

# RANDOM COVARIANCE MATRICES: UNIVERSALITY OF LOCAL STATISTICS OF EIGENVALUES

BY TERENCE TAO<sup>1</sup> AND VAN VU<sup>2</sup>

*UCLA and Rutgers University*

We study the eigenvalues of the covariance matrix  $\frac{1}{n}M^*M$  of a large rectangular matrix  $M = M_{n,p} = (\zeta_{ij})_{1 \leq i \leq p; 1 \leq j \leq n}$  whose entries are i.i.d. random variables of mean zero, variance one, and having finite  $C_0$ th moment for some sufficiently large constant  $C_0$ .

The main result of this paper is a Four Moment theorem for i.i.d. covariance matrices (analogous to the Four Moment theorem for Wigner matrices established by the authors in [*Acta Math.* (2011) Random matrices: Universality of local eigenvalue statistics] (see also [*Comm. Math. Phys.* **298** (2010) 549–572])). We can use this theorem together with existing results to establish universality of local statistics of eigenvalues under mild conditions.

As a byproduct of our arguments, we also extend our previous results on random Hermitian matrices to the case in which the entries have finite  $C_0$ th moment rather than exponential decay.

## 1. Introduction.

1.1. *The model.* The main purpose of this paper is to study the asymptotic local eigenvalue statistics of covariance matrices of large random matrices. Let us first fix the matrix ensembles that we will be studying.

DEFINITION 1 (Random covariance matrices). Let  $n$  be a large integer parameter going off to infinity, and let  $p = p(n)$  be another integer parameter such that  $p \leq n$  and  $\lim_{n \rightarrow \infty} p/n = y$  for some  $0 < y \leq 1$ . We let  $M = M_{n,p} = (\zeta_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  be a random  $p \times n$  matrix, whose distribution is allowed to depend on  $n$ . We say that the matrix ensemble  $M$  obeys condition C1 with some exponent  $C_0 \geq 2$  if the random variables  $\zeta_{ij}$  are jointly independent, have mean zero and variance 1, and obey the moment condition  $\sup_{i,j} \mathbf{E}|\zeta_{ij}|^{C_0} \leq C$  for some constant  $C$  independent of  $n, p$ . We say that the matrix  $M$  is *i.i.d.* if the  $\zeta_{ij}$  are identically and independently distributed with law independent of  $n, p$ .

Given such a matrix, we form the  $n \times n$  covariance matrix  $W = W_{n,p} := \frac{1}{n}M^*M$ . This matrix has rank  $p$  and so the first  $n - p$  eigenvalues are trivial; we

---

Received December 2009; revised January 2011.

<sup>1</sup>Supported by a grant from the MacArthur Foundation, by NSF Grant DMS-06-49473, and by the NSF Waterman award.

<sup>2</sup>Supported by research Grants DMS-09-01216 and AFOSAR-FA-9550-09-1-0167.  
*MSC2010 subject classifications.* 15B52, 62J10.

*Key words and phrases.* Four moment theorem, universality, covariance matrices.

order the (necessarily positive) remaining eigenvalues of these matrices (counting multiplicity) as

$$0 \leq \lambda_1(W) \leq \dots \leq \lambda_p(W).$$

We often abbreviate  $\lambda_i(W)$  as  $\lambda_i$ .

Note that the only distributional hypothesis we require on the entries  $\zeta_{ij}$ , besides the crucial joint independence hypothesis, are moment conditions. In particular, we make no distinction between continuous and discrete distributions here.

REMARK 2. In this paper, we will focus primarily on the case  $y = 1$ , but several of our results extend to other values of  $y$  as well. The case  $p > n$  can be easily deduced from the  $p < n$  case after some minor notational changes by transposing the matrix  $M$ , which does not affect the nontrivial eigenvalues of the covariance matrix. One can also easily normalise the variance of the entries to be some other quantity  $\sigma^2$  than 1 if one wishes. Observe that the quantities  $\sigma_i := \sqrt{n}\lambda_i^{1/2}$  can be interpreted as the nontrivial singular values of the original matrix  $M$ , and  $\lambda_1, \dots, \lambda_p$  can also be interpreted as the eigenvalues of the  $p \times p$  matrix  $\frac{1}{n}MM^*$ . It will be convenient to exploit all three of these spectral interpretations of  $\lambda_1, \dots, \lambda_p$  in this paper. condition C1 is analogous to condition C0 for Wigner-type matrices in [28], but with the exponential decay hypothesis relaxed to polynomial decay only.

The well-known *Marchenko–Pastur law* governs the bulk distribution of the eigenvalues  $\lambda_1, \dots, \lambda_p$  of  $W$ :

THEOREM 3 (Marchenko–Pastur law). *Assume condition C1 with  $C_0 > 2$ , and suppose that  $p/n \rightarrow y$  for some  $0 < y \leq 1$ . Then for any  $x > 0$ , the random variables*

$$\frac{1}{p} |\{1 \leq i \leq p : \lambda_i(W) \leq x\}|$$

converge in probability to  $\int_0^x \rho_{\text{MP},y}(x) dx$ , where

$$(1) \quad \rho_{\text{MP},y}(x) := \frac{1}{2\pi xy} \sqrt{(b-x)(x-a)} 1_{[a,b]}(x)$$

and

$$(2) \quad a := (1 - \sqrt{y})^2; \quad b := (1 + \sqrt{y})^2.$$

When furthermore  $M$  is i.i.d., one can also obtain the case  $C_0 = 2$ .

PROOF. For the case  $C_0 \geq 4$ , see [21, 23]; for the case  $C_0 > 2$ , see [29]; for the  $C_0 = 2$  i.i.d. case, see [30]. Further results are known on the rate of convergence: see [16].  $\square$

In this paper, we are concerned instead with the local eigenvalue statistics. A model case is the (complex) Wishart ensemble, in which the  $\zeta_{ij}$  are i.i.d. variables which are complex Gaussians with mean zero and variance 1. In this case, the distribution of the eigenvalues  $(\lambda_1, \dots, \lambda_n)$  of  $W$  can be explicitly computed (as a special case of the Laguerre unitary ensemble). For instance, when  $p = n$ , the joint distribution is given by the density function

$$(3) \quad \rho_n(\lambda_1, \dots, \lambda_n) = c(n) \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^2 \exp\left(-n \sum_{i=1}^n \lambda_i\right)$$

for some explicit normalization constant  $c(n)$  whose exact value is not important for this discussion.

Very similarly to the GUE case, one can use this explicit formula to directly compute several local statistics, including the distribution of the largest and smallest eigenvalues [3], the correlation functions [22] etc. Also in similarity to the GUE case, it is widely conjectured that these statistics hold for a much larger class of random matrices. For some earlier results in this direction, we refer to [2, 13, 25, 26] and the references therein.

The goal of this paper is to establish a Four Moment theorem for random covariance matrices, as an analogue of a recent result in [28]. Roughly speaking, this theorem asserts that the asymptotic behaviour of local statistics of the eigenvalues of  $W_n$  are determined by the first four moments of the entries.

1.2. *The Four Moment theorem.* To state the Four Moment theorem, we first need a definition.

DEFINITION 4 (Matching). We say that two complex random variables  $\zeta, \zeta'$  match to order  $k$  for some integer  $k \geq 1$  if one has  $\mathbf{ERe}(\zeta)^m \mathbf{Im}(\zeta)^l = \mathbf{ERe}(\zeta')^m \mathbf{Im}(\zeta')^l$  for all  $m, l \geq 0$  with  $m + l \leq k$ .

Our main result is the following.

THEOREM 5 (Four Moment theorem). For sufficiently small  $c_0 > 0$  and sufficiently large  $C_0 > 0$  ( $C_0 = 10^4$  would suffice) the following holds for every  $0 < \varepsilon < 1$  and  $k \geq 1$ . Let  $M = (\zeta_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  and  $M' = (\zeta'_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  be matrix ensembles obeying condition C1 with the indicated constant  $C_0$ , and assume that for each  $i, j$  that  $\zeta_{ij}$  and  $\zeta'_{ij}$  match to order 4. Let  $W, W'$  be the associated covariance matrices. Assume also that  $p/n \rightarrow y$  for some  $0 < y \leq 1$ .

Let  $G : \mathbb{R}^k \rightarrow \mathbb{R}$  be a smooth function obeying the derivative bounds

$$(4) \quad |\nabla^j G(x)| \leq n^{c_0}$$

for all  $0 \leq j \leq 5$  and  $x \in \mathbb{R}^k$ .

Then for any  $\varepsilon p \leq i_1 < i_2 < \dots < i_k \leq (1 - \varepsilon)p$ , and for  $n$  sufficiently large depending on  $\varepsilon, k, c_0$  we have

$$(5) \quad |\mathbf{E}(G(n\lambda_{i_1}(W), \dots, n\lambda_{i_k}(W))) - \mathbf{E}(G(n\lambda_{i_1}(W'), \dots, n\lambda_{i_k}(W')))| \leq n^{-c_0}.$$

If  $\zeta_{ij}$  and  $\zeta'_{ij}$  only match to order 3 rather than 4, the conclusion (5) still holds provided that one strengthens (4) to

$$|\nabla^j G(x)| \leq n^{-jc_1}$$

for all  $0 \leq j \leq 5$  and  $x \in \mathbb{R}^k$  and any  $c_1 > 0$ , provided that  $c_0$  is sufficiently small depending on  $c_1$ .

This is an analogue of [28], Theorem 15, for covariance matrices, with the main difference being that the exponential decay condition from [28], Theorem 15, has been weakened to the high moment condition in C1. This is achieved by an “exponential decay removing trick” that relies on using a truncated version of the four moment theorem to extend the range of validity of a key “gap condition” that is used in the proof of the above theorem. The same trick also allows one to obtain a similar strengthening of the main results of [27, 28], thus relaxing the exponential decay hypotheses in those results to high moment conditions. The value  $C_0 = 10^4$  is *ad hoc*, and we make no attempt to optimize this constant.

REMARK 6. The reason that we restrict the eigenvalues to the bulk of the spectrum [ $\varepsilon p \leq i \leq (1 - \varepsilon)p$ ] is to guarantee that the density function  $\rho_{\text{MP},y}$  is bounded away from zero. In view of the results in [27], we expect that the result extends to the edge of the spectrum as well. In particular, in view of the results in [2], it is likely that the hard edge asymptotics of Forrester [14] can be extended to a wider class of ensembles. We will pursue this issue elsewhere.

REMARK 7. As observed in [5], the requirement that the moments of  $\zeta_{ij}$  and  $\zeta'_{ij}$  match exactly can be relaxed slightly. Indeed, to obtain the desired conclusions, it suffices to require that for  $k = 1, 2, 3, 4$ , the  $k$ th moments of  $\zeta_{ij}$  and  $\zeta'_{ij}$  differ by  $O(n^{-(4-k)/2-\delta})$  for some  $\delta > 0$  independent of  $n$ . Indeed, if one inspects the proof of the four moment theorem, and specifically the step in which one performs a Taylor expansion argument to understand the effect of exchanging a single entry  $\zeta_{ij}$  with  $\zeta'_{ij}$  on the expectations in (5) (see [28], Section 3.2), the above near-matching property is sufficient to ensure that this effect has magnitude  $O(n^{-2-c})$  for some  $c > 0$ , and so the net effect on (5) after performing  $O(n^2)$  such exchange operations is acceptable. We omit the details. This relaxed version of the four moment theorem is particularly useful for dealing with Bernoulli distributions, which are completely determined by their first four moments; see [5] for further discussion.

1.3. *Applications.* One can apply Theorem 5 in a similar way as its counterpart [28], Theorem 15, in order to obtain universality results for large classes of random matrices. In many cases, one can combine this theorem with existing partial results for special ensembles to remove some of the moment assumptions. Let us demonstrate this through an example concerning the universality of the *sine kernel*.

Using the explicit formula (3), Nagao and Wadati [22] established the following result for the complex Wishart ensemble, which roughly speaking asserts that the spectrum of such an ensemble enjoys sine kernel statistics in the neighborhood of any bulk energy level  $0 < u < 4$ .

**THEOREM 8 (Sine kernel for Wishart ensemble).** [22] *Let  $k \geq 1$  be an integer, let  $f : \mathbb{R}^k \rightarrow \mathbb{C}$  be a continuous function with compact support and symmetric with respect to permutations, and let  $0 < u < 4$ ; we assume all these quantities are independent of  $n$ . Assume that<sup>3</sup>  $p = n + O(1)$  (thus  $y = 1$ ), and that  $W$  is given by the complex Wishart ensemble. Let  $\lambda_1, \dots, \lambda_p$  be the nontrivial eigenvalues of  $W$ . Then the quantity*

$$(6) \quad \mathbf{E} \sum_{1 \leq i_1, \dots, i_k \leq p} f(n\rho_{\text{MP},1}(u)(\lambda_{i_1} - u), \dots, n\rho_{\text{MP},1}(u)(\lambda_{i_k} - u))$$

converges as  $n \rightarrow \infty$  to

$$\int_{\mathbb{R}^k} f(t_1, \dots, t_k) \det(K(t_i, t_j))_{1 \leq i, j \leq k} dt_1 \cdots dt_k,$$

where  $K(x, y) := \frac{\sin(\pi(x-y))}{\pi(x-y)}$  is the sine kernel.

**REMARK 9.** The results in [22] allowed  $f$  to be bounded measurable rather than continuous, but when we consider discrete ensembles later, it will be important to keep  $f$  continuous.

Returning to the bulk, the following extension was established by Ben Arous and Peché [2], as a variant of Johansson’s result [19] for random hermitian matrices. We say that a complex random variable  $\zeta$  of mean zero and variance one is *Gauss divisible* if  $\zeta$  has the same distribution as  $\zeta = (1 - t)^{1/2}\zeta' + t^{1/2}\zeta''$  for some  $0 < t < 1$  and some independent random variables  $\zeta', \zeta''$  of mean zero and variance 1, with  $\zeta''$  distributed according to the complex Gaussian.

**THEOREM 10 (Sine kernel for Gaussian divisible ensemble).** [2] *Theorem 8 [which is for the Wishart ensemble and for  $p = n + O(1)$ ] can be extended to the case when  $p = n + O(n^{43/48})$  (so  $y$  is still 1), and when  $M$  is an i.i.d. matrix obeying condition C1 with  $C_0 = 2$ , and with the  $\zeta_{ij}$  gauss divisible.*

---

<sup>3</sup>See Section 1.5 for the asymptotic notation we will be using.

Using Theorem 5 and Theorem 10 (in exactly the same way we used [28], Theorem 15, and Johansson's theorem [19] to establish [28], Theorem 11), we can extend Theorem 10 from the gauss divisible case to a more general situation.

**COROLLARY 11** (Sine kernel for more general ensembles). *Theorem 8 can be extended to the case when  $p = n + O(n^{43/48})$  (so  $y$  is still 1), and when  $M$  is an i.i.d. matrix obeying condition C1 with  $C_0$  sufficiently large ( $C_0 = 10^4$  would suffice), and where the real and imaginary parts of  $\zeta_{ij}$  are i.i.d. and are supported on at least three points.*

**PROOF.** (Sketch) It was shown in [28], Corollary 30, that if the real and imaginary parts of a complex random variable  $\zeta$  were independent with mean zero and variance one, and both were supported on at least three points, then  $\zeta$  matched to order 4 with a gauss divisible random variable  $\zeta'$  with finite  $C_0$  moment (indeed, if one inspects the convexity argument used to solve the moment problem in [28], Lemma 28, the Gauss divisible random variable could be taken to be the sum of a Gaussian variable and a discrete variable, and in particular is thus exponentially decaying). If one lets  $M'$  be the i.i.d. matrix whose coefficients have entries  $\zeta'$ , then Theorem 10 asserts that the conclusions of Theorem 8 hold for  $M'$ . Using Theorem 5 exactly as in the proof of [28], Theorem 11, (and approximating  $f$  uniformly by smooth functions), we conclude that the conclusions of Theorem 8 hold for  $M$  also.  $\square$

One can also extend the above argument to cover cases in which the real and imaginary parts of  $\zeta_{ij}$  are not i.i.d. by an analysis of the moment matching problem for complex random variables (and in particular, by extending the three-moment analysis in Lemma 34 below to four moments), but we will not do so here.

The arguments in this paper will be a nonsymmetric version of those in [28]. The arguments in [28] started with analyzing the stability of the eigenvalue equation  $Mv_i = \lambda_i v_i$  where  $M$  is a random Hermitian matrix and  $\lambda_i$  is the  $i$ th eigenvalue with eigenvector  $v$ . For the situation considered in this paper, it is tempting to similarly analyze the eigenvalue equation  $Wv_i = \lambda_i v_i$  for the covariance matrix  $W$ . However, this does not work, since the covariance matrix  $W$ , while random, does not have independent entries. The new idea here is to work with a system of two equations

$$(7) \quad Mu_i = \sigma_i v_i$$

and

$$(8) \quad M^* v_i = \sigma_i u_i,$$

where  $u_i$  and  $v_i$  are the left and right singular vectors of  $M$ . This leads to a number of technical issues that need to be addressed through the paper.

One can combine the singular value equations (7), (8) into a single eigenvalue equation

$$\mathbf{M} \begin{pmatrix} v_i \\ u_i \end{pmatrix} = \sigma_i \begin{pmatrix} v_i \\ u_i \end{pmatrix},$$

where  $\mathbf{M}$  is the *augmented matrix*

$$(9) \quad \mathbf{M} := \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}.$$

Thus one can view the singular values of an i.i.d. matrix as being essentially given by the eigenvalues of a slightly larger Hermitian matrix which is of Wigner type except that the entries have been zeroed out on two diagonal blocks. We will take advantage of this augmented perspective in some parts of the paper (particularly when we wish to import results from [28] as black boxes), but in other parts it will in fact be more convenient to work with  $M$  directly. In particular, the fact that many of the entries in (9) are zero (and in particular, have zero mean and variance) seems to make it difficult to directly apply parts of the arguments from [28] (particularly those that are probabilistic in nature,<sup>4</sup> rather than deterministic) directly to the augmented matrix, and will instead work with  $M$  directly in these cases. Nevertheless, one can view this connection as a heuristic explanation as to why some (but not all) of the machinery in the Hermitian eigenvalue problem can be transferred to the non-Hermitian singular value problem.

1.4. *Extensions.* In a very recent work, Erdős et al. [10] extended<sup>5</sup> Theorem 10 to a large class of matrices, assuming that the distribution of the entries  $\zeta_{ij}$  is sufficiently smooth and obeys a log-Sobolev inequality. While their results do not apply for entries with discrete distributions, it allows one to extend Theorem 10 to the case when  $t$  is a negative power of  $n$ . Given this, one can use the argument in

---

<sup>4</sup>A typical instance of a probabilistic argument that encounters difficulty when there are many zero entries arises when one wants to estimate the distance  $\text{dist}(X, V)$  between a random vector  $X = (\xi_1, \dots, \xi_n)$  (which one should think of as something like a row of  $\mathbf{M}$ ) and a fixed subspace  $V$ . If all the entries of  $X$  are i.i.d. with mean zero and constant variance, then an easy second moment computation allows one to control  $\mathbf{E} \text{dist}(X, V)^2$  exactly in terms of the codimension of  $V$ ; in particular, no knowledge of the orientation of  $V$  is required. One also obtains reasonable upper and lower bounds on this quantity if the variance is not constant, but is also bounded above and below. However, if many of the entries of  $X$  have zero variance (i.e., they vanish), then one has difficulty lower bounding  $\mathbf{E} \text{dist}(X, V)^2$  because one has to somehow exclude the possibility that the normal vectors to  $V$  have almost all of their  $\ell^2$  mass supported on those zero variance entries. We do not know how to address this problem in general. *Note added in proof:* Several months after the submission of this paper, Erdős, Yau and Yin [11, 12] were able to obtain universality results for some classes of generalized Wigner matrices (such as band matrices) in which some entries are permitted to have zero variance. However, one of their key assumptions is that the matrix of (normalised) variances has a simple eigenvalue at 1, and this assumption does not hold for the augmented matrix (9).

<sup>5</sup>Even more recently, a similar result was also established by P ech e [24].

[5] to remove the requirement that the real and imaginary parts of  $\zeta_{ij}$  be supported on at least three points.

We can also have the following analogue of [5], Theorem 2.

**THEOREM 12 (Universality of averaged correlation function).** *Fix  $\varepsilon > 0$  and  $u$  such that  $0 < u - \varepsilon < u + \varepsilon < 4$ . Let  $k \geq 1$  and let  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  be a continuous, compactly supported function, and let  $W = W_{n,n}$  be a random covariance matrix, with  $n$  assumed large depending on  $u, \varepsilon, k$ . Then the quantity*

$$(10) \quad \frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} \int_{\mathbb{R}^k} f(t_1, \dots, t_k) \frac{1}{(n\rho_{MP,1}(u'))^k} P_n^{(k)} \times \left( u' + \frac{t_1}{n\rho_{MP,1}(u')}, \dots, u' + \frac{t_k}{n\rho_{MP,1}(u')} \right) dt_1 \cdots dt_k du'$$

converges as  $n \rightarrow \infty$  to

$$\int_{\mathbb{R}^k} f(t_1, \dots, t_k) \det(K(t_i, t_j))_{i,j=1}^k dt_1 \cdots dt_k,$$

where  $K(x, y)$  is the Dyson sine kernel

$$(11) \quad K(x, y) := \frac{\sin(\pi(x - y))}{\pi(x - y)},$$

and the  $k$ -point correlation function  $p_n^{(k)} : \mathbb{R}^k \rightarrow \mathbb{R}^+$  is the unique symmetric probability distribution such that

$$\int_{\mathbb{R}^k} f(\alpha_1, \dots, \alpha_k) p_n^{(k)}(\alpha_1, \dots, \alpha_k) := k! \sum_{1 \leq i_1 < \dots < i_k \leq n} f(\lambda_{i_1}, \dots, \lambda_{i_k})$$

for all symmetric test functions  $f$ . (If  $W$  is a discrete ensemble, one has to interpret  $p_n^{(k)}$  as a distribution or a probability measure rather than as a function.)

The detailed proof of Theorem 12 are essentially the same as the proof of [5], Theorem 2, and is omitted.

**REMARK 13.** The four moment theorem controls the distribution of individual eigenvalues (or singular values)  $\lambda_i$ , but as indicated above, this control can then be used to obtain control of correlation expressions such as (10). The local relaxation flow methods developed in [4, 6–10], by contrast, are focused on individual energy levels  $u$  rather than individual eigenvalues. As such, they provide an alternate approach to controlling correlation expressions such as (10), but we do not know how to convert such information back to control on individual eigenvalues or singular values in general, because the standard deviation of each eigenvalue can exceed (by a logarithmic factor, see [18]) the scale of the mean eigenvalue spacing, which is the scale at which the correlation estimates operate at.

1.5. *Notation.* Throughout this paper,  $n$  will be an asymptotic parameter going to infinity. Some quantities (e.g.,  $\varepsilon$ ,  $y$  and  $C_0$ ) will remain independent of  $n$ , while other quantities (e.g.,  $p$ , or the matrix  $M$ ) will depend on  $n$ . All statements here are understood to hold only in the asymptotic regime when  $n$  is sufficiently large depending on all quantities that are independent of  $n$ . We write  $X = O(Y)$ ,  $Y = \Omega(|X|)$ ,  $|X| \ll Y$ , or  $Y \gg |X|$  if one has  $|X| \leq CY$  for all sufficiently large  $n$  and some  $C$  independent of  $n$ . [Note however that  $C$  is allowed to depend on other quantities independent of  $n$ , such as  $\varepsilon$  and  $y$ , unless otherwise stated; we will sometimes emphasise this by using subscripts, thus, for instance,  $X = O_a(Y)$  denotes the estimate  $|X| \leq C_a Y$  for some constant  $C$  depending only on  $a$ .] We write  $X = o(Y)$  if  $|X| \leq c(n)Y$  where  $c(n) \rightarrow 0$  as  $n \rightarrow \infty$ . We write  $X = \Theta(Y)$  if  $X \ll Y \ll X$ , thus, for instance, if  $p/n \rightarrow y$  for some  $0 < y \leq 1$  then  $p = \Theta(n)$ .

We write  $\sqrt{-1}$  for the complex imaginary unit, in order to free up the letter  $i$  to denote an integer (usually between 1 and  $n$ ).

We write  $\|X\|$  for the length of a vector  $X$ ,  $\|A\| = \|A\|_{op}$  for the operator norm of a matrix  $A$ , and  $\|A\|_F = \text{tr}(AA^*)^{1/2}$  for the Frobenius (or Hilbert–Schmidt) norm.

We will need to quantify the intuitive assertion that a given event  $E$  occurs “frequently,” as follows.

DEFINITION 14 (Frequent events). [28] Let  $E$  be an event depending on  $n$ .

- $E$  holds with high probability if  $\mathbf{P}(E) \geq 1 - O(n^{-c})$  for some constant  $c > 0$  (independent of  $n$ ).
- $E$  holds with overwhelming probability if  $\mathbf{P}(E) \geq 1 - O_C(n^{-C})$  for every constant  $C > 0$ .
- $E$  holds almost surely if  $\mathbf{P}(E) = 1$ .

**2. The gap property and the exponential decay removing trick.** The following property, which roughly speaking asserts that unexpectedly small eigenvalue spacings are rare, plays an important role in proving the main results of [28].

DEFINITION 15 (Gap property). Let  $M$  be a matrix ensemble obeying condition C1. We say that  $M$  obeys the *gap property* if for every  $\varepsilon, c > 0$  (independent of  $n$ ), and for every  $\varepsilon p \leq i \leq (1 - \varepsilon)p$ , one has  $|\lambda_{i+1}(W) - \lambda_i(W)| \geq n^{-1-c}$  with high probability. (The implied constants in this statement are allowed to depend on  $\varepsilon$  and  $c$ .)

In the Wigner case, it was shown that exponential decay of the atom distribution implied the gap property, and the gap property was then used to establish deduce the four moment theorem from a “truncated four moment theorem.” As it turns out, the proof of this latter theorem does not require exponential decay of the atom distribution, relying instead on the weaker hypothesis that a sufficiently high moment of the atom distribution is finite. A new technical observation of this paper

is that one can use the truncated four moment theorem to extend the gap property from exponentially decaying atom distributions to distributions with sufficiently high moments finite, and as a consequence we can extend the full Four Moment theorem to this case also.

We turn to the details. First, as an analogue of [28], Theorem 19, we prove the following theorem, using a slight modification of the method in [28].

**THEOREM 16 (Gap theorem).** *Let  $M = (\zeta_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  obey condition C1 for some  $C_0$ , and suppose that the coefficients  $\zeta_{ij}$  are exponentially decaying in the sense that  $\mathbf{P}(|\zeta_{ij}| \geq t^C) \leq \exp(-t)$  for all  $t \geq C'$  for all  $i, j$  and some constants  $C, C' > 0$ . Then  $M$  obeys the gap property.*

Next, we have the following analogue of [28], Theorem 15.

**THEOREM 17 (Four Moment theorem with Gap assumption).** *For sufficiently small  $c_0 > 0$  and sufficiently large  $C_0 > 0$  ( $C_0 = 10^4$  would suffice) the following holds for every  $0 < \varepsilon < 1$  and  $k \geq 1$ . Let  $M = (\zeta_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  and  $M' = (\zeta'_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  be matrix ensembles obeying condition C1 with the indicated constant  $C_0$ , and assume that for each  $i, j$  that  $\zeta_{ij}$  and  $\zeta'_{ij}$  match to order 4. Let  $W, W'$  be the associated covariance matrices. Assume also that  $M$  and  $M'$  obeys the gap property, and that  $p/n \rightarrow y$  for some  $0 < y \leq 1$ .*

*Let  $G : \mathbb{R}^k \rightarrow \mathbb{R}$  be a smooth function obeying the derivative bounds*

$$(12) \quad |\nabla^j G(x)| \leq n^{c_0}$$

*for all  $0 \leq j \leq 5$  and  $x \in \mathbb{R}^k$ .*

*Then for any  $\varepsilon p \leq i_1 < i_2 < \dots < i_k \leq (1 - \varepsilon)p$ , and for  $n$  sufficiently large depending on  $\varepsilon, k, c_0$  we have*

$$(13) \quad |\mathbf{E}(G(n\lambda_{i_1}(W), \dots, n\lambda_{i_k}(W))) - \mathbf{E}(G(n\lambda_{i_1}(W'), \dots, n\lambda_{i_k}(W')))| \leq n^{-c_0}.$$

*If  $\zeta_{ij}$  and  $\zeta'_{ij}$  only match to order 3 rather than 4, the conclusion (13) still holds provided that one strengthens (12) to*

$$|\nabla^j G(x)| \leq n^{-jc_1}$$

*for all  $0 \leq j \leq 5$  and  $x \in \mathbb{R}^k$  and any  $c_1 > 0$ , provided that  $c_0$  is sufficiently small depending on  $c_1$ .*

This theorem is weaker than Theorem 5, as we assume the gap property. Besides the fact that we consider singular values here instead of eigenvalues, the main difference between this result and [28], Theorem 15, is that in the latter we assume exponential decay rather than the gap property. However, this difference is only a formality, since in the proof of [28], Theorem 15, the only place we used exponential decay is to prove the gap property (via [28], Theorem 19).

The core of the proof of Theorem 17 is a truncated four moment theorem (Theorem 32), which allows us to insert information such as the gap property into the test function  $G$ .

By combining Theorem 17 with Theorem 16, we obtain Theorem 5 in the case when the coefficients  $\zeta_{ij}$  are exponentially decaying. To remove the exponential decay hypothesis, we will apply the truncated four moment theorem (Theorem 32) a second time, together with a moment matching argument (Lemma 34) to eliminate this hypothesis from Theorem 16.

**THEOREM 18 (Gap theorem).** *Assume that  $M = (\zeta_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  satisfies condition C1 with  $C_0$  sufficiently large. Then  $M$  obeys the gap property.*

Theorem 5 follows directly from Theorems 17 and 18.

The rest of the paper is organized as follows. The next three sections are devoted to technical lemmas. The proofs of Theorems 17 and 18 are presented in Section 6, assuming Theorems 32 and 16. The proofs of these latter two theorems are presented in Sections 7 and 8, respectively.

**3. The main technical lemmas.** *Important note.* The arguments in this paper are very similar to, and draw heavily from, the previous paper [28] of the authors. We recommend therefore that the reader be familiar with that paper first, before reading the current one.

In the proof of the Four Moment theorem (as well as the Gap theorem) for  $n \times n$  Wigner matrices in [28], a crucial ingredient was a variant of the *Delocalization Theorem* of Erdős, Schlein and Yau [7–9]. This result asserts (assuming uniformly exponentially decaying distribution for the coefficients) that with overwhelming probability, all the unit eigenvectors of the Wigner matrix have coefficients  $O(n^{-1/2+o(1)})$  (thus, the “ $\ell^2$  energy” of the eigenvector is spread out more or less uniformly amongst the  $n$  coefficients). When one just assumes uniformly bounded  $C_0$  moment rather than uniform exponential decay, the bound becomes  $O(n^{-1/2+O(1/C_0)})$  instead (where the implied constant in the exponent is uniform in  $C_0$ ).

Similarly, to prove the Four Moment and Gap theorems in this paper, we will need a Delocalization theorem for the *singular vectors* of the matrix  $M$ . We define a *right singular vector*  $u_i$  (resp., *left singular vector*  $v_i$ ) with singular value  $\sigma_i(M) = \sqrt{n}\lambda_i(W)^{1/2}$  to be an eigenvector of  $W = \frac{1}{n}M^*M$  (resp.,  $\tilde{W} = \frac{1}{n}MM^*$ ) with eigenvalue  $\lambda_i$ . In the *generic case* when the singular values are simple (i.e.,  $0 < \sigma_1 < \dots < \sigma_p$ ), we observe from the singular value decomposition that one can find orthonormal bases  $u_1, \dots, u_p \in \mathbb{C}^n$  and  $v_1, \dots, v_p \in \mathbb{C}^p$  for the corange  $\ker(M)^\perp$  of  $M$  and of  $\mathbb{C}^p$ , respectively, such that

$$Mu_i = \sigma_i v_i$$

and

$$M^* v_i = \sigma_i u_i.$$

Furthermore, in the generic case the unit singular vectors  $u_i, v_i$  are determined up to multiplication by a complex phase  $e^{i\theta}$ .

We will establish the following Erdős–Schlein–Yau type delocalization theorem (analogous to [28], Proposition 62), which is an essential ingredient to Theorems 17, 16 and is also of some independent interest.

**THEOREM 19 (Delocalization theorem).** *Suppose that  $p/n \rightarrow y$  for some  $0 < y \leq 1$ , and let  $M$  obey condition C1 for some  $C_0 \geq 2$ . Suppose further that that  $|\zeta_{ij}| \leq K$  almost surely for some  $K > 1$  (which can depend on  $n$ ) and all  $i, j$ , and that the probability distribution of  $M$  is continuous. Let  $\varepsilon > 0$  be independent of  $n$ . Then with overwhelming probability, all the unit left and right singular vectors of  $M$  with eigenvalue  $\lambda_i$  in the interval  $[a + \varepsilon, b - \varepsilon]$  [with  $a, b$  defined in (2)] have all coefficients uniformly of size  $O(Kn^{-1/2} \log^{10} n)$ .*

The factors  $K \log^{10} n$  can probably be improved slightly, but anything which is polynomial in  $K$  and  $\log n$  will suffice for our purposes. Observe that if  $M$  obeys condition C1, then each event  $|\zeta_{ij}| \leq K$  with  $K := n^{10/C_0}$  (say) occurs with probability  $1 - O(n^{-10})$ . Thus, in practice, we will be able to apply the above theorem with  $K = n^{10/C_0}$  without difficulty. The continuity hypothesis is a technical one, imposed so that the singular values are almost surely simple, but in practice we will be able to eliminate this hypothesis by a limiting argument (as none of the bounds will depend on any quantitative measure of this continuity).

As with other proofs of delocalization theorems in the literature, Theorem 19 is in turn deduced from the following eigenvalue concentration bound (analogous to [28], Proposition 60).

**THEOREM 20 (Eigenvalue concentration theorem).** *Let the hypotheses be as in Theorem 19, and let  $\delta > 0$  be independent of  $n$ . Then for any interval  $I \subset [a + \varepsilon, b - \varepsilon]$  of length  $|I| \geq K^2 \log^{20} n/n$ , one has with overwhelming probability (uniformly in  $I$ ) that*

$$\left| N_I - p \int_I \rho_{MP,y}(x) dx \right| \leq \delta p,$$

where

$$(14) \quad N_I := \{1 \leq i \leq p : \lambda_i(W) \in I\}$$

is the number of eigenvalues in  $I$ .

We remark that a very similar result (with slightly different hypotheses on the parameters and on the underlying random variable distributions) was recently established in [10], Corollary 7.2.

We isolate one particular consequence of Theorem 20 (also established in [17]):

**COROLLARY 21** (Concentration of the bulk). *Let the hypotheses be as in Theorem 19. Then there exists  $\varepsilon' > 0$  independent of  $n$  such that with overwhelming probability, one has  $a + \varepsilon' \leq \lambda_i(W) \leq b - \varepsilon'$  for all  $\varepsilon p \leq i \leq (1 - \varepsilon)p$ .*

**PROOF.** From Theorem 20, we see with overwhelming probability that the number of eigenvalues in  $[a + \varepsilon', b - \varepsilon']$  is at least  $(1 - \varepsilon)p$ , if  $\varepsilon'$  is sufficiently small depending on  $\varepsilon$ . The claim follows.  $\square$

**4. Basic tools.**

**4.1. Tools from linear algebra.** In this section, we recall some basic identities and inequalities from linear algebra which will be used in this paper.

We begin with the Cauchy interlacing law and the Weyl inequalities.

**LEMMA 22** (Cauchy interlacing law). *Let  $1 \leq p \leq n$ .*

(i) *If  $A_n$  is an  $n \times n$  Hermitian matrix, and  $A_{n-1}$  is an  $(n - 1) \times (n - 1)$  minor, then  $\lambda_i(A_n) \leq \lambda_i(A_{n-1}) \leq \lambda_{i+1}(A_n)$  for all  $1 \leq i < n$ .*

(ii) *If  $M_{n,p}$  is a  $p \times n$  matrix, and  $M_{n,p-1}$  is an  $(p - 1) \times n$  minor, then  $\sigma_i(M_{n,p}) \leq \sigma_i(M_{n,p-1}) \leq \sigma_{i+1}(M_{n,p})$  for all  $1 \leq i < p$ .*

(iii) *If  $p < n$ , if  $M_{n,p}$  is a  $p \times n$  matrix, and  $M_{n-1,p}$  is a  $p \times (n - 1)$  minor, then  $\sigma_{i-1}(M_{n,p}) \leq \sigma_i(M_{n-1,p}) \leq \sigma_i(M_{n,p})$  for all  $1 \leq i \leq p$ , with the understanding that  $\sigma_0(M_{n,p}) = 0$ . [For  $p = n$ , one can also use the transpose of (ii) instead.]*

**PROOF.** Claim (i) follows from the minimax formula

$$\lambda_i(A_n) = \inf_{V: \dim(V)=i} \sup_{v \in V: \|v\|=1} v^* A_n v,$$

where  $V$  ranges over  $i$ -dimensional subspaces in  $\mathbb{C}^n$ . Similarly, (ii) and (iii) follow from the minimax formula

$$\sigma_i(M_{n,p}) = \inf_{V: \dim(V)=i+n-p} \sup_{v \in V: \|v\|=1} \|M_{n,p}v\|. \quad \square$$

**LEMMA 23** (Weyl inequality). *Let  $1 \leq p \leq n$ .*

- *If  $A, B$  are  $n \times n$  Hermitian matrices, then  $|\lambda_i(A) - \lambda_i(B)| \leq \|A - B\|_{op}$  for all  $1 \leq i \leq n$ .*
- *If  $M, N$  are  $p \times n$  matrices, then  $|\sigma_i(M) - \sigma_i(N)| \leq \|M - N\|_{op}$  for all  $1 \leq i \leq p$ .*

**PROOF.** This follows from the same minimax formulae used to establish Lemma 22.  $\square$

REMARK 24. One can also deduce the singular value versions of Lemmas 22, 23 from their Hermitian counterparts by using the augmented matrices (9). We omit the details.

We have the following elementary formula for a component of an eigenvector of a Hermitian matrix, in terms of the eigenvalues and eigenvectors of a minor.

LEMMA 25 (Formula for coordinate of an eigenvector). [7] *Let*

$$A_n = \begin{pmatrix} A_{n-1} & X \\ X^* & a \end{pmatrix}$$

*be a  $n \times n$  Hermitian matrix for some  $a \in \mathbb{R}$  and  $X \in \mathbb{C}^{n-1}$ , and let  $\begin{pmatrix} v \\ x \end{pmatrix}$  be a unit eigenvector of  $A_n$  with eigenvalue  $\lambda_i(A_n)$ , where  $x \in \mathbb{C}$  and  $v \in \mathbb{C}^{n-1}$ . Suppose that none of the eigenvalues of  $A_{n-1}$  are equal to  $\lambda_i(A_n)$ . Then*

$$|x|^2 = \frac{1}{1 + \sum_{j=1}^{n-1} (\lambda_j(A_{n-1}) - \lambda_i(A_n))^{-2} |u_j(A_{n-1})^* X|^2},$$

*where  $u_1(A_{n-1}), \dots, u_{n-1}(A_{n-1}) \in \mathbb{C}^{n-1}$  is an orthonormal eigenbasis corresponding to the eigenvalues  $\lambda_1(A_{n-1}), \dots, \lambda_{n-1}(A_{n-1})$  of  $A_{n-1}$ .*

PROOF. See, for example, [28], Lemma 41.  $\square$

This implies an analogous formula for singular vectors.

COROLLARY 26 (Formula for coordinate of a singular vector). *Let  $p, n \geq 1$ , and let*

$$M_{p,n} = \begin{pmatrix} M_{p,n-1} & X \end{pmatrix}$$

*be a  $p \times n$  matrix for some  $X \in \mathbb{C}^p$ , and let  $\begin{pmatrix} u \\ x \end{pmatrix}$  be a right unit singular vector of  $M_{p,n}$  with singular value  $\sigma_i(M_{p,n})$ , where  $x \in \mathbb{C}$  and  $u \in \mathbb{C}^{n-1}$ . Suppose that none of the singular values of  $M_{p,n-1}$  are equal to  $\sigma_i(M_{p,n})$ . Then*

$$|x|^2 = \left( 1 + \sum_{j=1}^{\min(p,n-1)} \frac{\sigma_j(M_{p,n-1})^2}{(\sigma_j(M_{p,n-1})^2 - \sigma_i(M_{p,n})^2)^2} |v_j(M_{p,n-1})^* X|^2 \right)^{-1},$$

*where  $v_1(M_{p,n-1}), \dots, v_{\min(p,n-1)}(M_{p,n-1}) \in \mathbb{C}^p$  is an orthonormal system of left singular vectors corresponding to the nontrivial singular values of  $M_{p,n-1}$ .*

*In a similar vein, if*

$$M_{p,n} = \begin{pmatrix} M_{p-1,n} \\ Y^* \end{pmatrix}$$

for some  $Y \in \mathbb{C}^n$ , and  $(v \ y)$  is a left unit singular vector of  $M_{p,n}$  with singular value  $\sigma_i(M_{p,n})$ , where  $y \in \mathbb{C}$  and  $v \in \mathbb{C}^{p-1}$ , and none of the singular values of  $M_{p-1,n}$  are equal to  $\sigma_i(M_{p,n})$ , then

$$|y|^2 = \left( 1 + \sum_{j=1}^{\min(p-1,n)} \frac{\sigma_j(M_{p-1,n})^2}{(\sigma_j(M_{p-1,n})^2 - \sigma_i(M_{p,n})^2)^2} |u_j(M_{p-1,n})^* Y|^2 \right)^{-1},$$

where  $u_1(M_{p-1,n}), \dots, u_{\min(p-1,n)}(M_{p-1,n}) \in \mathbb{C}^n$  is an orthonormal system of right singular vectors corresponding to the nontrivial singular values of  $M_{p-1,n}$ .

PROOF. We just prove the first claim, as the second is proven analogously (or by taking adjoints). Observe that  $\begin{pmatrix} u \\ x \end{pmatrix}$  is a unit eigenvector of the matrix

$$M_{p,n}^* M_{p,n} = \begin{pmatrix} M_{p,n-1}^* M_{p,n-1} & M_{p,n-1}^* X \\ X^* M_{p,n-1} & |X|^2 \end{pmatrix}$$

with eigenvalue  $\sigma_i(M_{p,n})^2$ . Applying Lemma 25, we obtain

$$|x|^2 = \left( 1 + \sum_{j=1}^{n-1} (\lambda_j(M_{p,n-1}^* M_{p,n-1}) - \sigma_i(M_{p,n})^2)^{-2} \times |u_j(M_{p,n-1}^* M_{p,n-1})^* M_{p,n-1}^* X|^2 \right)^{-1}.$$

But  $u_j(M_{p,n-1}^* M_{p,n-1})^* M_{p,n-1}^* = \sigma_j(M_{p,n-1}) v_j(M_{p,n-1})^*$  for the  $\min(p, n - 1)$  nontrivial singular values (possibly after relabeling the  $j$ ), and vanishes for trivial ones, and  $\lambda_j(M_{p,n-1}^* M_{p,n-1}) = \sigma_j(M_{p,n-1})^2$ , so the claim follows.  $\square$

The Stieltjes transform  $s(z)$  of a Hermitian matrix  $W$  is defined for complex  $z$  by the formula

$$s(z) := \frac{1}{n} \sum_{i=1}^n \frac{1}{\lambda_i(W) - z}.$$

It has the following alternate representation (see, e.g., [1], Chapter 11).

LEMMA 27. Let  $W = (\zeta_{ij})_{1 \leq i, j \leq n}$  be a Hermitian matrix, and let  $z$  be a complex number not in the spectrum of  $W$ . Then we have

$$s_n(z) = \frac{1}{n} \sum_{k=1}^n \frac{1}{\zeta_{kk} - z - a_k^*(W_k - zI)^{-1} a_k},$$

where  $W_k$  is the  $(n - 1) \times (n - 1)$  matrix with the  $k$ th row and column removed, and  $a_k \in \mathbb{C}^{n-1}$  is the  $k$ th column of  $W$  with the  $k$ th entry removed.

PROOF. By Schur’s complement,  $\frac{1}{\zeta_{kk} - z - a_k^*(W_k - zI)^{-1} a_k}$  is the  $k$ th diagonal entry of  $(W - zI)^{-1}$ . Taking traces, one obtains the claim.  $\square$

4.2. *Tools from probability theory.* We will rely frequently on the following concentration of measure result for projections of random vectors.

LEMMA 28 (Distance between a random vector and a subspace). *Let  $X = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$  be a random vector whose entries are independent with mean zero, variance 1, and are bounded in magnitude by  $K$  almost surely for some  $K$ , where  $K \geq 10(\mathbf{E}|\xi|^4 + 1)$ . Let  $H$  be a subspace of dimension  $d$  and  $\pi_H$  the orthogonal projection onto  $H$ . Then*

$$\mathbf{P}(|\|\pi_H(X)\| - \sqrt{d}| \geq t) \leq 10 \exp\left(-\frac{t^2}{10K^2}\right).$$

*In particular, one has*

$$\|\pi_H(X)\| = \sqrt{d} + O(K \log n)$$

*with overwhelming probability.*

PROOF. See [28], Lemma 43; the proof is a short application of Talagrand’s inequality [20].  $\square$

**5. Delocalization.** The purpose of this section is to establish Theorem 19 and Theorem 20. The material here is closely analogous to [28], Sections 5.2, 5.3, as well as that of the original results in [7–9] and can be read independently of the other sections of the paper. The recent paper [10] also contains arguments and results closely related to those in this section.

5.1. *Deduction of Theorem 19 from Theorem 20.* We begin by showing how Theorem 19 follows from Theorem 20. We shall just establish the claim for the right singular vectors  $u_i$ , as the claim for the left singular vectors is similar. We fix  $\varepsilon$  and allow all implied constants to depend on  $\varepsilon$  and  $y$ . We can also assume that  $K^2 \log^{20} n = o(n)$  as the claim is trivial otherwise.

As  $M$  is continuous, we see that the nontrivial singular values are almost surely simple and positive, so that the singular vectors  $u_i$  are well defined up to unit phases. Fix  $1 \leq i \leq p$ ; it suffices by the union bound and symmetry to show that the event that  $\lambda_i$  falls outside  $[a + \varepsilon, b - \varepsilon]$  or that the  $n$ th coordinate  $x$  of  $u_i$  is  $O(Kn^{-1/2} \log^{10} n)$  holds with (uniformly) overwhelming probability.

Applying Corollary 26, it suffices to show that with uniformly overwhelming probability, either  $\lambda_i \notin [a + \varepsilon, b - \varepsilon]$ , or

$$(15) \quad \sum_{j=1}^{\min(p, n-1)} \frac{\sigma_j(M_{p, n-1})^2}{(\sigma_j(M_{p, n-1})^2 - \sigma_i(M_{p, n})^2)^2} |v_j(M_{p, n-1})^* X|^2 \gg \frac{n}{K^2 \log^{20} n},$$

where  $M = (M_{p,n-1} \ X)$ . But if  $\lambda_i \in [a + \varepsilon, b - \varepsilon]$ , then by<sup>6</sup> Theorem 20, one can find (with uniformly overwhelming probability) a set  $J \subset \{1, \dots, \min(p, n - 1)\}$  with  $|J| \gg K^2 \log^{20} n$  such that  $\lambda_j(M_{p,n-1}) = \lambda_j(M_{p,n}) + O(K^2 \log^{20} n/n)$  for all  $j \in J$ ; since  $\lambda_i = \frac{1}{n} \sigma_i^2$ , we conclude that  $\sigma_j(M_{p,n-1})^2 = \sigma_j(M_{p,n})^2 + O(K^2 \log^{20} n)$ . In particular,  $\sigma_j(M_{p,n-1}) = \Theta(\sqrt{n})$ . By Pythagoras' theorem, the left-hand side of (15) is then bounded from below by

$$\gg n \frac{\|\pi_H X\|^2}{(K^2 \log^{20} n)^2},$$

where  $H \subset \mathbb{C}^p$  is the span of the  $v_j(M_{p,n-1})$  for  $j \in J$ . But from Lemma 28 (and the fact that  $X$  is independent of  $M_{p,n-1}$ ), one has

$$\|\pi_H X\|^2 \gg K^2 \log^{20} n$$

with uniformly overwhelming probability, and the claim follows.

It thus remains to establish Theorem 20.

5.2. *A crude upper bound.* Let the hypotheses be as in Theorem 20. We first establish a crude upper bound, which illustrates the techniques used to prove Theorem 20, and also plays an important direct role in that proof.

PROPOSITION 29 (Eigenvalue upper bound). *Let the hypotheses be as in Theorem 19. Then for any interval  $I \subset [a + \varepsilon, b - \varepsilon]$  of length  $|I| \geq K \log^2 n/n$ , one has with overwhelming probability (uniformly in  $I$ ) that*

$$|N_I| \ll n|I|,$$

where  $|I|$  denotes the length of  $I$ , and  $N_I$  was defined in (14).

To prove this proposition, we suppose for contradiction that

$$(16) \quad |N_I| \geq Cn|I|$$

for some large constant  $C$  to be chosen later. We will show that for  $C$  large enough, this leads to a contradiction with overwhelming probability.

We follow the standard approach (see, e.g., [1]) of controlling the eigenvalue counting function  $N_I$  via the Stieltjes transform

$$s(z) := \frac{1}{p} \sum_{j=1}^p \frac{1}{\lambda_j(W) - z}.$$

---

<sup>6</sup>In the case  $p = n$ , one would have to replace  $M_{p,n-1}$  by its transpose to return to the regime  $p \leq n$ .

Fix  $I$ . If  $x$  is the midpoint of  $I$ ,  $\eta := |I|/2$ , and  $z := x + \sqrt{-1}\eta$ , we see that

$$\operatorname{Im}s(z) \gg \frac{|N_I|}{\eta p}$$

[recall that  $p = \Theta(n)$ ] so from (16) one has

$$(17) \quad \operatorname{Im}(s(z)) \gg C.$$

Applying Lemma 27, with  $W$  replaced by the  $p \times p$  matrix  $\tilde{W} := \frac{1}{n}MM^*$  (which only has the nontrivial eigenvalues), we see that

$$(18) \quad s(z) = \frac{1}{p} \sum_{k=1}^p \frac{1}{\xi_{kk} - z - a_k^*(W_k - zI)^{-1}a_k},$$

where  $\xi_{kk}$  is the  $kk$  entry of  $\tilde{W}$ ,  $W_k$  is the  $(p-1) \times (p-1)$  matrix with the  $k$ th row and column of  $\tilde{W}$  removed, and  $a_k \in \mathbb{C}^{p-1}$  is the  $k$ th column of  $\tilde{W}$  with the  $k$ th entry removed.

Using the crude bound  $|\operatorname{Im}\frac{1}{z}| \leq \frac{1}{|\operatorname{Im}(z)|}$  and (17), one concludes

$$\frac{1}{p} \sum_{k=1}^p \frac{1}{|\eta + \operatorname{Im}a_k^*(W_k - zI)^{-1}a_k|} \gg C.$$

By the pigeonhole principle, there exists  $1 \leq k \leq p$  such that

$$(19) \quad \frac{1}{|\eta + \operatorname{Im}a_k^*(W_k - zI)^{-1}a_k|} \gg C.$$

The fact that  $k$  varies will cost us a factor of  $p$  in our failure probability estimates, but this will not be of concern since all of our claims will hold with overwhelming probability.

Fix  $k$ . Note that

$$(20) \quad a_k = \frac{1}{n}M_k X_k$$

and

$$W_k = \frac{1}{n}M_k M_k^*,$$

where  $X_k \in \mathbb{C}^n$  is the (adjoint of the)  $k$ th row of  $M$ , and  $M_k$  is the  $(p-1) \times n$  matrix formed by removing that row. Thus, if we let  $v_1(M_k), \dots, v_{p-1}(M_k) \in \mathbb{C}^{p-1}$  and  $u_1(M_k), \dots, u_{p-1}(M_k) \in \mathbb{C}^n$  be coupled orthonormal systems of left and right singular vectors of  $M_k$ , and let  $\lambda_j(W_k) = \frac{1}{n}\sigma_j(M_k)^2$  for  $1 \leq j \leq p-1$  be the associated eigenvalues, one has

$$(21) \quad a_k^*(W_k - zI)^{-1}a_k = \sum_{j=1}^{p-1} \frac{|a_k^* v_j(M_k)|^2}{\lambda_j(W_k) - z}.$$

and thus

$$\operatorname{Im} a_k^*(W_k - zI)^{-1} a_k \geq \eta \sum_{j=1}^{p-1} \frac{|a_k^* v_j(M_k)|^2}{\eta^2 + |\lambda_j(W_k) - x|^2}.$$

We conclude that

$$\sum_{j=1}^{p-1} \frac{|a_k^* v_j(M_k)|^2}{\eta^2 + |\lambda_j(W_k) - x|^2} \ll \frac{1}{C\eta}.$$

The expression  $a_k^* v_j(M_k)$  can be rewritten much more favorably using (20) as

$$(22) \quad a_k^* v_j(M_k) = \frac{\sigma_j(M_k)}{n} X_k^* u_j(M_k).$$

The advantage of this latter formulation is that the random variables  $X_k$  and  $u_j(M_k)$  are independent (for fixed  $k$ ).

Next, note that from (16) and the Cauchy interlacing law (Lemma 22) one can find an interval  $J \subset \{1, \dots, p - 1\}$  of length

$$(23) \quad |J| \gg C\eta n$$

such that  $\lambda_j(W_k) \in I$ . We conclude that

$$\sum_{j \in J} \frac{\sigma_j(M_k)^2}{n^2} |X_k^* u_j(M_k)|^2 \ll \frac{\eta}{C}.$$

Since  $\lambda_j(W_k) \in I$ , one has  $\sigma_j(M_k) = \Theta(\sqrt{n})$ , and thus

$$\sum_{j \in J} |X_k^* u_j(M_k)|^2 \ll \frac{\eta n}{C}.$$

The left-hand side can be rewritten using Pythagoras' theorem as  $\|\pi_H X_k\|^2$ , where  $H$  is the span of the eigenvectors  $u_j(M_k)$  for  $j \in J$ . But from Lemma 28 and (23), we see that this quantity is  $\gg \eta n$  with overwhelming probability, giving the desired contradiction with overwhelming probability (even after taking the union bound in  $k$ ). This concludes the proof of Proposition 29.

*5.3. Reduction to a Stieltjes transform bound.* We now begin the proof of Theorem 20 in earnest. We continue to allow all implied constants to depend on  $\varepsilon$  and  $y$ .

It suffices by a limiting argument (using Lemma 23) to establish the claim under the assumption that the distribution of  $M$  is continuous; our arguments will not use any quantitative estimates on this continuity.

The strategy is to compare  $s$  with the Marchenko–Pastur Stieltjes transform

$$s_{\text{MP},y}(z) := \int_{\mathbb{R}} \rho_{\text{MP},y}(x) \frac{1}{x - z} dx.$$

A routine application of (1) and the Cauchy integral formula yields the explicit formula

$$(24) \quad s_{\text{MP},y}(z) = -\frac{y+z-1-\sqrt{(y+z-1)^2-4yz}}{2yz},$$

where we use the branch of  $\sqrt{(y+z-1)^2-4yz}$  with cut at  $[a, b]$  that is asymptotic to  $y-z+1$  as  $z \rightarrow \infty$ . To put it another way, for  $z$  in the upper half-plane,  $s_{\text{MP},y}(z)$  is the unique solution to the equation

$$(25) \quad s_{\text{MP},y} = -\frac{1}{y+z-1+yzs_{\text{MP},y}(z)}$$

with  $\text{Im}s_{\text{MP},y}(z) > 0$ . (Details of these computations can also be found in [1].)

We have the following standard relation between convergence of Stieltjes transform and convergence of the counting function.

LEMMA 30 (Stieltjes transform controls counting function). *Let  $1/10 \geq \eta \geq 1/n$ , and  $L, \varepsilon, \delta > 0$ . Suppose that one has the bound*

$$(26) \quad |s_{\text{MP},y}(z) - s(z)| \leq \delta$$

*with overwhelming probability for each  $z$  with  $|\text{Re}(z)| \leq L$  and  $\text{Im}(z) \geq \eta$ , with the implied constants in the definition of overwhelming probability uniform in  $z$ . Then for any interval  $I$  in  $[a + \varepsilon, b - \varepsilon]$  with  $|I| \geq \max(2\eta, \frac{\eta}{8} \log \frac{1}{\delta})$ , one has*

$$\left| N_I - n \int_I \rho_{\text{MP},y}(x) dx \right| \ll \delta n |I|$$

*with overwhelming probability.*

PROOF. This follows from [28], Lemma 64; strictly speaking, that lemma was phrased for the semi-circular distribution rather than the Marchenko–Pastur distribution, but an inspection of the proof shows the proof can be modified without difficulty. See also [15] and [7], Corollary 4.2, for closely related lemmas.  $\square$

In view of this lemma, we see that to show Theorem 20, it suffices to show that for each complex number  $z$  in the region

$$\Omega := \left\{ z \in \mathbb{C} : a + \varepsilon/2 \leq \text{Re}(z) \leq b - \varepsilon/2; \text{Im}(z) \geq \eta := \frac{K^2 \log^{19} n}{n} \right\},$$

one has

$$(27) \quad s(z) - s_{\text{MP},y}(z) = o(1)$$

with (uniformly) overwhelming probability.

For this, we return to the formula (18). Inserting the identities (21), (22) into this formula, one obtains

$$(28) \quad s(z) = \frac{1}{p} \sum_{k=1}^p \frac{1}{\xi_{kk} - z - Y_k},$$

where  $Y_k = Y_k(z)$  is the quantity

$$Y_k := \sum_{j=1}^{p-1} \frac{\lambda_j(M_k)}{n} \frac{|X_k^* u_j(M_k)|^2}{\lambda_j(W_k) - z}.$$

Suppose we condition  $M_k$  (and thus  $W_k$ ) to be fixed; the entries of  $X_k$  remain independent with mean zero and variance 1, and thus (since the  $u_j$  are unit vectors)

$$\begin{aligned} \mathbf{E}(Y_k | M_k) &= \sum_{j=1}^{p-1} \frac{\lambda_j(M_k)}{n} \frac{1}{\lambda_j(W_k) - z} \\ &= \frac{p-1}{n} (1 + z s_k(z)), \end{aligned}$$

where

$$s_k(z) := \frac{1}{p-1} \sum_{j=1}^{p-1} \frac{1}{\lambda_j(W_k) - z}$$

is the Stieltjes transform of  $W_k$ .

From the Cauchy interlacing law (Lemma 22), we see that the difference

$$s(z) - \frac{p-1}{p} s_k(z) = \frac{1}{p} \left( \sum_{j=1}^p \frac{1}{\lambda_j(W) - z} - \sum_{j=1}^{p-1} \frac{1}{\lambda_j(W_k) - z} \right)$$

is bounded in magnitude by  $O(\frac{1}{p})$  times the total variation of the function  $\lambda \mapsto \frac{1}{\lambda-z}$  on  $[0, +\infty)$ , which is  $O(\frac{1}{\eta})$ . Thus,

$$\frac{p-1}{p} s_k(z) = s(z) + O\left(\frac{1}{p\eta}\right)$$

and thus

$$(29) \quad \begin{aligned} \mathbf{E}(Y_k | M_k) &= \frac{p-1}{n} + \frac{p}{n} z s(z) + O\left(\frac{1}{n\eta}\right) \\ &= y + o(1) + (y + o(1)) z s(z) \end{aligned}$$

since  $p/n = y + o(1)$  and  $1/\eta = o(n)$ .

We will shortly show a similar bound for  $Y_k$  itself.

LEMMA 31 (Concentration of  $Y_k$ ). *Let  $z \in \Omega$ . For each  $1 \leq k \leq p$ , one has  $Y_k = y + o(1) + (y + o(1))zs(z)$  with overwhelming probability (uniformly in  $k$  and  $I$ ).*

Meanwhile, we have

$$\xi_{kk} = \frac{1}{n} \|X_k\|^2$$

and hence by Lemma 28,  $\xi_{kk} = 1 + o(1)$  with overwhelming probability (again uniformly in  $k$  and  $I$ ). Inserting these bounds into (28), one obtains

$$s(z) = \frac{1}{p} \sum_{k=1}^p \frac{1}{1 - z - (y + o(1)) - (y + o(1))zs(z)}$$

with overwhelming probability; thus  $s(z)$  “almost solves” (25) in some sense. From the quadratic formula, the two solutions of (25) are  $s_{\text{MP},y}(z)$  and  $-\frac{y+z-1}{yz} - s_{\text{MP},y}(z)$ . One concludes that for each fixed  $z \in \Omega$ , it occurs with overwhelming probability that one has either

$$(30) \quad s(z) = s_{\text{MP},y}(z) + o(1)$$

or

$$(31) \quad s(z) = -\frac{y+z-1}{yz} + o(1)$$

or

$$(32) \quad s(z) = -\frac{y+z-1}{yz} - s_{\text{MP},y}(z) + o(1)$$

(with the convention that  $\frac{y+z-1}{yz} = 1$  when  $y = 1$ ). By using a  $n^{-100}$ -net of possible  $z$ 's in  $\Omega$  and using the union bound [and the fact that  $s(z)$  has a Lipschitz constant of at most  $O(n^{10})$  in  $\Omega$ ] we may assume (with overwhelming probability) that the above trichotomy holds for *all*  $z \in \Omega$ . In other words, if  $\delta > 0$  is a small number (which may depend on  $a, b, \varepsilon$ ) and  $n$  is sufficiently large depending on  $\delta$ , we may cover

$$\Omega \subset \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

where

$$\Omega_1 := \{z \in \Omega : |s(z) - s_{\text{MP},y}(z)| \leq \delta\},$$

$$\Omega_2 := \left\{z \in \Omega : \left|s(z) + \frac{y+z-1}{yz}\right| \leq \delta\right\},$$

$$\Omega_3 := \left\{z \in \Omega : \left|s(z) + \frac{y+z-1}{yz} + s_{\text{MP},y}(z)\right| \leq \delta\right\}.$$

When  $\text{Im}(z) = n^{10}$ , then  $s(z), s_{\text{MP},y}(z)$  are both  $o(1)$ , and so (for  $n$  sufficiently large) we see that  $z \in \Omega_1$  in this case. In particular,  $\Omega_1$  is empty. On the other hand,  $\Omega_1, \Omega_2, \Omega_3$  are closed subsets of  $\Omega$ . From (25), one has

$$s_{\text{MP},y}(z) \left( \frac{y+z-1}{yz} + s_{\text{MP},y}(z) \right) = -\frac{1}{yz},$$

which implies that the separation between  $s_{\text{MP},y}(z)$  from  $-\frac{y+z-1}{yz}$  is bounded from below, which implies that  $\Omega_1$  and  $\Omega_2$  are disjoint (for  $\delta$  small enough). Similarly, from (24), we see that

$$\frac{y+z-1}{yz} + 2s_{\text{MP},y}(z) = \frac{\sqrt{(y+z-1)^2 - 4yz}}{yz};$$

since  $(y+z-1)^2 - 4yz$  has zeroes only when  $z = a, b$ , and  $z$  is bounded away from these singularities, we see also that  $\Omega_1$  and  $\Omega_3$  are also disjoint.

The sets  $\Omega_1, \Omega_2 \cup \Omega_3$  are thus disjoint closed subsets of  $\Omega$ . As  $\Omega$  is connected and  $\Omega_1$  is nonempty, we conclude that  $\Omega_1 = \Omega$  (whenever  $n$  is sufficiently large depending on  $\delta$ ). Letting  $\delta \rightarrow 0$ , we conclude that (30) holds uniformly for  $z \in \Omega$  with overwhelming probability, which gives (27) and thus Theorem 20.

**6. Proof of Theorem 17 and Theorem 18.** We first prove Theorem 17. The arguments follow those in [28].

We begin by observing from Markov’s inequality and the union bound that one has  $|\zeta_{ij}|, |\zeta'_{ij}| \leq n^{10/C_0}$  (say) for all  $i, j$  with probability  $O(n^{-8})$ . Thus, by truncation (and adjusting the moments appropriately, using Lemma 23 to absorb the error), one may assume without loss of generality that

$$(33) \quad |\zeta_{ij}|, |\zeta'_{ij}| \leq n^{10/C_0}$$

almost surely for all  $i, j$ . Next, by a further approximation argument we may assume that the distribution of  $M, M'$  is continuous. This is a purely qualitative assumption, to ensure that the singular values are almost surely simple; our bounds will not depend on any quantitative measure on the continuity, and so the general case then follows by a limiting argument using Lemma 23.

The key technical step is the following theorem, whose proof is delayed to the next section.

**THEOREM 32 (Truncated Four Moment theorem).** *For sufficiently small  $c_0 > 0$  and sufficiently large  $C_0 > 0$ , the following holds for every  $0 < \varepsilon < 1$  and  $k \geq 1$ . Let  $M = (\zeta_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  and  $M' = (\zeta'_{ij})_{1 \leq i \leq p, 1 \leq j \leq n}$  be matrix ensembles obeying condition C1 for some  $C_0$ , as well as (33). Assume that  $p/n \rightarrow y$  for some  $0 < y \leq 1$ , and that  $\zeta_{ij}$  and  $\zeta'_{ij}$  match to order 4.*

*Let  $G : \mathbb{R}^k \times \mathbb{R}^k_+ \rightarrow \mathbb{R}$  be a smooth function obeying the derivative bounds*

$$(34) \quad |\nabla^j G(x_1, \dots, x_k, q_1, \dots, q_k)| \leq n^{c_0}$$

for all  $0 \leq j \leq 5$  and  $x_1, \dots, x_k \in \mathbb{R}, q_1, \dots, q_k \in \mathbb{R}$ , and such that  $G$  is supported on the region  $q_1, \dots, q_k \leq n^{c_0}$ , and the gradient  $\nabla$  is in all  $2k$  variables.

Then for any  $\varepsilon p \leq i_1 < i_2 < \dots < i_k \leq (1 - \varepsilon)p$ , and for  $n$  sufficiently large depending on  $\varepsilon, k, c_0$  we have

$$(35) \quad \left| \mathbf{E}(G(\sqrt{n}\sigma_{i_1}(M), \dots, \sqrt{n}\sigma_{i_k}(M), Q_{i_1}(M), \dots, Q_{i_k}(M))) - \mathbf{E}(G(\sqrt{n}\sigma_{i_1}(M'), \dots, \sqrt{n}\sigma_{i_k}(M'), Q_{i_1}(M'), \dots, Q_{i_k}(M'))) \right| \leq n^{-c_0}.$$

If  $\zeta_{ij}, \zeta'_{ij}$  match to order 3, then the conclusion still holds as long as one strengthens (34) to

$$(36) \quad |\nabla^j G(x_1, \dots, x_k, q_1, \dots, q_k)| \leq n^{-j c_1}$$

for some  $c_1 > 0$ , if  $c_0$  is sufficiently small depending on  $c_1$ .

Informally, Theorem 32 is a truncated version of Theorem 17 in which one has smoothly restricted attention to the event where eigenvalue gaps are not unexpectedly small.

Given a  $p \times n$  matrix  $M$  we form the augmented matrix  $\mathbf{M}$  defined in (9), whose eigenvalues are  $\pm\sigma_1(M), \dots, \pm\sigma_p(M)$ , together with the eigenvalue 0 with multiplicity  $n - p$  (if  $p < n$ ). For each  $1 \leq i \leq p$ , we introduce (in analogy with the arguments in [28]) the quantities

$$Q_i(\mathbf{M}) := \sum_{\lambda \neq \sigma_i(M)} \frac{1}{|\sqrt{n}(\lambda - \sigma_i(M))|^2} = \frac{1}{n} \left( \sum_{1 \leq j \leq p: j \neq i} \frac{1}{|\sigma_j(M) - \sigma_i(M)|^2} + \frac{n - p}{\sigma_i(M)^2} + \sum_{j=1}^p \frac{1}{|\sigma_j(M) + \sigma_i(M)|^2} \right).$$

(The factor of  $\frac{1}{n}$  in  $Q_i(\mathbf{M})$  is present to align the notation here with that in [28], in which one dilated the matrix by  $\sqrt{n}$ .) We set  $Q_i(\mathbf{M}) = \infty$  if the singular value  $\sigma_i$  is repeated, but this event occurs with probability zero since we are assuming  $M$  to be continuously distributed. One should view  $Q_i(\mathbf{M})$  as measuring the extent to which eigenvalue (or singular value) gaps near  $\sigma_i(M)$  are unexpectedly small.

The gap property on  $M$  ensures an upper bound on  $Q_i(\mathbf{M})$ .

LEMMA 33. *If  $M$  satisfies the gap property, then for any  $c_0 > 0$  (independent of  $n$ ), and any  $\varepsilon p \leq i \leq (1 - \varepsilon)p$ , one has  $Q_i(\mathbf{M}) \leq n^{c_0}$  with high probability.*

PROOF. Observe the upper bound

$$(37) \quad Q_i(\mathbf{M}) \leq \frac{2}{n} \sum_{1 \leq j \leq p: j \neq i} \frac{1}{|\sigma_j(M) - \sigma_i(M)|^2} + \frac{n - p + 1}{n\sigma_i(M)^2}.$$

From Corollary 21, we see that with overwhelming probability,  $\sigma_i(M)^2/n$  is bounded away from zero, and so  $\frac{n-p+1}{n\sigma_i(M)^2} = O(1/n)$ . To bound the other term in (37), one repeats the proof of [28], Lemma 49.  $\square$

By applying a truncation argument exactly as in [28], Section 3.3, one can now remove the hypothesis in Theorem 32 that  $G$  is supported in the region  $q_1, \dots, q_k \leq n^{c_0}$ . In particular, one can now handle the case when  $G$  is independent of  $q_1, \dots, q_k$ ; and Theorem 17 follows after making the change of variables  $\lambda = \frac{1}{n}\sigma^2$  and using the chain rule (and Corollary 21).

Next, we prove Theorem 18, assuming both Theorems 32 and 16. The main observation here is the following lemma.

**LEMMA 34 (Matching lemma).** *Let  $\zeta$  be a complex random variable with mean zero, unit variance, and third moment bounded by some constant  $a$ . Then there exists a complex random variable  $\tilde{\zeta}$  with support bounded by the ball of radius  $O_a(1)$  centered at the origin (and in particular, obeying the exponential decay hypothesis uniformly in  $\zeta$  for fixed  $a$ ) which matches  $\zeta$  to third order.*

**PROOF.** In order for  $\tilde{\zeta}$  to match  $\zeta$  to third order, it suffices that  $\tilde{\zeta}$  have mean zero, variance 1, and that  $\mathbf{E}\tilde{\zeta}^3 = \mathbf{E}\zeta^3$  and  $\mathbf{E}\tilde{\zeta}^2\bar{\tilde{\zeta}} = \mathbf{E}\zeta^3\bar{\zeta}$ .

Accordingly, let  $\Omega \subset \mathbb{C}^2$  be the set of pairs  $(\mathbf{E}\tilde{\zeta}^3, \mathbf{E}\tilde{\zeta}^2\bar{\tilde{\zeta}})$  where  $\tilde{\zeta}$  ranges over complex random variables with mean zero, variance one, and compact support. Clearly  $\Omega$  is convex. It is also invariant under the symmetry  $(z, w) \mapsto (e^{3i\theta}z, e^{i\theta}w)$  for any phase  $\theta$ . Thus, if  $(z, w) \in \Omega$ , then  $(-z, e^{i\pi/3}w) \in \Omega$ , and hence by convexity  $(0, \frac{\sqrt{3}}{2}e^{i\pi/6}w) \in \Omega$ , and hence by convexity and rotation invariance  $(0, w') \in \Omega$  whenever  $|w'| \leq \frac{\sqrt{3}}{2}w$ . Since  $(z, w)$  and  $(0, -\frac{\sqrt{3}}{2}w)$  both lie in  $\Omega$ , by convexity  $(cz, 0)$  lies in it also for some absolute constant  $c > 0$ , and so again by convexity and rotation invariance  $(z', 0) \in \Omega$  whenever  $|z'| \leq cz$ . One last application of convexity then gives  $(z'/2, w'/2) \in \Omega$  whenever  $|z'| \leq cz$  and  $|w'| \leq \frac{\sqrt{3}}{2}w$ .

It is easy to construct complex random variables with mean zero, variance one, compact support, and arbitrarily large third moment. Since the third moment is comparable to  $|z| + |w|$ , we thus conclude that  $\Omega$  contains all of  $\mathbb{C}^2$ , that is, every complex random variable with finite third moment with mean zero and unit variance can be matched to third order by a variable of compact support. An inspection of the argument shows that if the third moment is bounded by  $a$  then the support can also be bounded by  $O_a(1)$ .  $\square$

Now consider a random matrix  $M$  as in Theorem 18 with atom variables  $\zeta_{ij}$ . By the above lemma, for each  $i, j$ , we can find  $\zeta'_{ij}$  which satisfies the exponential decay hypothesis and match  $\zeta_{ij}$  to third order. Let  $\eta(q)$  be a smooth cutoff to the region  $q \leq n^{c_0}$  for some  $c_0 > 0$  independent of  $n$ , and let  $\varepsilon p \leq i \leq (1 - \varepsilon)p$ .

By Theorem 16, the matrix  $M'$  formed by the  $\zeta'_{ij}$  satisfies the gap property. By Lemma 33,

$$\mathbf{E}\eta(Q_i(M')) = 1 - O(n^{-c_1})$$

for some  $c_1 > 0$  independent of  $n$ , so by Theorem 32 one has

$$\mathbf{E}\eta(Q_i(M)) = 1 - O(n^{-c_2})$$

for some  $c_2 > 0$  independent of  $n$ . We conclude that  $M$  also obeys the gap property.

The next two sections are devoted to the proofs of Theorem 32 and Theorem 16, respectively.

**REMARK 35.** The above trick to remove the exponential decay hypothesis for Theorem 16 also works to remove the same hypothesis in [28], Theorem 19. The point is that in the analogue of Theorem 32 in that paper (implicit in [28], Section 3.3), the exponential decay hypothesis is not used anywhere in the argument; only a uniformly bounded  $C_0$  moment for  $C_0$  large enough is required, as is the case here. Because of this, one can replace all the exponential decay hypotheses in the results of [27, 28] by a hypothesis of bounded  $C_0$  moment; we omit the details.

**7. The proof of Theorem 32.** It remains to prove Theorem 32. By telescoping series, it suffices to establish a bound

$$(38) \quad \begin{aligned} & |\mathbf{E}(G(\sqrt{n}\sigma_{i_1}(M), \dots, \sqrt{n}\sigma_{i_k}(M), Q_{i_1}(M), \dots, Q_{i_k}(M))) \\ & - \mathbf{E}(G(\sqrt{n}\sigma_{i_1}(M'), \dots, \sqrt{n}\sigma_{i_k}(M'), Q_{i_1}(M'), \dots, Q_{i_k}(M'))) | \\ & \leq n^{-2-c_0} \end{aligned}$$

under the assumption that the coefficients  $\zeta_{ij}, \zeta'_{ij}$  of  $M$  and  $M'$  are identical except in one entry, say the  $qr$  entry for some  $1 \leq q \leq p$  and  $1 \leq r \leq n$ , since the claim then follows by interchanging each of the  $pn = O(n^2)$  entries of  $M$  into  $M'$  separately.

Write  $M(z)$  for the matrix  $M$  (or  $M'$ ) with the  $qr$  entry replaced by  $z$ . We apply the following proposition, which follows from a lengthy argument in [28]:

**PROPOSITION 36 (Replacement given a good configuration).** *Let the notation and assumptions be as in Theorem 32. There exists a positive constant  $C_1$  (independent of  $k$ ) such that the following holds. Let  $\varepsilon_1 > 0$ . We condition (i.e., freeze) all the entries of  $M(z)$  to be constant, except for the  $qr$  entry, which is  $z$ . We assume that for every  $1 \leq j \leq k$  and every  $|z| \leq n^{1/2+\varepsilon_1}$  whose real and imaginary parts are multiples of  $n^{-C_1}$ , we have*

- (Singular value separation) For any  $1 \leq i \leq n$  with  $|i - j| \geq n^{\varepsilon_1}$ , we have

$$(39) \quad |\sqrt{n}\sigma_i(M(z)) - \sqrt{n}\sigma_j(M(z))| \geq n^{-\varepsilon_1}|i - j|.$$

Also, we assume

$$(40) \quad \sqrt{n}\sigma_j(A(z)) \geq n^{-\varepsilon_1}n.$$

- (Delocalization at  $i_j$ ) If  $u_{i_j}(M(z)) \in \mathbb{C}^n$ ,  $v_{i_j}(M(z)) \in \mathbb{C}^p$  are unit right and left singular vectors of  $M(z)$ , then

$$(41) \quad |e_q^* v_{i_j}(M(z))|, |e_r^* u_{i_j}(M(z))| \leq n^{-1/2+\varepsilon_1}.$$

- For every  $\alpha \geq 0$

$$(42) \quad \|P_{i_j,\alpha}(M(z))e_q\|, \|P'_{i_j,\alpha}(M(z))e_r\| \leq 2^{\alpha/2}n^{-1/2+\varepsilon_1},$$

whenever  $P_{i_j,\alpha}$  (resp.,  $P'_{i_j,\alpha}$ ) is the orthogonal projection to the span of right singular vectors  $u_i(M(z))$  [resp., left singular vectors  $v_i(M(z))$ ] corresponding to singular values  $\sigma_i(A(z))$  with  $2^\alpha \leq |i - i_j| < 2^{\alpha+1}$ .

We say that  $M(0), e_q, e_r$  are a good configuration for  $i_1, \dots, i_k$  if the above properties hold. Assuming this good configuration, then we have (38) if  $\zeta_{ij}$  and  $\zeta'_{ij}$  match to order 4, or if they match to order 3 and (36) holds.

PROOF. This follows by applying [28], Proposition 46, to the  $p + n \times p + n$  Hermitian matrix  $A(z) := \sqrt{n}\mathbf{M}(z)$ , where  $\mathbf{M}(z)$  is the augmented matrix of  $M(z)$ , defined in (9). Note that the eigenvalues of  $A(z)$  are  $\pm\sqrt{n}\sigma_1(M(z)), \dots, \pm\sqrt{n}\sigma_p(M(z))$  and 0, and that the eigenvalues are given (up to unit phases) by  $\binom{v_j(M(z))}{\pm u_j(M(z))}$ . Note also that the analogue of (42) in [28], Proposition 46, is trivially true if  $2^\alpha$  is comparable to  $n$ , so one can restrict attention to the regime  $2^\alpha = o(n)$ . □

In view of the above proposition, we see that to conclude the proof of Theorem 32 (and thus Theorem 17) it suffices to show that for any  $\varepsilon_1 > 0$ , that  $M(0), e_q, e_r$  are a good configuration for  $i_1, \dots, i_k$  with overwhelming probability, if  $C_0$  is sufficiently large depending on  $\varepsilon_1$  (cf. [28], Proposition 48).

Our main tools for this are Theorem 19 and Theorem 20. Actually, we need a slight variant.

PROPOSITION 37. *The conclusions of Theorem 19 and Theorem 20 continue to hold if one replaces the  $qr$  entry of  $M$  by a deterministic number  $z = O(n^{1/2+O(1/C_0)})$ .*

This is proven exactly as in [28], Corollary 63, and is omitted.

We return to the task of establishing a good configuration with overwhelming probability. By the union bound, we may fix  $1 \leq j \leq k$ , and also fix the  $|z| \leq n^{1/2+\varepsilon_1}$  whose real and imaginary parts are multiples of  $n^{-C_1}$ . By the union bound again and Proposition 37, the eigenvalue separation condition (39) holds with overwhelming probability for every  $1 \leq i \leq n$  with  $|i - j| \geq n^{\varepsilon_1}$  (if  $C_0$  is sufficiently large), as does (41). A similar argument using Pythagoras' theorem and Corollary 21 gives (42) with overwhelming probability [noting as before that we may restrict attention to the regime  $2^\alpha = o(n)$ ]. Corollary 21 also gives (40) with overwhelming probability. This gives the claim, and Theorem 17 follows.

**8. Proof of Theorem 16.** We now prove Theorem 16, closely following the analogous arguments in [28]. Using the exponential decay condition, we may truncate the  $\zeta_{ij}$  (and renormalise moments, using Lemma 23) to assume that

$$(43) \quad |\zeta_{ij}| \leq \log^{O(1)} n$$

almost surely. By a limiting argument, we may assume that  $M$  has a continuous distribution, so that the singular values are almost surely simple.

We write  $i_0$  instead of  $i$ ,  $p_0$  instead of  $p$ , and write  $N_0 := p_0 + n$ . As in [28], the strategy is to propagate a narrow gap for  $M = M_{p_0,n}$  backwards in the  $p$  variable, until one can use Theorem 20 to show that the gap occurs with small probability.

More precisely, for any  $1 \leq i - l < i \leq p \leq p_0$ , we let  $M_{p,n}$  be the  $p \times n$  matrix formed using the first  $p$  rows of  $M_{p_0,n}$ , and we define (following [28]) the *regularized gap*

$$(44) \quad g_{i,l,p} := \inf_{1 \leq i_- \leq i-l < i \leq i_+ \leq p} \frac{\sqrt{N_0} \sigma_{i_+}(M_{p,n}) - \sqrt{N_0} \sigma_{i_-}(M_{p,n})}{\min(i_+ - i_-, \log^{C_1} N_0)^{\log^{0.9} N_0}},$$

where  $C_1 > 1$  is a large constant to be chosen later. It will suffice to show that

$$(45) \quad g_{i_0,1,p_0} \leq n^{-c_0}.$$

The main tool for this is the following lemma.

**LEMMA 38 (Backwards propagation of gap).** *Suppose that  $p_0/2 \leq p < p_0$  and  $l \leq \varepsilon p/10$  is such that*

$$(46) \quad g_{i_0,l,p+1} \leq \delta$$

for some  $0 < \delta \leq 1$  (which can depend on  $n$ ), and that

$$(47) \quad g_{i_0,l+1,p} \geq 2^m g_{i_0,l,p+1}$$

for some  $m \geq 0$  with

$$(48) \quad 2^m \leq \delta^{-1/2}.$$

Let  $X_{p+1}$  be the  $(p + 1)$ th row of  $M_{p_0,n}$ , and let  $u_1(M_{p,n}), \dots, u_p(M_{p,n})$  be an orthonormal system of right singular vectors of  $M_{p,n}$  associated to  $\sigma_1(M_{p,n}), \dots, \sigma_p(M_{p,n})$ . Then one of the following statements hold:

(i) (Macroscopic spectral concentration) There exists  $1 \leq i_- < i_+ \leq p + 1$  with  $i_+ - i_- \geq \log^{C_1/2} n$  such that  $|\sqrt{n} \sigma_{i_+}(M_{p+1,n}) - \sqrt{n} \sigma_{i_-}(M_{p+1,n})| \leq \delta^{1/4} \exp(\log^{0.95} n)(i_+ - i_-)$ .

(ii) (Small inner products) There exists  $\varepsilon p/2 \leq i_- \leq i_0 - l < i_0 \leq i_+ \leq (1 - \varepsilon/2)p$  with  $i_+ - i_- \leq \log^{C_1/2} n$  such that

$$(49) \quad \sum_{i_- \leq j < i_+} |X_{p+1}^* u_j(M_{p,n})|^2 \leq \frac{i_+ - i_-}{2^{m/2} \log^{0.01} n}.$$

(iii) (Large singular value) For some  $1 \leq i \leq p + 1$ , one has

$$|\sigma_i(M_{p+1,n})| \geq \frac{\sqrt{n} \exp(-\log^{0.95} n)}{\delta^{1/2}}.$$

(iv) (Large inner product in bulk) There exists  $\varepsilon p/10 \leq i \leq (1 - \varepsilon/10)p$  such that

$$|X_{p+1}^* u_i(M_{p,n})|^2 \geq \frac{\exp(-\log^{0.96} n)}{\delta^{1/2}}.$$

(v) (Large row) We have

$$\|X_{p+1}\|^2 \geq \frac{n \exp(-\log^{0.96} n)}{\delta^{1/2}}.$$

(vi) (Large inner product near  $i_0$ ) There exists  $\varepsilon p/10 \leq i \leq (1 - \varepsilon/10)p$  with  $|i - i_0| \leq \log^{C_1} n$  such that

$$|X_{p+1}^* u_i(M_{p,n})|^2 \geq 2^{m/2} n \log^{0.8} n.$$

PROOF. This follows by applying<sup>7</sup> [28], Lemma 51, to the  $p + n + 1 \times p + n + 1$  Hermitian matrix

$$A_{p+n+1} := \sqrt{n} \begin{pmatrix} 0 & M_{p+1,n}^* \\ M_{p+1,n} & 0 \end{pmatrix},$$

which after removing the bottom row and rightmost column (which is  $X_{p+1}$ , plus  $p + 1$  zeroes) yields the  $p + n \times p + n$  Hermitian matrix

$$A_{p+n} := \sqrt{n} \begin{pmatrix} 0 & M_{p,n}^* \\ M_{p,n} & 0 \end{pmatrix}$$

which has eigenvalues  $\pm\sqrt{n}\sigma_1(M_{p,n}), \dots, \pm\sqrt{n}\sigma_p(M_{p,n})$  and 0, and an orthonormal eigenbasis that includes the vectors  $\begin{pmatrix} u_j(M_{p,n}) \\ v_j(M_{p,n}) \end{pmatrix}$  for  $1 \leq j \leq p$ . (The “large coefficient” event in [28], Lemma 51(iii), cannot occur here, as  $A_{p+n+1}$  has zero diagonal.)  $\square$

By repeating the arguments in [28], Section 3.5, almost verbatim, it then suffices to show the following proposition.

PROPOSITION 39 (Bad events are rare). *Suppose that  $p_0/2 \leq p < p_0$  and  $l \leq \varepsilon p/10$ , and set  $\delta := n_0^{-\kappa}$  for some sufficiently small fixed  $\kappa > 0$ . Then:*

(a) *The events (i), (iii), (iv), (v) in Lemma 38 all fail with high probability.*

---

<sup>7</sup>Strictly speaking, there are some harmless adjustments by constant factors that need to be made to this lemma, ultimately coming from the fact that  $n, p, n + p$  are only comparable up to constants, rather than equal, but these adjustments make only a negligible change to the proof of that lemma.

(b) *There is a constant  $C'$  such that all the coefficients of the right singular vectors  $u_j(M_{p,n})$  for  $\varepsilon p/2 \leq j \leq (1 - \varepsilon/2)p$  are of magnitude at most  $n^{-1/2} \log^{C'} n$  with overwhelming probability. Conditioning  $M_{p,n}$  to be a matrix with this property, the events (ii) and (vi) occur with a conditional probability of at most  $2^{-\kappa m} + n^{-\kappa}$ .*

(c) *Furthermore, there is a constant  $C_2$  (depending on  $C', \kappa, C_1$ ) such that if  $l \geq C_2$  and  $M_{p,n}$  is conditioned as in (b), then (ii) and (vi) in fact occur with a conditional probability of at most  $2^{-\kappa m} \log^{-2C_1} n + n^{-\kappa}$ .*

But Proposition 39 can be proven by repeating the proof of [28], Proposition 53, with only cosmetic changes, the only significant difference being that Theorem 20 and Theorem 19 are applied instead of [28], Theorem 60, and [28], Proposition 62, respectively.

**Acknowledgments.** We thank Horng-Tzer Yau for references, and the anonymous referee for helpful comments.

## REFERENCES

- [1] BAI, Z. D. and SILVERSTEIN, J. (2006). *Spectral Analysis of Large Dimensional Random Matrices. Mathematics Monograph Series 2*. Science Press, Beijing.
- [2] BEN AROUS, G. and PÉCHÉ, S. (2005). Universality of local eigenvalue statistics for some sample covariance matrices. *Comm. Pure Appl. Math.* **58** 1316–1357. [MR2162782](#)
- [3] EDELMAN, A. (1988). Eigenvalues and condition numbers of random matrices. *SIAM J. Matrix Anal. Appl.* **9** 543–560. [MR0964668](#)
- [4] ERDŐS, L., PÉCHÉ, S., RAMÍREZ, J. A., SCHLEIN, B. and YAU, H.-T. (2010). Bulk universality for Wigner matrices. *Comm. Pure Appl. Math.* **63** 895–925. [MR2662426](#)
- [5] ERDŐS, L., RAMÍREZ, J., SCHLEIN, B., TAO, T., VU, V. and YAU, H.-T. (2010). Bulk universality for Wigner Hermitian matrices with subexponential decay. *Math. Res. Lett.* **17** 667–674. [MR2661171](#)
- [6] ERDŐS, L., RAMÍREZ, J. A., SCHLEIN, B. and YAU, H.-T. (2010). Universality of sine-kernel for Wigner matrices with a small Gaussian perturbation. *Electron. J. Probab.* **15** 526–603. [MR2639734](#)
- [7] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2009). Semicircle law on short scales and delocalization of eigenvectors for Wigner random matrices. *Ann. Probab.* **37** 815–852. [MR2537522](#)
- [8] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2009). Local semicircle law and complete delocalization for Wigner random matrices. *Comm. Math. Phys.* **287** 641–655. [MR2481753](#)
- [9] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2010). Wegner estimate and level repulsion for Wigner random matrices. *Int. Math. Res. Not. IMRN* **3** 436–479. [MR2587574](#)
- [10] ERDŐS, L., SCHLEIN, B., YAU, H. T. and YIN, J. (2009). The local relaxation flow approach to universality of the local statistics for random matrices. Available at [arXiv:0911.3687](#).
- [11] ERDŐS, L., YAU, H. T. and YIN, J. (2010). Bulk universality for generalized Wigner matrices. Preprint. Available at [arXiv:1001.3453](#).
- [12] ERDŐS, L., YAU, H. T. and YIN, J. (2010). Rigidity of eigenvalues of generalized Wigner matrices. Preprint. Available at [arXiv:1007.4652](#).
- [13] FELDHEIM, O. N. and SODIN, S. (2010). A universality result for the smallest eigenvalues of certain sample covariance matrices. *Geom. Funct. Anal.* **20** 88–123. [MR2647136](#)

- [14] FORRESTER, P. J. (1993). The spectrum edge of random matrix ensembles. *Nuclear Phys. B* **402** 709–728. [MR1236195](#)
- [15] GÖTZE, F. and TIKHOMIROV, A. (2003). Rate of convergence to the semi-circular law. *Probab. Theory Related Fields* **127** 228–276. [MR2013983](#)
- [16] GÖTZE, F. and TIKHOMIROV, A. (2004). Rate of convergence in probability to the Marchenko–Pastur law. *Bernoulli* **10** 503–548. [MR2061442](#)
- [17] GUIONNET, A. and ZEITOUNI, O. (2000). Concentration of the spectral measure for large matrices. *Electron. Commun. Probab.* **5** 119–136 (electronic). [MR1781846](#)
- [18] GUSTAVSSON, J. (2005). Gaussian fluctuations of eigenvalues in the GUE. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** 151–178. [MR2124079](#)
- [19] JOHANSSON, K. (2001). Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices. *Comm. Math. Phys.* **215** 683–705. [MR1810949](#)
- [20] LEDOUX, M. (2001). *The Concentration of Measure Phenomenon. Mathematical Surveys and Monographs* **89**. Amer. Math. Soc., Providence, RI. [MR1849347](#)
- [21] MARČENKO, V. A. and PASTUR, L. A. (1967). Distribution of eigenvalues in certain sets of random matrices. *Mat. Sb. (N.S.)* **72 (114)** 507–536. [MR0208649](#)
- [22] NAGAO, T. and WADATI, M. (1991). Correlation functions of random matrix ensembles related to classical orthogonal polynomials. *J. Phys. Soc. Japan* **60** 3298–3322. [MR1142971](#)
- [23] PASTUR, L. A. (1973). Spectra of random selfadjoint operators. *Russian Math. Surveys* **28** 1–67.
- [24] PÉCHÉ, S. (2009). Universality in the bulk of the spectrum for complex sample covariance matrices. Preprint. Available at [arXiv:0912.2493](#).
- [25] SOSHNIKOV, A. (2002). A note on universality of the distribution of the largest eigenvalues in certain sample covariance matrices. *J. Stat. Phys.* **108** 1033–1056. [MR1933444](#)
- [26] TAO, T. and VU, V. (2010). Random matrices: The distribution of the smallest singular values. *Geom. Funct. Anal.* **20** 260–297. [MR2647142](#)
- [27] TAO, T. and VU, V. (2010). Random matrices: Universality of local eigenvalue statistics up to the edge. *Comm. Math. Phys.* **298** 549–572. [MR2669449](#)
- [28] TAO, T. and VU, V. (2011). Random matrices: Universality of local eigenvalue statistics. *Acta Math.* **206** 127–204. [MR2784665](#)
- [29] WACHTER, K. W. (1978). The strong limits of random matrix spectra for sample matrices of independent elements. *Ann. Probab.* **6** 1–18. [MR0467894](#)
- [30] YIN, Y. Q. (1986). Limiting spectral distribution for a class of random matrices. *J. Multivariate Anal.* **20** 50–68. [MR0862241](#)

DEPARTMENT OF MATHEMATICS  
UCLA  
LOS ANGELES, CALIFORNIA 90095-1555  
USA  
E-MAIL: [tao@math.ucla.edu](mailto:tao@math.ucla.edu)

DEPARTMENT OF MATHEMATICS  
RUTGERS UNIVERSITY  
PISCATAWAY, NEW JERSEY 08854  
USA  
E-MAIL: [vanvu@math.rutgers.edu](mailto:vanvu@math.rutgers.edu)