. Suppose there is initially a single particle at x . The density of particles at the position y at time t can be calculated using the well-known many-to-one lemma. The density is a product of e^t , which represents the expected number of particles at time t , a Girsanov factor $e^{\mu(x-y) - \mu^2 t / 2}$ relating Brownian motion with drift $-\mu$ to ordinary Brownian motion, and the density of ordinary Brownian motion killed upon reaching 0 or K . Therefore, the density of particles at time t is

$$(15) \quad q_t(x, y) = e^{(1 - \mu^2 / 2)t + \mu(x - y)} \cdot \frac{2}{K} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t / 2K^2} \sin\left(\frac{n\pi x}{K}\right) \sin\left(\frac{n\pi y}{K}\right),$$

in the sense that if $B \subset (0, K)$, then the expected number of particles in B at time t is $\int_B q_t(x, y) dy$.

When $t \gg K^2$, the first term in the sum in (15) dominates. We make this more precise in Lemma 5 below. We first record the following trigonometric lemma.

LEMMA 4. *If $0 \leq y \leq \pi$ and $n \in \mathbb{N}$, then $|\sin ny| \leq n \sin y$.*

PROOF. We prove the result by induction. The result is trivial for $n = 1$. If it is true for $n - 1$, then

$$\begin{aligned}
 |\sin ny| &= |\sin((n - 1)y) \cos y + \cos((n - 1)y) \sin y| \\
 &\leq |\sin((n - 1)y)| |\cos y| + |\cos((n - 1)y)| |\sin y| \\
 &\leq |\sin((n - 1)y)| + |\sin y| \leq n \sin y,
 \end{aligned}$$

where the last step uses the induction hypothesis. \square

By applying Lemma 4 to each term in the sum on the right-hand side of (15), we easily get the following estimate. Note that $p_t(x, y)$ is simply the $n = 1$ term in the expression for $q_t(x, y)$. The error term $D_t(x, y)$ is small when $t \gg K^2$ and is bounded above by a constant when $t \geq C_1 K^2$ for some constant C_1 .

LEMMA 5. Consider branching Brownian motion in a strip in which each particle gives birth at rate one, drifts to the left at rate μ and is killed upon reaching 0 or K . Suppose there is initially a single particle at x . Let

$$p_t(x, y) = \frac{2}{K} e^{(1-\mu^2/2-\pi^2/2K^2)t} \cdot e^{\mu x} \sin\left(\frac{\pi x}{K}\right) \cdot e^{-\mu y} \sin\left(\frac{\pi y}{K}\right).$$

Then for all $x, y \in [0, K]$, define $D_t(x, y)$ by

$$\frac{q_t(x, y)}{p_t(x, y)} = 1 + D_t(x, y).$$

Then

$$(16) \quad |D_t(x, y)| \leq \frac{\sum_{n=2}^{\infty} n^2 e^{-\pi^2 n^2 t / 2K^2}}{e^{-\pi^2 t / 2K^2}}.$$

Therefore, if B is a Borel subset of $(0, K)$, then the expected number of particles in B at time t may be written as $(\int_B p_t(x, y) dy)(1 + D'_t(x, B))$, where $|D'_t(x, B)|$ is bounded by the right-hand side of (16).

Using these densities, we can estimate the expected values of certain functions of branching Brownian motion. Lemma 6, which is Lemma 2 of [38], gives a martingale for branching Brownian motion in which particles are killed only at zero. Lemma 7 estimates the expected values of three specific functions of branching Brownian motion in a strip.

LEMMA 6. Consider branching Brownian motion in which each particle gives birth at rate one, drifts to the left at rate μ and is killed upon reaching 0. Let $M(t)$ be the number of particles at time t , and denote the positions of the particles at time t by $X_1(t), \dots, X_{M(t)}(t)$. Let

$$V(t) = \sum_{i=1}^{M(t)} X_i(t) e^{\mu X_i(t) + (\mu^2/2-1)t}.$$

Then $(V(t), t \geq 0)$ is a martingale.

LEMMA 7. Consider branching Brownian motion in a strip in which each particle gives birth at rate one, drifts to the left at rate μ and is killed upon reaching 0 or K . Let $M(t)$ be the number of particles at time t , and denote the positions of the particles at time t by $X_1(t), \dots, X_{M(t)}(t)$. Let

$$Y(t) = \sum_{i=1}^{M(t)} e^{\mu X_i(t)}, \quad Z(t) = \sum_{i=1}^{M(t)} e^{\mu X_i(t)} \sin\left(\frac{\pi X_i(t)}{K}\right).$$

Then

$$(17) \quad E[M(t)] = \frac{2}{K} e^{(1-\mu^2/2-\pi^2/2K^2)t} (1 + D_1) Z(0) \int_0^K e^{-\mu y} \sin\left(\frac{\pi y}{K}\right) dy$$

and

$$(18) \quad E[Y(t)] = \frac{4}{\pi} e^{(1-\mu^2/2-\pi^2/2K^2)t} (1 + D_2) Z(0),$$

where $|D_1|$ and $|D_2|$ are bounded by the right-hand side of (16). Also,

$$(19) \quad E[Z(t)] = e^{(1-\mu^2/2-\pi^2/2K^2)t} Z(0).$$

PROOF. To prove (17), first suppose there is initially a single particle at x . Lemma 5 gives

$$\begin{aligned} E[M(t)] &= \left(\int_0^K p_t(x, y) dy \right) (1 + D_1) \\ &= \frac{2}{K} e^{(1-\mu^2/2-\pi^2/2K^2)t} (1 + D_1) e^{\mu x} \sin\left(\frac{\pi x}{K}\right) \int_0^K e^{-\mu y} \sin\left(\frac{\pi y}{K}\right) dy, \end{aligned}$$

where $|D_1|$ is bounded by the right-hand side of (16). The result now follows by summing over the particles at time zero.

Likewise, to prove (18), assume there is initially a single particle at x , and observe that Lemma 5 gives

$$E[Y(t)] = \left(\int_0^K e^{\mu y} p_t(x, y) dy \right) (1 + D_2),$$

where $|D_2|$ is bounded by the right-hand side of (16). Using

$$\int_0^K \sin\left(\frac{\pi y}{K}\right) dy = \frac{2K}{\pi},$$

we get

$$E[Y(t)] = \frac{4}{\pi} e^{(1-\mu^2/2-\pi^2/2K^2)t} e^{\mu x} \sin\left(\frac{\pi x}{K}\right) (1 + D_2).$$

The result again follows by summing over the particles at time zero.

To obtain (19), note that if n is a positive integer, then

$$\int_0^K \sin\left(\frac{\pi y}{K}\right) \sin\left(\frac{n\pi y}{K}\right) dy = \begin{cases} K/2, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

If at time zero there is just a single particle at x , then

$$\begin{aligned} E[Z(t)] &= \int_0^K e^{\mu y} \sin\left(\frac{\pi y}{K}\right) q_t(x, y) dy \\ &= e^{(1-\mu^2/2)t + \mu x} \cdot \frac{2}{K} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t / 2K^2} \sin\left(\frac{n\pi x}{K}\right) \int_0^K \sin\left(\frac{\pi y}{K}\right) \sin\left(\frac{n\pi y}{K}\right) dy \\ &= e^{\mu x} \sin\left(\frac{\pi x}{K}\right) e^{(1-\mu^2/2)t} e^{-\pi^2 t / 2K^2} = e^{(1-\mu^2/2-\pi^2/2K^2)t} e^{\mu x} \sin\left(\frac{\pi x}{K}\right). \end{aligned}$$

As before, the result now follows by summing over the particles at time zero. \square

For the next result, we will need the Green’s function for Brownian motion in a strip. Let $(B_t, t \geq 0)$ be one-dimensional Brownian motion without drift. Define the Green’s function $G(x, y)$ such that if $(B_t, t \geq 0)$ is Brownian motion started from $B_0 = x \in (0, K)$ and if $\tau = \inf\{t : B_t \notin (0, K)\}$, then for all bounded measurable functions g , we have

$$E\left[\int_0^\tau g(B_t) dt\right] = \int_0^K G(x, y)g(y) dy.$$

The Green’s function is given by (see, e.g., (4.4) on page 225 of [28])

$$(20) \quad G(x, y) = \begin{cases} 2x(K - y)/K, & \text{if } y \geq x, \\ 2y(K - x)/K, & \text{if } y \leq x. \end{cases}$$

To obtain this result from (4.4) in [28], observe that in the notation of [28], we have $\varphi(x) = x$ and $m(x) = 1$ for ordinary Brownian motion. If $y \leq x$, then $2y(K - x)/K \leq 2x(K - y)/K$. Therefore, for all $x, y \in [0, K]$,

$$(21) \quad G(x, y) \leq 2x(K - y)/K.$$

To control the fluctuations, we will also need a result about second moments. The following result, which is a slight extension of Lemma 3.1 of [43], will be a useful tool.

LEMMA 8. *Consider branching Brownian motion with particles killed at both 0 and K. Assume that at time zero there is just a single particle at x, and that the particles at time t are denoted by $X_1(t), \dots, X_{M(t)}(t)$. Let $f : (0, K) \rightarrow [0, \infty)$ be a measurable function. Then*

$$E\left[\left(\sum_{i=1}^{M(t)} f(X_i(t))\right)^2\right] = \int_0^K f(y)^2 q_t(x, y) dy + 2 \int_0^t \int_0^K q_s(x, z) \left(\int_0^K f(y) q_{t-s}(z, y) dy\right)^2 dz ds.$$

PROOF. For a Borel set $A \subset (0, K)$, let $N_A(t)$ be the number of particles in the set A at time t . Equation (2.8) of [63] gives

$$(22) \quad E[N_A(t)] = \int_A q_t(x, y) dy,$$

while equations (2.11) and (2.12) of [63] give

$$(23) \quad E[N_A(t)N_B(t)] = E[N_{A \cap B}(t)] + 2 \int_0^t \int_0^K q_s(x, z) \left(\int_A q_{t-s}(z, w) dw\right) \times \left(\int_B q_{t-s}(z, y) dy\right) dz ds.$$

Suppose f is a simple function, so that

$$f(x) = \sum_{i=1}^m a_i \mathbf{1}_{A_i},$$

where the A_i are disjoint Borel subsets of $(0, K)$ and the a_i are positive real numbers. In this case, we have

$$E \left[\left(\sum_{i=1}^{M(t)} f(X_i(t)) \right)^2 \right] = \sum_{i=1}^m \sum_{j=1}^m a_i a_j E[N_{A_i}(t) N_{A_j}(t)].$$

It is now straightforward to check, using (22) and (23), that the conclusion of Lemma 8 holds in this case. Since every nonnegative measurable function can be approximated from below by simple functions, the general result then follows from the monotone convergence theorem. \square

LEMMA 9. *Assume we are in the setting of Lemma 7. Assume that at time zero there is just a single particle at x . Suppose that $1 - \mu^2/2 - \pi^2/2K^2 \leq 0$. Also, assume there exist positive constants C_1 and C_2 such that $C_1 K^2 \leq t \leq C_2/(1 - \mu^2/2)$. Then there exists a constant C , depending on μ , C_1 and C_2 , but not on x or K , such that*

$$E[Z(t)^2] \leq C e^{\mu x} e^{\mu K} \left(\frac{1}{K^2} + \frac{t}{K^4} \right).$$

PROOF. We apply Lemma 8 with $f(y) = e^{\mu y} \sin(\pi y/K)$ to get

$$\begin{aligned} E[Z(t)^2] &= \int_0^K e^{2\mu y} \sin\left(\frac{\pi y}{K}\right)^2 q_t(x, y) dy \\ (24) \quad &+ 2 \int_0^t \int_0^K q_s(x, z) \left(\int_0^K e^{\mu y} \sin\left(\frac{\pi y}{K}\right) q_{t-s}(z, y) dy \right)^2 dz ds. \end{aligned}$$

We begin by bounding the first term in (24). By Lemma 5, for all $x, y \in [0, K]$ we have

$$(25) \quad q_t(x, y) \leq \frac{C}{K} e^{\mu(x-y)} \sin\left(\frac{\pi x}{K}\right) \sin\left(\frac{\pi y}{K}\right),$$

where we are using that $1 - \mu^2/2 - \pi^2/2K^2 \leq 0$. The assumption $t \geq C_1 K^2$ ensures that the error term from Lemma 5 can be bounded by a constant (throughout the proof, we allow the value of C to change from line to line). Note that

$$\begin{aligned} \int_0^K e^{\mu y} \sin\left(\frac{\pi y}{K}\right) dy &= \int_0^K e^{\mu(K-y)} \sin\left(\frac{\pi(K-y)}{K}\right) dy \\ (26) \quad &= e^{\mu K} \int_0^K e^{-\mu y} \sin\left(\frac{\pi y}{K}\right) dy \\ &\leq e^{\mu K} \int_0^K e^{-\mu y} \left(\frac{\pi y}{K}\right) dy \leq \frac{C e^{\mu K}}{K}. \end{aligned}$$

Here we are using that $\mu > 0$ and that C may depend on μ . Now using (25) and (26) and the bound $\sin(\pi y/K)^2 \leq 1$, we get

$$\begin{aligned}
 \int_0^K e^{2\mu y} \sin\left(\frac{\pi y}{K}\right)^2 q_t(x, y) dy &\leq \frac{C}{K} \int_0^K e^{2\mu y} e^{\mu(x-y)} \sin\left(\frac{\pi x}{K}\right) \sin\left(\frac{\pi y}{K}\right) dy \\
 (27) \qquad \qquad \qquad &\leq \frac{C e^{\mu x}}{K} \int_0^K e^{\mu y} \sin\left(\frac{\pi y}{K}\right) dy \\
 &\leq \frac{C e^{\mu x} e^{\mu K}}{K^2}.
 \end{aligned}$$

It remains to bound the second term in (24). Recall that $v_t(x, y)$, defined in (14), denotes the density at time t of Brownian motion started at x and killed when it reaches 0 or K . Note that

$$\int_0^\infty v_s(x, y) ds = G(x, y),$$

where $G(x, y)$ is Green’s function in (20). Since $t \leq C_2/(1 - \mu^2/2)$, we also have for $s \leq t$,

$$(28) \qquad q_s(x, y) = e^{\mu(x-y)+(1-\mu^2/2)s} v_s(x, y) \leq C e^{\mu(x-y)} v_s(x, y).$$

Since $t \geq C_1 K^2$, the bound (25) is valid for $q_{t-s}(x, y)$ when $s \leq t/2$. Using these results and (21),

$$\begin{aligned}
 &\int_0^{t/2} \int_0^K q_s(x, z) \left(\int_0^K e^{\mu y} \sin\left(\frac{\pi y}{K}\right) q_{t-s}(z, y) dy \right)^2 dz ds \\
 &\leq \int_0^{t/2} \int_0^K C e^{\mu(x-z)} v_s(x, z) \left(\int_0^K e^{\mu y} \sin\left(\frac{\pi y}{K}\right) \cdot \frac{C}{K} e^{\mu(z-y)} \right. \\
 &\qquad \qquad \qquad \left. \times \sin\left(\frac{\pi z}{K}\right) \sin\left(\frac{\pi y}{K}\right) dy \right)^2 dz ds \\
 (29) \qquad &\leq \frac{C e^{\mu x}}{K^2} \int_0^{t/2} \int_0^K e^{\mu z} v_s(x, z) \sin\left(\frac{\pi z}{K}\right)^2 \left(\int_0^K \sin\left(\frac{\pi y}{K}\right)^2 dy \right)^2 dz ds \\
 &\leq C e^{\mu x} \int_0^K e^{\mu z} \sin\left(\frac{\pi z}{K}\right)^2 \left(\int_0^{t/2} v_s(x, z) ds \right) dz \\
 &\leq C e^{\mu x} \int_0^K e^{\mu z} \sin\left(\frac{\pi z}{K}\right)^2 \frac{2x(K-z)}{K} dz \\
 &\leq C e^{\mu x} \int_0^K e^{\mu z} \frac{(K-z)^3}{K^2} dz \leq \frac{C e^{\mu x} e^{\mu K}}{K^2},
 \end{aligned}$$

where for the third inequality, we used that $\sin(\pi y/K)^2 \leq 1$, and for the next-to-last inequality, we used that $\sin(\pi z/K) = \sin(\pi(K-z)/K) \leq (K-z)/K$ and $x/K \leq 1$.

Next, let $v'_t(x, y)$ be the density at time t of Brownian motion started at x and killed when it hits 0. By the Reflection Principle, for $s \leq t$,

$$\begin{aligned} \int_0^K y v'_s(x, y) dy &= \frac{1}{\sqrt{2\pi s}} \int_0^K (ye^{-(x-y)^2/2s} - ye^{-(x+y)^2/2s}) dy \\ &= \frac{1}{\sqrt{2\pi s}} \int_{-x}^{K-x} (z+x)e^{-z^2/2s} dz \\ &\quad - \frac{1}{\sqrt{2\pi s}} \int_x^{K+x} (z-x)e^{-z^2/2s} dz \\ &\leq \frac{1}{\sqrt{2\pi s}} \int_{-x}^x ze^{-z^2/2s} dz + \frac{2x}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-z^2/2s} dz = 2x. \end{aligned}$$

Therefore, using that $v_s(x, y) = v_s(K - x, K - y) \leq v'_s(K - x, K - y)$,

$$\begin{aligned} &\int_{t/2}^t \int_0^K q_s(x, z) \left(\int_0^K e^{\mu y} \sin\left(\frac{\pi y}{K}\right) q_{t-s}(z, y) dy \right)^2 dz ds \\ &\leq \int_{t/2}^t \int_0^K \frac{C}{K} e^{\mu(x-z)} \sin\left(\frac{\pi x}{K}\right) \sin\left(\frac{\pi z}{K}\right) \\ &\quad \times \left(\int_0^K e^{\mu y} e^{\mu(z-y)} \sin\left(\frac{\pi y}{K}\right) v_{t-s}(z, y) dy \right)^2 dz ds \\ &\leq \frac{C e^{\mu x}}{K} \int_{t/2}^t \int_0^K e^{\mu z} \sin\left(\frac{\pi z}{K}\right) \left(\int_0^K \sin\left(\frac{\pi y}{K}\right) v_{t-s}(z, y) dy \right)^2 dz ds \\ (30) \quad &\leq \frac{C e^{\mu x}}{K} \int_{t/2}^t \int_0^K e^{\mu z} \sin\left(\frac{\pi z}{K}\right) \left(\int_0^K \left(\frac{K-y}{K}\right) v_{t-s}(z, y) dy \right)^2 dz ds \\ &\leq \frac{C e^{\mu x}}{K^3} \int_{t/2}^t \int_0^K e^{\mu z} \sin\left(\frac{\pi z}{K}\right) \left(\int_0^K y v'_{t-s}(K-z, y) dy \right)^2 dz ds \\ &\leq \frac{C e^{\mu x}}{K^3} \int_{t/2}^t \int_0^K e^{\mu z} \sin\left(\frac{\pi z}{K}\right) (K-z)^2 dz ds \\ &\leq \frac{C e^{\mu x} t}{K^4} \int_0^K e^{\mu z} (K-z)^3 dz \leq \frac{C e^{\mu x} e^{\mu K} t}{K^4}. \end{aligned}$$

The result follows from (27), (29) and (30). \square

3. Particles hitting the right-boundary. Recall that we are considering $(X_N(t), t \geq 0)$, which is a branching Brownian motion with drift $-\mu$ and killing at the origin. Recall also that Proposition 1 involves the processes $(Z_N(t), t \geq 0)$, where $Z_N(t)$ is a weighted sum of the positions of the particles at time t . Throughout this entire section, as well as Sections 5, 6 and 7, we assume that the hypotheses of Proposition 1 hold.

3.1. *The particles that never reach L_A .* To prove Proposition 1, we will need to consider these processes at two times u and $u + s$, where $0 \leq u < u + s$. Fix a small number $\theta > 0$ such that $\theta^{-1} \in \mathbb{N}$. For $0 \leq k \leq \theta^{-1}$, define the time $t_k = (u + \theta ks)(\log N)^3$. We will be interested in the value of the process Z_N at the times t_k . The assumption that $\theta^{-1} \in \mathbb{N}$ is useful for defining the sequence $\{t_k\}_{0 \leq k \leq \theta^{-1}}$. However, many of our results pertain to the state of the process at time t_k , conditional on the state of the process up to time t_{k-1} . For these results, the assumption $\theta^{-1} \in \mathbb{N}$ is not necessary.

Since $Y_N(0)/N(\log N)^3$ converges in probability to zero, there exists a nonrandom function $h : \mathbb{N} \rightarrow (0, \infty)$ such that $h(N) \rightarrow 0$ and $(\log N)h(N) \rightarrow \infty$ as $N \rightarrow \infty$, and $Y_N(0)/(N(\log N)^3 h(N))$ converges in probability to zero. [This is a simple consequence of the following fact: if $X_N \rightarrow 0$ in probability, then there exists a nonrandom sequence h_N such that $h_N \rightarrow 0$ as $N \rightarrow \infty$ and $P(X_N > h_N) \rightarrow 0$.] Let $\varepsilon > 0$. For $0 \leq k \leq \theta^{-1}$, let $G_{N,k}$ be the event that for $j = 0, 1, \dots, k$, the following two events occur:

- We have $Z_N(t_j) \leq \varepsilon^{-1/2} N(\log N)^2$.
- We have $Y_N(t_j) \leq N(\log N)^3 h(N)$.

Finally, let $G_N(\varepsilon) = G_{N,\theta^{-1}}$. Let $(\mathcal{F}_t, t \geq 0)$ be the natural filtration of $(X_N(t), t \geq 0)$. This filtration, of course, depends on N , but we suppress this dependence in the notation. We will need to consider the conditional distribution of $Z_N(t_k)$ given $\mathcal{F}_{t_{k-1}}$. Note that the event $G_{N,k-1}$ is in $\mathcal{F}_{t_{k-1}}$.

In this section, we will consider the particles that would still be alive if, between times t_{k-1} and t_k , we killed particles that hit L_A , where L_A was defined in (13). Recall that both L_A and the drift μ depend on N . We will always assume that N is large enough that $L_A > 0$ and

$$(31) \quad h(N) \leq \frac{e^{\mu L_A}}{N(\log N)^3} = e^{-A} e^{(\mu/\sqrt{2}-1)(\log N + 3 \log \log N - A)},$$

which is possible because, by (4), the right-hand side tends to e^{-A} as $N \rightarrow \infty$. Because $Y_N(t_k) \leq N(\log N)^3 h(N)$ on $G_{N,k}$, this ensures that on $G_{N,k}$, all particles at time t_k are to the left of L_A , a fact which will be invoked repeatedly in what follows.

Note that we have defined three constants: ε , A and θ . We think of ε as being small. Typically A will be a large positive constant, but we will also at times consider negative values of A . Finally, θ will always be a small positive constant. In particular, we will assume

$$(32) \quad \theta \leq 1,$$

$$(33) \quad |A|\theta \leq 1,$$

$$(34) \quad 4\pi^2 A\theta s \varepsilon^{-1/2} \leq e^{-A/4},$$

$$(35) \quad \theta e^A \varepsilon^{-1/2} \leq 1.$$

These assumptions will be in force through the rest of this section, except in Proposition 23 below, where it will be convenient to allow θ to be any number with $\theta^{-1} \in \mathbb{N}$. A stronger set of restrictions on θ will then be introduced at the beginning of Section 5.

For $t \in [t_{k-1}, t_k]$, we say $i \in S(t)$ if for all $v \in [t_{k-1}, t]$, the particle at time v that is the ancestor of $X_{i,N}(t)$ is in $(0, L_A)$. Consequently, for $t_{k-1} \leq t \leq t_k$, the positions of the particles in $S(t)$ follow a branching Brownian motion with drift $-\mu$, with particles killed when they reach 0 or L_A . Define

$$(36) \quad Z_{N,1}(t_k) = \sum_{i=1}^{M_N(t_k)} e^{\mu X_{i,N}(t_k)} \sin\left(\frac{\pi X_{i,N}(t_k)}{L}\right) \mathbf{1}_{\{i \in S(t_k)\}},$$

and for $t \in [t_{k-1}, t_k]$, define

$$(37) \quad Z'_{N,1}(t) = \sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)} \sin\left(\frac{\pi X_{i,N}(t)}{L_A}\right) \mathbf{1}_{\{i \in S(t)\}}.$$

Although our interest is in $Z_{N,1}(t_k)$, we will need to approximate this random variable by $Z'_{N,1}(t_k)$, which is defined in the same way except with L_A in place of L . The next result shows that the difference between these quantities is small.

LEMMA 10. *On $G_{N,k-1}$, both $|Z'_{N,1}(t_{k-1}) - Z_N(t_{k-1})|$ and $E[|Z'_{N,1}(t_k) - Z_{N,1}(t_k)| | \mathcal{F}_{t_{k-1}}]$ are $o(N(\log N)^2)$.*

PROOF. If $a > 0$, then

$$\left| \frac{d}{dx} \sin\left(\frac{a}{x}\right) \right| = \left| \frac{a}{x^2} \cos\left(\frac{a}{x}\right) \right| \leq \frac{a}{x^2}.$$

Therefore, if $0 \leq x \leq L_A$, then

$$\left| \sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{\pi x}{L_A}\right) \right| \leq \frac{|L - L_A| \pi x}{\min\{L_A, L\}^2} \leq \frac{\pi |A| L_A}{\sqrt{2} \min\{L_A, L\}^2}.$$

On $G_{N,k-1}$, all particles at time t_{k-1} are to the left of both L_A and L for sufficiently large N . The indicators are therefore not needed in (9) and (37) when $t = t_{k-1}$, and we get

$$(38) \quad \begin{aligned} |Z'_{N,1}(t_{k-1}) - Z_N(t_{k-1})| &\leq \frac{\pi |A| L_A}{\sqrt{2} \min\{L_A, L\}^2} \sum_{i=1}^{M_N(t_{k-1})} e^{\mu X_{i,N}(t_{k-1})} \\ &= \frac{\pi |A| L_A Y_N(t_{k-1})}{\sqrt{2} \min\{L_A, L\}^2}, \end{aligned}$$

which is $o(N(\log N)^2)$ on $G_{N,k-1}$. Applying the same reasoning at time t_k to the particles in $S(t_k)$, we get

$$E[|Z'_{N,1}(t_k) - Z_{N,1}(t_k)| | \mathcal{F}_{t_{k-1}}] \leq \frac{\pi |A| L_A E[Y_N(t_k) | \mathcal{F}_{t_{k-1}}]}{\sqrt{2} \min\{L_A, L\}^2}.$$

Note that

$$\begin{aligned}
 (39) \quad 1 - \frac{\mu^2}{2} - \frac{\pi^2}{2L_A^2} &= \frac{\pi^2}{(\log N + 3 \log \log N)^2} - \frac{\pi^2}{(\log N + 3 \log \log N - A)^2} \\
 &= -\frac{2\pi^2 A}{(\log N)^3} (1 + o(1)).
 \end{aligned}$$

Since $t_k - t_{k-1} = (\log N)^3 \theta s$ and (33) holds, equations (18) and (39) give $E[Y_N(t_k)|\mathcal{F}_{t_{k-1}}] \leq CZ_N(t_{k-1})(1 + o(1))$. It follows that

$$(40) \quad E[|Z'_{N,1}(t_k) - Z_{N,1}(t_k)||\mathcal{F}_{t_{k-1}}] \leq \frac{C|A|L_A Z_N(t_{k-1})(1 + o(1))}{\min\{L_A, L\}^2},$$

which is $o(N(\log N)^2)$ on $G_{N,k-1}$. \square

We now estimate the conditional mean and variance of $Z_{N,1}(t_k)$ given $\mathcal{F}_{t_{k-1}}$.

LEMMA 11. *On $G_{N,k-1}$, we have*

$$E[Z_{N,1}(t_k)|\mathcal{F}_{t_{k-1}}] = Z_N(t_{k-1})(1 - 2\pi^2 A \theta s + O(A^2 \theta^2)) + o(N(\log N)^2).$$

The same bound holds with $E[Z'_{N,1}(t_k)|\mathcal{F}_{t_{k-1}}]$ on the left-hand side.

PROOF. By (19) and the Markov property of branching Brownian motion with particles killed at 0 and L_A , we have for sufficiently large N on $G_{N,k-1}$,

$$(41) \quad E[Z'_{N,1}(t_k)|\mathcal{F}_{t_{k-1}}] = e^{(1-\mu^2/2-\pi^2/2L_A^2)(t_k-t_{k-1})} Z'_{N,1}(t_{k-1}),$$

using the fact that for sufficiently large N , on $G_{N,k-1}$ all particles at time t_{k-1} are to the left of L_A . Since $t_k - t_{k-1} = (\log N)^3 \theta s$, it follows from (39) that

$$\begin{aligned}
 (42) \quad e^{(1-\mu^2/2-\pi^2/2L_A^2)(t_k-t_{k-1})} &= e^{-2\pi^2 A \theta s(1+o(1))} \\
 &= 1 - 2\pi^2 A \theta s + O(A^2 \theta^2) + o(1),
 \end{aligned}$$

where assumption (33) ensures that the error term is $O(A^2 \theta^2)$. The result now follows from equations (41) and (42) together with the two bounds in Lemma 10. \square

LEMMA 12. *Assume $A \geq 0$. On $G_{N,k-1}$, we have*

$$\text{Var}(Z'_{N,1}(t_k)|\mathcal{F}_{t_{k-1}}) \leq C\theta N(\log N)^2 e^{-A} (Z_N(t_{k-1}) + o(N(\log N)^2)).$$

PROOF. For $t \in [t_{k-1}, t_k]$, define $Z'_{N,1}(t)$ as in (37), and define

$$Y'_N(t) = \sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)} \mathbf{1}_{\{i \in \mathcal{S}(t)\}}.$$

Define $t_{k-1} = s_0 < s_1 < \dots < s_M = t_k$ so that for some positive constants C_1 and C_2 , we have $C_1(\log N)^2 \leq s_n - s_{n-1} \leq C_2(\log N)^2$ for all n . Recall that for any random variable X and any σ -fields \mathcal{F} and \mathcal{G} with $\mathcal{F} \subset \mathcal{G}$, we have

$$\text{Var}(X|\mathcal{F}) = E[\text{Var}(X|\mathcal{G})|\mathcal{F}] + \text{Var}(E[X|\mathcal{G}]|\mathcal{F}).$$

Therefore, for $1 \leq n \leq M$, we have

$$\text{Var}(Z'_{N,1}(s_n)|\mathcal{F}_{s_0}) = E[\text{Var}(Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}})|\mathcal{F}_{s_0}] + \text{Var}(E[Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}}]|\mathcal{F}_{s_0}).$$

Equation (19) implies that $E[Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}}] = e^{(1-\mu^2/2-\pi^2/2L_A^2)(s_n-s_{n-1})} \times Z'_{N,1}(s_{n-1})$. Because $A \geq 0$ and thus $1 - \mu^2/2 - \pi^2/2L_A^2 \leq 0$, it follows that

$$\text{Var}(E[Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}}]|\mathcal{F}_{s_0}) \leq \text{Var}(Z'_{N,1}(s_{n-1})|\mathcal{F}_{s_0}).$$

Therefore,

$$\text{Var}(Z'_{N,1}(s_n)|\mathcal{F}_{s_0}) \leq E[\text{Var}(Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}})|\mathcal{F}_{s_0}] + \text{Var}(Z'_{N,1}(s_{n-1})|\mathcal{F}_{s_0}).$$

Now $\text{Var}(Z'_{N,1}(s_0)|\mathcal{F}_{s_0}) = 0$, so by induction,

$$(43) \quad \text{Var}(Z'_{N,1}(s_M)|\mathcal{F}_{s_0}) \leq \sum_{n=1}^M E[\text{Var}(Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}})|\mathcal{F}_{s_0}].$$

Because the particles at time s_{n-1} evolve independently between times s_{n-1} and s_n , the conditional variance $\text{Var}(Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}})$ is the sum of the conditional variances of the contributions to $Z'_{N,1}(s_n)$ from the individual particles at time s_{n-1} . We will use the inequality $\text{Var}(X|\mathcal{F}) \leq E[X^2|\mathcal{F}]$ and apply Lemma 9 with $K = L_A$ and $t = s_n - s_{n-1}$. The hypotheses are satisfied because $1 - \mu^2/2 - \pi^2/2L_A^2 \leq 0$, and both $s_n - s_{n-1}$ and $1/(1 - \mu^2/2)$ are of the order $(\log N)^2$. Therefore,

$$\text{Var}(Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}}) \leq C e^{\mu L_A} Y'_N(s_{n-1}) \left(\frac{1}{L_A^2} + \frac{s_n - s_{n-1}}{L_A^4} \right).$$

Now $e^{\mu L_A} \leq N(\log N)^3 e^{-A}$, so

$$(44) \quad \text{Var}(Z'_{N,1}(s_n)|\mathcal{F}_{s_{n-1}}) \leq C N(\log N)^3 e^{-A} \left(\frac{1}{L_A^2} + \frac{(\log N)^2}{L_A^4} \right).$$

From (18), we get

$$(45) \quad \begin{aligned} \max_{2 \leq n \leq M} E[Y'_N(s_{n-1})|\mathcal{F}_{s_0}] &\leq C Z'_{N,1}(s_0)(1 + o(1)) \\ &= C Z'_{N,1}(t_{k-1})(1 + o(1)). \end{aligned}$$

Finally, note that $M \leq C\theta(\log N)$. Combining this with (43), (44) and (45) gives that on $G_{N,k-1}$,

$$\begin{aligned}
 \text{Var}(Z'_{N,1}(t_k)|\mathcal{F}_{t_{k-1}}) &= \text{Var}(Z'_{N,1}(s_M)|\mathcal{F}_{s_0}) \\
 &\leq CN(\log N)^3 e^{-A} \left(\frac{1}{L_A^2} + \frac{(\log N)^2}{L_A^4} \right) \\
 (46) \quad &\quad \times (Y'_N(s_0) + C\theta(\log N)Z'_{N,1}(t_{k-1})(1 + o(1))) \\
 &\leq C\theta N(\log N)^2 e^{-A} \left(\frac{Y'_N(t_{k-1})}{\theta \log N} + Z'_{N,1}(t_{k-1}) \right) (1 + o(1)).
 \end{aligned}$$

The result now follows from Lemma 10 and the fact that $Y'_N(t_{k-1}) \leq Y_N(t_{k-1}) \leq N(\log N)^3 h(N)$ on $G_{N,k-1}$. \square

COROLLARY 13. Assume $A \geq 0$. On $G_{N,k-1}$, we have

$$P(|Z_{N,1}(t_k) - Z_N(t_{k-1})| > 4e^{-A/4} N(\log N)^2 | \mathcal{F}_{t_{k-1}}) \leq C\theta e^{-A/2} \varepsilon^{-1/2} (1 + o(1)).$$

PROOF. By the conditional form of Chebyshev's inequality and Lemma 12, on $G_{N,k-1}$ we have

$$\begin{aligned}
 P(|Z'_{N,1}(t_k) - E[Z'_{N,1}(t_k)|\mathcal{F}_{t_{k-1}}]| > e^{-A/4} N(\log N)^2 | \mathcal{F}_{t_{k-1}}) \\
 (47) \quad &\leq \frac{\text{Var}(Z'_{N,1}(t_k)|\mathcal{F}_{t_{k-1}})}{e^{-A/2} N^2 (\log N)^4} \\
 &\leq C\theta e^{-A/2} \varepsilon^{-1/2} (1 + o(1))
 \end{aligned}$$

because $Z_N(t_{k-1}) \leq \varepsilon^{-1/2} N(\log N)^2$ on $G_{N,k-1}$. Using (39), some calculus and the assumption that $A \geq 0$, we get that for N large enough that $A \leq 3 \log \log N$,

$$|e^{(1-\mu^2/2-\pi^2/2L_A^2)(t_k-t_{k-1})} - 1| \leq \left| 1 - \frac{\mu^2}{2} - \frac{\pi^2}{2L_A^2} \right| \theta s (\log N)^3 \leq 2\pi^2 A\theta s.$$

Therefore, by (41), if $A \leq 3 \log \log N$, then

$$|E[Z'_{N,1}(t_k)|\mathcal{F}_{t_{k-1}}] - Z'_{N,1}(t_{k-1})| \leq 2\pi^2 A\theta s Z'_{N,1}(t_{k-1}).$$

Because $Z'_{N,1}(t_{k-1}) = Z_N(t_{k-1}) + o(N(\log N)^2) \leq \varepsilon^{-1/2} N(\log N)^2 + o(N(\log N)^2)$ on $G_{N,k-1}$ by Lemma 10 and $2\pi^2 A\theta s \varepsilon^{-1/2} \leq e^{-A/4}/2$ by (34), it follows that for sufficiently large N ,

$$(48) \quad |E[Z'_{N,1}(t_k)|\mathcal{F}_{t_{k-1}}] - Z'_{N,1}(t_{k-1})| \leq e^{-A/4} N(\log N)^2$$

on $G_{N,k-1}$. By Lemma 10, on $G_{N,k-1}$, we have

$$(49) \quad |Z'_{N,1}(t_{k-1}) - Z_N(t_{k-1})| \leq e^{-A/4} N(\log N)^2$$

for sufficiently large N and

$$(50) \quad P(|Z_{N,1}(t_k) - Z'_{N,1}(t_k)| > e^{-A/4} N(\log N)^2 | \mathcal{F}_{t_{k-1}}) \rightarrow 0$$

uniformly as $N \rightarrow \infty$ on $G_{N,k-1}$. The result follows immediately from (47), (48), (49) and (50). \square

PROPOSITION 14. *Suppose $A = 0$. Let*

$$M'_N(t_k) = \sum_{i=1}^{M_N(t_k)} \mathbf{1}_{\{i \in \mathcal{S}(t_k)\}}$$

be the number of particles at time t_k whose ancestor at time t is in $(0, L)$ for all $t \in [t_{k-1}, t_k]$. On $G_{N,k-1}$, there exists a constant C such that

$$\text{Var}(M'_N(t_k) | \mathcal{F}_{t_{k-1}}) \leq C\theta\varepsilon^{-1/2} N^2(1 + o(1)).$$

PROOF. As in the proof of Lemma 12, the conditional variance can be bounded by the sum of the variances of the contributions to $M'_N(t_k)$ from the individual particles at time t_{k-1} . The variance of the contribution from a particle at x can be bounded by the expected square of the number of descendants of this particle at time t_k . This expectation is given by Lemma 8 with $f(x) = 1$ for all x and $t_k - t_{k-1}$ in place of t . Therefore,

$$\begin{aligned} &\text{Var}(M'_N(t_k) | \mathcal{F}_{t_{k-1}}) \\ &= \sum_{i=1}^{M_N(t_{k-1})} \int_0^L q_{t_k-t_{k-1}}(X_{i,N}(t_{k-1}), y) dy \\ &\quad + 2 \sum_{i=1}^{M_N(t_{k-1})} \int_{t_{k-1}}^{t_k} \int_0^L q_{t-t_{k-1}}(X_{i,N}(t_{k-1}), z) \left(\int_0^L q_{t_k-t}(z, y) dy \right)^2 dz dt. \end{aligned}$$

The first term is $E[M'_N(t_k) | \mathcal{F}_{t_{k-1}}]$, which by (17) with $K = L$ is at most $CZ_N(t_{k-1})(1 + o(1))/L^2$ because the integral on the right-hand side of (17) is of the order $1/K$. This expression is $o(N^2)$ on $G_{N,k-1}$.

The argument to bound the second term is similar to the proof of Lemma 9 but requires splitting the outer integral into four pieces. First consider the piece between t_{k-1} and $t_{k-1} + (\log N)^2$. If $t \leq (\log N)^2$, then (28) holds and

$$(51) \quad \int_0^\infty v_t(x, y) ds = G(x, y) \leq \frac{2x(L - y)}{L} \leq 2(L - y)$$

by (21). Since $1 - \mu^2/2 - \pi^2/2L^2 = 0$, Lemma 5 gives that on $G_{N,k-1}$,

$$\begin{aligned}
 & \sum_{i=1}^{M_N(t_{k-1})} \int_{t_{k-1}}^{t_{k-1}+(\log N)^2} \int_0^L q_{t-t_{k-1}}(X_{i,N}(t_{k-1}), z) \left(\int_0^L q_{t-t}(z, y) dy \right)^2 dz dt \\
 & \leq C \sum_{i=1}^{M_N(t_{k-1})} \int_0^{(\log N)^2} \int_0^L q_t(X_{i,N}(t_{k-1}), z) \\
 & \qquad \qquad \qquad \times \left(\int_0^L \frac{2}{L} e^{\mu z} \sin\left(\frac{\pi z}{L}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy \right)^2 dz dt \\
 & \leq \frac{C}{L^4} \sum_{i=1}^{M_N(t_{k-1})} \int_0^{(\log N)^2} \int_0^L q_t(X_{i,N}(t_{k-1}), z) e^{2\mu z} \sin\left(\frac{\pi z}{L}\right)^2 dz dt \\
 (52) \quad & \leq \frac{C}{L^4} \sum_{i=1}^{M_N(t_{k-1})} \int_0^L e^{2\mu z} \sin\left(\frac{\pi z}{L}\right)^2 \left(\int_0^{(\log N)^2} q_t(X_{i,N}(t_{k-1}), z) dt \right) dz \\
 & \leq \frac{C}{L^4} \sum_{i=1}^{M_N(t_{k-1})} \int_0^L e^{2\mu z} \sin\left(\frac{\pi z}{L}\right)^2 \\
 & \qquad \qquad \qquad \times e^{\mu(X_{i,N}(t_{k-1})-z)} \left(\int_0^\infty v_t(X_{i,N}(t_{k-1}), z) dt \right) dz \\
 & \leq \frac{C}{L^4} \left(\sum_{i=1}^{M_N(t_{k-1})} e^{\mu X_{i,N}(t_{k-1})} \right) \int_0^L e^{\mu z} \sin\left(\frac{\pi z}{L}\right)^2 (L-z) dz \\
 & \leq \frac{C}{L^4} \cdot Y_N(t_{k-1}) \cdot \frac{e^{\mu L}}{L^2} \leq CN^2 h(N).
 \end{aligned}$$

We next consider the case $t_{k-1} + (\log N)^2 \leq t \leq t_k - (\log N)^2$, and from Lemma 5, we get that on $G_{N,k-1}$,

$$\begin{aligned}
 & \sum_{i=1}^{M_N(t_{k-1})} \int_{t_{k-1}+(\log N)^2}^{t_k-(\log N)^2} \int_0^L q_{t-t_{k-1}}(X_{i,N}(t_{k-1}), z) \left(\int_0^L q_{t-t}(z, y) dy \right)^2 dz dt \\
 & \leq C \sum_{i=1}^{M_N(t_{k-1})} \int_{t_{k-1}+(\log N)^2}^{t_k-(\log N)^2} \int_0^L \frac{2}{L} e^{\mu X_{i,N}(t_{k-1})} \sin\left(\frac{\pi X_{i,N}(t_{k-1})}{L}\right) \\
 & \qquad \qquad \qquad \times e^{-\mu z} \sin\left(\frac{\pi z}{L}\right) \\
 (53) \quad & \qquad \qquad \times \left(\int_0^L \frac{2}{L} e^{\mu z} \sin\left(\frac{\pi z}{L}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy \right)^2 dz dt
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{C(t_k - t_{k-1})Z_N(t_{k-1})}{L^3} \left(\int_0^L e^{\mu z} \sin\left(\frac{\pi z}{L}\right)^3 dz \right) \left[\int_0^L e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy \right]^2 \\ &\leq \frac{C\theta(\log N)^3 Z_N(t_{k-1})}{L^3} \cdot \frac{e^{\mu L}}{L^3} \cdot \frac{1}{L^2} \leq C\theta\varepsilon^{-1/2} N^2. \end{aligned}$$

Consider now the case $t_k - (\log N)^2 \leq t \leq t_k - (\log N)^{7/4}$. Note that if $t \leq C(\log N)^2$, then $e^{(1-\mu^2/2)t} \leq C$, so by (15) and Lemma 4,

$$q_t(x, y) \leq \frac{C}{L} e^{\mu x} \sin\left(\frac{\pi x}{L}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) \sum_{n=1}^{\infty} n^2 e^{-\pi^2 n^2 t / L^2}.$$

Breaking up the sum into blocks of size $M = \lceil L/\sqrt{t} \rceil$ gives

$$\begin{aligned} \sum_{n=1}^{\infty} n^2 e^{-\pi^2 n^2 t / L^2} &\leq \sum_{\ell=0}^{\infty} M(M(\ell+1))^2 e^{-\pi^2 (M\ell)^2 t / L^2} \leq M^3 \sum_{\ell=0}^{\infty} (\ell+1)^2 e^{-\pi^2 \ell^2} \\ &\leq \frac{CL^3}{t^{3/2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^{M_N(t_{k-1})} \int_{t_k - (\log N)^2}^{t_k - (\log N)^{7/4}} \int_0^L q_{t-t_{k-1}}(X_{i,N}(t_{k-1}), z) \left(\int_0^L q_{t_k-t}(z, y) dy \right)^2 dz dt \\ &\leq C \sum_{i=1}^{M_N(t_{k-1})} \int_{(\log N)^{7/4}}^{(\log N)^2} \int_0^L \frac{2}{L} e^{\mu X_{i,N}(t_{k-1})} \sin\left(\frac{\pi X_{i,N}(t_{k-1})}{L}\right) e^{-\mu z} \sin\left(\frac{\pi z}{L}\right) \\ &\quad \times \left(\int_0^L \frac{L^2}{t^{3/2}} e^{\mu z} \sin\left(\frac{\pi z}{L}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy \right)^2 dz dt \\ (54) \quad &\leq CL^3 Z_N(t_{k-1}) \left(\int_{(\log N)^{7/4}}^{(\log N)^2} \frac{1}{t^3} dt \right) \left(\int_0^L e^{\mu z} \sin\left(\frac{\pi z}{L}\right)^3 dz \right) \\ &\quad \times \left[\int_0^L e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy \right]^2 \\ &\leq CL^3 Z_N(t_{k-1}) \cdot \frac{1}{(\log N)^{7/2}} \cdot \frac{e^{\mu L}}{L^3} \cdot \frac{1}{L^2} \leq \frac{C\varepsilon^{-1/2} N^2}{(\log N)^{1/2}}. \end{aligned}$$

Next, consider the case $t_k - (\log N)^{7/4} \leq t \leq t_k - 1$. Using (28), and the obvious fact that the density $v_t(x, y)$ of Brownian motion killed at 0 and L is dominated by the transition probabilities of standard Brownian motion, for $t \leq (\log N)^2$, we have

$$(55) \quad q_t(x, y) \leq C e^{\mu(x-y)} v_t(x, y) \leq C e^{\mu x} e^{-\mu y} \cdot \frac{1}{t^{1/2}} e^{-(x-y)^2/2t} \leq \frac{C e^{\mu x} e^{-\mu y}}{t^{1/2}}.$$

We split the integral over z into two pieces and obtain

$$\begin{aligned}
 & \sum_{i=1}^{M_N(t_{k-1})} \int_{t_k - (\log N)^{7/4}}^{t_k - 1} \int_0^{2L/3} q_{t-t_{k-1}}(X_{i,N}(t_{k-1}), z) \left(\int_0^L q_{t-t}(z, y) dy \right)^2 dz dt \\
 & \leq \frac{CZ_N(t_{k-1})}{L} \int_1^{(\log N)^{7/4}} \int_0^{2L/3} e^{-\mu z} \sin\left(\frac{\pi z}{L}\right) \left(\int_0^L q_t(z, y) dy \right)^2 dz dt \\
 (56) \quad & \leq \frac{CZ_N(t_{k-1})}{L} \int_1^{(\log N)^{7/4}} \int_0^{2L/3} e^{-\mu z} \sin\left(\frac{\pi z}{L}\right) \left(\int_0^L \frac{e^{\mu z} e^{-\mu y}}{t^{1/2}} dy \right)^2 dz dt \\
 & \leq \frac{CZ_N(t_{k-1})}{L} \left(\int_1^{(\log N)^{7/4}} \frac{1}{t} dt \right) \left(\int_0^{2L/3} e^{\mu z} \sin\left(\frac{\pi z}{L}\right) dz \right) \left(\int_0^L e^{-\mu y} dy \right) \\
 & \leq \frac{CZ_N(t_{k-1})}{L} \cdot \log \log N \cdot e^{2\mu L/3} = o(N^2)
 \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{i=1}^{M_N(t_{k-1})} \int_{t_k - (\log N)^{7/4}}^{t_k - 1} \int_{2L/3}^L q_{t-t_{k-1}}(X_{i,N}(t_{k-1}), z) \left(\int_0^L q_{t-t}(z, y) dy \right)^2 dz dt \\
 & \leq \frac{CZ_N(t_{k-1})}{L} \int_1^{(\log N)^{7/4}} \int_{2L/3}^L e^{-\mu z} \sin\left(\frac{\pi z}{L}\right) \left(\int_0^L q_t(z, y) dy \right)^2 dz dt \\
 & \leq \frac{CZ_N(t_{k-1})}{L} \\
 (57) \quad & \times \int_1^{(\log N)^{7/4}} \int_{2L/3}^L e^{\mu z} \left(\int_0^{L/3} e^{-\mu y} e^{-(z-y)^2/2t} dy + \int_{L/3}^L e^{-\mu y} dy \right)^2 dz dt \\
 & \leq \frac{CZ_N(t_{k-1})}{L} (\log N)^{7/4} e^{\mu L} \left(\int_0^{L/3} e^{-\mu y} e^{-(\log N)^{1/4}/36} dy + e^{-\mu L/3} \right)^2 \\
 & \leq CN(\log N)^{15/4} Z_N(t_{k-1}) (e^{-(\log N)^{1/4}/36} + e^{-\mu L/3})^2 = o(N^2).
 \end{aligned}$$

Finally, if $0 \leq t \leq 1$, then $\int_0^L q_t(z, y) dy \leq e^t \leq C$, so

$$\begin{aligned}
 & \sum_{i=1}^{M_N(t_{k-1})} \int_{t_{k-1}}^{t_k} \int_0^L q_{t-t_{k-1}}(X_{i,N}(t_{k-1}), z) \left(\int_0^L q_{t-t}(z, y) dy \right)^2 dz dt \\
 (58) \quad & \leq \frac{CZ_N(t_{k-1})}{L} \int_0^1 \int_0^L e^{-\mu z} \sin\left(\frac{\pi z}{L}\right) \left(\int_0^L q_t(z, y) dy \right)^2 dz dt \\
 & \leq \frac{CZ_N(t_{k-1})}{L} \int_0^L e^{-\mu z} \sin\left(\frac{\pi z}{L}\right) dz \leq \frac{CZ_N(t_{k-1})}{L^2} = o(N^2).
 \end{aligned}$$

The result now follows from (52)–(54) and (56)–(58). \square

3.2. *The number of particles that hit L_A .* For $k \in \mathbb{N}$, let R_k denote the number of times t between t_{k-1} and t_k that a particle reaches L_A at time t and, for all $u \in [t_{k-1}, t)$, the ancestor of this particle at time u was in $(0, L_A)$. Equivalently, R_k is the number of particles that are killed by hitting L_A between times t_{k-1} and t_k . Note that particles can reach L_A before time t_{k-1} and still contribute to R_k . Below we calculate the conditional mean and second moment of R_k given $\mathcal{F}_{t_{k-1}}$.

LEMMA 15. *Suppose there is a single particle at x at time zero, where $x \in (0, L_A)$. Suppose particles undergo branching Brownian motion with drift $-\mu$ and are killed when they reach 0 or L_A . Let R be the number of particles that hit L_A between times t and $t + \kappa$, where $0 < \kappa < 1$. Then*

$$(59) \quad E[R] = 2\pi e^{A\kappa} e^{(1-\mu^2/2-\pi^2/2L_A^2)t} \cdot e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right) \frac{(1+D)(1+o(1))(1+o(\kappa))}{N(\log N)^5},$$

where $|D|$ is bounded by the right-hand side of (16) with L_A in place of K , and $o(\kappa)$ is a term whose absolute value is bounded by $g(\kappa)$ for some bounded function $g: (0, 1) \rightarrow (0, \infty)$ with $\lim_{\kappa \rightarrow 0} g(\kappa) = 0$.

PROOF. Let $(B_t, t \geq 0)$ be standard Brownian motion started at the origin. Suppose that (for the branching Brownian motion), there is a particle at y at time t , and let $0 < \kappa < 1$. The expected number of descendants of the particle at time $t + \kappa$ is e^κ , and the drift of $-\mu$ can only reduce the probability that a Brownian particle reaches L_A . Therefore, an upper bound for the expected number of descendants that reach L_A at by time $t + \kappa$ is

$$(60) \quad e^\kappa P\left(\max_{0 \leq t \leq \kappa} B_t \geq L_A - y\right) = 2e^\kappa P(B_\kappa \geq L_A - y) = e^\kappa \sqrt{\frac{2}{\kappa\pi}} \int_{L_A-y}^\infty e^{-z^2/2\kappa} dz,$$

where the first equality follows from the reflection principle. To get a lower bound, we may ignore the branching, and bound the probability that Brownian motion with drift $-\mu$ reaches L_A by time κ without hitting the origin by the probability that ordinary Brownian motion reaches $L_A + \mu\kappa$ by time κ without hitting $\mu\kappa$. For $y \geq \mu\kappa$, this leads to a lower bound of

$$(61) \quad P\left(\max_{0 \leq t \leq \kappa} B_t \geq L_A - y + \mu\kappa\right) - P\left(\min_{0 \leq t \leq \kappa} B_t \leq -y + \mu\kappa\right) = \sqrt{\frac{2}{\kappa\pi}} \left(\int_{L_A-y+\mu\kappa}^\infty e^{-z^2/2\kappa} dz - \int_{y-\mu\kappa}^\infty e^{-z^2/2\kappa} dz \right).$$

From Lemma 5, the expected number of particles in the set B at time t is $(\int_B p_t(x, y) dy)(1 + D'_t(x, B))$. Now integrating over y and applying (60), we get

$$\begin{aligned}
 E[R] &\leq (1 + D) \int_0^{L_A} p_t(x, y) \cdot e^\kappa \sqrt{\frac{2}{\kappa\pi}} \left(\int_{L_A-y}^\infty e^{-z^2/2\kappa} dz \right) dy \\
 (62) \quad &= (1 + D) e^\kappa \sqrt{\frac{2}{\kappa\pi}} \cdot \frac{2}{L_A} e^{(1-\mu^2/2-\pi^2/2L_A^2)t} e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right) \\
 &\quad \times \int_0^{L_A} \int_{L_A-y}^\infty e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dz dy,
 \end{aligned}$$

where $|D|$ is bounded by the right-hand side of (16) with L_A in place of K . Interchanging the roles of y and $L_A - y$, then using Fubini's theorem followed by the bound $\sin y \leq y$ for $y \geq 0$ gives

$$\begin{aligned}
 &\int_0^{L_A} \int_{L_A-y}^\infty e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dz dy \\
 &= e^{-\mu L_A} \int_0^{L_A} \int_y^\infty e^{\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dz dy \\
 (63) \quad &= e^{-\mu L_A} \int_0^\infty \int_0^{\min\{z, L_A\}} e^{\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dy dz \\
 &\leq e^{-\mu L_A} \int_0^\infty \left(\int_0^z \frac{\pi y}{L_A} dy \right) e^{\mu z} e^{-z^2/2\kappa} dz \\
 &= \frac{\pi e^{-\mu L_A}}{2L_A} \int_0^\infty z^2 e^{\mu z} e^{-z^2/2\kappa} dz \\
 &= \frac{\pi e^{-\mu L_A}}{2L_A} \left(\kappa^{3/2} \sqrt{\frac{\pi}{2}} + \int_0^\infty z^2 (e^{\mu z} - 1) e^{-z^2/2\kappa} dz \right).
 \end{aligned}$$

The substitution $y = z/\sqrt{\kappa}$ gives

$$(64) \quad \int_0^\infty z^2 (e^{\mu z} - 1) e^{-z^2/2\kappa} dz = \kappa^{3/2} \int_0^\infty (e^{\mu y \sqrt{\kappa}} - 1) y^2 e^{-y^2/2} dy,$$

and the last integral goes to zero as $\kappa \rightarrow 0$ by the dominated convergence theorem. Therefore, combining (62), (63) and (64), we get

$$\begin{aligned}
 E[R] &\leq (1 + D) e^\kappa \sqrt{\frac{2}{\kappa\pi}} \cdot \frac{2}{L_A} e^{(1-\mu^2/2-\pi^2/2L_A^2)t} e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right) \\
 &\quad \times \frac{\pi^{3/2} e^{-\mu L_A} \kappa^{3/2}}{2^{3/2} L_A} (1 + o(\kappa)).
 \end{aligned}$$

Since $L_A = (2^{-1/2} \log N)(1 + o(1))$ and $e^{-\mu L_A} = e^A(1 + o(1))/(N(\log N)^3)$, it follows that $E[R]$ is bounded above by the right-hand side of (59).

We next establish the lower bound. Truncating the outer integral at $L_A/2$ and using (61), we get, for some D whose absolute value is bounded by the right-hand side of (16) with L_A in place of K ,

$$\begin{aligned}
 E[R] &\geq (1 + D) \int_{L_A/2}^{L_A} p_t(x, y) \sqrt{\frac{2}{\kappa\pi}} \\
 &\quad \times \left(\int_{L_A-y+\mu\kappa}^{\infty} e^{-z^2/2\kappa} dz - \int_{y-\mu\kappa}^{\infty} e^{-z^2/2\kappa} dz \right) dy \\
 &= (1 + D) \sqrt{\frac{2}{\kappa\pi}} \cdot \frac{2}{L_A} e^{(1-\mu^2/2-\pi^2/2L_A^2)t} e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right) \\
 &\quad \times \int_{L_A/2}^{L_A} e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) \left(\int_{L_A-y+\mu\kappa}^{\infty} e^{-z^2/2\kappa} dz - \int_{y-\mu\kappa}^{\infty} e^{-z^2/2\kappa} dz \right) dy \\
 (65) \quad &= (1 + D) \sqrt{\frac{2}{\kappa\pi}} \cdot \frac{2}{L_A} e^{(1-\mu^2/2-\pi^2/2L_A^2)t} e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right) \\
 &\quad \times \left(\int_0^{L_A} \int_{L_A-y+\mu\kappa}^{\infty} e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dz dy \right. \\
 &\quad \left. - \int_{L_A/2}^{L_A} e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) \left(\int_{y-\mu\kappa}^{\infty} e^{-z^2/2\kappa} dz \right) dy \right. \\
 &\quad \left. - \int_0^{L_A/2} \int_{L_A-y+\mu\kappa}^{\infty} e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dz dy \right).
 \end{aligned}$$

To bound the second term in (65), note that by substituting $w = z/\sqrt{\kappa}$ and using the fact that $\int_x^\infty e^{-w^2/2} dw \leq x^{-1} e^{-x^2/2}$, we get

$$\int_{(1/2)L_A-\mu\kappa}^{\infty} e^{-z^2/2\kappa} dz \leq \frac{2\sqrt{\kappa}}{L_A - 2\mu\kappa} e^{-(L_A-2\mu\kappa)^2/8\kappa}.$$

Therefore,

$$\begin{aligned}
 &\int_{L_A/2}^{L_A} e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) \left(\int_{y-\mu\kappa}^{\infty} e^{-z^2/2\kappa} dz \right) dy \\
 &\leq \int_{L_A/2}^{L_A} e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) \left(\int_{(1/2)L_A-\mu\kappa}^{\infty} e^{-z^2/2\kappa} dz \right) dy \\
 (66) \quad &\leq \frac{2\sqrt{\kappa}}{L_A - 2\mu\kappa} e^{-(L_A-2\mu\kappa)^2/8\kappa} \int_{L_A/2}^{L_A} e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) dy \\
 &= \left(\frac{e^{-\mu L_A}}{L_A} \right) o(1).
 \end{aligned}$$

To bound the third term in (65), note that

$$\begin{aligned}
 & \int_0^{L_A/2} \int_{L_A-y+\mu\kappa}^\infty e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dz dy \\
 &= e^{-\mu L_A} \int_{L_A/2}^{L_A} \int_{y+\mu\kappa}^\infty e^{\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dz dy \\
 (67) \quad &= e^{-\mu L_A} \int_{L_A/2}^\infty \int_{L_A/2}^{\min\{z-\mu\kappa, L_A\}} e^{\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dy dz \\
 &\leq \frac{\pi e^{-\mu L_A}}{L_A} \int_{L_A/2}^\infty z^2 e^{\mu z} e^{-z^2/2\kappa} dz = \left(\frac{e^{-\mu L_A}}{L_A}\right) o(1).
 \end{aligned}$$

For the first term, we argue as in the proof of the upper bound, and then use that $e^{\mu y} \geq 1$ and $\sin y \geq y - y^3/6$ for all $y \geq 0$ to get

$$\begin{aligned}
 & \int_0^{L_A} \int_{L_A-y+\mu\kappa}^\infty e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dz dy \\
 &= e^{-\mu L_A} \int_0^\infty \int_0^{\min\{z-\mu\kappa, L_A\}} e^{\mu y} \sin\left(\frac{\pi y}{L_A}\right) e^{-z^2/2\kappa} dy dz \\
 &\geq \frac{\pi e^{-\mu L_A}}{L_A} \int_0^\infty \int_0^{\min\{z-\mu\kappa, L_A\}} y e^{-z^2/2\kappa} dy dz \\
 &\quad - \frac{\pi^3 e^{-\mu L_A}}{6L_A^3} \int_0^\infty \int_0^z y^3 e^{-z^2/2\kappa} dy dz \\
 &= \frac{\pi e^{-\mu L_A}}{2L_A} \int_0^\infty \min\{z - \mu\kappa, L_A\}^2 e^{-z^2/2\kappa} dz \\
 &\quad - \frac{\pi^3 e^{-\mu L_A}}{24L_A^3} \int_0^\infty z^4 e^{-z^2/2\kappa} dz.
 \end{aligned}$$

The second integral is a constant times $\kappa^{5/2}$. The first integral would be $\kappa^{3/2} \sqrt{\pi/2}$ if we had z^2 in the integrand in place of $\min\{z - \mu\kappa, L_A\}^2$. Also,

$$\int_0^\infty (z^2 - \min\{z - \mu\kappa, L_A\}^2) e^{-z^2/2\kappa} dz \leq \int_0^\infty \max\{2\mu\kappa z, z^2 \mathbf{1}_{\{z \geq L_A\}}\} e^{-z^2/2\kappa} dz.$$

If we use $2\mu\kappa z$ in the integrand, the integral is bounded by $C\kappa^2$. If we use $z^2 \mathbf{1}_{\{z \geq L_A\}}$, the integral divided by $\kappa^{3/2}$ tends to zero uniformly over $\kappa \in (0, 1)$ as $N \rightarrow \infty$, so the integral is $\kappa^{3/2} o(1)$. These observations, combined with (65) and the bounds in (66) and (67), imply that $E[R]$ is bounded below by the right-hand side of (59). \square

PROPOSITION 16. *We have*

$$E[R_k | \mathcal{F}_{t_{k-1}}] = 2\pi e^A \cdot \frac{Z'_{N,1}(t_{k-1})\theta s}{N(\log N)^2} (1 + O(|A|\theta) + o(1)) + \frac{Ce^A Y_N(t_{k-1})(1 + o(1))}{N(\log N)^3}.$$

On $G_{N,k-1}$, we have

$$E[R_k | \mathcal{F}_{t_{k-1}}] = 2\pi e^A \cdot \frac{Z_N(t_{k-1})\theta s}{N(\log N)^2} (1 + O(|A|\theta)) + o(1).$$

PROOF. We first consider the particles that reach L_A between times t_{k-1} and $t_{k-1} + (\log N)^2$. Define $R_k(t)$ in the same manner as R_k , but counting only particles that reach L_A between times t_{k-1} and t . Let $R_{k,1} = R_k(t_{k-1} + (\log N)^2)$. We now consider the martingale from Lemma 6. Since $\mu^2/2 - 1 < 0$, this process will still be a supermartingale if particles are stopped, but not killed, when reaching L_A . More precisely, for $t_{k-1} \leq t \leq t_k$, let $X_{i,N}^{L_A}(t) = X_{i,N}(t)$ if, for all $u \in [t_{k-1}, t)$, the ancestor at time u of the individual $X_{i,N}(t)$ is in $(0, L_A)$, and let $X_{i,N}^{L_A}(t) = 0$ otherwise, and then for $t_{k-1} \leq t \leq t_k$, define

$$V_A(t) = R_k(t)L_A e^{\mu L_A + (\mu^2/2 - 1)(t - t_{k-1})} + \sum_{i=1}^{M_N(t)} X_{i,N}^{L_A}(t) e^{\mu X_{i,N}^{L_A}(t) + (\mu^2/2 - 1)(t - t_{k-1})}.$$

Then $(V_A(t), t_{k-1} \leq t \leq t_k)$ is a supermartingale with respect to $(\mathcal{F}_t, t_{k-1} \leq t \leq t_k)$. Therefore,

$$\begin{aligned} V_A(t_{k-1}) &\geq E[V_A(t_{k-1} + (\log N)^2) | \mathcal{F}_{t_{k-1}}] \\ &\geq E[R_{k,1} L_A e^{\mu L_A + (\mu^2/2 - 1)(\log N)^2} | \mathcal{F}_{t_{k-1}}] \\ &= L_A e^{\mu L_A + (\mu^2/2 - 1)(\log N)^2} E[R_{k,1} | \mathcal{F}_{t_{k-1}}]. \end{aligned}$$

Note that since $X_{i,N}^{L_A}(t) \leq L_A$, we have $V_A(t_{k-1}) \leq L_A Y_N(t_{k-1})$, which means

$$E[R_{k,1} | \mathcal{F}_{t_{k-1}}] \leq \frac{Y_N(t_{k-1})}{e^{\mu L_A + (\mu^2/2 - 1)(\log N)^2}}.$$

Since $(1 - \mu^2/2) \leq C/(\log N)^2$ and $e^{\mu L_A} \geq N(\log N)^3 e^{-A}(1 + o(1))$, we have

$$(68) \quad E[R_{k,1} | \mathcal{F}_{t_{k-1}}] \leq \frac{Ce^A Y_N(t_{k-1})(1 + o(1))}{N(\log N)^3}.$$

We next consider the particles that reach L_A between times $t_{k-1} + (\log N)^2$ and t_k . The strategy will be to choose a small number δ , break the time interval $[t_{k-1} + (\log N)^2, t_k]$ into time intervals of length δ and then use Lemma 15 to

estimate the number of particles that reach L_A in each of these intervals. We first make three remarks concerning the application of Lemma 15. First, note that if the interval starts at time t , then $t - t_{k-1}$ plays the role of t in Lemma 15. Since $t - t_{k-1} \leq (\log N)^3 \theta s$, equation (39) implies that

$$(69) \quad |e^{(1-\mu^2/2-\pi^2/2L_A^2)(t-t_{k-1})} - 1| \leq C|A|\theta + o(1).$$

Second, we need to consider all particles at time t rather than just a single particle at x , so in place of $e^{\mu x} \sin(\pi x/L_A)$, we have the expression $Z'_{N,1}(t_{k-1})$ from (37). Third, note that by (16) with $K = L_A$, the error term $|D|$ is bounded by $C(1 + o(1))$ for $t \geq t_{k-1} + (\log N)^2$, and $|D|$ is $o(1)$ for $t \geq t_{k-1} + (\log N)^{5/2}$.

Let $R_{k,2}$ be defined in the same way as R , but counting only particles that reach L_A between times $t_{k-1} + (\log N)^2$ and $t_{k-1} + (\log N)^{5/2}$. We can divide this time interval into at most $\delta^{-1}(\log N)^{5/2}$ time intervals of length δ , so by Lemma 15,

$$(70) \quad \begin{aligned} E[R_{k,2} | \mathcal{F}_{t_{k-1}}] &\leq C e^A \delta \cdot \frac{Z'_{N,1}(t_{k-1})}{N(\log N)^5} \cdot \frac{(\log N)^{5/2}}{\delta} \cdot (1 + o(1)) \\ &\leq \frac{C e^A Z'_{N,1}(t_{k-1})(1 + o(1))}{N(\log N)^{5/2}}. \end{aligned}$$

Let $R_{k,3}$ be defined in the same way as R , but counting only particles that reach L_A between times $t_{k-1} + (\log N)^{5/2}$ and t_k . This interval can be divided into $\delta^{-1}(\log N)^3 \theta s(1 + o(1))$ intervals of length δ , so by Lemma 15 and (69),

$$(71) \quad \begin{aligned} E[R_{k,3} | \mathcal{F}_{t_{k-1}}] &= 2\pi e^A \delta (1 + O(|A|\theta)) \\ &\times \frac{Z'_{N,1}(t_{k-1})(1 + o(1))(1 + o(\delta))}{N(\log N)^5} \cdot \frac{(\log N)^3 \theta s}{\delta} \\ &= 2\pi e^A \frac{Z'_{N,1}(t_{k-1})\theta s}{N(\log N)^2} (1 + O(|A|\theta))(1 + o(\delta))(1 + o(1)) + o(1). \end{aligned}$$

The first statement of the proposition follows from (68), (70) and (71) by choosing δ as a function of N so that $\delta \rightarrow 0$ as $N \rightarrow \infty$. On $G_{N,k-1}$, the second statement follows from Lemma 10. \square

The corollary below follows immediately from the above proof, because the number of particles that hit L_A between $t_k - (\log N)^{5/2}$ and t_k can be bounded in the same manner as $E[R_{k,3} | \mathcal{F}_{t_{k-1}}]$, and the number of intervals of length δ is only $(\log N)^{5/2}/\delta$.

COROLLARY 17. *Define \tilde{R}_k the same way as R_k , except only counting particles that reach L_A between $t_k - (\log N)^{5/2}$ and t_k . Then $E[\tilde{R}_k | \mathcal{F}_{t_{k-1}}]$ is $o(1)$ on $G_{N,k-1}$.*

PROPOSITION 18. Assume $A \geq 0$. On $G_{N,k-1}$, we have

$$E[R_k^2 | \mathcal{F}_{t_{k-1}}] \leq \frac{C\theta e^A Z_N(t_{k-1})}{N(\log N)^2} + o(1).$$

PROOF. For the purposes of this proof, we may assume that particles are killed upon reaching L_A . Note that $R_k^2 = R_k + 2Y$, where Y is the number of distinct pairs of particles that get killed upon reaching L_A . We may further write $Y = Y_1 + Y_2$, where Y_1 denotes the number of pairs of particles that get killed upon reaching L_A whose most recent common ancestor is before time t_{k-1} and Y_2 counts the other pairs of particles. Proposition 16 and (33) give that on $G_{N,k-1}$,

$$(72) \quad E[R_k | \mathcal{F}_{t_{k-1}}] \leq \frac{C\theta e^A Z_N(t_{k-1})}{N(\log N)^2} + o(1).$$

If there is a particle at x at time t_{k-1} and a descendant of this particle reaches L_A by time t_k , then the number of pairs in Y_1 involving this descendant will be precisely the number of particles descended from particles other than the particle at x at time t_{k-1} that reach L_A by time t_k , which is bounded by R_k . Because descendants of different particles evolve independently, it follows that

$$(73) \quad E[Y_1 | \mathcal{F}_{t_{k-1}}] \leq (E[R_k | \mathcal{F}_{t_{k-1}}])^2 \leq C \left(\frac{\theta e^A Z_N(t_{k-1})}{N(\log N)^2} \right)^2 + o(1).$$

It remains to consider Y_2 . Because pairs of particles contributing to Y_2 have the same ancestor at time t_{k-1} , we may consider separately the contributions of the particles at time t_{k-1} . Assume for now that there is a single particle at x at time t_{k-1} , and we denote the number of associated pairs of particles contributing to Y_2 by Y_2^x . Let $h(t, y)$ be the expected number of offspring of a single particle that is at y at time $t_{k-1} + t$ that will hit L_A before time t_k . A branching event at $(t_{k-1} + t, y)$ produces, on average, $h(t, y)^2$ pairs of particles that hit L_A and have their most recent common ancestor at time t . Since each particle branches at rate 1,

$$(74) \quad E[Y_2^x] = \int_0^{(\log N)^3 \theta s} \int_0^{L_A} q_t(x, y) h(t, y)^2 dy dt.$$

Since $h(t, y) \leq h(0, y)$, it follows from Proposition 16 that

$$(75) \quad \begin{aligned} h(t, y) &\leq \frac{C e^A \theta e^{\mu y}}{N(\log N)^2} \sin\left(\frac{\pi y}{L_A}\right) (1 + o(1)) + \frac{C e^A e^{\mu y} (1 + o(1))}{N(\log N)^3} \\ &\leq \frac{C e^A (1 + o(1))}{N(\log N)^2} \left(\theta e^{\mu y} \sin\left(\frac{\pi y}{L_A}\right) + \frac{e^{\mu y}}{\log N} \right), \end{aligned}$$

where the $o(1)$ term tends to zero uniformly in y as $N \rightarrow \infty$.

We first evaluate the portion of the integral in (74) when $t \leq (\log N)^2$. Recall from (28) that when $t \leq (\log N)^2$,

$$q_t(x, y) \leq C e^{\mu(x-y)} v_t(x, y),$$

where $v_t(x, y)$ is the density of Brownian motion in the strip $(0, L_A)$, defined as in (14) with L_A in place of K . Therefore, changing the order of integration,

$$\begin{aligned} & \int_0^{(\log N)^2} \int_0^{L_A} q_t(x, y)h(t, y)^2 dy dt \\ & \leq \frac{C e^{2A} e^{\mu x} (1 + o(1))}{N^2 (\log N)^4} \\ & \quad \times \int_0^{L_A} \left(\theta e^{\mu y} \sin\left(\frac{\pi y}{L_A}\right) + \frac{e^{\mu y}}{\log N} \right)^2 e^{-\mu y} \left(\int_0^{(\log N)^2} v_t(x, y) dt \right) dy. \end{aligned}$$

By (51), $\int_0^\infty v_t(x, y) dt \leq 2(L_A - y)$. Using also that $(a + b)^2 \leq C(a^2 + b^2)$, that $\sin(\pi y/L_A) = \sin(\pi(L_A - y)/L_A) \leq \pi(L_A - y)/L_A \leq C(L_A - y)/(\log N)$, that $e^{\mu L_A} = N(\log N)^3 e^{-A}(1 + o(1))$, and that (32) holds, we get

$$\begin{aligned} & \int_0^{(\log N)^2} \int_0^{L_A} q_t(x, y)h(t, y)^2 dy dt \\ (76) \quad & \leq \frac{C e^{2A} e^{\mu x} (1 + o(1))}{N^2 (\log N)^4} \int_0^{L_A} \left(\frac{\theta^2 e^{\mu y} (L_A - y)^3}{(\log N)^2} + \frac{e^{\mu y} (L_A - y)}{(\log N)^2} \right) dy \\ & \leq \frac{C e^{2A} e^{\mu x} (1 + o(1))}{N^2 (\log N)^6} (\theta^2 e^{\mu L_A} + e^{\mu L_A}) \leq \frac{C e^A e^{\mu x} (1 + o(1))}{N (\log N)^3}. \end{aligned}$$

When $t \geq (\log N)^2$, Lemma 5 implies that

$$q_t(x, y) \leq C p_t(x, y) (1 + o(1)) \leq \frac{C(1 + o(1))}{\log N} e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right)$$

because $A \geq 0$. Therefore,

$$\begin{aligned} & \int_{(\log N)^2}^{(\log N)^3 \theta s} \int_0^{L_A} q_t(x, y)h(t, y)^2 dy dt \\ & \leq \frac{C e^{2A} (1 + o(1))}{N^2 (\log N)^5} e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right) \\ & \quad \times \int_{(\log N)^2}^{(\log N)^3 \theta s} \int_0^{L_A} e^{-\mu y} \sin\left(\frac{\pi y}{L_A}\right) \left(\theta e^{\mu y} \sin\left(\frac{\pi y}{L_A}\right) + \frac{e^{\mu y}}{\log N} \right)^2 dy dt \\ (77) \quad & \leq \frac{C \theta e^{2A} (1 + o(1))}{N^2 (\log N)^2} e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right) \\ & \quad \times \int_0^{L_A} \left(\frac{\theta^2 e^{\mu y} (L_A - y)^3}{(\log N)^3} + \frac{e^{\mu y} (L_A - y)}{(\log N)^3} \right) dy \\ & \leq \frac{C \theta e^A (1 + o(1))}{N (\log N)^2} \cdot e^{\mu x} \sin\left(\frac{\pi x}{L_A}\right), \end{aligned}$$

using that the last integral can be bounded by $Ce^{\mu L_A}/(\log N)^3 \leq Ce^{-A}N$. By combining (76) and (77) and summing over the contributions from different particles, we get on $G_{N,k-1}$,

$$\begin{aligned}
 E[Y_2|\mathcal{F}_{t_{k-1}}] &\leq \left(\frac{Ce^A Y_N(t_{k-1})}{N(\log N)^3} + \frac{C\theta e^A Z'_{N,1}(t_{k-1})}{N(\log N)^2} \right) (1 + o(1)) \\
 &\leq \frac{C\theta e^A Z_N(t_{k-1})}{N(\log N)^2} + o(1),
 \end{aligned}
 \tag{78}$$

where the last inequality uses Lemma 10. The result now follows from (72), (73) and (78), using the assumption from (35) that $\theta e^A \varepsilon^{-1/2} \leq 1$. \square

REMARK 19. By Proposition 16 and Markov’s inequality, we see that when θ is small, during most intervals $[t_{k-1}, t_k]$, no particles reach L_A . Using Propositions 16 and 18 and the second moment method, we get that on $G_{N,k-1}$,

$$\begin{aligned}
 P(R_k > 0|\mathcal{F}_{t_{k-1}}) &\geq \frac{(E[R_k|\mathcal{F}_{t_{k-1}}])^2}{E[R_k^2|\mathcal{F}_{t_{k-1}}]} \\
 &\geq C \left(\frac{\theta e^A Z_N(t_{k-1})}{N(\log N)^2} \right)^2 \left(\frac{N(\log N)^2}{\theta e^A Z_N(t_{k-1})} \right) (1 + o(1)) \\
 &\geq \frac{C\theta e^A Z_N(t_{k-1})(1 + o(1))}{N(\log N)^2}.
 \end{aligned}
 \tag{79}$$

Thus, it follows that

$$E[R_k|R_k > 0, \mathcal{F}_{t_{k-1}}] = \frac{E[R_k|\mathcal{F}_{t_{k-1}}]}{P(R_k > 0|\mathcal{F}_{t_{k-1}})} \leq C(1 + o(1)).$$

That is, conditional on the event that at least one particle reaches L_A , the expected number of particles that reach L_A is bounded by a constant that does not depend on θ or A .

3.3. *The probability of $G_N(\varepsilon)$.* We have now acquired enough tools to prove that the probability of $G_N(\varepsilon)$ is close to 1 when ε is small and N is large. It is this result that allows us to work on the event G_N throughout much of the paper.

Throughout this section, we will assume that particles are killed upon reaching L_A . Define $\bar{Z}_N(t)$ and $\bar{Y}_N(t)$ in the same way as $Z_N(t)$ and $Y_N(t)$ in (9) and (10), but for this modified process in which particles are killed upon reaching L_A . Also, we use L_A rather than L in the definition of $\bar{Z}_N(t)$.

LEMMA 20. *For any fixed $A \in \mathbb{R}$ and any fixed $t \geq 0$, under the hypotheses of Proposition 1, we have*

$$\lim_{N \rightarrow \infty} P(\bar{Y}_N(t(\log N)^3) > N(\log N)^3 h(N)) = 0.$$

PROOF. Let B_N be the event that all particles at time zero are in $(0, L_A)$. We have $P(B_N) \rightarrow 1$ as $N \rightarrow \infty$ because on the event that there is a particle to the right of L_A at time zero, we have $Y_N(0) \geq N(\log N)^3 h(N)$ by (31), and $Y_N(0)/(N(\log N)^3 h(N))$ converges in probability to zero by the definition of $h(N)$. On B_N , we have $Y_N(0) = \bar{Y}_N(0)$, so the result holds when $t = 0$.

Suppose instead $t > 0$. By (18), on B_N we have

$$E[\bar{Y}_N(t(\log N)^3) | \mathcal{F}_0] = \frac{4}{\pi} e^{(1-\mu^2/2-\pi^2/2L_A^2)t(\log N)^3} (1 + D)Z_N(0),$$

where $|D|$ is bounded by the right-hand side of (16) with L_A in place of K . Therefore, using (39), on B_N we have

$$E[\bar{Y}_N(t(\log N)^3) | \mathcal{F}_0] \leq C e^{-2\pi^2 A t(1+o(1))} Z_N(0)(1 + o(1)).$$

Therefore, by Markov’s inequality, on B_N we have

$$\begin{aligned} P(\bar{Y}_N(t(\log N)^3) > N(\log N)^3 h(N) | \mathcal{F}_0) &\leq \frac{E[\bar{Y}_N(t(\log N)^3) | \mathcal{F}_0]}{N(\log N)^3 h(N)} \\ &\leq \frac{C e^{-2\pi^2 A t(1+o(1))} Z_N(0)(1 + o(1))}{N(\log N)^3 h(N)}. \end{aligned}$$

The right-hand side converges in probability to zero because $Z_N(0)/N(\log N)^2$ converges in distribution to ν and $(\log N)h(N) \rightarrow \infty$ as $N \rightarrow \infty$. The result follows. \square

For the rest of Section 3.3, we will assume that $A < 0$, so that $L_A > L$.

LEMMA 21. *Fix $A < 0$ and $t \geq 0$. For all $\kappa > 0$, there exists a positive constant C_1 , depending on κ but not on A or t , such that under the hypotheses of Proposition 1,*

$$P\left(\max_{0 \leq r \leq t(\log N)^3} \bar{Z}_N(r) > \frac{1}{2} \varepsilon^{-1/2} N(\log N)^2\right) \leq \kappa + C_1 \varepsilon^{1/2} e^{-2\pi^2 A t(1+o(1))}.$$

PROOF. By (19), if $U(r) = e^{-(1-\mu^2/2-\pi^2/2L_A^2)s} \bar{Z}_N(r)$, then $(U(r), r \geq 0)$ is a martingale. Since $A < 0$, we have $1 - \mu^2/2 - \pi^2/2L_A^2 > 0$. Therefore, by Doob’s maximal inequality (see, e.g., the $p = 1$ case of Theorem 1.4 in [24]) and (39),

$$\begin{aligned} &P\left(\max_{0 \leq r \leq t(\log N)^3} \bar{Z}_N(r) > \frac{1}{2} \varepsilon^{-1/2} N(\log N)^2 \mid \mathcal{F}_0\right) \\ &\leq P\left(\max_{0 \leq s \leq t(\log N)^3} U(r) > \frac{1}{2} \varepsilon^{-1/2} N(\log N)^2 e^{-(1-\mu^2/2-\pi^2/2L_A^2)t(\log N)^3} \mid \mathcal{F}_0\right) \\ &\leq \frac{2\bar{Z}_N(0)\varepsilon^{1/2} e^{(1-\mu^2/2-\pi^2/2L_A^2)t(\log N)^3}}{N(\log N)^2} \leq \frac{2\bar{Z}_N(0)\varepsilon^{1/2} e^{-2\pi^2 A t(1+o(1))}}{N(\log N)^2}. \end{aligned}$$

Because the distribution of $Z_N(0)/N(\log N)^2$, and therefore that of $\bar{Z}_N(0)/N(\log N)^2$, converges to ν , there exists a constant C_1 such that $P(\bar{Z}_N(0)/N(\log N)^2 > C_1) \leq \kappa$. The result follows. \square

LEMMA 22. *Let $\kappa > 0$ and $t > 0$. Under the hypotheses of Proposition 1, there exist positive constants C_2 and γ , depending on κ and t , such that for all $A < 0$, the probability that some particle reaches L_A before time $t(\log N)^3$ is at most $C_2e^{\gamma A} + \kappa$ for sufficiently large N .*

PROOF. Let $J = \lceil 4\pi^2 t \rceil$. For $1 \leq j \leq J$, let $A_j = 2^{j-J} A$ and $s_j = (j/4\pi^2) \times (\log N)^3$. Let $s_0 = 0$. Consider a modified branching Brownian motion, defined up to time $t(\log N)^3$, in which particles that reach L_{A_j} between times s_{j-1} and s_j are killed. Because $L_{A_j} \leq L_A$ for all j , it suffices to bound the probability that at least one particle gets killed in this new modified branching Brownian motion.

Let $\tilde{M}_N(r)$ denote the number of particles alive at time r , and denote the positions of these particles by $\tilde{X}_{1,N}(r) \geq \dots \geq \tilde{X}_{\tilde{M}_N(r),N}(r)$. For $r \in [s_{j-1}, s_j]$, define

$$\tilde{Z}_{N,j}(r) = \sum_{i=1}^{\tilde{M}_N(r)} e^{\mu \tilde{X}_{i,N}(r)} \sin\left(\frac{\pi \tilde{X}_{i,N}(r)}{L_{A_j}}\right)$$

and

$$\tilde{Y}_N(r) = \sum_{i=1}^{\tilde{M}_N(r)} e^{\mu \tilde{X}_{i,N}(r)}.$$

For all $r \in [s_{j-1}, s_j]$, we have, using (19) and (39),

$$\begin{aligned} E[\tilde{Z}_{N,j}(r) | \mathcal{F}_{s_{j-1}}] &= e^{(1-\mu^2/2-\pi^2/2L_{A_j}^2)(r-s_{j-1})} \tilde{Z}_{N,j}(s_{j-1}) \\ (80) \qquad \qquad \qquad &\leq e^{-2\pi^2 A_j (s_j - s_{j-1})(1+o(1))/(\log N)^3} \tilde{Z}_{N,j}(s_{j-1}) \\ &= e^{-A_j(1+o(1))/2} \tilde{Z}_{N,j}(s_{j-1}). \end{aligned}$$

This bound allows us to bound the probability that a particle reaches \tilde{L}_{A_j} between times s_{j-1} and s_j using Proposition 16. We divide the interval from s_{j-1} to s_j into smaller subintervals of length approximately $\theta(\log N)^3$. More precisely, define times $s_{j-1} = u_0 < u_1 < \dots < u_D = s_j$ such that $\theta(\log N)^3 \leq u_k - u_{k-1} \leq 2\theta(\log N)^3$ for all k , which is possible as long as we choose $\theta \leq 1/4\pi^2$. Letting $\tilde{R}_{j,k}$ denote the number of particles that reach L_{A_j} between times u_{k-1} and u_k , we get from Proposition 16,

$$E[\tilde{R}_{j,k} | \mathcal{F}_{u_{k-1}}] \leq \frac{Ce^{A_j} \tilde{Z}_{N,j}(u_{k-1})\theta(1+o(1))}{N(\log N)^2} + \frac{Ce^{A_j} \tilde{Y}_N(u_{k-1})(1+o(1))}{N(\log N)^3}.$$

By Markov's inequality,

$$P(\tilde{R}_{j,k} > 0 | \mathcal{F}_{u_{k-1}}) \leq \frac{C e^{A_j} \tilde{Z}_{N,j}(u_{k-1}) \theta (1 + o(1))}{N(\log N)^2} + \min \left\{ \frac{C e^{A_j} \tilde{Y}_N(u_{k-1}) (1 + o(1))}{N(\log N)^3}, 1 \right\}.$$

By Lemma 20 applied at the times u_0, \dots, u_D , the second term is $o_p(1)$, which means it tends to zero in probability as $N \rightarrow \infty$ for any fixed values of the parameters A and θ . Therefore,

$$P(\tilde{R}_{j,k} > 0 | \mathcal{F}_{s_{j-1}}) = E[P(\tilde{R}_{j,k} > 0 | \mathcal{F}_{u_{k-1}}) | \mathcal{F}_{s_{j-1}}] \leq \frac{C e^{A_j} E[\tilde{Z}_{N,j}(u_{k-1}) | \mathcal{F}_{s_{j-1}}] \theta (1 + o(1))}{N(\log N)^2} + o_p(1).$$

Let $\tilde{R}_j = \sum_{k=1}^D \tilde{R}_{j,k}$. By (80) and the fact that $D \leq C/\theta$,

$$(81) \quad P(\tilde{R}_j > 0 | \mathcal{F}_{s_{j-1}}) \leq \frac{C e^{A_j(1+o(1))/2} \tilde{Z}_{N,j}(s_{j-1}) (1 + o(1))}{N(\log N)^2} + o_p(1).$$

Let \tilde{G}_j be the event that $\tilde{Y}_N(s_k) \leq N(\log N)^3 h(N)$ for $k = 0, \dots, j$, and let $\tilde{G} = \tilde{G}_{J-1}$. We have $P(\tilde{G}) = 1 - o(1)$ by Lemma 20. We now show by induction that for $j = 1, \dots, J$, on \tilde{G}_{j-1} we have

$$(82) \quad E[\tilde{Z}_{N,j}(s_j) | \mathcal{F}_{s_0}] \leq C e^{-(A_1 + \dots + A_j)(1+o(1))/2} Z_N(0) + o(N(\log N)^2).$$

The $j = 1$ case follows from (80) and the fact that $|\tilde{Z}_{N,1}(0) - Z_N(0)| \leq C|A_1| \tilde{Y}_N(0)/(\log N)$ by (38), the difference between the expressions coming from the fact that L_{A_1} is used in the definition of $\tilde{Z}_{N,1}(0)$ and L is used in the definition of $Z_N(0)$. Suppose the result is true for $j - 1$. On \tilde{G}_{j-1} , we have, using (80) and the argument leading to (38),

$$\begin{aligned} E[\tilde{Z}_{N,j}(s_j) | \mathcal{F}_{s_{j-1}}] &\leq e^{-A_j(1+o(1))/2} \tilde{Z}_{N,j}(s_{j-1}) \\ &\leq e^{-A_j(1+o(1))/2} \tilde{Z}_{N,j-1}(s_{j-1}) \\ &\quad + e^{-A_j(1+o(1))/2} |\tilde{Z}_{N,j}(s_{j-1}) - \tilde{Z}_{N,j-1}(s_{j-1})| \\ &\leq e^{-A_j(1+o(1))/2} \tilde{Z}_{N,j-1}(s_{j-1}) \\ &\quad + C e^{-A_j(1+o(1))/2} |A_j - A_{j-1}| \tilde{Y}_N(s_{j-1}) / (\log N) \\ &\leq e^{-A_j(1+o(1))/2} \tilde{Z}_{N,j-1}(s_{j-1}) + o(N(\log N)^2). \end{aligned}$$

Taking conditional expectations with respect to \mathcal{F}_{s_0} and applying the induction hypothesis gives (82). The result (82) for all $j = 1, \dots, J$ on \tilde{G}_{j-1} follows by induction.

We now take conditional expectations with respect to \mathcal{F}_{s_0} on both sides of (81). Using that $|\tilde{Z}_{N,j}(s_{j-1}) - \tilde{Z}_{N,j-1}(s_{j-1})| = o(N(\log N)^2)$ on \tilde{G}_{j-1} as shown above and that

$$A_j - (A_1 + \dots + A_{j-1}) = 2^{j-J} A - (2^j - 2)2^{-J} A = A_1 = 2^{1-J} A = A_1,$$

we get

$$\begin{aligned} P(\tilde{R}_j > 0 | \mathcal{F}_{s_0}) &\leq \frac{C e^{A_j(1+o(1))/2} E[\tilde{Z}_{N,j}(s_{j-1}) | \mathcal{F}_{s_0}](1 + o(1))}{N(\log N)^2} + o_p(1) \\ &\leq \frac{C e^{(A_j - (A_1 + \dots + A_{j-1}))(1+o(1))/2} Z_N(0)(1 + o(1))}{N(\log N)^2} + o_p(1) \\ &\leq \frac{C e^{A_1(1+o(1))/2} Z_N(0)(1 + o(1))}{N(\log N)^2} + o_p(1). \end{aligned}$$

Therefore, the probability, conditional on \mathcal{F}_{s_0} , that some particle reaches $A = A_j$ by time $t(\log N)^3$ is at most

$$\frac{C J e^{A 2^{-J}(1+o(1))} Z_N(0)(1 + o(1))}{N(\log N)^2} + o_p(1).$$

Since $Z_N(0)/N(\log N)^2$ converges in distribution to ν as $N \rightarrow \infty$, there is a constant c such that $P(Z_N(0)/N(\log N)^2 > c) < \kappa/2$ for sufficiently large N . The result follows. \square

PROPOSITION 23. *Under the hypotheses of Proposition 1, we have*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\theta} \left(\limsup_{N \rightarrow \infty} (1 - P(G_N(\varepsilon))) \right) = 0,$$

where the supremum is taken over all values of $\theta > 0$ such that $\theta^{-1} \in \mathbb{N}$.

PROOF. Let $0 < \varepsilon < 1$, and let $\kappa > 0$. Choose C_1 as in Lemma 21, and choose γ and C_2 as in Lemma 22, with $u + s$ in place of t . Choose $A = (\log \varepsilon)/(8\pi^2(u + s)) < 0$. We now assume that θ is small enough that assumptions (32)–(35) hold for these choices of ε and A , so that previous results in this section may be applied. This assumption is permissible because dividing θ by a positive integer to make it small enough to satisfy these conditions can only reduce the value of $P(G_N(\varepsilon))$ by adding additional times at which conditions on Y_N and Z_N must hold.

By Lemma 22, the probability that some particle reaches L_A by time $(u + s) \times (\log N)^3$ is at most

$$(83) \quad \kappa + C_2 e^{\gamma(\log \varepsilon)/(8\pi^2(u+s))}$$

for sufficiently large N . By Lemma 20,

$$(84) \quad \lim_{N \rightarrow \infty} P(\bar{Y}_N(t_j) > N(\log N)^3 h(N) \text{ for some } j \leq \theta^{-1}) = 0.$$

By Lemma 21,

$$\begin{aligned} \limsup_{N \rightarrow \infty} P\left(\bar{Z}_N(t_j) > \frac{1}{2}\varepsilon^{-1/2}N(\log N)^2 \text{ for some } j \leq \theta^{-1}\right) \\ \leq \kappa + C_1\varepsilon^{1/2}e^{-(\log \varepsilon)(1+o(1))/4} \\ \leq \kappa + C_1\varepsilon^{(1+o(1))/4}. \end{aligned}$$

Using (38), on the event that $\bar{Y}_N(t_j) \leq N(\log N)^3h(N)$ and no particle reaches L_A by time $(u + s)(\log N)^3$, we have

$$Z_N(t_j) \leq \bar{Z}_N(t_j) + \frac{\pi|A|L_A N(\log N)^3h(N)}{\sqrt{2}L^2} \leq \bar{Z}_N(t_j) + \frac{1}{2}\varepsilon^{-1/2}N(\log N)^2$$

for sufficiently large N . Thus,

$$\begin{aligned} (85) \quad \limsup_{N \rightarrow \infty} P(Z_N(t_j) > \varepsilon^{-1/2}N(\log N)^2 \text{ for some } j \leq \theta^{-1}) \\ \leq 2\kappa + C_1\varepsilon^{1/4} + C_2e^{\gamma(\log \varepsilon)/(8\pi^2(u+s))}. \end{aligned}$$

As $\bar{Y}_N(t_j) = Y_N(t_j)$ and $\bar{Z}_N(t_j) = Z_N(t_j)$ when no particles reach L_A by time $(u + s)^3(\log N)^3$, we see that $1 - P(G_N(\varepsilon))$ is bounded by the sum of the probabilities in (83), (84) and (85). Since none of the bounds depends on θ , it follows that

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\theta} \left(\limsup_{N \rightarrow \infty} (1 - P(G_N(\varepsilon))) \right) \leq 3\kappa,$$

and the result follows by letting $\kappa \rightarrow 0$. \square

4. Critical branching Brownian motion with killing at $-y$. Consider a branching Brownian motion with drift $-\sqrt{2}$ started with a single particle at 0. From this process, a modified process can be constructed in which particles that reach $-y$ are killed. Let Z_y denote the number of particles that reach $-y$ and are killed. Note that Z_y has the same distribution as the number of particles that hit zero in branching Brownian motion with drift $-\sqrt{2}$ and absorption at zero, started with a single particle at y . Because this process almost surely goes extinct by Theorem 1.1 of [43], and it is easy to verify that infinitely many particles will not reach the origin within any finite time interval, we see that Z_y is almost surely finite for every $0 \leq y < \infty$. Furthermore, because each particle that reaches $-x$ behaves thereafter like another particle started at zero, the number of particles that reach $-(x + y)$ conditional on Z_x is the same as the distribution of the sum of Z_x independent random variables with the same distribution as Z_y . Consequently, the process $(Z_y)_{y \geq 0}$ is a continuous-time branching process, as is shown in Section 5 of [56]. As noted in [56] and in the more recent work of Maillard [53], this branching process is not in the $L \log L$ class. However, the following proposition appears on page 238 of [56].

PROPOSITION 24. *There exists a random variable W such that almost surely*

$$\lim_{y \rightarrow \infty} ye^{-\sqrt{2}y} Z_y = W.$$

Furthermore, for all $u \in \mathbb{R}$, we have

$$(86) \quad E[e^{-e^{\sqrt{2}u}W}] = \psi(u),$$

where $\psi : \mathbb{R} \rightarrow (0, 1)$ solves Kolmogorov's equation

$$\frac{1}{2}\psi'' - \sqrt{2}\psi' = \psi(1 - \psi).$$

COROLLARY 25. *Let $\eta > 0$. There exists y such that*

$$(87) \quad P(|ye^{-\sqrt{2}y} Z_y - W| > \eta) < \eta.$$

Moreover, there exists $\zeta > 0$ such that if particles are killed when they reach $-y$, the probability that any particle remains alive after time ζ is less than η .

PROOF. Equation (87) is immediate from Proposition 24. The second statement follows from the fact that Z_y is almost surely finite, and therefore so is the time when the last remaining particle hits $-y$. \square

Our goal in this section is to show that $P(W > x) \sim B/x$ as $x \rightarrow \infty$ for some constant B . The strategy will be to consider the Laplace transform $E[e^{-\lambda W}]$ for small values of λ , and then apply a Tauberian theorem. From (86), we see that this requires having asymptotic results for $\psi(u)$ as $u \rightarrow -\infty$. Equivalently, if we define $w(x) = \psi(-x)$, then

$$(88) \quad \frac{1}{2}w'' + \sqrt{2}w' + w(w - 1) = 0,$$

and we are looking for asymptotic results for $w(x)$ as $x \rightarrow \infty$. It is well known that

$$(89) \quad 1 - w(x) \sim Cxe^{-\sqrt{2}x};$$

see, for example, (11) in [47] or (1.13) in [16]. However, this result turns out to be insufficient for our purposes. The asymptotic result that we will need is given in the proposition below.

PROPOSITION 26. *Suppose that w is an increasing function satisfying (88) with $\lim_{x \rightarrow \infty} w(x) = 1$ and $\lim_{x \rightarrow -\infty} w(x) = 0$. For all x , let $u(x) = 1 - w(x)$ and $v(x) = u(x)/(xe^{-\sqrt{2}x})$. Then for all $c > 0$, we have*

$$\lim_{x \rightarrow \infty} x(v(x + c) - v(x)) = 0.$$

PROOF. Let $x > 0$. Let $(R_t, t \geq 0)$ be a three-dimensional Bessel process with $R_0 = x$. According to (2.6) of [40], the process

$$X_t = v(R_t) \exp\left(-\int_0^t u(R_s) ds\right)$$

is a positive local martingale, and therefore a supermartingale. Let $T = \inf\{t : R_t = x + c\}$. By the optional sampling theorem,

$$v(x) = E[X_0] \geq E[X_T] = v(x + c) E\left[\exp\left(-\int_0^T u(R_s) ds\right)\right],$$

which means

$$v(x + c) - v(x) \leq v(x + c) \left(1 - E\left[\exp\left(-\int_0^T u(R_s) ds\right)\right]\right).$$

Let $0 < \gamma < 1$, and let A be the event that $R_t \leq \gamma x$ for some $t \leq T$. That is, A is the event that the Bessel process reaches γx before reaching $x + c$. By Corollary 3.4 on page 253 of [61], we have

$$P(A) = \frac{(x + c)^{-1} - x^{-1}}{(x + c)^{-1} - (\gamma x)^{-1}} = \frac{c\gamma}{c + (1 - \gamma)x}.$$

In view of (89), there are constants C_1 and C_2 such that for sufficiently large x , we have $v(x + c) \leq C_1$ and

$$\max_{\gamma x \leq y \leq x+c} u(y) \leq C_2 x e^{-\sqrt{2}\gamma x}.$$

It follows that for sufficiently large x ,

$$\begin{aligned} v(x + c) - v(x) &\leq C_1 E\left[1 - \exp\left(-\int_0^T u(R_s) ds\right)\right] \\ &\leq C_1 E\left[\mathbf{1}_A + \left(\int_0^T u(R_s) ds\right) \mathbf{1}_{A^c}\right] \\ &\leq C_1 P(A) + C_1 C_2 x e^{-\sqrt{2}\gamma x} E[T]. \end{aligned}$$

To bound $E[T]$, note that using E_x to denote expectation for the Bessel process started at x , and τ_z to be the first time that the Bessel process hits z , we have $E_0[\tau_{x+c}] = E_0[\tau_x] + E_x[\tau_{x+c}]$ by the strong Markov property. Therefore, $E[T] = E_x[\tau_{x+c}] \leq E_0[\tau_{x+c}]$. Furthermore, the three-dimensional Bessel process is the Euclidean norm of three-dimensional Brownian motion, which is bounded below by the absolute value of the first coordinate, which is a one-dimensional Brownian motion. Therefore, $E_0[\tau_{x+c}]$ is at most the expected time for a one-dimensional Brownian motion to reach $-(x + c)$ or $x + c$, which for sufficiently large x is at

most C_3x^2 for some constant C_3 . It follows that

$$\begin{aligned} \limsup_{x \rightarrow \infty} x(v(x+c) - v(x)) &\leq \limsup_{x \rightarrow \infty} \left(x \cdot C_1 \frac{c\gamma}{c + (1-\gamma)x} + C_1 C_2 C_3 x^4 e^{-\sqrt{2}\gamma x} \right) \\ &= \frac{c\gamma C_1}{1-\gamma}. \end{aligned}$$

Because this holds for any $\gamma > 0$, and C_1 does not depend on γ , the result follows. □

PROPOSITION 27. *Let W be the limiting random variable in Proposition 24. Then, there exists a constant $B > 0$ such that as $x \rightarrow \infty$,*

$$P(W > x) \sim \frac{B}{x}.$$

PROOF. Let $\phi(\lambda) = E[e^{-\lambda W}]$. According to the discussion on page 335 of [12], the condition that $P(W > x) \sim B/x$ as $x \rightarrow \infty$ is equivalent to the condition that the function $f(z) = z(1 - \phi(1/z))$ has B -index 1, meaning (see page 128 of [12]) that for all $r \geq 1$, we have

$$\lim_{z \rightarrow \infty} (f(rz) - f(z)) = B \log r.$$

That is, $P(W > x) \sim B/x$ is equivalent to the condition that for all $r \geq 1$, we have

$$\lim_{z \rightarrow \infty} rz(1 - \phi(1/rz)) - z(1 - \phi(1/z)) = B \log r,$$

or equivalently, letting $\lambda = 1/z$,

$$(90) \quad \lim_{\lambda \rightarrow 0} \frac{r(1 - \phi(\lambda/r)) - (1 - \phi(\lambda))}{\lambda} = B \log r.$$

Consequently we need to show that (90) holds for all $r \geq 1$.

By (86), we have $\phi(\lambda) = \psi((\log \lambda)/\sqrt{2})$. Let $w(x) = \psi(-x)$, so w satisfies (88). For all x , let $u(x) = 1 - w(x)$ and $v(x) = u(x)/(xe^{-\sqrt{2}x})$, as in Proposition 26. Then

$$\begin{aligned} 1 - \phi(\lambda) &= 1 - \psi\left(\frac{\log \lambda}{\sqrt{2}}\right) = u\left(\frac{-\log \lambda}{\sqrt{2}}\right) = u\left(\frac{\log(1/\lambda)}{\sqrt{2}}\right) \\ &= v\left(\frac{\log(1/\lambda)}{\sqrt{2}}\right) \left(\frac{\log(1/\lambda)}{\sqrt{2}}\right) \lambda. \end{aligned}$$

Likewise,

$$1 - \phi(\lambda/r) = v\left(\frac{\log(r/\lambda)}{\sqrt{2}}\right) \left(\frac{\log(r/\lambda)}{\sqrt{2}}\right) \frac{\lambda}{r}.$$

Letting $x = (\log(1/\lambda))/\sqrt{2}$ and $c = (\log r)/\sqrt{2}$, it follows that

$$\begin{aligned} & \frac{r(1 - \phi(\lambda/r)) - (1 - \phi(\lambda))}{\lambda} \\ &= v\left(\frac{\log(r/\lambda)}{\sqrt{2}}\right)\left(\frac{\log(r/\lambda)}{\sqrt{2}}\right) - v\left(\frac{\log(1/\lambda)}{\sqrt{2}}\right)\left(\frac{\log(1/\lambda)}{\sqrt{2}}\right) \\ &= v(x+c)(x+c) - v(x)x = x(v(x+c) - v(x)) + cv(x+c). \end{aligned}$$

As $x \rightarrow \infty$, we have $v(x+c) \rightarrow C$, where C is the constant from (89), and $x(v(x+c) - v(x)) \rightarrow 0$ by Proposition 26. Therefore,

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{r(1 - \phi(\lambda/r)) - (1 - \phi(\lambda))}{\lambda} &= \lim_{x \rightarrow \infty} [x(v(x+c) - v(x)) + cv(x+c)] \\ &= \frac{C \log r}{\sqrt{2}}, \end{aligned}$$

so (90) holds with $B = C/\sqrt{2}$. \square

We will see later in the proof of Proposition 41 that $B = 1/\sqrt{2}$.

COROLLARY 28. *There is a constant C such that $P(W > x) \leq C/x$ for all x , and $E[W\mathbf{1}_{\{W \leq x\}}] \leq C \log x$ and $E[W^2\mathbf{1}_{\{W \leq x\}}] \leq Cx$ for all $x \geq 2$.*

PROOF. The first statement is immediate from Proposition 27. Since

$$E[W\mathbf{1}_{\{W \leq x\}}] \leq \int_0^x P(W \geq y) dy \leq 1 + \int_1^x \frac{C}{y} dy = 1 + C \log x$$

and

$$E[W^2\mathbf{1}_{\{W \leq x\}}] \leq \int_0^x 2yP(W \geq y) dy \leq 1 + 2 \int_1^x y \cdot \frac{C}{y} dy \leq 1 + 2Cx,$$

the other two statements follow. \square

5. The particles after hitting L_A . Recall that in Section 3.1, we obtained estimates on the number of particles in branching Brownian motion that never reach the level L_A , while in Section 3.2 we estimated the number of particles that reach L_A . In this section, we determine how much the descendants of the particles that reach L_A will contribute to the process at later times. The basic strategy will be to argue that if a particle reaches L_A , then the number of descendants that it will have in the population a long time into the future can be approximated by the number of its descendants that reach $L_A - y$, where y is some large constant. The number of descendants that reach $L_A - y$ can be approximated using the random variable W in Proposition 24.

5.1. *Notation and constants.* Recall from Section 3.2 that R_k particles reach L_A between times t_{k-1} and t_k . By Propositions 16 and 18, on $G_{N,k-1}$ we have

$$(91) \quad E[R_k | \mathcal{F}_{t_{k-1}}] \leq C\theta e^A \varepsilon^{-1/2} + o(1)$$

and

$$(92) \quad E[R_k^2 | \mathcal{F}_{t_{k-1}}] \leq C\theta e^A \varepsilon^{-1/2} + o(1).$$

These moment estimates will be used repeatedly in what follows. Denote by $u_1 < u_2 < \dots < u_{R_k}$ the times at which these particles reach L_A . Recalling (9) and (36), define

$$Z_{N,2}(t_k) = \sum_{i=1}^{M_N(t_k)} e^{\mu X_{i,N}(t_k)} \sin\left(\frac{\pi X_{i,N}(t_k)}{L}\right) \mathbf{1}_{\{i \notin S(t_k)\}} \mathbf{1}_{\{X_{i,N}(t_k) \leq L\}}.$$

Note that

$$Z_N(t_k) = Z_{N,1}(t_k) + Z_{N,2}(t_k),$$

and the particles contributing to $Z_{N,2}(t_k)$ are precisely the particles at time t_k that are descended from the particles that reach L_A at one of the times u_1, \dots, u_{R_k} .

Our aim in this section will be to estimate, on $G_{N,k-1}$, the expectation

$$(93) \quad E[(Z_N(t_k) - Z_N(t_{k-1})) \mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}} | \mathcal{F}_{t_{k-1}}],$$

as well as probabilities of the form

$$(94) \quad P(Z_N(t_k) - Z_N(t_{k-1}) > rN(\log N)^2 | \mathcal{F}_{t_{k-1}})$$

for $r \geq \varepsilon$. We apply the truncation at $\varepsilon N(\log N)^2$ to focus separately on particles reaching L_A that make a small addition to the value of the process, whose contributions are counted in (93), and particles reaching L_A that lead to large jumps in the value of the process, an event whose probability is estimated by (94).

Estimating these quantities precisely will involve manipulating seven constants. Recall that we have been already working with the three constants ε , A and θ . Throughout this section, ε will be a fixed number with $0 < \varepsilon < 1$. We will also introduce a new constant $\delta > 0$ and in fact will fix

$$(95) \quad \delta \leq \varepsilon^7.$$

By Proposition 27, one can choose x large enough that if $z \geq x$, then

$$(96) \quad \frac{(1 - \delta)B}{z} \leq P(W > z) \leq \frac{(1 + \delta)B}{z},$$

where B comes from Proposition 27. We will then choose $A \geq 1$ large enough that

$$(97) \quad 2\sqrt{2}\pi e^{-A}x \leq \varepsilon,$$

$$(98) \quad 4e^{-A/9} \leq \delta/6.$$

Once ε , δ and A are chosen, we will choose $\theta > 0$ small enough to satisfy the following equations:

- (99) $A\theta \leq 1;$
- (100) $4\pi^2 A\theta s\varepsilon^{-1/2} \leq e^{-A/4};$
- (101) $4\theta^{1/4} \leq \delta/6;$
- (102) $\theta^{1/4}e^A \leq \delta;$
- (103) $\theta A^2 \leq \delta^{1/2};$
- (104) $\theta A^2 e^A \varepsilon^{-1/2} \leq 1;$
- (105) $C_0 A \theta^{1/2} \leq 1,$

where C_0 is a constant to be defined later in (112). Note that (99) and (100) were already assumed in (33) and (34), while (104) implies (35) because $A \geq 1$. In this section, we will also work with the additional constants η , y and ζ from Corollary 25. We will choose $\eta = \theta$. We will then choose y to be large enough to satisfy both (87) and the equation

(106) $1 \leq \theta y.$

We finally choose ζ to satisfy the conditions of Corollary 25 for these values of η and y .

Consider the particle that reaches L_A at time $u_j \in (t_{k-1}, t_k]$. Denote by $V_{j,k}$ the number of descendants of this particle that, at some time $t > u_j$, reach $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ and have the property that, for all $u \in [u_j, t)$, the ancestor of this particle was in the interval $(L_A - y + (u - u_j)(\sqrt{2} - \mu), \infty)$. This is equivalent to the number of descendant particles that would reach $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ at time t for some t if particles were killed upon reaching this level. Denote the first times at which these $V_{j,k}$ particles reach level $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ by $r_{1,j,k} < r_{2,j,k} < \dots < r_{V_{j,k},j,k}$. Note that $V_{j,k}$ has the same distribution as the random variable Z_y of Proposition 24, and the adjustment of $(t - u_j)(\sqrt{2} - \mu)$ is necessary because particles drift to the left at rate μ , rather than at rate $\sqrt{2}$ as in the setting of Proposition 24. Now let

$$W'_{j,k} = ye^{-\sqrt{2}y} V_{j,k}.$$

By Corollary 25, there exists a random variable $W_{j,k}$ with the same distribution as the random variable W in Corollary 25 such that $P(|W'_{j,k} - W_{j,k}| > \eta) < \eta$. Furthermore, it is clear that for fixed k , conditional on $\mathcal{F}_{t_{k-1}}$ and conditional on $R_k = r$, the random variables $W'_{1,k}, \dots, W'_{r,k}$ are independent and have the same distribution as $ye^{-\sqrt{2}y} Z_y$. Likewise, the random variables $W_{j,k}$ can be chosen such that conditional on $\mathcal{F}_{t_{k-1}}$ and conditional on $R_k = r$, $W_{1,k}, \dots, W_{r,k}$ are independent and have the same distribution as the random variable W in Corollary 25.

5.2. *The contribution of one particle at L_A .* In this subsection, we show that the contribution to $Z_{N,2}(t_k)$ from the j th particle to hit L_A can be approximated by $\pi\sqrt{2}e^{-A}N(\log N)^2W_{j,k}$. As a result, typically $Z_N(t_k) - Z_N(t_{k-1}) > \varepsilon N(\log N)^2$ precisely when $W_{j,k} > \varepsilon/(\pi\sqrt{2}e^{-A})$ for some $j \leq R_k$. Establishing the validity of this approximation requires bounding the probabilities of several unlikely events.

LEMMA 29. *Let B_1 be the event that there exist $j_1, j_2 \leq R_k$ with $j_1 \neq j_2$ such that $W_{j_1,k} \geq e^{2A/3}$ and $W_{j_2,k} \geq e^{2A/3}$. Then on $G_{N,k-1}$, we have $P(B_1|\mathcal{F}_{t_{k-1}}) \leq C\theta e^{-A/3}\varepsilon^{-1/2} + o(1)$.*

PROOF. Conditional on $\mathcal{F}_{t_{k-1}}$ and R_k , the expected number of pairs (j_1, j_2) with $j_1 \neq j_2$ such that $W_{j_1,k} \geq e^{2A/3}$ and $W_{j_2,k} \geq e^{2A/3}$ is $\binom{R_k}{2}P(W \geq e^{2A/3})^2$, where W is the random variable defined in Corollary 25. By Proposition 27, $P(W \geq e^{2A/3}) \leq Ce^{-2A/3}$, so

$$P(B_1|\mathcal{F}_{t_{k-1}}) \leq CE[R_k^2|\mathcal{F}_{t_{k-1}}]e^{-4A/3} \leq C\theta e^{-A/3}\varepsilon^{-1/2} + o(1),$$

where the last inequality uses (92). \square

LEMMA 30. *Fix $r \geq \varepsilon$, and let B_2 be the event that*

$$\frac{r - 4e^{-A/4} - e^{-A/9} - 4\theta^{1/4}}{\pi\sqrt{2}e^{-A}} \leq W_{j,k} \leq \frac{r + 4e^{-A/4} + 4\theta^{1/4}}{\pi\sqrt{2}e^{-A}}$$

for some $j \leq R_k$. On $G_{N,k-1}$, we have $P(B_2|\mathcal{F}_{t_{k-1}}) \leq C\theta\delta\varepsilon^{-5/2} + o(1)$, where the constant C does not depend on r .

PROOF. Let $\gamma = 4e^{-A/4} + e^{-A/9} + 4\theta^{1/4}$. Note that $\gamma \leq \delta/2 \leq \varepsilon/2$ because $4e^{-A/4} \leq 4e^{-A/9} \leq \delta/6$ and $4\theta^{1/4} \leq \delta/6$ by (98) and (101). Assume x is chosen so that (96) holds for $z \geq x$. By (97), we have $(r - \gamma)/(\pi\sqrt{2}e^{-A}) \geq (\varepsilon - \gamma)/(\pi\sqrt{2}e^{-A}) \geq x$. Therefore,

$$\begin{aligned} P\left(\frac{r - \gamma}{\pi\sqrt{2}e^{-A}} \leq W \leq \frac{r + \gamma}{\pi\sqrt{2}e^{-A}}\right) &\leq \frac{B(1 + \delta)\pi\sqrt{2}e^{-A}}{r - \gamma} - \frac{B(1 - \delta)\pi\sqrt{2}e^{-A}}{r + \gamma} \\ &\leq Ce^{-A}\left(\frac{1 + \delta}{r - \gamma} - \frac{1 - \delta}{r + \gamma}\right) = Ce^{-A}\left(\frac{2\gamma + 2r\delta}{r^2 - \gamma^2}\right) \leq \frac{Ce^{-A}\delta}{\varepsilon^2}. \end{aligned}$$

It follows from this bound and Markov’s inequality that $P(B_2|\mathcal{F}_{t_{k-1}}) \leq Ce^{-A}\delta \times \varepsilon^{-2}E[R_k|\mathcal{F}_{t_{k-1}}]$. The result now follows from (91). \square

LEMMA 31. *Let B_3 be the event that for some j , the particle that reaches L_A at time u_j has a descendant that at some time $t \in (u_j, t_k]$ reaches $L_A - y + (t - u_j)(\sqrt{2} - \mu)$, and that this descendant itself has a descendant that reaches L_A before time t_k . Then on $G_{N,k-1}$, we have $P(B_3|\mathcal{F}_{t_{k-1}}) \leq Ce^A\theta^{3/2}\varepsilon^{-1/2} + o(1)$.*

PROOF. The particle that reaches L_A at time u_j has $V_{j,k}$ descendants that reach $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ at some time $t > u_j$. Let $A_{j,k}$ be the event that one of these particles reaches $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ at some time $t > u_j + \zeta$. By Corollary 25 and Proposition 27, since $\theta = \eta < 1$,

$$\begin{aligned}
 &P(A_{j,k} \cup \{W'_{j,k} > \theta^{-1/2}\} \text{ for some } j \leq R_k | \mathcal{F}_{t_{k-1}}) \\
 (107) \quad &\leq E[R_k | \mathcal{F}_{t_{k-1}}](2\eta + P(W > \theta^{-1/2} - \eta)) \\
 &\leq CE[R_k | \mathcal{F}_{t_{k-1}}](\eta + \sqrt{\theta}).
 \end{aligned}$$

At most $y^{-1}e^{\sqrt{2}y}W'_{j,k}$ descendants of the particle that reaches L_A at time u_j will hit $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ at some time $t \leq t_k$. This is an upper bound rather than an equality because some particles may reach this level after time t_k . We now consider N large enough that $y \geq \zeta(\sqrt{2} - \mu)$. On the event $A_{j,k}^c \cap \{W'_{j,k} \leq \theta^{-1/2}\}$, the probability that a descendant of one of these particles reaches L_A by time t_k can be bounded above by $y^{-1}e^{\sqrt{2}y}\theta^{-1/2}$ times the probability that a single particle at $L_A - y + \zeta(\sqrt{2} - \mu)$ has a descendant that reaches L_A by time $(\log N)^3\theta s$. Using Markov's inequality to bound this latter probability by the expectation of the number of such descendants, it follows from Proposition 16 that the probability is bounded above by

$$\begin{aligned}
 &\frac{Ce^A}{N(\log N)^2} \left(\theta e^{\mu(L_A - y + \zeta(\sqrt{2} - \mu))} \sin\left(\frac{\pi(L_A - y + \zeta(\sqrt{2} - \mu))}{L_A}\right) \right. \\
 &\qquad \qquad \qquad \left. + \frac{e^{\mu(L_A - y + \zeta(\sqrt{2} - \mu))}}{\log N} \right) (1 + o(1)).
 \end{aligned}$$

Note that we are applying Proposition 16 in the case when $k = 1$, and there is just a single particle initially at the location $L_A - y + \zeta(\sqrt{2} - \mu)$. Since $e^{\mu L_A} = N(\log N)^3 e^{-A}(1 + o(1))$, $\sin(\pi(L_A - y + \zeta(\sqrt{2} - \mu))/L_A) \leq (Cy/\log N)(1 + o(1))$, and $e^{\mu\zeta(\sqrt{2} - \mu)}$ is $1 + o(1)$ this expression can be bounded above by

$$\begin{aligned}
 &\frac{Ce^A}{N(\log N)^2} (\theta y e^{-\mu y} N(\log N)^2 e^{-A} + e^{-\mu y} N(\log N)^2 e^{-A})(1 + o(1)) \\
 &\leq Ce^{-\mu y} (\theta y + 1)(1 + o(1)).
 \end{aligned}$$

Combining these observations gives

$$\begin{aligned} P(B_3|\mathcal{F}_{t_{k-1}}) &\leq CE[R_k|\mathcal{F}_{t_{k-1}}](\eta + \sqrt{\theta} + y^{-1}e^{\sqrt{2}y}\theta^{-1/2} \cdot e^{-\mu y}(\theta y + 1))(1 + o(1)) \\ &\leq CE[R_k|\mathcal{F}_{t_{k-1}}](\eta + \sqrt{\theta} + \theta^{-1/2}y^{-1})(1 + o(1)). \end{aligned}$$

The result now follows from (91) and the assumptions that $\eta = \theta$ and $1 \leq \theta y$. \square

Recall that the particles at time t_k contributing to $Z_{N,2}(t_k)$ are precisely the particles at time t_k that are descended from the particles that reach L_A at one of the times u_1, \dots, u_{R_k} . To separate the contributions from each of these particles, write $i \in S_j$ if the particle at $X_{i,N}(t_k)$ at time t_k is descended from the particle that was at L_A at time u_j . Then for $1 \leq j \leq R_k$, define

$$(108) \quad Z_{N,2,j}(t_k) = \sum_{i=1}^{M_N(t_k)} e^{\mu X_{i,N}(t_k)} \sin\left(\frac{\pi X_{i,N}(t_k)}{L}\right) \mathbf{1}_{\{i \in S_j\}} \mathbf{1}_{\{X_{i,N}(t) \leq L\}}.$$

Note that $Z_{N,2}(t_k) = \sum_{j=1}^{R_k} Z_{N,2,j}(t_k)$. The next lemma shows that $Z_{N,2,j}(t_k)$ is approximately determined by the random variable $W_{j,k}$.

LEMMA 32. *Let B_4 be the event that for some $j \leq R_k$, we have*

$$|Z_{N,2,j}(t_k) - \pi\sqrt{2}e^{-A}N(\log N)^2W_{j,k}| > 4N(\log N)^2\theta^{1/4}.$$

On $G_{N,k-1}$, we have $P(B_4|\mathcal{F}_{t_{k-1}}) \leq Ce^A\theta^{5/4}\varepsilon^{-1/2} + o(1)$.

PROOF. Define a new random variable $Z'_{N,2,j}(t_k)$ by modifying $Z_{N,2,j}(t_k)$ in the following three ways:

- We set $Z'_{N,2,j}(t_k)$ to zero if $u_j > t_k - (\log N)^{5/2}$.
- We set $Z'_{N,2,j}(t_k)$ to zero if $rv_{j,k,j,k} > u_j + \zeta$.
- We modify S_j to exclude particles that, after time u_j , reach $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ at some time $t \in (u_j, t_k]$ but then reach L_A again before time t_k . [Note that this modification is equivalent to killing particles that reach L_A after they reach $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ at some time $t > u_j$.]

Then define $Z''_{N,2,j}(t_k)$ by making these three modifications and replacing L by L_A in the definition (108).

By Corollary 17 and Markov’s inequality, $P(u_{R_k} > t_k - (\log N)^{5/2}|\mathcal{F}_{t_{k-1}}) = o(1)$ on $G_{N,k-1}$. This implies that the first of the four modifications above is unlikely to occur. By Corollary 25 and (91),

$$P(rv_{j,k,j,k} > u_j + \zeta \text{ for some } j|\mathcal{F}_{t_{k-1}}) \leq \eta E[R_k|\mathcal{F}_{t_{k-1}}] \leq C\eta\theta e^A\varepsilon^{-1/2} + o(1),$$

which bounds the probability of the second type of modification. Lemma 31 bounds the probability of the third type of modification. These results and the fact that $\eta = \theta$ imply that on $G_{N,k-1}$,

$$(109) \quad \begin{aligned} P(Z'_{N,2,j}(t_k) \neq Z_{N,2,j}(t_k) \text{ for some } j \leq R_k | \mathcal{F}_{t_{k-1}}) \\ \leq C e^A \theta^{3/2} \varepsilon^{-1/2} + o(1). \end{aligned}$$

Let Γ_j be the event that $u_j \leq t_k - (\log N)^{5/2}$, that $r_{V_{j,k},j,k} \leq u_j + \zeta$ and that $W'_{j,k} \leq \theta^{-1/4}$. The probability that either of the first two of these events fails to occur has already been bounded, so using the argument given in (107), on $G_{N,t_{k-1}}$ we have

$$(110) \quad P\left(\bigcup_{j=1}^{R_k} \Gamma_j^c | \mathcal{F}_{t_{k-1}}\right) \leq C e^A \theta^{5/4} \varepsilon^{-1/2} + o(1).$$

Let $\mathcal{H}_{k-1} = \sigma(\mathcal{F}_{t_{k-1}}, V_{1,k}, \dots, V_{R_k,k}, u_1, \dots, u_{R_k}, (r_{i,j,k})_{1 \leq i \leq V_{j,k}, 1 \leq j \leq R_k})$. Note that $\Gamma_j \in \mathcal{H}_{k-1}$ for all j , and on Γ_j for sufficiently large N , the $V_{j,k}$ particles that reach $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ for some $t > u_j$ are all reaching a level between $L_A - y$ and $L_A - y + \zeta(\sqrt{2} - \mu)$ at some time between t_{k-1} and t_k . These particles and their descendants then evolve independently until time t_k , and we kill particles that return to L_A if we are evaluating $Z'_{N,2,j}(t_k)$ or $Z''_{N,2,j}(t_k)$.

By the argument leading to (40), with the times $r_{i,j,k}$ playing the role of t_{k-1} , on Γ_j we have

$$\begin{aligned} E[|Z'_{N,2,j}(t_k) - Z''_{N,2,j}(t_k)| | \mathcal{H}_{k-1}] \\ \leq V_{j,k} \frac{C A e^{\mu(L_A - y)}}{\log N} (1 + o(1)) \\ \leq C y^{-1} e^{\sqrt{2}y} W'_{j,k} \cdot \frac{A e^{-A} N (\log N)^3 e^{-\mu y}}{\log N} (1 + o(1)) \\ \leq C y^{-1} N (\log N)^2 \theta^{-1/4} (1 + o(1)). \end{aligned}$$

Therefore, by Markov's inequality and assumption (106), that $1 \leq \theta y$, on Γ_j , we have

$$(111) \quad \begin{aligned} P(|Z'_{N,2,j}(t_k) - Z''_{N,2,j}(t_k)| > N (\log N)^2 \theta^{1/4} | \mathcal{H}_{k-1}) \leq C y^{-1} \theta^{-1/2} + o(1) \\ \leq C \theta^{1/2} + o(1). \end{aligned}$$

Let

$$y_{i,j,k} = e^{\mu(L_A - y + (r_{i,j,k} - u_j)(\sqrt{2} - \mu))}$$

and

$$z_{i,j,k} = y_{i,j,k} \sin\left(\frac{\pi(L_A - y + (r_{i,j,k} - u_j)(\sqrt{2} - \mu))}{L_A}\right).$$

The i th of the $V_{j,k}$ particles that reach $L_A - y + (t - u_j)(\sqrt{2} - \mu)$ for some $t > u_j$ reaches this level at time $r_{i,j,k}$. Therefore, by (19) and (39), on Γ_j the expected contribution to $Z''_{N,2,j}(t_k)$ from descendants of this particle is given by

$$\begin{aligned} & e^{(1-\mu^2/2-\pi^2/2L_A^2)(t_k-r_{i,j,k})} z_{i,j,k} \\ &= (1 + O(A\theta) + o(1)) e^{\mu(L_A-y)} \frac{\pi y}{L_A} \left(1 - O\left(\frac{\zeta(\sqrt{2} - \mu)}{y}\right)\right) \\ &= e^{\mu(L_A-y)} \frac{\pi y}{L_A} (1 + O(A\theta) + o(1)). \end{aligned}$$

Thus, on Γ_j ,

$$\begin{aligned} E[Z''_{N,2,j}(t_k)|\mathcal{H}_{k-1}] &= V_{j,k} \left(e^{\mu(L_A-y)} \frac{\pi y}{L_A} (1 + O(A\theta) + o(1)) \right) \\ &= V_{j,k} \left(N(\log N)^3 e^{-A} e^{-\mu y} \frac{\pi y}{L_A} \right) (1 + O(A\theta) + o(1)) \\ &= W'_{j,k} (\pi \sqrt{2} e^{-A} N(\log N)^2) (1 + O(A\theta) + o(1)). \end{aligned}$$

This means there is a constant C_0 such that for sufficiently large N , on Γ_j ,

$$\begin{aligned} |E[Z''_{N,2,j}(t_k)|\mathcal{H}_{k-1}] - \pi \sqrt{2} e^{-A} N(\log N)^2 W'_{j,k}| &\leq C_0 N(\log N)^2 W'_{j,k} A\theta \\ &\leq C_0 N(\log N)^2 A\theta^{3/4}. \end{aligned}$$

Therefore, using (105),

$$(112) \quad |E[Z''_{N,2,j}(t_k)|\mathcal{H}_{k-1}] - \pi \sqrt{2} e^{-A} N(\log N)^2 W'_{j,k}| \leq N(\log N)^2 \theta^{1/4}$$

for sufficiently large N . On Γ_j we can similarly estimate the variance of the contribution of each of these particles. We apply (46), with the times $r_{i,j,k}$ playing the role of t_{k-1} . Since the descendants of these particles after times $r_{1,j,k}, \dots, r_{V_{j,k},j,k}$ evolve independently, we get

$$\text{Var}(Z''_{N,2,j}(t_k)|\mathcal{H}_{k-1}) \leq \sum_{i=1}^{V_{j,k}} C\theta N(\log N)^2 e^{-A} \left(z_{i,j,k} + \frac{y_{i,j,k}}{\theta \log N} \right) (1 + o(1)).$$

Arguing as above and using (106), we find that on Γ_j ,

$$\begin{aligned} & \text{Var}(Z''_{N,2,j}(t_k)|\mathcal{H}_{k-1}) \\ & \leq CV_{j,k}\theta N(\log N)^2 \\ & \quad \times e^{-A} \left(e^{\mu(L_A-y)} \sin\left(\frac{\pi(L_A-y)}{L_A}\right) + \frac{e^{\mu(L_A-y)}}{\theta \log N} \right) (1 + o(1)) \\ (113) \quad & \leq CV_{j,k}\theta N(\log N)^2 \end{aligned}$$

$$\begin{aligned} &\times e^{-2A}(ye^{-\mu y}N(\log N)^2 + \theta^{-1}e^{-\mu y}N(\log N)^2)(1 + o(1)) \\ &\leq CW'_{j,k}N^2(\log N)^4\theta e^{-2A}(1 + \theta^{-1}y^{-1})(1 + o(1)) \\ &\leq CN^2(\log N)^4\theta^{3/4}(1 + o(1)). \end{aligned}$$

By (112), (113) and the conditional form of Chebyshev’s inequality, on Γ_j we have

$$\begin{aligned} &P(|Z''_{N,2,j}(t_k) - \pi\sqrt{2}e^{-A}N(\log N)^2W'_{j,k}| > 2N(\log N)^2\theta^{1/4}|\mathcal{H}_{k-1}) \\ &\leq \frac{CN^2(\log N)^4\theta^{3/4}(1 + o(1))}{(N(\log N)^2\theta^{1/4})^2} \leq C\theta^{1/4} + o(1). \end{aligned}$$

Note that $\pi\sqrt{2}e^{-A}\eta \leq \theta^{1/4}$ because $A \geq 0, \eta = \theta, \delta \leq 1$ by (95), and thus $\theta^{3/4} \leq 1/24^3$ by (101). Therefore, since $P(|W'_{j,k} - W_{j,k}| > \eta) < \eta$, on Γ_j we have

$$\begin{aligned} &P(|Z''_{N,2,j}(t_k) - \pi\sqrt{2}e^{-A}N(\log N)^2W_{j,k}| > 3N(\log N)^2\theta^{1/4}|\mathcal{H}_{k-1}) \\ &\leq C\theta^{1/4} + \eta + o(1) \leq C\theta^{1/4} + o(1). \end{aligned}$$

Now (111) leads to

$$\begin{aligned} &P(|Z'_{N,2,j}(t_k) - \pi\sqrt{2}e^{-A}N(\log N)^2W_{j,k}| > 4N(\log N)^2\theta^{1/4}|\mathcal{H}_{k-1}) \\ &\leq \mathbf{1}_{\Gamma_j^c} + C\theta^{1/4} + o(1). \end{aligned}$$

Taking the union over over $j \leq R_k$ and then taking conditional expectations of both sides with respect to $\mathcal{F}_{t_{k-1}}$, we get

$$(114) \quad \begin{aligned} &P(|Z'_{N,2,j}(t_k) - \pi\sqrt{2}e^{-A}N(\log N)^2W_{j,k}| > 4N(\log N)^2\theta^{1/4} \\ &\text{for some } j \leq R_k | \mathcal{F}_{t_{k-1}}) \end{aligned}$$

$$\leq P\left(\bigcup_{j=1}^{R_k} \Gamma_j^c | \mathcal{F}_{t_{k-1}}\right) + (C\theta^{1/4} + o(1))E[R_k | \mathcal{F}_{t_{k-1}}].$$

The result now follows from (109), (110) and (91). \square

LEMMA 33. *Let B_5 be the event that*

$$\sum_{j=1}^{R_k} Z_{N,2,j}(t_k)\mathbf{1}_{\{W_{j,k} \leq e^{2A/3}\}} > e^{-A/9}N(\log N)^2$$

or

$$\sum_{j=1}^{R_k} W_{j,k}\mathbf{1}_{\{W_{j,k} \leq e^{2A/3}\}} > \frac{e^{8A/9}}{\pi\sqrt{2}}.$$

Then $P(B_5 | \mathcal{F}_{t_{k-1}}) \leq C(\theta^{5/4}e^A + \theta e^{-A/9})\varepsilon^{-1/2} + o(1)$ on $G_{N,k-1}$.

PROOF. We have

$$(115) \quad \begin{aligned} P(B_5|\mathcal{F}_{t_{k-1}}) &\leq P(B_4|\mathcal{F}_{t_{k-1}}) \\ &+ P\left(\sum_{j=1}^{R_k} (W_{j,k} + \beta)\mathbf{1}_{\{W_{j,k} \leq e^{2A/3}\}} > \frac{e^{8A/9}}{\pi\sqrt{2}} \middle| \mathcal{F}_{t_{k-1}}\right), \end{aligned}$$

where $\beta = 4e^A\theta^{1/4}/(\pi\sqrt{2})$, which by (102) is bounded by a constant. Let $\mathcal{G}_{k-1} = \sigma(\mathcal{F}_{t_{k-1}}, R_k)$. Using Corollary 28 with $x = e^{2A/3}$ and (91),

$$(116) \quad \begin{aligned} E\left[\text{Var}\left(\sum_{j=1}^{R_k} (W_{j,k} + \beta)\mathbf{1}_{\{W_{j,k} \leq e^{2A/3}\}} \middle| \mathcal{G}_{k-1}\right) \middle| \mathcal{F}_{t_{k-1}}\right] \\ &= E[R_k \text{Var}((W + \beta)\mathbf{1}_{\{W \leq e^{2A/3}\}}) \middle| \mathcal{F}_{t_{k-1}}] \\ &\leq E[(W + \beta)^2\mathbf{1}_{\{W \leq e^{2A/3}\}}]E[R_k|\mathcal{F}_{t_{k-1}}] \\ &\leq C e^{2A/3} \cdot C\theta e^A \varepsilon^{-1/2} + o(1) \\ &\leq C\theta e^{5A/3} \varepsilon^{-1/2} + o(1). \end{aligned}$$

Likewise, using Corollary 28 and (92),

$$(117) \quad \begin{aligned} \text{Var}\left(E\left[\sum_{j=1}^{R_k} (W_{j,k} + \beta)\mathbf{1}_{\{W_{j,k} \leq e^{2A/3}\}} \middle| \mathcal{G}_{k-1}\right] \middle| \mathcal{F}_{t_{k-1}}\right) \\ \leq \text{Var}(R_k E[(W + \beta)\mathbf{1}_{\{W \leq e^{2A/3}\}}] \middle| \mathcal{F}_{t_{k-1}}) \\ \leq \text{Var}(C A R_k \middle| \mathcal{F}_{t_{k-1}}) \leq C A^2 E[R_k^2 \middle| \mathcal{F}_{t_{k-1}}] \leq C\theta A^2 e^A \varepsilon^{-1/2} + o(1). \end{aligned}$$

Recall that for all random variables X and σ -fields \mathcal{F} and \mathcal{G} with $\mathcal{F} \subset \mathcal{G}$,

$$\text{Var}(X|\mathcal{F}) = E[\text{Var}(X|\mathcal{G})|\mathcal{F}] + \text{Var}(E[X|\mathcal{G}]|\mathcal{F}).$$

Therefore, summing (116) and (117) gives

$$\text{Var}\left(\sum_{j=1}^{R_k} (W_{j,k} + \beta)\mathbf{1}_{\{W_{j,k} \leq e^{2A/3}\}} \middle| \mathcal{F}_{t_{k-1}}\right) \leq C\theta e^{5A/3} \varepsilon^{-1/2} + o(1),$$

as $A^2 e^{-2A/3}$ is bounded by a constant. Also, using again Corollary 28 and since $A^2\theta e^A \varepsilon^{-1/2} \leq 1$ by (104),

$$\begin{aligned} E\left[\sum_{j=1}^{R_k} (W_{j,k} + \beta)\mathbf{1}_{\{W_{j,k} \leq e^{2A/3}\}} \middle| \mathcal{F}_{t_{k-1}}\right] &\leq C E(R_k|\mathcal{F}_{t_{k-1}})(\beta + 2A/3) \\ &\leq C\theta \varepsilon^{-1/2} A e^A + o(1) \leq C + o(1). \end{aligned}$$

Thus by the conditional form of Chebyshev’s inequality, we get

$$\begin{aligned}
 P\left(\sum_{j=1}^{R_k} (W_{j,k} + \beta) \mathbf{1}_{\{W_{j,k} \leq e^{2A/3}\}} > \frac{e^{8A/9}}{\pi\sqrt{2}} \middle| \mathcal{F}_{t_{k-1}}\right) &\leq \frac{C\theta e^{5A/3} \varepsilon^{-1/2}}{(e^{8A/9}/\pi\sqrt{2})^2} + o(1) \\
 &\leq C\theta e^{-A/9} \varepsilon^{-1/2} + o(1),
 \end{aligned}$$

which, combined with (115) and Lemma 32, gives the result. \square

LEMMA 34. Fix $r \geq \varepsilon$. Consider the event E that $Z_N(t_k) - Z_N(t_{k-1}) > rN(\log N)^2$, and consider the event F that $W_{j,k} > r/(\pi\sqrt{2}e^{-A})$ for some $j \leq R_k$. Let B_6 be the event that one of these two events occurs but not the other (i.e., the symmetric difference of these two events). Then $P(B_6|\mathcal{F}_{t_{k-1}}) \leq C\theta\delta\varepsilon^{-5/2} + o(1)$ on $G_{N,k-1}$, where the constant C does not depend on r .

PROOF. Let B_0 be the event that $|Z_{N,1}(t_k) - Z_N(t_{k-1})| > 4e^{-A/4}N(\log N)^2$. By Corollary 13 and Lemmas 29–33 as well as the assumptions (98) and (102), we have on $G_{N,k-1}$,

$$P\left(\bigcup_{i=0}^5 B_i \middle| \mathcal{F}_{t_{k-1}}\right) \leq C\delta\theta\varepsilon^{-5/2} + o(1).$$

Therefore, it suffices to show that

$$B_6 \subset B = \bigcup_{i=0}^5 B_i.$$

Thus, suppose first $\omega \in E^c \cap F$, and let us show that $\omega \in B$. We have $W_{j,k} > r/(\pi\sqrt{2}e^{-A})$ for some $j \leq R_k$. It follows that if $\omega \in B_2^c$, we have $W_{j,k} > (r + 4e^{-A/4} + 4\theta^{1/4})/(\pi\sqrt{2}e^{-A})$. If furthermore $\omega \in B_2^c \cap B_4^c$, we have $Z_{N,2,j}(t_k) > N(\log N)^2(r + 4e^{-A/4})$. Now if also $\omega \in B_0^c$, we have $Z_{N,1}(t_k) \geq Z_N(t_{k-1}) - 4e^{-A/4}N(\log N)^2$, so on $B_2^c \cap B_4^c \cap B_0^c$, we have $Z_N(t_k) \geq Z_{N,1}(t_k) + Z_{N,2,j}(t_k) > Z_N(t_{k-1}) + rN(\log N)^2$, and so E occurs. Since we have assumed that $\omega \notin E$, it must be that $\omega \in B_0 \cup B_2 \cup B_4 \subset B$.

Alternatively, suppose $\omega \in E \cap F^c$, hence $W_{j,k} \leq r/(\pi\sqrt{2}e^{-A})$ for all $j \leq R_k$. It follows that on B_2^c , we have $W_{j,k} \leq (r - 4e^{-A/4} - e^{-A/9} - 4\theta^{1/4})/(\pi\sqrt{2}e^{-A})$ for all $j \leq R_k$. Then on $B_2^c \cap B_4^c$, we have $Z_{N,2,j}(t_k) \leq N(\log N)^2(r - 4e^{-A/4} - e^{-A/9})$ for all $j \leq R_k$. On B_1^c , there exists at most one $j \leq R_k$ such that $W_{j,k} \geq e^{2A/3}$. Therefore, on $B_2^c \cap B_4^c \cap B_1^c \cap B_5^c$, we have

$$Z_{N,2}(t_k) = \sum_{j=1}^{R_k} Z_{N,2,j}(t_k) \leq N(\log N)^2(r - 4e^{-A/4}).$$

Finally, on B_0^c , we have $Z_{N,1}(t_k) \leq Z_N(t_{k-1}) + 4e^{-A/4}N(\log N)^2$, so on $\bigcap_{i=0}^5 B_i^c$, we have $Z_N(t_k) \leq Z_N(t_{k-1}) + rN(\log N)^2$ which means that E does not occur. Since we assumed $\omega \in E$, it must be that $\omega \in \bigcup_{i=0}^5 B_i = B$, which finishes the proof of the lemma. \square

5.3. *The small jumps.* In this subsection, we estimate the expectation in (93), which covers the case in which the process Z_N does not jump by more than $\varepsilon N(\log N)^2$ between times t_{k-1} and t_k . We have

$$Z_N(t_k) - Z_N(t_{k-1}) = (Z_{N,1}(t_k) - Z_N(t_{k-1})) + \sum_{j=1}^{R_k} Z_{N,2,j}(t_k).$$

Lemma 34 with $r = \varepsilon$ shows that with high probability, we have $Z_N(t_k) - Z_N(t_{k-1}) > \varepsilon N(\log N)^2$ if and only if one of the random variables $W_{1,k}, \dots, W_{R_k,k}$ is greater than $\varepsilon/(\pi\sqrt{2}e^{-A})$. Therefore, in view of Lemma 32, we can approximate the quantity in (93) by

$$(118) \quad \begin{aligned} S_k &= (Z_{N,1}(t_k) - Z_N(t_{k-1})) \\ &+ \pi\sqrt{2}e^{-A}N(\log N)^2 \sum_{j=1}^{R_k} W_{j,k} \mathbf{1}_{\{W_{j,k} \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}}, \end{aligned}$$

which omits the contributions from terms with $W_{j,k} > \varepsilon/(\pi\sqrt{2}e^{-A})$. We now calculate the expected value of S_k and will later justify in Lemma 38 that this is sufficiently close to the quantity in (93).

LEMMA 35. *On $G_{N,k-1}$, we have*

$$\begin{aligned} E[S_k | \mathcal{F}_{t_{k-1}}] &= Z_N(t_{k-1})\theta s(2\sqrt{2}\pi^2 E[W \mathbf{1}_{\{W \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}}] - 2\pi^2 A) \\ &+ O(A^2\theta^2\varepsilon^{-1/2}N(\log N)^2) + o(N(\log N)^2). \end{aligned}$$

PROOF. By Lemma 11, we have on $G_{N,k-1}$

$$(119) \quad \begin{aligned} &E[Z_{N,1}(t_k) - Z_N(t_{k-1}) | \mathcal{F}_{t_{k-1}}] \\ &= -Z_N(t_{k-1})(2\pi^2 A\theta s + O(A^2\theta^2)) + o(N(\log N)^2). \end{aligned}$$

Also, since the random variables $W_{j,k}$ are independent of one another, and of $\mathcal{F}_{t_{k-1}}$ and R_k , we have

$$E \left[\sum_{j=1}^{R_k} W_{j,k} \mathbf{1}_{\{W_{j,k} \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}} \middle| \mathcal{F}_{t_{k-1}} \right] = E[W \mathbf{1}_{\{W \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}}] E[R_k | \mathcal{F}_{t_{k-1}}].$$

Combining this result with Proposition 16, we get on $G_{N,k-1}$,

$$\begin{aligned}
 & E \left[\pi \sqrt{2} e^{-A} N (\log N)^2 \sum_{i=1}^{R_k} W_{j,k} \mathbf{1}_{\{W_{j,k} \leq \varepsilon / (\pi \sqrt{2} e^{-A})\}} \middle| \mathcal{F}_{t_{k-1}} \right] \\
 (120) \quad & = E [W \mathbf{1}_{\{W \leq \varepsilon / (\pi \sqrt{2} e^{-A})\}}] \\
 & \quad \times (2\sqrt{2}\pi^2\theta_s Z_N(t_{k-1})(1 + O(A\theta)) + o(N(\log N)^2)).
 \end{aligned}$$

Note that from Corollary 28,

$$\begin{aligned}
 (121) \quad & E [W \mathbf{1}_{\{W \leq \varepsilon / (\pi \sqrt{2} e^{-A})\}}] \leq 1 + C \log \left(\frac{\varepsilon}{\pi \sqrt{2} e^{-A}} \right) \\
 & \leq 1 + C(\log \varepsilon + A) \leq CA,
 \end{aligned}$$

since $\log \varepsilon < 0$. The result now follows by combining (119) and (120), and using (121) to help bound some of the error terms. \square

It remains to bound the expected error that is made when approximating the increment $(Z_N(t_k) - Z_N(t_{k-1})) \mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}$ by S_k .

LEMMA 36. *We have $E[(Z'_{N,1}(t_k) - Z_N(t_{k-1}))^2 | \mathcal{F}_{t_{k-1}}] \leq C\theta N^2(\log N)^4 \times (e^{-A}\varepsilon^{-1/2} + o(1))$ on $G_{N,k-1}$.*

PROOF. By Lemmas 11 and 12, on $G_{N,k-1}$,

$$\begin{aligned}
 & E[(Z'_{N,1}(t_k) - Z_N(t_{k-1}))^2 | \mathcal{F}_{t_{k-1}}] \\
 & = \text{Var}(Z'_{N,1}(t_k) | \mathcal{F}_{t_{k-1}}) + (E[Z'_{N,1}(t_k) - Z_N(t_{k-1}) | \mathcal{F}_{t_{k-1}}])^2 \\
 & \leq C\theta N(\log N)^2 e^{-A} (Z_N(t_{k-1}) + o(N(\log N)^2)) \\
 & \quad + (CA\theta Z_N(t_{k-1}) + o(N(\log N)^2))^2 \\
 & \leq C\theta N^2(\log N)^4 e^{-A} \varepsilon^{-1/2} + CA^2\theta^2 N^2(\log N)^4 \varepsilon^{-1} + o(N^2(\log N)^4) \\
 & \leq C\theta N^2(\log N)^4 (e^{-A} \varepsilon^{-1/2} + A^2\theta \varepsilon^{-1} + o(1)),
 \end{aligned}$$

and the result follows from (104). \square

LEMMA 37. *On $G_{N,k-1}$, we have*

$$E \left[\left(\sum_{j=1}^{R_k} W_{j,k} \mathbf{1}_{\{W_{j,k} \leq \varepsilon / (\pi \sqrt{2} e^{-A})\}} \right)^2 \middle| \mathcal{F}_{t_{k-1}} \right] \leq C\theta e^{2A} \varepsilon^{1/2} + o(1).$$

PROOF. Note that $e^{-A/9} \leq C\delta \leq C\varepsilon$ by (95) and (98). Because $A^2e^{-8A/9}$ is bounded above by a constant, it follows that $A^2 \leq C\varepsilon e^A$. Therefore, by (91), (92), (121) and Corollary 28, on $G_{N,k-1}$,

$$\begin{aligned} & E \left[\left(\sum_{j=1}^{R_k} W_{j,k} \mathbf{1}_{\{W_{j,k} \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}} \right)^2 \middle| \mathcal{F}_{t_{k-1}} \right] \\ &= E[R_k | \mathcal{F}_{t_{k-1}}] E[W^2 \mathbf{1}_{\{W \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}}] \\ &\quad + E[R_k(R_k - 1) | \mathcal{F}_{t_{k-1}}] (E[W \mathbf{1}_{\{W \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}}])^2 \\ &\leq (C\theta e^A \varepsilon^{-1/2} + o(1))(\varepsilon e^A + A^2) \leq C\theta e^{2A} \varepsilon^{1/2} + o(1) \end{aligned}$$

as claimed. \square

LEMMA 38. On $G_{N,k-1}$, we have

$$\begin{aligned} (122) \quad & E[|S_k - (Z_N(t_k) - Z_N(t_{k-1})) \mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}| | \mathcal{F}_{t_{k-1}}] \\ & \leq C\theta N(\log N)^2 \delta^{1/2} \varepsilon^{-3} + o(N(\log N)^2). \end{aligned}$$

PROOF. Throughout this proof, we work on the event $G_{N,k-1}$. Choose $r = \varepsilon$, and recall from the proof of Lemma 34 that the event $B = \bigcup_{i=0}^5 B_i$ can also be written as $B = \bigcup_{i=0}^6 B_i$ since $B_6 \subset \bigcup_{i=0}^5 B_i$. We will bound the following three terms:

$$(123) \quad E[\mathbf{1}_{B^c} |S_k - (Z_N(t_k) - Z_N(t_{k-1})) \mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}| | \mathcal{F}_{t_{k-1}}];$$

$$(124) \quad E[\mathbf{1}_B |(Z_N(t_k) - Z_N(t_{k-1})) \mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}| | \mathcal{F}_{t_{k-1}}];$$

$$(125) \quad E[\mathbf{1}_B |S_k| | \mathcal{F}_{t_{k-1}}].$$

We first bound (123). On B_6^c , we have $Z_N(t_k) - Z_N(t_{k-1}) > \varepsilon N(\log N)^2$ if and only if $W_{j,k} > \varepsilon/(\pi\sqrt{2}e^{-A})$ for some $j \leq R_k$. In this case, on the event that $W_{j_0,k} > \varepsilon/(\pi\sqrt{2}e^{-A})$ for some $j_0 \leq R_k$, the difference between S_k and $(Z_N(t_k) - Z_N(t_{k-1})) \mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}$ will simply be S_k , as the latter expression will be zero. However, on B_0^c , we have

$$|Z_{N,1}(t_k) - Z_N(t_{k-1})| \leq 4e^{-A/4} N(\log N)^2.$$

By (98) and the fact that $\delta \in (0, \varepsilon)$ we have $\varepsilon/(\pi\sqrt{2}e^{-A}) \geq e^{2A/3}$. Thus $W_{j_0,k} \geq e^{2A/3}$ and on B_1^c , for all $j \neq j_0$, $W_{j,k} \leq e^{2A/3}$. Thus, the definition of B_5 from Lemma 33 implies that on $B_1^c \cap B_5^c$, we have

$$\sum_{i=1}^{R_k} W_{j,k} \mathbf{1}_{\{W_{j,k} \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}} \leq \frac{e^{8A/9}}{\pi\sqrt{2}}.$$

Therefore, $|S_k| \leq (4e^{-A/4} + e^{-A/9})N(\log N)^2$ on $B^c \cap \{W_{j_0,k} > \varepsilon/(\pi\sqrt{2}e^{-A})$ for some $j_0 \leq R_k\}$. If, however, $W_{j,k} \leq \varepsilon/(\pi\sqrt{2}e^{-A})$ for all $j \leq R_k$, then on B_4^c , the difference between S_k and $(Z_N(t_k) - Z_N(t_{k-1}))\mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}$ is bounded by $4R_k N(\log N)^2 \theta^{1/4}$. Therefore,

$$\begin{aligned} & E[|S_k - (Z_N(t_k) - Z_N(t_{k-1}))\mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}| \mathbf{1}_{B^c} | \mathcal{F}_{t_{k-1}}] \\ & \leq ((4e^{-A/4} + e^{-A/9}) \\ & \quad \times P(W_{j_0,k} > \varepsilon/(\pi\sqrt{2}e^{-A}) \text{ for some } j_0 | \mathcal{F}_{t_{k-1}}) \\ & \quad + 4\theta^{1/4} E[R_k | \mathcal{F}_{t_{k-1}}]) N(\log N)^2. \end{aligned}$$

Now (91) gives $E[R_k | \mathcal{F}_{t_{k-1}}] \leq C\theta e^A \varepsilon^{-1/2} + o(1)$, and Proposition 27 implies

$$\begin{aligned} P(W_{j_0,k} \geq \varepsilon/(\pi\sqrt{2}e^{-A}) \text{ for some } j_0 | \mathcal{F}_{t_{k-1}}) & \leq C e^{-A} \varepsilon^{-1} E[R_k | \mathcal{F}_{t_{k-1}}] \\ & \leq C\theta \varepsilon^{-3/2} + o(1). \end{aligned}$$

Therefore,

$$\begin{aligned} (126) \quad & E[|S_k - (Z_N(t_k) - Z_N(t_{k-1}))\mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}| \mathbf{1}_{B^c} | \mathcal{F}_{t_{k-1}}] \\ & \leq (C\theta \varepsilon^{-3/2} e^{-A/9} + C\theta^{5/4} e^A \varepsilon^{-1/2} + o(1)) N(\log N)^2, \end{aligned}$$

which gives a bound on (123).

We next bound (124). By Lemma 34 and its proof, we have

$$(127) \quad P(B | \mathcal{F}_{t_{k-1}}) \leq C\theta \delta \varepsilon^{-5/2} + o(1).$$

The random variable in (124) is bounded in absolute value by $\max\{Z_N(t_{k-1}), \varepsilon N(\log N)^2\}$. Therefore, on $G_{N,k-1}$,

$$\begin{aligned} (128) \quad & E[|(Z_N(t_k) - Z_N(t_{k-1}))\mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N(\log N)^2\}}| \mathbf{1}_B | \mathcal{F}_{t_{k-1}}] \\ & \leq P(B | \mathcal{F}_{t_{k-1}}) N(\log N)^2 \varepsilon^{-1/2} \leq C\theta \delta \varepsilon^{-3} N(\log N)^2 + o(N(\log N)^2). \end{aligned}$$

It remains to bound (125). By the conditional Cauchy–Schwarz inequality, Lemma 36, and (127),

$$\begin{aligned} (129) \quad & E[|Z'_{N,1}(t_k) - Z_N(t_{k-1})| \mathbf{1}_B | \mathcal{F}_{t_{k-1}}] \\ & \leq \sqrt{E[(Z'_{N,1}(t_k) - Z_N(t_{k-1}))^2 | \mathcal{F}_{t_{k-1}}] P(B | \mathcal{F}_{t_{k-1}})} \\ & \leq \sqrt{C\theta N^2 (\log N)^4 e^{-A} \varepsilon^{-1/2} \cdot \theta \delta \varepsilon^{-5/2} (1 + o(1))} \\ & \leq C\theta e^{-A/2} \delta^{1/2} \varepsilon^{-3/2} N(\log N)^2 (1 + o(1)). \end{aligned}$$

Likewise, by the conditional Cauchy–Schwarz inequality and Lemma 37,

$$\begin{aligned}
 & E \left[\left| \pi \sqrt{2} e^{-A} N (\log N)^2 \sum_{j=1}^{R_k} W_{j,k} \mathbf{1}_{\{W_{j,k} \leq \varepsilon / (\pi \sqrt{2} e^{-A})\}} \right| \mathbf{1}_B \middle| \mathcal{F}_{t_{k-1}} \right] \\
 (130) \quad & \leq C e^{-A} N (\log N)^2 \sqrt{\theta e^{2A} \varepsilon^{1/2} \cdot \theta \delta \varepsilon^{-5/2} (1 + o(1))} \\
 & \leq C \theta N (\log N)^2 \delta^{1/2} \varepsilon^{-1} (1 + o(1)).
 \end{aligned}$$

Now Lemma 10, (129) and (130) imply

$$\begin{aligned}
 (131) \quad & E[|S_k \mathbf{1}_B| \mathcal{F}_{t_{k-1}}] \leq C \theta N (\log N)^2 \delta^{1/2} \\
 & \times (e^{-A/2} \varepsilon^{-3/2} + \varepsilon^{-1}) + o(N (\log N)^2).
 \end{aligned}$$

The result follows from (126), (128) and (131) in view of the inequality (98) and $\delta \leq \varepsilon$, as well as (102). \square

PROPOSITION 39. *There exists a real number c such that*

$$\begin{aligned}
 & E[(Z_N(t_k) - Z_N(t_{k-1})) \mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N (\log N)^2\}} | \mathcal{F}_{t_{k-1}}] \\
 & = Z_N(t_{k-1}) \theta s (c + 2\pi^2 \log \varepsilon + g(\varepsilon, A)) \\
 & + O(\theta N (\log N)^2 \delta^{1/2} \varepsilon^{-3}) + o(N (\log N)^2),
 \end{aligned}$$

where $g : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ is a function such that $\lim_{y \rightarrow \infty} g(x, y) = 0$ for all $x > 0$.

PROOF. By combining Lemmas 35 and 38 and using (103), we get

$$\begin{aligned}
 (132) \quad & E[(Z_N(t_k) - Z_N(t_{k-1})) \mathbf{1}_{\{Z_N(t_k) - Z_N(t_{k-1}) \leq \varepsilon N (\log N)^2\}} | \mathcal{F}_{t_{k-1}}] \\
 & = Z_N(t_{k-1}) \theta s (2\sqrt{2} \pi^2 E[W \mathbf{1}_{\{W \leq \varepsilon / (\pi \sqrt{2} e^{-A})\}}] - 2\pi^2 A) \\
 & + O(\theta N (\log N)^2 \delta^{1/2} \varepsilon^{-3}) + o(N (\log N)^2).
 \end{aligned}$$

Denote the conditional expectation on the left-hand side of this equation by $f(N, \varepsilon, \theta)$. Note that this expectation depends on N , ε and θ , but can not depend on δ or A , as these constants were introduced just for the proof. Assume for the moment that $k = 1$, and the initial conditions are chosen so that $Z_N(0) = N (\log N)^2$. Then there exists a positive constant C such that

$$\begin{aligned}
 & \limsup_{\theta \rightarrow 0} \limsup_{N \rightarrow \infty} \frac{f(N, \varepsilon, \theta)}{N (\log N)^2 \theta s} \\
 & \leq (2\sqrt{2} \pi^2 E[W \mathbf{1}_{\{W \leq \varepsilon / (\pi \sqrt{2} e^{-A})\}}] - 2\pi^2 A) + C \delta^{1/2} \varepsilon^{-3}
 \end{aligned}$$

and likewise

$$\begin{aligned} & \liminf_{\theta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{f(N, \varepsilon, \theta)}{N(\log N)^2 \theta s} \\ & \geq (2\sqrt{2}\pi^2 E[W\mathbf{1}_{\{W \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}}] - 2\pi^2 A) - C\delta^{1/2}\varepsilon^{-3}. \end{aligned}$$

We now simultaneously take $\delta \rightarrow 0$ and $A \rightarrow \infty$. This can be done without violating the constraints on the constants because once δ is chosen, we can pick A large enough to satisfy (97) and (98), and then only consider θ small enough that (99)–(105) are satisfied. The second term $C\delta^{1/2}\varepsilon^{-3}$ then tends to zero. Since the left-hand side does not depend on A , the first term must also tend to a limit as $A \rightarrow \infty$. That is, we know that

$$(133) \quad \lim_{A \rightarrow \infty} (2\sqrt{2}\pi^2 E[W\mathbf{1}_{\{W \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}}] - 2\pi^2 A) \quad \text{exists.}$$

Now let $r = \varepsilon/(\pi\sqrt{2}e^{-A})$, so

$$A = \log\left(\frac{\pi\sqrt{2}r}{\varepsilon}\right) = \log(\pi\sqrt{2}r) - \log \varepsilon.$$

Therefore, the limit in (133) is equal to

$$(134) \quad \lim_{r \rightarrow \infty} 2\pi^2(\sqrt{2}E[W\mathbf{1}_{\{W \leq r\}}] - \log(\pi\sqrt{2}r) + \log \varepsilon) = c + 2\pi^2 \log \varepsilon$$

for some real number c that does not depend on ε . The proposition follows. \square

REMARK 40. Equation (133) is a statement which concerns only critical branching Brownian motion with absorption and does not depend on N . It would be desirable to find a direct proof of this fact, but we were not able to obtain one. This would follow if one could show that

$$\int_1^\infty \left| P(W > x) - \frac{1}{\sqrt{2x}} \right| dx < \infty.$$

An explicit expression for the value of the limit in (133) would also make it possible to identify the constant a appearing in the statement of Proposition 1.

5.4. *The large jumps.* We now estimate the probability in (94) that the process Z_N makes a large jump between times t_{k-1} and t_k .

PROPOSITION 41. *For all $r \geq \varepsilon$, on $G_{N,k-1}$ we have*

$$\begin{aligned} & P(Z_N(t_k) - Z_N(t_{k-1}) > rN(\log N)^2 | \mathcal{F}_{t_{k-1}}) \\ & = \frac{2\pi^2 \theta s}{r} \cdot \frac{Z_N(t_{k-1})}{N(\log N)^2} + O(\theta \delta \varepsilon^{-5/2}) + o(1). \end{aligned}$$

PROOF. By Lemma 34, we have

$$\begin{aligned} &P(Z_N(t_k) - Z_N(t_{k-1}) > rN(\log N)^2 | \mathcal{F}_{t_{k-1}}) \\ &= P(W_{j,k} > r/(\pi\sqrt{2}e^{-A}) \text{ for some } j | \mathcal{F}_{t_{k-1}}) + O(\theta\delta\varepsilon^{-5/2}) + o(1). \end{aligned}$$

Recall that $\varepsilon/(\pi\sqrt{2}e^{-A}) \geq e^{2A/3}$ by (98) and the fact that $\delta \in (0, \varepsilon)$. By Lemma 29, for sufficiently large A the probability that $W_{j_1,k} > r/(\pi\sqrt{2}e^{-A})$ and $W_{j_2,k} > r/(\pi\sqrt{2}e^{-A})$ for some $j_1 \neq j_2$ is at most $C\theta e^{-A/3}\varepsilon^{-1/2} + o(1)$. Therefore,

$$\begin{aligned} &P(W_{j,k} > r/(\pi\sqrt{2}e^{-A}) \text{ for some } j | \mathcal{F}_{t_{k-1}}) \\ &= E[R_k | \mathcal{F}_{t_{k-1}}]P(W > r/(\pi\sqrt{2}e^{-A})) + O(\theta e^{-A/3}\varepsilon^{-1/2}) + o(1), \end{aligned}$$

and the error term is smaller than $O(\theta\delta\varepsilon^{-5/2})$ by (98).

By Proposition 27, if we use \sim to mean that the ratio of the two sides tends to one as $x \rightarrow \infty$, then

$$\begin{aligned} E[W\mathbf{1}_{\{W \leq x\}}] &= \int_0^x P(y \leq W \leq x) dy \\ &= \int_0^x P(W \geq y) dy - xP(W > x) \sim B \log x. \end{aligned}$$

Therefore, (134) implies that $B = 1/\sqrt{2}$. Therefore, by (96),

$$\frac{(1 - \delta)\pi}{re^A} \leq P\left(W > \frac{r}{\pi\sqrt{2}e^{-A}}\right) \leq \frac{(1 + \delta)\pi}{re^A}.$$

Combining this result with Proposition 16, we get on $G_{N,k-1}$,

$$\begin{aligned} (135) \quad &E[R_k | \mathcal{F}_{t_{k-1}}]P(W > r/(\pi\sqrt{2}e^{-A})) \\ &= \frac{2\pi^2\theta s}{r} \cdot \frac{Z_N(t_{k-1})}{N(\log N)^2} \cdot (1 + O(A\theta))(1 + O(\delta)) + o(1), \end{aligned}$$

which is enough to imply the result. Since $1/r \leq \varepsilon^{-1}$ and $Z_N(t_{k-1})/N(\log N)^2 \leq \varepsilon^{-1/2}$ on $G_{N,k-1}$, the dominant error term coming from (135) is $O(\theta\delta\varepsilon^{-3/2})$. \square

6. Convergence to the CSBP. In this section, we prove Proposition 1 and Theorem 2. Both of these results require proving that a sequence of processes converges to the continuous-state branching process $(Z(t), t \geq 0)$ with branching mechanism

$$\Psi(u) = au + 2\pi^2u \log u = -cu + 2\pi^2 \int_0^\infty (e^{-ux} - 1 + ux\mathbf{1}_{\{x \leq 1\}})x^{-2} dx,$$

where c is the constant defined in (134). We will first establish Proposition 1, and then use this result to deduce Theorem 2.

6.1. *The generator of the CSBP.* Let $C_0([0, \infty))$ be the set of continuous functions $f : [0, \infty) \rightarrow \mathbb{R}$ that vanish at infinity, endowed with the sup norm so that for $f \in C_0([0, \infty))$, we have

$$\|f\| = \sup_{x \geq 0} |f(x)|.$$

For $f \in C_0([0, \infty))$ and $x \in [0, \infty)$, let $T_t f(x) = E[f(Z(t)) | Z(0) = x]$. It is well-known (see, e.g., [23]) that $(T_t, t \geq 0)$ is a Feller semigroup. The following result describes the associated infinitesimal generator. This result is essentially well-known. The form of the generator appeared in [64], and later in [26] where a particle representation of continuous-state branching processes was constructed. The fact that the set \mathcal{E} defined below is a core for the generator was established for closely related families of processes in [51, 52]. However, we give a short proof of the result below for completeness.

PROPOSITION 42. *Let A be the infinitesimal generator for $(Z(t), t \geq 0)$. Let $\mathcal{E} \subset C_0([0, \infty))$ be the set of functions of the form*

$$(136) \quad f(x) = a_1 e^{-\lambda_1 x} + \dots + a_m e^{-\lambda_m x},$$

where $a_1, \dots, a_m \in \mathbb{R}$ and $\lambda_1, \dots, \lambda_m > 0$. Then \mathcal{E} is a core for A , and for $f \in \mathcal{E}$,

$$(137) \quad Af(x) = x \left(cf'(x) + 2\pi^2 \int_0^\infty (f(x+y) - f(x) - y \mathbf{1}_{\{y \leq 1\}} f'(x)) y^{-2} dy \right).$$

PROOF. If $f(x) = e^{-\lambda x}$, then by (5) and (6), we have

$$\begin{aligned} Af(x) &= \lim_{t \rightarrow 0} \frac{T_t f(x) - f(x)}{t} = \lim_{t \rightarrow 0} \frac{e^{-xu_t(\lambda)} - e^{-\lambda x}}{t} \\ &= \left. \frac{\partial}{\partial t} e^{-xu_t(\lambda)} \right|_{t=0} = x e^{-\lambda x} \Psi(\lambda), \end{aligned}$$

which equals the right-hand side of (137). The result (137) then follows for all $f \in \mathcal{E}$ by linearity. By the Stone–Weierstrass theorem, \mathcal{E} is dense in $C_0([0, \infty))$. By (5), we have $T_t f \in \mathcal{E}$ whenever $f \in \mathcal{E}$. It now follows from Proposition 3.3 in Chapter 1 of [31] that \mathcal{E} is a core for A . \square

6.2. *Proof of Proposition 1.* The next result is Theorem 8.2 in Chapter 4 of [31] in the present context.

PROPOSITION 43. *Suppose the distribution of $V_N(0)$ converges to the distribution of $Z(0)$ as $N \rightarrow \infty$. Then the finite-dimensional distributions of $(V_N(t), t \geq 0)$ converge to those of $(Z(t), t \geq 0)$ as $N \rightarrow \infty$ if and only if for all*

$j \geq 0$, all $0 \leq s_1 < s_2 < \dots < s_j \leq u < u + s$, all bounded continuous functions $h_1, \dots, h_j : [0, \infty) \rightarrow \mathbb{R}$, and all $f \in \mathcal{E}$, we have

$$(138) \quad \lim_{N \rightarrow \infty} E \left[\left(f(V_N(u + s)) - f(V_N(u)) - \int_u^{u+s} Af(V_N(t)) dt \right) \prod_{i=1}^j h_i(V_N(s_i)) \right] = 0.$$

In view of this result, we will aim to establish (138), which will imply Proposition 1. We will assume that $0 \leq s_1 < s_2 < \dots < s_j \leq u < u + s$. We also define the times

$$u = \tau_0 < \tau_1 < \dots < \tau_{\theta-1} = u + s,$$

where $\tau_k = t_k / (\log N)^3$ for all k . This means that $V_N(\tau_k) = Z_N(t_k) / (N(\log N)^2)$ for all k . We also assume that the function $f \in \mathcal{E}$ and the bounded continuous functions h_1, \dots, h_j are fixed throughout this subsection.

Since f is of the form given in (136), the norms $\|f\|$, $\|f'\|$ and $\|f''\|$ are finite and thus can be treated as constants. If $g(x) = xf(x)$ and $d(x) = xf'(x)$, then $\|g\|$, $\|g'\|$ and $\|d\|$ are likewise finite. Also, if we define

$$(139) \quad h(x) = \sup_{y \geq x} x|f''(y)|, \quad k(x) = \sup_{y \geq x} x|f(y)|,$$

then it is easy to check that $\|h\| < \infty$ and $\|k\| < \infty$. Finally, if $y \geq 0$, then by Taylor's theorem there is a $z \in [x, x + y]$ such that $f(x + y) = f(x) + yf'(x) + \frac{1}{2}y^2 f''(z)$. Therefore,

$$\left| x \int_0^1 (f(x + y) - f(x) - yf'(x))y^{-2} dy \right| \leq \frac{1}{2}|h(x)|$$

and

$$\left| x \int_1^\infty (f(x + y) - f(x))y^{-2} dy \right| \leq |k(x)| + |g(x)|.$$

It follows that

$$(140) \quad \|Af\| \leq |c|\|d\| + 2\pi^2(\|g\| + \|k\| + \frac{1}{2}\|h\|) < \infty.$$

LEMMA 44. *We have*

$$(141) \quad \begin{aligned} & E[(f(V_N(\tau_k)) - f(V_N(\tau_{k-1})))\mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) \leq \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \\ & = f'(V_N(\tau_{k-1}))V_N(\tau_{k-1})\theta s(c + 2\pi^2 \log \varepsilon) \mathbf{1}_{G_{N,k-1}} + O(\theta \varepsilon^{1/2}) + o(1). \end{aligned}$$

PROOF. Define

$$\bar{S}_k = \frac{Z'_{N,1}(t_k) - Z_N(t_{k-1})}{N(\log N)^2} + \pi\sqrt{2}e^{-A} \sum_{j=1}^{R_k} W_{j,k} \mathbf{1}_{\{W_{j,k} \leq \varepsilon/(\pi\sqrt{2}e^{-A})\}}.$$

Note that \bar{S}_k would be equal to $S_k/(N(\log N)^2)$, where S_k is defined in (118), if $Z'_{N,1}(t_k)$ were replaced in the definition by $Z_{N,1}(t_k)$. Therefore, by Lemma 10,

$$(142) \quad E \left[\left| \bar{S}_k - \frac{S_k}{N(\log N)^2} \right| \middle| \mathcal{F}_{t_{k-1}} \right] \mathbf{1}_{G_{N,k-1}} = o(1).$$

Thus, by Lemma 38,

$$(143) \quad \begin{aligned} E[|\bar{S}_k - (V_N(\tau_k) - V_N(\tau_{k-1}))| \mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) \leq \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \\ \leq C\theta\delta^{1/2}\varepsilon^{-3} + o(1). \end{aligned}$$

It follows from (143) that

$$(144) \quad \begin{aligned} E[|f(V_N(\tau_k)) - f(V_N(\tau_{k-1}) + \bar{S}_k)| \mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) \leq \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \\ \leq C\|f'\|\theta\delta^{1/2}\varepsilon^{-3} + o(1) \leq C\theta\delta^{1/2}\varepsilon^{-3} + o(1). \end{aligned}$$

By Taylor's theorem, there exists ξ between $V_N(\tau_{k-1})$ and $V_N(\tau_{k-1}) + \bar{S}_k$ such that

$$(145) \quad \begin{aligned} E[(f(V_N(\tau_{k-1}) + \bar{S}_k) - f(V_N(\tau_{k-1}))) \mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) \leq \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \\ = E[(f'(V_N(\tau_{k-1}))\bar{S}_k + f''(\xi)\bar{S}_k^2/2) \mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) \leq \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \\ = f'(V_N(\tau_{k-1}))E[\bar{S}_k \mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) \leq \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \\ + O(E[\bar{S}_k^2 | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}}). \end{aligned}$$

Lemma 38 and (142) give

$$(146) \quad E[|\bar{S}_k| \mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) > \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \leq C\theta\delta^{1/2}\varepsilon^{-3} + o(1).$$

Note that $\delta^{1/2}\varepsilon^{-3} \leq \varepsilon^{1/2}$ by (95), and A can be chosen large enough so that $g(\varepsilon, A) \leq \varepsilon$, where g is the function from Proposition 39. Therefore, (146) combined with Lemma 38, equation (142), and Proposition 39 implies

$$(147) \quad \begin{aligned} f'(V_N(\tau_{k-1}))E[\bar{S}_k \mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) \leq \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \\ = f'(V_N(\tau_{k-1}))E[\bar{S}_k | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} + O(\theta\varepsilon^{1/2}) + o(1) \\ = f'(V_N(\tau_{k-1}))V_N(\tau_{k-1})\theta s(c + 2\pi^2 \log \varepsilon) \mathbf{1}_{\{G_{N,k-1}\}} + O(\theta\varepsilon^{1/2}) + o(1). \end{aligned}$$

Since $e^{-A}\varepsilon^{-1/2} \leq \varepsilon^{1/2}$ by (95) and (98), it follows from Lemmas 36 and 37 that

$$(148) \quad \begin{aligned} E[\bar{S}_k^2 | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \leq C(\theta e^{-A}\varepsilon^{-1/2} + e^{-2A} \cdot \theta e^{2A}\varepsilon^{1/2}) + o(1) \\ \leq C\theta\varepsilon^{1/2} + o(1). \end{aligned}$$

The result follows from (144), (145), (147) and (148). \square

LEMMA 45. *We have*

$$\begin{aligned}
 & E[(f(V_N(\tau_k)) - f(V_N(\tau_{k-1})))\mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) > \varepsilon\}}|\mathcal{F}_{t_{k-1}}]\mathbf{1}_{G_{N,k-1}} \\
 &= 2\pi^2\theta s V_N(\tau_{k-1})\mathbf{1}_{G_{N,k-1}} \int_{\varepsilon}^{\infty} (f(V_N(\tau_{k-1}) + y) - f(V_N(\tau_{k-1})))y^{-2} dy \\
 &+ O(\theta\varepsilon^{1/2}) + o(1).
 \end{aligned}$$

PROOF. By Proposition 41 with $r = \varepsilon$,

$$\begin{aligned}
 & E[f(V_N(\tau_{k-1}))\mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) > \varepsilon\}}|\mathcal{F}_{t_{k-1}}]\mathbf{1}_{G_{N,k-1}} \\
 (149) \quad &= f(V_N(\tau_{k-1}))P(V_N(\tau_k) - V_N(\tau_{k-1}) > \varepsilon|\mathcal{F}_{t_{k-1}})\mathbf{1}_{G_{N,k-1}} \\
 &= f(V_N(\tau_{k-1}))V_N(\tau_{k-1}) \cdot \frac{2\pi^2\theta s}{\varepsilon}\mathbf{1}_{G_{N,k-1}} + O(\theta\delta\varepsilon^{-5/2}) + o(1) \\
 &= 2\pi^2\theta s V_N(\tau_{k-1})\mathbf{1}_{G_{N,k-1}} \int_{\varepsilon}^{\infty} f(V_N(\tau_{k-1}))y^{-2} dy + O(\theta\delta\varepsilon^{-5/2}) + o(1).
 \end{aligned}$$

To simplify notation, assume that ε^{-1} is an integer. Then

$$\begin{aligned}
 & E[f(V_N(\tau_k))\mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) > \varepsilon\}}|\mathcal{F}_{t_{k-1}}]\mathbf{1}_{G_{N,k-1}} \\
 &= \sum_{m=\varepsilon^{-1}}^{\infty} E[f(V_N(\tau_k))\mathbf{1}_{\{m\varepsilon^2 < V_N(\tau_k) - V_N(\tau_{k-1}) \leq (m+1)\varepsilon^2\}}|\mathcal{F}_{t_{k-1}}]\mathbf{1}_{G_{N,k-1}} \\
 &= \sum_{m=\varepsilon^{-1}}^{\varepsilon^{-3}-1} f(\varepsilon^2 m + V_N(\tau_{k-1})) \\
 (150) \quad &\times P(m\varepsilon^2 < V_N(\tau_k) - V_N(\tau_{k-1}) \leq (m+1)\varepsilon^2|\mathcal{F}_{t_{k-1}})\mathbf{1}_{G_{N,k-1}} \\
 &+ \sum_{m=\varepsilon^{-1}}^{\varepsilon^{-3}-1} E[(f(V_N(\tau_k)) - f(\varepsilon^2 m + V_N(\tau_{k-1}))) \\
 &\quad \times \mathbf{1}_{\{m\varepsilon^2 < V_N(\tau_k) - V_N(\tau_{k-1}) \leq (m+1)\varepsilon^2\}}|\mathcal{F}_{t_{k-1}}]\mathbf{1}_{G_{N,k-1}} \\
 &+ E[f(V_N(\tau_k))\mathbf{1}_{\{V_N(\tau_k) - V_N(\tau_{k-1}) > \varepsilon^{-1}\}}|\mathcal{F}_{t_{k-1}}]\mathbf{1}_{G_{N,k-1}}.
 \end{aligned}$$

Denote the three terms on the right-hand side of (150) by T_1 , T_2 and T_3 . Proposition 41 gives

$$\begin{aligned}
 & |T_2| \leq \varepsilon^2 \|f'\| P(V_N(\tau_k) - V_N(\tau_{k-1}) > \varepsilon|\mathcal{F}_{t_{k-1}})\mathbf{1}_{G_{N,k-1}} \\
 (151) \quad &\leq C\theta\varepsilon V_N(\tau_{k-1})\mathbf{1}_{G_{N,k-1}} + O(\theta\delta\varepsilon^{-1/2}) + o(1) \\
 &\leq C\theta(\varepsilon^{1/2} + \delta\varepsilon^{-1/2}) + o(1)
 \end{aligned}$$

and

$$\begin{aligned}
 |T_3| &\leq \|f\| P(V_N(\tau_k) - V_N(\tau_{k-1}) > \varepsilon^{-1} | \mathcal{F}_{t_{k-1}}) \mathbf{1}_{G_{N,k-1}} \\
 (152) \quad &\leq C\theta\varepsilon V_N(\tau_{k-1}) \mathbf{1}_{G_{N,k-1}} + O(\theta\delta\varepsilon^{-5/2}) + o(1) \\
 &\leq C\theta(\varepsilon^{1/2} + \delta\varepsilon^{-5/2}) + o(1).
 \end{aligned}$$

By Proposition 41 and the fact that

$$\frac{1}{\varepsilon^2} \left(\frac{1}{m} - \frac{1}{m+1} \right) = \frac{1}{\varepsilon^2 m(m+1)},$$

we have

$$\begin{aligned}
 P(m\varepsilon^2 < V_N(\tau_k) - V_N(\tau_{k-1}) \leq (m+1)\varepsilon^2 | \mathcal{F}_{t_{k-1}}) \mathbf{1}_{G_{N,k-1}} \\
 = \frac{2\pi^2\theta s V_N(\tau_{k-1})}{\varepsilon^2 m(m+1)} \mathbf{1}_{G_{N,k-1}} + O(\theta\delta\varepsilon^{-5/2}) + o(1).
 \end{aligned}$$

Adding up at most ε^{-3} error terms of order $\theta\delta\varepsilon^{-5/2}$ to get a single error term of order $\theta\delta\varepsilon^{-11/2}$, we get

$$\begin{aligned}
 T_1 &= 2\pi^2\theta s V_N(\tau_{k-1}) \mathbf{1}_{G_{N,k-1}} \sum_{m=\varepsilon^{-1}}^{\varepsilon^{-3}-1} \frac{f(\varepsilon^2 m + V_N(\tau_{k-1}))}{\varepsilon^2 m(m+1)} \\
 (153) \quad &+ O(\theta\delta\varepsilon^{-11/2}) + o(1) \\
 &= 2\pi^2\theta s V_N(\tau_{k-1}) \mathbf{1}_{G_{N,k-1}} \sum_{m=\varepsilon^{-1}}^{\varepsilon^{-3}-1} f(\varepsilon^2 m + V_N(\tau_{k-1})) \int_{\varepsilon^2 m}^{\varepsilon^2(m+1)} y^{-2} dy \\
 &+ O(\theta\delta\varepsilon^{-11/2}) + o(1).
 \end{aligned}$$

Because an error of at most $\|f'\|\varepsilon^2$ is made when replacing $f(\varepsilon^2 m + V_N(\tau_{k-1}))$ by $f(V_N(\tau_{k-1}) + y)$ with $\varepsilon^2 m \leq y \leq \varepsilon^2(m+1)$, we get

$$\begin{aligned}
 &\left| \sum_{m=\varepsilon^{-1}}^{\varepsilon^{-3}-1} f(\varepsilon^2 m + V_N(\tau_{k-1})) \int_{\varepsilon^2 m}^{\varepsilon^2(m+1)} y^{-2} dy - \int_{\varepsilon}^{\infty} f(V_N(\tau_{k-1}) + y) y^{-2} dy \right| \\
 &\leq C \left(\int_{\varepsilon}^{\varepsilon^{-1}} \varepsilon^2 y^{-2} dy + \int_{\varepsilon^{-1}}^{\infty} y^{-2} dy \right) \leq C\varepsilon.
 \end{aligned}$$

Combining this with (153) gives

$$\begin{aligned}
 T_1 &= 2\pi^2\theta s V_N(\tau_{k-1}) \mathbf{1}_{G_{N,k-1}} \int_{\varepsilon}^{\infty} f(V_N(\tau_{k-1}) + y) y^{-2} dy \\
 (154) \quad &+ O(\theta\varepsilon^{1/2}) + O(\theta\delta\varepsilon^{-11/2}) + o(1).
 \end{aligned}$$

Since we have chosen $\delta \leq \varepsilon^7 \leq \varepsilon^6$ by (95), the lemma now follows by summing (151), (152) and (154) and subtracting (149) from the result. \square

Note that

$$\int_{\varepsilon}^{\infty} y \mathbf{1}_{\{y \leq 1\}} f'(x) y^{-2} dy = f'(x) \int_{\varepsilon}^1 y^{-1} dy = -f'(x) \log \varepsilon.$$

Therefore, for every $\varepsilon > 0$ one can write

$$Af(x) = A_1 f(x) + A_2 f(x),$$

where

$$(155) \quad \begin{aligned} A_1 f(x) = x & \left((c + 2\pi^2 \log \varepsilon) f'(x) \right. \\ & \left. + 2\pi^2 \int_0^{\varepsilon} (f(x+y) - f(x) - yf'(x)) y^{-2} dy \right) \end{aligned}$$

and

$$A_2 f(x) = x \left(2\pi^2 \int_{\varepsilon}^{\infty} (f(x+y) - f(x)) y^{-2} dy \right).$$

LEMMA 46. *On $G_{N,k-1}$, we have*

$$E \left[\int_{\tau_{k-1}}^{\tau_k} \mathbf{1}_{\{|V_N(t) - V_N(\tau_{k-1})| > \varepsilon^2\}} dt \middle| \mathcal{F}_{\tau_{k-1}} \right] \leq C\theta\varepsilon^2 + o(1).$$

PROOF. Since $\theta \leq \theta^{1/4}$, and since $\delta < \varepsilon^{5/2}$ by (95), it follows from (102) that $\theta e^A \varepsilon^{-1/2} \leq \varepsilon^2$. Therefore, by Proposition 16 and Markov's inequality, on $G_{N,k-1}$,

$$P(R_k > 0 | \mathcal{F}_{\tau_{k-1}}) \leq C\theta e^A \varepsilon^{-1/2} + o(1) \leq C\varepsilon^2 + o(1).$$

$(Z_{N,1}(t_k) - Z_N(t_{k-1})) / (N(\log N)^2) = V_N(t_k) - V_N(t_{k-1})$ on $G_{N,k-1} \cap \{R_k = 0\}$ and $4e^{-A/4} \leq \varepsilon^2$ by (95) and (98), it follows from Corollary 13 that

$$(156) \quad \begin{aligned} P(|V_N(t_k) - V_N(t_{k-1})| > \varepsilon^2 | \mathcal{F}_{t_{k-1}}) & \leq C\varepsilon^2 + C\theta e^{-A/2} \varepsilon^{-1/2} + o(1) \\ & \leq C\varepsilon^2 + o(1). \end{aligned}$$

We claim that (156) also holds with τ_k replaced by any t such that $\tau_{k-1} < t < \tau_k$. Applying Corollary 13 requires specifying five parameters: u, s, ε, A and θ . To establish the claim, we apply Corollary 13 with new parameters $\tilde{u} = t_{k-1} / (\log N)^3$, $\tilde{s} = s, \tilde{\varepsilon} = \varepsilon, \tilde{A} = A$ and $\tilde{\theta} = (t - \tau_{k-1}) / s$. Note that $\tilde{\theta} \leq \theta$, so conditions (32)–(35) continue to hold with the new parameters. Also, using the new parameters, we get $\tilde{t}_0 = \tilde{u}(\log N)^3 = t_{k-1}$ and $\tilde{t}_1 = (\tilde{u} + \tilde{\theta}s)(\log N)^3 = t(\log N)^3$. It thus follows from Corollary 13 that

$$P(|V_N(t) - V_N(\tau_{k-1})| > \varepsilon^2 | \mathcal{F}_{t_{k-1}}) \leq C\varepsilon^2 + C\tilde{\theta} e^{-\tilde{A}/2} \tilde{\varepsilon}^{-1/2} + o(1) \leq C\varepsilon^2 + o(1).$$

Here the constant C does not depend upon the choice of t . The absolute value of the $o(1)$ can be bounded above by $B_N(t)$, where $B_N(t) \leq 1$ for all N and t , and $\lim_{N \rightarrow \infty} B_N(t) = 0$ for every fixed t . Thus, by Fubini's theorem and the dominated convergence theorem,

$$E \left[\int_{\tau_{k-1}}^{\tau_k} \mathbf{1}_{\{|V_N(t) - V_N(\tau_{k-1})| > \varepsilon^2\}} dt \middle| \mathcal{F}_{t_{k-1}} \right] \leq \int_{\tau_{k-1}}^{\tau_k} C\varepsilon^2 + B_N(t) dt \leq C\theta\varepsilon^2 + o(1)$$

as claimed. \square

LEMMA 47. *We have*

$$E \left[\int_{\tau_{k-1}}^{\tau_k} A_1 f(V_N(t)) dt \middle| \mathcal{F}_{t_{k-1}} \right] \mathbf{1}_{G_{N,k-1}} = f'(V_N(\tau_{k-1}))V_N(\tau_{k-1})\theta s(c + 2\pi^2 \log \varepsilon) \mathbf{1}_{G_{N,k-1}} + O(\theta\varepsilon^{1/2}) + o(1).$$

PROOF. If $0 \leq y \leq \varepsilon$, then

$$f(V_N(t) + y) = f(V_N(t)) + yf'(V_N(t)) + \frac{1}{2}f''(\xi_y)y^2$$

for some ξ_y satisfying $V_N(t) \leq \xi_y \leq V_N(t) + \varepsilon$. Therefore,

$$\begin{aligned} & \left| \int_{\tau_{k-1}}^{\tau_k} V_N(t) \int_0^\varepsilon (f(V_N(t) + y) - f(V_N(t)) - yf'(V_N(t)))y^{-2} dy dt \right| \\ &= \left| \int_{\tau_{k-1}}^{\tau_k} V_N(t) \left(\int_0^\varepsilon \frac{1}{2} f''(\xi_y) dy \right) dt \right| \\ (157) \quad & \leq \theta s \sup_{t \in [\tau_{k-1}, \tau_k]} \sup_{z \in [V_N(t), V_N(t) + \varepsilon]} \frac{\varepsilon}{2} V_N(t) |f''(z)| \\ & \leq C\varepsilon\theta s, \end{aligned}$$

where the last inequality follows from the fact that $\|h\| < \infty$, where h is defined in (139). Equations (155) and (157) give

$$\begin{aligned} & E \left[\int_{\tau_{k-1}}^{\tau_k} A_1 f(V_N(t)) dt \middle| \mathcal{F}_{t_{k-1}} \right] \mathbf{1}_{G_{N,k-1}} \\ (158) \quad & = (c + 2\pi^2 \log \varepsilon) E \left[\int_{\tau_{k-1}}^{\tau_k} V_N(t) f'(V_N(t)) dt \middle| \mathcal{F}_{t_{k-1}} \right] \mathbf{1}_{G_{N,k-1}} + O(\theta\varepsilon). \end{aligned}$$

Recall that $d(x) = xf'(x)$ for $x \geq 0$. Therefore,

$$\begin{aligned} & E \left[\int_{\tau_{k-1}}^{\tau_k} V_N(t) f'(V_N(t)) dt \middle| \mathcal{F}_{t_{k-1}} \right] \mathbf{1}_{G_{N,k-1}} \\ (159) \quad & = f'(V_N(\tau_{k-1}))V_N(\tau_{k-1})\theta s \mathbf{1}_{G_{N,k-1}} \\ & \quad + E \left[\int_{\tau_{k-1}}^{\tau_k} d(V_N(t)) - d(V_N(\tau_{k-1})) dt \middle| \mathcal{F}_{t_{k-1}} \right] \mathbf{1}_{G_{N,k-1}}. \end{aligned}$$

The absolute value of the second term on the right-hand side of (159) is at most

$$2\|d\|E\left[\int_{\tau_{k-1}}^{\tau_k} \mathbf{1}_{\{|V_N(t)-V_N(\tau_{k-1})|>\varepsilon^2\}} dt \middle| \mathcal{F}_{t_{k-1}}\right] \mathbf{1}_{G_{N,k-1}} + \theta s \|d'\| \varepsilon^2,$$

which is at most $C\theta\varepsilon^2 + o(1)$ by Lemma 46.

Therefore, the result follows from (158) and (159), since $\varepsilon|\log\varepsilon| < \varepsilon^{1/2}$ for sufficiently small ε . \square

LEMMA 48. *We have*

$$\begin{aligned} & E\left[\int_{\tau_{k-1}}^{\tau_k} A_2 f(V_N(t)) dt \middle| \mathcal{F}_{t_{k-1}}\right] \mathbf{1}_{G_{N,k-1}} \\ &= 2\pi^2\theta s V_N(\tau_{k-1}) \mathbf{1}_{G_{N,k-1}} \int_{\varepsilon}^{\infty} (f(V_N(\tau_{k-1}) + y) - f(V_N(\tau_{k-1}))) y^{-2} dy \\ &+ O(\theta\varepsilon) + o(1). \end{aligned}$$

PROOF. For $y \geq 0$, define the function $g_y(x) = xf(x + y)$. Note that $\sup_{y \geq 0} \|g_y\| < \infty$ and $\sup_{y \geq 0} \|g'_y\| < \infty$. We have

$$\begin{aligned} & E\left[\int_{\tau_{k-1}}^{\tau_k} A_2 f(V_N(t)) dt \middle| \mathcal{F}_{t_{k-1}}\right] \mathbf{1}_{G_{N,k-1}} \\ &= E\left[2\pi^2 \int_{\tau_{k-1}}^{\tau_k} V_N(t) \int_{\varepsilon}^{\infty} (f(V_N(t) + y) - f(V_N(t))) y^{-2} dy dt \middle| \mathcal{F}_{t_{k-1}}\right] \\ &\quad \times \mathbf{1}_{G_{N,k-1}} \\ (160) \quad &= 2\pi^2\theta s V_N(\tau_{k-1}) \mathbf{1}_{G_{N,k-1}} \int_{\varepsilon}^{\infty} (f(V_N(\tau_{k-1}) + y) - f(V_N(\tau_{k-1}))) y^{-2} dy \\ &\quad + 2\pi^2 E\left[\int_{\tau_{k-1}}^{\tau_k} \int_{\varepsilon}^{\infty} (g_y(V_N(t)) - g_0(V_N(t)) - g_y(V_N(\tau_{k-1})) \right. \\ &\quad \left. + g_0(V_N(\tau_{k-1}))) y^{-2} dy dt \middle| \mathcal{F}_{t_{k-1}}\right] \mathbf{1}_{G_{N,k-1}}. \end{aligned}$$

The absolute value of the second term on the right-hand side of (160) is at most

$$\begin{aligned} & \frac{2\pi^2 s}{\varepsilon} \left(2 \sup_{y \geq 0} \|g_y\| + 2\|g_0\|\right) E\left[\int_{\tau_{k-1}}^{\tau_k} \mathbf{1}_{\{|V_N(t)-V_N(\tau_{k-1})|>\varepsilon^2\}} dt \middle| \mathcal{F}_{t_{k-1}}\right] \mathbf{1}_{G_{N,k-1}} \\ &+ 2\pi^2\theta s \varepsilon \left(\sup_{y \geq 0} \|g'_y\| + \|g'_0\|\right), \end{aligned}$$

using that $\int_{\varepsilon}^{\infty} y^{-2} dy = \varepsilon^{-1}$. By Lemma 46, this expression is at most $C\theta\varepsilon + o(1)$, which, combined with (160), implies the result. \square

PROOF OF PROPOSITION 1. Recall that we need to establish (138). For $1 \leq k \leq \theta^{-1}$, define

$$J_k = f(V_N(\tau_k)) - f(V_N(\tau_{k-1})) - \int_{\tau_{k-1}}^{\tau_k} Af(V_N(t)) dt.$$

Then

$$(161) \quad f(V_N(u+s)) - f(V_N(s)) - \int_u^{u+s} Af(V_N(t)) dt = \sum_{k=1}^{\theta^{-1}} J_k.$$

Let $B_{N,0} = G_{N,0}^c$, and for $1 \leq k \leq \theta^{-1}$, let $B_{N,k} = G_{N,k-1} \cap G_{N,k}^c$. Then $G_N(\varepsilon)^c = \bigcup_{k=0}^{\theta^{-1}} B_{N,k}$ and

$$1 - P(G_N(\varepsilon)) = \sum_{k=0}^{\theta^{-1}} P(B_{N,k}).$$

Now

$$(162) \quad \begin{aligned} & E \left[\left(\sum_{k=1}^{\theta^{-1}} J_k \right) \prod_{i=1}^j h_i(V_N(s_i)) \right] \\ &= E \left[\left(\sum_{k=1}^{\theta^{-1}} J_k \left(\mathbf{1}_{G_{N,k-1}} + \sum_{\ell=0}^{k-1} \mathbf{1}_{B_{N,\ell}} \right) \right) \prod_{i=1}^j h_i(V_N(s_i)) \right] \\ &= E \left[\left(\sum_{k=1}^{\theta^{-1}} J_k \mathbf{1}_{G_{N,k-1}} \right) \prod_{i=1}^j h_i(V_N(s_i)) \right] \\ &\quad + \sum_{\ell=0}^{\theta^{-1}-1} E \left[\left(\sum_{k=\ell+1}^{\theta^{-1}} J_k \right) \mathbf{1}_{B_{N,\ell}} \prod_{i=1}^j h_i(V_N(s_i)) \right]. \end{aligned}$$

For $0 \leq \ell \leq \theta^{-1} - 1$,

$$\begin{aligned} \left| \sum_{k=\ell+1}^{\theta^{-1}} J_k \right| &= \left| f(V_N(u+s)) - f(V_N(\tau_\ell)) - \int_{\tau_\ell}^{u+s} Af(V_N(t)) dt \right| \\ &\leq 2\|f\| + s\|Af\|. \end{aligned}$$

Therefore, the absolute value of the second term on the right-hand side of (162) is at most

$$\left(\prod_{i=1}^j \|h_i\| \right) (2\|f\| + s\|Af\|) \sum_{\ell=0}^{\theta^{-1}-1} P(B_{N,\ell}) \leq C(1 - P(G_N(\varepsilon))),$$

using (140). To bound the first term on the right-hand side of (162), note that by conditioning on $\mathcal{F}_{t_{k-1}}$,

$$\begin{aligned} & E \left[\left(\sum_{k=1}^{\theta^{-1}} J_k \mathbf{1}_{G_{N,k-1}} \right) \prod_{i=1}^j h_i(V_N(s_i)) \right] \\ &= \sum_{k=1}^{\theta^{-1}} E \left[J_k \mathbf{1}_{G_{N,k-1}} \prod_{i=1}^j h_i(V_N(s_i)) \right] \\ &= \sum_{k=1}^{\theta^{-1}} E \left[\left(\prod_{i=1}^j h_i(V_N(s_i)) \right) E[J_k | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}} \right]. \end{aligned}$$

By Lemmas 44, 45, 47 and 48,

$$|E[J_k | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}}| \leq C\theta\varepsilon^{1/2} + o(1)$$

for all k . Therefore,

$$\begin{aligned} & \left| E \left[\left(\sum_{k=1}^{\theta^{-1}} J_k \mathbf{1}_{G_{N,k-1}} \right) \prod_{i=1}^j h_i(V_N(s_i)) \right] \right| \\ & \leq \left(\prod_{i=1}^j \|h_i\| \right) \left(\sum_{k=1}^{\theta^{-1}} E[|E[J_k | \mathcal{F}_{t_{k-1}}] \mathbf{1}_{G_{N,k-1}}|] \right) \\ & \leq C\varepsilon^{1/2} + o(1). \end{aligned}$$

It follows that

$$\left| E \left[\left(\sum_{k=1}^{\theta^{-1}} J_k \right) \prod_{i=1}^j h_i(V_N(s_i)) \right] \right| \leq C\varepsilon^{1/2} + C(1 - P(G_N(\varepsilon))) + o(1).$$

In view of (161) and Proposition 23, equation (138) now follows by letting $N \rightarrow \infty$ and then letting $\varepsilon \rightarrow 0$. \square

6.3. *The number of particles.* Because the value of $Z_N(t)$ approximately determines the number of particles a short time after time t , the fact that the number of particles converges to a continuous-state branching process follows rather simply from Proposition 1.

PROOF OF THEOREM 2. In view of Proposition 1, it suffices to show that for any fixed $t > 0$, we have

$$(163) \quad \left| \frac{1}{2\pi N} M_N((\log N)^3 t) - V_N(t) \right| \rightarrow_p 0.$$

Let $\gamma > 0$ be arbitrary. Set $u = 0$ and $s = t$. By Proposition 23, we can choose $\varepsilon \in (0, \gamma)$ sufficiently small that

$$(164) \quad \sup_{\theta} \left(\limsup_{N \rightarrow \infty} (1 - P(G_N(\varepsilon))) \right) < \frac{\gamma}{2},$$

where the supremum is taken over all θ such that $\theta^{-1} \in \mathbb{N}$. Proposition 41 implies that for sufficiently small θ ,

$$(165) \quad P(|V_N(t) - V_N(t(1 - \theta))| \geq \gamma) \leq C\theta\varepsilon^{-3/2} + (1 - P(G_N(\varepsilon))) + o(1).$$

It follows from (164) and (165) that for sufficiently small θ and sufficiently large N ,

$$(166) \quad P(|V_N(t) - V_N(t(1 - \theta))| < \gamma) > 1 - \gamma.$$

Let $M'_N((\log N)^3 t)$ denote the number of particles at time $(\log N)^3 t$ whose ancestor at time u is in $(0, L)$ for all $(\log N)^3(t(1 - \theta)) \leq u \leq (\log N)^3 t$. By Proposition 16 and (164), for sufficiently small $\theta > 0$ and sufficiently large N ,

$$(167) \quad P(M_N((\log N)^3 t) = M'_N((\log N)^3 t)) > 1 - \gamma.$$

By (17) and the fact that $1 - \mu^2/2 - \pi^2/2L^2 = 0$,

$$\begin{aligned} & E[M'_N((\log N)^3 t) | \mathcal{F}_{(\log N)^3(t(1-\theta))}] \\ &= \frac{2N(\log N)^2 V_N(t(1 - \theta))(1 + o(1))}{L} \int_0^L e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy. \end{aligned}$$

Now

$$\begin{aligned} \int_0^L e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy &= \int_0^\infty \frac{\pi y}{L} e^{-\mu y} dy + \int_0^L e^{-\mu y} \left(\sin\left(\frac{\pi y}{L}\right) - \frac{\pi y}{L} \right) dy \\ &\quad - \int_L^\infty \frac{\pi y}{L} e^{-\mu y} dy \\ &= \frac{\pi}{L\mu^2} + O\left(\int_0^L e^{-\mu y} \frac{y^3}{L^3} dy\right) + O(e^{-\mu L}) \\ &= \frac{\pi}{L\mu^2} + O\left(\frac{1}{L^3}\right) + O(e^{-\mu L}) = \frac{\pi}{2L}(1 + o(1)). \end{aligned}$$

It follows that

$$\begin{aligned} E[M'_N((\log N)^3 t) | \mathcal{F}_{(\log N)^3(t(1-\theta))}] &= \frac{\pi N(\log N)^2 V_N(t(1 - \theta))(1 + o(1))}{L^2} \\ &= 2\pi N V_N(t(1 - \theta))(1 + o(1)). \end{aligned}$$

Therefore, for sufficiently large N ,

$$(168) \quad P\left(\left| E\left[\frac{M'_N((\log N)^3 t)}{2\pi N} \middle| \mathcal{F}_{(\log N)^3(t(1-\theta))}\right] - V_N(t(1 - \theta)) \right| < \gamma\right) > 1 - \gamma.$$

By Proposition 14, we have

$$\text{Var}(M'_N((\log N)^3 t) | \mathcal{F}_{(\log N)^3(t(1-\theta))}) \leq C\theta \varepsilon^{-1/2} N^2(1 + o(1))$$

on an event defined in the same manner as $G_{N,k-1}$ but with $(\log N)^3(t(1-\theta))$ playing the role of t_{k-1} . Combining this result with (164) and the conditional form of Chebyshev’s inequality, we get for sufficiently small θ and sufficiently large N ,

$$(169) \quad P\left(\left|\frac{M'_N((\log N)^3 t)}{2\pi N} - E\left[\frac{M'_N((\log N)^3 t)}{2\pi N} \middle| \mathcal{F}_{(\log N)^3(t(1-\theta))}\right]\right| < \gamma\right) > 1 - \gamma.$$

It now follows from (166)–(169) that for sufficiently large N , we have

$$P\left(\left|\frac{1}{2\pi N} M_N((\log N)^3) - V_N(t)\right| < 3\gamma\right) > 1 - 4\gamma.$$

Result (163) follows. \square

7. Convergence to the Bolthausen–Sznitman coalescent. In this section, we prove Theorem 3. The strategy will be to show that a sequence of processes that describe the genealogy of branching Brownian motion converges to a flow of bridges defined in [8], which is known to be dual to the Bolthausen–Sznitman coalescent.

7.1. *The flow of bridges.* Consider the continuous state branching process of Proposition 1 and Theorem 2 with branching mechanism $\Psi(u) = au + 2\pi^2 u \log u$. Recall from [7] that we can define this as a two-parameter process $(Z(t, x), t \geq 0, x \geq 0)$, where t is the time parameter, and x is the initial population size. Also recall from [7] that we can associate with this continuous-state branching process a flow of subordinators. On some probability space, there exists a process $(S^{(s,t)}(x), 0 \leq s \leq t, x \geq 0)$ such that:

- For every $0 \leq s \leq t$, the process $S^{(s,t)} = (S^{(s,t)}(x), x \geq 0)$ is a subordinator with Laplace exponent u_{t-s} .
- For every integer $k \geq 2$ and every $0 \leq t_1 \leq \dots \leq t_k$, the subordinators $S^{(t_1,t_2)}, \dots, S^{(t_{k-1},t_k)}$ are independent, and

$$S^{(t_1,t_k)} = S^{(t_{k-1},t_k)} \circ \dots \circ S^{(t_1,t_2)}.$$

- The processes $(Z(t, x), t \geq 0, x \geq 0)$ and $(S^{(0,t)}(x), t \geq 0, x \geq 0)$ have the same finite-dimensional marginals.

Here $S^{(s,t)}(x)$ can be understood as the descendants in the population at time t of the first x individuals in the population at time s .

Suppose that we start with the initial population $Z(0) = z$. For each $s \leq t$, we can define the renormalized process $(B_{s,t}(x), 0 \leq x \leq 1)$ by

$$B_{s,t}(x) = S^{(s,t)}(x S^{(0,s)}(z)) / S^{(0,t)}(z).$$

It is easily seen that $B_{s,t}$ is a *bridge*, which we define as in [8] to be a nondecreasing, $[0, 1]$ -valued stochastic process $(B(r), 0 \leq r \leq 1)$ with exchangeable increments and right-continuous paths such that $B(0) = 0$ and $B(1) = 1$.

It follows from (7) that when $\Psi(u) = au + 2\pi^2u \log u$, the subordinator $S^{(s,t)}$ is a stable subordinator with index $e^{-2\pi(t-s)}$. Consequently, letting $R_{s,t}$ denote the range of $B_{s,t}$, the lengths of the disjoint open intervals whose union is $[0, 1] \setminus R_{s,t}$ are independent of $S^{(0,s)}(z)$ and have the Poisson–Dirichlet distribution with parameters $(e^{-2\pi(t-s)}, 0)$. See [60] for a definition and further discussion of the two-parameter Poisson–Dirichlet distribution and its connections with stable subordinators. It now follows (see Example 2 in [8]) that $(B_{s,t}(x), 0 \leq s \leq t, 0 \leq x \leq 1)$ is a *flow of bridges*, which is a collection $(B_{s,t}, 0 \leq s \leq t)$ of bridges such that if Id denotes the identity function from $[0, 1]$ to itself, then:

- For every $s < t < u$, we have $B_{s,u} = B_{t,u} \circ B_{s,t}$.
- The law of $B_{s,t}$ only depends on $t - s$.
- If $s_1 < s_2 < \dots < s_n$, then the bridges $B_{s_1,s_2}, \dots, B_{s_{n-1},s_n}$ are independent.
- $B_{0,0} = \text{Id}$ and $B_{0,t} \rightarrow \text{Id}$ as $t \rightarrow 0$ in probability, in the sense of Skorohod’s topology.

Note that we are using a different convention for the time parameters than in [8]. The bridge $B_{s,t}$ defined here would be called $B_{-t,-s}$ in [8].

If B is a bridge, define, for $u \in [0, 1]$,

$$(170) \quad B^{-1}(u) = \inf\{s \in [0, 1] : B(s) \geq u\}.$$

If $s < t < u$, then $B_{s,u}^{-1} = B_{s,t}^{-1} \circ B_{t,u}^{-1}$. Given independent random variables U_1, \dots, U_n with the uniform distribution on $[0, 1]$, we can define $\pi(B)$ to be the partition of $\{1, \dots, n\}$ such that i and j are in the same block of $\pi(B)$ if and only if $B^{-1}(U_i) = B^{-1}(U_j)$. Now, given a flow of bridges $(B_{s,t}, 0 \leq s \leq t)$ and independent uniform random variables U_1, \dots, U_n , we can fix a time $t > 0$ and consider the partition-valued process $(\Pi(s), 0 \leq s \leq t)$ defined by $\Pi(s) = \pi(B_{t-s,t})$. The main result of Bertoin and Le Gall [8] establishes that this process is a so-called exchangeable coalescent process and that there is in fact a one-to-one correspondence between flows of bridges and exchangeable coalescent processes. In the example above, in which the flow of bridges is defined from a continuous-state branching process with $\Psi(u) = au + 2\pi^2u \log u$, the process $(\pi(B_{t-s/2\pi,t}), 0 \leq s \leq 2\pi t)$ is the Bolthausen–Sznitman coalescent run for time $2\pi t$ (see, e.g., Example 2 in [8]).

7.2. Flows describing the genealogy of branching Brownian motion. To represent the genealogy of branching Brownian motion, we now introduce a sequence of discrete versions of these flows of bridges. We fix $K \in \mathbb{N}$ and the times $0 = t_0 < t_1 < \dots < t_K$. For $0 \leq i < j \leq K$ we will define a process $(B_{t_i,t_j}^N(s), 0 \leq s \leq 1)$.

We consider the branching Brownian motion X_N at the successive times $t_j(\log N)^3$. We assign labels to the particles at these times, and denote by $u_{i,j}$

the label of the i th largest particle at time $t_j(\log N)^3$, that is, the particle in position $X_{i,N}(t_j(\log N)^3)$. We first define a collection of independent random variables $(v_{i,j}, i \geq 0, 0 \leq j \leq K)$ having the uniform distribution on $[0, 1]$. For $i \leq M_N(0)$, we define $u_{i,0} = v_{i,0}$. That is, the individuals at time zero are labeled by independent uniform random variables. For $j \geq 1$, the $u_{i,j}$ are sequences of length $j + 1$ which are defined inductively by saying that $u_{i,j} = (u_{p(i),j-1}, v_{i,j})$, where $u_{p(i),j-1}$ is the label of the particle at time $t_{j-1}(\log N)^3$ from which the i th particle at time $t_j(\log N)^3$ has descended. That is, we concatenate $v_{i,j}$ with the label of the ancestor of the i th particle to obtain the label of the i th particle. The particles at time $t_j(\log N)^3$ can now be ordered using the lexicographical order of their labels. We denote by $x_{i,j}$ the position of the i th individual in this lexicographical order at time $t_j(\log N)^3$.

We now assign weights to the individuals. For $0 \leq j \leq K$ and $1 \leq i \leq M_N(t_j(\log N)^3)$, define

$$w(i, j) = \begin{cases} \frac{1}{Z_N(t_j(\log N)^3)} e^{\mu x_{i,j}} \sin\left(\frac{\pi x_{i,j}}{L}\right) \mathbf{1}_{\{x_{i,j} \leq L\}}, & \text{if } 0 \leq j \leq K - 1, \\ \frac{1}{M_N(t_j(\log N)^3)}, & \text{if } j = K. \end{cases}$$

That is, the particles are weighted proportional to their contribution to the sum in (9), except at time $t_K(\log N)^3$ when all particles are weighted equally. We use these weights because we will later sample particles uniformly at time $t_K(\log N)^3$, but the number of descendants that a particle at time $t_i(\log N)^3$ has at time $t_K(\log N)^3$ will be roughly proportional to the weight that it has been assigned. Also define $A_i(j, k)$ to be the set of descendants at time $t_k(\log N)^3$ of the i th individual at time $t_j(\log N)^3$. More precisely, $A_i(j, k)$ is the set of indices ℓ such that the individual at position $x_{\ell,k}$ at time $t_k(\log N)^3$ is descended from the individual in position $x_{i,j}$ at time $t_j(\log N)^3$. We are now ready to define the discrete bridges. First, for $0 \leq y \leq 1$, and $0 \leq j \leq K$, define

$$(171) \quad L_j(y) = \max \left\{ I \in \mathbb{N} : \sum_{i=1}^I w(i, j) \leq y \right\},$$

with the convention that the maximum of the empty set is 0. We think of $L_j(y)$ as being approximately the y th quantile of the population at time $t_j(\log N)^3$ when individuals are weighted as above and ordered according to their labels. Then for $0 \leq y \leq 1$ and $0 \leq j < k \leq K$, let

$$B_{t_j, t_k}^N(y) = \sum_{i=1}^{L_j(y)} \sum_{m \in A_i(j, k)} w(m, k).$$

Note that these discrete bridges B_{t_j, t_k}^N are not exactly bridges in the sense defined above; for example, their increments are not exactly exchangeable because there

are only finitely many particles at time t_j . However, we will show in Lemmas 52 and 54 below that these discrete bridges converge to the bridges B_{t_j, t_k} .

LEMMA 49. *If $0 \leq i < j < k \leq K$, then $B_{t_i, t_k}^N = B_{t_j, t_k}^N \circ B_{t_i, t_j}^N$ and $(B_{t_i, t_k}^N)^{-1} = (B_{t_i, t_j}^N)^{-1} \circ (B_{t_j, t_k}^N)^{-1}$, where the inverse functions are defined as in (170).*

PROOF. For $0 \leq y \leq 1$,

$$B_{t_i, t_k}^N(y) = \sum_{\ell=1}^{L_i(y)} \sum_{m \in A_\ell(i, k)} w(m, k).$$

Note that $m \in A_\ell(i, k)$ for some $\ell \leq L_i(y)$ if and only if $m \in A_\ell(j, k)$ for some $\ell \leq L_j(B_{t_i, t_j}^N(y))$. Therefore,

$$B_{t_i, t_k}^N(y) = \sum_{\ell=1}^{L_j(B_{t_i, t_j}^N(y))} \sum_{m \in A_\ell(j, k)} w(m, k) = B_{t_j, t_k}^N(B_{t_i, t_j}^N(y)).$$

That is, $B_{t_i, t_k}^N = B_{t_j, t_k}^N \circ B_{t_i, t_j}^N$. Also,

$$\begin{aligned} (B_{t_i, t_k}^N)^{-1}(y) &= \inf\{s : B_{t_i, t_k}^N(s) \geq y\} \\ &= \inf\{s : B_{t_j, t_k}^N(B_{t_i, t_j}^N(s)) \geq y\} \\ &= \inf\{s : B_{t_i, t_j}^N(s) \geq (B_{t_j, t_k}^N)^{-1}(y)\} \\ &= (B_{t_i, t_j}^N)^{-1}((B_{t_j, t_k}^N)^{-1}(y)), \end{aligned}$$

which implies that $(B_{t_i, t_k}^N)^{-1} = (B_{t_i, t_j}^N)^{-1} \circ (B_{t_j, t_k}^N)^{-1}$. \square

7.3. *Convergence of one bridge.* Let $(B_{s, t}, 0 \leq s \leq t)$ be the flow of bridges defined above from the continuous-state branching process with branching mechanism $\Psi(u) = au + 2\pi^2 u \log u$. We will now show that for $1 \leq i \leq K$, the sequence of discrete bridges $(B_{0, t_i}^N(u), 0 \leq u \leq 1)$ converges to $(B_{0, t_i}(u), 0 \leq u \leq 1)$ in the sense of finite-dimensional distributions. The first step is the following extension of Proposition 1.

LEMMA 50. *Assume that the initial population is subdivided into m possibly random subgroups S_1, \dots, S_m , and that given the initial positions of the particles, they evolve according to branching Brownian motion killed at 0. Assume that $Y_N(0)/(N(\log N)^3)$ converges to zero in probability. Let $Z_{i, N}(t)$ denote the contribution to the sum in (9) from particles descended from one of the particles that*

is in S_i at time zero, and let $M_{i,N}(t)$ denote the number of particles at time t descended from one of the particles that is in S_i at time zero. Assume that the initial joint distribution of

$$(172) \quad \left(\frac{Z_{i,N}(0)}{N(\log N)^2} \right)_{i=1}^m$$

converges as $N \rightarrow \infty$ to some probability measure ρ on $[0, \infty)^m$. Then the finite-dimensional distributions of the m -dimensional vector-valued processes

$$\left\{ \left(\frac{Z_{i,N}(t(\log N)^3)}{N(\log N)^2} \right)_{i=1}^m, t \geq 0 \right\} \quad \text{and} \quad \left\{ \left(\frac{M_{i,N}(t(\log N)^3)}{2\pi N} \right)_{i=1}^m, t > 0 \right\}$$

each converge as $N \rightarrow \infty$ to the finite-dimensional distributions of $\{(Z_i(t))_{i=1}^m, t \geq 0\}$, where $(Z_i(0))_{i=1}^m$ has distribution ρ , and conditional on $(Z_i(0))_{i=1}^m$, each Z_i evolves independently as a continuous-state branching process with branching mechanism $\Psi(u) = au + 2\pi u \log u$.

PROOF. While this is in principle a simple extension of Proposition 1, some care is needed in the proof because the components of the process are not independent but only conditionally independent given the initial configuration. To ease notation, we only show here the proof of the one-dimensional marginal convergence (which is all that is needed later), as the general result is conceptually identical but more cumbersome. Thus, let $t > 0$, and fix arbitrary bounded and continuous test functions $f_1, \dots, f_m : [0, \infty) \rightarrow \mathbb{R}$. By Skorohod’s Representation Theorem, we may assume that all the branching Brownian motions X_N are constructed on the same probability space in such a way that the expression in (172) converges almost surely to $(Z_i(0))_{i=1}^m$ having joint distribution ρ .

For $i = 1, \dots, m$, let $X_{i,N}$ denote the branching Brownian motion obtained by considering only the descendants of the particles in S_i . Let $\mathcal{F} = \sigma(X_{i,N}(0), i = 1, \dots, m, N = 1, 2, \dots)$ be the filtration generated by all the processes at time zero for all subgroups. Let also $\mathcal{G} = \sigma(Z_1(0), \dots, Z_m(0))$. Note that the random variables $Z_{1,N}(t(\log N)^3), \dots, Z_{m,N}(t(\log N)^3)$ are conditionally independent given \mathcal{F} . Therefore,

$$(173) \quad \begin{aligned} E \left[\prod_{i=1}^m f_i \left(\frac{Z_{i,N}(t(\log N)^3)}{N(\log N)^2} \right) \right] &= E \left[E \left[\prod_{i=1}^m f_i \left(\frac{Z_{i,N}(t(\log N)^3)}{N(\log N)^2} \right) \middle| \mathcal{F} \right] \right] \\ &= E \left[\prod_{i=1}^m E \left[f_i \left(\frac{Z_{i,N}(t(\log N)^3)}{N(\log N)^2} \right) \middle| \mathcal{F} \right] \right]. \end{aligned}$$

By Proposition 1, for $1 \leq i \leq m$ we have that $E[f_i(Z_{i,N}(t(\log N)^3)/(N(\log N)^2)) | \mathcal{F}]$ converges almost surely to the random variable $E_{Z_i(0)}[f_i(Z(t))]$, where $E_x[f_i(Z(t))]$ denotes the expected value for the continuous-state branching process started from the value $Z(0) = x$. The application of Proposition 1 is justified

here because the condition that $Y_N(0)/(N(\log N)^3)$ converges in probability to zero is satisfied for the entire process, and hence the analogous condition is satisfied for each of the m components. We may rewrite this random variable as

$$E_{Z_i(0)}[f_i(Z(t))] = E[f_i(Z_i(t))|\mathcal{G}].$$

Since all random variables on the right-hand side of (173) are bounded, we deduce by the dominated convergence theorem that

$$\begin{aligned} \lim_{N \rightarrow \infty} E \left[\prod_{i=1}^m f_i \left(\frac{Z_{i,N}(t(\log N)^3)}{N(\log N)^2} \right) \right] &= E \left[\prod_{i=1}^m E[f_i(Z_i(t))|\mathcal{G}] \right] \\ &= E \left[\prod_{i=1}^m f_i(Z_i(t)) \right], \end{aligned}$$

since the random variables $(Z_i(t))_{i=1}^m$ are conditionally independent given \mathcal{G} . This completes the proof of convergence for the processes $Z_{i,N}$. The proof for the processes $M_{i,N}$ is identical, except that we invoke Theorem 2 instead of Proposition 1. \square

Before proving the convergence of bridges, we establish the following lemma, which states that at a typical time t , no single particle makes too large a contribution to $Z_N(t)$.

LEMMA 51. *Let*

$$m_N(s) = \max_{1 \leq i \leq M_N(s(\log N)^3)} e^{\mu X_{i,N}(s(\log N)^3)} \sin \left(\frac{\pi X_{i,N}(s(\log N)^3)}{L} \right).$$

Then for all $s \geq 0$, we have $m_N(s)/(N(\log N)^2) \rightarrow 0$ in probability as $N \rightarrow \infty$.

PROOF. Suppose $(x_N)_{N=1}^\infty$ is a sequence such that $e^{\mu x_N}/(N(\log N)^3) \rightarrow 0$ as $N \rightarrow \infty$. Letting $w_N = L - x_N$, we have $w_N \rightarrow \infty$ as $N \rightarrow \infty$. Therefore,

$$\begin{aligned} (174) \quad e^{\mu x_N} \sin \left(\frac{\pi x_N}{L} \right) &= e^{\mu(L-w_N)} \sin \left(\frac{\pi w_N}{L} \right) \leq \frac{\pi e^{\mu L}}{L} \cdot w_N e^{-\mu w_N} \\ &= o(N(\log N)^2). \end{aligned}$$

Observe that $Y_N(s(\log N)^3)/(N(\log N)^3)$ converges in probability to zero, which is true by assumption when $s = 0$ and by Proposition 23 when $s > 0$. Therefore, given any subsequence $(N_j)_{j=1}^\infty$, there is a further subsequence $(N_{j_k})_{k=1}^\infty$ such that $Y_{N_{j_k}}(s(\log N_{j_k})^3)/(N_{j_k}(\log N_{j_k})^3)$ converges to zero almost surely. It follows from (174) that $m_{N_{j_k}}(s)/(N_{j_k}(\log N_{j_k})^2)$ converges to zero almost surely, which implies the result. \square

LEMMA 52. Assume the hypotheses of Theorem 3 hold. Recall that $0 = t_0 < t_1 < \dots < t_K$. Let $m \geq 1$ and let $0 = u_0 < u_1 < \dots < u_m = 1$. Then for each fixed i , with $1 \leq i \leq K$, we have

$$(B_{0,t_i}^N(u_j))_{j=1}^m \Rightarrow (B_{0,t_i}(u_j))_{j=1}^m,$$

where \Rightarrow denotes convergence in distribution as $N \rightarrow \infty$.

PROOF. It suffices to prove the joint convergence of the increments $(B_{0,t_i}^N(u_j) - B_{0,t_i}^N(u_{j-1}))_{j=1}^m$. Define L_0 as in (171), and for $1 \leq j \leq m$, let

$$S_j = \{L_0(u_{j-1}) + 1, L_0(u_{j-1}) + 2, \dots, L_0(u_j)\}$$

be the subset of particles in the population at time zero associated with the quantiles in $[u_{j-1}, u_j)$. Note that the S_j are disjoint, and divide the population at time zero into m subgroups. We treat the positions of the particles in these m subgroups as m random starting configurations, to which we will apply Lemma 50.

For $1 \leq j \leq m$, define the process $(Z_{j,N}(t), t \geq 0)$ as in Lemma 50. We claim that the distribution of

$$(175) \quad \left(\frac{Z_{j,N}(0)}{N(\log N)^2} \right)_{j=1}^m$$

converges as $N \rightarrow \infty$ to some probability measure ρ on $[0, \infty)^m$. Here ρ has the distribution of $(\delta_j X)_{j=1}^m$, where $\delta_j = u_j - u_{j-1}$ for $1 \leq j \leq m$ and X has distribution ν . To check that this convergence holds, note that

$$|Z_{j,N}(0) - \delta_j Z_N(0)| \leq 2m_N(0),$$

where $m_N(0)$ is defined as in Lemma 51 and the error term $2m_N(0)$ comes from the fact that $\sum_{k=1}^j e^{\mu x_k,0} \sin(\pi x_k,0/L)$ increases discontinuously with j . In view of Lemma 51, the convergence of the distribution of (175) to ρ follows by Slutsky's theorem (see Corollary 3.3 in Chapter 3 of [31]) and Proposition 1. Therefore, the hypotheses of Lemma 50 are satisfied.

Assume for now that $i \leq K - 1$. By Lemma 50,

$$\left(\frac{Z_{j,N}(t_i(\log N)^3)}{N(\log N)^2} \right)_{j=1}^m \Rightarrow (Z_j(t_i))_{j=1}^m,$$

where $\{(Z_j(t))_{j=1}^m, t \geq 0\}$ is defined as in Lemma 50. Thus for any $\alpha > 0$,

$$(176) \quad \left(\frac{Z_{j,N}(t_i(\log N)^3)}{Z_N(t_i(\log N)^3) \vee \alpha N(\log N)^2} \right)_{j=1}^m \Rightarrow \left(\frac{Z_j(t_i)}{\alpha \vee \sum_{k=1}^m Z_k(t_i)} \right)_{j=1}^m.$$

Choose $\gamma > 0$, and let α be such that $P(Z(t_i) < \alpha) \leq \gamma$, where $(Z(t), t \geq 0)$ is a continuous-state branching process with branching mechanism Ψ and initial distribution ν , which is possible because $\nu(\{0\}) = 0$ and $(Z(t), t \geq 0)$ never goes extinct. Thus, by Proposition 1 we have for N large enough,

$$P(Z_N(t_i(\log N)^3) < \alpha N(\log N)^2) \leq 2\gamma.$$

Now fix f_1, \dots, f_m , some arbitrary bounded and continuous test functions on $[0, 1]$ and let $M = \|f_1\| \cdots \|f_m\|$. Thus, we have

$$\begin{aligned} & \left| E \left[\prod_{j=1}^m f_j(B_{0,t_i}^N(u_j) - B_{0,t_i}^N(u_{j-1})) \right] - E \left[\prod_{j=1}^m f_j \left(\frac{Z_j(t_i)}{\sum_{k=1}^m Z_k(t_i)} \right) \right] \right| \\ & \leq \left| E \left[\prod_{j=1}^m f_j \left(\frac{Z_{j,N}(t_i(\log N)^3)}{\alpha N(\log N)^2 \vee Z_N(t_i(\log N)^3)} \right) \right] \right. \\ & \quad \left. - E \left[\prod_{j=1}^m f_j \left(\frac{Z_j(t_i)}{\alpha \vee \sum_{k=1}^m Z_k(t_i)} \right) \right] \right| \\ & \quad + MP(Z_N(t_i(\log N)^3) < \alpha N(\log N)^2) + MP(Z(t_i) < \alpha). \end{aligned}$$

Taking the limsup of both sides, we find that the first term in the right-hand side of the above inequality converges to 0 by (176), and the second and third terms are respectively smaller than $2M\gamma$ and $M\gamma$. Since $\gamma > 0$ is arbitrary, and since

$$\left(\frac{Z_j(t_i)}{\sum_{k=1}^m Z_k(t_i)} \right)_{j=1}^m$$

has the same distribution as $(B_{0,t_i}(u_j) - B_{0,t_i}(u_{j-1}))_{j=1}^m$, this finishes the proof when $1 \leq i \leq K - 1$.

The proof when $i = K$ is the same, except $Z_{j,N}(t_i(\log N)^3)/(N(\log N)^2)$ needs to be replaced throughout the argument by $M_{j,N}(t_i(\log N)^3)/(2\pi N)$, where the processes $(M_{j,N}(t), t \geq 0)$ are defined as in Lemma 50. \square

7.4. Joint convergence of bridges. In this subsection we extend the convergence obtained in Lemma 52 to the joint convergence of the finite-dimensional distributions of several bridges. We begin by establishing a result about the convergence of the distribution of a single bridge, conditional on the branching Brownian motion up to the starting point of the bridge.

LEMMA 53. *Assume the hypotheses of Theorem 3 hold. Recall that $0 = t_0 < t_1 < \dots < t_K$. Let $m \geq 1$, and let $0 = u_0 < u_1 < \dots < u_m$. Let $f : [0, 1]^{m+1} \rightarrow \mathbb{R}$ be bounded and continuous. For $0 \leq i \leq K - 1$, we have*

$$\begin{aligned} & E[f(B_{t_i,t_{i+1}}^N(u_0), \dots, B_{t_i,t_{i+1}}^N(u_m)) | \mathcal{F}_{t_i(\log N)^3}] \\ & \rightarrow_p E[f(B_{t_i,t_{i+1}}(u_0), \dots, B_{t_i,t_{i+1}}(u_m))], \end{aligned}$$

where \rightarrow_p denotes convergence in probability as $N \rightarrow \infty$.

PROOF. Let $(Z(t), t \geq 0)$ be a continuous-state branching process with branching mechanism Ψ and initial distribution ν . By Proposition 1,

$$\frac{Z_N(t_i(\log N)^3)}{N(\log N)^2} \Rightarrow Z(t_i).$$

Also, we have $Y_N(t_i(\log N)^3)/(N(\log N)^3) \rightarrow_p 0$ by assumption if $i = 0$ and by Proposition 23 if $i \geq 1$. Therefore, by Skorohod’s Representation Theorem, the branching Brownian motion processes $(X_N, N \geq 1)$ can be constructed on a single probability space in such a way that $Z_N(t_i(\log N)^3)/(N(\log N)^2) \rightarrow Z(t_i)$ a.s. and $Y_N(t_i(\log N)^3)/(N(\log N)^3) \rightarrow 0$ a.s. Furthermore, it can be arranged that the processes X_N evolve independently of one another after time $t_i(\log N)^3$.

Let $\tilde{\mathcal{F}}_t = \sigma(X_N(s), N \geq 1, 0 \leq s \leq t)$ be the σ -field generated by all the information up to time t by all processes. By the Markov property, conditional on $\tilde{\mathcal{F}}_{t_i(\log N)^3}$, the process X_N evolves after time $t_i(\log N)^3$ like a branching Brownian motion with absorption whose initial configuration is that of $X_N(t_i(\log N)^3)$. Therefore, we can apply Lemma 52, with $t_{i+1} - t_i$ playing the role of t_i in Lemma 52, to get that on this probability space

$$\begin{aligned} E[f(B_{t_i, t_{i+1}}^N(u_0), \dots, B_{t_i, t_{i+1}}^N(u_m)) | \tilde{\mathcal{F}}_{t_i(\log N)^3}] \\ \rightarrow E[f(B_{0, t_{i+1}-t_i}(u_0), \dots, B_{0, t_{i+1}-t_i}(u_m))] \quad \text{a.s.} \end{aligned}$$

The result follows because the bridges $B_{0, t_{i+1}-t_i}$ and $B_{t_i, t_{i+1}}$ have the same law. □

LEMMA 54. *Assume the hypotheses of Theorem 3 hold. Recall that $0 = t_0 \leq t_1 < \dots < t_K$ and let $0 = u_0 < u_1 \leq \dots < u_m \leq 1$. Then*

$$(177) \quad (B_{t_i, t_{i+1}}^N(u_j))_{\substack{0 \leq i \leq K-1 \\ 1 \leq j \leq m}} \Rightarrow (B_{t_i, t_{i+1}}(u_j))_{\substack{0 \leq i \leq K-1 \\ 1 \leq j \leq m}},$$

where the bridges $B_{t_i, t_{i+1}}, 0 \leq i \leq K - 1$, are independent.

PROOF. We proceed by induction. The convergence of $(B_{t_0, t_1}^N(u_j))_{1 \leq j \leq m}$ to $(B_{t_0, t_1}(u_j))_{1 \leq j \leq m}$ is a consequence of Lemma 52. Thus assume that the convergence (177) holds for $0 \leq i \leq k - 1$ with $2 \leq k \leq K - 1$. Let $f_1, \dots, f_k : [0, 1]^{m+1} \rightarrow \mathbb{R}$ be bounded continuous functions. By Proposition 1, we know that

$$(178) \quad (Z_N(t_1(\log N)^3), \dots, Z_N(t_k(\log N)^3)) \Rightarrow (Z(t_1), \dots, Z(t_k)),$$

where $(Z(t), t \geq 0)$ is a continuous-state branching process with branching mechanism Ψ and initial distribution ν . Let $\tilde{\mathcal{F}}_t = \sigma(X_N(s), N \geq 1, 0 \leq s \leq t)$ be the σ -field generated by the information up to time t . To simplify notation, we write $\beta_i^N = (B_{t_i, t_{i+1}}^N(u_j))_{1 \leq j \leq m}$ and $\beta_i = (B_{t_i, t_{i+1}}(u_j))_{j=1}^m$. Since β_k^N is conditionally independent of $\beta_1^N, \dots, \beta_{k-1}^N$ given $\mathcal{F}_{t_{k-1}(\log N)^3}$,

$$(179) \quad E\left[\prod_{i=1}^k f_i(\beta_i^N)\right] = E\left[\left(\prod_{i=1}^{k-1} f_i(\beta_i^N)\right) E[f_k(\beta_k^N) | \tilde{\mathcal{F}}_{t_{k-1}(\log N)^3}]\right].$$

Lemma 53 states that

$$(180) \quad E[f_k(\beta_k^N) | \tilde{\mathcal{F}}_{t_{k-1}(\log N)^3}] \rightarrow_p E[f_k(\beta_k)],$$

where \rightarrow_p denotes convergence in probability as $N \rightarrow \infty$. Using the identity of real numbers

$$x'y' - xy = x'(y' - y) + y(x' - x)$$

in (179) with $x' = \prod_{i=1}^{k-1} f_i(\beta_i^N)$, $y' = E[f_k(\beta_k^N) | \tilde{\mathcal{F}}_{t_{k-1}(\log N)^3}]$, $x = \prod_{i=1}^{k-1} E[f_i(\beta_i)]$ and $y = E[f_k(\beta_k)]$, and then taking the expectation, we get

$$\begin{aligned} & E \left[\prod_{i=1}^k f_i(\beta_i^N) \right] - \prod_{i=1}^k E[f_i(\beta_i)] \\ &= E \left[\prod_{i=1}^{k-1} f_i(\beta_i^N) (E[f_k(\beta_k^N) | \tilde{\mathcal{F}}_{t_{k-1}(\log N)^3}] - E[f_k(\beta_k)]) \right] \\ & \quad + E[f_k(\beta_k)] \left(E \left[\prod_{i=1}^{k-1} f_i(\beta_i^N) \right] - \prod_{i=1}^{k-1} E[f_i(\beta_i)] \right). \end{aligned}$$

The first term on the right-hand side converges to 0 by the dominated convergence theorem and (180) since f_1, \dots, f_k are bounded, and the second term converges to 0 by the induction hypothesis. This finishes the proof of Lemma 54. \square

7.5. *Tightness.* Our goal in this subsection is to prove the following tightness result.

LEMMA 55. *Assume the hypotheses of Theorem 3 hold. For $0 \leq i \leq K - 1$, the sequence of random discrete bridges $(B_{t_i, t_{i+1}}^N(u), 0 \leq u \leq 1)$ is a tight sequence with respect to the Skorohod topology.*

PROOF. For $\delta > 0$ and a function $B : [0, 1] \rightarrow [0, 1]$, define

$$w'(B, \delta) = \inf_{\{x_j\}} \max_j \sup_{x, y \in [x_j, x_{j+1}]} |B(x) - B(y)|,$$

where the infimum is taken over all subdivisions $\{x_j\}$ of $[0, 1]$ with $0 = x_0 < x_1 < \dots < x_m = 1$ and $\min(x_{j+1} - x_j) \geq \delta$. It suffices to show (see Chapter 13 of [11]) that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(181) \quad \limsup_{N \rightarrow \infty} P(w'(B_{t_i, t_{i+1}}^N, \delta) \geq \varepsilon) \leq \varepsilon.$$

Assume for now that $i \leq K - 2$. To prove (181), we need to show that two jumps do not occur very close to one another. Let $\varepsilon > 0$. Let $(Z(t), t \geq 0)$ be a continuous-state branching process with branching mechanism Ψ and initial distribution ν . Since $\nu(\{0\}) = 0$, the continuous-state branching process does not explode or go extinct, so we can fix $0 < a < 1$ such that

$$P(a < Z(t_{i+1}) < 1/a) \geq 1 - \varepsilon/4.$$

Let $A(a, N)$ be the event that $aN(\log N)^2 < Z_N(t_{i+1}(\log N)^3) < a^{-1}N(\log N)^2$. By Proposition 1, we can choose N_0 so that for all $N \geq N_0$, we have $P(A(a, N)) \geq 1 - \varepsilon/2$.

For $0 \leq x \leq 1$, let

$$\begin{aligned} Z_{t_i, t_{i+1}}^N(x) &= Z_N(t_{i+1}(\log N)^3) B_{t_i, t_{i+1}}^N(x) \\ &= \sum_{\ell=1}^{L_i(x)} \sum_{m \in A_\ell(i, i+1)} e^{\mu x_{m, i+1}} \sin\left(\frac{\pi x_{m, i+1}}{L}\right). \end{aligned}$$

We now define our subdivision $\{x_j\}$. Let $x_0 = 0$, and for $j \geq 1$ such that $x_{j-1} < 1$, let

$$x_j = 1 \wedge \min\{x \geq 0 : Z_{t_i, t_{i+1}}^N(x) - Z_{t_i, t_{i+1}}^N(x_{j-1}) \geq a\varepsilon N(\log N)^2\}.$$

Since $P(A(a, N)) \geq 1 - \varepsilon/2$, and since this subdivision ensures that $|B_{t_i, t_{i+1}}^N(x) - B_{t_i, t_{i+1}}^N(y)| < \varepsilon$ for all $x, y \in [x_j, x_{j+1})$ on the event $A(a, N)$, it remains only to show that there is a $\delta > 0$ such that

$$\limsup_{N \rightarrow \infty} P\left(A(a, N) \cap \left\{ \min_j (x_j - x_{j-1}) < \delta \right\}\right) \leq \varepsilon/2.$$

Let D_j be the event that $x_j \leq 1 - \delta$. On the event $A(a, N)$, there can be at most $1/\varepsilon a^2$ values of x_j less than 1. Also, on the event D_j , we have $x_j - x_{j-1} \leq \delta$ if and only if $Z_{t_i, t_{i+1}}^N(x_{j-1} + \delta) - Z_{t_i, t_{i+1}}^N(x_{j-1}) \geq a\varepsilon N(\log N)^2$. Therefore, it suffices to show that there exists $\delta > 0$ such that

$$\limsup_{N \rightarrow \infty} P(Z_{t_i, t_{i+1}}^N(1) - Z_{t_i, t_{i+1}}^N(1 - \delta) \geq a\varepsilon N(\log N)^2) \leq \varepsilon/4$$

and for all $0 \leq i \leq (1/\varepsilon a^2) - 1$,

$$\limsup_{N \rightarrow \infty} P(D_i \cap \{Z_{t_i, t_{i+1}}^N(x_{j-1} + \delta) - Z_{t_i, t_{i+1}}^N(x_{j-1}) \geq a\varepsilon N(\log N)^2\}) \leq \varepsilon^2 a^2/4.$$

In view of Lemma 51, both of these statements follow from an application of Proposition 1, in which the distribution of $\delta Z(t_i)$ plays the role of ν .

If $i = K - 1$, the proof proceeds in the same way, except that instead of working with $Z_{t_i, t_{i+1}}^N$, we define $M_{t_{K-1}, t_K}^N(x) = M_N(t_K(\log N)^3) B_{t_{K-1}, t_K}(x)$. The subdivision is defined by $x_0 = 0$ and, for $j \geq 1$,

$$x_j = 1 \wedge \min\{x \geq 0 : M_{t_{K-1}, t_K}^N(x) - M_{t_{K-1}, t_K}^N(x_{j-1}) \geq 2\pi a\varepsilon N\}.$$

The proof concludes with an application of Theorem 2 rather than Proposition 1. □

Because the tightness of each sequence $(B_{t_i, t_{i+1}}^N(u), 0 \leq u \leq 1)$ implies the joint tightness of the K sequences of bridges, Lemmas 54 and 55 combine to yield the following corollary.

COROLLARY 56. *The sequence of processes $((B_{t_0, t_1}^N(u), B_{t_1, t_2}^N(u), \dots, B_{t_{K-1}, t_K}^N(u)), 0 \leq u \leq 1)$ converges in the Skorohod topology to $((B_{t_0, t_1}(u), B_{t_1, t_2}(u), \dots, B_{t_{K-1}, t_K}(u)), 0 \leq u \leq 1)$.*

7.6. *Coalescence.* Let D be the set of functions $f : [0, 1] \rightarrow \mathbb{R}$ that are right continuous and have left limits. Let ρ denote the Skorohod metric on D . Let Λ denote the set of functions $\lambda : [0, 1] \rightarrow [0, 1]$ that are continuous and strictly increasing and satisfy $\lambda(0) = 0$ and $\lambda(1) = 1$. Recall (see Chapter 12 of [11]) that if f, f_1, f_2, \dots are functions in D , then $\lim_{n \rightarrow \infty} \rho(f_n, f) = 0$ if and only if there exists a sequence of functions $(\lambda_n)_{n=1}^\infty$ in Λ such that

$$(182) \quad \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} |f_n(\lambda_n(t)) - f(t)| = 0$$

and

$$(183) \quad \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} |\lambda_n(t) - t| = 0.$$

The lemma below is similar to Lemma 1 of [8] but differs in that we do not require the processes B_N to have exchangeable increments.

LEMMA 57. *Suppose b, b_1, b_2, \dots are functions from $[0, 1]$ to $[0, 1]$ that are nondecreasing and right continuous and have left limits at every point other than 0. Suppose $\lim_{N \rightarrow \infty} \rho(b_N, b) = 0$, where ρ denotes the Skorohod metric. Suppose $(x_N)_{N=1}^\infty$ and $(y_N)_{N=1}^\infty$ are sequences in $[0, 1]$ such that $x_N \rightarrow x$ and $y_N \rightarrow y$ as $N \rightarrow \infty$. Suppose x and y are not in the closure of the range of b . Then for sufficiently large N we have $b_N^{-1}(x_N) = b_N^{-1}(y_N)$ if and only if $b^{-1}(x) = b^{-1}(y)$. Furthermore,*

$$(184) \quad \lim_{N \rightarrow \infty} b_N^{-1}(x_N) = b^{-1}(x).$$

PROOF. Because x is not in the closure of the range of b , there exists some maximal open interval (u, v) with $u < x < v$ such that (u, v) does not intersect the range of b . For sufficiently small δ , we have $u + 2\delta < x < v - 2\delta$, which implies $u + \delta < x_N < v - \delta$ for sufficiently large N . By condition (182) applied to b_N and b , for sufficiently large N the interval $(u + \delta, v - \delta)$ does not intersect the range of b_N . Therefore, there exists γ_N such that $b_N(\gamma_N) \geq v - \delta$ and $b_N(\gamma_N -) \leq u + \delta$. Then $b_N^{-1}(x_N) = \gamma_N$ for sufficiently large N . Also, there is a sequence of functions $(\lambda_N)_{N=1}^\infty$ in Λ such that $\lambda_N(b^{-1}(x)) = \gamma_N$ for sufficiently large N by (182) and therefore $\lim_{N \rightarrow \infty} \gamma_N = b^{-1}(x)$ by (183). Result (184) follows.

Suppose $b^{-1}(x) = b^{-1}(y)$. Because b is right continuous with left limits, we have $u < y < v$. Arguing as above, we have $b_N^{-1}(y_N) = \gamma_N$ for sufficiently large N , and thus $b_N^{-1}(x_N) = b_N^{-1}(y_N)$ for sufficiently large N . Alternatively, suppose $b^{-1}(x) \neq b^{-1}(y)$. We may assume without loss of generality that $x < y$. Then

$y > v$, and there is some open interval (r, s) with $v < r < y < s$ such that (r, s) does not intersect the range of b . As above, there exists $\delta > 0$ and ξ_N such that for sufficiently large N , we have $b_N(\xi_N) \geq s - \delta$, $b_N(\xi_N -) \leq r + \delta$, and $b_N^{-1}(y_N) = \xi_N$. Therefore, $b_N^{-1}(x_N) \neq b_N^{-1}(y_N)$ for sufficiently large N , and the lemma follows. \square

PROOF OF THEOREM 3. Fix times $0 = t_0 < t_1 < \dots < t_K = t$. By Corollary 56 and Skorohod’s representation theorem, we may work on a probability space on which the sequence of discrete bridges $((B_{t_0, t_1}^N(u), B_{t_1, t_2}^N(u), \dots, B_{t_{K-1}, t_K}^N(u)), 0 \leq u \leq 1)$ converges almost surely to $((B_{t_0, t_1}(u), B_{t_1, t_2}(u), \dots, B_{t_{K-1}, t_K}(u)), 0 \leq u \leq 1)$. Note that in this setting, almost sure convergence means that $\rho(B_{t_i, t_{i+1}}^N, B_{t_i, t_{i+1}}) \rightarrow 0$ as $N \rightarrow \infty$ for $i = 0, 1, \dots, K - 1$, where ρ denotes the Skorohod metric.

Fix a positive integer n , and let U_1, \dots, U_n be independent random variables having the uniform distribution on $[0, 1]$. For $0 \leq i \leq K - 1$, define the partition $\pi(B_{t_i, t}^N) = \pi(B_{t_i, t_K}^N)$ to be the partition of $\{1, \dots, n\}$ such that i and j are in the same block of the partition if and only if $(B_{t_i, t_K}^N)^{-1}(U_i) = (B_{t_i, t_K}^N)^{-1}(U_j)$. Likewise, define $\pi(B_{t_i, t}) = \pi(B_{t_i, t_K})$ to be the partition of $\{1, \dots, n\}$ such that i and j are in the same block of the partition if and only if $B_{t_i, t_K}^{-1}(U_i) = B_{t_i, t_K}^{-1}(U_j)$. It follows from the definition of the processes B_{t_i, t_K}^N that i and j are in the same block of the partition if and only if the individuals who are in positions $\lceil U_i M_N(t_K (\log N)^3) \rceil$ and $\lceil U_j M_N(t_K (\log N)^3) \rceil$ in the lexicographical order at time $t_K (\log N)^3$ are descended from the same ancestor at time $t_i (\log N)^3$. As a result, we have the equality in distribution

$$(185) \quad (\pi(B_{t_{K-1}, t_K}^N), \dots, \pi(B_{t_0, t_K}^N)) \\ =_d (\Pi_N(2\pi(t - t_{K-1})), \dots, \Pi_N(2\pi(t - t_0))),$$

where Π_N is the process defined in Theorem 3. We note that the sampling scheme here using the random variables U_1, \dots, U_n corresponds to sampling with replacement from the individuals at time $t_K (\log N)^3$, but the difference between sampling with and without replacement is unimportant because the probability of sampling the same individual twice tends to zero as $N \rightarrow \infty$.

We claim that for $0 \leq i \leq K - 1$, almost surely

$$(186) \quad \pi(B_{t_i, t_K}^N) = \pi(B_{t_i, t_K})$$

for sufficiently large N . Because we know the process $(\pi(B_{t-s/2\pi, t}), 0 \leq s \leq 2\pi t)$ is the Bolthausen–Sznitman coalescent run for time t , this claim in combination with (185) will imply Theorem 3.

We now prove (186) by backward induction. Since the lengths of the intervals of the complement of the range of B_{t_{K-1}, t_K} have a Poisson–Dirichlet distribution and thus sum to 1 (see, e.g., Proposition 2 in [60]), the closure of the range of B_{t_{K-1}, t_K}

has Lebesgue measure zero almost surely. Therefore, almost surely U_1, \dots, U_n are not in the closure of the range of B_{t_{K-1}, t_K} . It follows from Lemma 57 that $\pi(B_{t_{K-1}, t_K}^N) = \pi(B_{t_{K-1}, t_K})$ for sufficiently large N almost surely. Furthermore,

$$\lim_{N \rightarrow \infty} (B_{t_{K-1}, t_K}^N)^{-1}(U_j) = B_{t_{K-1}, t_K}^{-1}(U_j)$$

almost surely for $j = 1, \dots, n$. Also, by Lemma 2 of [8], the random variables $B_{t_{K-1}, t_K}^{-1}(U_j)$ each have the uniform distribution on $[0, 1]$.

For the induction step, suppose that for some $i = 2, \dots, K - 1$, the following hold:

- We have $\lim_{N \rightarrow \infty} (B_{t_i, t_K}^N)^{-1}(U_j) = B_{t_i, t_K}^{-1}(U_j)$ almost surely for $j = 1, \dots, n$.
- The random variables $B_{t_i, t_K}^{-1}(U_j)$ each have the uniform distribution on $[0, 1]$.

Now $(B_{t_{i-1}, t_K}^N)^{-1}(U_j) = (B_{t_{i-1}, t_i}^N)^{-1}((B_{t_i, t_K}^N)^{-1}(U_j))$ by Lemma 49, and likewise $B_{t_{i-1}, t_K}^{-1}(U_j) = B_{t_{i-1}, t_i}^{-1}(B_{t_i, t_K}^{-1}(U_j))$. Because the random variables $B_{t_i, t_K}^{-1}(U_j)$ each have the uniform distribution on $[0, 1]$ and are independent of B_{t_{i-1}, t_i} , almost surely none of these random variables is in the closure of the range of B_{t_{i-1}, t_i} . Since also $(B_{t_i, t_K}^N)^{-1}(U_j) \rightarrow B_{t_i, t_K}^{-1}(U_j)$ almost surely for $j = 1, \dots, n$, Lemma 57 implies that $\pi(B_{t_{i-1}, t_K}^N) = \pi(B_{t_{i-1}, t_K})$ for sufficiently large N almost surely. Furthermore, $(B_{t_{i-1}, t_K}^N)^{-1}(U_j) \rightarrow B_{t_{i-1}, t_K}^{-1}(U_j)$ almost surely for $j = 1, \dots, n$. By Lemma 2 of [8], the random variables $B_{t_{i-1}, t_K}^{-1}(U_j)$ each have the uniform distribution on $[0, 1]$. The claim (186) now follows by induction. \square

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