A SUPER ORNSTEIN–UHLENBECK PROCESS INTERACTING WITH ITS CENTER OF MASS

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We construct a supercritical interacting measure-valued diffusion with representative particles that are attracted to, or repelled from, the center of mass. Using the historical stochastic calculus of Perkins, we modify a super Ornstein–Uhlenbeck process with attraction to its origin, and prove continuum analogues of results of Engländer [*Electron. J. Probab.* **15** (2010) 1938–1970] for binary branching Brownian motion.

It is shown, on the survival set, that in the attractive case the mass normalized interacting measure-valued process converges almost surely to the stationary distribution of the Ornstein–Uhlenbeck process, centered at the limiting value of its center of mass. In the same setting, it is proven that the normalized super Ornstein–Uhlenbeck process converges a.s. to a Gaussian random variable, which strengthens a theorem of Engländer and Winter [*Ann. Inst. Henri Poincaré Probab. Stat.* **42** (2006) 171–185] in this particular case. In the repelling setting, we show that the center of mass converges a.s., provided the repulsion is not too strong and then give a conjecture. This contrasts with the center of mass of an ordinary super Ornstein–Uhlenbeck process with repulsion, which is shown to diverge a.s.

A version of a result of Tribe [*Ann. Probab.* **20** (1992) 286–311] is proven on the extinction set; that is, as it approaches the extinction time, the normalized process in both the attractive and repelling cases converges to a random point a.s.

1. Introduction and main results. The existence and uniqueness of a self-interacting measure-valued diffusion that is either attracted to or repelled from its centre of mass is shown below. It is natural to consider a super Ornstein–Uhlenbeck (SOU) process with attractor (repeller) given by the centre of mass of the process as it is the simplest diffusion of this sort.

This type of model first appeared in a recent paper of Engländer [1] where a d-dimensional binary Brownian motion, with each parent giving birth to exactly two offspring and branching occurring at integral times, is used to construct a binary branching Ornstein–Uhlenbeck process where each particle is attracted (repelled) by the center of mass (COM). This is done by solving the appropriate SDE along each branch of the particle system and then stitching these solutions together.

This model can be generalized such that the underlying process is a branching Brownian motion (BBM), $\mathcal{T}$ (i.e., with a general offspring distribution). We might then solve an SDE on each branch of $\mathcal{T}$:

$$(1.1) \qquad Y_i^n(t) = Y_{p(i)}^{n-1}(n-1) + \gamma \int_{n-1}^{t} \bar{Y}_n(s) - Y_i^n(s) \, ds + \int_{n-1}^{t} dB_i^n(s)$$

for $n-1 < t \le n$, where B_i^n labels the ith particle of $\mathcal{T}$ alive from time $n-1$ to n, $p(i)$ is the parent of i and

$$\bar{Y}_n(s) = \frac{1}{\tau_n} \sum_{i=1}^{\tau_n} Y_i^n(s)$$

is the center of mass. Here, τ_n is the population of particles alive from time $n-1$ to n. This constructs a branching OU system with attraction to the COM when $\gamma > 0$ and repulsion when $\gamma < 0$.

It seems reasonable then, to take a scaling limit of branching particle systems of this form and expect it to converge in distribution to a measure-valued process where the representative particles behave like an OU process attracting to (repelling from) the COM of the process. Though viable, this approach will be avoided in lieu of a second method utilizing the historical stochastic calculus of Perkins [9] which is more convenient for both constructing the SOU interacting with its COM and for proving various properties. The idea is to use a supercritical historical Brownian motion to construct the interactive SOU process by solving a certain stochastic equation. This approach for constructing interacting measure-valued diffusions was pioneered in [8] and utilized in, for example, [6].

A supercritical historical Brownian motion, K, is a stochastic process taking values in the space of measures over the space of paths in $\mathbb{R}^d$. One can think of K as a supercritical superprocess which has a path-valued Brownian motion as the underlying process. That is, if B_t is a d-dimensional Brownian motion, then $\hat{B}_t = B_{\cdot \wedge t}$ is the underlying process of K. More information about K is provided in Section 2.

It can be shown that if a path $y : [0, \infty) \to \mathbb{R}^d$ is chosen according to K_t (loosely speaking—this is made rigorous below in Definition 2.1), then $y(s)$ is a Brownian motion stopped at t. Projecting down gives

$$X_t^K(\cdot) = \int \mathbf{1}(y_t \in \cdot) K_t(dy),$$

which is a (supercritical) super Brownian motion.

A key advantage to projecting down and constructing measure-valued processes is that it is possible to use the historical stochastic calculus to couple different projections together (and hence couple measure-valued diffusions).

One can sensibly define the Ornstein–Uhlenbeck SDE driven by y according to K_t as the solution of a stochastic equation.

DEFINITION 1.1.   Let $Z_0 : \mathbb{R}^d \to \mathbb{R}^d$ be Borel measurable. We say that (X, Z) is a solution to the strong equation $(SE)^1_{Z_0, K}$ if the pair satisfies

$$\text{(a)} \quad Z_t(y) = Z_0(y_0) + y_t - y_0 - \gamma \int_0^t Z_s(y) \, ds, \qquad K\text{-a.e.,}$$

$((SE)^1_{Z_0, K})$

$$\text{(b)} \quad X_t(A) = \int 1(Z_t \in A) K_t(dy) \qquad \forall A \in \mathcal{B}(\mathbb{R}^d) \ \forall t \geq 0,$$

where X and Z are appropriately adapted. We will henceforth call the projection X the ordinary super Ornstein–Uhlenbeck process.

If $\gamma > 0$, Z_t is attracted to the origin, and if $\gamma < 0$ it is repelled. The approximate meaning of K-a.e. in (a) is that the statement holds for K_t-a.a. y, for all K_t, $\mathbb{P}$-a.s. The exact definition is given in the next section. The projection X_t is a SOU process with attraction to (repulsion from) the origin at rate γ. Intuitively, K tracks the underlying branching structure and Z_t is a function transforming a typical Brownian path into a typical Ornstein–Uhlenbeck path.

Note that in the definitions of $(SE)^1_{Z_0, K}$ and $(SE)^2_{Y_0, K}$ given below, part (b) is unnecessary to solve the equations. It has been included to provide an easy comparison to the strong equation of Chapter V.1 of [9].

For all the results mentioned in the remainder of this work, the standing assumption (unless indicated otherwise) will be that

$$(1.2) \qquad\qquad \int 1 \, dK_0 < \infty.$$

THEOREM 1.2.   *There is a pathwise unique solution (X, Z) to $(SE)^1_{Z_0, K}$. That is, X is unique $\mathbb{P}$-a.s. and Z K-a.e. unique. Furthermore, the map $t \to X_t$ is continuous and X is a β-super-critical super Ornstein–Uhlenbeck process.*

Similar to the above, we establish a function of the path y that is a path of an OU process with attraction (repulsion) to the COM and project down.

DEFINITION 1.3.   Let $Y_0 : \mathbb{R}^d \to \mathbb{R}^d$ be Borel measurable. Define (X', Y) as the solution of

$$\text{(a)} \quad Y_t(y) = Y_0(y_0) + y_t - y_0 + \gamma \int_0^t \bar{Y}_s - Y_s(y) \, ds, \qquad K\text{-a.e.}$$

$((SE)^2_{Y_0, K})$

$$\text{(b)} \quad X'_t(A) = \int 1(Y_t \in A) K_t(dy) \qquad \forall A \in \mathcal{B}(\mathbb{R}^d) \ \forall t \geq 0,$$

where the COM is

$$\bar{Y}_s = \frac{\int x X'_s(dx)}{\int 1 X'_s(dx)}.$$

We will call the projection X' the super Ornstein–Uhlenbeck process with attraction (repulsion) to its COM, or the interacting super Ornstein–Uhlenbeck process.

Note that by our definitions of X and X' as solutions to $(\text{SE})^i_{Y_0, K}$, $i = 1, 2$, respectively, that (1.2) is the same as saying that $\int 1 \, dX_0, \int 1 \, dX'_0 < \infty$, as these quantities equal that in (1.2).

THEOREM 1.4.   *There is a pathwise unique solution to $(\text{SE})^2_{Y_0, K}$.*

One could prove this theorem using a combination of the proof of Theorem 1.2 and a localization argument. We find it more profitable however, to employ a correspondence with the ordinary SOU process X. This correspondence plays a central role in the analysis of X', and indeed reveals a very interesting structure: We have that for any γ,

$$(1.3) \qquad \int \phi(x) \, dX'_t(x) = \int \phi\left(x + \gamma \int_0^t \bar{Z}_s \, ds\right) dX_t(x),$$

where $\bar{Z}$ is the COM of X, defined as $\bar{Z}_s = \frac{\int x X_s(dx)}{\int 1 X_s(dx)}$. The correspondence essentially says that the SOU process with attraction (repulsion) to its COM is the same as the ordinary SOU process being dynamically pushed by its COM. From this equation, a relation between $\bar{Y}$ and $\bar{Z}$ can be established:

$$\bar{Y}_t = \bar{Z}_t + \gamma \int_0^t \bar{Z}_s \, ds.$$

Define $\tilde{X}_t = \frac{X_t}{X_t(1)}$ and $\tilde{X}'_t = \frac{X'_t}{X'_t(1)}$. As the goal of this work is to prove that $\tilde{X}'_t$ has interesting limiting behavior as t approaches infinity, (1.3) yields a method of approach: show first that the time integral in (1.3) converges in some sense and establish limiting behavior for $\tilde{X}$. One then hopes to combine these two facts with (1.3) to get the desired result.

Let S be the event that K_t survives indefinitely. Note that this implies that on S, both X and X' survive indefinitely by their definitions as solutions of the equations above. Let η be the time at which K_t goes extinct.

The next two theorems settle the question of what happens on the extinction set S^c.

THEOREM 1.5.   *On S^c, $\bar{Y}_t$ and $\bar{Z}_t$ converge as $t \uparrow \eta < \infty$, $\mathbb{P}$-a.s., for any $\gamma \in \mathbb{R}$.*

THEOREM 1.6.   *On the extinction set, S^c,*
$$\tilde{X}_t \to \delta_F \quad and \quad \tilde{X}'_t \to \delta_{F'}$$
as $t \uparrow \eta < \infty$ a.s., where F and F' are $\mathbb{R}^d$-valued random variables such that

$$F' = F + \gamma \int_0^\eta \bar{Z}_s \, ds.$$

This last theorem is an analogue of the result of Tribe [14] for ordinary critical superprocesses. Note that here it does not matter whether there is attraction or repulsion from the COM.

The following three theorems for the attractive case ($\gamma > 0$) form the main results of this work.

THEOREM 1.7. *On S the following hold*:

(a) *If $\gamma > 0$*

$$\bar{Z}_t \xrightarrow{a.s.} 0 \quad and \quad \bar{Y}_t \xrightarrow{a.s.} \gamma \int_0^\infty \bar{Z}_s\, ds,$$

and this integral is finite almost surely.

(b) *If $\gamma = 0$, then $\bar{Z}_t = \bar{Y}_t$ and this quantity converges almost surely.*

This says that the COMs of ordinary and interacting SOU process converge and is the result that allows us to fruitfully use the correspondence of (1.3) to show convergence of the interacting SOU process.

The next theorem shows that the mass normalized SOU process converges almost surely, which is a new result among superprocesses. Engländer and Winter in [3] have shown that this process converges in probability, and before them, Engländer and Turaev in [2] shown convergence in distribution.

One expects a result of this sort to hold since in the particle picture conditional on survival, at large time horizons, there are a very large number of particles that move as independent OU processes, each of which are located in the vicinity of the origin. Thus, we expect that in the limit the mass will be distributed according to the limiting distribution of an OU process.

THEOREM 1.8. *Suppose $\gamma > 0$. Then on S,*

$$d(\tilde{X}_t, P_\infty) \xrightarrow{a.s.} 0,$$

where P_t is the semigroup of an Ornstein–Uhlenbeck process with attraction to 0 at rate γ, and d is the Vasserstein metric on the space of finite measures on $\mathbb{R}^d$ (see Definition 2.6).

REMARK 1.9.   It is possible to show that Theorem 1.8 holds for a more general class of superprocesses. If the underlying process has an exponential rate of convergence to a stationary distribution, then the above theorem goes through. One can appeal to, for example, Theorem 4.2 of Tweedie and Roberts [12] for a class of such continuous time processes.

Using the correspondence of (1.3), one can then show that the mass normalized interacting SOU process with attraction converges to a Gaussian distribution, centered at the limiting value of the COM, $\bar{Y}_\infty$.

THEOREM 1.10.   *Suppose $\gamma > 0$. Then on S,*

(1.4) $$d(\tilde{X}'_t, P^{\bar{Y}_\infty}_\infty) \xrightarrow{a.s.} 0,$$

where $P^{\bar{Y}_\infty}_\infty$ is the OU-semigroup at infinity, with the origin shifted to $\bar{Y}_\infty$.

When there is repulsion, matters become more difficult on the survival set. It is no longer clear whether there exists a limiting random measure, or what the correct normalizing factor is. We can show however that in some cases the COM of the interacting SOU process still converges. That there should be a limiting measure comes from the fact that the ordinary repelling SOU process has been shown to converge in probability by Engländer and Winter in [3] to a multiple of Lebesgue measure. One may expect something similar to hold for the interacting SOU process, given (1.3). Unfortunately, the correspondence is rendered ineffectual in this case by part (b) of the following theorem.

THEOREM 1.11.   *The following hold on S if $0 > \gamma > -\frac{\beta}{2}$:*

(a) *The process, $\bar{Y}_t$ converges almost surely.*

(b) *However, $\bar{Z}_t$ diverges exponentially fast. That is, there is a random variable L, such that*

$$\mathbb{P}(e^{\gamma t}\bar{Z}_t \to L, L \neq 0 | S) = 1.$$

This reveals an interesting byplay between X and X' in the repelling case. That is, if one fixes a compact set $A \subset \mathbb{R}^d$, then for the ordinary SOU process, X_t, A is exponentially distant from the COM of the process. However, the COM of X' will possibly lie in the vicinity of A for all time. Therefore, one might expect that A is charged by a different amount of mass by X'_t than X_t, and thus we might need to renormalize X'_t differently to get a valid limit. It is also possible that the limit for each case is different (and not simply connected by a random translation).

The proofs for these theorems and more are contained in the following sections. In Section 2, we give some background information for the historical process K and some rigorous definitions. In Section 3, we prove Theorems 1.2 and 1.4 and state some important preliminary results regarding the nature of the support of a supercritical historical Brownian motion. These are consequently used to get moment bounds on the center of mass processes $\bar{Y}_t$ and $\bar{Z}_t$. We also derive a martingale problem for $\tilde{X}$ and $\tilde{X}'$. In Section 4, we give the proofs of the convergence theorems mentioned above and in the final section give the proofs of technical results that are crucial to prove these.

2. Definitions and background material.

NOTATION.   We collect some terms below:

E, E'	Metric spaces
$C_c(E, E')$	Compact, cont. functions from E to E'
$C_b(E, E')$	Bounded, cont. functions from E to E'
$(E, \mathcal{E})$	Arbitrary measure space over E
$b\mathcal{E}$	Bounded, $\mathcal{E}$-mble. real-valued functions
$M_F(E)$	Space of finite measures on E
$\mu(f) = \int f \, d\mu$	where $f : E \to \mathbb{R}$ and μ a measure on E
$\mu(f) = (\mu(f_1), \ldots, \mu(f_n))$	if $f = (f_1, \ldots, f_n)$, $f_i : E \to \mathbb{R}$
$|p|$	$p \in \mathbb{R}^d$, denotes Euclidean norm of p
$\|f\| = \sup_E \sum |f_i(x)|$	if $f = (f_1, \ldots, f_n)$, $f_i : E \to \mathbb{R}$
$C = C(\mathbb{R}_+, \mathbb{R}^d)$	Space of continuous paths in $\mathbb{R}^d$
$\mathcal{C}$	The Borel σ-field of C
$y^t = y_{\cdot \wedge t}$	The path y stopped at t
$C^t = \{y^t : y \in C\}$	Set of all paths stopped at t
$\mathcal{C}_t = \sigma(y^s, s \le t, y \in C)$	Natural filtration associated with $\mathcal{C}$

We take K to be a supercritical historical Brownian motion. Specifically, let K be a $(\Delta/2, \beta, 1)$-historical superprocess (here Δ is the d-dimensional Laplacian), where $\beta > 0$ constant, on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$. Here β corresponds to the branching bias in the offspring distribution, and the 1 to the variance of the offspring distribution. A martingale problem characterizing K is given below. For a more thorough explanation of historical Brownian motion than found here, see Section V.2 of [9].

It turns out that K_t is supported on $C^t \subset C$ a.s. and typically, K_t puts mass on those paths that are "Brownian" (until time t and fixed thereafter). As K takes values in $M_F(C)$, $K_t(\cdot)$ will denote integration over the y variable.

Let $\hat{B}_t = B(\cdot \wedge t)$ be the path-valued process associated with B, taking values in C^t. Then for $\phi \in b\mathcal{C}$, if $s \le t$ let $P_{s,t}\phi(y) = \mathbb{E}^{s,y}(\phi(\hat{B}_t))$, where the right-hand side denotes expectation at time t given that until time s, $\hat{B}$ follows the path y.

The weak generator, $\hat{A}$, of $\hat{B}$ is as follows. If $\phi : \mathbb{R}_+ \times C \to \mathbb{R}$ we say $\phi \in \mathcal{D}(\hat{A})$ if and only if ϕ is bounded, continuous and $(\mathcal{C}_t)$-predictable, and for some $\hat{A}_s \phi(y)$ with the same properties as ϕ,

$$\phi(t, \hat{B}) - \phi(s, \hat{B}) - \int_s^t \hat{A}_r \phi(\hat{B}) \, dr, \qquad t \ge s,$$

is a $(\mathcal{C}_t)$-martingale under $P_{s,y}$ for all $s \ge 0$, $y \in C^s$.

If $m \in M_F(\mathbb{R}^d)$, we will say K satisfies the historical martingale problem, (HMP)$_m$, if and only if $K_0 = m$ a.s. and $\forall \phi \in \mathcal{D}(\hat{A})$,

$$M_t(\phi) \equiv K_t(\phi_t) - K_0(\phi_0) - \int_0^t K_s(\hat{A}_s \phi) \, ds - \beta \int_0^t K_s(\phi_s) \, ds$$

((HMP)$_m$)   is a continuous $(\mathcal{F}_t)$-martingale

$$\text{with } \langle M(\phi) \rangle_t = \int_0^t K_s(\phi_s^2) \, ds \ \forall t \geq 0, \text{ a.s.}$$

Using the martingale problem (HMP)$_m$, one can construct an orthogonal martingale measure $M_t(\cdot)$ with the method of Walsh [15]. Denote by $\mathcal{P}$, the σ-field of $(\mathcal{F}_t)$-predictable sets in $\mathbb{R}_+ \times \Omega$. If $\psi : \mathbb{R}_+ \times \Omega \times C \to \mathbb{R}$ is $\mathcal{P} \times \mathcal{C}$-measurable and

$$(2.1) \qquad\qquad \int_0^t K_s(\psi_s^2) \, ds < \infty \qquad \forall t \geq 0,$$

then there exists a continuous local martingale $M_t(\psi)$ with quadratic variation $\langle M(\psi) \rangle_t = \int_0^t K_s(\psi_s^2) \, ds$. If the expectation of the term in (2.1) is finite, then $M_t(\psi)$ is an L^2-martingale.

DEFINITION 2.1. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t) = (\Omega \times C, \mathcal{F} \times \mathcal{C}, \mathcal{F}_t \times \mathcal{C}_t)$. Let $\hat{\mathcal{F}}_t^*$ denote the universal completion of $\hat{\mathcal{F}}_t$. If T is a bounded $(\mathcal{F}_t)$-stopping time, then the normalized Campbell measure associated with T is the measure $\hat{\mathbb{P}}_T$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ given by

$$\hat{\mathbb{P}}_T(A \times B) = \frac{\mathbb{P}(\mathbf{1}_A K_T(B))}{m_T(1)} \qquad \text{for } A \in \mathcal{F}, B \in \mathcal{C},$$

where $m_T(1) = \mathbb{P}(K_T(1))$. We denote sample points in $\hat{\Omega}$ by (ω, y). Therefore, under $\hat{\mathbb{P}}_T$, ω has law $K_T(1) \, d\mathbb{P} \cdot m_T^{-1}(1)$ and conditional on ω, y has law $K_T(\cdot)/K_T(1)$.

DEFINITION 2.2. For two $(\hat{\mathcal{F}}_t^*)$-measurable processes Z^1 and Z^2, we will say that $Z^1 = Z^2$, K-a.e. if

$$Z^1(s, \omega, y) = Z^2(s, \omega, y) \qquad \forall s \leq t, K_t\text{-a.a. } y$$

for all fixed times $t \geq 0$.

DEFINITION 2.3. We say that (X, Z) is a solution to the strong equation (SE)$^1_{Z_0, K}$ if both (a) and (b) of that equation are satisfied and where Z_t is an $(\hat{\mathcal{F}}_t^*)$-predictable process and X_t is an $(\mathcal{F}_t)$-predictable process.

We say (X', Y) is the solution to the stochastic equation (SE)$^2_{Y_0, K}$ if it satisfies (a) and (b) of that equation and where Y_t is an $(\hat{\mathcal{F}}_t^*)$-predictable process, X_t' is an $(\mathcal{F}_t)$-predictable process.

Unless stated otherwise, (X, Z) will refer to a solution of $(SE)^1_{Z_0, K}$ and (X', Y) to the solution of $(SE)^2_{Y_0, K}$.

DEFINITION 2.4. For an arbitrary $\hat{\mathcal{F}}^*_t$-adapted, $\mathbb{R}^d$-valued process z_t, define the centre of mass (COM), $\bar{z}_t$, with respect to K_t as follows:

$$\bar{z}_t = \frac{K_t(z_t)}{K_t(1)}.$$

We also let

$$\tilde{X}_t(\cdot) \equiv \frac{X_t(\cdot)}{X_t(1)} = \frac{K_t(Z_t \in \cdot)}{K_t(1)}, \qquad \tilde{X}'_t(\cdot) \equiv \frac{X'_t(\cdot)}{X'_t(1)} = \frac{K_t(Y_t \in \cdot)}{K_t(1)} \quad \text{and}$$

$$\tilde{K}_t(\cdot) = \frac{K_t(\cdot)}{K_t(1)}.$$

Note that as K is supercritical, K_t survives indefinitely on a set of positive probability S, and goes extinct on the set of positive probability S^c. Hence, we can only make sense of $\bar{z}_t$ for $t < \eta$ where η is the extinction time.

DEFINITION 2.5. Let the process M_t be defined for $t \in [0, \zeta)$ where $\zeta \leq \infty$ possibly random. M is called a local martingale on its lifetime if there exist stopping times $T_N \uparrow \zeta$ such that $M_{T_N \wedge \cdot}$ is a martingale for all N. The interval $[0, \zeta)$ is called the lifetime of M.

The following definition introduces a metric on the space of finite measures, which is equivalent to the metric of convergence in distribution on the space of probability measures.

DEFINITION 2.6. Let d denote the Vasserstein metric on the space of finite measures on $\mathbb{R}^d$. That is, for μ, ν finite measures,

$$d(\mu, \nu) = \sup_{\phi \in \text{Lip}_1} \int \phi(x) \, d(\mu - \nu)(x),$$

where $\text{Lip}_1 = \{\psi \in C(\mathbb{R}^d) : \forall x, y, |\psi(x) - \psi(y)| \leq |x - y|, \|\psi\| \leq 1\}$.

3. Proofs of existence and preliminary results.

PROOF OF THEOREM 1.2. Although closely related, this does not follow automatically from Theorem 4.10 of Perkins [8] where it is shown that equations like $(SE)^2$ [but with more general, interactive, drift and diffusion terms in (a)] have solutions if K is a *critical* historical Brownian motion.

Note that $\mathring{K}_t = e^{-\beta t} K_t$ defines a $(\Delta/2, 0, e^{-\beta t})$-Historical superprocess. Let $\hat{\mathbb{P}}^1_T$ be the Campbell measure associated with $\mathring{K}$ (note that if T is taken to be a

random stopping time, this measure differs from $\hat{\mathbb{P}}_T$). The proof of Theorem V.4.1 of [9] with minor modifications shows that $(SE)^1_{Z_0,\mathring{K}}$ has a pathwise unique solution. This is because (K3) of Theorem 2.6 of [8] shows that under $\hat{\mathbb{P}}^1_T$, y_t is a Brownian motion stopped at time T and Proposition 2.7 of the same memoir can be used to replace Proposition 2.4 and Remark 2.5(c) of [9] for the setting where the branching variance depends on time.

Once this is established, it is simple to deduce that if $(\mathring{X}, Z)$ is the solution of $(SE)^1_{Z_0,\mathring{K}}$, and we let

$$X_t(\cdot) \equiv e^{\beta t}\mathring{X}_t(\cdot) = \int 1(Z_t \in \cdot)K_t(dy)$$

then (X, Z) is the pathwise unique solution of $(SE)^1_{Z_0,K}$. The only thing to check this is that $Z_t(\omega, y) = Z_0(\omega, y_0) + y_t - y_0 - \gamma \int_0^t Z_s(\omega, y)\, ds$ K-a.e., but this follows from the fact that $\mathring{K}_t \ll K_t, \forall t$.

It can be shown by using by Theorem 2.14 of [8] that X satisfies the following martingale problem: For $\phi \in C_b^2(\mathbb{R}^d)$,

$$M_t(\phi) \equiv X_t(\phi) - X_0(\phi) + \int_0^t \int \gamma x \cdot \nabla \phi(x) - \frac{\Delta}{2}\phi(x)X_s(dx)\, ds - \beta \int_0^t X_s(\phi)\, ds$$

is a martingale where $\langle M(\phi)\rangle_t = \int_0^t X_s(\phi^2)\, ds$. Then by Theorem II.5.1 of [9] this implies that X is a version of a SOU process, with initial distribution given by $K_0(Z_0^{-1}(\cdot))$. $\quad\square$

REMARK 3.1. (a) Under the Lipschitz assumptions of Section V.1 of [9], one can in fact uniquely solve $(SE)^1_{Z_0,K}$ where K is a supercritical historical Brownian motion. The proof above can be extended with minor modifications.

(b) The proof of Theorem 1.2 essentially shows that under $\hat{\mathbb{P}}_T$, T fixed, the path process $y: \mathbb{R}^+ \times \hat{\Omega} \to \mathbb{R}^d$ such that $(t, (\omega, y)) \mapsto y_t$ is a d-dimensional Brownian motion stopped at T.

(c) Under $\hat{\mathbb{P}}_T$, T fixed, Z_t can be written explicitly as a function of the driving path: For $t \leq T$,

$$\begin{aligned}
(3.1)\qquad e^{\gamma t}Z_t &= Z_0 + \int_0^t e^{\gamma s}\, dZ_s + \int_0^t Z_s(\gamma e^{\gamma s})\, ds \\
&= Z_0 + \int_0^t e^{\gamma s}\, dy_s + \int_0^t e^{\gamma s}(-\gamma Z_s)\, ds + \int_0^t Z_s(\gamma e^{\gamma s})\, ds,
\end{aligned}$$

where we have used a differential form of $(SE)^1_{Z_0,K}$(a) for the second equality. Hence,

$$(3.2)\qquad Z_t(y) = e^{-\gamma t}Z_0 + \int_0^t e^{-\gamma(t-s)}\, dy_s.$$

Next, we show that there exists a unique solution to $(SE)^2_{Y_0, K}$.

PROOF OF THEOREM 1.4. Suppose there exists a solution Y satisfying $(SE)^2_{Y_0, K}$. Then under $\hat{\tilde{\mathbb{P}}}_T$, Y_t can be written as a function of the driving path y and $\bar{Y}$. Using integration by parts gives

$$e^{\gamma t} Y_t = Y_0 + \int_0^t e^{\gamma s} \, dY_s + \int_0^t \gamma e^{\gamma s} Y_s \, ds$$

$$= Y_0 + \int_0^t e^{\gamma s} \, dy_s + \int_0^t \gamma e^{\gamma s} \bar{Y}_s \, ds$$

and hence

$$(3.3) \qquad Y_t = e^{-\gamma t} Y_0 + \int_0^t e^{\gamma (s-t)} \, dy_s + \int_0^t \gamma e^{\gamma (s-t)} \bar{Y}_s \, ds.$$

If (X, Z) is the solution to $(SE)^1_{Z_0, K}$ where $Z_0 = Y_0$, then note that by Remark 3.1(c),

$$Y_t = Z_t + \int_0^t \gamma e^{\gamma (s-t)} \bar{Y}_s \, ds.$$

By taking the normalized measure $\tilde{K}_t$ on both sides of the above equation, we get

$$(3.4) \qquad \bar{Y}_t = \bar{Z}_t + \gamma \int_0^t e^{-\gamma (t-s)} \bar{Y}_s \, ds.$$

Hence, $\bar{Y}_t$ is seen to satisfy a Volterra Integral Equation of the second kind (see equation (2.2.1) of [11]) and therefore can be solved pathwise to give

$$\bar{Y}_t = \bar{Z}_t + \gamma \int_0^t \bar{Z}_s \, ds,$$

which is easily verified using integration by parts. Also, if $\bar{Y}^1_t$ is a second process which solves (3.4), then

$$|\bar{Y}_t - \bar{Y}^1_t| = \left| \gamma \int_0^t e^{-\gamma (t-s)} (\bar{Y}_s - \bar{Y}^1_s) \, ds \right|$$

$$\leq |\gamma| \int_0^t e^{-\gamma (t-s)} |\bar{Y}_s - \bar{Y}^1_s| \, ds.$$

By Gronwall's inequality, this implies $\bar{Y}_t = \bar{Y}^1_t$, for all t and ω. Pathwise uniqueness of X' follows from the uniqueness of the solution to $(SE)^1_{Z_0, K}$ and the uniqueness of the process $\bar{Y}_t$ solving (3.4).

We have shown that if there exists a solution to $(SE)^2_{Y_0, K}$, then it is necessarily pathwise unique. Turning now to existence to complete the proof, we work in the

opposite order and define Y and X' as functions of the pathwise unique solution to $(SE)^1_{Z_0, K}$ where $Z_0 = Y_0$:

$$Y_t = Z_t + \gamma \int_0^t \bar{Z}_s \, ds,$$

$$X'_t(\cdot) = K_t(Y_t \in \cdot).$$

Then $\bar{Y}_t$ satisfies the integral equation (3.4), and hence

$$\int_0^t \bar{Z}_s \, ds = \int_0^t e^{-\gamma(t-s)} \bar{Y}_s \, ds.$$

Therefore

$$Y_t = Z_t + \gamma \int_0^t e^{-\gamma(t-s)} \bar{Y}_s \, ds$$

$$= e^{-\gamma t} Y_0 + \int_0^t e^{-\gamma(t-s)} \, dy_s + \gamma \int_0^t e^{-\gamma(t-s)} \bar{Y}_s \, ds,$$

by equation (3.2), and so

$$e^{\gamma t} Y_t = Y_0 + \int_0^t e^{\gamma s} \, dy_s + \gamma \int_0^t e^{\gamma s} \bar{Y}_s \, ds.$$

Multiplying by $e^{-\gamma t}$ and using integration by parts shows

$$Y_t = Y_0 + y_t - y_0 + \gamma \int_0^t (\bar{Y}_s - Y_s) \, ds$$

which holds for K-a.e. y, thereby showing (X', Y) satisfies $(SE)^2_{Y_0, K}$. □

REMARK 3.2. Some useful equivalences in the above proof are collected below. If $Y_0 = Z_0$, then for $t < \eta$,

(a) $$Y_t = Z_t + \gamma \int_0^t \bar{Z}_s \, ds,$$

(b) $$\bar{Y}_t = \bar{Z}_t + \gamma \int_0^t \bar{Z}_s \, ds,$$

(c) $$Y_t - \bar{Y}_t = Z_t - \bar{Z}_t,$$

(d) $$\int_0^t e^{-\gamma(t-s)} \bar{Y}_s \, ds = \int_0^t \bar{Z}_s \, ds.$$

These equations intimately tie the behaviour of the interacting and ordinary SOU processes. Part (a) says that the interacting SOU process with attraction to the center of mass is the same as the ordinary SOU process pushed by the position of its center of mass.

We now consider the martingale problem for X'. For $\phi : \mathbb{R}^d \to \mathbb{R}$, recall that $\bar{\phi}_t \equiv K_t(\phi(Y_t))/K_t(1)$ and that the lifetime of the process $\bar{\phi}$ is $[0, \eta)$. Then the following theorem holds:

THEOREM 3.3. *For $\phi \in C_b^2(\mathbb{R}^d, \mathbb{R})$, and $t < \eta$,*

$$\bar{\phi}_t = \bar{\phi}_0 + N_t + \int_0^t \bar{b}_s \, ds,$$

where

$$b_s = \gamma \nabla \phi(Y_s) \cdot (\bar{Y}_s - Y_s) + \tfrac{1}{2} \Delta \phi(Y_s)$$

and N_t is a continuous local martingale on its lifetime such that

$$N_t = \int_0^t \int \frac{\phi(Y_s) - \bar{\phi}_s}{K_s(1)} \, dM(s, y)$$

and hence has quadratic variation given by

$$[N]_t = \int_0^t \frac{\overline{\phi_s^2} - (\bar{\phi}_s)^2}{K_s(1)} \, ds.$$

Similarly, the following is true.

REMARK 3.4. The method of Theorem 3.3 can be used to show that for the β-supercritical SOU process, X, for $\phi \in C_b^2(\mathbb{R}^d, \mathbb{R})$ and $t < \eta$,

$$\tilde{X}_t(\phi) = \tilde{K}_t(\phi(Z_t)) = \tilde{K}_0(\phi(Z_0)) + N_t + \int_0^t \tilde{K}_s(L\phi(Z_s)) \, ds,$$

where

$$L\phi(x) = -\gamma x \cdot \nabla \phi(x) + \tfrac{1}{2} \Delta \phi(x)$$

and N_t is a continuous local martingale on its lifetime such that

$$N_t = \int_0^t \int \frac{\phi(Z_s) - \tilde{K}_s(\phi(Z_s))}{K_s(1)} \, dM(s, y)$$

and hence has quadratic variation given by

$$[N]_t = \int_0^t \frac{\tilde{K}_s(\phi^2(Z_s)) - \tilde{K}_s(\phi(Z_s))^2}{K_s(1)} \, ds.$$

PROOF OF THEOREM 3.3. The proof is not very difficult; one need only use Itô's lemma followed by some slight modifications of theorems in Chapter V of [9] to deal with the drift introduced in the historical martingale problem due to the supercritical branching.

Let T be a fixed time and $t \leq T$. Recall that under $\hat{\mathbb{P}}_T$, y is a stopped Brownian motion by Remark 3.1(b), and hence $Y_t(y)$ is a stopped OU process (attracting

to $\bar{Y}_t$). Therefore, under $\hat{\mathbb{P}}_T$:

$$\phi(Y_t) = \phi(Y_0) + \int_0^t \nabla\phi(Y_s) \cdot dY_s + \frac{1}{2}\sum_{i,j\leq d}\int_0^t \phi_{ij}(Y_s)\, d[Y^i, Y^j]_s$$

$$= \phi(Y_0) + \int_0^t \nabla\phi(Y_s) \cdot dy_s + \int_0^t \gamma\nabla\phi(Y_s) \cdot (\bar{Y}_s - Y_s) + \frac{1}{2}\Delta\phi(Y_s)\, ds$$

by the classical Itô's lemma. Then

$$K_t(\phi(Y_t)) = K_t(\phi(Y_0)) + K_t\left(\int_0^t \nabla\phi(Y_s) \cdot dy_s\right) + K_t\left(\int_0^t b_s\, ds\right)$$

$$= K_0(\phi(Y_0)) + \int_0^t \int \phi(Y_0)\, dM(s, y) + \beta\int_0^t K_s(\phi(Y_0))\, ds$$

$$+ \int_0^t \int \left[\int_0^s \nabla\phi(Y_r) \cdot dy_r\right] dM(s, y)$$

$$+ \beta\int_0^t K_s\left[\int_0^s \nabla\phi(Y_r) \cdot dy_r\right] ds$$

$$+ \int_0^t \int b_s\, dM(s, y) + \beta\int_0^t K_s(b_s)\, ds + \int_0^t K_s(b_s)\, ds$$

$$= K_0(\phi(Y_0)) + \int_0^t \int \phi(Y_s)\, dM(s, y)$$

$$+ \beta\int_0^t K_s(\phi(Y_s))\, ds + \int_0^t K_s(b_s)\, ds.$$

The equality in the third line to $K_t(\int_0^t \nabla\phi(Y_s) \cdot dy_s)$ follows from Proposition 2.13 of [8]. The equality of the fourth line to the last term in the first line follows from a generalization of Proposition V.2.4 (b) of [9]. The last equality then follows by collecting like terms and using the definition of $\bar{Y}$.

Note that $K_t(1)$ is Feller's β-supercritical branching diffusion and hence

$$K_t(1) = K_0(1) + M_t^0 + \beta\int_0^t K_s(1)\, ds,$$

where M_t^0 is a martingale such that $[M^0]_t = \int_0^t K_s(1)\, ds$.

Therefore for $t < \eta$, Itô's formula and properties of $K_t(1)$ and $K_t(\phi(Y_t))$ imply that

$$\bar{\phi}_t = \frac{K_t(\phi(Y_t))}{K_t(1)}$$

$$= \bar{\phi}_0 + \int_0^t \int \left[\frac{\phi(Y_s)}{K_s(1)} - \frac{K_s(\phi(Y_s))}{K_s(1)^2}\right] dM(s, y) + \int_0^t \frac{K_s(b_s)}{K_s(1)}\, ds.$$

Since ϕ is bounded, the stochastic integral term can be localized using the stopping times $T_N \equiv \min\{t : K_t(1) \geq N \text{ or } K_t(1) \leq 1/N\} \wedge N$ and hence it is a local martingale on $[0, \eta)$. It is easy to check that it has the appropriate quadratic variation. $\square$

The following lemmas will be used extensively in Section 4, but will be proven in Section 5.

LEMMA 3.5.   *There is a nonnegative random variable W such that*

$$e^{-\beta t} K_t(1) \to W \qquad a.s.$$

and $\{\eta < \infty\} = \{W = 0\}$ almost surely.

Note that as X and X' are defined as projections of K, their mass processes are the same as that of K, and thus grow at the same rate.

DEFINITION 3.6.   Let

$$h(\delta) = \left(\delta \ln^+(1/\delta)\right)^{1/2},$$

where $\ln^+(x) = \ln x \vee 1$. Let

$$S(\delta, c) = \{y : |y_t - y_s| < ch(|t - s|), \forall t, s \text{ with } |t - s| \leq \delta\}.$$

LEMMA 3.7.   *Let K be a supercritical historical Brownian motion, with drift β, branching variance 1, and initial measure X_0. For $c_0 > 6$ fixed, $c(t) = \sqrt{t + c_0}$, there exists a.s. $\delta(\omega) > 0$ such that $\mathrm{Supp}(K_t(\omega)) \subset S(\delta(\omega), c(t))$ for all t. Further, given c_0, $\mathbb{P}(\delta < \lambda) < p_{c_0}(\lambda)$ where $p_{c_0}(\lambda) \downarrow 0$ as $\lambda \downarrow 0$ and for any $\alpha > 0$, c_0 can be chosen large enough so that $p_{c_0}(\lambda) = C(d, c_0)\lambda^\alpha$ for $\lambda \in [0, 1]$.*

The following moment estimates are useful in establishing the convergence of $\bar{Y}_t$. Recall that η is the extinction time of K.

LEMMA 3.8.   *Assume $\mathbb{P}(\tilde{K}_0(|Y_0|^2 + |y_0|^2)) < \infty$. Then,*

$$\mathbb{P}(\overline{|Y_t|^2}; t < \eta) < A(\gamma, t),$$

where

$$A(\gamma, t) = \begin{cases} O(1 + t^6 e^{-2\gamma t}), & \text{if } \gamma < 0, \\ O(1 + t^5), & \text{if } \gamma \geq 0. \end{cases}$$

REMARK 3.9.   (a) The proof of Lemma 3.8, under the same hypotheses (if $Z_0 = Y_0$) yields

$$\mathbb{P}(\overline{|Z_t|^2}; t < \eta) < A(\gamma, t).$$

(b) Lemma 3.8 and its proof can be extended to show that for any positive integer k if $\mathbb{P}(\tilde{K}_0(|Z_0|^k + |y_0|^k)) < \infty$ (and $Z_0 = Y_0$), then there exists a function $B(\gamma, t, k)$ polynomial in t if $\gamma \geq 0$, exponential if $\gamma < 0$ such that

$$\mathbb{P}(\overline{|Y_t|^k}; t < \eta) < B(\gamma, t, k) \quad \text{and} \quad \mathbb{P}(\overline{|Z_t|^k}; t < \eta) < B(\gamma, t, k).$$

4. Proofs of convergence. We will henceforth, unless specified otherwise, assume that for a path $y \in C$, $Z_0(y_0) = Y_0(y_0) = y_0$. Recall that, by the construction of solutions to $(\text{SE})^i$,

$$\int \phi(y_0) K_0(dy) = \int \phi(x) X_0(dx) = \int \phi(x) X_0'(dx).$$

Also recall that our standing hypothesis is that K_0 has finite initial mass [and hence $X_0'(1) = X_0(1) < \infty$].

In this section, we will first settle what happens to X and X' when there is extinction, and then the case when there is survival, under the attractive regime and lastly address the interacting repelling SOU process on the survival set.

4.1. *On the extinction set.*

PROOF OF THEOREM 1.5.   Assume for now that $\mathbb{P}(\tilde{K}_0(|y_0|^2)) < \infty$ (the case where $K_0 = 0$ can be ignored without loss of generality). By Theorem 3.3,

$$\bar{Y}_t = \bar{Y}_0 + \int_0^t \int \frac{Y_s - \bar{Y}_s}{K_s(1)} \, dM(s, y)$$

and therefore is a local martingale on its lifetime with reducing sequence $\{T_N\}$ as defined in the proof of the same theorem. Using Doob's weak inequality and Lemma 3.8,

$$\mathbb{P}\left(\sup_{s < t \wedge \eta} |\bar{Y}_s| > n \right) = \lim_{N \to \infty} \mathbb{P}\left(\sup_{s < t \wedge T_N} |\bar{Y}_s| > n \right)$$

$$\leq \lim_{N \to \infty} \frac{1}{n^2} \mathbb{E}(|\bar{Y}_{t \wedge T_N}|^2)$$

$$< \frac{A(\gamma, t)}{n^2}.$$

By the first Borel–Cantelli lemma,

$$\mathbb{P}\left(\sup_{s < t \wedge \eta} |\bar{Y}_t| > n \text{ i.o.} \right) = 0.$$

It follows that

$$\liminf_{s \to t \wedge \eta} \bar{Y}_s > -\infty \quad \text{and} \quad \limsup_{s \to t \wedge \eta} \bar{Y}_s < \infty$$

which implies that on the set $\{\eta < t\}$, $\bar{Y}_s$ converges, by Theorem IV.34.12 of Rogers and Williams [13]. This shows convergence on the extinction set as $S^c = \bigcup_t \{\eta < t\}$.

Note that if $\nu(\cdot) = \mathbb{P}(K_0 \in \cdot)$. Theorem II.8.3 of [9] gives

$$\mathbb{P}(K \in \cdot) = \int \mathbb{P}_{K_0}(K \in \cdot)\, d\nu(K_0),$$

where $\mathbb{P}_{K_0}$ is the law of a historical Brownian motion with initial distribution δ_{K_0}. Hence, the a.s. convergence of $\bar{Y}_t$ in the case where $\tilde{K}_0(|y_0|^2)$ is finite in mean imply

$$\mathbb{P}\Big(\lim_{t\uparrow\eta} \bar{Y}_t \text{ exists}; S^c\Big) = \int \mathbb{P}_{K_0}\Big(\lim_{t\uparrow\eta} \bar{Y}_t \text{ exists}; S^c\Big) d\nu(K_0)$$

(4.1)
$$= \int \mathbb{P}_{K_0}(S^c)\, d\nu(K_0)$$

$$= \mathbb{P}(S^c)$$

if $K_0(|y_0|^2 + 1) < \infty$, ν-a.s.

Finally, to get rid of the assumption that $K_0(|y_0|^2) < \infty$ note that Corollary 3.4 of [8] ensures that if $K_0(1) < \infty$, then at any time $t > 0$, K_t (and hence X_t, X_t') is compactly supported. Therefore, letting $S_r = \{K_r \neq 0\}$ we see that

$$\mathbb{P}\Big(\lim_{t\uparrow\eta} \bar{Y}_t \text{ does not exist}, S^c\Big) = \mathbb{P}\Big(\bigcup_{r\in\mathbb{N}} \Big\{\lim_{t\uparrow\eta} \bar{Y}_t \text{ does not exist}, S_{1/r}, S^c\Big\}\Big)$$

$$\leq \sum_{r\in\mathbb{N}} \mathbb{P}\Big(\mathbb{P}_{K_{1/r}}\Big(\lim_{t\uparrow\eta} \bar{Y}_t \text{ does not exist}, S^c\Big) \mathbf{1}_{S_{1/r}}\Big)$$

$$= 0$$

by (4.1) since $K_{1/r}$ a.s. compact implies that $K_{1/r}(|y_{1/r}|^2) < \infty$ holds. This completes the proof for the convergence of $\bar{Y}$ on S^c in its full generality.

The convergence of $\bar{Z}_t$ now follows from the convergence of $\bar{Y}_t$ and equation (3.4). $\square$

PROOF OF THEOREM 1.6.   As in the previous proof, note that we need only consider the case that $\mathbb{P}(\tilde{K}_0(|y_0|^2)) < \infty$.

We will follow the proof of Theorem 1 of Tribe [14] here. Define

$$\zeta(t) = \int_0^t \frac{1}{K_s(1)}\, ds, \qquad t < \eta.$$

It is known by the work of Konno and Shiga [7] in the case where $\beta = 0$, that $\zeta : [0, \eta) \to [0, \infty)$ homeomorphically (recall that $\eta < \infty$ a.s. in that case). This latter result also holds when $\beta > 0$ on the extinction set S^c by a Girsanov argument.

Define $D : [0, \zeta(\eta-)) \to [0, \eta)$ as the unique inverse of ζ (on S^c, this defines the inverse on $[0, \infty)$) and for $t \geq \zeta(\eta-)$, let $D_t = \infty$. Let

$$X_t^D = X'_{D_t}, \qquad \tilde{X}_t^D = \frac{X_t^D}{X_t^D(1)} \quad \text{and} \quad \mathcal{G}_t = \mathcal{F}_{D_t}$$

and define

$$L_t \phi(x) = \gamma (\bar{Y}_t - x) \cdot \nabla \phi(x) + \tfrac{1}{2} \Delta \phi(x).$$

Let

$$T_N = \int_0^{\eta_N} \frac{1}{K_s(1)} \, ds,$$

where $\eta_N = \inf\{s : K_s(1) \leq 1/N\}$. Then note that $T_N \uparrow \zeta(\eta-)$ and each T_N is a $\mathcal{G}_t$-stopping time. On S^c for $\phi \in C_b^2$, Theorem 3.3 implies

$$\tilde{X}_{t \wedge T_N}^D(\phi) = \tilde{X}_0(\phi) + \int_0^{D_{t \wedge T_N}} \tilde{X}'_s(L_s \phi) \, ds + M_{D_{t \wedge T_N}}(\phi)$$

$$(4.2) \qquad\qquad = \tilde{X}_0(\phi) + \int_0^{t \wedge T_N} \tilde{X}_s^D(L_{D_s} \phi) X_s^D(1) \, ds + N_{t \wedge T_N}(\phi)$$

$$= \tilde{X}_0(\phi) + \int_0^{t \wedge T_N} X_s^D(L_{D_s} \phi) \, ds + N_{t \wedge T_N}(\phi)$$

since $dD_t = X_t^D(1) \, dt$ and where $N_t = M_{D_t}$. It follows that $N_{t \wedge T_N}$ is a $\mathcal{G}_t$-local martingale. Then, by Theorem 3.3,

$$[\tilde{X}^D(\phi)]_{t \wedge T_N} = \int_0^{D_{t \wedge T_N}} \frac{\tilde{X}'_s(\phi^2) - \tilde{X}'_s(\phi)^2}{X_s(1)} \, ds$$

$$= \int_0^{t \wedge T_N} \tilde{X}_s^D(\phi^2) - \tilde{X}_s^D(\phi)^2 \, ds,$$

which is uniformly bounded in N. Hence, sending $N \to \infty$, one sees that $N_{t \wedge \zeta(\eta-)}$ is a $\mathcal{G}_t$-martingale.

Note that on S^c, $\zeta(\eta-) = \infty$ and hence on that event,

$$\int_0^\infty X_s^D(|L_{D_s} \phi|) \, ds \leq \int_0^\infty \int |\gamma (\bar{Y}_{D_s} - x) \cdot \nabla \phi(x)| + \frac{1}{2} |\Delta \phi(x)| X_s^D(dx) \, ds$$

$$\leq \int_0^\infty |\gamma| \|\nabla \phi\| K_{D_s}(|\bar{Y}_{D_s} - Y_{D_s}|) + \frac{1}{2} \|\Delta \phi\| X_s^D(1) \, ds$$

$$= \int_0^\infty X_s^D(1) \left(|\gamma| \|\nabla \phi\| \tilde{K}_{D_s}(|\bar{Y}_{D_s} - Y_{D_s}|) + \frac{1}{2} \|\Delta \phi\| \right) ds$$

$$\leq \int_0^\infty X_s^D(1) \left(|\gamma| \|\nabla \phi\| (\overline{|Y_{D_s}|^2})^{1/2} + \frac{1}{2} \|\Delta \phi\| \right) ds,$$

where in the second line we have used the definition of X' and the Cauchy–Schwarz inequality in the fourth. Using the definition of D_s yields

$$\int_0^\infty X_s^D(|L_{D_s}\phi|)\,ds \leq \left(|\gamma|\,\|\nabla\phi\|\sup_{s<\eta}(\overline{|Y_s|^2})^{1/2} + \frac{1}{2}\|\Delta\phi\|\right)\int_0^\infty X_s^D(1)\,ds$$

(4.3)
$$= \left(|\gamma|\,\|\nabla\phi\|\sup_{s<\eta}(\overline{|Y_s|^2})^{1/2} + \frac{1}{2}\|\Delta\phi\|\right)\eta$$

$$< \infty$$

as $\phi \in C_b^2$ and $\overline{|Y_s|^2}$ is continuous on $[0, \eta)$ (which follows from Theorem 3.3). Hence, this implies that for ϕ positive, on S^c

$$N_t(\phi) > -\tilde{X}_0(\phi) - \int_0^\infty X_s^D(|L_{D_s}\phi|)\,ds$$

for all t and hence by Corollary IV.34.13 of [13], N_t converges as $t \to \infty$. Therefore by (4.2) and (4.3), $\tilde{X}_t^D(\phi)$ converges a.s. as well.

Denote by $\tilde{X}_\infty^D(\phi)$ the limit of $\tilde{X}_t^D(\phi)$. It is immediately evident that $\tilde{X}_\infty^D(\cdot)$ is a probability measure on $\mathbb{R}^d$. To show that $\tilde{X}_\infty^D(\cdot) = \delta_{F'}$ where F' is a random point in $\mathbb{R}^d$, we now defer to the proof of Theorem 1 in Tribe [14], as it is identical from this point forward.

Similar (but simpler) reasoning holds to show $\tilde{X}_t \to \delta_F$ a.s. on S^c where F is a random point in $\mathbb{R}^d$. Let $f(t) = \gamma\int_0^t \bar{Z}_s\,ds$. Note that f is independent of y and that $f(t) \to f(\eta)$ a.s. when $t \uparrow \eta$ because $\bar{Y}_t = \bar{Z}_t + f(t)$ and both $\bar{Y}_t$ and $\bar{Z}_t$ converge a.s. by Theorem 1.5. Then for ϕ bounded and Lipschitz,

$$\left|\int \phi(x - f(t))\tilde{X}_t'(dx) - \int \phi(x - f(\eta))\tilde{X}_t'(dx)\right| \leq C|f(\eta) - f(t)|$$

$$\xrightarrow{\text{a.s.}} 0$$

as $t \uparrow \eta$. Therefore it is enough to note that since $f(\eta)$ depends only on ω, the convergence of $\tilde{X}_t'$ gives

$$\int \phi(x - f(\eta))\tilde{X}_t'(dx) \xrightarrow{\text{a.s.}} \phi(F' - f(\eta))$$

and hence

$$\int \phi(x - f(t))\tilde{X}_t'(dx) \xrightarrow{\text{a.s.}} \phi(F' - f(\eta)).$$

By Remark 3.2(a),

$$\int \phi(x - f(t))\tilde{X}_t'(dx) = \int \phi(Y_t(y) - f(t))\tilde{K}_t(dy)$$

$$= \int \phi(Z_t(y))\tilde{K}_t(dy)$$

$$= \int \phi(x)\tilde{X}_t(dx)$$

$$\xrightarrow{\text{a.s.}} \phi(F)$$

as $t \uparrow \eta$. Since there exists a countable separating set of bounded Lipschitz functions $\{\phi_n\}$, and the above holds for each ϕ_n,

$$F' = F + \gamma \int_0^\eta \bar{Z}_s \, ds \qquad \text{a.s.} \qquad \qquad \square$$

REMARK 4.1.   (a) Theorem 1.6 holds in the critical branching case. That is, if $\beta = 0$,

$$X_t \xrightarrow{\text{a.s.}} \delta_F \quad \text{and} \quad X'_t \xrightarrow{\text{a.s.}} \delta_{F'},$$

where

$$F' = F + \int_0^\eta \bar{Z}_s \, ds.$$

The convergence of the critical ordinary SOU process to a random point follows directly from Tribe's result. That this holds for the SOU process with attraction to the COM follows from the calculations above.

(b) The distribution of the random point F has been identified in Tribe [14] by approximating with branching particle systems. In fact, the law of F can be identified as x_η, where x_t is an Ornstein–Uhlenbeck process with initial distribution given by $\tilde{X}_0$ and η is the extinction time. Finding the distribution of F' remains an open problem however.

4.2. *On the survival set, the attractive case.*

PROOF OF THEOREM 1.7.   Let $\gamma \geq 0$ and as in the proof of Theorem 1.5, assume $\mathbb{P}(\tilde{K}_0(|y_0|^2)) < \infty$. Also, without loss of generality, assume that $d = 1$ for this proof.

By Theorem 3.3, $\bar{Y}_t$ is a continuous local martingale with decomposition given by $\bar{Y}_t = \bar{Y}_0 + M_t(Y)$ where

$$[M(Y)]_t = \int_0^t \frac{\overline{Y_s^2} - \bar{Y}_s^2}{K_s(1)} \, ds = \int_0^t \frac{V(Y_s)}{K_s(1)} \, ds,$$

with $V(Y_s) = \overline{Y_s^2} - \bar{Y}_s^2$. Theorem IV.34.12 of [13] shows that on the set $\{[M(Y)]_\infty < \infty\} \cap S$, $M_t(Y)$ a.s. converges.

Note that by Lemma 3.5, for a.e. $\omega \in S$, $W(\omega) > 0$, recalling that $W = \lim_{t \to \infty} e^{-\beta t} K_t(1)$. Hence, it follows that $[M(Y)]_\infty < \infty$ on S if

$$\int_0^\infty e^{-\beta s} V(Y_s) \, ds < \infty.$$

Then

$$\mathbb{P}\left(\int_0^\infty e^{-\beta s} V(Y_s)\,ds; S\right) \leq \mathbb{P}\left(\int_0^\infty e^{-\beta s}\overline{Y_s^2}\,ds; S\right)$$

$$\leq \int_0^\infty e^{-\beta s} A(\gamma, s)\,ds$$

$$< \infty$$

by Cauchy–Schwarz and Lemma 3.8 since $\gamma \geq 0$. Therefore on S, $\bar{Y}_t$ converges a.s. to some limit $\bar{Y}_\infty$. Note that if $\gamma = 0$, Remark 3.2(b) gives $\bar{Y}_t = \bar{Z}_t$ and so (b) holds.

That $\bar{Z}_t$ converges on S for $\gamma > 0$ follows from the fact that $\bar{Y}_t$ converges and equation (3.4) by setting

$$\bar{Z}_t = \bar{Y}_t - \gamma \int_0^t e^{-\gamma(t-s)}\bar{Y}_s\,ds$$

$$= \bar{Y}_t - \bar{Y}_\infty + \gamma \int_0^t e^{-\gamma(t-s)}(\bar{Y}_\infty - \bar{Y}_s)\,ds + e^{-\gamma t}\bar{Y}_\infty$$

$$\to 0 \qquad \text{as } t \to \infty.$$

By Remark 3.2(b), we see that for $\gamma > 0$ since $\bar{Y}_t = \bar{Z}_t + \gamma \int_0^t \bar{Z}_s\,ds$, $\bar{Z}_t \xrightarrow{\text{a.s.}} 0$ and $\bar{Y}_t \xrightarrow{\text{a.s.}} \gamma \int_0^\infty \bar{Z}_s\,ds$.

Now argue by conditioning as in the end of Theorem 1.5 to get the full result. □

The next few results are necessary to establish the almost sure convergence of $\tilde{X}_t$ on the survival set. This will in turn be used to show the almost sure convergence of $\tilde{X}_t'$ using the correspondence of Remark 3.2(a).

Let P_t be the standard Ornstein–Uhlenbeck semigroup (with attraction to the origin). Note that $P_t \to P_\infty$ in norm where

$$P_\infty \phi(x) = \int \phi(z)\left(\frac{\gamma}{\pi}\right)^{d/2} e^{-\gamma|z|^2}\,dz,$$

which is independent of x. Recall that $W = \lim_{t\to\infty} e^{-\beta t} X_t(1)$ and $S = \{W > 0\}$ a.s. from Lemma 3.5.

LEMMA 4.2. *If $\gamma > 0$, $\mathbb{P}(X_0(|x|^4)) < \infty$ and $\mathbb{P}(X_0(1)^4) < \infty$, then on S, for any $\phi \in \mathrm{Lip}_1$, $e^{-\beta t} X_t(\phi) \xrightarrow{L^2} W P_\infty \phi$ and*

$$\mathbb{P}\big(|e^{-\beta t} X_t(\phi) - W P_\infty \phi|^2\big) \leq C e^{-\zeta t},$$

where C depends only on d and X_0, and ζ is a positive constant dependent only on β and γ.

REMARK 4.3.   As the L^2 convergence in Lemma 4.2 is exponentially fast, it follows from the Borel–Cantelli lemma and Chebyshev inequality that for a strictly increasing sequence $\{t_n\}_{n=0}^{\infty}$ where $|\{t_n\} \cap [k, k+1)| = \lfloor e^{\zeta k/2} \rfloor$, for $\phi \in \mathrm{Lip}_1$,

$$e^{-\beta t_n} X_{t_n}(\phi) \to W P_{\infty} \phi \qquad \text{a.s. as } n \to \infty.$$

The idea is to use the above remark to bootstrap up to almost sure convergence in Lemma 4.2 with some estimates on the modulus of continuity of the process $e^{-\beta t} X_t(\phi)$.

LEMMA 4.4.   *Suppose $\gamma > 0$, $\mathbb{P}(X_0(|x|^8)) < \infty$ and $\mathbb{P}(X_0(1)^8) < \infty$. If $\phi \in$ Lip_1 and $h > 0$, then*

$$(4.4) \qquad \mathbb{P}\big([e^{-\beta(t+h)} X_{t+h}(\phi) - e^{-\beta t} X_t(\phi)]^4 \big) \leq C(t) h^2 e^{-\zeta^* t},$$

where ζ^* is a positive constant depending only on β and γ and C is polynomial in t, and depends on γ, β and d.

Let $\Psi : \mathbb{R}^d \to \mathbb{R}$, $p : \mathbb{R}_+ \to \mathbb{R}$ be positive, continuous functions. Further, suppose Ψ is symmetric about 0 and convex with $\lim_{|x| \to \infty} \Psi(x) = \infty$ and $p(x)$ is increasing with $p(0) = 0$. The following is a very useful result of Garsia, Rodemich and Rumsey [5].

PROPOSITION 4.5.   *If f is a measurable function on $[0, 1]$ such that*

$$(4.5) \qquad \int \int_{[0,1]^2} \Psi \left(\frac{f(t) - f(s)}{p(|t - s|)} \right) ds \, dt = B < \infty$$

then there is a set K of measure 0 such that if $s, t \in [0, 1] \setminus K$ then

$$(4.6) \qquad |f(t) - f(s)| \leq 8 \int_0^{|t-s|} \Psi^{-1} \left(\frac{B}{u^2} \right) dp(u).$$

If f is also continuous, then K is in fact the empty set.

With this result in hand, we can now bring everything together to prove convergence of $\tilde{X}_t$.

PROOF OF THEOREM 1.8.   The strategy for this proof is simple: We use Remark 4.3 to see that we can lay down an increasingly (exponentially) dense sequence $e^{-\beta t_n} X_{t_n}(\phi)$ which converges almost surely, and that we can use Lemma 4.4 to get a modulus of continuity on the process $e^{-\beta t} X_t(\phi)$, which then implies that if the sequence is converging, then the entire process must be converging.

Assume that $\mathbb{P}(K_0(|y_0|^8)) = \mathbb{P}(X_0(|x|^8)) < \infty$ and $\mathbb{P}(X_0(1)^8) < \infty$ and argue as in Theorem 1.5 in the general case. Let $\phi \in \mathrm{Lip}_1$. Denote $e^{-\beta t} X_t$ by

$\mathring{X}_t$ for the remainder of the proof. Let $T > 0$, and let $\Psi(x) = |x|^4$ and $p(t) = |t|^{3/4}(\log(\frac{\lambda}{t}))^{1/2}$ where $\lambda = e^4$. Let $B_T(\omega)$ be the constant B that appears in Proposition 4.5, with aforementioned functions Ψ and p, for the path $\mathring{X}_{Tt}(\omega)$, $t \in [0, 1]$.

Then note that

$$\mathbb{P}(B_T) \equiv \mathbb{P}\left[\int\int_{[0,1]^2} \Psi\left(\frac{\mathring{X}_{Tt} - \mathring{X}_{Ts}}{p(|t-s|)}\right) ds\, dt\right]$$

$$= \int\int_{[0,1]^2} \frac{\mathbb{P}[|\mathring{X}_{Tt} - \mathring{X}_{Ts}|^4]}{|t-s|^3 \log^2(\lambda/|t-s|)} ds\, dt$$

$$\leq \int\int_{[0,1]^2} \frac{C(T(s \wedge t))e^{-\zeta^*(s \wedge t)}T^2|t-s|^2}{|t-s|^3 \log^2(\lambda/|t-s|)} ds\, dt$$

$$= 2T^2 \int_0^1 \int_0^t \frac{C(Ts)e^{-\zeta^*(Ts)}|t-s|^2}{|t-s|^3 \log^2(\lambda/|t-s|)} ds\, dt$$

$$\leq 2C(T)T^2 \int_0^1 \int_0^t \frac{1}{|t-s| \log^2(\lambda/|t-s|)} ds\, dt$$

$$\leq \frac{C(T)T^2}{2e^4},$$

where C is the polynomial term that appears in Lemma 4.4. Since $\mathring{X}_t$ is continuous, by Garsia–Rodemech–Rumsey [5], for all $s, t \leq 1$,

$$|\mathring{X}_{Tt} - \mathring{X}_{Ts}| \leq 8 \int_0^{|t-s|} \left(\frac{B_T}{u^2}\right)^{1/4} dp(u)$$

$$\leq AB_T^{1/4}|t-s|^{1/4}\left(\log\frac{\lambda}{|t-s|}\right)^{1/2},$$

where A is a constant independent of T (see Corollary 1.2 of Walsh [15] for this calculation). Rewriting the above,

$$(4.7) \qquad |\mathring{X}_t - \mathring{X}_s| \leq D_T|t-s|^{1/4}\left(\log\frac{\lambda T}{|t-s|}\right)^{1/2} \qquad \forall s < t \leq T,$$

where $D_T \equiv A(\frac{B_T}{T})^{1/4}$. Note that $\mathbb{P}(D_T^4) = \frac{A^4 T C(T)}{2e^4}$, which is polynomial in T of fixed degree $d_0 > 1$. Let Ω_0 be the set of probability 1 such that for all positive integers T equation (4.7) holds and $D_T \leq T^{d_0}$ for T large enough. To see $P(\Omega_0) = 1$, use Borel–Cantelli:

$$\mathbb{P}(D_T \geq T^{d_0}) = \mathbb{P}(D_T^4 \geq T^{4d_0})$$

$$\leq \frac{\mathbb{P}(D_T^4)}{T^{4d_0}}$$

$$\leq \frac{c}{T^{3d_0}}$$

which is summable over all positive integers T.

Suppose $\omega \in \Omega_0$. Let $\delta^T(\omega)$ be such that $\delta^{-1/8}(\log \frac{\lambda}{\delta})^{-1/2} = T^{d_0}$. Then for all integral $T > T_0(\omega)$, and $s, t \leq T$ with $|t - s| \leq \delta$, $|\mathring{X}_t - \mathring{X}_s| \leq |t - s|^{1/8}$.

Now let $\{\mathring{X}_{t_n}\}$ be a sequence of the form in Remark 4.3, with the additional condition that $\{t_n\} \cap [k, k+1)$ are evenly spaced within $[k, k+1)$ for each $k \in \mathbb{Z}_+$ (i.e., $t_{n+1} - t_n = ce^{-\zeta k/2}$ for $t_n \in \{t_n\} \cap [k, k+1)$). Evidently $\mathring{X}_{t_n}$ converges a.s. to a limit $\mathring{X}_\infty$. Without loss of generality, assume convergence of the sequence on the set Ω_0.

There exists $T_1(\omega)$ such that for all $T > T_1$, $ce^{-\zeta T/2} < \delta$. Hence, for all t such that $T_1 \vee T_0 < t \leq T$ there exists $t'_n \in \{t_n\}$ such that $|t - t'_n| < ce^{-\zeta \lfloor t \rfloor/2} < \delta$ and hence

$$|\mathring{X}_t - \mathring{X}_\infty| \leq |\mathring{X}_t - \mathring{X}_{t'_n}| + |\mathring{X}_{t'_n} - \mathring{X}_\infty|$$

$$\leq |t - t'_n|^{1/8} + |\mathring{X}_{t'_n} - \mathring{X}_\infty|$$

$$\leq ce^{-\zeta \lfloor t \rfloor/16} + |\mathring{X}_{t'_n} - \mathring{X}_\infty|.$$

Sending $t \to \infty$ gives almost sure convergence of $e^{-\beta t} X_t(\phi)$ to $\mathring{X}_\infty = W P_\infty(\phi)$ by Theorem 4.2, since $t'_n \to \infty$ with t. Note that this implies for $\phi \in \mathrm{Lip}_1$

(4.8) $$\tilde{X}_t(\phi) \xrightarrow{\text{a.s.}} P_\infty(\phi)$$

since on S, $\frac{e^{\beta t}}{K_t(1)} \to W^{-1}$ a.s.

By Exercise 2.2 of [4] on $\mathcal{M}_1(\mathbb{R}^d)$, the space of probability measures on $\mathbb{R}^d$, the Prohorov metric of weak convergence is equivalent to the Vasserstein metric. It is easy to construct a class Θ that is a countable algebra of Lipschitz functions and that therefore is strongly separating (see page 113 of [4]). Hence by Theorem 3.4.5(b) of [4], Θ is convergence determining. Since there exists a set $S_0 \subset S$ with $\mathbb{P}(S \setminus S_0) = 0$ such that on S_0, equation (4.8) holds simultaneously for all $\phi \in \Theta$,

$$\tilde{X}_t(\cdot) \to P_\infty(\cdot)$$

in the Vasserstein metric, for $\omega \in S_0$ because Θ is convergence determining.

To drop the dependence on the eighth moment, we argue as in the proof of Theorem 1.5, where we make use of the Markov Property and the Compact support property for Historical Brownian Motion. $\square$

PROOF OF THEOREM 1.10. This follows almost immediately from Theorem 1.8 and the representation given in Remark 3.2(a). Let $\phi \in \mathrm{Lip}_1$, then

$$\tilde{X}'_t(\phi) = \tilde{K}_t(\phi(Y_t)) = \tilde{K}_t\left(\phi\left(Z_t + \gamma \int_0^t \bar{Z}_s \, ds\right)\right)$$

$$= \int \phi(x + f(t, \omega)) \, d\tilde{X}_t(dx),$$

where $f(t) = \gamma \int_0^t \bar{Z}_s \, ds$. Remark 3.2(b) gives $f(t) = \bar{Y}_t - \bar{Z}_t$, and hence $f(t) \xrightarrow{a.s.} \bar{Y}_\infty$ follows from Theorem 1.7. Note that

$$|\tilde{X}'_t(\phi) - P^{\bar{Y}_\infty}_\infty(\phi)|$$

$$\leq \left| \int \phi(x + f(t)) \, d\tilde{X}_t(dx) - \int \phi(x + f(\infty)) \, d\tilde{X}_t(dx) \right|$$

$$+ \left| \int \phi(x + f(\infty)) \, d\tilde{X}_t(dx) - \int \phi(x + f(\infty)) \, d\tilde{X}_\infty(dx) \right|$$

$$\leq |f(t) - f(\infty)| + d(\tilde{X}_t, \tilde{X}_\infty)$$

since $\phi \in \text{Lip}_1$. Taking the supremum over ϕ and the previous theorem give

$$d(\tilde{X}'_t, \tilde{X}'_\infty) \leq |f(t, \omega) - f(\infty, \omega)| + d(\tilde{X}_t, \tilde{X}_\infty) \xrightarrow{a.s.} 0. \qquad \square$$

4.3. *The repelling case, on the survival set.* Much less can be said for the SOU process repelling from its center of mass than in the attractive case. We can, however, show that the center of mass converges, provided the rate of repulsion is not too strong, which we recall was the first step toward showing the a.s. convergence of the normalized interacting SOU process in the attractive case. The situation here is more complicated since we prove that the COM of the ordinary SOU process with repulsion diverges almost surely, implying that results for convergence of $\tilde{X}'$ will not simply be established through the correspondence. We finish with some conjectures on the limiting measure for the repelling case.

As in the previous section, assume $Z_0 = Y_0 = y_0$, unless stated otherwise.

PROOF OF THEOREM 1.11. Assume that $\mathbb{P}(\tilde{K}(|y_0|^2)) < \infty$, like in the proof of Theorem 1.5. As in that theorem, this condition can be weakened to just the finite initial mass condition using similar reasoning.

For part (a), note that as in Theorem 1.7, $\bar{Y}_t$ will converge if $\mathbb{P}([\bar{Y}]_t) < \infty$ and which holds if the following quantity is bounded:

$$\mathbb{P}\left(\int_0^\infty \frac{\overline{|Y_s|^2} - \bar{Y}_s^2}{e^{\beta s}} \, ds; S \right) \leq \mathbb{P}\left(\int_0^\infty \frac{\overline{Y_s^2}}{e^{\beta s}} \, ds; S \right)$$

$$\leq c \int_0^\infty \frac{1 + s^6 e^{-2\gamma s}}{e^{\beta s}} \, ds$$

$$< \infty,$$

by Lemma 3.8 and by the conditions on γ.

For (b), we require the following lemma.

LEMMA 4.6. *Let* $-\beta/2 < \gamma < 0$ *and* $X_0 \neq 0$. *For a measure* m *on* $\mathbb{R}^d$, *let* $\tau_a(m)$ *be* m *translated by* $a \in \mathbb{R}^d$. *That is,* $\tau_a(m)(\phi) = \int \phi(x + a) m(dx)$. *Then:*

(i) *For all but at most countably many a,*

$$(4.9) \qquad P_{\tau_a(X_0)}(e^{\gamma t} \bar{Z}_t \to L \neq 0 | S) = 1.$$

(ii) *For all but at most one value of a*

$$P_{\tau_a(X_0)}(e^{\gamma t} \bar{Z}_t \to L \neq 0 | S) > 0.$$

PROOF.   We first note that, by the correspondence (3.4), we have that

$$\bar{Z}_t = \bar{Y}_t - \gamma e^{-\gamma t} \int_0^t e^{\gamma s} \bar{Y}_s \, ds.$$

Under our hypotheses $\bar{Y}_t$ converges by Theorem 1.11(a), and hence

$$(4.10) \qquad \lim_{t \to \infty} e^{\gamma t} \bar{Z}_t + \int_0^t \gamma e^{\gamma s} \bar{Y}_s \, ds = 0$$

a.s. on S. Therefore, on S,

$$(4.11) \qquad \lim_{t \to \infty} e^{\gamma t} \bar{Z}_t \qquad \text{exists a.s.}$$

Note that one can build a solution of $(SE)^2$ with initial conditions given by $\tau_a(X_0)$ by seeing that if Y_t gives the solution of $(SE)^2_{Y_0, K}$, then $Y_t + a$ gives the solution of $(SE)^2_{Y_0 + a, K}$, and that the projection

$$X'_t(\cdot) = \int \mathbf{1}(Y_t + a \in \cdot) K_t(dy)$$

gives the appropriate interacting SOU process.

By (4.10) and (4.11),

$$\mathbb{P}_{X_0}(\{e^{\gamma t} \bar{Z}_t \to L \neq 0\}^c | S) = \mathbb{P}_{\tau_a(X_0)} \left(\lim_{t \to \infty} \int_0^t e^{\gamma s} \bar{Y}_s \, ds = 0 \Big| S \right)$$

$$= \mathbb{P}_{X_0} \left(\lim_{t \to \infty} \int_0^t e^{\gamma s} \frac{K_s(a + Y_s)}{K_s(1)} \, ds = 0 \Big| S \right)$$

$$= \mathbb{P}_{X_0} \left(-\frac{a}{\gamma} + \lim_{t \to \infty} \int_0^t e^{\gamma s} \bar{Y}_s \, ds = 0 \Big| S \right).$$

The random variable $\int_0^\infty e^{\gamma s} \bar{Y}_s \, ds$ is finite a.s. and so only a countable number of values a exist with the latter expression positive, implying the first result. The second result also follows as well since the last expression in the above display can be 1 for at most 1 value of a. $\square$

To complete the proof of Theorem 1.11(b), choose a value $a \in \mathbb{R}^d$ such that (4.9) holds. By Theorem III.2.2 of [9] and the fact that $X_0 P_s \ll \tau_a(X_0) P_t$, for all $0 < s \leq t$, for the OU semigroup P_t, we have that for all $0 < s \leq t$

$$(4.12) \qquad \mathbb{P}_{X_0}(X_{s+\cdot} \in \cdot) \ll \mathbb{P}_{\tau_a(X_0)}(X_{t+\cdot} \in \cdot).$$

By our choice of a,

$$\mathbb{P}_{\tau_a(X_0)}\Big(\mathbb{P}_{X_1}\Big(\lim_{t\to\infty} e^{\gamma t}\bar{Z}_t = 0, S\Big)\Big) = 0,$$

holds, and hence by (4.12) we have

$$\mathbb{P}_{X_0}\Big(\lim_{t\to\infty} e^{\gamma t}\bar{Z}_t = 0, S\Big) = \mathbb{P}_{X_0}\Big(\mathbb{P}_{X_1}\Big(\lim_{t\to\infty} e^{\gamma t}\bar{Z}_t = 0, S\Big)\Big) = 0.$$

Recalling from (4.11) that $\lim_{t\to\infty} e^{\gamma t}\bar{Z}_t$ exists a.s., we are done. $\quad\square$

Note that for $0 > \gamma > -\frac{\beta}{2}$, this implies that even if mass is repelled at rate γ, the COM of the interacting SOU process still settles down in the long run. That is, driving Y_t away from $\bar{Y}_t$ seems to have the effect of stabilizing it. One can think of this as a situation where the mass is growing quickly enough that the law of large numbers overcomes the repelling force.

More surprising is that the COM of the ordinary SOU process diverges exponentially fast, even while the COM of the interacting one settles down. This follows from the correspondence

$$\bar{Y}_t = \bar{Z}_t + \gamma \int_0^t \bar{Z}_s \, ds,$$

and the cancellation that occurs in it due to the exponential rate of $\bar{Z}_t$.

The next lemma shows that Theorem 1 of Engländer and Winter [3] can be reformulated to yield a result for the SOU process with repulsion at rate γ (where γ is taken to be a negative parameter in our setting).

LEMMA 4.7. *On S, for the SOU process, X, with repulsion rate $-\frac{\beta}{d} < \gamma < 0$ and compactly supported initial measure μ, and any $\psi \in C_c^+(R^d)$*

$$e^{d|\gamma|t}\tilde{X}_t(\psi) \xrightarrow{P} \xi \int_{\mathbb{R}^d} \psi(x)\,dx,$$

where ξ is a positive random variable on the set S.

PROOF. Note that by Example 2 of Pinsky [10] it is shown that the hypotheses of Theorem 1 of [3] hold for the SOU process with repulsion from the origin at rate $0 < -\gamma < \frac{\beta}{d}$. The theorem says that there is a function $\phi_c \in C_b^\infty(\mathbb{R}^d)$ such that

(4.13) $$\frac{X_t(\psi)}{\mathbb{E}^\mu(X_t(\psi))} \xrightarrow{P} \frac{W\xi}{\mu(\phi_c)},$$

where W is as in Lemma 3.5.

Example 2 also shows that for $\psi \in C_c^+(\mathbb{R}^d)$,

$$\lim_{t\to\infty} e^{-(\beta+\gamma d)t}\mathbb{E}^\mu(X_t(\psi)) = \mu(\phi_c)m(\psi),$$

where m is Lebesgue measure on $\mathbb{R}^d$. Hence, manipulating the expression in (4.13) by using the previous equation and Lemma 3.5 gives

$$
\begin{aligned}
e^{|\gamma|dt} \tilde{X}_t(\psi) \xrightarrow{P} \ & \frac{\xi W}{\mu(\phi_c)} \lim_{t \to \infty} \frac{e^{|\gamma|dt} \mathbb{E}^\mu(X_t(\psi))}{X_t(1)} \\
= \ & \frac{\xi W}{\mu(\phi_c)} \lim_{t \to \infty} \frac{e^{-(\beta+\gamma d)t} \mathbb{E}^\mu(X_t(\psi))}{e^{-\beta t} X_t(1)} \\
= \ & \frac{\xi W}{\mu(\phi_c)} \frac{\mu(\phi_c) m(\psi)}{W}. \qquad\qquad \square
\end{aligned}
$$

This lemma indicates that on the survival set, when $\gamma < 0$, one cannot naively normalize X_t by its mass since the probability measures $\{\tilde{X}_t\}$ are not tight. That is, a proportion of mass is escaping to infinity and is not seen by compact sets. Note that the lemma above implies that for X_t, the right normalizing factor is $e^{(\beta+\gamma d)t}$.

DEFINITION 4.8. We say a measure-valued process $\mathring{X}$ goes locally extinct if for any finite initial measure $\mathring{X}_0$ and any bounded $A \in \mathcal{B}(\mathbb{R}^d)$, there is a $\mathbb{P}_{\mathring{X}_0}$-a.s. finite stopping time τ_A so that $\mathring{X}_t(A) = 0$ for all $t \geq \tau_A$ a.s.

REMARK 4.9. Example 2 of Pinsky [10] also shows that for $\gamma \leq -\beta/d$ the SOU undergoes local extinction (all the mass escapes). Hence for $\psi \in C_c(\mathbb{R}^d)$, there is no normalization where $X_t(\psi)$ can be expected to converge to something nontrivial.

From Remark 3.4, one can show that

$$
\bar{Z}_t = \bar{Z}_0 + N_t - \gamma \int_0^t \bar{Z}_s \, ds,
$$

where N is a martingale. Therefore, you can think of the COM of X, the SOU process repelling from origin, as being given by an exponential drift term plus fluctuations. The correspondence of Remark 3.2(b) implies then that $\bar{Y}_t = \bar{Y}_0 + N_t$, or in other words, the center of mass of the SOU process repelling from its COM is given by simply the fluctuations.

We finish with some conjectures.

CONJECTURE 4.10. On the survival set, if X_0' is fixed and compactly supported, then the following is conjectured to hold:

(a) If $0 < -\gamma < \frac{\beta}{d}$, then there exists constant $\beta + \gamma d \leq \alpha < \beta$ so that for $\phi \in C_c(\mathbb{R}^d)$,

$$
e^{-\alpha t} X_t'(\phi) \xrightarrow{P} \nu(\psi),
$$

where ν is a random measure depending on $\bar{Y}_\infty$.

(b) If $\beta/d \le -\gamma$, then X'_t undergoes local extinction.

We expect that $\alpha < \beta$ simply because of the repulsion from the COM built in to the model results in a proportion of mass being lost to infinity. One would expect that the limiting measure ν is a random multiple of Lebesgue measure as in the ordinary SOU process case, due to the correspondence, but it is conceivable that it is some other measure which has, for example, a dearth of mass near the limiting COM.

As stated earlier, it is difficult to use Lemma 4.7 to prove this conjecture as the correspondence becomes much less useful in the repulsive case. The problem is that while the equation

$$\int \phi(x)\, dX'_t(x) = \int \phi\left(x + \gamma \int_0^t \bar{Z}_s\, ds\right) dX_t(x)$$

still holds for t finite, the time integral of $\bar{Z}_s$ now diverges.

5. Proofs of technical lemmas. We now prove the lemmas first stated in Section 3.

PROOF OF LEMMA 3.5. We first note that $\mathring{K}_t = e^{-\beta t} K_t$ is a $(\Delta/2, 0, e^{-\beta t})$-historical process. The martingale problem then shows that $\mathring{K}_t(1)$ is a nonnegative $(\mathcal{F}_t)$-martingale and therefore converges almost surely by the Martingale Convergence theorem to a random variable W. It follows that $\{\eta < \infty\} \subset \{W = 0\}$, since 0 is an absorbing state for $K_t(1)$. Exercise II.5.3 in [9] shows that

$$\mathbb{P}(\eta < \infty) = e^{-2\beta K_0(1)}.$$

The same exercise also shows

$$\mathbb{P}_{K_0}(\exp(-\lambda \mathring{K}_t(1))) = \exp\left(-\frac{2\lambda\beta \mathring{K}_0(1)}{2\beta + \lambda(1 - e^{-\beta t})}\right).$$

Now sending $t \to \infty$ gives

$$\mathbb{P}_{K_0}(e^{-\lambda W}) = \exp\left(-\frac{2\beta\lambda \mathring{K}_0(1)}{2\beta + \lambda}\right),$$

and sending $\lambda \to \infty$ gives

$$\mathbb{P}(W = 0) = e^{-2\beta \mathring{K}_0(1)} = \mathbb{P}(\eta < \infty)$$

since $K_0 = \mathring{K}_0$. Therefore, $\{\eta < \infty\} = \{W = 0\}$ almost surely. □

PROOF OF LEMMA 3.7. We follow the proof of Theorem III.1.3(a) in Perkins [9]. First, note that if H is another supercritical historical Brownian motion starting at time τ with initial measure m under $\mathbb{Q}_{\tau,m}$ and A a Borel subset of C, then the process defined by

$$H'_t(\cdot) = H_t(\cdot \cap \{y : y_\tau \in A\})$$

is also a supercritical historical Brownian motion starting at time τ with initial measure m' given by $m'(\cdot) = m(\cdot \cap A)$ under $\mathbb{Q}_{\tau,m}$. Then using the extinction probabilities for H' (refer, e.g., to Exercise II.5.3 of [9]) we have

$$(5.1) \qquad \mathbb{Q}_{\tau,m}\big(H_t(\{y : y_\tau \in A\}) = 0 \ \forall t \geq s\big) = \exp\left\{-\frac{2\beta m(A)}{1 - e^{-\beta(s-\tau)}}\right\}.$$

Using the Markov property for K at time $\frac{j}{2^n}$ and (5.1) gives

$$\mathbb{P}\left[\exists t > \frac{j+1}{2^n} \text{ s.t. } K_t\left(\left\{y : \left|y\left(\frac{j}{2^n}\right) - y\left(\frac{j-1}{2^n}\right)\right| > c\left(\frac{j}{2^n}\right)h(2^{-n})\right\}\right) > 0\right]$$

$$= \mathbb{P}\big[1 - \exp\{-(2\beta K_{j/2^n}(\{y : |y(j/2^n) - y((j-1)/2^n)|$$
$$> c(j/2^n)h(2^{-n})\}))$$
$$\big/(1 - e^{-\beta/2^n})\}\big]$$

$$(5.2)$$

$$\leq \mathbb{P}\left[\frac{2\beta K_{j/2^n}(\{y : |y(j/2^n) - y((j-1)/2^n)| > c(j/2^n)h(2^{-n})\})}{1 - e^{-\beta/2^n}}\right]$$

$$\leq \mathbb{P}\left[\frac{2\beta K_{j/2^n}(\{y : |y(j/2^n) - y((j-1)/2^n)| > c(j/2^n)h(2^{-n})\})}{(\beta/2^n) - (\beta^2/2^{2n+1})}\right]$$

$$\leq \frac{2^{n+1}}{1 - (\beta/2^{n+1})}$$

$$\times \mathbb{P}(K_{j/2^n}(1))\hat{\mathbb{P}}_{j/2^n}\left(\left|y\left(\frac{j}{2^n}\right) - y\left(\frac{j-1}{2^n}\right)\right| > c\left(\frac{j}{2^n}\right)h(2^{-n})\right),$$

where in the last step we have simply rearranged the constants and multiplied and divided by the mean mass at time $j2^{-n}$ and used the definition of the Campbell measure (Definition 2.1).

Since under the normalized mean measure, y is a stopped Brownian motion by Remark 3.1(b), we use tail estimates to see that the last quantity in (5.2) is

$$\leq \frac{2^{n+1}}{1 - (\beta/2^{n+1})}\mathbb{P}(K_{j/2^n}(1))c_d n^{d/2-1}2^{-nc(j/2^n)^2/2}$$

$$\leq 2^{n+2}2^{\beta j \ln 2/2^n}\mathbb{P}(X_0(1))c_d n^{d/2-1}2^{-n((j/2^n)+c_0)/2}.$$

Hence, summing over j from 1 to $n2^n$ gives

$$\mathbb{P}\bigg[\text{There exists } 1 \leq j \leq n2^n \text{ s.t. } \exists t > \frac{j+1}{2^n} \text{ s.t.}$$

$$K_t\left(\left\{y : \left|y\left(\frac{j}{2^n}\right) - y\left(\frac{j-1}{2^n}\right)\right| > c\left(\frac{j}{2^n}\right)h(2^{-n})\right\}\right) > 0\bigg]$$

$$\leq \mathbb{P}(X_0(1))c_d n^{d/2-1}2^{n+2-nc_0/2}\sum_{j=1}^{n2^n}2^{\beta j \ln 2/2^n - n(j/2^n)}$$

$$\leq \mathbb{P}(X_0(1)) c_d n^{d/2-1} 2^{-2n+2} \sum_{j=1}^{n2^n} 2^{-j/2^n}$$

$$\leq \mathbb{P}(X_0(1)) c_d n^{d/2-1} 2^{-n+2},$$

where we have used the fact that $c_0 > 6$. Hence, the sum over n of the above shows by Borel–Cantelli that there exists for almost sure ω, $N(\omega)$ such that for all $n > N$, for all $1 \leq j \leq n2^n$, for all $t \geq \frac{j+1}{2^n}$, for K_t-a.a. y,

$$\left| y\left(\frac{j-1}{2^n}\right) - y\left(\frac{j}{2^n}\right) \right| < c\left(\frac{j}{2^n}\right) h(2^{-n}).$$

Letting $\delta(\omega) = 2^{-N(\omega)}$, note that on the dyadics, by above, we have that

$$\mathbb{P}(\delta < \lambda) = \mathbb{P}\left(N > -\frac{\ln\lambda}{\ln 2}\right)$$

$$= \mathbb{P}\left[\exists n > -\frac{\ln\lambda}{\ln 2}, \exists j \leq n2^n, \exists t > \frac{j+1}{2^n}, \text{ s.t.}\right.$$

$$K_t\left(\left\{y: \left| y\left(\frac{j}{2^n}\right) - y\left(\frac{j-1}{2^n}\right) \right| > c\left(\frac{j}{2^n}\right) h(2^{-n})\right\}\right) > 0\right]$$

$$\leq \sum_{n=\lfloor -\ln\lambda/\ln 2\rfloor} C'(d, c_0) n^{d/2-1} 2^{2n-nc_0/2}$$

$$\leq C(d, c_0, \varepsilon) \lambda^{(c_0/2)-\varepsilon},$$

where ε can be chosen to be arbitrarily small (though the constant C will increase as it decreases). The rest of the proof follows as in Theorem III.1.3(a) of [9], via an argument similar to Levy's proof for the modulus of continuity for Brownian motion. $\square$

PROOF OF LEMMA 3.8. Assume that $Z_0 = Y_0$ and $t < \eta$. Recall

$$Z_t(\omega, y) \equiv e^{-\gamma t} Z_0 + \int_0^t e^{\gamma(s-t)} \, dy_s.$$

Note that below "$\lesssim$" denotes less than up to multiplicative constants independent of t and y. Suppose that $y \in S(\delta, c(t))$, where $S(\delta, c(t))$ is the same as in the previous lemma. Then, as $Y_t = Z_t + \gamma \int_0^t \bar{Z}_s \, ds$,

$$|Y_t|^2 \lesssim |Z_t|^2 + \gamma^2 t \int_0^t |\bar{Z}_s|^2 \, ds \lesssim |Z_t|^2 + \gamma^2 t \int_0^t \overline{|Z_s|^2} \, ds$$

by Cauchy–Schwarz and Jensen's inequality. Therefore, integrating with respect to the normalized measure gives

$$(5.3) \qquad \overline{|Y_t|^2} \leq \overline{|Z_t|^2} + \gamma^2 t \int_0^t \overline{|Z_s|^2} \, ds$$

and therefore we need only find the appropriate bounds for expectation of $\overline{|Z_t|^2}$ to get the result.

After another few applications of Cauchy–Schwarz and integrating by parts,

$$|Z_t|^2 \lesssim e^{-2\gamma t}|Z_0|^2 + \left|e^{-\gamma t}\int_0^t e^{\gamma s}\,dy_s\right|^2$$

$$\lesssim e^{-2\gamma t}|Z_0|^2 + e^{-2\gamma t}\left|e^{\gamma t}y_t - y_0 - \gamma\int_0^t y_s e^{\gamma s}\,ds\right|^2$$

$$\lesssim e^{-2\gamma t}(|Z_0|^2 + |y_0|^2) + |y_t|^2 + \gamma^2 t\int_0^t |y_s|^2 e^{-2\gamma(t-s)}\,ds.$$

As $y \in S(\delta, c(t))$,

$$|Z_t|^2 \lesssim e^{-2\gamma t}(|Z_0|^2 + |y_0|^2) + |y_0|^2 + \left(\frac{tc(t)h(\delta)}{\delta}\right)^2$$

$$+ \gamma^2 t\int_0^t\left[|y_0|^2 + \left(\frac{sc(s)h(\delta)}{\delta}\right)^2\right]e^{-2\gamma(t-s)}\,ds$$

$$\lesssim e^{-2\gamma t}(|Z_0|^2 + |y_0|^2) + |y_0|^2 + |y_0|^2\gamma t(1 - e^{-2\gamma t})/2$$

$$+ c(t)^2\left(\frac{h(\delta)}{\delta}\right)^2(t^2 + \gamma t^3(1 - e^{-2\gamma t}))$$

$$\lesssim (1 + |\gamma|t)(1 + e^{-2\gamma t})(|Z_0|^2 + |y_0|^2)$$

$$+ c(t)^2\left(\frac{h(\delta)}{\delta}\right)^2(t^2 + \gamma t^3(1 - e^{-2\gamma t})).$$

Integrating by the normalized measure $\tilde{K}_t$,

$$\overline{|Z_t|^2} \lesssim (1 + |\gamma|t)(1 + e^{-2\gamma t})\tilde{K}_t(|Z_0|^2 + |y_0|^2)$$

$$+ c(t)^2\left(\frac{h(\delta)}{\delta}\right)^2(t^2 + \gamma t^3(1 - e^{-2\gamma t})).$$

Then using (5.3) and using the above bound on $\overline{|Z_t|^2}$ gives

$$\overline{|Y_t|^2} \leq \overline{|Z_t|^2} + \gamma^2 t\int_0^t \overline{|Z_s|^2}\,ds$$

$$(5.4)\quad \lesssim (1 + t^2)(1 + e^{-2\gamma t})\left(\tilde{K}_t(|Z_0|^2 + |y_0|^2) + \int_0^t \tilde{K}_s(|Z_0|^2 + |y_0|^2)\,ds\right)$$

$$+ (1 + t)c(t)^2\left(\frac{h(\delta)}{\delta}\right)^2(t^2 + \gamma t^3(1 - e^{-2\gamma t})).$$

Note that $\phi(y) \equiv |Z_0(y_0)|^2 + |y_0|^2$ and $\phi_n(y) \equiv \phi(y)1(|y| \leq n)$ are $\hat{\mathcal{F}}_0$ measurable. By applying Itô's formula to $K_t(\phi_n)K_t^{-1}(1)$ and using the decomposi-

tion $K_t(\phi_n) = K_0(\phi_n) + \int_0^t \phi_n(y)\,dM(s,y) + \beta \int_0^t K_s(\phi_n)\,ds$ (which follows from Proposition 2.7 of [8]), we get

$$\tilde{K}_t(\phi_n) = \tilde{K}_0(\phi_n) + N_t(\phi_n),$$

where $N_t(\phi_n)$ is a local martingale until time η, for each n. In fact, the sequence of stopping times $\{T_N\}$ appearing in Theorem 3.3 can be used to localize each $N_t(\phi_n)$. Applying first the monotone convergence theorem and then localizing gives

$$
\begin{aligned}
\mathbb{P}(\tilde{K}_t(\phi); t < \eta) &= \lim_{n\to\infty} \mathbb{P}(\tilde{K}_t(\phi_n); t < \eta) \\
&= \lim_{n\to\infty} \lim_{N\to\infty} \mathbb{P}(\tilde{K}_t(\phi_n)\mathbf{1}(t < T_N)) \\
&= \lim_{n\to\infty} \lim_{N\to\infty} \mathbb{P}(\tilde{K}_{t\wedge T_N}(\phi_n) - \tilde{K}_{T_N}(\phi_n)\mathbf{1}(t \geq T_N)) \\
&\leq \lim_{n\to\infty} \lim_{N\to\infty} \mathbb{P}(\tilde{K}_{t\wedge T_N}(\phi_n)) \\
&= \lim_{n\to\infty} \lim_{N\to\infty} \mathbb{P}(\tilde{K}_0(\phi_n) + N_{t\wedge T_N}(\phi_n)) \\
&= \lim_{n\to\infty} \mathbb{P}(\tilde{K}_0(\phi_n)) \\
&= \mathbb{P}(\tilde{K}_0(\phi)),
\end{aligned}
$$

where we have used the positivity of ϕ_n to get the fourth line and the monotone convergence theorem in the last line. Further, note that

$$
\begin{aligned}
\mathbb{P}\left(\int_0^t \tilde{K}_s(\phi)\,ds; t < \eta\right) &\leq \mathbb{P}\left(\int_0^t \tilde{K}_s(\phi)\mathbf{1}(s < \eta)\,ds\right) \\
&= \int_0^t \mathbb{P}(\tilde{K}_s(\phi); s < \eta)\,ds \\
&\leq t\mathbb{P}(\tilde{K}_0(\phi)),
\end{aligned}
$$

by the calculation immediately above. Thus, taking expectations in (5.4) and plugging in $c(t) = \sqrt{c_0 + t}$ gives

$$\mathbb{P}(\overline{|Y_t|^2}; t < \eta) \lesssim (1 + t^3)(1 + e^{-2\gamma t})\mathbb{P}(\tilde{K}_0(\phi)) + t^5(1 + te^{-2\gamma t})\mathbb{P}\left(\frac{h(\delta)^2}{\delta^2}\right).$$

Now let c_0 be chosen so that $\text{Supp}(K_t) \subset S(\delta, c(t))$ and $p_{c_0}(\lambda) = C\lambda^\alpha$ for $\lambda \in [0,1]$. Note that

$$
\begin{aligned}
\mathbb{P}\left(\frac{h(\delta)^2}{\delta^2}\right) = \mathbb{P}\left(\frac{\ln^+(1/\delta)}{\delta}\right) &= \int_0^\infty \left(\frac{\ln^+(1/\lambda)}{\lambda}\right)d\mathbb{P}(\delta < \lambda) \\
&\leq \int_0^1 \left(\frac{\ln^+(1/\lambda)}{\lambda}\right)d\mathbb{P}(\delta < \lambda) + \mathbb{P}(\delta > 1)
\end{aligned}
$$

$$= \lim_{\lambda \downarrow 0} \left(\frac{\ln^+(1/\lambda)}{\lambda} \mathbb{P}(\delta < \lambda) \right) - \mathbb{P}(\delta < 1)$$

$$- \int_0^1 \mathbb{P}(\delta < \lambda) \, d\left(\frac{\ln^+(1/\lambda)}{\lambda} \right) + \mathbb{P}(\delta > 1)$$

$$< \infty,$$

by choosing the constant c_0 so that α is large ($\alpha \geq 2$ is enough). □

Remark 3.9 follows from the above proof, after noting that the exponent α in Lemma 3.7 can be made arbitrarily large by choosing a sufficiently large constant c_0. Hence by choosing α appropriately, we can show that

$$\mathbb{P}\left[\left(\frac{h(\delta)}{\delta} \right)^k \right] < \infty,$$

which can then be used to adapt the proof above.

Recall that $\text{Lip}_1 = \{ \psi \in C(\mathbb{R}^d) : \forall x, y, |\psi(x) - \psi(y)| \leq |x - y|, \|\psi\| \leq 1 \}$. We will, with a slight abuse of notation, allow M to denote the orthogonal martingale measure generated by the martingale problem for X. Let A be the infinitesimal generator for an OU process, and hence recall that for $\phi \in C^2(\mathbb{R}^d)$,

$$A\phi(x) = -\gamma x \cdot \nabla \phi(x) + \frac{\Delta}{2} \phi(x).$$

The next two proofs are for lemmas stated in Section 4.2.

PROOF OF LEMMA 4.2.   Let $\phi \in \text{Lip}_1$. By the extension of the martingale problem for X given in Proposition II.5.7 of [9], for functions $\psi : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that ψ satisfies the definition before that proposition,

$$X_t(\psi_t) = X_0(\psi_0) + \int_0^t \int \psi_s(x) \, dM(x, s) + \int_0^t X_s(A\psi_s + \beta\psi_s + \dot{\psi}_s) \, ds,$$

where M is the orthogonal martingale measure derived from the martingale problem for the SOU process. It is not difficult to show that $\psi_s = P_{t-s}\phi$ where ϕ as above satisfies requirements for Proposition II.5.7 of [9]. Plugging this in gives

$$X_t(\phi) = X_0(P_t\phi) + \int_0^t \int P_{t-s}\phi(x) \, dM(s, x) + \int_0^t \beta X_s(P_{t-s}\phi) \, ds$$

since $\frac{\partial}{\partial s} P_s \phi = A P_s \phi$. Multiplying by $e^{-\beta t}$ and integrating by parts gives

$$e^{-\beta t} X_t(\phi) = e^{-\beta t} X_t(\psi_t)$$

(5.5)
$$= X_0(\psi_0) - \int_0^t \beta e^{-\beta s} X_s(\psi_s) \, ds + \int_0^t e^{-\beta s} \, dX_s(\psi_s)$$

$$= X_0(P_t\phi) + \int_0^t \int e^{-\beta s} P_{t-s}\phi(x) \, dM(s, x).$$

Note that as the OU-process has a stationary distribution P_∞ where $P_t \to P_\infty$ in norm. When s is large in (5.5), $P_{t-s}\phi(x)$ does not contribute much to the stochastic integral and hence we expect the limit of $e^{-\beta t} X_t(\phi)$ to be

$$(5.6) \qquad X_0(P_\infty\phi) + \int_0^\infty \int e^{-\beta s} P_\infty\phi(x)\, dM(s, x),$$

which is a well defined, finite random variable as

$$\left[\int_0^\cdot \int e^{-\beta s} P_\infty\phi(x)\, dM(s, x) \right]_\infty < \|\phi\|^2 \int_0^\infty e^{-2\beta s} X_s(1)\, ds,$$

which is finite in expectation. As $P_\infty\phi(x)$ does not depend on x, it follows that

$$(5.6) = (P_\infty\phi) X_0(1) + (P_\infty\phi) \int_0^\infty \int e^{-\beta s}\, dM(s, x) = W P_\infty\phi.$$

Given this decomposition for $W P_\infty\phi$, we write

$$\mathbb{P}\big((e^{-\beta t} X_t(\phi) - W P_\infty\phi)^2 \big)$$

$$\leq 3\mathbb{P}\left(\left(\int_t^\infty e^{-\beta s} P_\infty\phi(x)\, dM(s, x) \right)^2 \right)$$

$$+ 3\mathbb{P}\left(\left(\int_0^t \int e^{-\beta s} (P_{t-s}\phi(x) - P_\infty\phi(x))\, dM(s, x) \right)^2 \right.$$

$$\left. + X_0(P_\infty\phi - P_t\phi)^2 \right).$$

If z_t is a d-dimensional OU process satisfying $dz_t = -\gamma z_t dt + dB_t$, where B_t is a d-dimensional Brownian motion, then

$$z_t = e^{-\gamma t} z_0 + \int_0^t e^{-(t-s)\gamma}\, dB_t$$

and hence z_t is Gaussian, with mean $e^{-\gamma t} z_0$ and covariance matrix $\frac{1}{2\gamma}(1 - e^{-2\gamma t})I$. Evidently, z_∞ is also Gaussian, mean 0 and variance $\frac{1}{2\gamma}I$. We use a simple coupling: suppose that w_t is a random variable independent of z_t such that $z_\infty = z_t + w_t$ (i.e., w_t is Gaussian with mean $-e^{-\gamma t} z_0$ and covariance $\frac{1}{2\gamma} e^{-2\gamma t} I$). Then using the fact that $\phi \in \mathrm{Lip}_1$ and the Cauchy–Schwarz inequality, followed by our coupling with $z_0 = x$ gives

$$X_0(P_\infty\phi - P_t\phi)^2 = \left(\int \mathbb{E}^x(\phi(z_\infty) - \phi(z_t)) X_0(dx) \right)^2$$

$$\leq \int \mathbb{E}^x(|z_\infty - z_t|)^2 X_0(dx) X_0(1)$$

$$= \int \mathbb{E}^x(|w_t|^2) X_0(dx) X_0(1)$$

$$= \int e^{-2\gamma t} \left(|x|^2 + \frac{d}{2\gamma} \right) X_0(dx) X_0(1)$$

$$\leq c e^{-2\gamma t} \left(\int |x|^2 X_0(dx) X_0(1) + X_0(1)^2 \right).$$

Taking expectations and using Cauchy–Schwarz and the assumptions on X_0 gives exponential rate of convergence for the above term.

Since we can think of $\int_0^r \int e^{-\beta s} P_{t-s}\phi(x) \, dM(s,x)$ as a martingale in r up until time t, various martingale inequalities can be applied to get bounds for the terminal element, $\int_0^t \int e^{-\beta s} P_{t-s}\phi(x) \, dM(s,x)$. Note that this process is not in general a martingale in t. Therefore, we have

$$\mathbb{P}\left[\left(\int_0^t \int e^{-\beta s} P_\infty \phi(x) \, dM(s,x) - \int_0^t \int e^{-\beta s} P_{t-s}\phi(x) \, dM(s,x) \right)^2 \right]$$

$$(5.7) \qquad = \mathbb{P}\left[\left(\int_0^t \int e^{-\beta s} \left(P_\infty \phi(x) - P_{t-s}\phi(x) \right) dM(s,x) \right)^2 \right]$$

$$\leq \mathbb{P}\left[\int_0^t e^{-2\beta s} \int \left(P_\infty \phi(x) - P_{t-s}\phi(x) \right)^2 X_s(dx) \, ds \right].$$

Then as ϕ Lipschitz, by the coupling above,

$$(5.7) \leq \mathbb{P}\left[\int_0^t e^{-2\beta s} \int e^{-2\gamma(t-s)} \left(|x|^2 + \frac{d}{2\gamma} \right) X_s(dx) \, ds \right]$$

$$= \int_0^t e^{-2\beta s - 2\gamma(t-s)} \mathbb{P}\left[\int |x|^2 + \frac{d}{2\gamma} X_s(dx) \right] ds$$

$$= \int_0^t e^{-2\beta s - 2\gamma(t-s)} \mathbb{P}\left[K_s(|Z_s|^2) + \frac{d}{2\gamma} X_s(1) \right] ds.$$

Applying the Cauchy–Schwarz inequality followed by Remark 3.9(b) gives

$$\mathbb{P}(K_s(|Z_s|^2))$$

$$= \mathbb{P}(\overline{|Z_s|^2} K_s(1); s < \eta)$$

$$\leq \mathbb{P}(\overline{|Z_s|^2}^2; s < \eta)^{1/2} \mathbb{P}(X_s(1)^2)^{1/2}$$

$$\leq B(s, \gamma, 4)^{1/2} \mathbb{P}(X_s(1)^2)^{1/2}$$

$$\leq c B(s, \gamma, 4)^{1/2} e^{\beta s} \mathbb{P}\left(X_0(1)^2 + \frac{1}{\beta} X_0(1) \right)^{1/2},$$

where the last line follows by first noting that

$$e^{-\beta t} X_t(1) = X_0(1) + \int_0^t \int e^{-\beta s} \, dM(s,x)$$

is a martingale. That is,

$$
\begin{aligned}
e^{-2\beta s}\mathbb{P}(X_s(1)^2) &\leq 2\mathbb{P}\left(X_0(1)^2 + \left(\int_0^s e^{-\beta r}\, dM(r,x)\right)^2\right) \\
&= 2\mathbb{P}\left(X_0(1)^2 + \left[\int_0^\cdot e^{-\beta r}\, dM(r,x)\right]_s\right) \\
&= 2\mathbb{P}\left(X_0(1)^2 + \int_0^s e^{-2\beta r} X_r(1)\, dr\right) \\
&= 2\mathbb{P}(X_0(1)^2) + 2\int_0^s e^{-\beta r}\mathbb{P}(e^{-\beta r} X_r(1))\, dr \\
&= 2\mathbb{P}(X_0(1)^2) + 2\int_0^s e^{-\beta r}\mathbb{P}(X_0(1))\, dr \\
&\leq 2\mathbb{P}\left(X_0(1)^2 + \frac{1}{\beta}X_0(1)\right).
\end{aligned}
$$

(5.8)

Therefore,

$$
\begin{aligned}
(5.7) &\leq \int_0^t e^{-2\beta s - 2\gamma(t-s)}\left[e^{\beta s} B(s,\gamma,4)^{1/2}\mathbb{P}\left(X_0(1)^2 + \frac{1}{\beta}X_0(1)\right)^{1/2}\right] ds \\
&\quad + \int_0^t e^{-2\beta s - 2\gamma(t-s)} e^{\beta s}\frac{d}{2\gamma}\mathbb{P}(X_0(1))\, ds \\
&\leq \int_0^t e^{-\beta s - 2\gamma(t-s)}\left[B(s,\gamma,4)^{1/2}\mathbb{P}\left(X_0(1)^2 + \frac{1}{\beta}X_0(1)\right)^{1/2} \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + \frac{d}{2\gamma}\mathbb{P}(X_0(1))\right] ds \\
&< Ce^{-\zeta_1 t},
\end{aligned}
$$

where $\zeta_1 = \min(\beta, 2\gamma) - \varepsilon$ where ε is arbitrary small and comes from the polynomial term in the integral.

Finally,

$$
\begin{aligned}
\mathbb{P}\left(\left(\int_t^\infty e^{-\beta s} P_\infty\phi(x)\, dM(s,x)\right)^2\right) &= (P_\infty\phi)^2\mathbb{P}\left(\int_t^\infty e^{-2\beta s} X_s(1)\, ds\right) \\
&\leq (P_\infty\phi)^2\int_t^\infty e^{-\beta s}\mathbb{P}(e^{-\beta s} X_s(1))\, ds \\
&\leq \frac{(P_\infty\phi)^2}{\beta} e^{-\beta t}\mathbb{P}(X_0(1)),
\end{aligned}
$$

since $e^{-\beta s} X_s(1)$ is a martingale. Therefore, since $\zeta_1 < \beta$, we see that $\zeta = \zeta_1$ gives the correct exponent. $\quad\square$

PROOF OF LEMMA 4.4.   The proof will follow in a manner very similar to the proof of the previous lemma. From the calculations above, we see that

$$e^{-\beta(t+h)}X_{t+h}(\phi) - e^{-\beta t}X_t(\phi)$$

$$= X_0(P_{t+h}\phi - P_t\phi)$$

$$+ \int_0^{t+h}\int e^{-\beta s}P_{t+h-s}\phi(x)\,dM(s,x) - \int_0^t\int e^{-\beta s}P_{t-s}\phi(x)\,dM(s,x)$$

$$= X_0(P_{t+h}\phi - P_t\phi) + \int_0^t\int e^{-\beta s}((P_{t+h-s} - P_{t-s})\phi(x))\,dM(s,x)$$

$$+ \int_t^{t+h}\int e^{-\beta s}P_{t+h-s}\phi(x)\,dM(s,x)$$

$$\equiv I_1 + I_2 + I_3.$$

Using the Cauchy–Schwarz inequality, we can find bounds for $\mathbb{P}(|I_k|^4)$, $k = 1, 2, 3$, separately:

$$|I_1|^4 = X_0(P_{t+h}\phi - P_t\phi)^4$$

$$\leq \left[X_0((P_{t+h}\phi - P_t\phi)^2)X_0(1)\right]^2.$$

Recalling the simple coupling in the previous lemma to see that

$$\left(P_{t+h}\phi(x) - P_t\phi(x)\right)^2 \leq \mathbb{E}^x\left(|\phi(z_{t+h}) - \phi(z_t)|^2\right)$$

$$\leq \mathbb{E}^x\left(|z_{t+h} - z_t|^2\right)$$

$$\leq \mathbb{E}^x\left(|w_{t,t+h}|^2\right),$$

where z is as above, an OU process started at x, and $w_{s,t}$ is independent of z_s but such that $z_t = z_s + w_{s,t}$. Hence, $w_{s,t}$ is Gaussian with mean $x(e^{-\gamma t} - e^{-\gamma s})$ and covariance matrix $\frac{I}{2\gamma}(e^{-2s\gamma} - e^{-2t\gamma})$. Therefore,

$$\left(P_{t+h}\phi(x) - P_t\phi(x)\right)^2 \leq |x|^2\left(e^{-\gamma(t+h)} - e^{-\gamma t}\right)^2 + \frac{d}{2\gamma}\left(e^{-2t\gamma} - e^{-2(t+h)\gamma}\right)$$

$$= e^{-2\gamma t}\left(|x|^2(1 - e^{-\gamma h})^2 + \frac{d}{2\gamma}(1 - e^{-2h\gamma})\right).$$

Hence,

$$\mathbb{P}(|I_1|^4) = \mathbb{P}\left[\left(e^{-2\gamma t}(1 - e^{-\gamma h})^2\int |x|^2 X_0(dx)X_0(1)\right.\right.$$

$$\left.\left. + \frac{d}{2\gamma}e^{-2\gamma t}(1 - e^{2\gamma h})X_0(1)^2\right)^2\right]$$

$$\leq C_1(d,\gamma)h^2 e^{-4\gamma t},$$

where C_1 is a constant that is finite by assumptions on the initial measure.

To get bounds on the expectation of I_2, we use martingale inequalities. Note that $\int_0^{\cdot} \int e^{-\beta s}((P_{t+h-s} - P_{t-s})\phi(x)) \, dM(s, x) = N(\cdot)$ is a martingale until time t. Therefore, using the Burkholder–Davis–Gundy inequality and the coupling above gives

$$\mathbb{P}(|I_2|^4) \leq c\mathbb{P}([N]_t^2)$$

$$= c\mathbb{P}\left[\left(\int_0^t \int e^{-2\beta s}((P_{t+h-s} - P_{t-s})\phi(x))^2 X_s(dx) \, ds\right)^2\right]$$

$$\leq c\mathbb{P}\left[\left(\int_0^t e^{-2\beta s}(e^{-\gamma(t+h-s)} - e^{-\gamma(t-s)})^2 \int |x|^2 X_s(dx) \, ds\right)^2\right.$$

$$\left. + \left(\int_0^t \frac{de^{-2\beta s}}{2\gamma}(e^{-2(t-s)\gamma} - e^{-2(t+h-s)\gamma}) X_s(1) \, ds\right)^2\right]$$

$$\leq c\mathbb{P}\left[\left(t\int_0^t e^{-4\beta s}(e^{-\gamma(t+h-s)} - e^{-\gamma(t-s)})^4\left(\int |x|^2 X_s(dx)\right)^2 ds\right)\right.$$

$$\left. + t\left(\frac{d}{2\gamma}\right)^2 \int_0^t e^{-4\beta s}(e^{-2(t-s)\gamma} - e^{-2(t+h-s)\gamma})^2 X_s(1)^2 \, ds\right]$$

$$= ct\int_0^t e^{-2\beta s}(e^{-\gamma(t+h-s)} - e^{-\gamma(t-s)})^4 \mathbb{P}\left[\left(\int |x|^2 e^{-\beta s} X_s(dx)\right)^2\right] ds$$

$$+ t\left(\frac{d}{2\gamma}\right)^2 \int_0^t e^{-2\beta s}(e^{-2(t-s)\gamma} - e^{-2(t+h-s)\gamma})^2 \mathbb{P}[(e^{-\beta s} X_s(1))^2] \, ds.$$

Since $X_s(|x|^2) = K_s(|Z_s|^2)$, by Remark 3.9(b),

$$\mathbb{P}(e^{-2\beta s} X_s(|x|^2)^2) \leq \mathbb{P}[\overline{Z_s^2}^2(e^{-\beta s} X_s(1))^2; s < \eta]$$

$$\leq \mathbb{P}(\overline{Z_s^2}^4; s < \eta)^{1/2} \mathbb{P}(e^{-4\beta s} X_s(1)^4)^{1/2}$$

$$\leq \mathbb{P}(\overline{Z_s^8}; s < \eta)^{1/2} \mathbb{P}(e^{-4\beta s} X_s(1)^4)^{1/2}$$

$$\leq cB(\gamma, s, 8)^{1/2}\left(\mathbb{P}(X_0(1)^4 + sX_0(1)^2 + sX_0(1))\right)^{1/2}.$$

The bound on the expectation of $e^{-4\beta s} X_s(1)^4$ follows by an application of the BDG Inequality to $e^{-\beta s} X_s(1) = X_0(1) + \int_0^s e^{-\beta r} \, dM(r, x)$ and similar calculations used to determine the bound on (5.8). Therefore,

$$\mathbb{P}(|I_2|^4) \leq ct\int_0^t e^{-2\beta s} B(\gamma, s, 8)^{1/2}(e^{-\gamma(t+h-s)} - e^{-\gamma(t-s)})^4$$

$$\times \left(\mathbb{P}(X_0(1)^4 + sX_0(1)^2 + sX_0(1))\right)^{1/2} ds$$

$$+ \frac{ct}{4\gamma^2}\int_0^t e^{-2\beta s}(e^{-2(t-s)\gamma} - e^{-2(t+h-s)\gamma})^2 \mathbb{P}\left(X_0(1)^2 + \frac{1}{\beta}X_0(1)\right) ds$$

$$\leq ct B(\gamma, t, 8)^{1/2} \big(\mathbb{P}\big(X_0(1)^4 + t X_0(1)^2 + t X_0(1) \big) \big)^{1/2}$$

$$\times (e^{-\gamma h} - 1)^4 \int_0^t e^{-2\beta s} e^{-4\gamma(t-s)} \, ds$$

$$+ \frac{ct}{4\gamma^2} (e^{-2\gamma h} - 1)^2 \mathbb{P}\bigg(X_0(1)^2 + \frac{1}{\beta} X_0(1) \bigg) \int_0^t e^{-2\beta s} e^{-4(t-s)\gamma} \, ds$$

$$\leq C_2(t, \gamma, \beta) h^2 e^{-\zeta_1 t},$$

where C_2 is polynomial in t. By another application of the BDG inequality, and noting that $\|\phi\| = 1$,

$$\mathbb{P}(|I_3|^4) = \mathbb{P}\bigg[\bigg(\int_t^{t+h} \int e^{-\beta s} P_{t+h-s} \phi(x) \, dM(s, x) \bigg)^4 \bigg]$$

$$\leq c \mathbb{P}\bigg[\bigg(\int_t^{t+h} \int (e^{-\beta s} P_{t+h-s} \phi(x))^2 X_s(dx) \, ds \bigg)^2 \bigg]$$

$$\leq ch \mathbb{P}\bigg[\int_t^{t+h} e^{-4\beta s} X_s(1)^2 \, ds \bigg]$$

$$= che^{-2\beta t} \int_t^{t+h} \mathbb{P}[e^{-2\beta s} X_s(1)^2] \, ds$$

$$\leq che^{-2\beta t} \int_t^{t+h} \mathbb{P}[X_0(1)^2 + X_0(1)/\beta] \, ds$$

$$\leq C_3(\beta) h^2 e^{-2\beta t},$$

where the second last line follows from the same calculations performed in estimating moments of I_2. Note that the constant C_3 does not depend on t here.

Putting the pieces together shows that there exists a function C polynomial in t and a positive constant ζ^* such that (4.4) holds. $\quad\square$

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