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# SHARP HEAT KERNEL ESTIMATES FOR RELATIVISTIC STABLE PROCESSES IN OPEN SETS

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In this paper, we establish sharp two-sided estimates for the transition densities of relativistic stable processes [i.e., for the heat kernels of the operators $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$] in $C^{1,1}$ open sets. Here $m>0$ and $\alpha\in(0,2)$. The estimates are uniform in $m\in(0,M]$ for each fixed $M>0$. Letting $m\downarrow 0$, we recover the Dirichlet heat kernel estimates for $\Delta^{\alpha/2}:=-(-\Delta)^{\alpha/2}$ in $C^{1,1}$ open sets obtained in [14]. Sharp two-sided estimates are also obtained for Green functions of relativistic stable processes in bounded $C^{1,1}$ open sets.

**1. Introduction.** Throughout this paper we assume that $d\geq 1$ and $\alpha\in(0,2)$. For any $m>0$, a relativistic $\alpha$-stable process $X^m$ on $\mathbb{R}^d$ with mass $m$ is a Lévy process with characteristic function given by

$$\mathbb{E}\big[\exp\big(i\xi\cdot(X_t^m-X_0^m)\big)\big]=\exp\big(-t\big((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\big)\big),\qquad \xi\in\mathbb{R}^d. \tag{1.1}$$

The limiting case $X^0$, corresponding to $m=0$, is a (rotationally) symmetric $\alpha$-stable (Lévy) process on $\mathbb{R}^d$ which we will simply denote as $X$. The infinitesimal generator of $X^m$ is $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$. Note that when $m=1$, this infinitesimal generator reduces to $1-(1-\Delta)^{\alpha/2}$. Thus the 1-resolvent kernel of the relativistic $\alpha$-stable process $X^1$ on $\mathbb{R}^d$ is just the Bessel potential kernel. (See [7] for more on this connection.) When $\alpha=1$, the infinitesimal generator reduces to the so-called free relativistic Hamiltonian $m-\sqrt{-\Delta+m^2}$. The operator $m-\sqrt{-\Delta+m^2}$ is very important in mathematical physics due to its application to relativistic quantum mechanics. Physical models related to this operator have been much studied over the past 30 years, and there exists a huge literature on the properties of relativistic Hamiltonians (see, e.g., [8, 26, 31, 36, 37, 42]). For recent papers in the mathematical physics literature related to the relativistic Hamiltonian, we refer the readers to [25, 27, 28, 38] and the references therein. Various fine properties of relativistic $\alpha$-stable processes have been studied recently in [7, 17, 20, 30, 32, 33, 35, 39].

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The objective of this paper is to establish (quantitatively) sharp two-sided estimates on the transition density $p_D^m(t,x,y)$ of the subprocess of $X^m$ killed upon exiting any $C^{1,1}$ open set $D \subset \mathbb{R}^d$. The density function $p_D^m(t,x,y)$ is also the heat kernel of the restriction of $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$ in $D$ with zero exterior condition. Recall that an open set $D$ in $\mathbb{R}^d$ (when $d \geq 2$) is said to be a (uniform) $C^{1,1}$ open set if there are (localization radius) $R > 0$ and $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $|\nabla\varphi(x) - \nabla\varphi(z)| \leq \Lambda_0|x - z|$ and an orthonormal coordinate system $CS_z : y = (y_1, \dots, y_{d-1}, y_d) := (\widetilde{y}, y_d)$ with origin at $z$ such that $B(z,R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS_z : y_d > \varphi(\widetilde{y})\}$. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be expressed as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive, and the minimum of the distances between these intervals is positive. For $x \in \mathbb{R}^d$, let $\delta_D(x)$ denote the Euclidean distance between $x$ and $D^c$ and $\delta_{\partial D}(x)$ the Euclidean distance between $x$ and $\partial D$. It is well known that a $C^{1,1}$ open set $D$ satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition*: there exists $r_0 < R$ such that for every $x \in D$ with $\delta_{\partial D}(x) < r_0$ and $y \in \mathbb{R}^d \setminus \overline{D}$ with $\delta_{\partial D}(y) < r_0$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$, $|y - z_y| = \delta_{\partial D}(y)$ and that $B(x_0, r_0) \subset D$ and $B(y_0, r_0) \subset \mathbb{R}^d \setminus \overline{D}$, where $x_0 = z_x + r_0(x - z_x)/|x - z_x|$ and $y_0 = z_y + r_0(y - z_y)/|y - z_y|$. In fact, $D$ is $C^{1,1}$ if and only if $D$ satisfies the uniform interior ball condition and the uniform exterior ball condition (see [1], Lemma 2.2). In this paper we call the pair $(r_0, \Lambda_0)$ the characteristics of the $C^{1,1}$ open set $D$.

The main result of this paper is Theorem 1.1 below. The open set $D$ below is not necessarily bounded or connected. In this paper, we use ":=" as a way of definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

THEOREM 1.1. *Suppose that $D$ is a $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics $(r_0, \Lambda_0)$.*

(i) *For any $M > 0$ and $T > 0$, there exists $C_1 = C_1(\alpha, r_0, \Lambda_0, M, T) > 1$ such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$\begin{aligned} &\frac{1}{C_1}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right) \\ &\quad \leq p_D^m(t,x,y) \\ &\quad \leq C_1\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|/16)}{|x-y|^{d+\alpha}}\right), \end{aligned} \tag{1.2}$$

*where $\phi(r) = e^{-r}(1 + r^{(d+\alpha-1)/2})$.*

(ii) *Suppose in addition that $D$ is bounded. For any $M > 0$ and $T > 0$, there exists $C_2 = C_2(\alpha, r_0, \Lambda_0, M, T, \mathrm{diam}(D)) > 1$ such that for any $m \in (0, M]$ and*

*$(t, x, y) \in [T, \infty) \times D \times D$,*

$$C_2^{-1} e^{-t\lambda_1^{\alpha,m,D}} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \le p_D^m(t, x, y) \le C_2 e^{-t\lambda_1^{\alpha,m,D}} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2},$$

*where $\lambda_1^{\alpha,m,D} > 0$ is the smallest eigenvalue of the restriction of $(m^{2/\alpha} - \Delta)^{\alpha/2} - m$ in $D$ with zero exterior condition.*

REMARK 1.2. (i) Note that the estimates in Theorem 1.1 are uniform in $m \in (0, M]$. When $m \downarrow 0$, $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$ converges to the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ in the distributional sense, and it is easy to check that $X^m$ converges weakly to $X$ in the Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$. It follows from the uniform Hölder continuity result of [16], Theorem 4.14, that $p_D^m(t, x, y)$ converges pointwise to $p_D(t, x, y)$, the transition density function of the subprocess $X^D$ of $X$ in $D$. Furthermore, when $D$ is bounded, by [22], Theorem 1.1, $\lim_{m\downarrow 0} \lambda_1^{\alpha,m,D} = \lambda_1^{\alpha,D}$, the smallest eigenvalue of $(-\Delta)^{\alpha/2}$ in $D$ with zero exterior condition. So letting $m \downarrow 0$ in Theorem 1.1 recovers the sharp two-sided estimates of $p_D(t, x, y)$ for $C^{1,1}$ open set $D$, which were obtained recently in [14]. We emphasize here that the proof of Theorem 1.1 of this paper uses the main results of [14], so the above remark should not be interpreted as that passing $\alpha \to 0$ gives a new proof of the main results of [14].

(ii) When $D$ is bounded, the functions $(x, y) \mapsto \phi(m^{1/\alpha}|x - y|/16)$ and $(x, y) \mapsto \phi(m^{1/\alpha}|x - y|)$ on $D \times D$ are bounded between two positive constants independent of $m \in (0, M]$. Thus it follows from Theorem 1.1(i) above and [14], Theorem 1.1(i), that, for each $T > 0$, the heat kernel $p_D^m(t, x, y)$ is uniformly comparable to the heat kernel $p_D(t, x, y)$ on $(0, T] \times D \times D$ when $D$ is a bounded $C^{1,1}$ open set. However when $D$ is unbounded, these two are not comparable.

(iii) In fact, the upper bound estimates in both Theorem 1.1 and Theorem 1.3 below hold for any open set $D$ satisfying (a weak version of) the *uniform exterior ball condition* in place of the $C^{1,1}$ condition, while the lower bound estimates in both Theorem 1.1 and Theorem 1.3 below hold for any open set $D$ satisfying the *uniform interior ball condition* in place of the $C^{1,1}$ condition. (See the paragraph before Lemma 4.3 for the definition of the weak version of the uniform exterior ball condition.)

(iv) Let $p^m(t, x, y)$ denote the transition density function for $X^m$. Then in view of (2.4) and the estimates on $p^m(t, x, y)$ to be given below in Theorem 4.1, the estimate (1.2) can be restated as

$$\begin{aligned} &\frac{1}{C_1}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) p^m(t, x, y) \\ &\quad \le p_D^m(t, x, y) \le C_1 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) p^m(t, x/16, y/16). \end{aligned} \tag{1.3}$$

Though the heat kernel estimates for symmetric diffusions (such as Aronson's estimates) have a long history, the study of sharp two-sided estimates on the transition densities of jump processes in $\mathbb{R}^d$ started quite recently. See [9, 10, 16, 17] and the references therein. Due to the complication near the boundary, the investigation of sharp two-sided estimates on the transition densities of jump processes in open sets is even more recent. In [14], we obtained sharp two-sided estimates for the transition density of the symmetric $\alpha$-stable process killed upon exiting any $C^{1,1}$ open set $D \subset \mathbb{R}^d$. That was the first time sharp two-sided estimates were established for Dirichlet heat kernels of nonlocal operators. Subsequently, we obtained in [15] sharp two-sided heat kernel estimates for censored stable processes in $C^{1,1}$ open sets. Chen and Tokle [23] derived two-sided global heat kernel estimates for symmetric stable processes in two classes of unbounded $C^{1,1}$ open sets. See [6] for Varopoulos-type two-sided heat kernel estimates for symmetric stable processes in a general class of domains including Lipschitz domains expressed in terms of the surviving probability function $\mathbb{P}_x(\tau_D > t)$.

This paper can be viewed as a natural continuation of our previous works [14, 15]. We point out that, although this paper adopts its main strategy from [14], there are many new difficulties and differences between obtaining estimates on the transition densities of relativistic stable processes in open sets and those of symmetric stable processes and censored stable processes in open sets studied [14, 15]. For example, unlike symmetric stable processes and censored stable processes, relativistic stable processes do not have the scaling property, which is one of the main ingredients used in the approaches of [14, 15]. As in [14, 15], the Lévy system of $X^m$, which describes how the process jumps [see (2.6)], is the basic tool used throughout our argument because $X^m$ moves by "pure jumps." However, the Lévy density of $X^m$ does not have a simple form and has exponential decay at infinity as opposed to the polynomial decay of the Lévy density of symmetric stable processes. [See (2.1)–(2.4) and (2.10)–(2.11) below.] Moreover, in this paper we aim at obtaining sharp estimates that are uniform in $m \in (0, M]$; that is, the constants $C_1$ and $C_2$ in Theorem 1.1 are independent of $m \in (0, M]$. This requires very careful and detailed estimates throughout our proofs.

The approach of this paper uses a combination of probabilistic and analytic techniques, but it is mainly probabilistic. It was first established in [39], and then in [20] by using a different method, that the Green function of $X^m$ in a bounded $C^{1,1}$ open set $D$ is comparable to that of $X$ in $D$. We show in Theorem 2.6 below, following the approach of [20], that such a comparison is uniform in $m \in (0, M]$ for small balls. This uniform Green function estimate is then used to get the boundary decay rate of $p^m_D(t, x, y)$. When $x$ and $y$ are far from the boundary in a scale given by $t$, the near diagonal lower bound estimate of $p^m_D(t, x, y)$ is derived from the uniform parabolic inequality (Theorem 2.9), the uniform exit time estimate (Theorem 2.8) and the fact that $X^m_t$ moves from $x$ to a neighborhood of $y$ by one single jump with positive probability. These estimates can be used to get the lower bound estimate on the global heat kernel $p^m(t, x, y)$. The upper bound estimate on $p^m(t, x, y)$

is obtained from the heat kernel of Brownian motion through subordination. This sharp two-sided estimates on the transition density $p^m(t,x,y)$ in bounded time interval are presented in Theorem 4.1 and will be used to derive upper bound estimates on $p_D^m(t,x,y)$. The estimates in Theorem 4.1 sharpen the corresponding estimates established earlier in [17] that are applicable for more general jump processes with exponentially decaying jump kernels. After the first version of this paper was written and posted on the arXiv, the authors were informed that the estimates in Theorem 4.1 are also obtained in [41]. Since $X^m$ can be obtained from $X$ by pruning jumps in a suitable way (see [2], Remarks 3.4 and 3.5), we can conclude that $p_D^m(t,x,y) \le e^{Mt} p_D(t,x,y)$ for all $m \in (0, M]$. The upper bound estimate on $p_D^m(t,x,y)$ (Theorem 4.4) is then obtained by using the Lévy system formula, comparison with the heat kernel estimates on exterior balls (Lemma 4.2), the estimates on $p_D(t,x,y)$ from [14] and the two-sided estimates on $p^m(t,x,y)$ (Theorem 4.1).

When $D$ is a bounded $C^{1,1}$ open set, integrating the estimates on $p_D^m(t,x,y)$ from Theorem 1.1 over $t$ yields sharp two-sided sharp estimates on the Green function $G_D^m(x,y) := \int_0^\infty p_D^m(t,x,y)\,dt$. To state this result, we define a function $V_D^\alpha$ on $D \times D$ by

$$V_D^\alpha(x,y) := \begin{cases} \left(1 \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right)|x-y|^{\alpha-d}, \\ \qquad \text{when } d > \alpha, \\ \log\left(1 + \dfrac{\delta_D(x)^{1/2}\delta_D(y)^{1/2}}{|x-y|}\right), \\ \qquad \text{when } d = 1 = \alpha, \\ (\delta_D(x)\delta_D(y))^{(\alpha-1)/2} \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|}, \\ \qquad \text{when } d = 1 < \alpha. \end{cases} \tag{1.4}$$

THEOREM 1.3. *Let $M > 0$ be a constant and $D$ a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. Then there exists a constant $C_3 > 1$ depending only on $d, \alpha, r_0, \Lambda_0, M, T$,* diam$(D)$ *such that for every $m \in (0, M]$ and $(x,y) \in D \times D$,*

$$C_3^{-1} V_D^\alpha(x,y) \le G_D^m(x,y) \le C_3 V_D^\alpha(x,y).$$

The proof of Theorem 1.3 is the same as that of [14], Corollary 1.2. Theorem 1.3 extends and improves the Green function estimates obtained in [20, 33, 39] in the sense that our estimates are uniform in $m \in (0, M]$ and the case $d = 1$ is now covered. Although we do not yet have large time heat kernel estimates when $D$ is unbounded, the short time heat kernel estimates in Theorem 1.1(i) can be used together with the two-sided Green function estimates on the upper half space from [30] and a comparison idea from [23] to obtain sharp two-sided estimates on the Green function $G_D^m(x, y)$ when $D$ is a half-space-like $C^{1,1}$ open set. We will address this in a separate paper [13].

The rest of the paper is organized as follows. In Section 2 we recall some basic facts about the relativistic stable process $X^m$ and prove some preliminary uniform results on $X^m$ including uniform estimates on the Green function $G_D^m$ for small balls and annuli, and uniform parabolic Harnack inequality. Some preliminary lower bound of $p_D^m(t, x, y)$ is proved in Section 3, while the proof of Theorem 1.1 is given in Section 4.

In the remainder of this paper, we assume that $m > 0$. We will use capital letters $C_1, C_2, \dots$ to denote constants in the statements of results, and their labeling will be fixed. The lower case constants $c_1, c_2, \dots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in every proof. The dependence of the lower case constants on the dimension $d$ will not be mentioned explicitly. We will use $\partial$ to denote a cemetery point and for every function $f$, we extend its definition to $\partial$ by setting $f(\partial) = 0$. We will use $dx$ to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure.

**2. Relativistic stable processes and some uniform estimates.** The Lévy measure of the relativistic $\alpha$-stable process $X^m$, defined in (1.1), has a density

$$\begin{aligned} J^m(x) &= j^m(|x|) \\ &:= \frac{\alpha}{2\Gamma(1-\alpha/2)} \int_0^\infty (4\pi u)^{-d/2} e^{-|x|^2/4u} e^{-m^{2/\alpha} u} u^{-(1+\alpha/2)}\, du, \end{aligned} \tag{2.1}$$

which is continuous and radially decreasing on $\mathbb{R}^d \setminus \{0\}$ (see [39], Lemma 2). Here and in the rest of this paper, $\Gamma$ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t}\, dt$ for every $\lambda > 0$. Put $J^m(x, y) := j^m(|x - y|)$. Let $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$. Using change of variables twice, first with $u = |x|^2 v$ then with $v = 1/s$, we get

$$J^m(x, y) = \mathcal{A}(d, -\alpha) |x - y|^{-d-\alpha} \psi(m^{1/\alpha} |x - y|), \tag{2.2}$$

where

$$\psi(r) := 2^{-(d+\alpha)} \Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{(d+\alpha)/2-1} e^{-s/4 - r^2/s}\, ds, \tag{2.3}$$

which satisfies $\psi(0) = 1$ and

$$c_1^{-1} e^{-r} r^{(d+\alpha-1)/2} \le \psi(r) \le c_1 e^{-r} r^{(d+\alpha-1)/2} \qquad \text{on } [1, \infty) \tag{2.4}$$

for some $c_1 > 1$ (see [20], pages 276–277, for details). In particular, we see that for $m > 0$,

$$J^m(x, y) = m^{(d+\alpha)/\alpha} J^1(m^{1/\alpha} x, m^{1/\alpha} y). \tag{2.5}$$

We denote the Lévy density of $X$ by

$$J(x, y) := J^0(x, y) = \mathcal{A}(d, -\alpha)|x - y|^{-(d+\alpha)}.$$

The Lévy density gives rise to a Lévy system for $X^m$, which describes the jumps of the process $X^m$: for any $x \in \mathbb{R}^d$, stopping time $T$ (with respect to the filtration of $X^m$) and nonnegative measurable function $f$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$ and $s \geq 0$,

$$\mathbb{E}_x\biggl[\sum_{s\leq T} f(s, X^m_{s-}, X^m_s)\biggr] = \mathbb{E}_x\biggl[\int_0^T \biggl(\int_{\mathbb{R}^d} f(s, X^m_s, y) J^m(X^m_s, y)\, dy\biggr) ds\biggr]. \tag{2.6}$$

(See, e.g., [16], proof of Lemma 4.7, and [17], Appendix A.)

For $r \in (0, \infty)$, we define

$$\xi(r) := \begin{cases} r^2, & \text{when } d + \alpha > 2, \\ r^{1+\alpha}, & \text{when } d = 1 > \alpha, \\ r^2 \ln\Bigl(1 + \dfrac{1}{r}\Bigr), & \text{when } d = 1 = \alpha. \end{cases} \tag{2.7}$$

We start with an elementary inequality.

LEMMA 2.1. *For any $R_0 > 0$, there exists $C_4 = C_4(d, \alpha, R_0) > 0$ such that for all $r \in (0, R_0]$,*

$$1 - \psi(r) \leq C_4 \xi(r).$$

PROOF. We have

$$1 - \psi(r) = 2^{-(d+\alpha)}\Gamma\Bigl(\frac{d+\alpha}{2}\Bigr)^{-1}\Bigl(\int_0^{r^2} + \int_{r^2}^\infty\Bigr) s^{(d+\alpha)/2-1} e^{-s/4}(1 - e^{-r^2/s})\, ds.$$

Note that

$$\int_0^{r^2} s^{(d+\alpha)/2-1} e^{-s/4}(1 - e^{-r^2/s})\, ds \leq \int_0^{r^2} s^{(d+\alpha)/2-1}\, ds \leq c_1 r^{d+\alpha}, \tag{2.8}$$

and that, by the inequality $1 - e^{-z} \leq z$ for $z \geq 0$,

$$\begin{aligned} \int_{r^2}^\infty s^{(d+\alpha)/2-1} e^{-s/4}(1 - e^{-r^2/s})\, ds &\leq r^2 \int_{r^2}^\infty s^{(d+\alpha)/2-2} e^{-s/4}\, ds \\ &\leq c_2 \xi(r). \end{aligned} \tag{2.9}$$

We arrive at the conclusion of this lemma by combining (2.8) and (2.9). □

The next two inequalities, which can be seen easily from the monotonicity of $\psi$ and (2.4), will be used several times in this paper. For any $a > 0$ and $M > 0$, there exist positive constants $C_5$ and $C_6$ depending only on $a$ and $M$ such that for any $m \in (0, M]$,

$$j^m(r) \le C_5 j^m(2r) \qquad \text{for every } r \in (0, a] \tag{2.10}$$

and

$$j^m(r) \le C_6 j^m(r+a) \qquad \text{for every } r > a. \tag{2.11}$$

We will use $p^m(t, x, y) = p^m(t, x - y)$ to denote the transition density of $X^m$ and use $p(t, x, y)$ to denote the transition density of $X$. It is well known that (cf. [16])

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \qquad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.12}$$

Here and in the sequel, for two nonnegative functions $f, g$, $f \asymp g$ means that there is a positive constant $c_0 > 1$ so that $c_0^{-1} f \le g \le c_0 f$ on their common domain of definitions. It is also known that

$$p^1(t, x) = e^t \int_0^\infty (4\pi u)^{-d/2} e^{-|x|^2/(4u)} e^{-u} \theta_\alpha(t, u)\, du, \tag{2.13}$$

where $\theta_\alpha(t, u)$ is the transition density of an $\frac{\alpha}{2}$-stable subordinator with the Laplace transform $e^{-t\lambda^{\alpha/2}}$. It follows from [3], Theorem 2.1, and [43], (2.5.17), (2.5.18), that

$$\theta_\alpha(t, u) \le ctu^{-1-\alpha/2} \qquad \text{for every } t > 0, u > 0.$$

Thus by (2.1) and (2.13), there exists $L = L(\alpha) > 0$ such that

$$p^1(t, x, y) \le Lte^t J^1(x, y) \qquad \text{for all } (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.14}$$

From (1.1), one can easily see that $X^m$ has the following approximate scaling property: $\{m^{-1/\alpha}(X^1_{mt} - X^1_0), t \ge 0\}$ has the same distribution as that of $\{X^m_t - X^m_0, t \ge 0\}$. In terms of transition densities, this approximate scaling property can be written as

$$p^m(t, x, y) = m^{d/\alpha} p^1(mt, m^{1/\alpha}x, m^{1/\alpha}y). \tag{2.15}$$

Thus by (2.5), (2.14) and (2.15), we have

$$p^m(t, x, y) \le Lte^{mt} J^m(x, y) \qquad \text{for } (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.16}$$

On the other hand, by [39], Lemma 3, there exists $c = c(\alpha) > 0$ such that

$$p^m(t, x, y) \le c(m^{d/\alpha - d/2} t^{-d/2} + t^{-d/\alpha}). \tag{2.17}$$

For any open set $D$, we use $\tau^m_D$ to denote the first exit time from $D$ for $X^m$, that is, $\tau^m_D = \inf\{t > 0 : X^m_t \notin D\}$ and let $\tau_D$ be the first exit time from $D$ for $X$. We define $X^{m,D}$ by $X^{m,D}_t(\omega) = X^m_t(\omega)$ if $t < \tau^m_D(\omega)$ and $X^{m,D}_t(\omega) = \partial$ if $t \ge \tau^m_D(\omega)$. We define $X^D$ similarly. $X^{m,D}$ is called the subprocess of $X^m$ killed upon exiting

$D$ (or, the killed relativistic stable process in $D$ with mass $m$), and $X^D$ is called the killed symmetric $\alpha$-stable process in $D$.

It is known (see [17]) that $X^{m,D}$ has a transition density $p_D^m(t,x,y)$, which is continuous on $(0,\infty)\times D\times D$ with respect to the Lebesgue measure. Note that the transition density $p_D^m(t,x,y)$ may not be continuous on $\overline{D}\times\overline{D}$ if the boundary of $D$ is irregular.

We will use $G_D^m(x,y) := \int_0^\infty p_D^m(t,x,y)\,dt$ to denote the Green function of $X^{m,D}$. We use $p_D(t,x,y)$ and $G_D(x,y)$ to denote the transition density and the Green function of $X^D$, respectively.

The Dirichlet heat kernel $p_D^m(t,x,y)$ also has the following approximate scaling property:

$$p_D^m(t,x,y) = m^{d/\alpha} p^1_{m^{1/\alpha}D}(mt, m^{1/\alpha}x, m^{1/\alpha}y). \tag{2.18}$$

Thus the Green function $G_D^m(x,y)$ of $X^{m,D}$ satisfies

$$G_D^m(x,y) = m^{(d-\alpha)/\alpha} G^1_{m^{1/\alpha}D}(m^{1/\alpha}x, m^{1/\alpha}y) \qquad \text{for every } x,y\in D. \tag{2.19}$$

REMARK 2.2. We point out here that the uniform heat kernel estimates in Theorem 1.1 do not follow from a combination of the sharp heat kernel estimates of $p_D^1(t,x,y)$ and the scaling property (2.18). This is because if $D$ is a $C^{1,1}$ open sets with $C^{1,1}$ characteristics $(r_0,\Lambda_0)$, then $m^{1/\alpha}D$ is a $C^{1,1}$ open sets with different $C^{1,1}$ characteristics $(m^{1/\alpha}r_0, m^{-1/\alpha}\Lambda_0)$.

Let

$$J_m(x,y) = J(x,y) - J^m(x,y) = \mathcal{A}(d,-\alpha)|x-y|^{-d-\alpha}\bigl(1-\psi(m^{1/\alpha}|x-y|)\bigr).$$

Then

$$\int_{\mathbb{R}^d} J_m(x,y)\,dy = m \qquad \text{for all } x\in\mathbb{R}^d. \tag{2.20}$$

(See [39], Lemma 2.) Thus $X^m$ can be constructed from $X$ by reducing jumps via Meyer's construction (see [2], Remarks 3.4 and 3.5). By [39], Lemma 5, or [2], (3.18), we have

$$p_D^m(t,x,y) \le e^{mt} p_D(t,x,y) \qquad \text{for every } (t,x,y)\in(0,\infty)\times D\times D. \tag{2.21}$$

In the next two results, we discuss the Green function of one-dimensional symmetric $\alpha$-stable processes killed upon exiting $B=(0,2)\subset\mathbb{R}$. Define for $x,y\in B$,

$$f(x,y) := \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}. \tag{2.22}$$

LEMMA 2.3. *Suppose that $B=(0,2)\subset\mathbb{R}$ and $\alpha>1$.*

(i) *There exists* $C_7 = C_7(\alpha) > 0$ *such that*

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le C_7 \qquad \textit{for every } x,y,z \in B.$$

(ii) *If* $f(x,w) \ge 4$, *there exists* $C_8 = C_8(\alpha) > 0$ *such that*

$$\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \le C_8 \delta_B(y)^{(\alpha-1)/2} \delta_B(z)^{(\alpha-1)/2} \le C_8 \qquad \textit{for } y,z \in B.$$

PROOF. (i) follows from [5], (3.5). So we only need to prove (ii).

Note that (see [24], page 187) $|x-y| \le \delta_B(x) \wedge \delta_B(y)$ and $\delta_B(x) \wedge \delta_B(y) \ge \frac{1}{2}(\delta_B(x) \vee \delta_B(y))$ if $f(x,y) \ge 4$. We know from [5], Corollary 3.2, or [14], Corollary 1.2, that

$$G_B(x,y) \asymp (\delta_B(x)\delta_B(y))^{(\alpha-1)/2} \wedge \frac{\delta_B(x)^{\alpha/2}\delta_B(y)^{\alpha/2}}{|x-y|}. \tag{2.23}$$

So when $f(x,w) \ge 4$, we have by (2.23) that

$$\begin{aligned}\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} &\le c_1 \frac{(\delta_B(x)\delta_B(y))^{(\alpha-1)/2}(\delta_B(z)\delta_B(w))^{(\alpha-1)/2}}{(\delta_B(x)\delta_B(w))^{(\alpha-1)/2}} \\ &= c_1 \delta_B(y)^{(\alpha-1)/2}\delta_B(z)^{(\alpha-1)/2}.\end{aligned}$$ □

The second part of the next result strengthens [5], (3.4).

LEMMA 2.4. *Suppose that* $B = (0,2) \subset \mathbb{R}$ *and* $\alpha = 1$. *Let* $f$ *be as in* (2.22) *and define* $F(x,y) := \log(1 + f(x,y)^{1/2})$.

(i) *If* $f(x,w) \ge 4$, *there exists* $C_9 > 0$ *such that*

$$\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \le C_9 F(x,y)F(z,w), \qquad y,z \in B.$$

(ii) *There exists* $C_{10} > 0$ *such that*

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le C_{10}\big(1 + F(x,y) + F(y,z)\big), \qquad x,y,z \in B.$$

PROOF. (i) is an immediate consequence of [14], Corollary 1.2. Using [14], Corollary 1.2, (ii) can be proved by following the argument of the proof of [24], Theorem 6.24. We omit the details. □

For $r \in (0,1]$, we define

$$\sigma(r) = \begin{cases} r^{2-\alpha-d}, & \text{when } d+\alpha > 2, \\ 1, & \text{when } d = 1 > \alpha, \\ \ln(1+1/r), & \text{when } d = 1 = \alpha. \end{cases}$$

The following result will be used to prove Theorem 2.6. Note that the case $d = 1 \le \alpha$ in Lemma 2.5(i) does not follow from [29], Lemma 3.14.

LEMMA 2.5. (i) *If $B$ is a ball of radius* 1 *in* $\mathbb{R}^d$, *then*,

$$\sup_{x,y\in B, x\neq y} \int_{B\times B} \frac{G_B(x,w)\sigma(|w-z|)G_B(z,y)}{G_B(x,y)}\,dw\,dz < \infty.$$

(ii) *If $d \geq 2$ and $U$ is an annulus of inner radius* 1 *and outer radius* $3/2$ *in* $\mathbb{R}^d$, *then*

$$\sup_{x,y\in U, x\neq y} \int_{U\times U} \frac{G_U(x,w)\sigma(|w-z|)G_U(z,y)}{G_U(x,y)}\,dw\,dz < \infty.$$

PROOF. We only present the proof of (i). The proof of (ii) is similar to the proof of (i) for the case $d > \alpha$. We prove (i) by dealing with two separate cases.

Case 1: $d > \alpha$. In this case, by repeating the argument in [19], Example 2 (also see [29], Lemma 3.14), we know that there exists $c_1 = c_1(d, \alpha) > 0$ such that

$$\begin{aligned}
&\frac{G_B(x,w)\sigma(|w-z|)G_B(z,y)}{G_B(x,y)}\\
&\quad\leq c_4\biggl(\frac{1}{|z-y|^{d-\alpha/2}|w-z|^{d+\alpha-\beta}} + \frac{1}{|x-w|^{d-\alpha/2}|w-z|^{d+\alpha-\beta}}\\
&\qquad + \frac{1}{|z-y|^{d-\alpha}|w-z|^{d+\alpha-\beta}} + \frac{1}{|x-w|^{d-\alpha}|w-z|^{d+\alpha-\beta}}\\
&\qquad + \frac{1}{|x-w|^{d-\alpha/2}|z-y|^{d-\alpha/2}|w-z|^{3\alpha/2-\beta}}\\
&\qquad + \frac{1}{|x-w|^{d-\alpha/2}|z-y|^{d-\alpha}|w-z|^{2\alpha-\beta}}\biggr),
\end{aligned}$$

where $\beta = 2$ when $d \geq 2$ and $\beta = 1 + \alpha$ when $d = 1 > \alpha$. The conclusion now follows immediately.

Case 2: $d = 1 \leq \alpha$. In this case, it follows from the first part of the proof of [29], Proposition 3.17, that

$$\sup_{x,y\in B, x\neq y, f(x,y)\leq 4} \int_{B\times B} \frac{G_B(x,w)\sigma(|w-z|)G_B(z,y)}{G_B(x,y)}\,dw\,dz < \infty,$$

where the $f$ is the function defined in (2.22). The inequality

$$\sup_{x,y\in B, x\neq y, f(x,y)\geq 4} \int_{B\times B} \frac{G_B(x,w)\sigma(|w-z|)G_B(z,y)}{G_B(x,y)}\,dw\,dz < \infty$$

follows easily from Lemmas 2.3 and 2.4. □

The following result will be used later in this paper. Note that this result does not follow from the main result in [29], since the constants in the following results are uniform in $m \in (0, \infty)$ and $r \in (0, R_0 m^{-1/\alpha}]$. It is known that (see [18] and [34] for the case $d \geq 2$ and [5] for the case $d = 1$) that $G_B(x, y)$ is comparable to $V_B^\alpha(x, y)$ of (1.4).

THEOREM 2.6. *There exist positive constants $R_0 \in (0,1]$ and $C_{11} > 1$ depending only on $d$ and $\alpha$ such that for any $m \in (0,\infty)$, any ball $B$ of radius $r \le R_0 m^{-1/\alpha}$,*

$$C_{11}^{-1} G_B(x,y) \le G_B^m(x,y) \le C_{11} G_B(x,y), \qquad x,y \in B.$$

*Furthermore, in the case $d \ge 2$, there exists a constant $C_{12} = C_{12}(d,\alpha) > 1$ such that for any $m \in (0,\infty)$, $r \in (0, R_0 m^{-1/\alpha}]$ and any annulus $U$ of inner radius $r$ and outer radius $3r/2$,*

$$C_{12}^{-1} G_U(x,y) \le G_U^m(x,y) \le C_{12} G_U(x,y), \qquad x,y \in U.$$

PROOF. We only present the proof for balls, the case of annuli is similar. By [5, 14, 34], $G_B(x,y) \asymp V_B^\alpha(x,y)$. Hence by (2.19), we only need to prove the theorem for $m=1$. In this proof we will use $B_r$ to denote the ball $B(0,r)$.

Put

$$F(x,y) := \frac{J^1(x,y)}{J(x,y)} - 1 = \psi(|x-y|) - 1, \qquad x,y \in \mathbb{R}^d.$$

Then it follows from (2.1)–(2.3) that there exists $c_1 = c_1(d,\alpha) > 0$ such that for any $r \in (0,1]$, $\inf_{x,y\in B_r} F(x,y) \ge c_1 - 1$. It follows from Lemma 2.1 that there exists $c_2 = c_2(d,\alpha) > 0$ such that for any $r \in (0,1]$ and $x,y \in B_1$,

$$\begin{aligned} &|F(rx,ry)| + \bigl|\ln\bigl(1+F(rx,ry)\bigr)\bigr| + \bigl(e^{4|\ln(1+F(rx,ry))|} - 1\bigr) \\ &\quad \le c_2 \xi(r|x-y|). \end{aligned} \tag{2.24}$$

For $x \in B_r$, put

$$q_{B_r}(x) := \int_{B_r^c} J_1(x,y)\,dy = \mathcal{A}(d,-\alpha) \int_{B_r^c} |x-y|^{-d-\alpha}\bigl(1-\psi(|x-y|)\bigr)\,dy.$$

Then it follows from [20], Section 3, that

$$G_{B_r}^1(x,y) = G_{B_r}(x,y)\mathbb{E}_x^y[K^{B_r}(\tau_{B_r})] \qquad \text{for every } x,y \in B_r,$$

where

$$\begin{aligned} K^{B_r}(t) := \exp\biggl(&\sum_{0<s\le t} \ln\bigl(1+F(X_{s-}^{B_r}, X_s^{B_r})\bigr) \\ &- \int_0^t \int_{B_r} F(X_s^{B_r}, y) J(X_s^{B_r}, y)\,dy\,ds + \int_0^t q_{B_r}(X_s^{B_r})\,ds\biggr). \end{aligned}$$

Using the scaling property of $G_{B_r}$, we get

$$\begin{aligned} &\sup_{x,y\in B_r, x\ne y} \int_{B_r\times B_r} \frac{G_{B_r}(x,w)(e^{4|\ln(1+F(w,z))|}-1)G_{B_r}(z,y)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}\,dw\,dz \\ &\quad = \sup_{x,y\in B_1, x\ne y} \int_{B_1\times B_1} \frac{G_{B_1}(x,w)(e^{4|\ln(1+F(rw,rz))|}-1)G_{B_1}(z,y)}{G_{B_1}(x,y)|w-z|^{d+\alpha}}\,dw\,dz, \end{aligned} \tag{2.25}$$

$$\begin{aligned}(2.26)\quad &\sup_{x,y\in B_r,x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)|F(w,z)|G_{B_r}(z,y)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}\,dw\,dz\\ &\quad=\sup_{x,y\in B_1,x\neq y}\int_{B_1\times B_1}\frac{G_{B_1}(x,w)|F(rw,rz)|G_{B_1}(z,y)}{G_{B_1}(x,y)|w-z|^{d+\alpha}}\,dw\,dz\end{aligned}$$

and

$$\begin{aligned}(2.27)\quad &\sup_{x,y\in B_r,x\neq y}\int_{B_r}\frac{G_{B_r}(x,w)G_{B_r}(w,y)}{G_{B_r}(x,y)}q_{B_r}(w)\,dw\\ &\quad=r^\alpha\cdot\sup_{x,y\in B_1,x\neq y}\int_{B_1}\frac{G_{B_1}(x,w)G_{B_1}(w,y)}{G_{B_1}(x,y)}q_{B_r}(rw)\,dw.\end{aligned}$$

Using (2.24)–(2.26) and Lemma 2.5, we have for $r\in(0,1]$,

$$\sup_{x,y\in B_r,x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)(e^{4|\ln(1+F(w,z))|}-1)G_{B_r}(z,y)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}\,dw\,dz\le c_3r$$

and

$$\sup_{x,y\in B_r,x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)|F(w,z)|G_{B_r}(z,y)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}\,dw\,dz\le c_3r.$$

By applying (2.20), the 3G inequality [Lemma 2.3(ii) and Lemma 2.4(ii) for $d=1$] and (2.27), we also have

$$\sup_{x,y\in B_r,x\neq y}\int_{B_r}\frac{G_{B_r}(x,w)G_{B_r}(w,y)}{G_{B_r}(x,y)}q_{B_r}(w)\,dw\le c_3r^\alpha.$$

Now choose $R_0>0$ small enough so that for $r\le R_0$,

$$\sup_{x,y\in B_r,x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)(e^{4|\ln(1+F(w,z))|}-1)G_{B_r}(z,y)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}\,dw\,dz\le\frac{1}{2},$$

$$\sup_{x,y\in B_r,x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)|F(w,z)|G_{B_r}(z,y)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}\,dw\,dz\le\frac{1}{8}$$

and

$$\sup_{x,y\in B_r,x\neq y}\int_{B_r}\frac{G_{B_r}(x,w)G_{B_r}(w,y)}{G_{B_r}(x,y)}\,dw\le\frac{1}{8}.$$

Using the three displays above, we can follow the argument in [19], Proposition 2.3 (with the constants involved there taken to be $\alpha=\gamma=2,\theta=1/2$) to conclude that for all $r\le R_0$,

$$\sup_{x,y\in B_r,x\neq y}\mathbb{E}_x^y[K^{B_r}(\tau_{B_r})]\le 2^{3/4}.$$

Now the upper bound on $G^1_{B_r}$ follows immediately. The lower bound on $G^1_{B_r}$ is an easy consequence of Jensen's inequality (see [19], Remark 2, for details). $\square$

In the remainder of this paper, $R_0 \in (0, 1]$ will always stand for the constant in Theorem 2.6. The next corollary will be used in Section 4.

COROLLARY 2.7. *There exist positive constants $C_{13} > 1$ and $C_{14} < 1$ depending only on $d$ and $\alpha$ such that for any $m \in (0, \infty)$, any $r \le R_0 m^{-1/\alpha}$, any ball $B$ of radius $r$ and, when $d \ge 2$, any annulus $U = B(x_0, 3r/2) \setminus \overline{B(x_0, r)}$*

$$\mathbb{P}_x(X^m_{\tau^m_B} \in A) \le C_{13}\mathbb{P}_x(X_{\tau_B} \in A) \qquad \text{for every } x \in B \text{ and } A \subset B^c, \tag{2.28}$$

$$\mathbb{P}_x(X^m_{\tau^m_U} \in A) \le C_{13}\mathbb{P}_x(X_{\tau_U} \in A) \qquad \text{for every } x \in U \text{ and } A \subset U^c. \tag{2.29}$$

*In addition, if $N \ge 2R_0$, then for every $x \in U$ and $A \subset B(x_0, Nm^{-1/\alpha}) \setminus B(x_0, 3r/2)$,*

$$\mathbb{P}_x(X^m_{\tau^m_U} \in A) \ge C_{14}\psi(2R_0 + N)\mathbb{P}_x(X_{\tau_U} \in A). \tag{2.30}$$

PROOF. By (2.6) and [40],

$$\mathbb{P}_x(X^m_{\tau^m_B} \in A) = \int_A \int_B G^m_B(x, y) J^m(y, z)\, dy\, dz$$

and

$$\mathbb{P}_x(X^m_{\tau^m_U} \in A) = \int_A \int_U G^m_U(x, y) J^m(y, z)\, dy\, dz.$$

Thus, using Theorem 2.6 and the fact $J^m \le J^0$, (2.28) and (2.29) follow immediately.

Moreover, when $y \in B(x_0, 3r/2) \setminus \overline{B(x_0, r)}$ and $z \in A \subset B(x_0, Nm^{-1/\alpha}) \setminus B(x_0, 3r/2)$, $m^{1/\alpha}|y - z| \le 2R_0 + N$. Thus $J^m(y, z) \ge \psi(2R_0 + N)J(y, z)$ and, using Theorem 2.6, (2.30) follows. $\square$

Later in this paper, we will also need the following exit time estimate and parabolic Harnack inequality that are uniform in $m \in (0, M]$. These results are extensions of Proposition 4.9 and Theorem 4.12 of [17], respectively.

THEOREM 2.8. *For any $M > 0$, $R > 0$, $A > 0$ and $B \in (0, 1)$, there exists $\gamma = \gamma(A, B, M, R) \in (0, 1/2)$ such that for every $m \in (0, M]$, $r \in (0, R]$ and $x \in \mathbb{R}^d$,*

$$\mathbb{P}_x\big(\tau^m_{B(x,Ar)} < \gamma r^\alpha\big) \le B.$$

PROOF. Let $Y^m$ be a symmetric pure jump process on $\mathbb{R}^d$ with jump kernel given by

$$J_0^m(x,y) = \begin{cases} j^m(|x-y|), & \text{if } |x-y| \le 1, \\ j^m(1)|x-y|^{-(d+\alpha)}, & \text{if } |x-y| > 1. \end{cases}$$

Note that $J_0^m(x,y) \ge J^m(x,y)$. In view of (2.1)–(2.4) and (2.20), there are constants $c_i = c_i(M,\alpha) > 0$, $i = 1, 2$, such that

$$\frac{c_2}{|x-y|^{d+\alpha}} \le J_0^m(x,y) \le \frac{c_1}{|x-y|^{d+\alpha}} \tag{2.31}$$

for every $m \in (0, M]$ and $x, y \in \mathbb{R}^d$, and

$$\sup_{z\in\mathbb{R}^d} \mathcal{J}_0^m(z) \le M \qquad \text{for every } m \in (0, M], \tag{2.32}$$

where $\mathcal{J}_0^m(z) := \int_{\mathbb{R}^d} (J_0^m(z,w) - J^m(z,w))\,dw$. In view of (2.31), it follows from [16], Proposition 4.1, that for each $M > 0$, $R > 0$, $A > 0$ and $B \in (0,1)$, there is $\gamma = \gamma(A, B, M, R) \in (0,1)$ such that for every $m \in (0, M]$, $r \in (0, R]$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\big(\tau^{Y^m}_{B(x,Ar)} < \gamma r^\alpha\big) \le B/2,$$

where $\tau^{Y^m}_{B(x,Ar)}$ is the first time the process $Y^m$ exits the set $B(x, Ar)$. On the other hand, in view of (2.32), $Y^m$ can be obtained from $X^m$ by adding new jumps according to the jump kernel $J_0^m(x,y) - J^m(x,y)$ through Meyer's construction (see [2], Remark 3.4). Hence we have for every $m \in (0, M]$, $r \in (0, R]$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} &\mathbb{P}_x\big(\tau^m_{B(x,Ar)} < \gamma r^\alpha\big) \\ &\quad \le \mathbb{P}_x\big(\tau^{Y^m}_{B(x,Ar)} < \gamma r^\alpha \text{ and there is no new jumps added to } X^m \text{ by time } \gamma r^\alpha\big) \\ &\qquad + \mathbb{P}_x(\text{there is at least one new jump added to } X^m \text{ by time } \gamma r^\alpha) \\ &\quad \le B/2 + \big(1 - e^{-\gamma r^\alpha \|\mathcal{J}_0^m\|_\infty}\big) \le B/2 + (1 - e^{-\gamma R^\alpha M}) < B, \end{aligned}$$

where the last inequality is achieved by decreasing the value of $\gamma$ if necessary. $\square$

We now introduce the space–time process $Z_s^m := (V_s, X_s^m)$, where $V_s = V_0 - s$. The filtration generated by $Z^m$ and satisfying the usual condition will be denoted as $\{\widetilde{\mathcal{F}}_s; s \ge 0\}$. The law of the space–time process $s \mapsto Z_s^m$ starting from $(t,x)$ will be denoted as $\mathbb{P}^{(t,x)}$ and as usual, $\mathbb{E}^{(t,x)}(\cdot) = \int \cdot\,\mathbb{P}^{(t,x)}(d\omega)$.

We say that a nonnegative Borel function $h(t,x)$ on $[0,\infty) \times \mathbb{R}^d$ is *parabolic* with respect to the process $X^m$ in a relatively open subset $E$ of $[0,\infty) \times \mathbb{R}^d$ if for every relatively compact open subset $E_1$ of $E$, $h(t,x) = \mathbb{E}^{(t,x)}[h(Z^m_{\widetilde{\tau}^m_{E_1}})]$ for every $(t,x) \in E_1$, where $\widetilde{\tau}^m_{E_1} = \inf\{s > 0 : Z_s^m \notin E_1\}$. Note that $p_D^m(\cdot,\cdot,y)$ is parabolic with respect to the process $X^m$.

THEOREM 2.9. *For any $R > 0$ and $M > 0$, there exists $C_{15} > 0$ such that for every $m \in (0, M]$, $\delta \in (0, 1)$, $x_0 \in \mathbb{R}^d$, $t_0 \geq 0$, $r \in (0, R]$ and every nonnegative function $u$ on $[0, \infty) \times \mathbb{R}^d$ that is parabolic with respect to the process $X^m$ on $(t_0, t_0 + 4\delta r^\alpha] \times B(x_0, 4r)$,*

$$\sup_{(t_1, y_1) \in Q_-} u(t_1, y_1) \leq C_{15} \inf_{(t_2, y_2) \in Q_+} u(t_2, y_2),$$

*where $Q_- = [t_0 + \delta r^\alpha, t_0 + 2\delta r^\alpha] \times B(x_0, r)$ and $Q_+ = [t_0 + 3\delta r^\alpha, t_0 + 4\delta r^\alpha] \times B(x_0, r)$.*

PROOF. Since $\psi$ is decreasing, by the change of variable $z = |y|w$, we have for any $|y| \geq 2r$,

$$\begin{aligned}\frac{1}{r^d} \int_{B(0,r)} \frac{\psi(m^{1/\alpha}|z - y|)\, dz}{|z - y|^{d+\alpha}} &\geq \frac{\psi(m^{1/\alpha}|y|)}{r^d |y|^\alpha} \int_{\{|w| \leq r/|y|, |w - y/|y|| \leq 1\}} \frac{dw}{|w - y/|y||^d} \\ &\geq c_0 \frac{\psi(m^{1/\alpha}|y|)}{|y|^{d+\alpha}}.\end{aligned}$$

Thus there is a constant $c > 0$ so that for every $m > 0$,

$$J^m(x, y) \leq \frac{c}{r^d} \int_{B(x,r)} J^m(z, y)\, dz \qquad \text{for every } r \leq \frac{|x - y|}{2}.$$

The above property is called UJS (see [10, 11]). Using Theorem 2.8 and UJS, the conclusion of the theorem now follows from [10], Theorem 4.5, or [11], Theorem 5.2. $\square$

**3. Preliminary lower bound estimates.** In this section, we give some preliminary lower bounds on $p^m_D(t, x, y)$, which will be used in Section 4 to derive the sharp two-sided estimates for $p^m(t, x, y)$ as well as for $p^m_D(t, x, y)$.

LEMMA 3.1. *For any positive constants $M$, $T$, $b$ and $a$, there exists $C_{16} = C_{16}(a, b, M, \alpha, T) > 0$ such that for all $m \in (0, M]$, $z \in \mathbb{R}^d$ and $\lambda \in (0, T]$,*

$$\inf_{\substack{y \in \mathbb{R}^d \\ |y - z| \leq b\lambda^{1/\alpha}}} \mathbb{P}_y\big(\tau^m_{B(z, 2b\lambda^{1/\alpha})} > a\lambda\big) \geq C_{16}.$$

PROOF. By Theorem 2.8, there exists $\varepsilon = \varepsilon(b, \alpha, M, T) > 0$ such that for all $m \in (0, M]$ and $\lambda \in (0, T]$,

$$\inf_{y \in \mathbb{R}^d} \mathbb{P}_y\big(\tau^m_{B(y, b\lambda^{1/\alpha}/2)} > \varepsilon\lambda\big) \geq \frac{1}{2}.$$

We may assume that $\varepsilon < a$. Applying Theorem 2.9 at most $1 + [(a - \varepsilon)(4/b)^\alpha]$ times, we get that there exists $c_1 = c_1(\alpha, M, a, T) > 0$ such that for all $m \in (0, M]$,

$$c_1 p^m_{B(y, b\lambda^{1/\alpha})}(\varepsilon\lambda, y, w) \leq p^m_{B(y, b\lambda^{1/\alpha})}(a\lambda, y, w) \qquad \text{for } w \in B(y, b\lambda^{1/\alpha}/2).$$

Thus for any $m \in (0, M]$,

$$\begin{aligned}\mathbb{P}_y\big(\tau^m_{B(y,b\lambda^{1/\alpha})} > a\lambda\big) &= \int_{B(y,b\lambda^{1/\alpha})} p^m_{B(y,b\lambda^{1/\alpha})}(a\lambda, y, w)\,dw \\ &\geq \int_{B(y,b\lambda^{1/\alpha}/2)} p^m_{B(y,b\lambda^{1/\alpha})}(a\lambda, y, w)\,dw \\ &\geq c_1 \int_{B(y,b\lambda^{1/\alpha}/2)} p^m_{B(y,b\lambda^{1/\alpha}/2)}(\varepsilon\lambda, y, w)\,dw \geq c_1/2.\end{aligned}$$

This proves the lemma. □

For the next four results, $D$ is an arbitrary nonempty open set and we use the convention that $\delta_D(\cdot) \equiv \infty$ when $D = \mathbb{R}^d$.

PROPOSITION 3.2. *Let $M$ and $T$ be positive constants. Suppose that $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) \geq t^{1/\alpha} \geq 2|x - y|$. Then there exists a positive constant $C_{17} = C_{17}(M, \alpha, T)$ such that for any $m \in (0, M]$,*

$$p^m_D(t, x, y) \geq C_{17} t^{-d/\alpha}. \tag{3.1}$$

PROOF. Let $t \leq T$ and $x, y \in D$ with $\delta_D(x) \geq t^{1/\alpha} \geq 2|x - y|$. Note that, since $t \leq T$, we have $|x - y| \leq 2^{-1} t^{1/\alpha} \leq 2^{-1} T^{1/\alpha}$. Thus, by the uniform parabolic Harnack inequality (Theorem 2.9), there exists $c_1 = c_1(\alpha, M, T) > 0$ such that for any $m \in (0, M]$,

$$p^m_D(t/2, x, w) \leq c_1 p^m_D(t, x, y) \qquad \text{for every } w \in B(x, 2t^{1/\alpha}/3).$$

This together with Lemma 3.1 yields that for any $m \in (0, M]$,

$$\begin{aligned}p^m_D(t, x, y) &\geq \frac{1}{c_1|B(x, t^{1/\alpha}/2)|} \int_{B(x,t^{1/\alpha}/2)} p^m_D(t/2, x, w)\,dw \\ &\geq c_2 t^{-d/\alpha} \int_{B(x,t^{1/\alpha}/2)} p^m_{B(x,t^{1/\alpha}/2)}(t/2, x, w)\,dw \\ &= c_2 t^{-d/\alpha} \mathbb{P}_x\big(\tau^m_{B(x,t^{1/\alpha}/2)} > t/2\big) \geq c_3 t^{-d/\alpha},\end{aligned}$$

where $c_i = c_i(T, \alpha, M) > 0$ for $i = 2, 3$. □

LEMMA 3.3. *Let $M > 0$ and $T > 0$ be constants. Suppose that $(t, x, y) \in (0, T] \times D \times D$ with $\min\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$ and $t^{1/\alpha} \leq 2|x - y|$. Then there exists a constant $C_{18} = C_{18}(\alpha, T, M) > 0$ such that for all $m \in (0, M]$,*

$$\mathbb{P}_x\big(X^{m,D}_t \in B(y, 2^{-1}t^{1/\alpha})\big) \geq C_{18} t^{d/\alpha+1} J^m(x, y).$$

PROOF. By Lemma 3.1, starting at $z \in B(y, 4^{-1}t^{1/\alpha})$, with probability at least $c_1 = c_1(\alpha, M, T) > 0$, for any $m \in (0, M]$, the process $X^m$ does not move more than $6^{-1}t^{1/\alpha}$ by time $t$. Thus, it is sufficient to show that there exists a constant $c_2 = c_2(\alpha, M, T) > 0$ such that for any $m \in (0, M]$, $t \in (0, T]$ and $(x, y)$ with $t^{1/\alpha} \le 2|x - y|$,

$$\mathbb{P}_x(X^{m,D} \text{ hits the ball } B(y, 4^{-1}t^{1/\alpha}) \text{ by time } t) \ge c_2 t^{d/\alpha+1} J^m(x, y). \tag{3.2}$$

Let $B_x := B(x, 6^{-1}t^{1/\alpha})$, $B_y := B(y, 6^{-1}t^{1/\alpha})$ and $\tau_x^m := \tau_{B_x}^m$. It follows from Lemma 3.1, there exists $c_3 = c_3(\alpha, M, T) > 0$ such that for all $m \in (0, M]$,

$$\mathbb{E}_x[t \wedge \tau_x^m] \ge t\mathbb{P}_x(\tau_x^m \ge t) \ge c_3 t \qquad \text{for } t > 0. \tag{3.3}$$

By the Lévy system in (2.6),

$$\begin{aligned} &\mathbb{P}_x(X^{m,D} \text{ hits the ball } B(y, 4^{-1}t^{1/\alpha}) \text{ by time } t) \\ &\quad \ge \mathbb{P}_x\big(X^m_{t\wedge\tau_x^m} \in B(y, 4^{-1}at^{1/\alpha}) \text{ and } t \wedge \tau_x^m \text{ is a jumping time}\big) \\ &\quad \ge \mathbb{E}_x\left[\int_0^{t\wedge\tau_x^m} \int_{B_y} J^m(X_s^m, u)\, du\, ds\right]. \end{aligned} \tag{3.4}$$

We consider two cases separately.

(i) Suppose $|x - y| \le T^{1/\alpha}$. Since $|x - y| \ge 2^{-1}t^{1/\alpha}$, we have for $s < \tau_x^m$ and $u \in B_y$,

$$|X_s^m - u| \le |X_s^m - x| + |x - y| + |y - u| \le 2|x - y|.$$

Thus from (3.4), for any $m \in (0, M]$,

$$\begin{aligned} &\mathbb{P}_x(X^{m,D} \text{ hits the ball } B(y, 4^{-1}t^{1/\alpha}) \text{ by time } t) \\ &\quad \ge \mathbb{E}_x[t \wedge \tau_x^m] \int_{B_y} j^m(2|x - y|)\, du \\ &\quad \ge c_4 t |B_y| j^m(2|x - y|) \ge c_5 t^{d/\alpha+1} j^m(2|x - y|) \end{aligned}$$

for some positive constants $c_i = c_i(\alpha, M, T)$, $i = 4, 5$. Here in the second inequality above, we used (3.3). Therefore in view of (2.10), the assertion of the lemma holds when $|x - y| \le T^{1/\alpha}$.

(ii) Suppose $|x - y| > T^{1/\alpha}$. In this case, for $s < \tau_x^m$ and $u \in B_y$,

$$\begin{aligned} |X_s^m - u| &\le |X_s^m - x| + |x - y| + |y - u| \\ &\le |x - y| + 3^{-1}t^{1/\alpha} \le |x - y| + 3^{-1}T^{1/\alpha}. \end{aligned}$$

Thus from (3.4), for any $m \in (0, M]$,

$$\begin{aligned} &\mathbb{P}_x(X^{m,D} \text{ hits the ball } B(y, 4^{-1}t^{1/\alpha}) \text{ by time } t) \\ &\quad \ge \mathbb{E}_x[t \wedge \tau_x^m] \int_{B_y} j^m(|x - y| + 3^{-1}T^{1/\alpha})\, du \\ &\quad \ge c_6 t |B_y| j^m(|x - y| + 3^{-1}T^{1/\alpha}) \\ &\quad \ge c_7 t^{d/\alpha+1} j^m(|x - y| + 3^{-1}T^{1/\alpha}) \end{aligned}$$

for some positive constants $c_i = c_i(\alpha, M, T)$, $i = 6, 7$. Here in the second inequality, (3.3) is used. Since $|x-y| > T^{1/\alpha}$, by (2.11), we see that the assertion of the lemma is valid for $|x-y| > T^{1/\alpha}$ as well. $\square$

PROPOSITION 3.4. *Let $M$ and $T$ be positive constants. Suppose that $(t, x, y) \in (0, T] \times D \times D$ with $\min\{\delta_D(x), \delta_D(y)\} \ge (t/2)^{1/\alpha}$ and $(t/2)^{1/\alpha} \le 2|x-y|$. Then there exists a constant $C_{19} = C_{19}(\alpha, M, T) > 0$ such that for all $m \in (0, M]$,*

$$p_D^m(t, x, y) \ge C_{19} t J^m(x, y).$$

PROOF. By the semigroup property, Proposition 3.2 and Lemma 3.3, there exist positive constants $c_1 = c_1(\alpha, T, M)$ and $c_2 = c_2(\alpha, T, M)$ such that for all $m \in (0, M]$,

$$\begin{aligned} p_D^m(t, x, y) &= \int_D p_D^m(t/2, x, z) p_D^m(t/2, z, y)\, dz \\ &\ge \int_{B(y, 2^{-1}(t/2)^{1/\alpha})} p_D^m(t/2, x, z) p_D^m(t/2, z, y)\, dz \\ &\ge c_1 t^{-d/\alpha} \mathbb{P}_x\big(X_{t/2}^{m,D} \in B(y, 2^{-1}(t/2)^{1/\alpha})\big) \\ &\ge c_2 t J^m(x, y). \end{aligned}$$
$\square$

Combining Propositions 3.2 and 3.4, we have the following preliminary lower bound for $p_D^m(t, x, y)$.

PROPOSITION 3.5. *Let $M$ and $T$ be positive constants. Suppose that $(t, x, y) \in (0, T] \times D \times D$ with $\min\{\delta_D(x), \delta_D(y)\} \ge t^{1/\alpha}$. Then there exists a constant $C_{20} = C_{20}(\alpha, M, T) > 0$ such that for all $m \in (0, M]$,*

$$p_D^m(t, x, y) \ge C_{20}\big(t^{-d/\alpha} \wedge t J^m(x, y)\big).$$

**4. Sharp two-sided Dirichlet heat kernel estimates.** The goal of this section is to establish the sharp two-sided estimates for $p_D^m(t, x, y)$ as stated in Theorem 1.1.

First, combining (2.16) and (2.17) with Proposition 3.5, we have the following sharp two-sided estimates for $p^m(t, x, y)$.

THEOREM 4.1. *Let $M$ and $T$ be positive constants. Then there exists a constant $C_{21} = C_{21}(\alpha, M, T) > 1$ such that for all $m \in (0, M]$, $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,*

$$C_{21}^{-1}\big(t^{-d/\alpha} \wedge t J^m(x, y)\big) \le p^m(t, x, y) \le C_{21}\big(t^{-d/\alpha} \wedge t J^m(x, y)\big).$$

The two-sided estimates in Theorem 4.1 will be used in the proof of Theorem 4.4 to derive sharp uniform upper bound on the Dirichlet heat kernel $p_D^m(t, x, y)$.

LEMMA 4.2. *Suppose* $M > 0$ *and* $r_0 \le R_0 M^{-1/\alpha}$. *Let* $E = \{x \in \mathbb{R}^d : |x| > r_0\}$. *For every* $T > 0$, *there is a constant* $C_{22} = C_{22}(r_0, \alpha, M, T) > 0$ *such that*

$$p_E^m(t, x, y) \le C_{22}\sqrt{t}\,\delta_E(x)^{\alpha/2} j^m(|x - y|/16)$$

*for all* $m \in (0, M]$, $r_0 < |x| < 5r_0/4$, $|y| \ge 2r_0$ *and* $t \le T$.

PROOF. Define

$$U := \begin{cases} \{z \in \mathbb{R}^d : r_0 < |z| < 3r_0/2\}, & \text{if } d \ge 2, \\ \{z \in \mathbb{R}^1 : r_0 < z < 3r_0/2\}, & \text{if } d = 1. \end{cases}$$

It is well known (see, e.g., [40]) that $X^m_{\tau_U} \notin \partial U$. For $r_0 < |x| < 5r_0/4$, $|y| \ge 2r_0$ and $t \in (0, T]$, it follows from the strong Markov property and (2.6) that

$$\begin{aligned}
&p_E^m(t, x, y) \\
&\quad = \mathbb{E}_x[p_E^m(t - \tau_U^m, X^m_{\tau_U^m}, y); \tau_U^m < t, (3r_0/4) + (|y|/2) \ge |X^m_{\tau_U^m}| > 3r_0/2] \\
&\qquad + \mathbb{E}_x[p_E^m(t - \tau_U^m, X^m_{\tau_U^m}, y); \tau_U^m < t, |X^m_{\tau_U^m}| > (3r_0/4) + (|y|/2)] \\
&\quad \le \Bigl( \sup_{\substack{s : s \in (0,t) \\ w : (3r_0/4) + (|y|/2) \ge |w| > 3r_0/2}} p_E^m(t - s, w, y) \Bigr) \\
&\qquad \times \mathbb{P}_x\bigl(\tau_U^m < t, (3r_0/4) + (|y|/2) \ge |X^m_{\tau_U^m}| > 3r_0/2\bigr) \\
&\qquad + \int_0^t \int_U p_U(s, x, z) \\
&\qquad\qquad \times \Bigl( \int_{\{w : |w| > (3r_0/4) + (|y|/2)\}} J^m(z, w) p_E^m(t - s, w, y)\, dw \Bigr) dz\, ds \\
&\quad =: I + II.
\end{aligned}$$

If $|w| \le (3r_0/4) + (|y|/2)$, then $|w - y| \ge |y| - |w| \ge \frac{1}{2}(|y| - \frac{3r_0}{2}) \ge \frac{|y|}{8} \ge \frac{|x-y|}{16}$. Thus by (2.16) and the fact that $p_E^m \le p^m$, we have for $|w| \le (3r_0/4) + (|y|/2)$ and $0 < s < t < T$,

$$p_E^m(t - s, w, y) \le p^m(t - s, x/16, y/16) \le Lte^{mT} j^m(|x - y|/16).$$

Therefore

$$I \le Lte^{MT} j^m(|x - y|/16)\mathbb{P}_x(|X^m_{\tau_U^m}| > 3r_0/2).$$

By Corollary 2.7,

$$\mathbb{P}_x(|X^m_{\tau_U^m}| > 3r_0/2) \le C_{13}\mathbb{P}_x(|X_{\tau_U}| > 3r_0/2) \le c_1\delta_U(x)^{\alpha/2} = c_1\delta_E(x)^{\alpha/2}$$

for some positive constant $c_1 = c_1(M, r_0, \alpha)$. Here the last inequality is due to the boundary Harnack inequality for $X$ on $U$ proved in [4] (see the proof of [14], Lemma 2.2). Thus we have

$$I \le c_2 te^{MT}\delta_E(x)^{\alpha/2} j^m(|x - y|/16), \qquad m \in (0, M], \tag{4.1}$$

for some positive constant $c_2 = c_2(r_0, \alpha, M)$.

On the other hand, for $z \in U$ and $w \in \mathbb{R}^d$ with $|w| > (3r_0/4) + (|y|/2)$, we have

$$|z - w| \geq |w| - |z| \geq \frac{1}{2}\left(|y| - \frac{3r_0}{2}\right) \geq \frac{|y|}{8} \geq \frac{|x-y|}{16}.$$

Thus by the symmetry of $p_E^m(t-s, w, y)$ in $(w, y)$, we have that there exists $c_3 = c_3(M, r_0, \alpha) > 0$ such that for any $m \in (0, M]$,

$$\begin{aligned} II &\leq \int_0^t \biggl(\int_U p_U^m(s,x,z) \\ &\qquad \times \biggl(\int_{\{w : |w| > (3r_0/4)+(|y|/2)\}} J^m(x/16, y/16) p_E^m(t-s,y,w)\,dw\biggr)\,dz\biggr)\,ds \\ &\leq c_3 j^m(|x-y|/16) \int_0^t \biggl(\int_U p_U^m(s,x,z)\,dz\biggr)\,ds. \end{aligned}$$

By (2.21), there exists $c_4 = c_4(\alpha, T) > 0$ such that for every $s \leq T$,

$$p_U^m(s,x,z) \leq e^{ms} p_U(s,x,z) \leq c_4 e^{ms} \frac{\delta_U(x)^{\alpha/2}}{\sqrt{s}}\left(s^{-d/\alpha} \wedge \frac{s}{|x-z|^{d+\alpha}}\right).$$

The last inequality above comes from [14], Theorem 1.1. Thus

$$\begin{aligned} &\int_0^t \biggl(\int_U p_U^m(s,x,z)\,dz\biggr)\,ds \\ &\quad\leq c_4 e^{mT} \delta_U(x)^{\alpha/2} \\ &\qquad \times \biggl(\int_0^t \int_{\{|z| \leq s^{1/\alpha}\}} s^{-d/\alpha - 1/2}\,dz\,ds + \int_0^t \int_{\{|z| > s^{1/\alpha}\}} \frac{\sqrt{s}}{|z|^{d+\alpha}}\,dz\,ds\biggr) \\ &\quad\leq c_5 \delta_E(x)^{\alpha/2} \sqrt{t}. \end{aligned}$$

This together with our estimate on $I$ above completes the proof the lemma. □

Recall that an open set $D$ is said to satisfy the weak uniform exterior ball condition with radius $r_0 > 0$ if, for every $z \in \partial D$, there is a ball $B^z$ of radius $r_0$ such that $B^z \subset \mathbb{R}^d \setminus \overline{D}$ and $z \in \partial B^z$.

LEMMA 4.3. *Let $M > 0$ be a constant and $D$ an open set satisfying the weak uniform exterior ball condition with radius $r_0 > 0$. For every $T > 0$, there exists a positive constant $C_{23} = C_{23}(T, r_0, \alpha, M)$ such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D^m(t,x,y) \leq C_{23}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) p^m(t, x/16, y/16).$$

PROOF. Let $r_1 = r_0 \wedge (R_0 M^{-1/\alpha})$. In view of Theorem 4.1, it suffices to prove the theorem for $x \in D$ with $\delta_D(x) < r_1/4$. By (2.21) and [14], Theorem 1.1, there exists $c_1 = c_1(\alpha, T, D) > 0$ such that on $(0, T] \times D \times D$

$$p_D^m(t,x,y) \le e^{mt} p_D(t,x,y) \le c_1 e^{MT} \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \left( t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right). \tag{4.2}$$

For $x, y \in D$, let $z \in \partial D$ so that $|x - z| = \delta_D(x)$. Let $B_z \subset D^c$ be the ball with radius $r_1$ so that $\partial B_z \cap \partial D = \{z\}$. When $\delta_D(x) < r_1/4$ and $|x - y| \ge 5r_1$, we have $\delta_{B_z^c}(y) > 2r_1$ and so by Lemma 4.2, there is a constant $c_2 = c_2(r_1, T, M, \alpha) > 0$ such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,

$$\begin{aligned} p_D^m(t,x,y) &\le p_{(\overline{B}_z)^c}^m(t,x,y) \le c_2 \delta_{(\overline{B}_z)^c}(x)^{\alpha/2} \sqrt{t}\, j^m(|x-y|/16) \\ &= c_2 \delta_D(x)^{\alpha/2} \sqrt{t}\, j^m(|x-y|/16). \end{aligned} \tag{4.3}$$

Since there exist constants $c_3$ and $c_4$ depending only on $M, \alpha$ and $r_1$ such that

$$\frac{c_3}{|x-y|^{d+\alpha}} \le j^m(|x-y|/16) \le \frac{c_4}{|x-y|^{d+\alpha}} \qquad \text{for } m \in (0, M] \text{ and } |x-y| < 5r_1,$$

combining (4.2) and (4.3) with Theorem 4.1, we arrive at the conclusion of the theorem. $\square$

THEOREM 4.4. *Let $M$ and $T$ be positive constants. Suppose that $D$ is an open set satisfying the weak uniform exterior ball condition with radius $r_0 > 0$. Then there exists a constant $C_{24} = C_{24}(T, r_0, M, \alpha) > 0$ such that for all $m \in (0, M]$, $t \in (0, T]$ and $x, y \in D$,*

$$p_D^m(t,x,y) \le C_{24} \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) p^m(t, x/16, y/16). \tag{4.4}$$

PROOF. Fix $T > 0$ and $M > 0$. By Lemma 4.3, symmetry and the semigroup property, we have for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,

$$\begin{aligned} p_D^m(t,x,y) &= \int_D p_D^m(t/2,x,z) p_D^m(t/2,z,y)\, dz \\ &\le c_1 \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \\ &\quad \times \int_{\mathbb{R}^d} p^m(t/2, x/16, z/16) p^m(t/2, z/16, y/16)\, dz \\ &\le c_2 \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) p^m(t, x/16, y/16). \end{aligned}$$ $\square$

In the next two results, the open set $D$ is assumed to satisfy the uniform interior ball condition with radius $r_0 > 0$ in the following sense: For every $x \in D$ with $\delta_D(x) < r_0$, there is $z_x \in \partial D$ so that $|x - z_x| = \delta_D(x)$ and $B(x_0, r_0) \subset D$ for $x_0 := z_x + r_0(x - z_x)/|x - z_x|$. Note that this condition is strictly stronger than the weak uniform interior ball condition with radius $r_0$ defined as follows: For every $z \in \partial D$, there is a ball $B^z$ of radius $r_0$ such that $B^z \subset D$ and $z \in \partial B^z$. Here is an example. In $\mathbb{R}^2$, let $x_k = (2k, 0) \in \mathbb{R}^2$ and define $D = \mathbb{R}^2 \setminus \bigcup_{k=1}^{\infty} \partial B(x_k, 1/k)$. Then $D$ satisfies the weak uniform interior ball condition but not the uniform interior ball condition.

Under the uniform interior ball condition, we will prove the following lower bound for $p_D^m(t, x, y)$.

THEOREM 4.5. *For any $M > 0$ and $T > 0$ there exists positive constant $C_{25} = C_{25}(\alpha, T, M, r_0)$ such that for all $m \in (0, M]$, $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D^m(t, x, y) \geq C_{25}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge t j^m(|x - y|)\right).$$

In order to prove the theorem, for $M > 0$, we let

$$T_0 = T_0(r_0, R_0, M) := \left(\frac{r_0 \wedge R_0 M^{-1/\alpha}}{16}\right)^{\alpha}. \tag{4.5}$$

In the remainder of this section, for any $x \in D$ with $\delta_D(x) < r_0$, $z_x$ is a point on $\partial D$ such that $|z_x - x| = \delta_D(x)$ and $\mathbf{n}(z_x) := (x - z_x)/|z_x - x|$.

LEMMA 4.6. *Let $M > 0$ be a constant. Suppose that $(t, x) \in (0, T_0] \times D$ with $\delta_D(x) \leq 3t^{1/\alpha} < r_0/4$ and $\kappa \in (0, 1)$. Put $x_0 = z_x + 4.5t^{1/\alpha}\mathbf{n}(z_x)$. Then for any $a > 0$, there exists a constant $C_{26} = C_{26}(M, \kappa, \alpha, r_0, a) > 0$ such that for all $m \in (0, M]$,*

$$\mathbb{P}_x\left(X_{at}^{m,D} \in B(x_0, \kappa t^{1/\alpha})\right) \geq C_{26}\frac{\delta_D(x)^{\alpha/2}}{t^{1/2}}. \tag{4.6}$$

PROOF. Let $0 < \kappa_1 \leq \kappa$ and assume first that $2^{-4}\kappa_1 t^{1/\alpha} < \delta_D(x) \leq 3t^{1/\alpha}$. As in the proof of Lemma 3.3, we get that, in this case, using the fact that $|x - x_0| \in [1.5\kappa t^{1/\alpha}, 6t^{1/\alpha}]$, there exist constants $c_i = c_i(\alpha, \kappa_1, M, r_0, a) > 0$, $i = 1, 2$, such that for all $m \in (0, M]$ and $t \leq T_0$,

$$\mathbb{P}_x\left(X_{at}^{m,D} \in B(x_0, \kappa_1 t^{1/\alpha})\right) \geq c_1 t^{d/\alpha+1} J^m(x, x_0) \geq c_2 > 0. \tag{4.7}$$

By taking $\kappa_1 = \kappa$, this shows that (4.6) holds for all $a > 0$ in the case when $2^{-4}\kappa t^{1/\alpha} < \delta_D(x) \leq 3t^{1/\alpha}$.

So it suffices to consider the case that $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$. We now show that there is some $a_0 > 1$ so that (4.6) holds for every $a \geq a_0$ and $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$.

For simplicity, we assume without loss of generality that $x_0 = 0$ and let $\widehat{B} := B(0, \kappa t^{1/\alpha})$. Let $x_1 = z_x + 4^{-1}\kappa \mathbf{n}(z_x) t^{1/\alpha}$ and $B_1 := B(x_1, 4^{-1}\kappa t^{1/\alpha})$. By the strong Markov property of $X^{m,D}$ at the first exit time $\tau^m_{B_1}$ from $B_1$ and Lemma 3.1, there exists $c_3 = c_3(a, \kappa, \alpha, M, T) > 0$ such that for all $m \in (0, M]$,

$$
\begin{aligned}
&\mathbb{P}_x(X^{m,D}_{at} \in \widehat{B}) \\
&\quad \geq \mathbb{P}_x\big(\tau^m_{B_1} < at,\, X^m_{\tau^m_{B_1}} \in B(0, 2^{-1}\kappa t^{1/\alpha}) \text{ and} \\
&\qquad\qquad |X^{m,D}_s - X^m_{\tau_{B_1}}| < 2^{-1}\kappa t^{1/\alpha} \text{ for } s \in [\tau^m_{B_1}, \tau^m_{B_1} + at]\big) \\
&\quad \geq c_3 \mathbb{P}_x\big(\tau^m_{B_1} < at \text{ and } X^m_{\tau^m_{B_1}} \in B(0, 2^{-1}\kappa t^{1/\alpha})\big).
\end{aligned}
\tag{4.8}
$$

It follows from the first display in Theorem 2.6 and the explicit formula for the Poisson kernel of balls with respect to $X$ that there exist $c_4 = c_4(\alpha, M) > 0$ and $c_5 = c_5(\alpha, M, \kappa, r_0) > 0$ such that for all $m \in (0, M]$,

$$
\begin{aligned}
\mathbb{P}_x\big(X^m_{\tau^m_{B_1}} \in B(0, 2^{-1}\kappa t^{1/\alpha})\big) &\geq c_4 \mathbb{P}_x\big(X_{\tau_{B_1}} \in B(0, 2^{-1}\kappa t^{1/\alpha})\big) \\
&\geq c_5 \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}.
\end{aligned}
\tag{4.9}
$$

Applying Theorem 2.6 and the estimates for $G_{B_1}$ (see, e.g., [18], (1.4)), we get that there exist $c_6 = c_6(\alpha, M) > 0$ and $c_7 = c_7(\alpha, M, \kappa, r_0) > 0$ such that for all $m \in (0, M]$,

$$
\mathbb{P}_x(\tau^m_{B_1} \geq at) \leq (at)^{-1}\mathbb{E}_x[\tau^m_{B_1}] \leq c_6 (at)^{-1}\mathbb{E}_x[\tau_{B_1}] \leq a^{-1} c_7 \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}.
$$

Define $a_0 = 2c_7/(c_5)$. We have by (4.8) and (4.9) and the display above that for $a \geq a_0$ and $m \in (0, M]$,

$$
\begin{aligned}
\mathbb{P}_x(X^{m,D}_{at} \in \widehat{B}) &\geq c_3\big(\mathbb{P}_x\big(X^m_{\tau^m_{B_1}} \in B(0, 2^{-1}\kappa t^{1/\alpha})\big) - \mathbb{P}_x(\tau^m_{B_1} \geq at)\big) \\
&\geq c_3(c_5/2)\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}.
\end{aligned}
\tag{4.10}
$$

Equations (4.7) and (4.10) show that (4.6) holds for every $a \geq a_0$ and for every $x \in D$ with $\delta_D(x) \leq 3t^{1/\alpha}$.

Now we deal with the case $0 < a < a_0$ and $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$. If $\delta_D(x) \leq 3(at/a_0)^{1/\alpha}$, we have from (4.6) for the case of $a = a_0$ that there exist $c_8 = c_8(\kappa, \alpha, M) > 0$ and $c_9 = c_9(\kappa, \alpha, M, a) > 0$ such that for all $m \in (0, M]$,

$$
\begin{aligned}
\mathbb{P}_x\big(X^{m,D}_{at} \in B(x_0, \kappa t^{1/\alpha})\big) &\geq \mathbb{P}_x\big(X^{m,D}_{a_0(at/a_0)} \in B(x_0, \kappa (at/a_0)^{1/\alpha})\big) \\
&\geq c_8 \left(\frac{\delta_D(x)}{(at/a_0)^{1/\alpha}}\right)^{\alpha/2} = c_9 \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}.
\end{aligned}
$$

If $3(at/a_0)^{1/\alpha} < \delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$ [in this case $\kappa > 3 \cdot 2^4 (a/a_0)^{1/\alpha}$], we get (4.6) from (4.7) by taking $\kappa_1 = (a/a_0)^{1/\alpha}$. The proof of the lemma is now complete. $\square$

PROOF OF THEOREM 4.5. In the first part of this proof, we adapt some arguments from [6].

Assume first that $t \le T_0$. Since $D$ satisfies the uniform interior ball condition with radius $r_0$ and $0 < t \le T_0$, we can choose $\xi_x^t$ as follows: if $\delta_D(x) \le 3t^{1/\alpha}$, let $\xi_x^t = z_x + (9/2)t^{1/\alpha}\mathbf{n}(z_x)$ [so that $B(\xi_x^t, (3/2)t^{1/\alpha}) \subset B(z_x + 3t^{1/\alpha}\mathbf{n}(z_x), 3t^{1/\alpha}) \setminus \{x\}$ and $\delta_D(z) \ge 3t^{1/\alpha}$ for every $z \in B(\xi_x^t, (3/2)t^{1/\alpha})$]. If $\delta_D(x) > 3t^{1/\alpha}$, choose $\xi_x^t \in B(x, \delta_D(x))$ so that $|x - \xi_x^t| = (3/2)t^{1/\alpha}$. Note that in this case, $B(\xi_x^t, (3/2)t^{1/\alpha}) \subset B(x, \delta_D(x)) \setminus \{x\}$ and $\delta_D(z) \ge t^{1/\alpha}$ for every $z \in B(\xi_x^t, 2^{-1}t^{1/\alpha})$. We also define $\xi_y^t$ the same way.

If $\delta_D(x) \le 3t^{1/\alpha}$, by Lemma 4.6 (with $a = 3^{-1}, \kappa = 2^{-1}$),

$$\mathbb{P}_x\big(X_{t/3}^{m,D} \in B(\xi_x^t, 2^{-1}t^{1/\alpha})\big) \ge c_0 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.$$

If $\delta_D(x) > 3t^{1/\alpha}$, by Proposition 3.5,

$$\begin{aligned}
\mathbb{P}_x\big(X_{t/3}^{m,D} \in B(\xi_x^t, 2^{-1}t^{1/\alpha})\big) &= \int_{B(\xi_x^t, 2^{-1}t^{1/\alpha})} p_D^m(t/3, x, u)\, du \\
&\ge c_1 t^{-d/\alpha}\big(1 \wedge \psi((MT)^{1/\alpha})\big)|B(\xi_x^t, 2^{-1}t^{1/\alpha})| \\
&\ge c_2 \ge c_3 \bigg(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\bigg).
\end{aligned}\tag{4.11}$$

Similarly,

$$\mathbb{P}_y\big(X_{t/3}^{m,D} \in B(\xi_y^t, 2^{-1}t^{1/\alpha})\big) \ge c_3 \bigg(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\bigg).\tag{4.12}$$

Note that by the semigroup property, Proposition 3.5 and (4.11) and (4.12),

$$\begin{aligned}
&p_D^m(t, x, y) \\
&\quad \ge \int_{B(\xi_y^t, 2^{-1}t^{1/\alpha})} \int_{B(\xi_x^t, 2^{-1}t^{1/\alpha})} p_D^m(t/3, x, u) p_D^m(t/3, u, v) \\
&\qquad\qquad \times p_D^m(t/3, v, y)\, du\, dv \\
&\quad \ge c_4 \int_{B(\xi_y^t, 2^{-1}t^{1/\alpha})} \int_{B(\xi_x^t, 2^{-1}t^{1/\alpha})} p_D^m(t/3, x, u)\big(tJ^m(u, v) \wedge t^{-d/\alpha}\big) \\
&\qquad\qquad \times p_D^m(1/3, v, y)\, du\, dv \\
&\quad \ge c_5 \Big(\inf_{\substack{u \in B(\xi_x^t, 2^{-1}t^{1/\alpha}) \\ v \in B(\xi_y^t, 2^{-1}t^{1/\alpha})}} \big(tJ^m(u, v) \wedge t^{-d/\alpha}\big)\Big) \bigg(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\bigg) \bigg(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\bigg).
\end{aligned}\tag{4.13}$$

For $(u,v)\in B(\xi^t_x, 2^{-1}t^{1/\alpha})\times B(\xi^t_y, 2^{-1}t^{1/\alpha})$, since $|u-v|\le t^{1/\alpha}+|\xi^t_x-\xi^t_y|\le 10t^{1/\alpha}+|x-y|$, by considering the cases $|x-y|\ge t^{1/\alpha}$ and $|x-y|<t^{1/\alpha}$ separately using (2.10) and (2.11), we have

$$\begin{aligned}\inf_{(u,v)\in B(\xi^t_x,2^{-1}t^{1/\alpha})\times B(\xi^t_y,2^{-1}t^{1/\alpha})}&\big(tJ^m(u,v)\wedge t^{-d/\alpha}\big)\\ &\ge c_6\big(tJ^m(x,y)\wedge t^{-d/\alpha}\big).\end{aligned}\tag{4.14}$$

Thus combining (4.13) and (4.14), we conclude that for $t\in(0,T_0]$,

$$p^m_D(t,x,y)\ge c_7\left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\wedge 1\right)\big(tJ^m(x,y)\wedge t^{-d/\alpha}\big).\tag{4.15}$$

Next assume $T=2T_0$. Recall that $T_0=((r_0\wedge R_0M^{-1/\alpha})/16)^\alpha$. For $(t,x,y)\in(T_0,2T_0]\times D\times D$, let $x_0,y_0\in D$ be such that $\max\{|x-x_0|,|y-y_0|\}<r_0$ and $\min\{\delta_D(x_0),\delta_D(y_0)\}\ge r_0/2$. Note that, if $|x-y|\ge 4r_0$, then $|x-y|-2r_0\le |x_0-y_0|\le|x-y|+2r_0$, so by (2.11), $c_8^{-1}J^m(x_0,y_0)\le J^m(x,y)\le c_8J^m(x_0,y_0)$ for some constant $c_8=c_8(M)>1$. Thus we have

$$(t/2)^{-d/\alpha}\wedge\frac{t}{2}J^m(x_0,y_0)\ge c_9\big(t^{-d/\alpha}\wedge tJ^m(x,y)\big).\tag{4.16}$$

Similarly, there is a positive constant $c_{10}$ such that for every $z,w\in D$,

$$\begin{aligned}(t/3)^{-d/\alpha}\wedge(t/3)J^m(x,z)&\ge c_{10}\left((t/12)^{-d/\alpha}\wedge\frac{t}{12}J^m(x_0,z)\right),\\ (t/3)^{-d/\alpha}\wedge(t/3)J^m(w,y)&\ge c_{10}\left((t/12)^{-d/\alpha}\wedge\frac{t}{12}J^m(w,y_0)\right).\end{aligned}\tag{4.17}$$

By (4.17) and the lower bound estimate in Theorem 4.5 for $p^m_D$ on $(0,T_0]\times D\times D$, we have

$$\begin{aligned}&p^m_D(t,x,y)\\ &\quad=\int_{D\times D}p^m_D(t/3,x,z)p^m_D(t/3,z,w)p^m_D(t/3,w,y)\,dz\,dw\\ &\quad\ge c_{11}\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t/3}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t/3}}\right)\int_{D\times D}\big((t/3)^{-d/\alpha}\wedge(t/3)J^m(x,z)\big)\\ &\qquad\times\left(1\wedge\frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right)p^m_D(t/3,z,w)\left((t/3)^{-d/\alpha}\wedge\frac{t}{3}J^m(w,y)\right)\\ &\qquad\times\left(1\wedge\frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right)dz\,dw\\ &\quad\ge c_{12}\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\int_{D\times D}\left((t/12)^{-d/\alpha}\wedge\frac{t}{12}J^m(x_0,z)\right)\end{aligned}$$

$$\times\left(1\wedge\frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right)p^m_D(t/3,z,w)\left((t/12)^{-d/\alpha}\wedge\frac{t}{12}J^m(w,y_0)\right)$$
$$\times\left(1\wedge\frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right)dz\,dw$$

for some positive constants $c_i, i=11,12$. Let $D_1:=\{z\in D:\delta_D(z)>r_0/4\}$. Clearly, $x_0, y_0\in D_1$ and

$$\min\{\delta_{D_1}(x_0),\delta_{D_1}(y_0)\}\ge r_0/4=4(T_0)^{1/\alpha}\ge 4(t/2)^{1/\alpha}. \tag{4.18}$$

We have by Theorem 4.1, (4.16) and Lemma 4.3 that

$$\begin{aligned}
&\int_{D\times D}\left(\left(t/(12)\right)^{-d/\alpha}\wedge\frac{t}{12}J^m(x_0,z)\right)\left(1\wedge\frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right)\\
&\qquad\times p^m_D(t/3,z,w)\left((t/12)^{-d/\alpha}\wedge\frac{t}{12}J^m(w,y_0)\right)\left(1\wedge\frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right)dz\,dw\\
&\quad\ge c_{13}\int_{D_1\times D_1}p^m_{D_1}(t/12,x_0,z)p^m_{D_1}(t/3,z,w)p^m_{D_1}\big(t/(12),w,y_0\big)\,dz\,dw\\
&\quad=c_{13}p^m_{D_1}(t/2,x_0,y_0)\ge c_{14}\left((t/2)^{-d/\alpha}\wedge\frac{t}{2}J^m(x_0,y_0)\right)\\
&\quad\ge c_{15}\big(t^{-d/\alpha}\wedge tJ^m(x,y)\big)
\end{aligned}$$

for some positive constants $c_i, i=13,\dots,15$. Here Proposition 3.5 is used in the third inequality in view of (4.18). Iterating the above argument one can deduce that Theorem 4.5 holds for $T=kT_0$ for any integer $k\ge 2$. This completes the proof of the theorem. □

PROOF OF THEOREM 1.1. Theorem 1.1(i) is a combination of Theorems 4.4 and 4.5, so we only need to prove Theorem 1.1(ii).

Let $D$ be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics $(r_0,\Lambda_0)$. Clearly there is a ball $B\subset D$ whose radius depends only on $r_0$ and $\Lambda_0$. For each $m\ge 0$, the semigroup of $X^{m,D}$ is Hilbert–Schmidt as, by Theorem 1.1(i)

$$\int_{D\times D}p^m_D(t,x,y)^2\,dx\,dy=\int_D p^m_D(2t,x,x)\,dx\le C_1(2t)^{-d/\alpha}|D|<\infty,$$

and hence is compact. Let $\{\lambda_k^{\alpha,m,D}, k=1,2,\dots\}$ be the eigenvalues of $((m^{2/\alpha}-\Delta)^{\alpha/2}-m)|_D$, arranged in increasing order and repeated according to multiplicity, and let $\{\phi_k^{\alpha,m,D}, k=1,2,\dots\}$ be the corresponding eigenfunctions normalized to have unit $L^2$-norm on $D$. It is well known that $\lambda_1^{\alpha,m,D}$ is strictly positive and simple, and that $\phi_1^{\alpha,m,D}$ can be chosen to be strictly positive on $D$, and that $\{\phi_k^{\alpha,m,D}:k=1,2,\dots\}$ forms an orthonormal basis of $L^2(D;dx)$.

We also let $\{\lambda_k^{\alpha,m,B} : k = 1, 2, \ldots\}$ be the eigenvalues of $((m^{2/\alpha} - \Delta)^{\alpha/2} - m)|_B$, arranged in increasing order and repeated according to multiplicity. From the domain monotonicity of the first eigenvalue, it is easy to see that $\lambda_1^{\alpha,m,B} \geq \lambda_1^{\alpha,m,D}$. Thus, using [21], Theorem 3.4, we have that for every $m \in (0, M]$,

$$\lambda_1^{\alpha,m,D} \leq \lambda_1^{\alpha,m,B} \leq (\lambda_1^B + m^{2/\alpha})^{\alpha/2} - m \leq (\lambda_1^B + M^{2/\alpha})^{\alpha/2} =: c_1, \tag{4.19}$$

where $\lambda_1^B$ the first eigenvalue of $-\Delta|_B$. Moreover, by the Cauchy–Schwarz inequality,

$$\int_D (1 \wedge \delta_D(x)^{\alpha/2}) \phi_1^{\alpha,m,D}(x)\,dx \leq \left( \int_D (1 \wedge \delta_D(x)^{\alpha})\,dx \right)^{1/2} =: c_2. \tag{4.20}$$

Since $p_D^m(t, x, y)$ admits the following eigenfunction expansion:

$$p_D^m(t, x, y) = \sum_{k=1}^{\infty} e^{-t\lambda_k^{\alpha,m,D}} \phi_k^{\alpha,m,D}(x) \phi_k^{\alpha,m,D}(y) \qquad \text{for } t > 0 \text{ and } x, y \in D,$$

we have

$$\begin{aligned} &\int_{D \times D} (1 \wedge \delta_D(x)^{\alpha/2}) p_D^m(t, x, y) (1 \wedge \delta_D(y)^{\alpha/2})\,dx\,dy \\ &\qquad = \sum_{k=1}^{\infty} e^{-t\lambda_k^{\alpha,m,D}} \left( \int_D (1 \wedge \delta_D(x)^{\alpha/2}) \phi_k^{\alpha,m,D}(x)\,dx \right)^2. \end{aligned} \tag{4.21}$$

Consequently, using the fact that $\{\phi_k^{\alpha,m,D} : k = 1, 2, \ldots\}$ forms an orthonormal basis of $L^2(D; dx)$, we have

$$\begin{aligned} &\int_{D \times D} (1 \wedge \delta_D(x)^{\alpha/2}) p_D^m(t, x, y) (1 \wedge \delta_D(y)^{\alpha/2})\,dx\,dy \\ &\qquad \leq e^{-t\lambda_1^{\alpha,m,D}} \int_D (1 \wedge \delta_D(x)^{\alpha})\,dx \end{aligned} \tag{4.22}$$

for all $m > 0$ and $t > 0$. On the other hand, since

$$\phi_1^{\alpha,m,D}(x) = e^{\lambda_1^{\alpha,m,D}} \int_D p_D^m(1, x, y) \phi_1^{\alpha,m,D}(y)\,dy,$$

by the upper bound estimate in Theorem 1.1(i) and (4.20), we see that for every $m \in (0, M]$ and $x \in D$,

$$\begin{aligned} \phi_1^{\alpha,m,D}(x) &\leq e^{\lambda_1^{\alpha,m,D}} C_1 (1 \wedge \delta_D(x)^{\alpha/2}) \int_D (1 \wedge \delta_D(y)^{\alpha/2}) \phi_1^{\alpha,m,D}(y)\,dy \\ &\leq e^{\lambda_1^{\alpha,m,D}} c_2 C_1 (1 \wedge \delta_D(x)^{\alpha/2}). \end{aligned}$$

Hence

$$\begin{aligned} &\int_D (1 \wedge \delta_D(x)^{\alpha/2}) \phi_1^{\alpha,m,D}(x)\,dx \\ &\qquad \geq e^{-\lambda_1^{\alpha,m,D}} (c_2 C_1)^{-1} \int_D \phi_1^{\alpha,m,D}(x)^2\,dx = e^{-\lambda_1^{\alpha,m,D}} (c_2 C_1)^{-1}. \end{aligned}$$

It now follows from (4.21) that for every $m \in (0, M]$ and $t > 0$

$$\begin{aligned}
&\int_{D\times D} \big(1 \wedge \delta_D(x)^{\alpha/2}\big) p_D^m(t, x, y) \big(1 \wedge \delta_D(y)^{\alpha/2}\big)\, dx\, dy \\
&\quad \ge e^{-t\lambda_1^{\alpha,m,D}} \left( \int_D \big(1 \wedge \delta_D(x)^{\alpha/2}\big) \phi_1^{\alpha,m,D}(x)\, dx \right)^2 \qquad (4.23) \\
&\quad \ge e^{-(t+2)\lambda_1^{\alpha,m,D}} (c_2 C_1)^{-2}.
\end{aligned}$$

It suffices to prove Theorem 1.1(ii) for $T \ge 3$. For $t \ge T$ and $x, y \in D$, observe that

$$p_D^m(t, x, y) = \int_{D\times D} p_D^m(1, x, z) p_D^m(t-2, z, w) p_D^m(1, w, y)\, dz\, dw. \qquad (4.24)$$

Since $D$ is bounded, we have by the upper bound estimate in Theorem 1.1(i), (4.19) and (4.22) that for every $m \in (0, M]$, $t \ge T$ and $x, y \in D$,

$$\begin{aligned}
&p_D^m(t, x, y) \\
&\quad \le C_1^2 \big(1 \wedge \delta_D(x)^{\alpha/2}\big) \big(1 \wedge \delta_D(y)^{\alpha/2}\big) \\
&\qquad \times \int_{D\times D} \big(1 \wedge \delta_D(z)^{\alpha/2}\big) p_D^m(t-2, z, w) \big(1 \wedge \delta_D(w)^{\alpha/2}\big)\, dz\, dw \\
&\qquad \le C_1^2 \big(1 \wedge \delta_D(x)^{\alpha/2}\big) \big(1 \wedge \delta_D(y)^{\alpha/2}\big) e^{-(t-2)\lambda_1^{\alpha,m,D}} \int_D 1 \wedge \delta_D(x)^{\alpha}\, dx \\
&\qquad \le c_3 \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} e^{t\lambda_1^{\alpha,m,D}}.
\end{aligned}$$

Similarly, by the lower bound estimate in Theorem 1.1(i) and (4.23) that for every $m \in (0, M]$, $t \ge T$ and $x, y \in D$,

$$\begin{aligned}
&p_D^m(t, x, y) \\
&\quad \ge c_4 \big(1 \wedge \delta_D(x)^{\alpha/2}\big) \big(1 \wedge \delta_D(y)^{\alpha/2}\big) \\
&\qquad \times \int_{D\times D} \big(1 \wedge \delta_D(z)^{\alpha/2}\big) p_D^m(t-2, z, w) \big(1 \wedge \delta_D(w)^{\alpha/2}\big)\, dz\, dw \\
&\quad \ge c_5 \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} e^{-t\lambda_1^{\alpha,m,D}}.
\end{aligned}$$

This establishes Theorem 1.1(ii). □

REMARK 4.7. (i) In this paper, we do not use the boundary Harnack inequality for $X^m$. The boundary decay rate is obtained by comparing the Green function of $X^m$ in balls and annulus with that of $X^0$ through drift transform (see Theorem 2.6).

(ii) Let $Y$ be a relativistic stable-like process on $\mathbb{R}^d$, as studied in [12]. If one can establish scale invariant boundary Harnack inequality for $Y$ in bounded $C^{1,1}$

open sets with explicit boundary decay rate $\delta_D(x)^{\alpha/2}$, then one can easily modify the approach of this paper to show that Theorem 1.1 holds for $Y$ with $\phi(m^{1/\alpha}|x-y|)$ and $\phi(m^{1/\alpha}|x-y|/16)$ being replaced by $\phi(c_1|x-y|)$ and $\phi(c_2|x-y|)$, respectively, for some positive constant $c_1$ and $c_2$.

(iii) By integrating (1.2) with respect to $y$, we see that for each fixed $M, T>0$,

$$\mathbb{P}_x(t<\tau_D^m)\asymp 1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\qquad\text{for } m\in(0,M] \text{ and } (t,x)\in(0,T]\times D.$$

Hence both (1.2) and (1.3) can be restated as, for each fixed $T>0$ and every $(t,x,y)\in(0,T]\times D\times D$,

$$\begin{aligned}&\frac{1}{C_1}\mathbb{P}_x(t<\tau_D^m)\mathbb{P}_y(t<\tau_D^m)p^m(t,x,y)\\&\qquad\le p_D^m(t,x,y)\le C_1\mathbb{P}_x(t<\tau_D^m)\mathbb{P}_y(t<\tau_D^m)p^m(t,x/16,y/16).\end{aligned}$$

It is possible to establish the above estimates by adapting the approach in [6], using the scale invariant boundary Harnack inequality for $X^m$, uniform on $m\in(0,M]$, in arbitrary $\kappa$-fat open sets which is recently established in [13], Theorem 2.6. We omit the details here.

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