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CROSSING RANDOM WALKS AND STRETCHED POLYMERS AT WEAK DISORDER

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We consider a model of a polymer in $\mathbb{Z}^{d+1}$, constrained to join 0 and a hyperplane at distance N. The polymer is subject to a quenched nonnegative random environment. Alternatively, the model describes crossing random walks in a random potential (see Zerner [*Ann Appl. Probab.* **8** (1998) 246–280] or Chapter 5 of Sznitman [*Brownian Motion, Obstacles and Random Media* (1998) Springer] for the original Brownian motion formulation). It was recently shown [*Ann. Probab.* **36** (2008) 1528–1583; *Probab. Theory Related Fields* **143** (2009) 615–642] that, in such a setting, the quenched and annealed free energies coincide in the limit $N \to \infty$, when $d \geq 3$ and the temperature is sufficiently high. We first strengthen this result by proving that, under somewhat weaker assumptions on the distribution of disorder which, in particular, enable a small probability of traps, the ratio of quenched and annealed partition functions actually converges. We then conclude that, in this case, the polymer obeys a diffusive scaling, with the same diffusivity constant as the annealed model.

1. Notation and results. For simplicity[3] we shall consider stretched polymers which are represented by nearest-neighbor paths on $\mathbb{Z}^{d+1}$. Due to the presence of a preferred direction, it is convenient to decompose $\mathsf{x} \in \mathbb{Z}^{d+1}$ into transverse and longitudinal parts: $\mathsf{x} = (\mathsf{x}^{\perp}, \mathsf{x}^{\parallel})$ with $\mathsf{x}^{\perp} \in \mathbb{Z}^d$ and $\mathsf{x}^{\parallel} \in \mathbb{Z}$. Given $N \in \mathbb{N}$, we define

$$\mathcal{H}_N^- \stackrel{\Delta}{=} \{\mathsf{x} \in \mathbb{Z}^{d+1} : \mathsf{x}^{\parallel} < N\}$$

and its outer vertex boundary $\mathcal{L}_N \stackrel{\Delta}{=} \partial \mathcal{H}_N^-$. We shall consider the family $\mathcal{D}_N$ of nearest-neighbor paths from the origin 0 to $\mathcal{L}_N$. The name *stretched* stipulates that although the second endpoint of $\gamma \in \mathcal{D}_N$ is constrained to lie on $\mathcal{L}_N$, there are no other restrictions on the geometry of polymers, which can bend and self-intersect. In the Brownian version of this problem [9], an alternative designation often used in the literature is *crossing Brownian motion*.

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[3]We emphasize that our techniques can deal in the same way with any finite-range step distribution. Similarly, the particular geometric setting used, with the arrival hyperplane orthogonal to some lattice direction, can easily be generalized.

The weight $W^{\omega}_{\lambda,\beta}(\gamma)$ of a polymer $\gamma = (\gamma(0), \dots, \gamma(n)) \in \mathcal{D}_N$ is given by

$$W^{\omega}_{\lambda,\beta}(\gamma) \stackrel{\Delta}{=} \exp\left\{-\lambda n - \beta \sum_{l=1}^{n} V^{\omega}(\gamma(l))\right\}. \tag{1.1}$$

Here $\lambda > \lambda_0 \stackrel{\Delta}{=} \log(2d+2)$, $\beta > 0$ and the random environment $\{V^{\omega}(x)\}_{x \in \mathbb{Z}^{d+1}}$, $\omega \in \Omega$, is assumed to be i.i.d., $V^{\omega}(x) \stackrel{\mathrm{d}}{\sim} V$, and such that:

ASSUMPTION (A). $0 \in \mathrm{supp}(V) \subseteq [0, \infty]$ and $p \stackrel{\Delta}{=} \mathbb{P}(V = \infty)$ is sufficiently small.

That the potential V be bounded below is essential, since it guarantees ballistic behavior (spatial extension) of stretched polymers.

The condition on the smallness of p is also essential, since it guarantees that we never meet situations when $\{\mathsf{x} : V^{\omega}(\mathsf{x}) < \infty\}$ does not percolate. On the other hand, the condition $\inf \mathrm{supp}(V) = 0$ is just a normalization.

The corresponding quenched and annealed partition functions are defined as

$$\mathfrak{D}^{\omega}_N = \mathfrak{D}^{\omega}_N(\lambda, \beta) \stackrel{\Delta}{=} \sum_{\gamma \in \mathcal{D}_N} W^{\omega}_{\lambda,\beta}(\gamma) \quad \text{and} \quad \mathbf{D}_N \stackrel{\Delta}{=} \mathbb{E}\mathfrak{D}^{\omega}_N.$$

Note that the annealed potential is always attractive: For any pair of paths γ_1 and γ_2,

$$\mathbb{E}(W^{\omega}_{\lambda,\beta}(\gamma_1) W^{\omega}_{\lambda,\beta}(\gamma_2)) \geq \mathbb{E}(W^{\omega}_{\lambda,\beta}(\gamma_1))\mathbb{E}(W^{\omega}_{\lambda,\beta}(\gamma_2)). \tag{1.2}$$

(This can be most easily deduced from the fact that decreasing functions on $\mathbb{R}$ are always positively correlated.)

It has recently been proved by Flury [5] (under the additional assumption that $\mathbb{E}V^{d+1} < \infty$), and then reproved by Zygouras [11] [for arbitrary directions, under the additional assumption that $\mathrm{supp}(V)$ be bounded] that, in four and higher dimensions (i.e., for $d \geq 3$ in our notation) and for any $\lambda > \lambda_0$, the annealed and quenched free energies coincide when β is small enough. Namely, for all β sufficiently small, there exists $\xi = \xi(\lambda, \beta) > 0$ such that

$$-\lim_{N \to \infty} \frac{1}{N} \log \mathfrak{D}^{\omega}_N = \xi = -\lim_{N \to \infty} \frac{1}{N} \log \mathbf{D}_N. \tag{1.3}$$

This is an important result: In sharp contrast with models of directed polymers, the model of stretched polymers does not have an immediate underlying martingale structure, and this makes it necessary to find different (and arguably more intrinsic) approaches to its analysis. The condition $\mathbb{E}V^{d+1} < \infty$, under which (1.3) was derived, is inherited from [10], where it was shown to be sufficient to guarantee the existence of the quenched free energy, that is, the left-most limit in (1.3).

In the sequel, we shall prove the following sharp version of (1.3): Let $\mathrm{Cl}_{\infty}(V)$ be the (unique) infinite connected cluster of sites x with $V(\mathsf{x}) < \infty$. Under Assumption (A), such a cluster $\mathbb{P}$-a.s. exists and is unique.

THEOREM A. *Let $d \geq 3$. Then, for every $\lambda > \lambda_0$, there exists $\beta_0 = \beta_0(\lambda, d)$ and $p_\infty > 0$, such that, if Assumption (A) holds with $p \leq p_\infty$, then, for every $\beta \in [0, \beta_0)$, the limit*

$$\mathfrak{d}^\omega \triangleq \lim_{N\to\infty} \frac{\mathfrak{D}_N^\omega}{\mathbf{D}_N} \tag{1.4}$$

exists $\mathbb{P}$-a.s. and in $L^2(\Omega)$. In particular, the quenched free energy $-\lim_{N\to\infty} \frac{1}{N} \times \log \mathfrak{D}_N^\omega$ is well defined, and (1.3) holds. Furthermore, $\mathfrak{d}^\omega > 0$ $\mathbb{P}$-a.s. on the event $\{0 \in \mathrm{Cl}_\infty(V)\}$.

Our work was inspired by [5, 11]; however, our proof of Theorem A does not rely on the results therein. In particular, in addition to strengthening their conclusion, Theorem A lifts some of the restrictions imposed on the potential V in these works. In fact, under our assumptions, which do not impose any moment conditions on the distribution of V and even enable a small probability of traps, the existence of the quenched free energy needs a justification: as we have already mentioned, the corresponding existence results in [10], which is a reference work for both [5] and [11], have been established under the additional assumption $\mathbb{E}V^{d+1} < \infty$.

Our second result confirms the prediction that stretched polymers should be diffusive at weak disorder: On the event $0 \in \mathrm{Cl}_\infty(V)$, the random weights (1.1) induce a (random) probability distribution μ_N^ω on $\mathcal{D}_N$. For a polymer $\gamma = (\gamma(0), \ldots, \gamma(n)) \in \mathcal{D}_N$, we define $\pi^\perp(\gamma)$ as the ($\mathbb{Z}^d$-valued) transverse component of its endpoint, and $\pi^\parallel(\gamma) = N$ as its longitudinal component, so that $\gamma(n) = (\pi^\perp(\gamma), \pi^\parallel(\gamma))$.

THEOREM B. *Let $d \geq 3$. Then, for every $\lambda > \lambda_0$, there exist $\hat{\beta}_0 = \hat{\beta}_0(\lambda, d)$ and $\hat{p}_\infty > 0$ such that, if Assumption (A) holds with $p \leq \hat{p}_\infty$, then, for every $\beta \in [0, \hat{\beta}_0)$, the distribution of $\pi^\perp$ displays diffusive scaling with a nonrandom nondegenerate diffusivity matrix Σ and, accordingly, a positive diffusivity constant $\sigma^2 = \sigma^2(\beta, \lambda) \triangleq \mathsf{Tr}(\Sigma) > 0$. Namely, define $\mathbb{P}^*(\cdot) \triangleq \mathbb{P}(\cdot | 0 \in \mathrm{Cl}_\infty(V))$. Then,*

$$\mathbb{P}^*\text{-}\lim_{N\to\infty} \mu_N^\omega\left(\frac{|\pi^\perp(\gamma)|^2}{N}\right) = \sigma^2, \tag{1.5}$$

where $\mathbb{P}^*$-$\lim$ denotes convergence in $\mathbb{P}^*$-probability. Furthermore, for any bounded continuous function f on $\mathbb{R}^d$,

$$\begin{aligned} &\mathbb{P}^*\text{-}\lim_{N\to\infty} \sum_{\mathsf{x}\in\mathbb{Z}^d} \mu_N^\omega(\pi^\perp(\gamma) = \mathsf{x}) f\left(\frac{\mathsf{x}}{\sqrt{N}}\right) \\ &\quad = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \int_{\mathbb{R}^d} f(\mathsf{x}) e^{-1/2(\Sigma^{-1}\mathsf{x},\mathsf{x})}\, d\mathsf{x}. \end{aligned} \tag{1.6}$$

Σ and σ^2 above are precisely the diffusion matrix and the diffusivity constant of the corresponding annealed polymer model; see (2.11) below.

We expect both (1.5) and (1.6) to hold not only in $\mathbb{P}^*$-probability, but also in $L^2(\Omega)$ and $\mathbb{P}^*$-a.s.

1.1. *Some open problems.* In this subsection, we briefly discuss some points that are left untouched in the present work.

Stronger modes of convergence. As already mentioned above, we expect our diffusivity results to hold also a.s. in the environment and in $L^2(\Omega)$. Such results are known in the directed case, as a consequence of the much simpler martingale structure [1]. Furthermore, we expect the $\mathbb{P}^*$-a.s. validity of a local CLT, or equivalently, of a (random) Ornstein–Zernike-type formula for long-range quenched connections; see the discussion at the end of Section 3.5.

Invariance principle. Once equipped with a local CLT and thanks to our good control on the path geometry, it should be mostly straightforward to obtain a full invariance principle for the path.

"Real" stretched polymer. In the present work, we have focused on ensembles of paths of "point-to-plane" type (the set $\mathcal{D}_N$). It would be physically quite interesting to analyze also the case of fixed-length polymers, stretched by an external force (notice that in the directed case there is no difference between "point-to-plane" and "fixed-length" scenarios); in particular, it would be interesting to obtain a local limit theorem for the free endpoint. Such questions have been investigated in the annealed setting in our previous work [6]. In the quenched setting coincidence of Lyapunov exponents (under the additional $\mathbb{E}V^{d+1} < \infty$ assumption) has been established in [5].

Nonperturbative proof. Our results are only valid at very high temperatures. It would be quite interesting (and probably challenging) to push them to the full weak-disorder regime. Results of that type have been obtained in the directed case [4].

Strong disorder. We only consider the weak disorder case here. Obtaining some information on the behavior of typical paths in the strong disorder regime would also be quite interesting, and is the subject of some work in progress. See [3] for such results, in the full strong disorder regime, in the directed case.

1.2. *A remark on notational conventions.* Given two sequences $\{a_n(w)\}$ and $\{b_n(w)\}$ of positive real numbers indexed by w from some set of parameters $\mathfrak{W}_n$, we say that $a_n(w) \lesssim b_n(w)$, if

$$\limsup_{n\to\infty} \frac{a_n(w)}{b_n(w)} < \infty,$$

uniformly in $w \in \mathfrak{W}_n$.

Given $\mathsf{z}, \mathsf{w} \in \mathbb{C}^{d+1}$, we use

$$(\mathsf{z}, \mathsf{w})_{d+1} \triangleq \sum_{i=1}^{d+1} \mathsf{z}_i \bar{\mathsf{w}}_i \quad \text{and} \quad (\mathsf{z}, \mathsf{w})_d \triangleq \sum_{i=1}^{d} \mathsf{z}_i \bar{\mathsf{w}}_i .$$

With a slight abuse of notation, we shall also write $(z, \mathsf{w})_d$ for the same expression with $z \in \mathbb{C}^d$.

2. Convergence of partition functions.

2.1. *Irreducible decomposition of paths $\gamma \in \mathcal{D}_N$.* Given $\delta > 0$, we define a positive cone along the $\vec{\mathsf{e}} \triangleq \vec{\mathsf{e}}_{d+1}$-direction by

$$\mathcal{Y}_\delta \triangleq \{\mathsf{x} \in \mathbb{R}^{d+1} : \|\mathsf{x}^\perp\| < \delta x^\parallel\},$$

where $\|\cdot\|$ denotes the Euclidean norm. We say that a trajectory $\gamma = (\gamma(0), \dots, \gamma(n))$ of length $|\gamma| = n$ is *cone-confined* if

$$\gamma \subseteq (\gamma(0) + \mathcal{Y}_\delta) \cap (\gamma(n) - \mathcal{Y}_\delta).$$

Although paths $\gamma \in \mathcal{D}_N$ always satisfy $0 = (\gamma(0), \vec{\mathsf{e}})_{d+1} < (\gamma(n), \vec{\mathsf{e}})_{d+1}$, evidently not all of them are cone-confined. For $1 \le k < n = |\gamma|$, let us say that $\gamma(k)$ is a *cone-point* of γ if

$$(\gamma(0), \vec{\mathsf{e}})_{d+1} < (\gamma(k), \vec{\mathsf{e}})_{d+1} < (\gamma(n), \vec{\mathsf{e}})_{d+1},$$

and, in addition, if

$$\gamma \subseteq (\gamma(k) - \mathcal{Y}_\delta) \cup (\gamma(k) + \mathcal{Y}_\delta).$$

We say that a trajectory γ is *irreducible* if it contains less than two cone-points. We say that it is *strongly irreducible* if it does not contain cone-points at all.

The following mass-separation property of irreducible trajectories, proved in [6], is crucial to our analysis: There exists $\nu > 0$ such that, for all N large enough,

$$\frac{1}{\mathbf{D}_N} \sum_{\substack{\gamma \in \mathcal{D}_N \\ \text{irreducible}}} \mathbb{E} W^{\omega}_{\lambda,\beta}(\gamma) \le e^{-\nu N}. \tag{2.1}$$

On the other hand, reducible trajectories are unambiguously represented as concatenation of strongly irreducible pieces (as induced by the collection of all the cone-points of γ; see Figure 1),

$$\gamma = \gamma_l \cup \gamma_1 \cup \cdots \cup \gamma_n \cup \gamma_r . \tag{2.2}$$

By construction, $\gamma_1, \dots, \gamma_n$ above are also cone-confined, and so is their concatenation $\gamma_1 \cup \cdots \cup \gamma_n$. Thus, (2.1) and (2.2) suggest that the asymptotics of $\mathbf{D}_N$

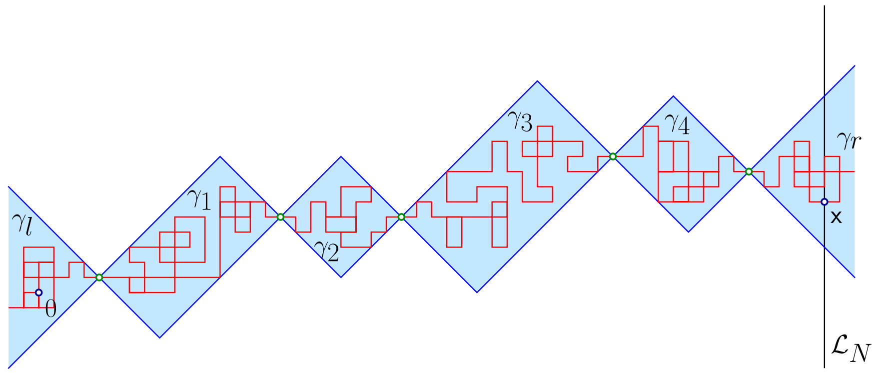

FIG. 1. *The decomposition of a path $\gamma \in \mathcal{D}_N$ into a concatenation of strongly irreducible pieces.*

and $\mathfrak{D}_N^\omega$ should be closely related to the asymptotics of the corresponding cone-confined quantities. This intuition turns out to be correct.

Let $\mathcal{T}_N$ be the family of all cone-confined trajectories from 0 to $\mathcal{L}_N$. Set

$$\mathfrak{T}_N^\omega(\lambda, \beta) \triangleq \sum_{\gamma \in \mathcal{T}_N} W_{\lambda,\beta}^\omega(\gamma) \quad \text{and} \quad \mathbf{T}_N \triangleq \mathbb{E}\mathfrak{T}_N^\omega.$$

The following statement as well as the understanding one needs to develop for its proof are crucial: In the notation and under the conditions of Theorem A, for every $\beta \in [0, \beta_0)$, the limit

$$\lim_{N\to\infty} \frac{\mathfrak{T}_N^\omega}{\mathbf{T}_N} \tag{2.3}$$

exists $\mathbb{P}$-a.s. and in $L^2(\Omega)$. For a while we shall focus on the ensembles of cone-confined trajectories and on proving (2.3). We shall return to $\mathcal{D}_N$ and prove the full statement (1.4) only in Section 2.7.

Notation for scaled quantities. Recall the definition of the Lyapunov exponent ξ in (1.3). Given $N \geq 1$ and $\gamma \in \mathcal{T}_N$, we define the scaled random path weights

$$w_{\lambda,\beta}^\omega(\gamma) \triangleq e^{N\xi} W_{\lambda,\beta}^\omega(\gamma).$$

For $\mathsf{x} \in \mathcal{L}_N$, we define

$$\mathsf{t}_\mathsf{x}^\omega \triangleq \sum_{\substack{\gamma:0\to\mathsf{x}\\ \gamma\in\mathcal{T}_N}} w_{\lambda,\beta}^\omega(\gamma), \qquad \mathsf{q}_\mathsf{x}^\omega \triangleq \sum_{\substack{\gamma:0\to\mathsf{x}\\ \gamma\in\mathcal{T}_N^0}} w_{\lambda,\beta}^\omega(\gamma) \quad \text{and}$$
$$\mathbf{t}_\mathsf{x} \triangleq \mathbb{E}\mathsf{t}_\mathsf{x}^\omega, \qquad \mathbf{q}_\mathsf{x} \triangleq \mathbb{E}\mathsf{q}_\mathsf{x}^\omega, \tag{2.4}$$

where $\mathcal{T}_N^0$ denotes the set of all strongly irreducible $\gamma \in \mathcal{T}_N$. Similarly, we define

$$\mathsf{t}_N^\omega \triangleq \sum_{\mathsf{x}\in\mathcal{L}_N} \mathsf{t}_\mathsf{x}^\omega, \qquad \mathsf{q}_N^\omega \triangleq \sum_{\mathsf{x}\in\mathcal{L}_N} \mathsf{q}_\mathsf{x}^\omega \quad \text{and} \quad \mathbf{t}_N \triangleq \mathbb{E}\mathsf{t}_N^\omega, \qquad \mathbf{q}_N \triangleq \mathbb{E}\mathsf{q}_N^\omega.$$

We also set $\mathfrak{t}_0^{\omega} = \mathbf{t}_0 \stackrel{\Delta}{=} 1$.

2.2. *Renewal analysis of annealed partition function* $\mathbf{T}_N$. With the above notation, the sequence $\{\mathbf{t}_N\}$ satisfies the renewal relation

$$\mathbf{t}_0 = 1 \quad \text{and} \quad \mathbf{t}_N = \sum_{M=0}^{N-1} \mathbf{t}_M \mathbf{q}_{N-M}, \qquad N \geq 1. \tag{2.5}$$

We fix λ and β and set

$$\mu = \mu(\lambda, \beta) \stackrel{\Delta}{=} \sum_{M \geq 1} M \mathbf{q}_M. \tag{2.6}$$

Note that the above series converges since, by our basic mass-separation estimate for annealed quantities (2.1),

$$\mathbf{q}_M \leq e^{-\nu M} e^{M\xi} \mathbf{D}_M \leq e^{-\nu M}, \tag{2.7}$$

where we used the fact that $\mathbf{D}_M \leq e^{-M\xi}$, which follows from subadditivity.

LEMMA 2.1. *For any $\beta \geq 0$ and $\lambda > \lambda_0$,*

$$\lim_{N\to\infty} e^{N\xi} \mathbf{T}_N = \lim_{N\to\infty} \mathbf{t}_N = \frac{1}{\mu(\lambda, \beta)}. \tag{2.8}$$

Moreover, the convergence in (2.8) *is exponentially fast.*

PROOF. This is a standard renewal argument which we shall briefly sketch for completeness. As a consequence of our scaling and the mass separation property (2.1), the radius of convergence of the generating function

$$\hat{\mathbf{t}}(u) \stackrel{\Delta}{=} \sum_{N \geq 0} u^N \mathbf{t}_N$$

is equal to 1 (see Section 3.3.6 in [6] for details). On the other hand, it follows from (2.7) that the irreducible generating function

$$\hat{\mathbf{q}}(u) \stackrel{\Delta}{=} \sum_{N \geq 1} u^N \mathbf{q}_N$$

has radius of convergence at least $1 + \nu$. This implies, via standard arguments based on (2.5), that $\hat{\mathbf{q}}(1) = 1$. Of course, $\mu = \hat{\mathbf{q}}'(1)$. Fix $\rho \in (0, 1)$. By Cauchy's formula,

$$\begin{aligned}\mathbf{t}_N - \frac{1}{\mu} &= \frac{1}{2\pi i} \int_{\mathbb{S}_\rho} \left\{ \frac{du}{u^{N+1}(1 - \hat{\mathbf{q}}(u))} - \frac{du}{u^{N+1}(1-u)\mu} \right\} \\ &= \frac{1}{2\pi i} \int_{\mathbb{S}_\rho} \frac{\Delta(u)}{u^{N+1}} \, du,\end{aligned}$$

where $\mathbb{S}_\rho = \partial\mathbb{B}_\rho$ and

$$\Delta(u) = \frac{(\hat{\mathbf{q}}(u) - \hat{\mathbf{q}}(1)) - \hat{\mathbf{q}}'(1)(u-1)}{(\hat{\mathbf{q}}(u) - \hat{\mathbf{q}}(1))\hat{\mathbf{q}}'(1)(u-1)}. \tag{2.9}$$

Since Δ is analytic on $\mathbb{B}_{1+\nu'}$ for some $\nu' \in (0, \nu)$, the result follows. $\square$

2.3. *Complex tilts and annealed diffusivity.* For δ small enough, let $\mathsf{P}^d_{2\delta} \subset \mathbb{C}^d$ be the complex polydisc with all d radii equal to 2δ. By the implicit function theorem (see, e.g., [7]) and in view of the mass-gap estimate (2.7), the relations

$$\varphi[0] = 0 \quad \text{and} \quad \sum_{M\geq 1}\sum_{\mathsf{x}\in\mathcal{L}_M} \mathbf{q}_{\mathsf{x}} e^{-M\varphi[z]+(z,\mathsf{x})_d} \stackrel{\Delta}{=} \sum_{M\geq 1} \mathbf{q}_M[z] = 1$$

define a holomorphic function $\varphi : \mathsf{P}^d_{2\delta} \to \mathbb{C}$. We shall assume that δ is so small that

$$|\mathbf{q}_M[z]| \lesssim e^{-\nu M/2}, \tag{2.10}$$

uniformly in $M \geq 1$ and $z \in \mathsf{P}^d_{2\delta}$.

The analysis of the previous subsection can be readily extended to obtain the asymptotic expansion of the moment generating functions

$$\mathbf{t}_0[z] \stackrel{\Delta}{=} 1 \quad \text{and} \quad \text{for } N \geq 1, \qquad \mathbf{t}_N[z] \stackrel{\Delta}{=} \sum_{\mathsf{x}\in\mathcal{L}_N} \mathbf{t}_{\mathsf{x}} e^{-N\varphi[z]+(z,\mathsf{x})_d}$$

for $z \in \mathsf{P}^d_{2\delta}$. Indeed, $\mathbf{t}_N[z]$ satisfies the renewal relation

$$\mathbf{t}_N[z] = \sum_{M=0}^{N-1} \mathbf{t}_M[z]\mathbf{q}_{N-M}[z].$$

Furthermore, if δ is sufficiently small, then not only does (2.10) hold, but there also exists $\nu' > 0$ such that, for all $z \in \mathsf{P}^d_{2\delta}$, $u = 1$ is the unique solution of the equation

$$\sum_{M\geq 1} u^M \mathbf{q}_M[z] \stackrel{\Delta}{=} \hat{\mathbf{q}}[z](u) = 1,$$

on $\mathbb{B}_{1+\nu'} \subset \mathbb{C}$. We define $\mu[z]$ exactly as in (2.6) by

$$\mu[z] \stackrel{\Delta}{=} \sum_{N\geq 1} N\mathbf{q}_N[z].$$

Relying on (2.10), we can choose δ so small that $\mu[\cdot]$ is analytic and nonzero on $\mathsf{P}^d_{2\delta}$. It then follows that

$$\lim_{N\to\infty} \mathbf{t}_N[z] = \frac{1}{\mu[z]},$$

uniformly exponentially fast on $\mathsf{P}^d_{2\delta}$.

The annealed diffusion matrix Σ and the corresponding diffusivity constant σ^2 in (1.5) are defined by

$$\Sigma \triangleq \mathrm{D}_d^2\varphi[0] \quad \text{and} \quad \sigma^2 \triangleq \mathrm{Tr}(\Sigma), \tag{2.11}$$

where $D_d^2\varphi$ denotes the Hessian of φ. Now, since we have chosen δ sufficiently small to ensure that $\mu[\cdot]$ is analytic and does not vanish on $\mathsf{P}_{2\delta}^d$, the functions $\log \mathbf{t}_N[\cdot] + \log \mu[\cdot]$ are analytic and exponentially small (in N) on $\mathsf{P}_{2\delta}^d$. In particular,

$$\mathrm{Tr}\big(\mathrm{D}_d^2(\log \mathbf{t}_N[z] + \log \mu[z])\big)$$

is also exponentially small. This shows that the leading contribution (in N) to the log-moment generating function $\log(\mathbf{t}_N[z]e^{N\varphi[z]})$ of $\pi^\perp(\gamma)$ under the induced measure is given by $N\varphi[z]$. We have thus proved that

LEMMA 2.2.

$$\left|\frac{1}{N\mathbf{t}_N}\sum_{\mathsf{x}\in\mathcal{L}_N}\|\mathsf{x}^\perp\|^2\mathbf{t}_{\mathsf{x}} - \sigma^2\right| \lesssim \frac{1}{N}.$$

Furthermore, $\pi^\perp(\gamma)/\sqrt{N} \Rightarrow \mathcal{N}(0,\Sigma)$ *under the sequence of annealed polymer measures* μ_N.

2.4. *Multi-dimensional renewal relation for quenched partition functions*. We continue to employ the notation introduced in (2.4). It is immediate to check that the following analogs of (2.5) hold:

$$\mathfrak{t}_{\mathsf{x}}^\omega = \sum_{\mathsf{y}} \mathfrak{t}_{\mathsf{y}}^\omega \mathfrak{q}_{\mathsf{x}-\mathsf{y}}^{\theta_{\mathsf{y}}\omega} \quad \text{and} \quad \mathfrak{t}_N^\omega = \sum_{M=0}^{N-1}\sum_{\mathsf{x}\in\mathcal{L}_M} \mathfrak{t}_{\mathsf{x}}^\omega \mathfrak{q}_{N-M}^{\theta_{\mathsf{x}}\omega} \tag{2.12}$$

for all $\mathsf{x} \in \mathcal{H}_0^+ \triangleq \{\mathsf{x} \in \mathbb{Z}^{d+1} : \mathsf{x}^\| > 0\}$ and $N \geq 1$. Set $\mathfrak{t}_0^\omega \triangleq 1$, and define the generating functions

$$\hat{\mathfrak{t}}^\omega(u) \triangleq \sum_{N=0}^\infty u^N \mathfrak{t}_N^\omega$$

and

$$\hat{\mathfrak{q}}^\omega(u) \triangleq \sum_{N=1}^\infty u^N \mathfrak{q}_N^\omega.$$

Since $|\hat{\mathfrak{t}}^\omega(u)| \leq \hat{\mathfrak{t}}^\omega(|u|)$ and $\mathbb{E}\hat{\mathfrak{t}}^\omega(\rho) = \hat{\mathbf{t}}(\rho)$, the random generating function $\hat{\mathfrak{t}}^\omega(u)$ is $\mathbb{P}$-a.s. defined and analytic in the interior of the unit disc $\mathbb{B}_1 \subset \mathbb{C}$. Similarly, the random generating function $\hat{\mathfrak{q}}^\omega(u)$ is $\mathbb{P}$-a.s. analytic on $\mathbb{B}_{1+\nu}$ for some $\nu > 0$.

We can rewrite (2.12) in terms of the generating function as

$$\begin{aligned}\hat{\mathfrak{t}}^{\omega}(u) &= 1 + \sum_{M=0}^{\infty} u^M \sum_{\mathsf{x}\in\mathcal{L}_M} \mathfrak{t}^{\omega}_{\mathsf{x}} \hat{\mathfrak{q}}^{\theta_{\mathsf{x}}\omega}(u) \\ &= 1 + \hat{\mathbf{q}}(u) \sum_{M=0}^{\infty} u^M \sum_{\mathsf{x}\in\mathcal{L}_M} \mathfrak{t}^{\omega}_{\mathsf{x}} \\ &\quad + \sum_{M=0}^{\infty} u^M \sum_{\mathsf{x}\in\mathcal{L}_M} \mathfrak{t}^{\omega}_{\mathsf{x}} \big(\hat{\mathfrak{q}}^{\theta_{\mathsf{x}}\omega}(u) - \hat{\mathbf{q}}(u)\big) \\ &\stackrel{\Delta}{=} 1 + \hat{\mathbf{q}}(u)\hat{\mathfrak{t}}^{\omega}(u) + \hat{\Psi}^{\omega}(u).\end{aligned} \tag{2.13}$$

Since $|\hat{\mathbf{q}}(u)| < 1$ whenever $|u| = \rho < 1$, we can record the last computation as

$$\hat{\mathfrak{t}}^{\omega}(u) = \frac{1 + \hat{\Psi}^{\omega}(u)}{1 - \hat{\mathbf{q}}(u)}.$$

Therefore,

$$\mathfrak{t}^{\omega}_N = \frac{1}{2\pi i} \int_{\mathbb{S}_\rho} \frac{1 + \hat{\Psi}^{\omega}(u)}{(1 - \hat{\mathbf{q}}(u))u^{N+1}}\, du, \tag{2.14}$$

$\mathbb{P}$-a.s. for all $\rho \in (0, 1)$.

2.5. *Recursion under L^2-weak disorder.* Equation (2.14) is the starting point for proving Theorem A. In fact, we are going to develop a recursion for the limit in (2.3) whenever the conditions of the latter theorem are satisfied.

Let us decompose

$$\mathfrak{t}^{\omega}_N = \frac{1}{\mu}\mathfrak{s}^{\omega}_N + \left(\mathfrak{t}^{\omega}_N - \frac{1}{\mu}\mathfrak{s}^{\omega}_N\right),$$

where[4]

$$\mathfrak{s}^{\omega}_N \stackrel{\Delta}{=} \frac{1}{2\pi i} \int_{\mathbb{S}_\rho} \frac{1 + \hat{\Psi}^{\omega}(u)}{u^{N+1}(1 - u)}\, du$$

and, accordingly,

$$\mathfrak{t}^{\omega}_N - \frac{1}{\mu}\mathfrak{s}^{\omega}_N = \frac{1}{2\pi i} \int_{\mathbb{S}_\rho} \frac{(1 + \hat{\Psi}^{\omega}(u))\Delta(u)}{u^{N+1}}\, du, \tag{2.15}$$

with $\Delta(u)$ defined in (2.9).

[4]Note that the definition does not depend on the particular choice of $\rho \in (0, 1)$.

After examining the definition of $\hat{\Psi}^\omega$ in (2.13), we arrive at the following expression for $\mathfrak{s}_N^\omega$:

$$\begin{aligned}\mathfrak{s}_N^\omega = \left[\frac{1+\hat{\Psi}^\omega(u)}{1-u}\right]_N &= 1 + \sum_{M=0}^{N-1}\sum_{\mathsf{x}\in\mathcal{L}_M} \mathfrak{t}_\mathsf{x}^\omega(\mathfrak{q}_{1,N-M}^{\theta_\mathsf{x}\omega} - \mathbf{q}_{1,N-M}) \\ &= 1 + \sum_{\mathsf{x}\in\mathcal{H}_N^-,\mathsf{y}\in\mathcal{H}_{N+1}^-} \mathfrak{t}_\mathsf{x}^\omega(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_\mathsf{x}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}}),\end{aligned} \tag{2.16}$$

where

$$\mathfrak{q}_{1,l}^\omega \triangleq \sum_{k=1}^{l}\mathfrak{q}_k^\omega \quad\text{and}\quad \mathbf{q}_{1,l}\triangleq \mathbb{E}\mathfrak{q}_{1,l}^\omega$$

and we used the standard notation $[\sum_{k\geq 0} a_k u^k]_N = a_N$ for expansion coefficients.

The following theorem is proved in Sections 3.2 and 3.3.

THEOREM 2.3. *For every $\lambda > \lambda_0$, there exist $\beta_0 = \beta_0(\lambda, d)$ and $p_\infty > 0$, such that if Assumption (A) holds with $p \leq p_\infty$, then, for every $\beta \in [0, \beta_0)$:*

(1) *The sequence $\mathfrak{t}_N^\omega - \mathfrak{s}_N^\omega/\mu$ converges to zero $\mathbb{P}$-a.s. and in $L^2(\Omega)$.*
(2) *The sequence $\mathfrak{s}_N^\omega$ converges $\mathbb{P}$-a.s. and in $L^2(\Omega)$ to*

$$\mathfrak{s}^\omega \triangleq 1 + \sum_{\mathsf{x}\in\mathcal{H}_0^+} \mathfrak{t}_\mathsf{x}^\omega(\mathfrak{q}_{1,\infty}^{\theta_\mathsf{x}\omega} - 1), \tag{2.17}$$

the latter sum also converging in $L^2(\Omega)$.

Theorem 2.3 implies that the limit in (2.3) indeed exists and, furthermore, that it is equal to the random variable $\mathfrak{s}^\omega$

$$\lim_{N\to\infty}\frac{\mathfrak{T}_N^\omega}{\mathbf{T}_N} = \lim_{N\to\infty}\frac{\mathfrak{t}_N^\omega}{\mathbf{t}_N} = \lim_{N\to\infty}\mathfrak{s}_N^\omega = \mathfrak{s}^\omega.$$

Note that if $0 \notin \mathrm{Cl}_\infty(V)$, then $\mathfrak{t}_N^\omega = 0$ for all N sufficiently large, say $N \geq N_0(\omega)$. Consequently, in this case $\mathfrak{s}^\omega$ is a difference of two convergent series,

$$\mathfrak{s}^\omega = 1 + \sum_{\mathsf{x},\mathsf{y}\in\mathcal{H}_{N_0}^-} \mathfrak{t}_\mathsf{x}^\omega \mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_\mathsf{x}\omega} - \sum_{\mathsf{x}\in\mathcal{H}_{N_0}^-}\mathfrak{t}_\mathsf{x}^\omega = 0.$$

Positivity of $\mathfrak{s}^\omega$ [or rather of the full limit $\mathfrak{d}^\omega$ in (1.4)] on the event $\{0 \in \mathrm{Cl}_\infty(V)\}$ is established in the concluding Section 4.5 of the paper.

2.6. *Relation with Sinai's representation.* Our representation (2.16) can be seen as an effective random walk version of the high-temperature expansion employed by Sinai in [8]. Indeed, let $\mathsf{x} \in \mathcal{L}_N$. Then

$$\begin{aligned}\mathfrak{t}^{\omega}_{\mathsf{x}} &= \sum_{n\geq 0}\sum_{\mathsf{x}_1,\dots,\mathsf{x}_n}\prod_{k=0}^{n}\mathfrak{q}^{\omega}_{\mathsf{x}_k,\mathsf{x}_{k+1}} \\ &= \sum_{n\geq 0}\sum_{\mathsf{x}_1,\dots,\mathsf{x}_n}\prod_{k=0}^{n}\mathbf{q}_{\mathsf{x}_k,\mathsf{x}_{k+1}}\Phi^{\omega}(\mathsf{x}_0,\dots,\mathsf{x}_{n+1}),\end{aligned}$$

where we have set $\mathsf{x}_0 = 0$, $\mathsf{x}_{n+1} = \mathsf{x}$, and

$$\Phi^{\omega}(\mathsf{x}_0,\dots,\mathsf{x}_{n+1}) \triangleq \prod_{k=0}^{n}\frac{\mathfrak{q}^{\omega}_{\mathsf{x}_k,\mathsf{x}_{k+1}}}{\mathbf{q}_{\mathsf{x}_k,\mathsf{x}_{k+1}}} \triangleq \prod_{k=0}^{n}\bigl(1+\phi^{\omega}(\mathsf{x}_k,\mathsf{x}_{k+1})\bigr).$$

Using the expansion

$$\Phi^{\omega}(\mathsf{x}_0,\dots,\mathsf{x}_{n+1}) = \sum_{A\subset\{0,\dots,n\}}\prod_{k\in A}\phi^{\omega}(\mathsf{x}_k,\mathsf{x}_{k+1}),$$

we obtain the representation

$$\mathfrak{t}^{\omega}_{\mathsf{x}} = \sum_{n\geq 0}\sum_{\mathsf{x}_1,\dots,\mathsf{x}_n}\prod_{k=0}^{n}\mathbf{q}_{\mathsf{x}_k,\mathsf{x}_{k+1}}\sum_{A\subset\{0,\dots,n\}}\prod_{\ell\in A}\phi^{\omega}(\mathsf{x}_\ell,\mathsf{x}_{\ell+1}).$$

Given n, $\mathsf{x}_1,\dots,\mathsf{x}_n$ and $\varnothing \neq A \subset \{0,\dots,n\}$, let us say that $(\mathsf{x}_{k^*},\mathsf{x}_{k^*+1})$ is the last perturbed segment if $k^* = \max\{k : k \in A\}$. Keeping the last perturbed segment fixed and resumming all the rest, we arrive at

$$\mathfrak{t}^{\omega}_{\mathsf{x}} = \mathbf{t}_{\mathsf{x}} + \sum_{\mathsf{y},\mathsf{z}}\mathfrak{t}^{\omega}_{\mathsf{y}}(\mathfrak{q}^{\theta_{\mathsf{y}}\omega}_{\mathsf{z}-\mathsf{y}} - \mathbf{q}_{\mathsf{z}-\mathsf{y}})\mathbf{t}_{\mathsf{x}-\mathsf{z}}. \tag{2.18}$$

Similarly, keeping the first perturbed segment fixed and resumming all the rest, we arrive at

$$\mathfrak{t}^{\omega}_{\mathsf{x}} = \mathbf{t}_{\mathsf{x}} + \sum_{\mathsf{y},\mathsf{z}}\mathbf{t}_{\mathsf{y}}(\mathfrak{q}^{\theta_{\mathsf{y}}\omega}_{\mathsf{z}-\mathsf{y}} - \mathbf{q}_{\mathsf{z}-\mathsf{y}})\mathfrak{t}^{\theta_{\mathsf{z}}\omega}_{\mathsf{x}-\mathsf{z}}. \tag{2.19}$$

It would have been possible to work directly with the above representations of $\mathfrak{t}$-quantities. In fact, Theorem 2.3(1) can be considered as the first step along these lines: it enables us to substitute and control the $\mathfrak{t}$-quantities by the more tractable $\mathfrak{s}$-quantities, as appears in (2.16).

Notice though that it is not clear how to prove the almost-sure convergence in Theorem A without having recourse to martingale arguments as developed in Section 3.1.

2.7. *Extension to the full $\mathcal{D}_N$-ensemble.* Let us go back to Theorem A. In view of (2.1), there is no loss in redefining $\mathcal{D}_N$ as the set of all *reducible* paths from 0 to $\mathcal{L}_N$. Thus, any $\gamma \in \mathcal{D}_N$ automatically satisfies (2.2). By construction (decomposition with respect to all cone-points), none of the paths $\gamma_l, \gamma_1, \dots, \gamma_n, \gamma_r$ in (2.2) has cone-points. Recall that we use the notation $\mathcal{T}^0$ for cone-confined paths without cone-points. Thus, $\gamma_1, \dots, \gamma_n \in \mathcal{T}^0$.

Paths $\gamma_l = (\gamma_l(0), \dots, \gamma_l(m))$ satisfy $\gamma_l \subseteq \gamma_l(m) - \mathcal{Y}_\delta$, and, similarly, paths $\gamma_r = (\gamma_r(0), \dots, \gamma_r(k))$ satisfy $\gamma_r \subseteq \gamma_r(0) + \mathcal{Y}_\delta$. We denote by $\mathcal{T}_l^0$ and $\mathcal{T}_r^0$ the sets of such paths; in this way, $\mathcal{T}^0 = \mathcal{T}_l^0 \cap \mathcal{T}_r^0$.

Following (2.4), define

$$\mathfrak{l}_{\mathsf{x}}^{\omega} \stackrel{\Delta}{=} \sum_{\substack{\gamma:0\mapsto \mathsf{x}\\ \gamma\in\mathcal{T}_l^0}} w_{\lambda,\beta}^{\omega}(\gamma) \quad \text{and} \quad \mathfrak{r}_{\mathsf{x}}^{\omega} \stackrel{\Delta}{=} \sum_{\substack{\gamma:0\mapsto \mathsf{x}\\ \gamma\in\mathcal{T}_r^0}} w_{\lambda,\beta}^{\omega}(\gamma).$$

As usual, we denote the corresponding annealed quantities by $\mathbf{l}_{\mathsf{x}}$ and $\mathbf{r}_{\mathsf{x}}$. The scaled full $\mathcal{D}_N$ partition function satisfies

$$\mathfrak{d}_N^{\omega} \stackrel{\Delta}{=} e^{N\xi}\mathfrak{D}_N^{\omega} = \sum_{\mathsf{x}\in\mathcal{L}_N} \sum_{\substack{\gamma:0\mapsto \mathsf{x}\\ \gamma\in\mathcal{D}_N}} w_{\lambda,\beta}^{\omega}(\gamma) = \sum_{0\le M_l<M_r\le N} \sum_{\substack{\mathsf{x}\in\mathcal{L}_{M_l}\\ \mathsf{y}\in\mathcal{L}_{M_r}}} \mathfrak{l}_{\mathsf{x}}^{\omega}\mathfrak{t}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega}\mathfrak{r}_{N-M_r}^{\theta_{\mathsf{y}}\omega}$$

$$\begin{aligned}
&= \sum_{0\le M_l<M_r\le N} \sum_{\mathsf{x}\in\mathcal{L}_{M_l}} \mathfrak{l}_{\mathsf{x}}^{\omega}\mathfrak{t}_{M_r-M_l}^{\theta_{\mathsf{x}}\omega}\mathbf{r}_{N-M_r} \\
&\quad + \sum_{0\le M_l<M_r\le N} \sum_{\substack{\mathsf{x}\in\mathcal{L}_{M_l}\\ \mathsf{y}\in\mathcal{L}_{M_r}}} \mathfrak{l}_{\mathsf{x}}^{\omega}\mathfrak{t}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega}(\mathfrak{r}_{N-M_r}^{\theta_{\mathsf{y}}\omega} - \mathbf{r}_{N-M_r}).
\end{aligned} \tag{2.20}$$

By the mass separation property (2.1), the annealed point-to-plane functions $\mathbf{l}_M$ and $\mathbf{r}_M$ have exponentially decaying tails, and in particular both are summable. Define $\mathbf{c}_r \stackrel{\Delta}{=} \sum_M \mathbf{r}_M < \infty$. The following theorem is proved in Section 3.4.

THEOREM 2.4. *For every $\lambda > \lambda_0$, there exist $\beta_0 = \beta_0(\lambda, d)$ and $p_\infty > 0$ such that, if Assumption* (A) *holds with $p \le p_\infty$, then, for every $\beta \in [0, \beta_0)$*:

(1) *The second term on the right-hand side of* (2.20) *converges to zero $\mathbb{P}$-a.s. and in $L^2(\Omega)$.*

(2) *The first term on the right-hand side of* (2.20) *converges to*

$$\frac{\mathbf{c}_r}{\mu} \sum_{\mathsf{x}} \mathfrak{l}_{\mathsf{x}}^{\omega}\mathfrak{s}^{\theta_{\mathsf{x}}\omega}, \tag{2.21}$$

$\mathbb{P}$-a.s. and in $L^2(\Omega)$.

Consequently, (1.4) of Theorem A follows with

$$\mathfrak{d}^\omega = \lim_{N\to\infty} \frac{\mathfrak{D}_N^\omega}{\mathbf{D}_N^\omega} = \mathbf{c}_r \sum_{\mathsf{x}} \mathfrak{l}_{\mathsf{x}}^\omega \mathfrak{s}^{\theta_{\mathsf{x}}\omega}.$$

Positivity of $\mathfrak{d}^\omega$ on the event $\{0 \in \mathrm{Cl}_\infty(V)\}$ is established in the concluding Section 4.5.

3. Proofs.

3.1. *The key computation.* Below, we formulate the key statement, essential for all our results in this paper. It heavily relies on the assumptions of weak disorder. We relegate the proof of Proposition 3.1 to the concluding Section 4.

PROPOSITION 3.1. *For every $\lambda > \lambda_0$, there exist $\beta_0 = \beta_0(\lambda, d)$ and $p_\infty > 0$ such that, if Assumption (A) holds with $p \le p_\infty$, then, for every $\beta \in [0, \beta_0)$,*

$$\sup_{N\ge 1} \mathbb{E}\Bigg[\sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_K^+} \mathfrak{t}_{\mathsf{x}}^\omega (\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}}) g(\mathsf{y})\Bigg]^2 \lesssim (K+1)^{1-d/2}\|g\|_\infty^2, \tag{3.1}$$

uniformly in $K \ge 0$ and in bounded functions g on $\mathbb{Z}^{d+1}$.

Furthermore,

$$\mathbb{E}\Bigg[\sum_{\mathsf{x}\in\mathcal{H}_K^-}\sum_{\mathsf{y}\in\mathcal{H}_K^+} \mathfrak{t}_{\mathsf{x}}^\omega (\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}}) g(\mathsf{y})\Bigg]^2 \lesssim (K+1)^{-d/2}\|g\|_\infty^2, \tag{3.2}$$

uniformly in $K \ge 0$ and in bounded functions g on $\mathbb{Z}^{d+1}$. Similarly,

$$\mathbb{E}\Bigg[\sum_{\mathsf{x},\mathsf{y}\in\mathcal{H}_K^-}\sum_{\mathsf{z}\in\mathcal{H}_K^+} \mathfrak{l}_{\mathsf{x}}^\omega \mathfrak{t}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} (\mathfrak{r}_{\mathsf{z}-\mathsf{y}}^{\theta_{\mathsf{y}}\omega} - \mathbf{r}_{\mathsf{z}-\mathsf{y}}) g(\mathsf{z})\Bigg]^2 \lesssim (K+1)^{-d/2}\|g\|_\infty^2 \tag{3.3}$$

also uniformly in $K \ge 0$ and in bounded functions g on $\mathbb{Z}^{d+1}$.

3.2. *Proof of Theorem 2.3*(1). Recall that

$$\mathfrak{t}_N^\omega - \frac{1}{\mu}\mathfrak{s}_N^\omega = \frac{1}{2\pi i}\int_{\mathbb{S}_\rho} \frac{(1+\hat{\Psi}^\omega(u))\Delta(u)}{u^{N+1}}\,du \tag{3.4}$$

for each $\rho \in (0,1)$. We are going to show that

LEMMA 3.2. (3.4) *still holds at $\rho = 1$ and $\hat{\Psi}^\omega(e^{i\theta}) \in L^2(\Omega \times [0, 2\pi])$.*

In particular, $\hat{\Psi}^\omega(e^{i\theta}) \in L^2([0, 2\pi])$ $\mathbb{P}$-a.s. Consequently, the right-hand side of (2.15) is $\mathbb{P}$-a.s. equal to the Nth Fourier coefficient of $(1 + \hat{\Psi}^\omega(e^{i\theta}))\Delta(e^{i\theta})$. Therefore, by Parseval's theorem,

$$\mathbb{E}\sum_N \left(\mathfrak{t}_N^\omega - \frac{1}{\mu}\mathfrak{s}_N^\omega\right)^2 = \frac{1}{2\pi}\mathbb{E}\int_0^{2\pi} \left|(1 + \hat{\Psi}^\omega(e^{i\theta}))\Delta(e^{i\theta})\right|^2 d\theta < \infty.$$

It thus follows from Fubini's theorem that

$$\lim_{N\to\infty}\left(\mathfrak{t}_N^\omega - \frac{1}{\mu}\mathfrak{s}_N^\omega\right) = 0,$$

$\mathbb{P}$-a.s. and in $L^2(\Omega)$.

It remains to prove Lemma 3.2. First of all, $\hat{\Psi}^\omega(e^{i\theta})$ can be rewritten as

$$\hat{\Psi}^\omega(e^{i\theta}) = \sum_{\mathsf{x},\mathsf{y}} \mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}})e^{i\theta\mathsf{y}^\parallel}.$$

Applying Proposition 3.1 with $K = 0$, $N = \infty$ and $g(y) = e^{i\theta(\vec{e}_{d+1},\mathsf{y})_{d+1}}$, we conclude that

$$\sup_\theta \mathbb{E}(\hat{\Psi}^\omega(e^{i\theta}))^2 \lesssim 1,$$

and hence $\Psi^\omega(e^{i\theta}) \in L_2(\Omega \times [0, 2\pi])$ indeed.

In a completely similar fashion, one concludes from Proposition 3.1 that

$$\lim_{K\to\infty}\sup_{|u|\le 1}\mathbb{E}\left(\sum_{M=K}^{\infty} u^M\psi_M^\omega\right)^2 = 0, \tag{3.5}$$

where $\{\psi_N^\omega\}$ are the expansion coefficients of $\hat{\Psi}^\omega(u) = \sum_M u^M\psi_M^\omega$, that is, explicitly,

$$\psi_M^\omega = \sum_{\mathsf{y}\in\mathcal{L}_M}\sum_{\mathsf{x}} \mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}}).$$

Obviously, for each K fixed,

$$\lim_{\rho\to 1}\sum_{N=1}^{K}(\rho e^{i\theta})^N\psi_N^\omega = \sum_{N=1}^{K}(e^{i\theta})^N\psi_N^\omega$$

in $L^2(\Omega \times [0, 2\pi])$. In view of (3.5), the latter implies that

$$\lim_{\rho\to 1}\mathbb{E}\int_0^{2\pi}(\hat{\Psi}^\omega(\rho e^{i\theta}) - \hat{\Psi}^\omega(e^{i\theta}))^2\, d\theta = 0.$$

As a result one can indeed pass to the limit $\rho \to 1$ in (3.4).

3.3. *Proof of Theorem* 2.3(2). Let $\mathcal{F}_N$ be the σ-algebra generated by $\{V_{\mathsf{x}}\}_{\mathsf{x}\in\mathcal{H}_{N+1}^-}$, and let us introduce

$$\mathcal{A}_N^\omega \triangleq 1+\sum_{M=0}^{N-1}\sum_{\mathsf{x}\in\mathcal{L}_M}\mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{1,\infty}^{\theta_{\mathsf{x}}\omega}-1),$$

$$\mathcal{B}_N^\omega \triangleq \sum_{M=0}^{N-1}\sum_{\mathsf{x}\in\mathcal{L}_M}\mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{N-M+1,\infty}^{\theta_{\mathsf{x}}\omega}-\mathbf{q}_{N-M+1,\infty}),$$

$$\mathcal{C}_N^\omega \triangleq \mathbb{E}(\mathcal{A}_N^\omega|\mathcal{F}_N).$$

We can then express $\mathfrak{s}_N^\omega$ as

$$\mathfrak{s}_N^\omega=\mathcal{C}_N^\omega+(\mathcal{A}_N^\omega-\mathcal{C}_N^\omega)-\mathcal{B}_N^\omega.$$

The $\mathbb{P}$-a.s. and $L^2(\Omega)$ convergence in (2.17) follows from the next two lemmas, since they imply that, $\mathbb{P}$-a.s. and in $L^2(\Omega)$, $\mathcal{B}_N^\omega$ and $\mathcal{A}_N^\omega-\mathcal{C}_N^\omega$ tend to 0, while $\mathcal{C}_N^\omega$ converges to $\mathfrak{s}^\omega$.

LEMMA 3.3. *For every $\lambda>\lambda_0$, there exist $\beta_0>0$ and $\hat{p}_\infty>0$ such that, if Assumption* (A) *holds with $p\le\hat{p}_\infty$, then, for each $\beta\in[0,\beta_0]$, the sequence $\{\mathcal{C}_N\}$ is an L^2-bounded martingale.*

LEMMA 3.4. *For every $\lambda>\lambda_0$, there exist $\beta_0>0$ and $\hat{p}_\infty>0$ such that, if Assumption* (A) *holds with $p\le\hat{p}_\infty$, then*

$$\sum_N\mathbb{E}(\mathcal{B}_N^\omega)^2<\infty\quad\textit{and}\quad\sum_N\mathbb{E}(\mathcal{A}_N^\omega-\mathcal{C}_N^\omega)^2<\infty$$

for each $\beta\in[0,\beta_0]$.

PROOF OF LEMMA 3.3. The fact that $\mathcal{C}_N^\omega$ is a martingale is straightforward: for any N and each $\mathsf{x}\in\mathcal{L}_N$,

$$\mathbb{E}(\mathcal{C}_{N+1}^\omega|\mathcal{F}_N)=\mathbb{E}(\mathcal{A}_N^\omega|\mathcal{F}_N)+\sum_{\mathsf{x}\in\mathcal{L}_N}\mathbb{E}\big(\mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{1,\infty}^{\theta_{\mathsf{x}}\omega}-1)|\mathcal{F}_N\big),$$

and

$$\mathbb{E}\big(\mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{1,\infty}^{\theta_{\mathsf{x}}\omega}-1)|\mathcal{F}_N\big)=\mathfrak{t}_{\mathsf{x}}^\omega\mathbb{E}(\mathfrak{q}_{1,\infty}^{\theta_{\mathsf{x}}\omega}-1)=0,$$

since $\mathsf{x}\in\mathcal{L}_N$.

It remains to check that $\mathcal{C}_N^\omega$ is $L^2(\Omega)$-bounded. We first deduce from Jensen's inequality that $\mathbb{E}(\mathcal{C}_N^\omega)^2\le\mathbb{E}(\mathcal{A}_N^\omega)^2$. However, uniform $L^2(\Omega)$-boundedness of the latter quantities follows immediately from Proposition 3.1 by taking $K=0$ and $g\equiv 1$. $\square$

PROOF OF LEMMA 3.4. Note that

$$\mathcal{A}_N^\omega - \mathcal{C}_N^\omega = \sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_N^+} \mathfrak{t}_{\mathsf{x}}^\omega\big(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbb{E}(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega}|\mathcal{F}_N)\big)$$

and

$$\mathcal{B}_N^\omega = \sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_N^+} \mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}}).$$

Thus, $\mathcal{A}_N^\omega - \mathcal{C}_N^\omega$ and $\mathcal{B}_N^\omega$ have very similar forms. In fact,

$$\mathbb{E}(\mathcal{A}_N^\omega - \mathcal{C}_N^\omega)^2 \le 4\mathbb{E}(\mathcal{B}_N^\omega)^2.$$

On the other hand, taking $g \equiv 1$ in the second of the statements of Proposition 3.1, we readily conclude that $\sum_N \mathbb{E}(\mathcal{B}_N^\omega)^2 < \infty$. $\square$

3.4. *Proof of Theorem* 2.4. The claim (2) of the theorem follows from the $\mathbb{P}$-a.s. and $L^2(\Omega)$ convergence to $\mathfrak{s}^\omega$ in (2.17) and from the fact that

$$\mathbb{E}\mathfrak{l}_{\mathsf{x}}^\omega = \mathbf{l}_{\mathsf{x}} \le e^{-\nu|\mathsf{x}|}\mathbf{1}_{\{\mathsf{x}\in\mathcal{Y}_\delta\}}.$$

The first claim (1) follows by an application of (3.3) with $K = N$ and $g(\mathsf{z}) = \mathbf{1}_{\{\mathsf{z}\in\mathcal{L}_N\}}$.

3.5. *Proof of Theorem* B. Let f be a bounded continuous function on $\mathbb{R}^d$. Using (2.18), we can write, for any $K \ge 0$,

$$\begin{aligned}
\sum_{\mathsf{z}\in\mathcal{L}_N} \mathfrak{t}_{\mathsf{z}}^\omega f\left(\frac{\mathsf{z}^\perp}{\sqrt{N}}\right) &= \sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}} \mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}}) \sum_{\mathsf{z}\in\mathcal{L}_N} \mathbf{t}_{\mathsf{z}-\mathsf{y}} f\left(\frac{\mathsf{z}^\perp}{\sqrt{N}}\right) \\
&= \sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_K^-} \mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}}) \sum_{\mathsf{z}\in\mathcal{L}_N} \mathbf{t}_{\mathsf{z}-\mathsf{y}} f\left(\frac{\mathsf{z}^\perp}{\sqrt{N}}\right) \qquad (3.6)\\
&\quad + \sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_{K-1}^+} \mathfrak{t}_{\mathsf{x}}^\omega(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}}) \sum_{\mathsf{z}\in\mathcal{L}_N} \mathbf{t}_{\mathsf{z}-\mathsf{y}} f\left(\frac{\mathsf{z}^\perp}{\sqrt{N}}\right).
\end{aligned}$$

Choosing $K = K(N) = N^\varepsilon$ for some $\varepsilon < 1/2$ and setting

$$g(\mathsf{y}) = g_N(\mathsf{y}) = \sum_{\mathsf{z}\in\mathcal{L}_N} \mathbf{t}_{\mathsf{z}-\mathsf{y}} f\left(\frac{\mathsf{z}^\perp}{\sqrt{N}}\right),$$

we can infer from Proposition 3.1 that the second sum on the right-hand side of (3.6) converges to zero in $L^2(\Omega)$. As for the first sum on the right-hand side of

(3.6), it follows from the annealed central limit theorem (and the continuity of f) that

$$\lim_{N\to\infty} \max_{\mathsf{y}\in\mathcal{H}^-_{N^\varepsilon}\cap\mathcal{Y}_\delta} \biggl| \sum_{\mathsf{z}\in\mathcal{L}_N} \mathbf{t}_{\mathsf{z}-\mathsf{y}} f\Bigl(\frac{\mathsf{z}^\perp}{\sqrt{N}}\Bigr) - \frac{1}{\mu}\frac{1}{\sqrt{\det(2\pi\Sigma)}}\int_{\mathbb{R}^d} f(\mathsf{x}) e^{-1/2(\Sigma^{-1}\mathsf{x},\mathsf{x})}\,d\mathsf{x}\biggr| = 0.$$

By another application of Proposition 3.1, this time with $K = 0$ and

$$g(\mathsf{y}) = \biggl(\sum_{\mathsf{z}\in\mathcal{L}_N} \mathbf{t}_{\mathsf{z}-\mathsf{y}} f\Bigl(\frac{\mathsf{z}^\perp}{\sqrt{N}}\Bigr) - \frac{1}{\mu}\frac{1}{\sqrt{\det(2\pi\Sigma)}}\int_{\mathbb{R}^d} f(\mathsf{x}) e^{-1/2(\Sigma^{-1}\mathsf{x},\mathsf{x})}\,d\mathsf{x}\biggr) \times \mathbf{1}_{\{\mathsf{y}\in\mathcal{H}^-_{N^\varepsilon}\cap\mathcal{Y}_\delta\}},$$

we conclude, in view of Theorem 2.3, that the first sum in (3.6) converges in $L^2(\Omega)$ to

$$\frac{\mathfrak{s}^\omega}{\mu}\frac{1}{\sqrt{\det(2\pi\Sigma)}}\int_{\mathbb{R}^d} f(\mathsf{x}) e^{-1/2(\Sigma^{-1}\mathsf{x},\mathsf{x})}\,d\mathsf{x} = \frac{\mathfrak{t}^\omega}{\sqrt{\det(2\pi\Sigma)}}\int_{\mathbb{R}^d} f(\mathsf{x}) e^{-1/2(\Sigma^{-1}\mathsf{x},\mathsf{x})}\,d\mathsf{x}.$$

Extension to the full $\mathcal{D}_N$-ensemble. Proceeding as in Section 2.7, we conclude that, for any bounded continuous f, the series

$$\sum_{\mathsf{z}\in\mathcal{L}_N} \mathfrak{d}^\omega_{\mathsf{z}} f\Bigl(\frac{\mathsf{z}^\perp}{\sqrt{N}}\Bigr)$$

converges in $L^2(\Omega)$ to

$$\frac{\mathbf{c}_r \sum_{\mathsf{x}} \mathfrak{l}^\omega_{\mathsf{x}} \mathfrak{s}^{\theta_{\mathsf{x}}\omega}}{\mu\sqrt{\det(2\pi\Sigma)}}\int_{\mathbb{R}^d} f(\mathsf{x}) e^{-1/2(\Sigma^{-1}\mathsf{x},\mathsf{x})}\,d\mathsf{x}.$$

Together with (2.21), this implies (1.6).

Local limit description. As in [8], equations (2.18) and (2.19) suggest the following quenched Ornstein–Zernike asymptotics for $\mathfrak{t}^\omega_{\mathsf{x}}$ (as inherited from the annealed OZ-asymptotics of $\mathbf{t}_{\mathsf{x}}$ in [6]): Given $\mathsf{x}\in\mathbb{Z}^{d+1}$, let $\hat{\theta}_{\mathsf{x}}\omega$ be the reflection with respect to the hyperplane $\mathcal{L}_N$ of the shifted environment $\theta_{\mathsf{x}}\omega$. In other words, $\hat{\theta}_{\mathsf{x}}\omega$ is the environment as seen backwards from x. Of course, the reflected environment has the very same averaged polymer connectivity functions. We conjecture that

$$\frac{\mathfrak{t}^\omega_{\mathsf{x}}}{\mathbf{t}_{\mathsf{x}}} = (1+\mathfrak{s}^\omega)(1+\mathfrak{s}^{\hat{\theta}_{\mathsf{x}}\omega})\bigl(1+o(1)\bigr). \tag{3.7}$$

Clearly, the strength of the above conjecture depends on what is meant by $o(1)$ in (3.7). A $\mathbb{P}$-a.s. statement would be a refinement of a $\mathbb{P}$-a.s. CLT, which is, as we already mentioned, an open problem by itself. Weaker statements, on the other hand, are feasible via an appropriate refinement of Proposition 3.1.

4. $L^2(\Omega)$ estimates at weak disorder.

4.1. *Preliminaries*. Our proof of Proposition 3.1 is based on a comparison with weakly interacting random walks on $\mathbb{Z}^d$. The bottom line is that, under Assumption (A), transience wins over attraction. From a technical point of view the approach is similar to [2].

Since, in all the estimates below, only the supremum norm of g in Proposition 3.1 would matter, we can assume, without loss of generality, that $g \equiv 1$. It is convenient to use the alternative notation

$$\mathsf{q}^{\omega}_{\mathsf{x},\mathsf{u}} \stackrel{\Delta}{=} \sum_{\gamma \in \mathcal{T}^0_{\mathsf{x},\mathsf{u}}} W^{\omega}_{\lambda,\beta}(\gamma) = \sum_{\gamma \in \mathcal{T}^0_{\mathsf{u}-\mathsf{x}}} W^{\theta_{\mathsf{x}}\omega}_{\lambda,\beta}(\gamma),$$

and $\mathbf{q}_{\mathsf{x},\mathsf{u}} = \mathbf{q}_{\mathsf{u}-\mathsf{x}} \stackrel{\Delta}{=} \mathbb{E}\mathsf{q}^{\omega}_{\mathsf{x},\mathsf{u}}$. Above, $\mathcal{T}^0_{\mathsf{u}}$ is the set of irreducible cone-confined paths from 0 to u and $\mathcal{T}^0_{\mathsf{x},\mathsf{u}} \stackrel{\Delta}{=} \mathsf{x} + \mathcal{T}^0_{\mathsf{u}-\mathsf{x}}$.

Given x and u, we define the diamond shape

$$D(\mathsf{x},\mathsf{u}) \stackrel{\Delta}{=} (\mathsf{x} + \mathcal{Y}_\delta) \cap (\mathsf{u} - \mathcal{Y}_\delta).$$

By construction, any path $\gamma \in \mathcal{T}_{\mathsf{x},\mathsf{u}}$ satisfies $\gamma \subset D(\mathsf{x},\mathsf{u})$. Hence, $\mathsf{q}^{\omega}_{\mathsf{x},\mathsf{u}}$ only depends on the environment inside $D(\mathsf{x},\mathsf{u})$.

Here is a useful observation (see Figure 2): If $D(\mathsf{x},\mathsf{u}) \cap D(\mathsf{y},\mathsf{v}) = \varnothing$, then

$$\mathbb{E}\{\mathsf{t}^{\omega}_{\mathsf{x}}(\mathsf{q}^{\omega}_{\mathsf{x},\mathsf{u}} - \mathbf{q}_{\mathsf{x},\mathsf{u}})\mathsf{t}^{\omega}_{\mathsf{y}}(\mathsf{q}^{\omega}_{\mathsf{y},\mathsf{v}} - \mathbf{q}_{\mathsf{y},\mathsf{v}})\} = 0.$$

Indeed, unless $\mathsf{x} = \mathsf{y}$, it is always true that either $D(\mathsf{x},\mathsf{u}) \cap (\mathsf{y} - \mathcal{Y}_\delta) = \varnothing$ or $D(\mathsf{y},\mathsf{v}) \cap (\mathsf{x} - \mathcal{Y}_\delta) = \varnothing$. If, in addition, the diamond shapes do not intersect, then in the former case $(\mathsf{q}^{\omega}_{\mathsf{x},\mathsf{u}} - \mathbf{q}_{\mathsf{x},\mathsf{u}})$ is independent of $\mathsf{t}^{\omega}_{\mathsf{x}}\mathsf{t}^{\omega}_{\mathsf{y}}(\mathsf{q}^{\omega}_{\mathsf{y},\mathsf{v}} - \mathbf{q}_{\mathsf{y},\mathsf{v}})$, and similarly for the latter case.

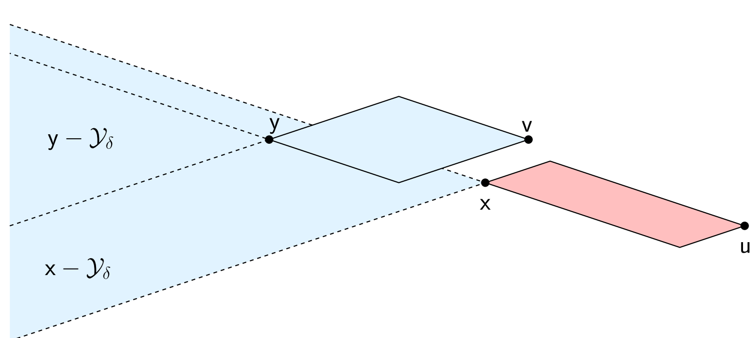

FIG. 2. *With* $\mathsf{x},\mathsf{y},\mathsf{u},\mathsf{v}$ *as in the picture, the paths contributing to* $\mathsf{t}^{\omega}_{\mathsf{x}}\mathsf{t}^{\omega}_{\mathsf{y}}(\mathsf{q}^{\omega}_{\mathsf{y},\mathsf{v}} - \mathbf{q}_{\mathsf{y},\mathsf{v}})$ *lie inside the blue region, while the paths contributing to* $\mathsf{q}^{\omega}_{\mathsf{x},\mathsf{u}} - \mathbf{q}_{\mathsf{x},\mathsf{u}}$ *lie inside the red region. These two quantities are thus independent* (*w.r.t. the disorder*).

Consequently, neglecting nonpositive terms, we obtain

$$\mathbb{E}\Big\{\sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_K^+}\mathsf{t}_{\mathsf{x}}^{\omega}(\mathsf{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega}-\mathbf{q}_{\mathsf{y}-\mathsf{x}})\Big\}^2 \le \sum_{\substack{\mathsf{x},\mathsf{y}\in\mathcal{H}_N^-\\ \mathsf{u},\mathsf{v}\in\mathcal{H}_K^+}}\mathbb{E}\{\mathsf{t}_{\mathsf{x}}^{\omega}\mathsf{q}_{\mathsf{x},\mathsf{u}}^{\omega}\mathsf{t}_{\mathsf{y}}^{\omega}\mathsf{q}_{\mathsf{y},\mathsf{v}}^{\omega}+\mathsf{t}_{\mathsf{x}}^{\omega}\mathbf{q}_{\mathsf{x},\mathsf{u}}\mathsf{t}_{\mathsf{y}}^{\omega}\mathbf{q}_{\mathsf{y},\mathsf{v}}\}\mathbf{1}_{\{D(\mathsf{x},\mathsf{u})\cap D(\mathsf{y},\mathsf{v})\neq\varnothing\}}.$$

Now, it follows from the attractiveness (1.2) of the interaction that

$$\mathbb{E}\{\mathsf{t}_{\mathsf{x}}^{\omega}\mathsf{q}_{\mathsf{x},\mathsf{u}}^{\omega}\mathsf{t}_{\mathsf{y}}^{\omega}\mathsf{q}_{\mathsf{y},\mathsf{v}}^{\omega}\}\ge\mathbb{E}\{\mathsf{t}_{\mathsf{x}}^{\omega}\mathbf{q}_{\mathsf{x},\mathsf{u}}\mathsf{t}_{\mathsf{y}}^{\omega}\mathbf{q}_{\mathsf{y},\mathsf{v}}\},$$

and thus

$$\mathbb{E}\Big\{\sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_K^+}\mathsf{t}_{\mathsf{x}}^{\omega}(\mathsf{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega}-\mathbf{q}_{\mathsf{y}-\mathsf{x}})\Big\}^2 \le 2\sum_{\substack{\mathsf{x},\mathsf{y}\in\mathcal{H}_N^-\\ \mathsf{u},\mathsf{v}\in\mathcal{H}_K^+}}\mathbb{E}\{\mathsf{t}_{\mathsf{x}}^{\omega}\mathsf{q}_{\mathsf{x},\mathsf{u}}^{\omega}\mathsf{t}_{\mathsf{y}}^{\omega}\mathsf{q}_{\mathsf{y},\mathsf{v}}^{\omega}\}\mathbf{1}_{\{D(\mathsf{x},\mathsf{u})\cap D(\mathsf{y},\mathsf{v})\neq\varnothing\}}. \tag{4.1}$$

The latter expression sets up the stage for an analysis in terms of weakly interacting random walks.

4.2. *Weakly interacting random walks.* Let P_{RW} be the path measure of a random walk on $\mathbb{Z}^{d+1}$ whose independent steps are distributed according to $\{\mathbf{q}_\ell\}$. We shall use notation $\underline{X}=(X_0,X_1,\ldots)$ for the path of this random walk. Let us say that $(\mathsf{x},\mathsf{u})\in\underline{\mathsf{X}}$ if there exists n such that $\mathsf{X}_n=\mathsf{x}$ and $\mathsf{X}_{n+1}=\mathsf{u}$. In this way, $\mathrm{P}_{\mathrm{RW}}((\mathsf{x},\mathsf{u})\in\underline{\mathsf{X}})=\mathbf{t}_{\mathsf{x}}\mathbf{q}_{\mathsf{x},\mathsf{u}}$. Let also $\mathrm{P}_{\mathrm{RW}}^{\otimes}$ be the product measure for a couple of such random walks.

Given a path $\underline{\mathsf{x}}=(0=\mathsf{x}_0,\mathsf{x}_1,\mathsf{x}_2,\ldots)$, we define the random functionals

$$\mathcal{Q}_n^{\omega}(\underline{\mathsf{x}})\triangleq\prod_{i=1}^{n}\mathsf{q}_{\mathsf{x}_{i-1},\mathsf{x}_i}^{\omega}.$$

Note that $\mathbb{E}\mathcal{Q}_n^{\omega}(\underline{\mathsf{x}})=\mathrm{P}_{\mathrm{RW}}(\underline{\mathsf{X}}_n=\underline{\mathsf{x}})$, where the event $\{\underline{\mathsf{X}}_n=\underline{\mathsf{x}}\}$ means that the first n steps of $\underline{\mathsf{X}}$ are given by the corresponding steps of $\underline{\mathsf{x}}$.

Consider now two admissible trajectories $\underline{\mathsf{x}}$ and $\underline{\mathsf{y}}$. For any $n\in\mathbb{N}$, we define the diamond sausage $D(\underline{\mathsf{x}}_n)$ around the first n steps of $\underline{\mathsf{x}}$ by

$$D(\underline{\mathsf{x}}_n)\triangleq\bigcup_{1}^{n}D(\mathsf{x}_{i-1},\mathsf{x}_i).$$

By definition, $D(\underline{\mathsf{x}})\triangleq D(\underline{\mathsf{x}}_\infty)$. If $D(\underline{\mathsf{x}}_n)\cap D(\underline{\mathsf{y}}_m)=\varnothing$, then

$$\mathbb{E}\mathcal{Q}_n^{\omega}(\underline{\mathsf{x}})\mathcal{Q}_m^{\omega}(\underline{\mathsf{y}})=\mathrm{P}_{\mathrm{RW}}^{\otimes}(\underline{\mathsf{X}}_n=\underline{\mathsf{x}};\underline{\mathsf{Y}}_m=\underline{\mathsf{y}}).$$

If, however, the above diamond sausages intersect, then, by the positive association (1.2) of one-dimensional random variables,

$$\mathbb{E}\mathcal{Q}_n^{\omega}(\underline{\mathsf{x}})\mathcal{Q}_m^{\omega}(\underline{\mathsf{y}}) \geq \mathrm{P}_{\mathrm{RW}}^{\otimes}(\underline{\mathsf{X}}_n = \underline{\mathsf{x}}; \underline{\mathsf{Y}}_m = \underline{\mathsf{y}}), \tag{4.2}$$

which means that the random weights $\mathcal{Q}^{\omega}$ produce attraction between the two paths. In particular, all terms which contribute to the right-hand side of (4.1) satisfy

$$\mathbb{E}\{\mathfrak{t}_{\mathsf{x}}^{\omega}\mathsf{q}_{\mathsf{x},\mathsf{u}}^{\omega}\mathfrak{t}_{\mathsf{y}}^{\omega}\mathsf{q}_{\mathsf{y},\mathsf{v}}^{\omega}\} \geq \mathrm{P}_{\mathrm{RW}}^{\otimes}\big((\mathsf{x},\mathsf{u}) \in \underline{\mathsf{X}}; (\mathsf{y},\mathsf{v}) \in \underline{\mathsf{Y}}\big).$$

Let now $\underline{\mathsf{x}}$ and $\underline{\mathsf{y}}$ be two infinite admissible paths. We define the corresponding diamond intersection number

$$\#(\underline{\mathsf{x}},\underline{\mathsf{y}}) \stackrel{\Delta}{=} \#\{(k,\ell) : D(\mathsf{x}_{k-1},\mathsf{x}_k) \cap D(\mathsf{y}_{\ell-1},\mathsf{y}_\ell) \neq \varnothing\}.$$

Let also $\mathcal{E}$ be the event that there exist k,ℓ such that $D(\mathsf{x}_{k-1},\mathsf{x}_k) \cap D(\mathsf{y}_{\ell-1},\mathsf{y}_\ell) \neq \varnothing$, $\mathsf{x}_{k-1},\mathsf{y}_{\ell-1} \in \mathcal{H}_N^-$ and $\mathsf{x}_k,\mathsf{y}_\ell \in \mathcal{H}_K^+$. Expanding $\mathfrak{t}_{\mathsf{x}}^{\omega}$ and $\mathfrak{t}_{\mathsf{y}}^{\omega}$ as in the first line of (2.18), we infer that the sum on the right-hand side of (4.1) is bounded above by

$$\begin{aligned}\sum_{\underline{\mathsf{x}},\underline{\mathsf{y}}} &\mathbb{E}\mathcal{Q}^{\omega}(\underline{\mathsf{x}})\mathcal{Q}^{\omega}(\underline{\mathsf{y}})\#(\underline{\mathsf{x}},\underline{\mathsf{y}})\mathbf{1}_{\mathcal{E}}(\underline{\mathsf{x}},\underline{\mathsf{y}}) \\ &\stackrel{\Delta}{=} \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\underline{\mathsf{x}},\underline{\mathsf{y}}}\mathbb{E}\mathcal{Q}_m^{\omega}(\underline{\mathsf{x}})\mathcal{Q}_n^{\omega}(\underline{\mathsf{y}})\#(\underline{\mathsf{x}},\underline{\mathsf{y}})\mathbf{1}_{\mathcal{E}}(\underline{\mathsf{x}},\underline{\mathsf{y}}).\end{aligned} \tag{4.3}$$

Existence of the above limit follows by monotonicity from (4.2). We thus obtain

$$\mathbb{E}\bigg\{\sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_K^+}\mathfrak{t}_{\mathsf{x}}^{\omega}(\mathsf{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}})\bigg\}^2 \leq 2\sum_{\underline{\mathsf{x}},\underline{\mathsf{y}}}\mathbb{E}\{\mathcal{Q}^{\omega}(\underline{\mathsf{x}})\mathcal{Q}^{\omega}(\underline{\mathsf{y}})\#(\underline{\mathsf{x}},\underline{\mathsf{y}})\mathbf{1}_{\mathcal{E}}\}. \tag{4.4}$$

Of course, in order to apply the latter upper bound one needs to control the statistics of $\#(\underline{\mathsf{x}},\underline{\mathsf{y}})$. The point is that, under Assumption (A), the $\mathcal{Q}^{\omega}$-induced interaction between the paths $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$ is so weak that it does not destroy transient behavior. This phenomenon is stated in Lemma 4.1 below, in a form which happens to be particularly convenient for the latter use.

Given t, $\mathsf{u}_0,\mathsf{v}_0 \in \mathcal{L}_0$ and $\mathsf{u}_1,\mathsf{v}_1 \in \mathcal{L}_t$ consider two pieces $\underline{\mathsf{x}}_n$ and $\underline{\mathsf{y}}_m$ of of admissible trajectories (assuming that they exist): $\underline{\mathsf{x}} = (\mathsf{u}_0 = \mathsf{x}_0,\ldots,\mathsf{x}_n = \mathsf{u}_1,\ldots)$ from u_0 to u_1, and $\underline{\mathsf{y}} = (\mathsf{v}_0 = \mathsf{y}_0,\ldots,\mathsf{y}_m = \mathsf{v}_1,\ldots)$ from v_0 to v_1.

LEMMA 4.1. *Once $\lambda > \lambda_0$ is fixed, for every $\eta > 0$ there exists $\beta_0 > 0$ and $p_\infty > 0$ such that*

$$\mathbb{E}\mathcal{Q}_n^{\omega}(\underline{\mathsf{x}})\mathcal{Q}_m^{\omega}(\underline{\mathsf{y}}) \leq \exp\{\tfrac{1}{2}\eta t\mathbf{1}_{\{D(\underline{\mathsf{x}}_n)\cap D(\underline{\mathsf{y}}_m)\neq\varnothing\}}\}\mathrm{P}_{\mathrm{RW}}^{\otimes}(\underline{\mathsf{X}}_n = \underline{\mathsf{x}}, \underline{\mathsf{Y}}_m = \underline{\mathsf{y}}), \tag{4.5}$$

uniformly in $\beta \in [0,\beta_0)$, provided that Assumption (A) is satisfied with $p < p_\infty$. The inequality (4.5) holds simultaneously for all t, $\mathsf{u}_0,\mathsf{v}_0 \in \mathcal{L}_0$ and $\mathsf{u}_1,\mathsf{v}_1 \in \mathcal{L}_t$ and the corresponding admissible trajectories $\underline{\mathsf{x}}$, $\underline{\mathsf{y}}$.

PROOF. The left-hand side of (4.5) equals to $\mathsf{P}^{\otimes}_{\mathrm{RW}}(\underline{\mathsf{X}}_n = \underline{\mathsf{x}}, \underline{\mathsf{Y}}_m = \underline{\mathsf{y}})$ whenever $D(\underline{\mathsf{x}}_n) \cap D(\underline{\mathsf{y}}_m) = \varnothing$. Indeed, in such a situation, $\mathcal{Q}^{\omega}_n(\underline{\mathsf{x}})$ and $\mathcal{Q}^{\omega}_m(\underline{\mathsf{y}})$ are independent.

We proceed to consider the case when $D(\underline{\mathsf{x}}_n) \cap D(\underline{\mathsf{y}}_m) \neq \varnothing$. Let us say that a path $\gamma \in \mathcal{T}_{\mathsf{u}_0,\mathsf{u}_1}$ is compatible with $\underline{\mathsf{x}}_n$; $\gamma \sim \underline{\mathsf{x}}_n$, if $\underline{\mathsf{x}}_n \setminus \{\mathsf{x}_0, \mathsf{x}_n\}$ is precisely the collection of all the cone points of γ. Similarly for $\gamma' \sim \underline{\mathsf{y}}_m$. The left-hand side in (4.5) is

$$e^{2t\xi} \sum_{\substack{\gamma\sim\underline{\mathsf{x}}_n \\ \gamma'\sim\underline{\mathsf{y}}_n}} \mathbb{E} W^{\omega}_{\lambda,\beta}(\gamma) W^{\omega}_{\lambda,\beta}(\gamma') = \sum_{\substack{\gamma\sim\underline{\mathsf{x}}_n \\ \gamma'\sim\underline{\mathsf{y}}_m}} \exp\{2t\xi - \lambda(|\gamma| + |\gamma'|) - \Phi_\beta(\gamma, \gamma')\},$$

where the annealed interaction potential $\Phi_\beta(\gamma, \gamma')$ is given by

$$\Phi_\beta(\gamma, \gamma') = \sum_{\mathsf{w}\in\mathbb{Z}^{d+1}} \phi_\beta(\ell_{\gamma\cup\gamma'}(\mathsf{w})) \qquad \text{with } \phi_\beta(\ell) \stackrel{\Delta}{=} -\log \mathbb{E} e^{-\ell V^{\omega}}.$$

Above, $\ell_{\gamma\cup\gamma'}(\mathsf{w})$ is the total combined local time of the couple (γ, γ') in w. Therefore, ignoring the interaction, one derives the following upper bound:

$$\mathbb{E} \mathcal{Q}^{\omega}_n(\underline{\mathsf{x}}) \mathcal{Q}^{\omega}_m(\underline{\mathsf{y}}) \leq \sum_{\substack{\gamma\sim\underline{\mathsf{x}}_n \\ \gamma'\sim\underline{\mathsf{y}}_n}} \exp\{2t\xi - \lambda(|\gamma| + |\gamma'|)\},$$

that is, in terms of the corresponding expression for the simple symmetric random walk on $\mathbb{Z}^{d+1}$ with the constant killing rate $\lambda - \lambda_0 = \lambda - \log(2d) > 0$.

Similarly,

$$\mathsf{P}^{\otimes}_{\mathrm{RW}}(\underline{\mathsf{X}}_n = \underline{\mathsf{x}}, \underline{\mathsf{Y}}_m = \underline{\mathsf{y}}) = \sum_{\substack{\gamma\sim\underline{\mathsf{x}}_n \\ \gamma'\sim\underline{\mathsf{y}}_m}} \exp\{2t\xi - \lambda(|\gamma| + |\gamma'|) - \Phi_\beta(\gamma) - \Phi_\beta(\gamma')\}.$$

The function ϕ_β is subadditive [5, 6]. Consequently $\phi_\beta(\ell) \leq \ell\phi_\beta(1)$. We conclude that the following lower bound on $\mathsf{P}^{\otimes}_{\mathrm{RW}}(\underline{\mathsf{X}}_n = \underline{\mathsf{x}}, \underline{\mathsf{Y}}_m = \underline{\mathsf{y}})$ holds for any $c > 0$:

$$e^{-ct\phi_\beta(1)} \sum_{\substack{\gamma\sim\underline{\mathsf{x}}_n \\ \gamma'\sim\underline{\mathsf{y}}_m}} \exp\{2t\xi - \lambda(|\gamma| + |\gamma'|)\} \mathbb{1}_{\{|\gamma|+|\gamma'|\leq ct\}}.$$

Recall that t is the horizontal span of both γ and γ' and that $\lambda > \lambda_0 = \log(2d)$ is fixed. Thus, as directly follows from the properties of the simple random walk on $\mathbb{Z}^{d+1}$ subject to a constant killing potential $\lambda - \lambda_0$, there exists $\varepsilon = \varepsilon(c)$, tending to zero as $c \to \infty$, such that

$$\sum_{\substack{\gamma\sim\underline{\mathsf{x}}_n \\ \gamma'\sim\underline{\mathsf{y}}_m}} \exp\{-\lambda(|\gamma| + |\gamma'|)\} \mathbb{1}_{\{|\gamma|+|\gamma'|\leq ct\}} \geq \bigl(1 - \varepsilon(c)\bigr) \sum_{\substack{\gamma\sim\underline{\mathsf{x}}_n \\ \gamma'\sim\underline{\mathsf{y}}_m}} \exp\{-\lambda(|\gamma| + |\gamma'|)\}.$$

Altogether, we conclude that, for any $c > 0$,

$$\frac{\mathbb{E}\mathcal{Q}_n^{\omega}(\underline{\mathsf{x}})\mathcal{Q}_m^{\omega}(\underline{\mathsf{y}})}{\mathrm{P}_{\mathrm{RW}}^{\otimes}(\underline{\mathsf{X}}_n = \underline{\mathsf{x}}, \underline{\mathsf{Y}}_m = \underline{\mathsf{y}})} \le \exp\{ct\phi_\beta(1) - \log(1 - \varepsilon(c))\}.$$

In its turn, the smallness of $\phi_\beta(1)$ is controlled through

$$\lim_{\beta\to 0} \phi_\beta(1) = -\log(1 - p).$$

Consequently, the claim of the Lemma follows first by taking c sufficiently large and then by choosing β and p appropriately small. $\square$

4.3. *Upper bounds in terms of synchronized random walks.* Let us explain how Lemma 4.1 is put to work in order to control (4.3). At this stage, it happens to be convenient to synchronize the two trajectories $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$, by expressing all the above quantities in terms of another induced $\mathbb{Z} \times \mathbb{Z}^d \times \mathbb{Z}^d$-valued random walk $(\underline{\mathsf{U}}, \underline{\mathsf{V}})$: Let $\underline{\mathsf{x}}$ and $\underline{\mathsf{y}}$ be realizations of $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$. Let us label all the $\mathcal{L}_n$-hyperplanes which are simultaneously hit by both the $\underline{\mathsf{x}}$ and $\underline{\mathsf{y}}$ trajectories as $n_1, n_2, \ldots,$ with $\mathsf{u}_1, \mathsf{u}_2, \ldots$ and $\mathsf{v}_1, \mathsf{v}_2, \ldots$ the corresponding hitting points (see Figure 3). Then the induced trajectory of $(\underline{\mathsf{U}}, \underline{\mathsf{V}})$ is $(\underline{\mathsf{u}}, \underline{\mathsf{v}})$. We denote by $t_1, t_2, \ldots$ the horizontal spans of the steps of $(\underline{\mathsf{u}}, \underline{\mathsf{v}})$. We shall use $\widehat{\mathrm{P}}$ for the path measure of $(\underline{\mathsf{U}}, \underline{\mathsf{V}})$. The distribution of a single step under $\widehat{\mathrm{P}}$ is given by

$$\begin{aligned}\widehat{\mathrm{P}}(\mathsf{u}, \mathsf{v}) &= \widehat{\mathrm{P}}(t, u, v)\\ &= \sum_{\substack{n=1\\ m=1}}^{t} \sum_{\substack{0<t_1<\ldots<t_n=t\\ 0<s_1<\ldots<s_m=t}} \sum_{\substack{\mathsf{x}_i\in\mathcal{L}_{t_i}\\ \mathsf{y}_j\in\mathcal{L}_{s_j}}} \prod_1^n \mathbf{q}_{\mathsf{x}_i-\mathsf{x}_{i-1}} \prod_1^m \mathbf{q}_{\mathsf{y}_i-\mathsf{y}_{i-1}} \prod_{\substack{0<i<n\\ 0<j<m}} \mathbf{1}_{\{t_i\neq s_j\}},\end{aligned}$$

where we have set $\mathsf{x}_0 = \mathsf{y}_0 = 0$ and $\mathsf{x}_n = \mathsf{u}$, $\mathsf{y}_m = \mathsf{v}$. Alternatively,

$$\widehat{\mathrm{P}}(\mathsf{u}, \mathsf{v}) = \widehat{\mathrm{P}}(t, u, v) = \mathrm{P}_{\mathrm{RW}}^{\otimes}(T(\underline{\mathsf{X}}, \underline{\mathsf{Y}}) = t; \mathsf{u} \in \mathrm{Range}(\underline{\mathsf{X}}); \mathsf{v} \in \mathrm{Range}(\underline{\mathsf{Y}})),$$

where

$$T(\underline{\mathsf{X}}, \underline{\mathsf{Y}}) \stackrel{\Delta}{=} \inf\{n : \mathrm{Range}(\underline{\mathsf{X}}) \cap \mathcal{L}_n \neq \varnothing \text{ and } \mathrm{Range}(\underline{\mathsf{Y}}) \cap \mathcal{L}_n \neq \varnothing\}$$

is the (random) horizontal span of a step of the $(\underline{\mathsf{U}}, \underline{\mathsf{V}})$-random walk. In view of the uniform exponential tails of $\{\mathbf{q}_N\}$, there exists $\kappa = \kappa(\lambda) > 0$ such that

$$\widehat{\mathrm{P}}(T > \ell) \lesssim e^{-\kappa\ell}, \tag{4.6}$$

uniformly in l and in $\beta \ge 0$.

Let us go back to (4.3). The i.i.d. horizontal spans of $(\underline{\mathsf{U}}, \underline{\mathsf{V}})$-steps will be denoted by $T_1, T_2, \ldots$. To ease notation, set $D_k(\underline{\mathsf{u}}) \stackrel{\Delta}{=} D(\mathsf{u}_k, \mathsf{u}_{k+1})$ and similarly for $D_k(\underline{\mathsf{v}})$. Obviously, if a pair $(\underline{\mathsf{x}}, \underline{\mathsf{y}})$ of $(\underline{\mathsf{X}}, \underline{\mathsf{Y}})$-paths is compatible with a synchronized $(\underline{\mathsf{U}}, \underline{\mathsf{V}})$-path $(\underline{\mathsf{u}}, \underline{\mathsf{v}})$; $(\underline{\mathsf{x}}, \underline{\mathsf{y}}) \sim (\underline{\mathsf{u}}, \underline{\mathsf{v}})$, then

$$\#(\underline{\mathsf{x}}, \underline{\mathsf{y}}) \le \sum_k T_k \mathbf{1}_{\{D_k(\underline{\mathsf{u}})\cap D_k(\underline{\mathsf{v}})\neq\varnothing\}}.$$

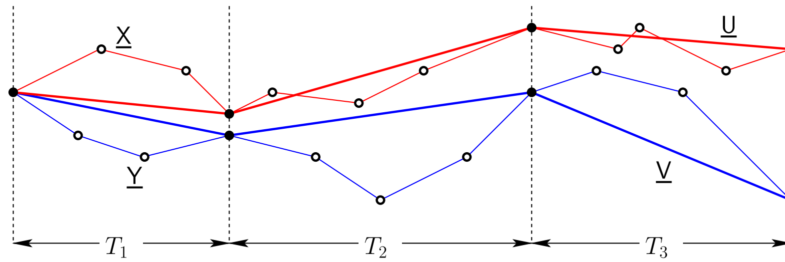

FIG. 3. *The* $\underline{\mathsf{X}}$, $\underline{\mathsf{Y}}$ *and* $(\underline{\mathsf{U}}, \underline{\mathsf{V}})$ *random walks.*

By Lemma 4.1, once $\lambda > \lambda_0$ is fixed, for every $\eta > 0$ there exist $\beta_0 > 0$ and $p_\infty > 0$ such that[5]

$$\sum_{(\underline{\mathsf{x}},\underline{\mathsf{y}})\sim(\underline{\mathsf{u}},\underline{\mathsf{v}})} \mathbb{E}\mathcal{Q}^\omega(\underline{\mathsf{x}})\mathcal{Q}^\omega(\underline{\mathsf{y}}) \le \exp\left\{\frac{1}{2}\eta\sum_k T_k \mathbf{1}_{\{D_k(\underline{\mathsf{u}})\cap D_k(\underline{\mathsf{v}})\neq\varnothing\}}\right\}\widehat{\mathrm{P}}(\underline{\mathsf{u}},\underline{\mathsf{v}}).$$

Therefore, since $xe^x \le e^{2x}$ for all $x \ge 0$, (4.4) implies that

$$\mathbb{E}\left\{\sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_K^+} \mathsf{t}_{\mathsf{x}}^\omega(\mathsf{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega} - \mathbf{q}_{\mathsf{y}-\mathsf{x}})\right\}^2 \le \frac{2}{\eta}\widehat{\mathrm{E}}\exp\left\{\eta\sum_k T_k\mathbf{1}_{\{D_k(\underline{\mathsf{U}})\cap D_k(\underline{\mathsf{V}})\neq\varnothing\}}\right\}\mathbf{1}_{\widehat{\mathcal{E}}}, \tag{4.7}$$

where $\widehat{\mathcal{E}}$ is the analog of $\mathcal{E}$ for the synchronized random walks, that is,

$$\widehat{\mathcal{E}} \triangleq \{\exists k : D_k(\underline{\mathsf{U}}) \cap D_k(\underline{\mathsf{V}}) \neq \varnothing; \mathsf{U}_k, \mathsf{V}_k \in \mathcal{H}_N^- \text{ and } \mathsf{U}_{k+1}, \mathsf{V}_{k+1} \in \mathcal{H}_K^+\}.$$

Of course $\mathcal{E} \subset \widehat{\mathcal{E}}$, in the sense that if $\mathcal{E}$ holds for $(\underline{\mathsf{x}}, \underline{\mathsf{y}})$, then $\widehat{\mathcal{E}}$ also holds for the synchronized $(\underline{\mathsf{u}}, \underline{\mathsf{v}})$ path.

Let us now bound the expectation in the right-hand side of (4.7), uniformly in η sufficiently small. Let $\mathsf{Z}_k \triangleq \mathsf{U}_k - \mathsf{V}_k$, and notice that there exists a constant $\alpha = \alpha(d, \delta)$ such that

$$\exp\{\eta T_k\mathbf{1}_{\{D_k(\underline{\mathsf{U}})\cap D_k(\underline{\mathsf{V}})\neq\varnothing\}}\} \le \exp\{\eta T_k\mathbf{1}_{\{T_k>\alpha\|\mathsf{Z}_{k-1}^\perp\|\}}\}.$$

Writing $\exp\{\eta T_k\mathbf{1}_{\{T_k>\alpha\|\mathsf{Z}_{k-1}^\perp\|\}}\} = ((e^{\eta T_k} - 1)\mathbf{1}_{\{T_k>\alpha\|\mathsf{Z}_{k-1}^\perp\|\}} + 1)$ and expanding, we obtain

$$\exp\left\{\sum_{k=1}^M \eta T_k\mathbf{1}_{\{T_k>\alpha\|\mathsf{Z}_{k-1}^\perp\|\}}\right\} = \sum_{A\subset\{1,\dots,M\}}\prod_{k\in A}(e^{\eta T_k} - 1)\mathbf{1}_{\{T_k>\alpha\|\mathsf{Z}_{k-1}^\perp\|\}}.$$

---

[5] Strictly speaking, the inequality makes sense for restrictions to any finite number of steps of the $(\underline{\mathsf{u}}, \underline{\mathsf{v}})$-trajectory.

Since $(e^{\eta T_k}-1)/(e^{\eta}-1) \le T_k e^{\eta T_k}$, we can bound the right-hand side from above by

$$\sum_{n\ge 0}(e^{\eta}-1)^n \sum_{\substack{A\subset\{1,\dots,M\}\\ |A|=n}} \prod_{k\in A} T_k e^{\eta T_k}\mathbf{1}_{\{T_k>\alpha\|\mathsf{Z}_{k-1}^{\perp}\|\}}\mathbf{1}_{\widehat{\mathcal{E}}}. \tag{4.8}$$

Let us write $A=\{a_1,a_2,\dots,a_n\}$, with $a_1<a_2<\cdots<a_n$, and let us set $a_0=0$. We are going to split the trajectories into n "bubbles," the ith bubble being composed of the steps $\mathsf{Z}_{a_{i-1}+1},\dots,\mathsf{Z}_{a_i}$. The horizontal span B_i of the ith bubble is thus

$$B_i \stackrel{\Delta}{=} \sum_{k=a_{i-1}+1}^{a_i} T_k, \qquad 1\le i\le n.$$

4.4. *Proof of Proposition* 3.1. We only prove (3.1) and (3.2), the third claim, (3.3), being a variant of the latter.

We first prove (3.1). In this case, we only retain from the event $\widehat{\mathcal{E}}$ the constraint that $\sum_i B_i > K$. More precisely, we bound above the $\widehat{\mathrm{E}}$-expectation of the sum in (4.8) by

$$\widehat{\mathrm{E}}\sum_{n\ge 1}(e^{\eta}-1)^{n-1}\sum_{|A|=n}\mathbf{1}_{\{\sum_i B_i>K\}}\prod_{k\in A}T_k e^{\eta T_k}\mathbf{1}_{\{T_k>\alpha\|\mathsf{Z}_{k-1}^{\perp}\|\}}. \tag{4.9}$$

Therefore, by the Markov property, (4.7) implies

$$\begin{aligned}&\sup_N \mathbb{E}\Biggl\{\sum_{\mathsf{x}\in\mathcal{H}_N^-}\sum_{\mathsf{y}\in\mathcal{H}_K^+}\mathfrak{t}_{\mathsf{x}}^{\omega}(\mathfrak{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega}-\mathbf{q}_{\mathsf{y}-\mathsf{x}})\Biggr\}^2\\ &\qquad\le\frac{2}{\eta}\sum_{n\ge 1}(e^{\eta}-1)^{n-1}\sum_{\substack{B_1,\dots,B_n\\ \sum_i B_i>K}}\prod_{i=1}^n I(B_i),\end{aligned} \tag{4.10}$$

where, for $B\in\mathbb{N}$,

$$I(B)\stackrel{\Delta}{=}\sup_{\mathsf{z}\in\mathbb{Z}^d}\sum_{m=1}^{B}\widehat{\mathrm{E}}\Biggl(T_m e^{\eta T_m};\sum_{k=1}^m T_k=B,\,T_m>\alpha\|Z_{m-1}^{\perp}\|\,|\mathsf{Z}_0^{\perp}=\mathsf{z}\Biggr).$$

We need a reasonable upper bound on the latter quantities. Recall that we can choose η as small as we wish. Observe first that (4.6) and a standard large deviation estimate imply the existence of $\varepsilon>0$ and $c>0$ such that, uniformly in $B\in\mathbb{N}$,

$$\sup_{\mathsf{z}\in\mathbb{Z}^d}\sum_{m=1}^{\varepsilon B}\widehat{\mathrm{E}}\Biggl(T_m e^{\eta T_m};\sum_{k=1}^m T_k=B\Big|\mathsf{Z}_0^{\perp}=\mathsf{z}\Biggr)\lesssim e^{-cB}.$$

On the other hand, relying again on (4.6) and using the local limit theorem for i.i.d. random variables with exponential tails, we obtain that

$$
\begin{aligned}
&\sup_{\mathsf{z}\in\mathbb{Z}^d}\sum_{m=\varepsilon B}^{B}\widehat{\mathrm{E}}\Bigl(T_m e^{\eta T_m};\sum_{k=1}^{m}T_k=B, T_m>\alpha\|Z_{m-1}^{\perp}\||\mathsf{Z}_0^{\perp}=\mathsf{z}\Bigr)\\
&\quad\lesssim\sum_{t\ge 1}te^{-(\nu-\eta)t}\sup_{\mathsf{z}\in\mathbb{Z}^d}\sum_{m=\varepsilon B}^{B}\widehat{\mathrm{P}}(\|Z_{m-1}^{\perp}\|<t/\alpha|\mathsf{Z}_0^{\perp}=\mathsf{z})\\
&\quad\lesssim\sum_{t\ge 1}e^{-(\nu-\eta)t}\frac{t^{d+1}}{B^{d/2}}\\
&\quad\lesssim B^{-d/2}.
\end{aligned}
\tag{4.11}
$$

We therefore conclude that, for any $B\in\mathbb{N}$,

$$
I(B)\lesssim B^{-d/2}.
$$

Let us now use this bound to control the right-hand side of (4.10). For fixed n, let $L=\sum_{i=1}^{n}B_i$; then there must be an index j such that $\prod_{i=1}^{n}I(B_i)\lesssim(n/L)^{d/2}\prod_{i\neq j}I(B_i)$. Therefore, choosing η small enough, we have

$$
\begin{aligned}
&\sum_{n\ge 1}(e^{\eta}-1)^{n-1}\sum_{\substack{B_1,\dots,B_n\\ \sum_i B_i>K}}\prod_{i=1}^{n}I(B_i)\\
&\quad\lesssim\sum_{L>K}L^{-d/2}\sum_{n\ge 0}n^{1+d/2}\Bigl((e^{\eta}-1)\sum_{B\ge 1}I(B)\Bigr)^{n}\\
&\quad\lesssim\sum_{L>K}L^{-d/2}\lesssim(1+K)^{1-d/2}.
\end{aligned}
\tag{4.12}
$$

Let us now turn to the proof of (3.2).

We proceed to bound the right-hand side of (3.2) in terms of the synchronized random walks $\underline{\mathsf{U}}$ and $\underline{\mathsf{V}}$. As before, $\mathsf{Z}_k=\mathsf{U}_k-\mathsf{V}_k$. Let j_0 be such that $\mathsf{Z}_{j_0-1}\in\mathcal{H}_K^-$ and $\mathsf{Z}_{j_0}\in\mathcal{H}_K^+$. We need to derive a bound on

$$
\widehat{\mathrm{E}}\exp\Bigl\{\sum_k\eta T_k\mathbf{1}_{\{T_k>\alpha\|\mathsf{Z}_{k-1}^{\perp}\|\}}\Bigr\}\mathbf{1}_{\{D_{j_0-1}(\underline{\mathsf{U}})\cap D_{j_0-1}(\underline{\mathsf{V}})\neq\varnothing\}}.
$$

Expanding as in (4.9), we may restrict attention to sets A which contain an element a_{i_0} such that $a_{i_0}=j_0$. This implies that, if $\sum_{i=1}^{i_0}B_i=K+t$, the excess t must be entirely due to the j_0th step of Z. In particular, this quantity has exponential tails, and, following the derivation of (4.11),

$$
I(B_{i_0})\lesssim e^{-(\nu-\eta)t}B_{i_0}^{-d/2}.
$$

We can thus write, proceeding as in (4.12),

$$
\begin{aligned}
&\mathbb{E}\Big\{\sum_{\mathsf{x}\in\mathcal{H}_K^-}\sum_{\mathsf{y}\in\mathcal{H}_K^+}\mathfrak{t}_{\mathsf{x}}^{\omega}(\mathsf{q}_{\mathsf{y}-\mathsf{x}}^{\theta_{\mathsf{x}}\omega}-\mathbf{q}_{\mathsf{y}-\mathsf{x}})\Big\}^2\\
&\quad\le\frac{2}{\eta}\sum_{t\ge1}\sum_{i_0\ge1}\sum_{n\ge0}(e^{\eta}-1)^{n+i_0-1}\sum_{\substack{B_1,\dots,B_{n+i_0}\\ \sum_{i=1}^{i_0}B_i=K+t}}\prod_{i=1}^{n+i_0}I(B_i)\\
&\quad\lesssim\sum_{t\ge1}e^{-(\nu-\eta)t}\sum_{i_0\ge1}(e^{\eta}-1)^{i_0}\sum_{\substack{B_1,\dots,B_{i_0}\\ \sum_{i=1}^{i_0}B_i=K+t}}\prod_{i=1}^{i_0}B_i^{-d/2}\\
&\quad\lesssim\sum_{t\ge1}e^{-(\nu-\eta)t}(K+t)^{-d/2}\\
&\quad\lesssim(1+K)^{-d/2}.
\end{aligned}
$$

REMARK 4.2. The above computations readily imply the following: Let $\mathsf{u},\mathsf{v}\in\mathcal{L}_0$ and let $\widehat{\mathrm{P}}_{\mathsf{u},\mathsf{v}}$ be the distribution of the synchronized $(\underline{\mathsf{U}},\underline{\mathsf{V}})$ random walk starting from (u,v). Then, under Assumption (A),

$$
\widehat{\mathrm{E}}_{\mathsf{u},\mathsf{v}}\exp\Big\{\eta\sum_k T_k\mathbf{1}_{\{D_k(\underline{\mathsf{U}})\cap D_k(\underline{\mathsf{V}})\neq\varnothing\}}\Big\}\lesssim1, \tag{4.13}
$$

uniformly in u,v and in all η sufficiently small.

4.5. *Positivity of* $\mathfrak{d}^{\omega}$ *on the event* $\{0\in\mathrm{Cl}_\infty(V)\}$. Let $0\in\mathrm{Cl}_\infty(V)$. Then $\mathfrak{d}^{\omega}>0$ if there exists $\mathsf{x}=(x,t)$ such that $\mathfrak{d}^{\theta_{\mathsf{x}}\omega}>0$. Indeed, such x should necessarily satisfy $\mathsf{x}\in\mathrm{Cl}_\infty(V)$. Hence, there exists a nearest-neighbor *finite* path $\gamma=(\gamma(0),\dots,\gamma(n))$ from 0 to x such that $\gamma(l)\in\mathrm{Cl}_\infty(V)$ for all $l=0,\dots,n$ and, consequently, such that $W^{\omega}_{\lambda,\beta}(\gamma)>0$. However,

$$
\mathfrak{D}_N^{\omega}\ge W^{\omega}_{\lambda,\beta}(\gamma)\mathfrak{D}_{N-t}^{\theta_{\mathsf{x}}\omega}.
$$

It follows that

$$
\liminf_{N\to\infty}e^{N\xi}\mathfrak{D}_N^{\omega}\gtrsim e^{t\xi}W^{\omega}_{\lambda,\beta}(\gamma)\mathfrak{s}^{\theta_{\mathsf{x}}\omega}.
$$

It remains to show that

$$
\mathbb{P}(\exists\mathsf{x}:\mathfrak{s}^{\theta_{\mathsf{x}}\omega}>0)=1.
$$

In fact, an ostensibly stronger claim holds:

LEMMA 4.3. *Under conditions of Theorem 2.3,*

$$\mathbb{P}(\exists \mathsf{x} \in \mathcal{L}_0 : \mathfrak{s}^{\theta_{\mathsf{x}}\omega} > 0) = 1.$$

PROOF. The proof is by the second moment method, and based on L^2-estimates at weak disorder as developed in the preceding subsection. Let $B_n \subset \mathcal{L}_0$ be the d-dimensional lattice box of side-length n,

$$B_n \stackrel{\Delta}{=} \{\mathsf{x} = (x_1, \dots, x_d, 0) : x_l \in \{0, \dots, n-1\} \text{ for } l = 1, \dots, d\}.$$

By Theorem 2.3, $\mathbb{E}\mathfrak{s}^{\theta_{\mathsf{x}}\omega} \equiv 1$. We claim that the variance

$$\mathbb{V}\mathrm{ar}\left(\frac{1}{n^d} \sum_{\mathsf{x} \in B_n} (\mathfrak{s}^{\theta_{\mathsf{x}}\omega} - 1)\right) \lesssim \frac{1}{n^{d/2-1}}. \tag{4.14}$$

The conclusion of the lemma would then follow by Chebyshev's estimate and a Borel–Cantelli argument. Now, the estimates developed in the preceding subsections imply that, under Assumption (A), the extra attraction stemming from integration of the factors $\mathcal{Q}^{\omega}$ over intersecting diamonds does not alter the statistical properties of the effective d-dimensional random walks $(\underline{\mathsf{X}}, \underline{\mathsf{Y}})$, or, equivalently, of the synchronized random walks $(\underline{\mathsf{U}}, \underline{\mathsf{V}})$. In particular, for any $\mathsf{x}, \mathsf{y} \in B_n$,

$$|\mathbb{E}(\mathfrak{s}^{\theta_{\mathsf{x}}\omega} - 1)(\mathfrak{s}^{\theta_{\mathsf{y}}\omega} - 1)| \lesssim \widehat{\mathrm{P}}_{\mathsf{x},\mathsf{y}}\big(D(\underline{\mathsf{U}}) \cap D(\underline{\mathsf{V}}) \neq \varnothing\big) \lesssim \frac{1}{|\mathsf{x} - \mathsf{y}|^{d/2-1}}. \tag{4.15}$$

Indeed, the second inequality above is straightforward. As for the first inequality in (4.15), proceeding as in the proof of (4.7), we infer that

$$|\mathbb{E}(\mathfrak{s}^{\theta_{\mathsf{x}}\omega} - 1)(\mathfrak{s}^{\theta_{\mathsf{y}}\omega} - 1)| \lesssim \widehat{\mathrm{E}} \exp\left\{\eta \sum_k T_k \mathbf{1}_{\{D_k(\underline{\mathsf{U}}) \cap D_k(\underline{\mathsf{V}}) \neq \varnothing\}}\right\} \mathbf{1}_{\{D(\underline{\mathsf{U}}) \cap D(\underline{\mathsf{V}}) \neq \varnothing\}}.$$

By the strong Markov property and in view of (4.13),

$$\widehat{\mathrm{E}} \exp\left\{\eta \sum_k T_k \mathbf{1}_{\{D_k(\underline{\mathsf{U}}) \cap D_k(\underline{\mathsf{V}}) \neq \varnothing\}}\right\} \mathbf{1}_{\{D(\underline{\mathsf{U}}) \cap D(\underline{\mathsf{V}}) \neq \varnothing\}} \lesssim \widehat{\mathrm{P}}_{\mathsf{x},\mathsf{y}}\big(D(\underline{\mathsf{U}}) \cap D(\underline{\mathsf{V}}) \neq \varnothing\big),$$

and (4.15) follows.

The variance decay estimate (4.14) is a direct consequence of (4.15)

$$\mathbb{V}\mathrm{ar}\left(\frac{1}{n^d} \sum_{\mathsf{x} \in B_n} (\mathfrak{s}^{\theta_{\mathsf{x}}\omega} - 1)\right) \lesssim \frac{1}{n^{2d}} \cdot n^d \cdot \sum_{k=1}^{n} \frac{k^{d-1}}{k^{d/2-1}}.$$

$\square$

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REFERENCES

[1] BOLTHAUSEN, E. (1989). A note on the diffusion of directed polymers in a random environment. *Comm. Math. Phys.* **123** 529–534. MR1006293

[2] CAMPANINO, M., IOFFE, D. and LOUIDOR, O. (2010). Finite connections for supercritical Bernoulli bond percolation in 2D. *Markov Process. Related Fields* **16** 225–266. MR2666853

[3] CARMONA, P. and HU, Y. (2006). Strong disorder implies strong localization for directed polymers in a random environment. *ALEA Lat. Am. J. Probab. Math. Stat.* **2** 217–229. MR2249669

[4] COMETS, F. and YOSHIDA, N. (2006). Directed polymers in random environment are diffusive at weak disorder. *Ann. Probab.* **34** 1746–1770. MR2271480

[5] FLURY, M. (2008). Coincidence of Lyapunov exponents for random walks in weak random potentials. *Ann. Probab.* **36** 1528–1583. MR2435858

[6] IOFFE, D. and VELENIK, Y. (2008). Ballistic phase of self-interacting random walks. In *Analysis and Stochastics of Growth Processes and Interface Models* 55–79. Oxford Univ. Press, Oxford. MR2603219

[7] KAUP, L. and KAUP, B. (1983). *Holomorphic Functions of Several Variables: An Introduction to the Fundamental Theory. de Gruyter Studies in Mathematics* **3**. de Gruyter, Berlin. MR0716497

[8] SINAI, Y. G. (1995). A remark concerning random walks with random potentials. *Fund. Math.* **147** 173–180. MR1341729

[9] SZNITMAN, A.-S. (1998). *Brownian Motion, Obstacles and Random Media*. Springer, Berlin. MR1717054

[10] ZERNER, M. P. W. (1998). Directional decay of the Green's function for a random nonnegative potential on $\mathbf{Z}^d$. *Ann. Appl. Probab.* **8** 246–280. MR1620370

[11] ZYGOURAS, N. (2009). Lyapounov norms for random walks in low disorder and dimension greater than three. *Probab. Theory Related Fields* **143** 615–642. MR2475675

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