

A FUNCTIONAL LIMIT THEOREM FOR DEPENDENT SEQUENCES WITH INFINITE VARIANCE STABLE LIMITS

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Under an appropriate regular variation condition, the affinely normalized partial sums of a sequence of independent and identically distributed random variables converges weakly to a non-Gaussian stable random variable. A functional version of this is known to be true as well, the limit process being a stable Lévy process. The main result in the paper is that for a stationary, regularly varying sequence for which clusters of high-threshold excesses can be broken down into asymptotically independent blocks, the properly centered partial sum process still converges to a stable Lévy process. Due to clustering, the Lévy triple of the limit process can be different from the one in the independent case. The convergence takes place in the space of càdlàg functions endowed with Skorohod's M_1 topology, the more usual J_1 topology being inappropriate as the partial sum processes may exhibit rapid successions of jumps within temporal clusters of large values, collapsing in the limit to a single jump. The result rests on a new limit theorem for point processes which is of independent interest. The theory is applied to moving average processes, squared GARCH(1, 1) processes and stochastic volatility models.

1. Introduction. Consider a stationary sequence of random variables $(X_n)_{n \geq 1}$ and its accompanying sequence of partial sums $S_n = X_1 + \cdots + X_n, n \geq 1$. The main goal of this paper is to investigate the asymptotic distributional behavior of the $D[0, 1]$ valued process

$$V_n(t) = a_n^{-1}(S_{\lfloor nt \rfloor} - \lfloor nt \rfloor b_n), \quad t \in [0, 1],$$

under the properties of weak dependence and regular variation with index $\alpha \in (0, 2)$, where $(a_n)_n$ is a sequence of positive real numbers such that

$$(1.1) \quad nP(|X_1| > a_n) \rightarrow 1$$

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as $n \rightarrow \infty$ and

$$b_n = E(X_1 1_{\{|X_1| \leq a_n\}}).$$

Here, $[x]$ represents the integer part of the real number x , and $D[0, 1]$ is the space of real-valued càdlàg functions on $[0, 1]$.

Recall that if the sequence $(X_n)_n$ is i.i.d. and if there exist real sequences $a'_n > 0$ and b'_n and a nondegenerate random variable S such that as $n \rightarrow \infty$

$$(1.2) \quad \frac{S_n - b'_n}{a'_n} \xrightarrow{d} S,$$

then S is necessarily an α -stable random variable. In standard terminology, the law of X_1 belongs to the domain of attraction of S . The domain of attraction of non-Gaussian stable random variables can be completely characterized by an appropriate regular variation condition; see (2.1) below. Classical references in the i.i.d. case are the books by Gnedenko and Kolmogorov [20], Feller [19] and Petrov [38]. In LePage et al. [32] one can find an elegant probabilistic proof of sufficiency and a nice representation of the limiting distribution.

Weakly dependent sequences can exhibit very similar behavior. The first results in this direction were rooted in martingale theory (see Durrett and Resnick [17]). In [11], Davis proved that if a regularly varying sequence of random variables $(X_n)_n$ has tail index $0 < \alpha < 2$ and satisfies a strengthened version of Leadbetter's D and D' conditions familiar from extreme value theory, then (1.2) holds for some α -stable random variable S and properly chosen normalizing sequences. These conditions are quite restrictive, however, even excluding m -dependent sequences. Extensions to Davis's results were provided in Denker and Jakubowski [16] and Jakubowski and Kobus [26], the latter paper being the first one in which clustering of big values is allowed. Using classical blocking techniques, necessary and sufficient conditions for convergence of sums of weakly dependent random variables to stable laws are given in two papers by Jakubowski [24, 25]. The case of associated sequences was treated in Dabrowski and Jakubowski [9]. In [12], Davis and Hsing showed that sequences which satisfy a regular variation condition for some $\alpha \in (0, 2)$ and certain mixing conditions also satisfy (1.2) with an α -stable limit. Building upon the same point process approach, Davis and Mikosch [13] generalized these results to multivariate sequences. A survey of these results is to be found in Bartkiewicz et al. [3], providing a detailed study of the conditions for the convergence of the partial sums of a strictly stationary process to an infinite variance stable distribution. In this paper, the parameters of the limiting distribution are determined in terms of some tail characteristics of the underlying stationary sequence.

The asymptotic behavior of the processes V_n as $n \rightarrow \infty$ is an extensively studied subject in the probability literature, too. As the index of regular variation α is assumed to be less than 2, the variance of X_1 is infinite. In the finite-variance case,

functional central limit theorems differ considerably and have been investigated in greater depth (see, e.g., Billingsley [7], Herrndorf [23], Merlevède and Peligrad [34], and Peligrad and Utev [37]).

A functional limit theorem for the processes V_n for infinite variance i.i.d. regularly varying sequences (X_n) was established in Skorohod [45], a very readable proof of which can be found in Resnick [41]. For stationary sequences, this question was studied by Leadbetter and Rootzén [31] and Tyran-Kamińska [48]. Essentially, what they showed is that the functional limit theorem holds in Skorohod's J_1 topology if and only if certain point processes of extremes converge to a Poisson random measure, which in turn is equivalent to a kind of nonclustering property for extreme values. The implication is that for many interesting models, convergence in the J_1 topology cannot hold. This fact led Avram and Taqqu [2] and Tyran-Kamińska [49] to opt for Skorohod's M_1 topology instead, a choice which turns out to work for linear processes with regularly varying innovations and nonnegative coefficients; see Section 3 for the definition of the M_1 topology. For some more recent articles with related but somewhat different subjects we refer to Sly and Heyde [46] who obtained nonstandard limit theorems for functionals of regularly varying sequences with long-range Gaussian dependence structure, and also to Aue et al. [1] who investigated the limit behavior of the functional CUSUM statistic and its randomly permuted version for i.i.d. random variables which are in the domain of attraction of a strictly α -stable law, for $\alpha \in (0, 2)$.

The main theorem of our article shows that for a stationary, regularly varying sequence for which clusters of high-threshold excesses can be broken down into asymptotically independent blocks, the properly centered partial sum process $(V_n(t))_{t \in [0,1]}$ converges to an α -stable Lévy process in the space $D[0, 1]$ endowed with Skorohod's M_1 metric under the condition that all extremes within one such cluster have the same sign. Our method of proof combines some ideas used in the i.i.d. case by Resnick [40, 41] with a new point process convergence result and some particularities of the M_1 metric on $D[0, 1]$ that can be found in Whitt [50]. The theorem can be viewed as a generalization of results in Leadbetter and Rootzén [31] and Tyran-Kamińska [48], where clustering of extremes is essentially prohibited, and in Avram and Taqqu [2] and Tyran-Kamińska [49], which are restricted to linear processes.

The paper is organized as follows. In Section 2 we determine precise conditions needed to separate clusters of extremes asymptotically. We also prove a new limit theorem for point processes which is the basis for the rest of the paper and which is of independent interest, too. In Section 3 we state and prove our main functional limit theorem. We also discuss possible extensions of this result to other topologies. Finally, in Section 4 several examples of stationary sequences covered by our main theorem are discussed, in particular moving averages and squared GARCH(1, 1) processes.

2. Stationary regularly varying sequences. The extremal dynamics of a regularly varying stationary time series can be captured by its tail process, which is the conditional distribution of the series, given that at a certain moment, it is far away from the origin (Section 2.1). In particular, the tail process allows explicit descriptions of the limit distributions of various point processes of extremes (Section 2.2). The main result in this section is Theorem 2.3, providing the weak limit of a sequence of time-space point processes, recording both the occurrence times and the values of extreme values.

2.1. *Tail processes.* Denote $\mathbb{E} = \overline{\mathbb{R}} \setminus \{0\}$ where $\overline{\mathbb{R}} = [-\infty, \infty]$. The space \mathbb{E} is equipped with the topology which makes it homeomorphic to $[-1, 1] \setminus \{0\}$ (Euclidean topology) in the obvious way. In particular, a set $B \subset \mathbb{E}$ has compact closure if and only if it is bounded away from zero, that is, if there exists $u > 0$ such that $B \subset \mathbb{E}_u = \mathbb{E} \setminus [-u, u]$. Denote by $C_K^+(\mathbb{E})$ the class of all nonnegative, continuous functions on \mathbb{E} with compact support.

We say that a strictly stationary process $(X_n)_{n \in \mathbb{Z}}$ is (jointly) regularly varying with index $\alpha \in (0, \infty)$ if for any nonnegative integer k , the k -dimensional random vector $\mathbf{X} = (X_1, \dots, X_k)$ is multivariate regularly varying with index α ; that is, for some (and then for every) norm $\|\cdot\|$ on \mathbb{R}^k there exists a random vector Θ on the unit sphere $\mathbb{S}^{k-1} = \{x \in \mathbb{R}^k : \|x\| = 1\}$ such that for every $u \in (0, \infty)$ and as $x \rightarrow \infty$,

$$(2.1) \quad \frac{\mathbb{P}(\|\mathbf{X}\| > ux, \mathbf{X}/\|\mathbf{X}\| \in \cdot)}{\mathbb{P}(\|\mathbf{X}\| > x)} \xrightarrow{w} u^{-\alpha} \mathbb{P}(\Theta \in \cdot),$$

the arrow “ \xrightarrow{w} ” denoting weak convergence of finite measures. For an extensive and highly-readable account of (multivariate) regular variation, see the monograph by Resnick [41].

Theorem 2.1 in Basrak and Segers [5] provides a convenient characterization of joint regular variation: it is necessary and sufficient that there exists a process $(Y_n)_{n \in \mathbb{Z}}$ with $\mathbb{P}(|Y_0| > y) = y^{-\alpha}$ for $y \geq 1$ such that as $x \rightarrow \infty$,

$$(2.2) \quad ((x^{-1} X_n)_{n \in \mathbb{Z}} \mid |X_0| > x) \xrightarrow{\text{fidi}} (Y_n)_{n \in \mathbb{Z}},$$

where “ $\xrightarrow{\text{fidi}}$ ” denotes convergence of finite-dimensional distributions. The process $(Y_n)_{n \in \mathbb{Z}}$ is called the *tail process* of $(X_n)_{n \in \mathbb{Z}}$. Writing $\Theta_n = Y_n/|Y_0|$ for $n \in \mathbb{Z}$, we also have

$$((|X_0|^{-1} X_n)_{n \in \mathbb{Z}} \mid |X_0| > x) \xrightarrow{\text{fidi}} (\Theta_n)_{n \in \mathbb{Z}}$$

(see Corollary 3.2 in [5]). The process $(\Theta_n)_{n \in \mathbb{Z}}$ is independent of $|Y_0|$ and is called the *spectral (tail) process* of $(X_n)_{n \in \mathbb{Z}}$. The law of $\Theta_0 = Y_0/|Y_0| \in \mathbb{S}^0 = \{-1, 1\}$ is the spectral measure of the common marginal distribution of the random variables X_i . Regular variation of this marginal distribution can be expressed in terms of vague convergence of measures on \mathbb{E} : for a_n as in (1.1) and as $n \rightarrow \infty$,

$$(2.3) \quad n\mathbb{P}(a_n^{-1} X_i \in \cdot) \xrightarrow{v} \mu(\cdot),$$

the Radon measure μ on \mathbb{E} being given by

$$(2.4) \quad \mu(dx) = (p1_{(0,\infty)}(x) + q1_{(-\infty,0)}(x))\alpha|x|^{-\alpha-1} dx,$$

where

$$p = P(\Theta_0 = +1) = \lim_{x \rightarrow \infty} \frac{P(X_i > x)}{P(|X_i| > x)},$$

$$q = P(\Theta_0 = -1) = \lim_{x \rightarrow \infty} \frac{P(X_i < -x)}{P(|X_i| > x)}.$$

2.2. *Point process convergence.* Define the time-space point processes

$$(2.5) \quad N_n = \sum_{i=1}^n \delta_{(i/n, X_i/a_n)} \quad \text{for all } n \in \mathbb{N}$$

with a_n as in (1.1). The aim of this section is to establish weak convergence of N_n in the state space $[0, 1] \times \mathbb{E}_u$ for $u > 0$, where $\mathbb{E}_u = [-\infty, -u) \cup (u, \infty]$. The limit process is a Poisson superposition of cluster processes, whose distribution is determined by the tail process $(Y_i)_{i \in \mathbb{Z}}$. Convergence of N_n was already alluded to without proof in Davis and Hsing [12] with a reference to Mori [36].

To control the dependence in the sequence $(X_n)_{n \in \mathbb{Z}}$ we first have to assume that clusters of large values of $|X_n|$ do not last for too long.

CONDITION 2.1 (Finite mean cluster size). There exists a positive integer sequence $(r_n)_{n \in \mathbb{N}}$ such that $r_n \rightarrow \infty$ and $r_n/n \rightarrow 0$ as $n \rightarrow \infty$ and such that for every $u > 0$,

$$(2.6) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left(\max_{m \leq i \leq r_n} |X_i| > ua_n \mid |X_0| > ua_n\right) = 0.$$

Put $M_{1,n} = \max\{|X_i| : i = 1, \dots, n\}$ for $n \in \mathbb{N}$. In Proposition 4.2 in [5], it has been shown that under Condition 2.1 we have $\theta > 0$, where

$$(2.7) \quad \theta = \lim_{r \rightarrow \infty} \lim_{x \rightarrow \infty} P(M_{1,r} \leq x \mid |X_0| > x)$$

$$= P\left(\sup_{i \geq 1} |Y_i| \leq 1\right) = P\left(\sup_{i \leq -1} |Y_i| \leq 1\right).$$

Moreover $P(\lim_{|n| \rightarrow \infty} |Y_n| = 0) = 1$, and, for every $u \in (0, \infty)$ and as $n \rightarrow \infty$,

$$(2.8) \quad P(M_{1,r_n} \leq a_n u \mid |X_0| > a_n u) = \frac{P(M_{1,r_n} > a_n u)}{r_n P(|X_0| > a_n u)} + o(1) \rightarrow \theta,$$

and thus

$$\lim_{n \rightarrow \infty} E \left[\sum_{t=1}^{r_n} 1_{(a_n u, \infty)}(X_t) \mid M_{1,r_n} > a_n u \right] = \frac{1}{\theta} < \infty.$$

This explains why we call Condition 2.1 the “finite mean cluster size” condition. In the setting of Theorem 2.3 below, the quantity θ in (2.7) is the *extremal index* of the sequence $(|X_n|)_{n \in \mathbb{Z}}$ (see [5], Remark 4.7): for all $u \in (0, \infty)$ and as $n \rightarrow \infty$,

$$P(M_{1,n} \leq a_n u) = (P(|X_1| \leq a_n u))^{n\theta} + o(1) \rightarrow e^{-\theta u^{-\alpha}}.$$

See Section 3.4.2 for further discussion.

Since $P(M_{1,r_n} > a_n u) \rightarrow 0$ as $n \rightarrow \infty$, we call the point process

$$\sum_{i=1}^{r_n} \delta_{(a_n u)^{-1} X_i} \quad \text{conditionally on } M_{1,r_n} > a_n u$$

a *cluster process*, to be thought of as a cluster of exceptionally large values occurring in a relatively short time span. Theorem 4.3 in [5] yields the weak convergence of the sequence of cluster processes in the state space \mathbb{E} ,

$$(2.9) \quad \left(\sum_{i=1}^{r_n} \delta_{(a_n u)^{-1} X_i} \mid M_{1,r_n} > a_n u \right) \xrightarrow{d} \left(\sum_{n \in \mathbb{Z}} \delta_{Y_n} \mid \sup_{i \leq -1} |Y_i| \leq 1 \right).$$

Note that since $|Y_n| \rightarrow 0$ almost surely as $|n| \rightarrow \infty$, the point process $\sum_n \delta_{Y_n}$ is well defined in \mathbb{E} . By (2.7), the probability of the conditioning event on the right-hand side of (2.9) is nonzero.

To establish convergence of N_n in (2.5), we need to impose a certain mixing condition denoted by $\mathcal{A}'(a_n)$ which is slightly stronger than the condition $\mathcal{A}(a_n)$ introduced in Davis and Hsing [12].

CONDITION 2.2 ($\mathcal{A}'(a_n)$). There exists a sequence of positive integers $(r_n)_n$ such that $r_n \rightarrow \infty$ and $r_n/n \rightarrow 0$ as $n \rightarrow \infty$ and such that for every $f \in C_K^+([0, 1] \times \mathbb{E})$, denoting $k_n = \lfloor n/r_n \rfloor$, as $n \rightarrow \infty$,

$$(2.10) \quad \mathbb{E} \left[\exp \left\{ - \sum_{i=1}^n f \left(\frac{i}{n}, \frac{X_i}{a_n} \right) \right\} \right] - \prod_{k=1}^{k_n} \mathbb{E} \left[\exp \left\{ - \sum_{i=1}^{r_n} f \left(\frac{kr_n}{n}, \frac{X_i}{a_n} \right) \right\} \right] \rightarrow 0.$$

It can be shown that Condition 2.2 is implied by the strong mixing property (see Krizmanić [30]). Recall $\mathbb{E}_u = \mathbb{E} \setminus [-u, u]$.

THEOREM 2.3. *If Conditions 2.1 and 2.2 hold, then for every $u \in (0, \infty)$ and as $n \rightarrow \infty$,*

$$N_n|_{[0,1] \times \mathbb{E}_u} \xrightarrow{d} N^{(u)} = \sum_i \sum_j \delta_{(T_i^{(u)}, u Z_{ij})} |_{[0,1] \times \mathbb{E}_u}$$

in $[0, 1] \times \mathbb{E}_u$, where:

- (1) $\sum_i \delta_{T_i^{(u)}}$ is a homogeneous Poisson process on $[0, 1]$ with intensity $\theta u^{-\alpha}$;

(2) $(\sum_j \delta_{Z_{ij}})_i$ is an i.i.d. sequence of point processes in \mathbb{E} , independent of $\sum_i \delta_{T_i^{(u)}}$, and with common distribution equal to the weak limit in (2.9).

It can be shown that Theorem 2.3 is still valid if \mathbb{E}_u is replaced by $\overline{\mathbb{E}_u} = [-\infty, -u] \cup [u, \infty]$.

PROOF OF THEOREM 2.3. Let $(X_{k,j})_{j \in \mathbb{N}}$, with $k \in \mathbb{N}$, be independent copies of $(X_j)_{j \in \mathbb{N}}$, and define

$$\hat{N}_n = \sum_{k=1}^{k_n} \hat{N}_{n,k} \quad \text{with} \quad \hat{N}_{n,k} = \sum_{j=1}^{r_n} \delta_{(kr_n/n, X_{k,j}/a_n)}.$$

By Condition 2.2, the weak limits of N_n and \hat{N}_n must coincide. By Kallenberg [27], Theorem 4.2, it is enough to show that the Laplace functionals of \hat{N}_n converge to those of $N^{(u)}$. Take $f \in C_K^+([0, 1] \times \mathbb{E}_u)$. We extend f to the whole of $[0, 1] \times \mathbb{E}$ by setting $f(t, x) = 0$ whenever $|x| \leq u$; in this way, f becomes a bounded, nonnegative and continuous function on $[0, 1] \times \mathbb{E}$. There exists $M \in (0, \infty)$ such that $0 \leq f(t, x) \leq M 1_{[-u, u]^c}(x)$. Hence as $n \rightarrow \infty$,

$$\begin{aligned} 1 &\geq \mathbb{E} e^{-\hat{N}_{n,k} f} \geq \mathbb{E} e^{-M \sum_{i=1}^{r_n} 1_{(|X_i| > a_n u)}} \\ &\geq 1 - M r_n \mathbb{P}(|X_0| > a_n u) = 1 - O(k_n^{-1}). \end{aligned}$$

In combination with the elementary bound $0 \leq -\log z - (1 - z) \leq (1 - z)^2/z$ for $z \in (0, 1]$, it follows that as $n \rightarrow \infty$,

$$-\log \mathbb{E} e^{-\hat{N}_n f} = -\sum_{k=1}^{k_n} \log \mathbb{E} e^{-\hat{N}_{n,k} f} = \sum_{k=1}^{k_n} (1 - \mathbb{E} e^{-\hat{N}_{n,k} f}) + O(k_n^{-1}).$$

By (2.8), $k_n \mathbb{P}(M_{1,r_n} > a_n u) \rightarrow \theta u^{-\alpha}$ for $u \in (0, \infty)$ and as $n \rightarrow \infty$. Hence

$$\begin{aligned} &\sum_{k=1}^{k_n} (1 - \mathbb{E} e^{-\hat{N}_{n,k} f}) \\ (2.11) \quad &= k_n \mathbb{P}(M_{1,r_n} > a_n u) \frac{1}{k_n} \sum_{k=1}^{k_n} \mathbb{E}[1 - e^{-\sum_{j=1}^{r_n} f(kr_n/n, X_j/a_n)} | M_{1,r_n} > a_n u] \\ &= \theta u^{-\alpha} \frac{1}{k_n} \sum_{k=1}^{k_n} \mathbb{E}[1 - e^{-\sum_{j=1}^{r_n} f(kr_n/n, X_j/a_n)} | M_{1,r_n} > a_n u] + o(1). \end{aligned}$$

Let the random variable T_n be uniformly distributed on $\{kr_n/n : k = 1, \dots, k_n\}$ and independent of $(X_j)_{j \in \mathbb{Z}}$. By the previous display, as $n \rightarrow \infty$,

$$\sum_{k=1}^{k_n} (1 - \mathbb{E} e^{-\hat{N}_{n,k} f}) = \theta u^{-\alpha} \mathbb{E}[1 - e^{-\sum_{j=1}^{r_n} f(T_n, u X_j/(ua_n))} | M_{1,r_n} > a_n u] + o(1).$$

The sequence T_n converges in law to a uniformly distributed random variable T on $(0, 1)$. By (2.9) and by independence of sequences (T_n) and (X_n)

$$\left(T_n, \sum_{i=1}^{r_n} \delta_{a_n^{-1} X_i} \mid M_{1,r_n} > a_n u\right) \xrightarrow{d} \left(T, \sum_{n \in \mathbb{Z}} \delta_{u Z_n}\right),$$

where $\sum_n \delta_{Z_n}$ is a point process on \mathbb{E} , independent of the random variable T and with distribution equal to the weak limit in (2.9). Thus the expressions in (2.11) converge as $n \rightarrow \infty$ to

$$(2.12) \quad \theta u^{-\alpha} \mathbb{E}[1 - e^{-\sum_j f(T, u Z_j)}] = \int_0^1 \mathbb{E}[1 - e^{-\sum_j f(t, u Z_j)}] \theta u^{-\alpha} dt.$$

It remains to be shown that the right-hand side above equals $-\log \mathbb{E} e^{-N^{(u)} f}$ for $N^{(u)}$ as in the theorem.

Define $g(t) = \mathbb{E} \exp\{-\sum_j f(t, u Z_j)\}$ for $t \in [0, 1]$. Since $\sum_i \delta_{T_i^{(u)}}$ is independent of the i.i.d. sequence $(\sum_j \delta_{Z_{ij}})_i$,

$$\begin{aligned} \mathbb{E} e^{-N^{(u)} f} &= \mathbb{E} e^{-\sum_i \sum_j f(T_i^{(u)}, u Z_{ij})} \\ &= \mathbb{E} \left[\prod_i \mathbb{E}(e^{-\sum_j f(T_i^{(u)}, u Z_{ij})} \mid (T_k^{(u)})_k) \right] = \mathbb{E} e^{\sum_i \log g(T_i^{(u)})}. \end{aligned}$$

The right-hand side is the Laplace functional of a homogeneous Poisson process on $[0, 1]$ with intensity $\theta u^{-\alpha}$ evaluated in the function $-\log g$. Therefore, it is equal to

$$\exp\left(-\int_0^1 \{1 - g(t)\} \theta u^{-\alpha} dt\right)$$

(see, e.g., Embrechts et al. [18], Lemma 5.1.12; note that $0 \leq g \leq 1$). By the definition of g , the integral in the exponent is equal to the one in (2.12). This completes the proof of the theorem. \square

3. Functional limit theorem. The main result in the paper states convergence of the partial sum process V_n to a stable Lévy process in the space $D[0, 1]$ equipped with Skorohod’s M_1 topology. The core of the proof rests on an application of the continuous mapping theorem: the partial sum process V_n is represented as the image of the time-space point process N_n in (2.5) under a certain summation functional. This summation functional enjoys the right continuity properties by which the weak convergence of N_n in Theorem 2.3 transfers to weak convergence of V_n .

The definition and basic properties of the M_1 topology are recalled in Section 3.1. In Section 3.2, the focus is on the summation functional and its continuity properties. The main result of the paper then comes in Section 3.3. The conditions entering this theorem are discussed in Section 3.4, while Section 3.5 provides some simplifications.

3.1. *The M_1 topology.* The metric d_{M_1} that generates the M_1 topology on $D[0, 1]$ is defined using completed graphs. For $x \in D[0, 1]$ the *completed graph* of x is the set

$$\Gamma_x = \{(t, z) \in [0, 1] \times \mathbb{R} : z = \lambda x(t-) + (1 - \lambda)x(t) \text{ for some } \lambda \in [0, 1]\},$$

where $x(t-)$ is the left limit of x at t . Besides the points of the graph $\{(t, x(t)) : t \in [0, 1]\}$, the completed graph of x also contains the vertical line segments joining $(t, x(t))$ and $(t, x(t-))$ for all discontinuity points t of x . We define an *order* on the graph Γ_x by saying that $(t_1, z_1) \leq (t_2, z_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|x(t_1-) - z_1| \leq |x(t_2-) - z_2|$. A *parametric representation* of the completed graph Γ_x is a continuous nondecreasing function (r, u) mapping $[0, 1]$ onto Γ_x , with r being the time component and u being the spatial component. Let $\Pi(x)$ denote the set of parametric representations of the graph Γ_x . For $x_1, x_2 \in D[0, 1]$ define

$$d_{M_1}(x_1, x_2) = \inf\{\|r_1 - r_2\|_{[0,1]} \vee \|u_1 - u_2\|_{[0,1]} : (r_i, u_i) \in \Pi(x_i), i = 1, 2\},$$

where $\|x\|_{[0,1]} = \sup\{|x(t)| : t \in [0, 1]\}$. This definition introduces d_{M_1} as a metric on $D[0, 1]$. The induced topology is called Skorohod’s M_1 topology and is weaker than the more frequently used J_1 topology which is also due to Skorohod.

The M_1 topology allows for a jump in the limit function $x \in D[0, 1]$ to be approached by multiple jumps in the converging functions $x_n \in D[0, 1]$. Let, for instance,

$$x_n(t) = \frac{1}{2}1_{[1/2-1/n, 1/2)}(t) + 1_{[1/2, 1]}(t), \quad x(t) = 1_{[1/2, 1]}(t)$$

for $n \geq 3$ and $t \in [0, 1]$. Then $d_{M_1}(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, although $(x_n)_n$ does not converge to x in either the uniform or the J_1 metric. For more discussion of the M_1 topology we refer to Avram and Taqqu [2] and Whitt [50].

3.2. *Summation functional.* Fix $0 < v < u < \infty$. The proof of our main theorem depends on the continuity properties of the summation functional

$$\psi^{(u)} : \mathbf{M}_p([0, 1] \times \mathbb{E}_v) \rightarrow D[0, 1]$$

defined by

$$\psi^{(u)}\left(\sum_i \delta_{(t_i, x_i)}\right)(t) = \sum_{t_i \leq t} x_i 1_{\{u < |x_i| < \infty\}}, \quad t \in [0, 1].$$

Observe that $\psi^{(u)}$ is well defined because $[0, 1] \times \mathbb{E}_u$ is a relatively compact subset of $[0, 1] \times \mathbb{E}_v$. The space \mathbf{M}_p of Radon point measures is equipped with the vague topology, and $D[0, 1]$ is equipped with the M_1 topology.

We will show that $\psi^{(u)}$ is continuous on the set $\Lambda = \Lambda_1 \cap \Lambda_2$, where

$$\Lambda_1 = \{\eta \in \mathbf{M}_p([0, 1] \times \mathbb{E}_v) : \eta(\{0, 1\} \times \mathbb{E}_u) = 0 = \eta([0, 1] \times \{\pm\infty, \pm u\})\},$$

$$\Lambda_2 = \{\eta \in \mathbf{M}_p([0, 1] \times \mathbb{E}_v) : \eta(\{t\} \times (v, \infty]) \wedge \eta(\{t\} \times [-\infty, -v)) = 0$$

for all $t \in [0, 1]\}$;

we write $s \wedge t$ for $\min(s, t)$. Observe that the elements of Λ_2 have the property that atoms with the same time coordinate are all on the same side of the time axis.

LEMMA 3.1. *Assume that with probability one, the tail process $(Y_i)_{i \in \mathbb{Z}}$ in (2.2) has no two values of the opposite sign. Then $P(N^{(v)} \in \Lambda) = 1$.*

The assumption that the tail process cannot switch sign will reappear in our main result, Theorem 3.4. For linear processes, for instance, it holds as soon as all coefficients are of the same sign.

PROOF. From the definition of the tail process $(Y_i)_{i \in \mathbb{Z}}$ we know that $P(Y_i = \pm\infty) = 0$ for any $i \in \mathbb{Z}$. Moreover, by the spectral decomposition $Y_i = |Y_0|\Theta_i$ into independent components $|Y_0|$ and Θ_i with $|Y_0|$ a Pareto random variable, it follows that Y_i cannot have any atoms except possibly at the origin. As a consequence, it holds with probability one that $\sum_j \delta_{vY_j}(\{\pm u\}) = 0$ and thus that $\sum_j \delta_{vZ_{ij}}(\{\pm u\}) = 0$ as well. Together with the fact that $P(\sum_i \delta_{T_i^{(v)}}(\{0, 1\}) = 0) = 1$ this implies $P(N^{(v)} \in \Lambda_1) = 1$.

Second, the assumption that with probability one the tail process $(Y_i)_{i \in \mathbb{Z}}$ has no two values of the opposite sign yields $P(N^{(v)} \in \Lambda_2) = 1$. \square

LEMMA 3.2. *The summation functional $\psi^{(u)} : \mathbf{M}_p([0, 1] \times \mathbb{E}_v) \rightarrow D[0, 1]$ is continuous on the set Λ , when $D[0, 1]$ is endowed with Skorohod’s M_1 metric.*

PROOF. Suppose that $\eta_n \xrightarrow{v} \eta$ in \mathbf{M}_p for some $\eta \in \Lambda$. We will show that $\psi^{(u)}(\eta_n) \rightarrow \psi^{(u)}(\eta)$ in $D[0, 1]$ according to the M_1 topology. By Whitt [50], Corollary 12.5.1, M_1 convergence for monotone functions amounts to pointwise convergence in a dense subset of points plus convergence at the endpoints. Our proof is based on an extension of this criterion to piecewise monotone functions. This cut-and-paste approach is justified in view of [50], Lemma 12.9.2, provided that the limit function is continuous at the cutting points.

As $[0, 1] \times \mathbb{E}_u$ is relatively compact in $[0, 1] \times \mathbb{E}_v$ there exists a nonnegative integer $k = k(\eta)$ such that

$$\eta([0, 1] \times \mathbb{E}_u) = k < \infty.$$

By assumption, η does not have any atoms on the horizontal lines at u or $-u$. As a consequence, by Resnick [41], Lemma 7.1, there exists a positive integer n_0 such that for all $n \geq n_0$ it holds that

$$\eta_n([0, 1] \times \mathbb{E}_u) = k.$$

If $k = 0$, there is nothing to prove, so assume $k \geq 1$, and let (t_i, x_i) for $i \in \{1, \dots, k\}$ be the atoms of η in $[0, 1] \times \mathbb{E}_u$. By the same lemma, the k atoms $(t_i^{(n)}, x_i^{(n)})$ of η_n

in $[0, 1] \times \mathbb{E}_u$ (for $n \geq n_0$) can be labeled in such a way that for $i \in \{1, \dots, k\}$, we have

$$(t_i^{(n)}, x_i^{(n)}) \rightarrow (t_i, x_i) \quad \text{as } n \rightarrow \infty.$$

In particular, for any $\delta > 0$ we can find a positive integer n_δ such that for all $n \geq n_\delta$,

$$(3.1) \quad \eta_n([0, 1] \times \mathbb{E}_u) = k, \\ |t_i^{(n)} - t_i| < \delta \quad \text{and} \quad |x_i^{(n)} - x_i| < \delta \quad \text{for } i = 1, \dots, k.$$

Let the sequence

$$0 < \tau_1 < \tau_2 < \dots < \tau_p < 1$$

be such that the sets $\{\tau_1, \dots, \tau_p\}$ and $\{t_1, \dots, t_k\}$ coincide. Note that $p \leq k$ always holds, but since η can have several atoms with the same time coordinate, equality does not hold in general. Put $\tau_0 = 0$, $\tau_{p+1} = 1$, and take

$$0 < r < \frac{1}{2} \min_{0 \leq i \leq p} |\tau_{i+1} - \tau_i|.$$

For any $t \in [0, 1] \setminus \{\tau_1, \dots, \tau_p\}$ we can find $\delta \in (0, u)$ such that

$$\delta < r \quad \text{and} \quad \delta < \min_{1 \leq i \leq p} |t - \tau_i|.$$

Then relation (3.1), for $n \geq n_\delta$, implies that $t_i^{(n)} \leq t$ is equivalent to $t_i \leq t$, and we obtain

$$|\psi^{(u)}(\eta_n)(t) - \psi^{(u)}(\eta)(t)| = \left| \sum_{t_i^{(n)} \leq t} x_i^{(n)} - \sum_{t_i \leq t} x_i \right| \leq \sum_{t_i \leq t} \delta \leq k\delta.$$

Therefore

$$\lim_{n \rightarrow \infty} |\psi^{(u)}(\eta_n)(t) - \psi^{(u)}(\eta)(t)| \leq k\delta,$$

and if we let $\delta \rightarrow 0$, it follows that $\psi^{(u)}(\eta_n)(t) \rightarrow \psi^{(u)}(\eta)(t)$ as $n \rightarrow \infty$. Put

$$v_i = \tau_i + r, \quad i \in \{1, \dots, p\}.$$

For any $\delta < u \wedge r$, relation (3.1) and the fact that $\eta \in \Lambda$ imply that the functions $\psi^{(u)}(\eta)$ and $\psi^{(u)}(\eta_n)$ ($n \geq n_\delta$) are monotone on each of the intervals $[0, v_1], [v_1, v_2], \dots, [v_p, 1]$. A combination of Corollary 12.5.1 and Lemma 12.9.2 in [50] yields $d_{M_1}(\psi^{(u)}(\eta_n), \psi^{(u)}(\eta)) \rightarrow 0$ as $n \rightarrow \infty$. The application of Lemma 12.9.2 is justified by continuity of $\psi^{(u)}(\eta)$ in the boundary points v_1, \dots, v_p . We conclude that $\psi^{(u)}$ is continuous at η . \square

3.3. *Main theorem.* Let $(X_n)_n$ be a strictly stationary sequence of random variables, jointly regularly varying with index $\alpha \in (0, 2)$ and tail process $(Y_i)_{i \in \mathbb{Z}}$. The theorem below gives conditions under which its partial sum process satisfies a nonstandard functional limit theorem with a non-Gaussian α -stable Lévy process as a limit. Recall that the distribution of a Lévy process $V(\cdot)$ is characterized by its *characteristic triple*, that is, the characteristic triple of the infinitely divisible distribution of $V(1)$. The characteristic function of $V(1)$ and the characteristic triple (a, ν, b) are related in the following way:

$$\mathbb{E}[e^{izV(1)}] = \exp\left(-\frac{1}{2}az^2 + ibz + \int_{\mathbb{R}} (e^{izx} - 1 - izx 1_{[-1,1]}(x))\nu(dx)\right)$$

for $z \in \mathbb{R}$; here $a \geq 0, b \in \mathbb{R}$ are constants, and ν is a measure on \mathbb{R} satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty;$$

that is, ν is a Lévy measure. For a textbook treatment of Lévy processes we refer to Bertoin [6] and Sato [42]. The description of the Lévy triple of the limit process will be in terms of the measures $\nu^{(u)}$ ($u > 0$) on \mathbb{E} defined for $x > 0$ by

$$(3.2) \quad \begin{aligned} \nu^{(u)}(x, \infty) &= u^{-\alpha} \mathbb{P}\left(u \sum_{i \geq 0} Y_i 1_{\{|Y_i| > 1\}} > x, \sup_{i \leq -1} |Y_i| \leq 1\right), \\ \nu^{(u)}(-\infty, -x) &= u^{-\alpha} \mathbb{P}\left(u \sum_{i \geq 0} Y_i 1_{\{|Y_i| > 1\}} < -x, \sup_{i \leq -1} |Y_i| \leq 1\right). \end{aligned}$$

In case $\alpha \in [1, 2)$, we will need to assume that the contribution of the smaller increments of the partial sum process is close to its expectation. The name of the condition is borrowed from Bartkiewicz et al. [3], Section 2.4; see Section 3.4.4 for a discussion on this assumption.

CONDITION 3.3 (Vanishing small values). For all $\delta > 0$,

$$\lim_{u \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\left[\max_{0 \leq k \leq n} \left| \sum_{i=1}^k \left(\frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}} - \mathbb{E}\left(\frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}}\right)\right) \right| > \delta\right] = 0.$$

THEOREM 3.4. Let $(X_n)_{n \in \mathbb{N}}$ be a strictly stationary sequence of random variables, jointly regularly varying with index $\alpha \in (0, 2)$, and of which the tail process $(Y_i)_{i \in \mathbb{Z}}$ almost surely has no two values of the opposite sign. Suppose that Conditions 2.1 and 2.2 hold. If $1 \leq \alpha < 2$, also suppose that Condition 3.3 holds. Then the partial sum stochastic process

$$V_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \frac{X_k}{a_n} - \lfloor nt \rfloor \mathbb{E}\left(\frac{X_1}{a_n} 1_{\{|X_1|/a_n \leq 1\}}\right), \quad t \in [0, 1],$$

satisfies

$$V_n \xrightarrow{d} V, \quad n \rightarrow \infty,$$

in $D[0, 1]$ endowed with the M_1 topology, where $V(\cdot)$ is an α -stable Lévy process with Lévy triple $(0, \nu, b)$ given by the limits

$$\nu^{(u)} \xrightarrow{\nu} \nu, \quad \int_{x: u < |x| \leq 1} x \nu^{(u)}(dx) - \int_{x: u < |x| \leq 1} x \mu(dx) \rightarrow b$$

as $u \downarrow 0$, with $\nu^{(u)}$ as in (3.2) and μ as in (2.4).

The condition that the tail process cannot switch sign is needed to ensure continuity of the summation functional; see Lemma 3.1. See Section 3.4.5 for some discussion of this condition.

PROOF. Note that from Theorem 2.3 and the fact that $|Y_n| \rightarrow 0$ almost surely as $|n| \rightarrow \infty$, the random variables

$$u \sum_j Z_{1j} 1_{\{|Z_{1j}| > 1\}}$$

are i.i.d. and almost surely finite. Define

$$\widehat{N}^{(u)} = \sum_i \delta_{(T_i^{(u)}, u \sum_j Z_{1j} 1_{\{|Z_{1j}| > 1\}})}$$

Then by Proposition 5.3 in Resnick [41], $\widehat{N}^{(u)}$ is a Poisson process (or a Poisson random measure) with mean measure

$$(3.3) \quad \theta u^{-\alpha} \lambda \times F^{(u)},$$

where λ is the Lebesgue measure, and $F^{(u)}$ is the distribution of the random variable $u \sum_j Z_{1j} 1_{\{|Z_{1j}| > 1\}}$. But for $0 \leq s < t \leq 1$ and $x > 0$, using the fact that the distribution of $\sum_j \delta_{Z_{1j}}$ is equal to the one of $\sum_j \delta_{Y_j}$ conditionally on the event $\{\sup_{i \leq -1} |Y_i| \leq 1\}$, we have

$$\begin{aligned} & \theta u^{-\alpha} \lambda \times F^{(u)}([s, t] \times (x, \infty)) \\ &= \theta u^{-\alpha} (t - s) F^{(u)}((x, \infty)) \\ &= \theta u^{-\alpha} (t - s) \mathbb{P}\left(u \sum_j Z_{1j} 1_{\{|Z_{1j}| > 1\}} > x\right) \\ &= \theta u^{-\alpha} (t - s) \mathbb{P}\left(u \sum_j Y_j 1_{\{|Y_j| > 1\}} > x \mid \sup_{i \leq -1} |Y_i| \leq 1\right) \\ &= \theta u^{-\alpha} (t - s) \frac{\mathbb{P}(u \sum_j Y_j 1_{\{|Y_j| > 1\}} > x, \sup_{i \leq -1} |Y_i| \leq 1)}{\mathbb{P}(\sup_{i \leq -1} |Y_i| \leq 1)} \end{aligned}$$

$$\begin{aligned}
 &= u^{-\alpha}(t-s)\mathbb{P}\left(u\sum_j Y_j 1_{\{|Y_j|>1\}} > x, \sup_{i\leq -1} |Y_i| \leq 1\right) \\
 &= \lambda \times \nu^{(u)}([s, t] \times (x, \infty)).
 \end{aligned}$$

The same can be done for the set $[s, t] \times (-\infty, -x)$, so that the mean measure in (3.3) is equal to $\lambda \times \nu^{(u)}$.

Consider now $0 < u < v$ and

$$\psi^{(u)}(N_n|_{[0,1]\times\mathbb{E}_u})(\cdot) = \psi^{(u)}(N_n|_{[0,1]\times\mathbb{E}_v})(\cdot) = \sum_{i/n\leq\cdot} \frac{X_i}{a_n} 1_{\{|X_i|/a_n > u\}},$$

which by Lemma 3.2 converges in distribution in $D[0, 1]$ under the M_1 metric to

$$\psi^{(u)}(N^{(v)})(\cdot) = \psi^{(u)}(N^{(v)}|_{[0,1]\times\mathbb{E}_u})(\cdot).$$

However, by the definition of the process $N^{(u)}$ in Theorem 2.3, it holds that

$$N^{(u)} \stackrel{d}{=} N^{(v)}|_{[0,1]\times\mathbb{E}_u}$$

for every $v \in (0, u)$. Therefore the last expression above is equal in distribution to

$$\psi^{(u)}(N^{(u)})(\cdot) = \sum_{T_i^{(u)}\leq\cdot} \sum_j u Z_{ij} 1_{\{|Z_{ij}|>1\}}.$$

But since $\psi^{(u)}(N^{(u)}) = \psi^{(u)}(\widehat{N}^{(u)}) \stackrel{d}{=} \psi^{(u)}(\widetilde{N}^{(u)})$, where

$$\widetilde{N}^{(u)} = \sum_i \delta_{(T_i, K_i^{(u)})}$$

is a Poisson process with mean measure $\lambda \times \nu^{(u)}$, we obtain

$$\sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} 1_{\{|X_i|/a_n > u\}} \xrightarrow{d} \sum_{T_i \leq \cdot} K_i^{(u)} \quad \text{as } n \rightarrow \infty$$

in $D[0, 1]$ under the M_1 metric. From (2.3) we have, for any $t \in [0, 1]$, as $n \rightarrow \infty$,

$$\begin{aligned}
 \lfloor nt \rfloor \mathbb{E}\left(\frac{X_1}{a_n} 1_{\{u < |X_1|/a_n \leq 1\}}\right) &= \frac{\lfloor nt \rfloor}{n} \int_{\{x: u < |x| \leq 1\}} x n \mathbb{P}\left(\frac{X_1}{a_n} \in dx\right) \\
 &\rightarrow t \int_{\{x: u < |x| \leq 1\}} x \mu(dx).
 \end{aligned}$$

This convergence is uniform in t , and hence

$$\lfloor n \cdot \rfloor \mathbb{E}\left(\frac{X_1}{a_n} 1_{\{u < |X_1|/a_n \leq 1\}}\right) \rightarrow (\cdot) \int_{\{x: u < |x| \leq 1\}} x \mu(dx)$$

in $D[0, 1]$. Since the latter function is continuous, we can apply Corollary 12.7.1 in Whitt [50], giving a sufficient criterion for addition to be continuous. We obtain, as $n \rightarrow \infty$,

$$(3.4) \quad \begin{aligned} V_n^{(u)}(\cdot) &= \sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} 1_{\{|X_i|/a_n > u\}} - \lfloor n \cdot \rfloor \mathbb{E} \left(\frac{X_1}{a_n} 1_{\{u < |X_1|/a_n \leq 1\}} \right) \\ &\xrightarrow{d} V^{(u)}(\cdot) := \sum_{T_i \leq \cdot} K_i^{(u)} - (\cdot) \int_{\{x: u < |x| \leq 1\}} x \mu(dx). \end{aligned}$$

Limit (3.4) can be rewritten as

$$\begin{aligned} &\sum_{T_i \leq \cdot} K_i^{(u)} - (\cdot) \int_{\{x: u < |x| \leq 1\}} x v^{(u)}(dx) \\ &+ (\cdot) \left(\int_{\{x: u < |x| \leq 1\}} x v^{(u)}(dx) - \int_{\{x: u < |x| \leq 1\}} x \mu(dx) \right). \end{aligned}$$

Note that the first two terms represent a Lévy–Itô representation of the Lévy process with characteristic triple $(0, v^{(u)}, 0)$ (see Resnick [41], page 150). The remaining term is just a linear function of the form $t \mapsto t b_u$. As a consequence, the process $V^{(u)}$ is a Lévy process for each $u < 1$, with characteristic triple $(0, v^{(u)}, b_u)$, where

$$b_u = \int_{\{x: u < |x| \leq 1\}} x v^{(u)}(dx) - \int_{\{x: u < |x| \leq 1\}} x \mu(dx).$$

By Theorem 3.1 in Davis and Hsing [12], for $t = 1$, $V^{(u)}(1)$ converges to an α -stable random variable. Hence by Theorem 13.17 in Kallenberg [28], there is a Lévy process $V(\cdot)$ such that, as $u \rightarrow 0$,

$$V^{(u)}(\cdot) \xrightarrow{d} V(\cdot)$$

in $D[0, 1]$ with the M_1 metric. It has characteristic triple $(0, v, b)$, where v is the vague limit of $v^{(u)}$ as $u \rightarrow 0$ and $b = \lim_{u \rightarrow 0} b_u$ (see Theorem 13.14 in [28]). Since the random variable $V(1)$ has an α -stable distribution, it follows that the process $V(\cdot)$ is α -stable.

If we show that

$$\lim_{u \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}[d_{M_1}(V_n^{(u)}, V_n) > \delta] = 0$$

for any $\delta > 0$, then by Theorem 3.5 in Resnick [41] we will have, as $n \rightarrow \infty$,

$$V_n \xrightarrow{d} V$$

in $D[0, 1]$ with the M_1 metric. Since the Skorohod M_1 metric on $D[0, 1]$ is bounded above by the uniform metric on $D[0, 1]$, it suffices to show that

$$\lim_{u \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq t \leq 1} |V_n^{(u)}(t) - V_n(t)| > \delta \right) = 0.$$

Recalling the definitions, we have

$$\begin{aligned}
 & \lim_{u \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq t \leq 1} |V_n^{(u)}(t) - V_n(t)| > \delta \right) \\
 &= \lim_{u \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left[\sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}} - \lfloor nt \rfloor \mathbb{E} \left(\frac{X_1}{a_n} 1_{\{|X_1|/a_n \leq u\}} \right) \right| > \delta \right] \\
 &= \lim_{u \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left[\sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\lfloor nt \rfloor} \left\{ \frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}} - \mathbb{E} \left(\frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}} \right) \right\} \right| > \delta \right] \\
 &= \lim_{u \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left[\max_{1 \leq k \leq n} \left| \sum_{i=1}^k \left\{ \frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}} - \mathbb{E} \left(\frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}} \right) \right\} \right| > \delta \right].
 \end{aligned}$$

Therefore we have to show

$$(3.5) \quad \lim_{u \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left[\max_{1 \leq k \leq n} \left| \sum_{i=1}^k \left\{ \frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}} - \mathbb{E} \left(\frac{X_i}{a_n} 1_{\{|X_i|/a_n \leq u\}} \right) \right\} \right| > \delta \right] = 0.$$

For $\alpha \in [1, 2)$ this relation is simply Condition 3.3. The proof that (3.5) automatically holds in case $\alpha \in (0, 1)$ is given at the end of the proof of Theorem 4.1 in Tyran-Kamińska [48], page 1640. \square

3.4. *Discussion of the conditions.* Here we revisit in detail all the conditions of Theorem 3.4.

3.4.1. *On joint regular variation.* As we mentioned in the Introduction, regular variation of the marginal distribution with index $\alpha \in (0, 2)$ is both necessary and sufficient for the existence of an α -stable limit for partial sums of i.i.d. random variables. In Tyran-Kamińska [48], only marginal regular variation is assumed from the outset, but in combination with the asymptotic independence condition on the finite-dimensional distributions, this actually implies joint regular variation.

The joint regular variation assumption (2.2) which underlies our main result frequently appears in limit theorems for partial sums [3, 12, 13]. The assumption is relatively straightforward to verify for many applied models; see, for instance, Section 4. The joint regular variation is the basis for the point process result of Theorem 2.3. In particular, it allows us to build on the theory developed in [5, 12] to determine the asymptotic behavior of partial sums over shorter blocks of indices. On the other hand, we note that there are published examples of bounded sequences whose partial sums have an infinite variance α -stable limit (e.g., see Gouëzel [22]).

3.4.2. *On the finite mean cluster size Condition 2.1.* This assumption, which appears frequently in the literature [5, 12, 43, 44, 47], restricts the length of clusters of extremes. It implies that the (max)-stable attractors of appropriately normalized partial sums and maxima have the same index α as the ones for the associated i.i.d. sequence. Alternative assumptions of this kind also exist, most of which are stronger; see, for instance, [3] for a short review.

3.4.3. *On the $\mathcal{A}(a_n)$ mixing Condition 2.2.* Extremely rich literature exists on mixing conditions and their relation with limit theorems. Our assumption $\mathcal{A}'(a_n)$ is a recognizable extension of the mixing condition $\mathcal{A}(a_n)$ due to [12]. Like the latter condition it is implied by the more frequently used strong mixing property (see [30]). However, if one is only interested in the limiting behavior of partial sums, weaker assumptions suffice (see [3]).

3.4.4. *On the vanishing small values Condition 3.3.* The name of the condition is borrowed from [3]. Similar conditions are ubiquitous in the related literature on the limit theory for partial sums [2, 17, 31, 48]. In case $\alpha \in (0, 1)$, it is simply a consequence of regular variation, and in the i.i.d. case, it also holds for $\alpha \in [1, 2)$ (see Resnick [40]). More generally, Tyran-Kamińska [48] showed that the condition holds if the sequence has ρ -mixing coefficients which satisfy $\sum_{j \geq 1} \rho(2^j) < \infty$. For linear processes of which the coefficients decay sufficiently fast, Tyran-Kamińska [49] showed that the condition can be omitted.

3.4.5. *About the no sign switching condition.* The assumption that the tail process has no two values of the opposite sign is crucial to obtain weak convergence of the partial sum process in the M_1 topology. It is admittedly restrictive but unavoidable since the M_1 topology, roughly speaking, can handle several (asymptotically) instantaneous jumps only if they are in the same direction (see Avram and Taqqu [2], Section 1, and Whitt [50], Chapter 12). Note that unlike Dabrowski and Jakubowski [9], our assumption does not exclude nonassociated sequences in general because it involves only the tail dependence in the process.

In Avram and Taqqu [2], Section 1, a conjecture is formulated concerning convergence in Skorohod's M_2 topology, which is somewhat weaker than the M_1 topology. Rather than being all of the same sign, extremes values within a cluster should be such that the values of the partial sums during the cluster are all contained in the interval formed by the partial sums at the beginning and the end of a cluster.

There appear to be some ways of omitting the no sign switching condition altogether. Neither of them is pursued here, however. First, one could opt for a much weaker topology on $D[0, 1]$, like L_1 , for instance. Another possibility is to avoid the within-cluster fluctuations in the partial sum process, for example, by smoothing out its trajectories or by considering the process $t \mapsto S_{r_n \lfloor k_n t \rfloor}$. If we do so, then convergence actually holds in the stronger J_1 topology (see Krizmanić [30], Chapter 3).

3.5. *Simplifications.* In certain cases, the formula for the Lévy measure can be simplified. Moreover, if $\alpha \in (0, 1)$, then no centering is needed.

3.5.1. *A closed form expression for the limiting Lévy measure.* It turns out that if the spectral tail process $(\Theta_i)_{i \in \mathbb{Z}}$ satisfies an additional integrability condition, the formula for the Lévy measure ν simplifies considerably. Note that in our case the Lévy measure ν satisfies the scaling property

$$\nu(s \cdot) = s^{-\alpha} \nu(\cdot)$$

(see Theorem 14.3 in Sato [42]). In particular, ν can be written as

$$\nu(dx) = (c_+ 1_{(0, \infty)}(x) + c_- 1_{(-\infty, 0)}(x)) \alpha |x|^{-\alpha-1} dx$$

for some nonnegative constants c_+ and c_- , and therefore $\nu(\{x\}) = 0$ for every $x \in \mathbb{E}$. Thus, from Theorem 3.2 in Resnick [41] we have

$$\begin{aligned} c_+ &= \nu(1, \infty) \\ &= \lim_{u \rightarrow 0} \nu^{(u)}(1, \infty) \\ &= \lim_{u \rightarrow 0} u^{-\alpha} \mathbb{P}\left(u \sum_{i \geq 0} Y_i 1_{\{|Y_i| > 1\}} > 1, \sup_{i \leq -1} |Y_i| \leq 1\right) \\ &= \lim_{u \rightarrow 0} u^{-\alpha} \int_1^\infty \mathbb{P}\left(u \sum_{i \geq 0} r \Theta_i 1_{\{r|\Theta_i| > 1\}} > 1, \sup_{i \leq -1} r|\Theta_i| \leq 1\right) d(-r^{-\alpha}) \\ &= \lim_{u \rightarrow 0} \int_u^\infty \mathbb{P}\left(\sum_{i \geq 0} r \Theta_j 1_{\{r|\Theta_j| > u\}} > 1, \sup_{i \leq -1} r|\Theta_i| \leq u\right) d(-r^{-\alpha}), \end{aligned}$$

and similarly

$$c_- = \lim_{u \rightarrow 0} \int_u^\infty \mathbb{P}\left(\sum_{i \geq 0} r \Theta_j 1_{\{r|\Theta_j| > u\}} < -1, \sup_{i \leq -1} r|\Theta_i| \leq u\right) d(-r^{-\alpha}).$$

Now suppose further that

$$(3.6) \quad \mathbb{E}\left[\left(\sum_{i \geq 0} |\Theta_i|\right)^\alpha\right] < \infty.$$

Then by the dominated convergence theorem,

$$(3.7) \quad \begin{aligned} c_+ &= \int_0^\infty \mathbb{P}\left(\sum_{i \geq 0} r \Theta_i > 1; \forall i \leq -1: \Theta_i = 0\right) d(-r^{-\alpha}) \\ &= \mathbb{E}\left[\left\{\max\left(\sum_{i \geq 0} \Theta_i, 0\right)\right\}^\alpha 1_{\{\forall i \leq -1: \Theta_i = 0\}}\right], \end{aligned}$$

$$(3.8) \quad c_- = \mathbb{E}\left[\left\{\max\left(-\sum_{i \geq 0} \Theta_i, 0\right)\right\}^\alpha 1_{\{\forall i \leq -1: \Theta_i = 0\}}\right].$$

These relations can be applied to obtain the Lévy measure ν for certain heavy-tailed moving average processes (Example 4.3).

3.5.2. *About centering.* If $\alpha \in (0, 1)$, the centering function in the definition of the stochastic process $V_n(\cdot)$ can be removed. This affects the characteristic triple of the limiting process in the way we describe here.

By Karamata’s theorem, as $n \rightarrow \infty$,

$$nE\left(\frac{X_1}{a_n}1_{\{|X_1|/a_n \leq 1\}}\right) \rightarrow (p - q)\frac{\alpha}{1 - \alpha}$$

with p and q as in (2.4). Thus, as $n \rightarrow \infty$,

$$\lfloor n \cdot \rfloor E\left(\frac{X_1}{a_n}1_{\{|X_1|/a_n \leq 1\}}\right) \rightarrow (\cdot)(p - q)\frac{\alpha}{1 - \alpha}$$

in $D[0, 1]$, which leads to

$$\sum_{k=1}^{\lfloor n \cdot \rfloor} \frac{X_k}{a_n} \xrightarrow{d} V(\cdot) + (\cdot)(p - q)\frac{\alpha}{1 - \alpha}$$

in $D[0, 1]$ endowed with the M_1 topology. The characteristic triple of the limiting process is therefore $(0, \nu, b')$ with $b' = b + (p - q)\alpha/(1 - \alpha)$.

4. Examples. In case of asymptotic independence, the limiting stable Lévy process is the same as in the case of an i.i.d. sequence with the same marginal distribution (Examples 4.1 and 4.2). Heavy-tailed moving averages and GARCH(1, 1) processes (Examples 4.3 and 4.4, respectively) yield more interesting limits.

EXAMPLE 4.1 (Isolated extremes models). Suppose (X_n) is a strictly stationary and strongly mixing sequence of regularly varying random variables with index $\alpha \in (0, 2)$ that satisfies the dependence condition D' in Davis [11], that is,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{i=1}^{\lfloor n/k \rfloor} P\left(\frac{|X_0|}{a_n} > x, \frac{|X_i|}{a_n} > x\right) = 0 \quad \text{for all } x > 0,$$

where $(a_n)_n$ is a positive real sequence such that $nP(|X_0| > a_n) \rightarrow 1$ as $n \rightarrow \infty$. Condition D' implies

$$P(|X_i| > a_n \mid |X_0| > a_n) = \frac{nP(|X_0| > a_n, |X_i| > a_n)}{nP(|X_0| > a_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all positive integer i ; that is, the variables $|X_0|$ and $|X_i|$ are asymptotically independent. As a consequence, the series $(X_n)_n$ is regularly varying and its tail process is the same as that for an i.i.d. sequence; that is, $Y_n = 0$ for $n \neq 0$, and Y_0 is as described in Section 2.1. It is trivially satisfied that no two values of $(Y_n)_n$ are of the opposite sign.

Since the sequence (X_n) is strongly mixing, Condition 2.2 is verified. The finite mean cluster size Condition 2.1 follows from condition D' , for the latter implies

$$\lim_{n \rightarrow \infty} n \sum_{i=1}^{r_n} \mathbb{P}\left(\frac{|X_0|}{a_n} > x, \frac{|X_i|}{a_n} > x\right) = 0 \quad \text{for all } x > 0$$

for any positive integer sequence $(r_n)_n$ such that $r_n \rightarrow \infty$ and $r_n/n \rightarrow 0$ as $n \rightarrow \infty$.

If we additionally assume that the sequence (X_n) satisfies the vanishing small values Condition 3.3 in case $\alpha \in [1, 2)$, then by Theorem 3.4 the sequence of partial sum stochastic processes $V_n(\cdot)$ converges in $D[0, 1]$ with the M_1 topology to an α -stable Lévy process $V(\cdot)$ with characteristic triple $(0, \mu, 0)$ with μ as in (2.4), just as in the i.i.d. case. It can be shown that the above convergence holds also in the J_1 topology (see Krizmanić [30]).

Condition 3.3 applies, for instance, if the series $(X_n)_n$ is a function of a Gaussian causal ARMA process, that is, $X_n = f(A_n)$, for some Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$ and some Gaussian causal ARMA process $(A_n)_n$. From the results in Brockwell and Davis [8] and Pham and Tran [39] (see also Davis and Mikosch [15]) it follows that the sequence $(A_n)_n$ satisfies the strong mixing condition with geometric rate. In this particular case this implies that the sequence $(A_n)_n$ satisfies the ρ -mixing condition with geometric rate (see Kolmogorov and Rozanov [29], Theorem 2), a property which transfers immediately to the series $(X_n)_n$. Hence by Tyran-Kamińska [48], Lemma 4.8, the vanishing small values Condition 3.3 holds.

EXAMPLE 4.2 (Stochastic volatility models). Consider the stochastic volatility model

$$X_n = \sigma_n Z_n, \quad n \in \mathbb{Z},$$

where the noise sequence (Z_n) consists of i.i.d. regularly varying random variables with index $\alpha \in (0, 2)$, whereas the volatility sequence $(\sigma_n)_n$ is strictly stationary, is independent of the sequence $(Z_n)_n$ and consists of positive random variables with finite moment of the order 4α .

Since the random variables Z_i are independent and regularly varying, it follows that the sequence $(Z_n)_n$ is regularly varying with index α . By an application of the multivariate version of Breiman’s lemma, the sequence $(X_n)_n$ is regularly varying with index α too; see Basrak et al. [4], Proposition 5.1.

From the results in Davis and Mikosch [14], it follows that

$$(4.1) \quad n \sum_{i=1}^{r_n} \mathbb{P}(|X_i| > ta_n, |X_0| > ta_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any $t > 0$, where $(r_n)_n$ is a sequence of positive integers such that $r_n \rightarrow \infty$ and $r_n/n \rightarrow 0$ as $n \rightarrow \infty$, and $(a_n)_n$ is a positive real sequence such that $n\mathbb{P}(|X_1| > a_n) \rightarrow 1$ as $n \rightarrow \infty$. From this relation, as in Example 4.1, it follows that the finite

mean cluster size Condition 2.1 holds. Moreover, the tail process $(Y_n)_n$ is the same as in the case of an i.i.d. sequence, that is, $Y_n = 0$ for $n \neq 0$. In particular, the tail process has no two values of the opposite sign.

Assume that $(\log \sigma_n)_n$ is a Gaussian casual ARMA process. Then $(X_n)_n$ satisfies the strong mixing condition with geometric rate (see Davis and Mikosch [15]). Hence the $\mathcal{A}'(a_n)$ mixing Condition 2.2 holds.

In case $\alpha \in [1, 2)$, we also assume the vanishing small values Condition 3.3 holds. Then all conditions in Theorem 3.4 are satisfied, and we obtain the convergence of the partial sum stochastic process toward an α -stable Lévy process with characteristic triple $(0, \mu, 0)$, with μ as in (2.4).

EXAMPLE 4.3 (Moving averages). Consider the finite-order moving average defined by

$$X_n = \sum_{i=0}^m c_i Z_{n-i}, \quad n \in \mathbb{Z},$$

where $(Z_i)_{i \in \mathbb{Z}}$ is an i.i.d. sequence of regularly varying random variables with index $\alpha \in (0, 2)$, $m \in \mathbb{N}$, c_0, \dots, c_m are nonnegative constants and at least c_0 and c_m are not equal to 0. Take a sequence of positive real numbers (a_n) such that

$$(4.2) \quad n\mathbb{P}(|Z_1| > a_n) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

The finite-dimensional distributions of the series $(X_n)_n$ can be seen to be multivariate regularly varying by an application of Proposition 5.1 in Basrak et al. [4] (see also Davis and Resnick [10]). Moreover, if we assume (without loss of generality) that $\sum_{i=0}^m c_i^\alpha = 1$, then also

$$n\mathbb{P}(|X_0| > a_n) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

The tail process $(Y_n)_n$ in (2.2) of the series $(X_n)_n$ can be found by direct calculation (see also Meinguet and Segers [33], Proposition 8.1, for an extension to infinite-order moving averages). First, $Y_0 = |Y_0|\Theta_0$ where $|Y_0|$ and $\Theta_0 = \text{sign}(Y_0)$ are independent with $P(|Y_0| > y) = y^{-\alpha}$ for $y \geq 1$ and $P(\Theta_0 = 1) = p = 1 - P(\Theta_0 = -1)$. Next, let K denote a random variable with values in the set $\{0, \dots, m\}$, independent of Y_0 and such that $P(K = k) = |c_k|^\alpha$ (recall the assumption $\sum_{i=0}^m c_i^\alpha = 1$). To simplify notation, put $c_i := 0$ for $i \notin \{0, \dots, m\}$. Then

$$Y_n = (c_{n+K}/c_K)Y_0, \quad \Theta_n = (c_{n+K}/c_K)\Theta_0, \quad n \in \mathbb{Z},$$

represents the tail process and spectral process of $(X_n)_n$, respectively. Clearly, at most $m + 1$ values Y_n and Θ_n are different from 0 and all have the same sign.

Since the sequence $(X_n)_n$ is m -dependent, it is also strongly mixing, and therefore the $\mathcal{A}'(a_n)$ mixing Condition 2.2 holds. By the same property it is easy to see that the finite mean cluster size Condition 2.1 holds. Moreover, in view of

Lemma 4.8 in Tyran-Kamińska [48], the vanishing small values Condition 3.3 holds as well when $\alpha \in [1, 2)$.

As a consequence, the sequence $(X_n)_n$ satisfies all the conditions of Theorem 3.4, and the partial sum process converges toward a stable Lévy process $V(\cdot)$. The Lévy measure ν can be derived from Section 3.5.1: since (3.6) is trivially fulfilled, we obtain from (3.7) and (3.8),

$$\nu(dx) = \left(\sum_{i=0}^m c_i\right)^\alpha (p1_{(0,\infty)}(x) + q1_{(-\infty,0)}(x))\alpha|x|^{-1-\alpha} dx,$$

which corresponds with the results in Davis and Resnick [10] and Davis and Hsing [12]. Further, if $\alpha \in (0, 1) \cup (1, 2)$, then in the latter two references it is shown that

$$b = (p - q) \frac{\alpha}{1 - \alpha} \left\{ \left(\sum_{i=0}^m c_i\right)^\alpha - 1 \right\},$$

with $q = 1 - p$. The case when $\alpha = 1$ can be treated similarly, but the corresponding expressions are much more complicated and are omitted here (see Davis and Hsing [12], Theorem 3.2 and Remark 3.3).

Infinite-order moving averages with nonnegative coefficients are considered in Avram and Taqqu [2] and Tyran-Kamińska [49]. The idea is to approximate such processes by a sequence of finite-order moving averages, for which Theorem 3.4 applies, and to show that the error of approximation is negligible in the limit.

EXAMPLE 4.4 (ARCH/GARCH models). We consider the GARCH(1, 1) model

$$X_n = \sigma_n Z_n,$$

where $(Z_n)_{n \in \mathbb{Z}}$ is a sequence of i.i.d. random variables with $E(Z_1) = 0$ and $\text{var}(Z_1) = 1$, and

$$(4.3) \quad \sigma_n^2 = \alpha_0 + (\alpha_1 Z_{n-1}^2 + \beta_1) \sigma_{n-1}^2,$$

with $\alpha_0, \alpha_1, \beta_1$ being nonnegative constants. Assume that $\alpha_0 > 0$ and

$$-\infty \leq E \ln(\alpha_1 Z_1^2 + \beta_1) < 0.$$

Then there exists a strictly stationary solution to the stochastic recurrence equation (4.3) (see Goldie [21] and Mikosch and Stărică [35]). The process (X_n) is then strictly stationary too. If $\alpha_1 > 0$ and $\beta_1 > 0$ it is called a GARCH(1, 1) process, while if $\alpha_1 > 0$ and $\beta_1 = 0$ it is called an ARCH(1) process.

In the rest of the example we consider a stationary squared GARCH(1, 1) process $(X_n^2)_n$. Assume that Z_1 is symmetric, has a positive Lebesgue density on \mathbb{R}

and there exists $\alpha \in (0, 2)$ such that

$$E[(\alpha_1 Z_1^2 + \beta_1)^\alpha] = 1 \quad \text{and} \quad E[(\alpha_1 Z_1^2 + \beta_1)^\alpha \ln(\alpha_1 Z_1^2 + \beta_1)] < \infty.$$

Then it is known that the processes $(\sigma_n^2)_n$ and $(X_n^2)_n$ are regularly varying with index α and strongly mixing with geometric rate [4, 35]. Therefore the sequence $(X_n^2)_n$ satisfies the $\mathcal{A}'(a_n)$ mixing Condition 2.2. The finite mean cluster size Condition 2.1 for the sequence $(X_n^2)_n$ follows immediately from the results in Basrak et al. [4].

The (forward) tail process of the bivariate sequence $((\sigma_n^2, X_n^2))_n$ is not too difficult to characterize (see Basrak and Segers [5]). Obviously, the tail process of $(X_n^2)_n$ cannot have two values of the opposite sign.

If additionally the vanishing small values Condition 3.3 holds when $\alpha \in [1, 2)$, then by Theorem 3.4, the sequence of partial sum stochastic processes $(V_n(\cdot))_n$, defined by

$$V_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \frac{X_k^2}{a_n} - \lfloor nt \rfloor E\left(\frac{X_1^2}{a_n} 1_{\{X_1^2/a_n \leq 1\}}\right), \quad t \in [0, 1],$$

converges weakly to an α -stable Lévy process $V(\cdot)$ in $D[0, 1]$ under the M_1 topology. Here $(a_n)_n$ is a positive sequence such that $nP(X_0^2 > a_n) \rightarrow 1$ as $n \rightarrow \infty$.

In case $\alpha \in (0, 1) \cup (1, 2)$, the characteristic triple $(0, \nu, b)$ of the stable random variable $V(1)$ and thus of the stable Lévy process $V(\cdot)$ can be determined from Bartkiewicz et al. [3], Proposition 4.8, Davis and Hsing [12], Remark 3.1, and Section 3.5.2: after some calculations, we find

$$\nu(dx) = c_+ 1_{(0, \infty)}(x) \alpha x^{-\alpha-1} dx, \quad b = \frac{\alpha}{1-\alpha} (c_+ - 1),$$

where

$$c_+ = \frac{E[(Z_0^2 + \tilde{T}_\infty)^\alpha - \tilde{T}_\infty^\alpha]}{E(|Z_1|^{2\alpha})}, \quad \tilde{T}_\infty = \sum_{t=1}^{\infty} Z_{t+1}^2 \prod_{i=1}^t (\alpha_1 Z_i^2 + \beta_1).$$

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