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SCALING FOR A ONE-DIMENSIONAL DIRECTED POLYMER WITH BOUNDARY CONDITIONS

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We study a $(1+1)$-dimensional directed polymer in a random environment on the integer lattice with log-gamma distributed weights. Among directed polymers, this model is special in the same way as the last-passage percolation model with exponential or geometric weights is special among growth models, namely, both permit explicit calculations. With appropriate boundary conditions, the polymer with log-gamma weights satisfies an analogue of Burke's theorem for queues. Building on this, we prove the conjectured values for the fluctuation exponents of the free energy and the polymer path, in the case where the boundary conditions are present and both endpoints of the polymer path are fixed. For the polymer without boundary conditions and with either fixed or free endpoint, we get the expected upper bounds on the exponents.

1. Introduction. The *directed polymer in a random environment* represents a polymer (a long chain of molecules) by a random walk path that interacts with a random environment. Let $x_{\bullet} = (x_k)_{k\geq 0}$ denote a nearest-neighbor path in $\mathbb{Z}^d$ started at the origin: $x_k \in \mathbb{Z}^d$, $x_0 = 0$, and $|x_k - x_{k-1}| = 1$. The environment $\omega = (\omega(s,u) : s \in \mathbb{N}, u \in \mathbb{Z}^d)$ puts a real-valued weight $\omega(s,u)$ at space–time point $(u,s) \in \mathbb{Z}^d \times \mathbb{N}$. For a path segment $x_{0,n} = (x_0, \dots, x_n)$, $H_n(x_{0,n})$ is the total weight collected by the walk up to time $n : H_n(x_{0,n}) = \sum_{s=1}^n \omega(s, x_s)$. The quenched polymer distribution on paths, in environment ω and at inverse temperature $\beta > 0$, is the probability measure defined by

$$Q_n^{\omega}(dx_{\bullet}) = \frac{1}{Z_n^{\omega}} \exp\{\beta H_n(x_{0,n})\} \tag{1.1}$$

with normalization factor (partition function) $Z_n^{\omega} = \sum_{x_{0,n}} e^{\beta H_n(x_{0,n})}$. The environment ω is taken as random with probability distribution $\mathbb{P}$, typically such that the weights $\{\omega(s,u)\}$ are i.i.d. random variables.

At $\beta = 0$, the model is standard simple random walk. The general objective is to understand how the model behaves as $\beta > 0$ and the dimension d varies.

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A key question is whether the diffusive behavior of the walk is affected. "Diffusive behavior" refers to the fluctuation behavior of standard random walk, characterized by $n^{-1}E(x_n^2) \to c$ and convergence of diffusively rescaled walks $n^{-1/2}x_{\lfloor nt \rfloor}$ to Brownian motion.

The directed polymer model was introduced in the statistical physics literature by Huse and Henley in 1985 [17]. The first rigorous mathematical work was by Imbrie and Spencer [18] in 1988. They proved with an elaborate expansion that in dimensions $d \geq 3$ and with small enough β, the walk is diffusive in the sense that, for a.e. environment ω, $n^{-1}E^{Q^\omega}(|x_n|^2) \to c$. Bolthausen [10] strengthened the result to a central limit theorem for the endpoint of the walk, still $d \geq 3$, small β and for a.e. ω, through the observation that $W_n = Z_n/\mathbb{E}(Z_n)$ is a martingale. Since then martingale techniques have been a standard fixture in much of the work on directed polymers.

The limit $W_\infty = \lim W_n$ is either almost surely 0 or almost surely > 0. The case $W_\infty = 0$ has been termed *strong disorder* and $W_\infty > 0$ *weak disorder*. There is a critical value β_c such that weak disorder holds for $\beta < \beta_c$ and strong for $\beta > \beta_c$. It is known that $\beta_c = 0$ for $d \in \{1, 2\}$ and $0 < \beta_c \leq \infty$ for $d \geq 3$. In $d \geq 3$ and weak disorder the walk converges to a Brownian motion, and the limiting diffusion matrix is the same as for standard random walk [15]. There is a further refinement of strong disorder into strong and very strong disorder. Sharp recent results appear in [23].

One way to phrase questions about the polymer model is to ask about two scaling exponents, ζ and χ, defined somewhat informally as follows:

$$\text{fluctuations of the path } x_{0,n} \text{ are of order } n^\zeta \tag{1.2}$$

and

$$\text{fluctuations of } \log Z_n \text{ are of order } n^\chi. \tag{1.3}$$

Let us restrict ourselves to the case $d = 1$ for the remainder of the paper. By the results mentioned above the model is in strong disorder for all $\beta > 0$. It is expected that the one-dimensional exponents are $\chi = 1/3$ and $\zeta = 2/3$ [22]. Precise values have not been obtained in the past, but during the last decade and a half nontrivial rigorous bounds have appeared in the literature for some models with Gaussian ingredients. For a Gaussian random walk in a Gaussian potential, Petermann [29] proved the lower bound $\zeta \geq 3/5$ and Mejane [26] provided the upper bound $\zeta \leq 3/4$. Petermann's proof was adapted to a certain continuous setting in [9]. For an undirected Brownian motion in a Poissonian potential, Wüthrich obtained $3/5 \leq \zeta \leq 3/4$ and $\chi \geq 1/8$ [34, 35]. For a directed Brownian motion in a Poissonian potential, Comets and Yoshida derived $\zeta \leq 3/4$ and $\chi \geq 1/8$ [14].

Piza [30] showed generally that the fluctuations of $\log Z_n$ diverge at least logarithmically, and bounds on exponents under curvature assumptions on the limiting free energy. Related results for first passage percolation appeared in [24, 27].

Exact exponents and even limit distributions have recently been derived for the so-called *continuum directed random polymer*. The partition function $\mathcal{Z}(t, x)$ is the solution of a stochastic heat equation $\mathcal{Z}_t = \frac{1}{2}\mathcal{Z}_{xx} - \mathcal{Z}\dot{W}$ where $\dot{W}$ is space–time white noise. In [7], the exact scaling exponent is determined for initial data $\mathcal{Z}(0, x) = e^{-B(x)}$ where B is a two-sided Brownian motion: $\mathrm{Var}(\log \mathcal{Z}(t, 0))$ is of order $t^{2/3}$. The result comes from corresponding bounds for the current of the weakly asymmetric simple exclusion process (WASEP). The techniques are related to the ones used in the present paper. The link from WASEP to $\log \mathcal{Z}$ that enables this transfer of estimates is originally due to [8]. Reference [2] obtains the probability distribution of $\log \mathcal{Z}$ for an initial delta function $\mathcal{Z}(0, x) = \delta_0(x)$ and proves a Tracy–Widom limit under the appropriate scaling. The WASEP connection is used again in [2], together with asymptotic analysis of a determinantal formula from [33]. There is no methodological overlap between [2] and the present paper.

Let us return to the $(1 + 1)$-dimensional lattice polymer. For the rest of the discussion we turn the picture 45 degrees clockwise so that the model lives in the nonnegative quadrant $\mathbb{Z}_+^2$ of the plane, instead of the space–time wedge $\{(u, s) \in \mathbb{Z} \times \mathbb{N} : |u| \le s\}$. The weights are i.i.d. variables $\{\omega(i, j) : i, j \ge 0\}$. The polymer $x_{\cdot}$ becomes a nearest-neighbor *up-right* path (see Figure 1). We also fix both endpoints of the path. So, given the endpoint (m, n), the partition function is

$$Z^{\omega}_{m,n} = \sum_{x_{0,m+n}} \exp\left\{\beta \sum_{k=1}^{m+n} \omega(x_k)\right\}, \tag{1.4}$$

where the sum is over paths $x_{0,m+n}$ that satisfy $x_0 = (0, 0)$, $x_{m+n} = (m, n)$ and $x_k - x_{k-1} = (1, 0)$ or $(0, 1)$. The polymer measure of such a path is

$$Q^{\omega}_{m,n}(x_{0,m+n}) = \frac{1}{Z^{\omega}_{m,n}} \exp\left\{\beta \sum_{k=1}^{m+n} \omega(x_k)\right\}. \tag{1.5}$$

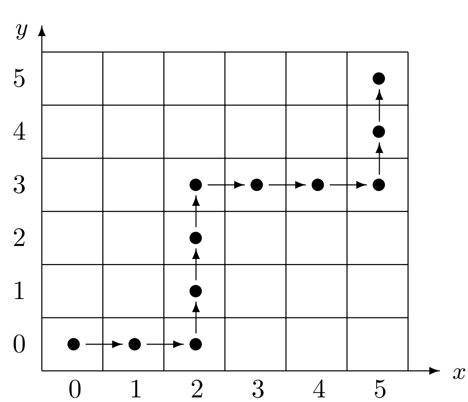

FIG. 1. *An up-right path from* $(0, 0)$ *to* $(5, 5)$ *in* $\mathbb{Z}_+^2$.

If we take the "zero temperature limit" $\beta \nearrow \infty$ in (1.5), then the measure $Q^{\omega}_{m,n}$ concentrates on the paths $x_{0,m+n}$ that maximize the sum $\sum_{k=1}^{m+n} \omega(x_k)$. Thus, the polymer model has become a *last-passage percolation model*, also called the *corner growth model*. The quantity that corresponds to $\log Z_{m,n}$ is the *passage time*

$$G_{m,n} = \max_{x_{0,m+n}} \sum_{k=1}^{m+n} \omega(x_k). \tag{1.6}$$

For certain last-passage growth models, notably for (1.6) with exponential or geometric weights $\omega(i, j)$, not only have the predicted exponents been confirmed but also limiting Tracy–Widom fluctuations for $G_{m,n}$ have been proved [5, 6, 13, 16, 19, 20]. The recent article [3] verifies a complete picture proposed in [31] that characterizes the scaling limits of $G_{m,n}$ with exponential weights as a function of the parameters of the boundary weights and the ratio m/n.

In the present paper, we study the polymer model (1.4) and (1.5) with fixed endpoints, with fixed $\beta = 1$, and for a particular choice of weight distribution. Namely, the weights $\{\omega(i, j)\}$ are independent random variables with log-gamma distributions. Precise definitions follow in the next section. This particular polymer model turns out to be amenable to explicit computation, similarly to the case of exponential or geometric weights among the corner growth models (1.6).

We introduce a polymer model with boundary conditions that possesses a two-dimensional stationarity property. By boundary conditions, we mean that the weights on the boundaries of $\mathbb{Z}_+^2$ are distributionally different from the weights in the interior, or bulk. For the model with boundary conditions, we prove that the fluctuation exponents take exactly their conjectured values $\chi = 1/3$ and $\zeta = 2/3$ when the endpoint (m, n) is taken to infinity along a characteristic direction. This characteristic direction is a function of the parameters of the weight distributions. In other directions, $\log Z_{m,n}$ satisfies a central limit theorem in the model with boundary conditions. As a corollary, we get the correct upper bounds for the exponents in the model without boundary and with either fixed or free endpoint, but still with i.i.d. log-gamma weights $\{\omega(i, j)\}$.

In addition to the $\beta \nearrow \infty$ limit, there is another formal connection between the polymer model and the corner growth model. Namely, the definitions of $Z_{m,n}$ and $G_{m,n}$ imply the equations

$$Z_{m,n} = e^{\beta\omega(m,n)}(Z_{m-1,n} + Z_{m,n-1}) \tag{1.7}$$

and

$$G_{m,n} = \omega(m, n) + \max(G_{m-1,n}, G_{m,n-1}). \tag{1.8}$$

These equations can be paraphrased by saying that $G_{m,n}$ obeys max-plus algebra, while $Z_{m,n}$ obeys the familiar algebra of addition and multiplication.

This observation informs the approach of the paper. It is not that we can convert results for G into results for Z. Rather, after the proofs have been found, one can detect a kinship with the arguments of [6], but transformed from (max, +) to $(+, \cdot)$. The ideas in [6] were originally adapted from the seminal paper [13]. The purpose was to give an alternative proof of the scaling exponents of the corner growth model, without the asymptotic analysis of Fredholm determinants utilized in [19].

Frequently used notation. $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Rectangles on the planar integer lattice are denoted by $\Lambda_{m,n} = \{0, \ldots, m\} \times \{0, \ldots, n\}$ and more generally $\Lambda_{(k,\ell),(m,n)} = \{k, \ldots, m\} \times \{\ell, \ldots, n\}$. $\mathbb{P}$ is the probability distribution on the random environments or weights ω, and under $\mathbb{P}$ the expectation of a random variable X is $\mathbb{E}(X)$ and variance $\mathbb{V}\mathrm{ar}(X)$. Overline means centering: $\overline{X} = X - \mathbb{E}X$. Q^ω is the quenched polymer measure in a rectangle. The annealed measure is $P(\cdot) = \mathbb{E}Q^\omega(\cdot)$ with expectation $E(\cdot)$. $\mathbf{P}$ is used for a generic probability measure that is not part of the polymer model. Paths can be written $x_{k,\ell} = (x_k, x_{k+1}, \ldots, x_\ell)$ but also $x_{\centerdot}$ when k, ℓ are understood. Occasionally A and B denote gamma-distributed random variables. The more usual random variable symbols X, Y, Z and W have specific meanings in the polymer model.

2. The model and results. We begin with the definition of the polymer model with boundaries and then state the results. As stated in the Introduction, relative to the standard description of the polymer model, we turn the picture 45 degrees clockwise so that the polymer lives in the nonnegative quadrant $\mathbb{Z}_+^2$ of the planar lattice. The inverse temperature parameter $\beta = 1$ throughout. We replace the exponentiated weights with multiplicative weights $Y_{i,j} = e^{\omega(i,j)}$, $(i, j) \in \mathbb{Z}_+^2$. Then the partition function for paths whose endpoint is constrained to lie at (m, n) is given by

$$Z_{m,n} = \sum_{x_{\centerdot} \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}, \tag{2.1}$$

where $\Pi_{m,n}$ denotes the collection of up-right paths $x_{\centerdot} = (x_k)_{0 \le k \le m+n}$ inside the rectangle $\Lambda_{m,n} = \{0, \ldots, m\} \times \{0, \ldots, n\}$ that go from $(0, 0)$ to $(m, n) : x_0 = (0, 0)$, $x_{m+n} = (m, n)$ and $x_k - x_{k-1} = (1, 0)$ or $(0, 1)$. We adopt the convention that $Z_{m,n}$ does not include the weight at the origin, and if a value is needed then set $Z_{0,0} = Y_{0,0} = 1$. The symbol ω will denote the entire random environment: $\omega = (Y_{i,j} : (i, j) \in \mathbb{Z}_+^2)$. When necessary the dependence of $Z_{m,n}$ on ω will be expressed by $Z^\omega_{m,n}$, with a similar convention for other ω-dependent quantities.

We assign distinct weight distributions on the boundaries $(\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})$ and in the bulk $\mathbb{N}^2$. To highlight this, the symbols U and V will denote weights on the horizontal and vertical boundaries:

$$U_{i,0} = Y_{i,0} \quad \text{and} \quad V_{0,j} = Y_{0,j} \qquad \text{for } i, j \in \mathbb{N}. \tag{2.2}$$

However, in formulas such as (2.1) it is obviously convenient to use a single symbol $Y_{i,j}$ for all the weights.

Our results rest on the assumption that the weights are reciprocals of gamma variables. Let us recall some basics. The gamma function is $\Gamma(s) = \int_0^\infty x^{s-1} \times e^{-x}\,dx$. We shall need it only for positive real s. The Gamma(θ, r) distribution has density $\Gamma(\theta)^{-1} r^\theta x^{\theta-1} e^{-rx}$ on $\mathbb{R}_+$, mean θ/r and variance θ/r^2.

The logarithm $\log\Gamma(s)$ is convex and infinitely differentiable on $(0,\infty)$. The derivatives are the polygamma functions $\Psi_n(s) = (d^{n+1}/ds^{n+1})\log\Gamma(s)$, $n \in \mathbb{Z}_+$, ([1], Section 6.4). For $n \ge 1$, Ψ_n is nonzero and has sign $(-1)^{n-1}$ throughout $(0,\infty)$ ([32], Theorem 7.71). Throughout the paper, we make use of the digamma and trigamma functions Ψ_0 and Ψ_1, on account of the connections

$$\Psi_0(\theta) = \mathbb{E}(\log A) \quad \text{and} \quad \Psi_1(\theta) = \mathbb{V}\mathrm{ar}(\log A) \tag{2.3}$$

for $A \sim \text{Gamma}(\theta, 1)$.

Here is the assumption on the distributions. Let $0 < \theta < \mu < \infty$.

(2.4) Weights $\{U_{i,0}, V_{0,j}, Y_{i,j} : i, j \in \mathbb{N}\}$ are independent with distributions $U_{i,0}^{-1} \sim \text{Gamma}(\theta, 1)$, $V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta, 1)$, and $Y_{i,j}^{-1} \sim \text{Gamma}(\mu, 1)$.

We fixed the scale parameter $r = 1$ in the gamma distributions above for the sake of convenience. We could equally well fix it to any value and our results would not change, as long as all three gamma distributions above have the same scale parameter.

A key property is that under (2.4) each ratio $U_{m,n} = Z_{m,n}/Z_{m-1,n}$ and $V_{m,n} = Z_{m,n}/Z_{m,n-1}$ has the same marginal distribution as U and V in (2.4). This is a Burke's theorem of sorts, and appears as Theorem 3.3 below. From this we can compute the mean exactly: for $m, n \ge 0$,

$$\mathbb{E}[\log Z_{m,n}] = m\mathbb{E}(\log U) + n\mathbb{E}(\log V) = -m\Psi_0(\theta) - n\Psi_0(\mu - \theta). \tag{2.5}$$

Together with the choice of the parameters θ, μ goes a choice of "characteristic direction" $(\Psi_1(\mu - \theta), \Psi_1(\theta))$ for the polymer. Let N denote the scaling parameter we take to ∞. We assume that the coordinates (m, n) of the endpoint of the polymer satisfy

$$|m - N\Psi_1(\mu - \theta)| \le \gamma N^{2/3} \quad \text{and} \quad |n - N\Psi_1(\theta)| \le \gamma N^{2/3} \tag{2.6}$$

for some fixed constant γ. Now we can state the variance bounds for the free energy.

THEOREM 2.1. *Assume* (2.4) *and let* (m, n) *be as in* (2.6). *Then there exist constants* $0 < C_1, C_2 < \infty$ *such that, for* $N \ge 1$,

$$C_1 N^{2/3} \le \mathbb{V}\mathrm{ar}(\log Z_{m,n}) \le C_2 N^{2/3}.$$

The constants C_1, C_2 in the theorem depend on $0 < \theta < \mu$ and on γ of (2.6), and they can be taken the same for (θ, μ, γ) that vary in a compact set. This holds for all the constants in the theorems of this section: they depend on the parameters of the assumptions, but for parameter values in a compact set the constants themselves can be fixed.

The upper bound on the variance is good enough for Borel–Cantelli to give the strong law of large numbers: with (m, n) as in (2.6),

$$\lim_{N\to\infty} N^{-1} \log Z_{m,n} = -\Psi_0(\theta)\Psi_1(\mu-\theta) - \Psi_0(\mu-\theta)\Psi_1(\theta) \qquad \mathbb{P}\text{-a.s.} \tag{2.7}$$

As a further corollary, we deduce that if the direction of the polymer deviates from the characteristic one by a larger power of N than allowed by (2.6), then $\log Z$ satisfies a central limit theorem. For the sake of concreteness, we treat the case where the horizontal direction is too large.

COROLLARY 2.2. *Assume* (2.4). *Suppose $m, n \to \infty$. Define parameter N by $n = \Psi_1(\theta)N$, and assume that*

$$N^{-\alpha}\big(m - \Psi_1(\mu-\theta)N\big) \to c_1 > 0 \qquad \textit{as } N \to \infty$$

for some $\alpha > 2/3$. Then as $N \to \infty$,

$$N^{-\alpha/2}\{\log Z_{m,n} - \mathbb{E}(\log Z_{m,n})\}$$

converges in distribution to a centered normal distribution with variance $c_1\Psi_1(\theta)$.

The quenched polymer measure $Q^{\omega}_{m,n}$ is defined on paths $x_\centerdot \in \Pi_{m,n}$ by

$$Q^{\omega}_{m,n}(x_\centerdot) = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} \tag{2.8}$$

remembering convention (2.2). Integrating out the random environment ω gives the annealed measure

$$P_{m,n}(x_\centerdot) = \int Q^{\omega}_{m,n}(x_\centerdot)\mathbb{P}(d\omega).$$

When the rectangle $\Lambda_{m,n}$ is understood, we drop the subscripts and write $P = \mathbb{E}Q^{\omega}$. Notation will be further simplified by writing Q for Q^{ω}.

We describe the fluctuations of the path $x_\centerdot$ under P. The next result shows that $N^{2/3}$ is the correct order of magnitude of the fluctuations of the path. Let $v_0(j)$ and $v_1(j)$ denote the left- and rightmost points of the path on the horizontal line with ordinate j:

$$v_0(j) = \min\big\{i \in \{0, \dots, m\} : \exists k \text{ such that } x_k = (i, j)\big\} \tag{2.9}$$

and

$$v_1(j) = \max\big\{i \in \{0, \dots, m\} : \exists k \text{ such that } x_k = (i, j)\big\}. \tag{2.10}$$

THEOREM 2.3. *Assume* (2.4) *and let* (m,n) *be as in* (2.6). *Let* $0 \le \tau < 1$. *Then there exist constants* $C_1, C_2 < \infty$ *such that for* $N \ge 1$ *and* $b \ge C_1$,

$$P\{v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \text{ or } v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3}\} \le C_2 b^{-3}. \tag{2.11}$$

The same bound holds for the vertical counterparts of v_0 *and* v_1.

Let $0 < \tau < 1$. *Then given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\varlimsup_{N\to\infty} P\{\exists k \text{ such that } |x_k - (\tau m, \tau n)| \le \delta N^{2/3}\} \le \varepsilon. \tag{2.12}$$

Presently we do not have sharp quenched results. From Lemma 4.3 and the proof of Theorem 2.3 in Section 6, one can extract estimates on the $\mathbb{P}$-tails of the quenched probabilities of the events in (2.11) and (2.12).

We turn to results for the model without boundaries, by restricting ourselves to the positive quadrant $\mathbb{N}^2$. Define the partition function of a general rectangle $\{k,\dots,m\} \times \{\ell,\dots,n\}$ by

$$Z_{(k,\ell),(m,n)} = \sum_{x_\centerdot \in \Pi_{(k,\ell),(m,n)}} \prod_{i=1}^{m-k+n-\ell} Y_{x_i}, \tag{2.13}$$

where $\Pi_{(k,\ell),(m,n)}$ is the collection of up-right paths $x_\centerdot = (x_i)_{i=0}^{m-k+n-\ell}$ from $x_0 = (k,\ell)$ to $x_{m-k+n-\ell} = (m,n)$. Admissible steps are always $x_{i+1} - x_i = e_1 = (1,0)$ or $x_{i+1} - x_i = e_2 = (0,1)$. We have chosen not to include the weight of the southwest corner (k,ℓ). The earlier definition (2.1) is the special case $Z_{m,n} = Z_{(0,0),(m,n)}$. Also we stipulate that when the rectangle reduces to a point, $Z_{(k,\ell),(k,\ell)} = 1$.

In particular, $Z_{(1,1),(m,n)}$ gives us partition functions that only involve the bulk weights $\{Y_{i,j} : i,j \in \mathbb{N}\}$. The assumption on their distribution is as before, with a fixed parameter $0 < \mu < \infty$:

$$\{Y_{i,j} : i,j \in \mathbb{N}\} \text{ are i.i.d. with common distribution } Y_{i,j}^{-1} \sim \text{Gamma}(\mu, 1). \tag{2.14}$$

We define the limiting free energy. The identity (see, e.g., (2.11) in [4] or Section 6.4 in [1])

$$\Psi_1(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}$$

shows that $\Psi_1(0+) = \infty$. Thus, given $0 < s,t < \infty$, there is a unique $\theta = \theta_{s,t} \in (0,\mu)$ such that

$$\frac{\Psi_1(\mu - \theta)}{\Psi_1(\theta)} = \frac{s}{t}. \tag{2.15}$$

Define

$$f_{s,t}(\mu) = -\big(s\Psi_0(\theta_{s,t}) + t\Psi_0(\mu - \theta_{s,t})\big). \tag{2.16}$$

It can be verified that for a fixed $0 < \mu < \infty$, $f_{s,t}(\mu)$ is a continuous function of $(s,t) \in \mathbb{R}_+^2$ with boundary values

$$f_{0,t}(\mu) = f_{t,0}(\mu) = -t\Psi_0(\mu).$$

Here is the result for the free energy of the polymer without boundary but still with fixed endpoint. The constants in this theorem depend on (s, t, μ).

THEOREM 2.4. *Assume* (2.14) *and let* $0 < s, t < \infty$. *We have the law of large numbers*

$$\lim_{N\to\infty} N^{-1} \log Z_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} = f_{s,t}(\mu) \qquad \mathbb{P}\text{-}a.s. \tag{2.17}$$

There exist finite constants N_0 *and* C_0 *such that, for* $b \geq 1$ *and* $N \geq N_0$,

$$\mathbb{P}\big[\big|\log Z_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} - Nf_{s,t}(\mu)\big| \geq bN^{1/3}\big] \leq C_0 b^{-3/2}. \tag{2.18}$$

In particular, we get the moment bound

$$\mathbb{E}\left\{\left|\frac{\log Z_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} - Nf_{s,t}(\mu)}{N^{1/3}}\right|^p\right\} \leq C(s,t,\mu,p) < \infty \tag{2.19}$$

for $N \geq N_0(s,t,\mu)$ and $1 \leq p < 3/2$. The theorem is proved by relating $Z_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}$ to a polymer with a boundary. Equation (2.15) picks the correct boundary parameter θ. Presently we do not have a matching lower bound for (2.18).

In a general rectangle the quenched polymer distribution of a path $x_{\cdot} \in \Pi_{(k,\ell),(m,n)}$ is

$$Q_{(k,\ell),(m,n)}(x_{\cdot}) = \frac{1}{Z_{(k,\ell),(m,n)}} \prod_{i=1}^{m-k+n-\ell} Y_{x_i}. \tag{2.20}$$

As before, the annealed distribution is $P_{(k,\ell),(m,n)}(\cdot) = \mathbb{E}Q_{(k,\ell),(m,n)}(\cdot)$. The upper fluctuation bounds for the path in the model with boundaries can be extended to the model without boundaries. Here we can again allow the endpoint (m, n) to deviate from the characteristic direction:

$$|m - Ns| \leq \gamma N^{2/3} \quad \text{and} \quad |n - Nt| \leq \gamma N^{2/3} \tag{2.21}$$

for a constant γ. The constants in this theorem depend on (s, t, μ, γ).

THEOREM 2.5. *Assume* (2.14), *fix* $0 < s, t < \infty$, *and assume* (2.21). *Let* $0 \leq \tau < 1$. *Then there exist finite constants* C, C_0 *and* N_0 *such that for* $N \geq N_0$ *and* $b \geq C_0$,

$$P_{(1,1),(m,n)}\{v_0(\lfloor\tau n\rfloor) < \tau m - bN^{2/3} \text{ or } v_1(\lfloor\tau n\rfloor) > \tau m + bN^{2/3}\} \leq Cb^{-3}. \tag{2.22}$$

The same bound holds for the vertical counterparts of v_0 *and* v_1.

Next we drop the restriction on the endpoint, and extend the upper bounds to the polymer with unrestricted endpoint and no boundaries. Given the value of the parameter $N \in \mathbb{N}$, the set of admissible paths is $\bigcup_{1\le k\le N-1} \Pi_{(1,1),(k,N-k)}$, namely the set of all up-right paths $x_{\centerdot} = (x_k)_{0\le k\le N-2}$ that start at $x_0 = (1,1)$ and whose endpoint x_{N-2} lies on the line $x + y = N$. The quenched polymer probability of such a path is

$$Q_N^{\text{tot}}(x_{\centerdot}) = \frac{1}{Z_N^{\text{tot}}} \prod_{k=1}^{N-2} Y_{x_k}$$

with the "total" partition function

$$Z_N^{\text{tot}} = \sum_{k=1}^{N-1} Z_{(1,1),(k,N-k)}.$$

The annealed measure is $P_N^{\text{tot}}(\cdot) = \mathbb{E}Q_N^{\text{tot}}(\cdot)$. We collect all the results in one theorem, proved in Section 8. In particular, (2.25) below shows that the fluctuations of the endpoint of the path are of order at most $N^{2/3}$. Statement (8.20) in the proof gives bounds on the quenched probability of a deviation.

THEOREM 2.6. *Fix $0 < \mu < \infty$ and assume weight distribution* (2.14). *We have the law of large numbers*

$$\lim_{N\to\infty} N^{-1} \log Z_N^{\text{tot}} = f_{1/2,1/2}(\mu) = -\Psi_0(\mu/2) \qquad \mathbb{P}\text{-a.s.} \tag{2.23}$$

Let $C(\mu)$ be a constant that depends on μ alone. For $b \ge 1$, there exists $N_0(\mu, b) < \infty$ such that

$$\sup_{N\ge N_0(\mu,b)} \mathbb{P}[|\log Z_N^{\text{tot}} - N f_{1/2,1/2}(\mu)| \ge bN^{1/3}] \le C(\mu) b^{-3/2} \tag{2.24}$$

and

$$\sup_{N\ge N_0(\mu,b)} P_N^{\text{tot}}\left\{\left| x_{N-2} - \left(\frac{N}{2}, \frac{N}{2}\right)\right| \ge bN^{2/3}\right\} \le C(\mu) b^{-3}. \tag{2.25}$$

The last case to address is the polymer with boundaries but free endpoint. This case is perhaps of less interest than the others for the free energy scales diffusively, but we record it for the sake of completeness. Fix $0 < \theta < \mu$ and let assumption (2.4) on the weight distributions be in force. The fixed-endpoint partition function $Z_{m,n} = Z_{(0,0),(m,n)}$ is the one defined in (2.1). Define the partition function of all paths from $(0,0)$ to the line $x + y = N$ by

$$Z_N^{\text{tot}}(\theta, \mu) = \sum_{\ell=0}^{N} Z_{\ell,N-\ell}.$$

Define a limiting free energy

$$g(\theta,\mu)=\max_{0\le s\le 1}\bigl(-s\Psi_0(\theta)-(1-s)\Psi_0(\mu-\theta)\bigr)=\begin{cases}-\Psi_0(\theta), & \theta\le\mu/2,\\ -\Psi_0(\mu-\theta), & \theta\ge\mu/2.\end{cases}$$

Set also

$$\sigma^2(\theta,\mu)=\begin{cases}\Psi_1(\theta), & \theta\le\mu/2,\\ \Psi_1(\mu-\theta), & \theta\ge\mu/2,\end{cases}$$

and define random variables $\zeta(\theta,\mu)$ as follows: for $\theta\ne\mu/2$, $\zeta(\theta,\mu)$ has centered normal distribution with variance $\sigma^2(\theta,\mu)$, while

$$\zeta(\mu/2,\mu)=\sqrt{2\Psi_1(\mu/2)}\,(M_{1/2}\vee M'_{1/2}), \tag{2.26}$$

where $M_t=\sup_{0\le s\le t}B(s)$ is the running maximum of a standard Brownian motion and M'_t is an independent copy of it.

THEOREM 2.7. *Let* $0<\theta<\mu$ *and assume* (2.4). *We have the law of large numbers*

$$\lim_{N\to\infty}N^{-1}\log Z_N^{\mathrm{tot}}(\theta,\mu)=g(\theta,\mu)\qquad \mathbb{P}\text{-}a.s. \tag{2.27}$$

and the distributional limit

$$N^{-1/2}\bigl(\log Z_N^{\mathrm{tot}}(\theta,\mu)-Ng(\mu/2,\mu)\bigr)\xrightarrow{d}\zeta(\theta,\mu). \tag{2.28}$$

When $\theta\ne\mu/2$, the axis with the larger $-\Psi_0$ value completely dominates, while if $\theta=\mu/2$ all directions have the same limiting free energy. This accounts for the results in the theorem.

Organization of the paper. Before we begin the proofs of the main theorems, Section 3 collects basic properties of the model, including the Burke-type property. The upper and lower bounds of Theorem 2.1 are proved in Sections 4 and 5. Corollary 2.2 is proved at the end of Section 4. The bounds for the fixed-endpoint path with boundaries are proved in Section 6, and the results for the fixed-endpoint polymer model without boundaries in Section 7. The results for the polymer with free endpoint are proved in Section 8.

3. Basic properties of the polymer model with boundaries. This section sets the stage for the proofs with some preliminaries. The main results of this section are the Burke property in Theorem 3.3 and identities that tie together the variance of the free energy and the exit points from the axes in Theorem 3.7.

Occasionally we will use notation for the partition function that includes the weight at the starting point, which we write as

$$Z^{\square}_{(i,j),(k,\ell)}=\sum_{x_\centerdot\in\Pi_{(i,j),(k,\ell)}}\prod_{r=0}^{k-i+\ell-j}Y_{x_r}=Y_{i,j}Z_{(i,j),(k,\ell)}. \tag{3.1}$$

Let the initial weights $\{U_{i,0}, V_{0,j}, Y_{i,j} : i, j \in \mathbb{N}\}$ be given. Starting from the lower left corner of $\mathbb{N}^2$, define inductively for $(i, j) \in \mathbb{N}^2$

$$U_{i,j} = Y_{i,j}\biggl(1 + \frac{U_{i,j-1}}{V_{i-1,j}}\biggr), \qquad V_{i,j} = Y_{i,j}\biggl(1 + \frac{V_{i-1,j}}{U_{i,j-1}}\biggr) \quad \text{and}$$
$$X_{i-1,j-1} = \biggl(\frac{1}{U_{i,j-1}} + \frac{1}{V_{i-1,j}}\biggr)^{-1}. \tag{3.2}$$

The partition function satisfies

$$Z_{m,n} = Y_{m,n}(Z_{m-1,n} + Z_{m,n-1}) \qquad \text{for } (m,n) \in \mathbb{N}^2 \tag{3.3}$$

and one checks inductively that

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \quad \text{and} \quad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \tag{3.4}$$

for $(m,n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$. Equations (3.3) and (3.4) are also valid for $Z^{\square}_{m,n}$ because the weight at the origin cancels from the equations.

It is also natural to associate the U- and V-variables to undirected edges of the lattice $\mathbb{Z}_+^2$. If $f = \{x - e_1, x\}$ is a horizontal edge, then $T_f = U_x$, while if $f = \{x - e_2, x\}$ then $T_f = V_x$.

The following monotonicity property can be proved inductively.

LEMMA 3.1. *Consider two sets of positive initial values* $\{U_{i,0}, V_{0,j}, Y_{i,j} : i, j \in \mathbb{N}\}$ *and* $\{\widetilde{U}_{i,0}, \widetilde{V}_{0,j}, \widetilde{Y}_{i,j} : i, j \in \mathbb{N}\}$ *that satisfy* $U_{i,0} \ge \widetilde{U}_{i,0}$, $V_{0,j} \le \widetilde{V}_{0,j}$, *and* $Y_{i,j} = \widetilde{Y}_{i,j}$. *From these define inductively the values* $\{U_{i,j}, V_{i,j} : (i,j) \in \mathbb{N}^2\}$ *and* $\{\widetilde{U}_{i,j}, \widetilde{V}_{i,j} : (i,j) \in \mathbb{N}^2\}$ *by equation* (3.2). *Then* $U_{i,j} \ge \widetilde{U}_{i,j}$ *and* $V_{i,j} \le \widetilde{V}_{i,j}$ *for all* $(i,j) \in \mathbb{N}^2$.

3.1. *Propagation of boundary conditions*. The next lemma gives a reversibility property that we can regard as an analogue of reversibility properties of M/M/1 queues and their last-passage versions. (A basic reference for queues is [21]. Related work appears in [6, 12, 13, 28].)

LEMMA 3.2. *Let* U, V *and* Y *be independent positive random variables. Define*

$$U' = Y(1 + UV^{-1}), \qquad V' = Y(1 + VU^{-1}) \quad \textit{and}$$
$$Y' = (U^{-1} + V^{-1})^{-1}. \tag{3.5}$$

Then the triple (U', V', Y') *has the same distribution as* (U, V, Y) *iff there exist positive parameters* $0 < \theta < \mu$ *and* r *such that*

$$U^{-1} \sim \text{Gamma}(\theta, r), \qquad V^{-1} \sim \text{Gamma}(\mu - \theta, r) \quad \textit{and}$$
$$Y^{-1} \sim \text{Gamma}(\mu, r). \tag{3.6}$$

PROOF. Assuming (3.6), define independent gamma variables $A = U^{-1}$, $B = V^{-1}$ and $Z = Y^{-1}$. Then set

$$A' = \frac{ZA}{A+B}, \qquad B' = \frac{ZB}{A+B} \quad \text{and} \quad Z' = A + B.$$

We need to show that $(A', B', Z') \overset{d}{=} (A, B, Z)$. Direct calculation with Laplace transforms is convenient. Alternatively, one can reason with basic properties of gamma distributions as follows. The pair $(A/(A+B), B/(A+B))$ has distributions $\text{Beta}(\theta, \mu-\theta)$ and $\text{Beta}(\mu-\theta, \theta)$, and is independent of the $\text{Gamma}(\mu, r)$-distributed sum $A + B = Z'$. Hence, A' and B' are a pair of independent variables with distributions $\text{Gamma}(\theta, r)$ and $\text{Gamma}(\mu-\theta, r)$, and by construction also independent of Z'.

Assuming $(A', B', Z') \overset{d}{=} (A, B, Z)$, $A'/B' = A/B$ is independent of $Z' = A + B$. By Theorem 1 of [25] A and B are independent gamma variables with the same scale parameter r. Then $Z \overset{d}{=} Z' = A + B$ determines the distribution of Z. □

From this lemma, we get a Burke-type theorem. Let $z_{\bullet} = (z_k)_{k\in\mathbb{Z}}$ be a nearest-neighbor down-right path in $\mathbb{Z}_+^2$, that is, $z_k \in \mathbb{Z}_+^2$ and $z_k - z_{k-1} = e_1$ or $-e_2$. Denote the undirected edges of the path by $f_k = \{z_{k-1}, z_k\}$, and let

$$T_{f_k} = \begin{cases} U_{z_k}, & \text{if } f_k \text{ is a horizontal edge,} \\ V_{z_{k-1}}, & \text{if } f_k \text{ is a vertical edge.} \end{cases}$$

Let the (lower left) *interior* of the path be the vertex set $\mathcal{I} = \{(i, j) \in \mathbb{Z}_+^2 : \exists m \in \mathbb{N} : (i+m, j+m) \in \{z_k\}\}$ (see Figure 2). $\mathcal{I}$ is finite if the path $z_{\bullet}$ coincides with the axes for all but finitely many edges. Recall the definition of $X_{i,j}$ from (3.2).

THEOREM 3.3. *Assume* (2.4). *For any down-right path* $(z_k)_{k\in\mathbb{Z}}$ *in* $\mathbb{Z}_+^2$, *the variables* $\{T_{f_k}, X_z : k \in \mathbb{Z}, z \in \mathcal{I}\}$ *are mutually independent with marginal distributions*

$$U^{-1} \sim \text{Gamma}(\theta, 1), \qquad V^{-1} \sim \text{Gamma}(\mu-\theta, 1) \quad \textit{and} \qquad (3.7)$$
$$X^{-1} \sim \text{Gamma}(\mu, 1).$$

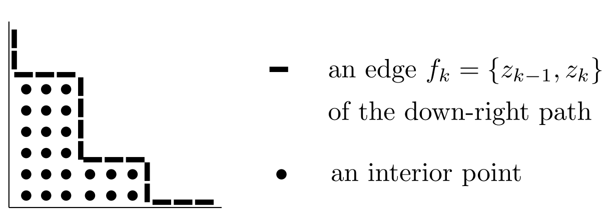

FIG. 2. *Illustration of a down-right path* (z_k) *and its set* $\mathcal{I}$ *of interior points. Interior point* (i, j) *is represented by a dot centered at* $(i + 1/2, j + 1/2)$.

PROOF. This is proved first by induction for down-right paths with finite interior $\mathcal{I}$. If $z_{\bullet}$ coincides with the x- and y-axes then $\mathcal{I}$ is empty, and the statement follows from assumption (2.4). The inductive step consists of adding a "growth corner" to $\mathcal{I}$ and an application of Lemma 3.2, namely, suppose $z_{\bullet}$ goes through the three points $(i-1, j)$, $(i-1, j-1)$ and $(i, j-1)$. Flip the corner over to create a new path $z'_{\bullet}$ that goes through $(i-1, j)$, (i, j) and $(i, j-1)$. The new interior is $\mathcal{I}' = \mathcal{I} \cup \{(i-1, j-1)\}$. Apply Lemma 3.2 with

$$U = U_{i,j-1}, \qquad V = V_{i-1,j}, \qquad Y = Y_{i,j}, \qquad U' = U_{i,j}, \qquad V' = V_{i,j}$$

and

$$Y' = X_{i-1,j-1}$$

to see that the conclusion continues to hold for $z'_{\bullet}$ and $\mathcal{I}'$.

To prove the theorem for an arbitrary down-right path it suffices to consider a finite portion of $z_{\bullet}$ and $\mathcal{I}$ inside some large square $B = \{0, \dots, M\}^2$. Apply the first part of the proof to the modified path that coincides with $z_{\bullet}$ inside B but otherwise follows the coordinate axes and connects up with $z_{\bullet}$ on the north and east boundaries of B. □

To understand the sense in which Theorem 3.3 is a "Burke property," note its similarity with Lemma 4.2 in [6] whose connection with M/M/1 queues in series is immediate through the last-passage representation.

3.2. *Reversal.* In a fixed rectangle $\Lambda = \{0, \dots, m\} \times \{0, \dots, n\}$, define the reversed partition function

$$Z^*_{i,j} = \frac{Z_{m,n}}{Z_{m-i,n-j}} \qquad \text{for } (i, j) \in \Lambda. \tag{3.8}$$

Note that for the partition function of the entire rectangle,

$$Z^*_{m,n} = Z_{m,n}.$$

Recalling (3.2) make these further definitions:

$$\begin{aligned} U^*_{i,j} &= U_{m-i+1,n-j} && \text{for } (i, j) \in \{1, \dots, m\} \times \{0, \dots, n\}, \\ V^*_{i,j} &= V_{m-i,n-j+1} && \text{for } (i, j) \in \{0, \dots, m\} \times \{1, \dots, n\}, \\ Y^*_{i,j} &= X_{m-i,n-j} && \text{for } (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}. \end{aligned} \tag{3.9}$$

The mapping $*$ is an involution, that is, inside the rectangle Λ, $Z^{**}_{i,j} = Z_{i,j}$ and similarly for U, V and Y.

PROPOSITION 3.4. *Assume distributions* (2.4). *Then inside the rectangle* Λ *the system* $\{Z^*_{i,j}, U^*_{i,j}, V^*_{i,j}, Y^*_{i,j}\}$ *replicates the properties of the original system* $\{Z_{i,j}, U_{i,j}, V_{i,j}, Y_{i,j}\}$. *Namely, we have these facts*:

(a) $\{U^*_{i,0}, V^*_{0,j}, Y^*_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ *are independent with distributions*

$$(U^*_{i,0})^{-1} \sim \text{Gamma}(\theta, 1), \qquad (V^*_{0,j})^{-1} \sim \text{Gamma}(\mu - \theta, 1) \quad \textit{and} \quad (Y^*_{i,j})^{-1} \sim \text{Gamma}(\mu, 1). \tag{3.10}$$

(b) *These identities hold*: $Z^*_{0,0} = 1$, $Z^*_{i,j} = Y^*_{i,j}(Z^*_{i-1,j} + Z^*_{i,j-1})$,

$$U^*_{i,j} = \frac{Z^*_{i,j}}{Z^*_{i-1,j}}, \qquad V^*_{i,j} = \frac{Z^*_{i,j}}{Z^*_{i,j-1}},$$

$$U^*_{i,j} = Y^*_{i,j}\left(1 + \frac{U^*_{i,j-1}}{V^*_{i-1,j}}\right) \quad \textit{and} \quad V^*_{i,j} = Y^*_{i,j}\left(1 + \frac{V^*_{i-1,j}}{U^*_{i,j-1}}\right).$$

PROOF. Part (a) is a consequence of Theorem 3.3. Part (b) follows from definitions (3.8) and (3.9) of the reverse variables and properties (3.2), (3.3) and (3.4) of the original system. □

Define a dual measure on paths $x_{0,m+n} \in \Pi_{m,n}$ by

$$Q^{*,\omega}(x_{0,m+n}) = \frac{1}{Z_{m,n}} \prod_{k=0}^{m+n-1} X_{x_k} \tag{3.11}$$

with the conventions $X_{i,n} = U_{i+1,n}$ for $0 \le i < m$ and $X_{m,j} = V_{m,j+1}$ for $0 \le j < n$. This convention is needed because inside the fixed rectangle Λ, (3.2) defines the X-weights only away from the north and east boundaries. The boundary weights are of the U- and V-type.

Define a reversed environment ω^* as a function of ω in Λ by

$$\omega^* = (U^*_{i,0}, V^*_{0,j}, Y^*_{i,j} : (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}).$$

Part (a) of Proposition 3.4 says that $\omega^* \stackrel{d}{=} \omega$. As before, utilize also the definitions $Y^*_{i,0} = U^*_{i,0}$ and $Y^*_{0,j} = V^*_{0,j}$. Write

$$x^*_k = (m, n) - x_{m+n-k}$$

for the reversed path. For an event $A \subseteq \Pi_{m,n}$ on paths let $A^* = \{x_{0,m+n} : x^*_{0,m+n} \in A\}$.

LEMMA 3.5. *$Q^{*,\omega}(A)$ and $Q^{\omega}(A^*)$ have the same distribution under $\mathbb{P}$.*

PROOF. By the definitions,

$$\begin{aligned} Q^{*,\omega}(A) &= \frac{1}{Z_{m,n}} \sum_{x_{0,m+n} \in A} \prod_{k=0}^{m+n-1} X_{x_k} = \frac{1}{Z^*_{m,n}} \sum_{x_{0,m+n} \in A} \prod_{j=1}^{m+n} Y^*_{x^*_j} \\ &= Q^{\omega^*}(A^*). \end{aligned} \tag{3.12}$$

By Proposition 3.4, $Q^{\omega^*}(A^*) \stackrel{d}{=} Q^{\omega}(A^*)$. □

REMARK 3.6. $Q^{*,\omega}(A)$ and $Q^{\omega}(A)$ do not in general have the same distribution because their boundary weights are different.

Under the dual measure the path $x_{0,m+n}$ is a Markov chain. This can be seen by rewriting (3.11) as

$$Q^{*,\omega}(x_{0,m+n}) = \prod_{k=0}^{m+n-1} \frac{X_{x_k} Z_{x_k}}{Z_{x_{k+1}}} = \prod_{k=0}^{m+n-1} \pi^*_{x_k,x_{k+1}}, \tag{3.13}$$

where the last equality defines the Markov kernel $\pi^*_{x,y}$ on the state space Λ. At points x away from the north and east boundaries we can write the kernel as

$$\pi^*_{x,x+e} = \frac{X_x Z_x}{Z_{x+e}} = \frac{Z^{-1}_{x+e}}{Z^{-1}_{x+e_1} + Z^{-1}_{x+e_2}}, \qquad e \in \{e_1, e_2\}. \tag{3.14}$$

On the north and east boundaries [i.e., either $x = (i, n)$ for some $0 \le i < m$ or $x = (m, j)$ for some $0 \le j < n$] the kernel is degenerate because there is only one admissible step.

3.3. *Variance and exit point*. Let

$$\xi_x = \max\{k \ge 0 : x_i = (i, 0) \text{ for } 0 \le i \le k\} \tag{3.15}$$

and

$$\xi_y = \max\{k \ge 0 : x_j = (0, j) \text{ for } 0 \le j \le k\} \tag{3.16}$$

denote the exit points of a path from the x- and y-axes. For any given path, exactly one of ξ_x and ξ_y is zero. In terms of (2.10), $\xi_x = v_1(0)$.

For $\theta, x > 0$ define the function

$$L(\theta, x) = \int_0^x \bigl(\Psi_0(\theta) - \log y\bigr) x^{-\theta} y^{\theta-1} e^{x-y}\, dy. \tag{3.17}$$

The observation

$$L(\theta, x) = -\Gamma(\theta) x^{-\theta} e^x \,\mathbb{C}\mathrm{ov}[\log A, \mathbf{1}\{A \le x\}]$$

for $A \sim \text{Gamma}(\theta, 1)$ shows that $L(\theta, x) > 0$. Furthermore, $\mathbb{E}L(\theta, A) = \Psi_1(\theta)$.

THEOREM 3.7. *Assume* (2.4). *Then for $m, n \in \mathbb{Z}_+$ we have these identities*:

$$\mathbb{V}\mathrm{ar}[\log Z_{m,n}] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2E_{m,n}\left[\sum_{i=1}^{\xi_x} L(\theta, Y^{-1}_{i,0})\right] \tag{3.18}$$

and

$$\begin{aligned}\mathbb{V}\mathrm{ar}[\log Z_{m,n}] &= -n\Psi_1(\mu - \theta) + m\Psi_1(\theta) \\ &\quad + 2E_{m,n}\left[\sum_{j=1}^{\xi_y} L(\mu - \theta, Y^{-1}_{0,j})\right].\end{aligned} \tag{3.19}$$

When $\xi_x = 0$ *the sum* $\sum_{i=1}^{\xi_x}$ *is interpreted as* 0, *and similarly for* $\xi_y = 0$.

PROOF. We prove (3.18). Identity (3.19) then follows by a reflection across the diagonal. Let us abbreviate temporarily, according to the compass directions of the rectangle $\Lambda_{m,n}$,

$$S_{\mathcal{N}} = \log Z_{m,n} - \log Z_{0,n}, \qquad S_{\mathcal{S}} = \log Z_{m,0},$$
$$S_{\mathcal{E}} = \log Z_{m,n} - \log Z_{m,0}, \qquad S_{\mathcal{W}} = \log Z_{0,n}.$$

Then

$$\begin{aligned}\mathbb{V}\mathrm{ar}[\log Z_{m,n}] &= \mathbb{V}\mathrm{ar}(S_{\mathcal{W}} + S_{\mathcal{N}})\\ &= \mathbb{V}\mathrm{ar}(S_{\mathcal{W}}) + \mathbb{V}\mathrm{ar}(S_{\mathcal{N}}) + 2\,\mathbb{C}\mathrm{ov}(S_{\mathcal{W}}, S_{\mathcal{N}})\\ &= \mathbb{V}\mathrm{ar}(S_{\mathcal{W}}) + \mathbb{V}\mathrm{ar}(S_{\mathcal{N}}) + 2\,\mathbb{C}\mathrm{ov}(S_{\mathcal{S}} + S_{\mathcal{E}} - S_{\mathcal{N}}, S_{\mathcal{N}})\\ &= \mathbb{V}\mathrm{ar}(S_{\mathcal{W}}) - \mathbb{V}\mathrm{ar}(S_{\mathcal{N}}) + 2\,\mathbb{C}\mathrm{ov}(S_{\mathcal{S}}, S_{\mathcal{N}}).\end{aligned} \tag{3.20}$$

The last equality came from the independence of $S_{\mathcal{E}}$ and $S_{\mathcal{N}}$, from Theorem 3.3 and (3.4). By assumption (2.4) $\mathbb{V}\mathrm{ar}(S_{\mathcal{W}}) = n\Psi_1(\mu - \theta)$, and by Theorem 3.3 $\mathbb{V}\mathrm{ar}(S_{\mathcal{N}}) = m\Psi_1(\theta)$.

To prove (3.18) it remains to work on $\mathbb{C}\mathrm{ov}(S_{\mathcal{S}}, S_{\mathcal{N}})$. In the remaining part of the proof, we wish to differentiate with respect to the parameter θ of the weights $Y_{i,0}$ on the x-axis (term $S_{\mathcal{S}}$) without involving the other weights. Hence, now think of a system with three independent parameters θ, ρ and μ and with weight distributions (for $i, j \in \mathbb{N}$)

$$Y_{i,0}^{-1} \sim \mathrm{Gamma}(\theta, 1), \qquad Y_{0,j}^{-1} \sim \mathrm{Gamma}(\rho, 1)$$

and

$$Y_{i,j}^{-1} \sim \mathrm{Gamma}(\mu, 1).$$

We first show that

$$\mathbb{C}\mathrm{ov}(S_{\mathcal{S}}, S_{\mathcal{N}}) = -\frac{\partial}{\partial\theta}\mathbb{E}(S_{\mathcal{N}}). \tag{3.21}$$

The variable $S_{\mathcal{S}}$ is a sum

$$S_{\mathcal{S}} = \sum_{i=1}^{m} \log U_{i,0}.$$

The joint density of the vector of summands $(\log U_{1,0}, \dots, \log U_{m,0})$ is

$$g_\theta(y_1, \dots, y_m) = \Gamma(\theta)^{-m} \exp\left(-\theta \sum_{i=1}^{m} y_i - \sum_{i=1}^{m} e^{-y_i}\right)$$

on $\mathbb{R}^m$. This comes from the product of Gamma$(\theta, 1)$ distributions. The density of $S_{\mathcal{S}}$ is

$$f_\theta(s) = \Gamma(\theta)^{-m} e^{-\theta s} \int_{\mathbb{R}^{m-1}} \exp\left(-\sum_{i=1}^{m-1} e^{-y_i} - e^{-s+y_1+\cdots+y_{m-1}}\right) dy_{1,m-1}.$$

We also see that, given $S_{\mathcal{S}}$, the joint distribution of $(\log U_{1,0}, \dots, \log U_{m,0})$ does not depend on θ. Consequently, in the calculation below, the conditional expectation does not depend on θ,

$$\begin{aligned} \frac{\partial}{\partial\theta}\mathbb{E}(S_{\mathcal{N}}) &= \frac{\partial}{\partial\theta} \int_{\mathbb{R}} \mathbb{E}(S_{\mathcal{N}} | S_{\mathcal{S}} = s) f_\theta(s)\, ds \\ &= \int_{\mathbb{R}} \mathbb{E}(S_{\mathcal{N}} | S_{\mathcal{S}} = s) \frac{\partial f_\theta(s)}{\partial\theta}\, ds \\ &= \int_{\mathbb{R}} \mathbb{E}(S_{\mathcal{N}} | S_{\mathcal{S}} = s) \left(-s - m\frac{\Gamma'(\theta)}{\Gamma(\theta)}\right) f_\theta(s)\, ds \\ &= -\mathbb{E}(S_{\mathcal{N}} S_{\mathcal{S}}) + \mathbb{E}(S_{\mathcal{N}}) m \mathbb{E}(\log U) \\ &= -\mathbb{E}(S_{\mathcal{N}} S_{\mathcal{S}}) + \mathbb{E}(S_{\mathcal{N}}) \mathbb{E}(S_{\mathcal{S}}) \\ &= -\mathbb{C}\mathrm{ov}(S_{\mathcal{N}}, S_{\mathcal{S}}). \end{aligned} \tag{3.22}$$

To justify taking $\partial/\partial\theta$ inside the integral, we check that for all $0 < \theta_0 < \theta_1$,

$$\int_{\mathbb{R}} \mathbb{E}(|S_{\mathcal{N}}| | S_{\mathcal{S}} = s) \sup_{\theta \in [\theta_0, \theta_1]} \left| \frac{\partial f_\theta(s)}{\partial\theta} \right| ds < \infty. \tag{3.23}$$

Since

$$\sup_{\theta \in [\theta_0, \theta_1]} \left| \frac{\partial f_\theta(s)}{\partial\theta} \right| \le C(1 + |s|)\big(f_{\theta_0}(s) + f_{\theta_1}(s)\big)$$

it suffices to get a bound for a fixed $\theta > 0$:

$$\begin{aligned} &\int_{\mathbb{R}} \mathbb{E}(|S_{\mathcal{N}}| | S_{\mathcal{S}} = s)(1 + |s|) f_\theta(s)\, ds \\ &\qquad = \mathbb{E}[|S_{\mathcal{N}}|(1 + |S_{\mathcal{S}}|)] \le \|S_{\mathcal{N}}\|_{L^2(\mathbb{P})} \|1 + S_{\mathcal{S}}\|_{L^2(\mathbb{P})} < \infty, \end{aligned}$$

because $S_{\mathcal{N}}$ and $S_{\mathcal{S}}$ are sums of i.i.d. random variables with all moments. Dominated convergence and this integrability bound (3.23) also give the continuity of $\theta \mapsto \mathbb{C}\mathrm{ov}(S_{\mathcal{N}}, S_{\mathcal{S}})$.

The next step is to calculate $(\partial/\partial\theta)\mathbb{E}(S_{\mathcal{N}})$ by a coupling. Sometimes we add a sub- or superscript θ to expectations and covariances to emphasize their dependence on the parameter θ of the distribution of the initial weights on the x-axis. We also introduce a direct functional dependence on θ in $Z_{m,n}$ by realizing the weights $U_{i,0}$ as functions of uniform random variables. Let

$$F_\theta(x) = \int_0^x \frac{y^{\theta-1}e^{-y}}{\Gamma(\theta)}\,dy, \qquad x \ge 0, \tag{3.24}$$

be the c.d.f. of the Gamma$(\theta, 1)$ distribution and H_θ its inverse function, defined on $(0, 1)$, that satisfies $\eta = F_\theta(H_\theta(\eta))$ for $0 < \eta < 1$. Then if η is a Uniform(0, 1) random variable, $U^{-1} = H_\theta(\eta)$ is a Gamma$(\theta, 1)$ random variable. Let $\eta_{1,m} = (\eta_1, \dots, \eta_m)$ be a vector of Uniform(0, 1) random variables. We redefine $Z_{m,n}$ as a function of the random variables $\{\eta_{1,m}; Y_{i,j} : (i, j) \in \mathbb{Z}_+ \times \mathbb{N}\}$ without changing its distribution:

$$Z_{m,n}(\theta) = \sum_{x_\bullet \in \Pi_{m,n}} \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}. \tag{3.25}$$

Next, we look for the derivative:

$$\begin{aligned}\frac{\partial}{\partial\theta} \log Z_{m,n}(\theta) = \frac{1}{Z_{m,n}(\theta)} \sum_{x_\bullet \in \Pi_{m,n}} &\left(-\sum_{i=1}^{\xi_x} \frac{\partial H_\theta(\eta_i)}{\partial\theta} H_\theta(\eta_i)^{-1}\right) \\ &\times \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}.\end{aligned}$$

Differentiate implicitly $\eta = F(\theta, H(\theta, \eta))$ to find

$$\frac{\partial H(\theta, \eta)}{\partial\theta} = -\frac{(\partial F/\partial\theta)(\theta, H(\theta, \eta))}{(\partial F/\partial x)(\theta, H(\theta, \eta))}. \tag{3.26}$$

[We write $F(\theta, x) = F_\theta(x)$ and $H(\theta, \eta) = H_\theta(\eta)$ when subscripts are not convenient.] If we define

$$L(\theta, x) = -\frac{1}{x} \cdot \frac{\partial F(\theta, x)/\partial\theta}{\partial F(\theta, x)/\partial x}, \qquad \theta, x > 0, \tag{3.27}$$

we can write

$$\begin{aligned}\frac{\partial}{\partial\theta} \log Z_{m,n}(\theta) = \frac{1}{Z_{m,n}(\theta)} \sum_{x_\bullet \in \Pi_{m,n}} &\left\{-\sum_{i=1}^{\xi_x} L(\theta, H_\theta(\eta_i))\right\} \\ &\times \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}.\end{aligned} \tag{3.28}$$

Direct calculation shows that (3.27) agrees with the earlier definition (3.17) of L.

Since $\Psi_0(\theta) = \Gamma(\theta)^{-1} \int_0^\infty (\log y) y^{\theta-1} e^{-y}\, dy$, we also have

$$L(\theta, x) = \int_x^\infty \bigl(-\Psi_0(\theta) + \log y\bigr) x^{-\theta} y^{\theta-1} e^{x-y}\, dy. \tag{3.29}$$

For $x \le 1$ drop e^{-y} and compute the integrals in (3.17), while for $x \ge 1$ apply Hölder's inequality judiciously to (3.29). This shows

$$0 < L(\theta, x) \le \begin{cases} C(\theta)(1 - \log x), & \text{for } 0 < x \le 1, \\ C(\theta) x^{-1/4}, & \text{for } x \ge 1. \end{cases} \tag{3.30}$$

In particular, $L(\theta, H_\theta(\eta))$ with $\eta \sim \text{Uniform}(0, 1)$ has an exponential moment: for small enough $t > 0$,

$$\mathbb{E}\bigl[e^{tL(\theta, H_\theta(\eta))}\bigr] = \int_0^\infty e^{tL(\theta,x)} \frac{x^{\theta-1} e^{-x}}{\Gamma(\theta)}\, dx < \infty. \tag{3.31}$$

Let $\widetilde{\mathbb{E}}$ denote expectation over the variables $\{Y_{i,j}\}_{(i,j)\in\mathbb{Z}_+\times\mathbb{N}}$ (i.e., excluding the weights on the x-axis). From (3.22), we get

$$\begin{aligned} -\int_{\theta_0}^{\theta_1} \mathbb{C}\mathrm{ov}^\theta(S_{\mathcal{N}}, S_{\mathcal{S}})\, d\theta &= \mathbb{E}^{\theta_1}(S_{\mathcal{N}}) - \mathbb{E}^{\theta_0}(S_{\mathcal{N}}) \\ &= \widetilde{\mathbb{E}} \int_{(0,1)^m} d\eta_{1,m} \bigl(\log Z_{m,n}(\theta_1) - \log Z_{m,n}(\theta_0)\bigr) \\ &= \widetilde{\mathbb{E}} \int_{(0,1)^m} d\eta_{1,m} \int_{\theta_0}^{\theta_1} \frac{\partial}{\partial\theta} \log Z_{m,n}(\theta)\, d\theta \\ &= \int_{\theta_0}^{\theta_1} d\theta\, \widetilde{\mathbb{E}} \int_{(0,1)^m} d\eta_{1,m} \frac{\partial}{\partial\theta} \log Z_{m,n}(\theta). \end{aligned} \tag{3.32}$$

The last equality above came from Tonelli's theorem, justified by (3.28) which shows that $(\partial/\partial\theta) \log Z_{m,n}(\theta)$ is always negative.

From (3.28), upon replacing $H(\theta, \eta_i)$ with $Y_{i,0}^{-1}$,

$$\begin{aligned} \frac{\partial}{\partial\theta} \log Z_{m,n}(\theta) &= \frac{1}{Z_{m,n}(\theta)} \sum_{x_\bullet \in \Pi_{m,n}} \Biggl\{-\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Biggr\} \prod_{k=1}^{m+n} Y_{x_k} \\ &= -E^{Q_{m,n}^\omega}\Biggl[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Biggr]. \end{aligned} \tag{3.33}$$

Consequently from (3.32),

$$\int_{\theta_0}^{\theta_1} \mathbb{C}\mathrm{ov}^\theta(S_{\mathcal{N}}, S_{\mathcal{S}})\, d\theta = \int_{\theta_0}^{\theta_1} \mathbb{E}^\theta E^{Q_{m,n}^\omega}\Biggl[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Biggr] d\theta.$$

Earlier we justified the continuity of $\mathbb{C}\mathrm{ov}^{\theta}(S_{\mathcal{N}}, S_{\mathcal{S}})$ as a function of $\theta > 0$. The same is true for the integrand on the right. Hence, we get

$$\mathbb{C}\mathrm{ov}^{\theta}(S_{\mathcal{N}}, S_{\mathcal{S}}) = E^{\theta}_{m,n}\left[\sum_{i=1}^{\xi_x} L(\theta, Y^{-1}_{i,0})\right]. \tag{3.34}$$

Putting this back into (3.20) completes the proof. □

4. Upper bound for the model with boundaries. In this section, we prove the upper bound of Theorem 2.1. Assumption (2.4) is in force, with $0 < \theta < \mu$ fixed. While keeping μ fixed, we shall also consider an alternative value $\lambda \in (0, \mu)$ and then assumption (2.4) is in force but with λ replacing θ. Since μ remains fixed, we omit dependence on μ from all notation. At times dependence on λ and θ has to be made explicit, as, for example, in the next lemma where $\mathbb{V}\mathrm{ar}^{\lambda}$ denotes variance computed under assumption (2.4) with λ replacing θ.

LEMMA 4.1. *Consider $0 < \delta_0 < \theta < \mu$ fixed. Then there exists a constant $C < \infty$ such that for all $\lambda \in [\delta_0, \theta]$,*

$$\mathbb{V}\mathrm{ar}^{\lambda}[\log Z_{m,n}] \le \mathbb{V}\mathrm{ar}^{\theta}[\log Z_{m,n}] + C(m+n)(\theta - \lambda). \tag{4.1}$$

A single constant C works for all $\delta_0 < \theta < \mu$ that vary in a compact set.

PROOF. Identity (3.19) will be convenient for $\lambda < \theta$:

$$\begin{aligned} &\mathbb{V}\mathrm{ar}^{\lambda}[\log Z_{m,n}] - \mathbb{V}\mathrm{ar}^{\theta}[\log Z_{m,n}] \\ &\quad = -n\Psi_1(\mu - \lambda) + m\Psi_1(\lambda) + n\Psi_1(\mu - \theta) - m\Psi_1(\theta) \qquad (4.2) \\ &\qquad + 2\mathbb{E}^{\lambda} E^{Q^{\omega}_{m,n}}\left[\sum_{j=1}^{\xi_y} L(\mu - \lambda, Y^{-1}_{0,j})\right] \\ &\qquad - 2\mathbb{E}^{\theta} E^{Q^{\omega}_{m,n}}\left[\sum_{j=1}^{\xi_y} L(\mu - \theta, Y^{-1}_{0,j})\right]. \qquad (4.3) \end{aligned}$$

Ψ_1 is continuously differentiable and so

$$\text{line (4.2)} \ \le C(m+n)(\theta - \lambda).$$

We work on the difference (4.3). As in the proof of Theorem 3.7 we replace the weights on the x- and y-axes with functions of uniform random variables. We need explicitly only the ones on the y-axis, denote these by η_j. Write $\widetilde{\mathbb{E}}$ for the expectation over the uniform variables and the bulk weights $\{Y_{i,j} : i, j \ge 1\}$. This expectation no longer depends on λ or θ. The quenched measure Q^ω does carry dependence on these parameters, and we express that by a superscript θ or λ:

$$
\begin{aligned}
&\text{Difference (4.3) without the factor 2}\\
&\quad = \widetilde{\mathbb{E}} E^{Q^{\lambda,\omega}_{m,n}}\Bigg[\sum_{j=1}^{\xi_y} L\big(\mu-\lambda, H_{\mu-\lambda}(\eta_j)\big)\Bigg]\\
&\qquad - \widetilde{\mathbb{E}} E^{Q^{\theta,\omega}_{m,n}}\Bigg[\sum_{j=1}^{\xi_y} L\big(\mu-\theta, H_{\mu-\theta}(\eta_j)\big)\Bigg]\\
&\quad = \widetilde{\mathbb{E}} E^{Q^{\lambda,\omega}_{m,n}}\Bigg[\sum_{j=1}^{\xi_y} L\big(\mu-\lambda, H_{\mu-\lambda}(\eta_j)\big)\Bigg]\\
&\qquad - \widetilde{\mathbb{E}} E^{Q^{\lambda,\omega}_{m,n}}\Bigg[\sum_{j=1}^{\xi_y} L\big(\mu-\theta, H_{\mu-\theta}(\eta_j)\big)\Bigg] \qquad (4.4)\\
&\qquad + \widetilde{\mathbb{E}} E^{Q^{\lambda,\omega}_{m,n}}\Bigg[\sum_{j=1}^{\xi_y} L\big(\mu-\theta, H_{\mu-\theta}(\eta_j)\big)\Bigg]\\
&\qquad - \widetilde{\mathbb{E}} E^{Q^{\theta,\omega}_{m,n}}\Bigg[\sum_{j=1}^{\xi_y} L\big(\mu-\theta, H_{\mu-\theta}(\eta_j)\big)\Bigg]. \qquad (4.5)
\end{aligned}
$$

We first show that difference (4.5) is ≤ 0, by showing that, as the parameter ρ in $Q^{\omega,\rho}_{m,n}$ increases, the random variable ξ_y increases stochastically. Write $B_j = H_{\mu-\rho}(\eta_j)$ for the Gamma$(\mu-\rho, 1)$ variable that gives the weight $Y_{0,j} = B_j^{-1}$ in the definition of $Q^{\omega,\rho}_{m,n}$. For a given μ, B_j decreases as ρ increases. Thus, it suffices to show that, for $1 \le k, \ell \le n$,

$$
(\partial/\partial B_\ell) Q^\omega\{\xi_y \ge k\} \le 0. \tag{4.6}
$$

Write $W = \prod_{j=1}^{\xi_y} B_j^{-1} \cdot \prod_{k=\xi_y+1}^{m+n} Y_{x_k}$ for the total weight of a path x (the numerator of the quenched polymer probability of the path)

$$
\begin{aligned}
\frac{\partial}{\partial B_\ell} Q^\omega\{\xi_y \ge k\} &= \frac{\partial}{\partial B_\ell}\Bigg(\frac{1}{Z_{m,n}}\sum_{x_\cdot} \mathbf{1}\{\xi_y \ge k\} W\Bigg)\\
&= \frac{1}{Z_{m,n}}\sum_{x_\cdot} \mathbf{1}\{\xi_y \ge k\}\mathbf{1}\{\xi_y \ge \ell\}(-B_\ell^{-1}) W\\
&\quad - \frac{1}{Z_{m,n}^2}\Bigg(\sum_{x_\cdot} \mathbf{1}\{\xi_y \ge k\} W\Bigg)\cdot\Bigg(\sum_{x_\cdot} \mathbf{1}\{\xi_y \ge \ell\}(-B_\ell^{-1}) W\Bigg)\\
&= -B_\ell^{-1}\operatorname{Cov}^{Q^\omega}[\mathbf{1}\{\xi_y \ge k\}, \mathbf{1}\{\xi_y \ge \ell\}] < 0.
\end{aligned}
$$

Thus, we can bound difference (4.5) above by 0.

In difference (4.4) inside the brackets only ξ_y is random under $Q^{\omega,\lambda}_{m,n}$. We replace ξ_y with its upper bound n and then we are left with integrating over uniform variables η_j,

$$\begin{aligned} |\text{line (4.4)}| &\le \widetilde{\mathbb{E}} E^{Q^{\lambda,\omega}_{m,n}}\Bigg[\sum_{j=1}^{\xi_y} \big|L\big(\mu-\lambda, H_{\mu-\lambda}(\eta_j)\big) - L\big(\mu-\theta, H_{\mu-\theta}(\eta_j)\big)\big|\Bigg] \\ &\le n\int_0^1 \big|L\big(\mu-\lambda, H_{\mu-\lambda}(\eta)\big) - L\big(\mu-\theta, H_{\mu-\theta}(\eta)\big)\big|\, d\eta \qquad (4.7)\\ &= n\int_0^1\int_{\mu-\theta}^{\mu-\lambda}\left|\frac{d}{d\rho}L(\rho, H_\rho(\eta))\right| d\rho\, d\eta. \end{aligned}$$

From (3.26) and (3.27),

$$\begin{aligned} \frac{d}{d\rho}L(\rho, H_\rho(\eta)) &= \frac{\partial L}{\partial \rho} + \frac{\partial L}{\partial x}\frac{\partial H_\rho(\eta)}{\partial \rho} \\ &= \left(\frac{\partial L(\rho,x)}{\partial \rho} + xL(\rho,x)\frac{\partial L(\rho,x)}{\partial x}\right)\bigg|_{x=H_\rho(\eta)}. \end{aligned}$$

Utilizing (3.30) and explicit computations leads to bounds

$$\begin{aligned} &\left|\frac{\partial L(\rho,x)}{\partial \rho} + xL(\rho,x)\frac{\partial L(\rho,x)}{\partial x}\right| \\ &\quad\le \begin{cases} C(\rho)\big(1+(\log x)^2\big), & \text{for } 0 < x \le 1, \\ C(\rho)x^{1/2}, & \text{for } x \ge 1. \end{cases} \end{aligned} \qquad (4.8)$$

With ρ restricted to a compact subinterval of $(0,\infty)$, these bounds are valid for a fixed constant C. Continue from (4.7), letting B_ρ denote a Gamma$(\rho, 1)$ random variable:

$$\begin{aligned} \text{difference (4.4)} &\le n\int_{\mu-\theta}^{\mu-\lambda}\int_0^1 \left|\frac{d}{d\rho}L(\rho, H_\rho(\eta))\right| d\eta\, d\rho \\ &\le Cn\int_{\mu-\theta}^{\mu-\lambda}\mathbb{E}[1+(\log B_\rho)^2 + B_\rho^{1/2}]\, d\rho \\ &\le Cn(\theta-\lambda). \end{aligned}$$

To summarize, we have shown that difference (4.3) $\le Cn(\theta-\lambda)$ and thereby completed the proof of the lemma. □

The preliminaries are ready and we turn to the upper bound. Let the scaling parameter $N \ge 1$ be real valued. We assume that the dimensions $(m,n) \in \mathbb{N}^2$ of the rectangle satisfy

$$|m - N\Psi_1(\mu - \theta)| \le \kappa_N \quad \text{and} \quad |n - N\Psi_1(\theta)| \le \kappa_N \tag{4.9}$$

for a sequence $\kappa_N \le CN^{2/3}$ with a fixed constant $C < \infty$.

For a walk $x_\centerdot$ such that $\xi_x > 0$, weights at distinct parameter values are related by

$$\begin{aligned} W(\theta) &= \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k} \\ &= W(\lambda) \cdot \prod_{i=1}^{\xi_x} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}. \end{aligned}$$

For $\lambda < \theta$, $H_\lambda(\eta) \le H_\theta(\eta)$ and consequently

$$\begin{aligned} Q^{\theta,\omega}\{\xi_x \ge u\} &= \frac{1}{Z(\theta)} \sum_{x_\centerdot} \mathbf{1}\{\xi_x \ge u\} W(\theta) \\ &\le \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}. \end{aligned} \tag{4.10}$$

We bound the $\mathbb{P}$-tail of $Q^\omega\{\xi_x \ge u\}$ separately for two ranges of a positive real u. Let $c, \delta > 0$ be constants. Their values will be determined in the course of the proof. For future use of the estimates developed here, it is to be noted that c and δ, and the other constants introduced in this upper bound proof, are functions of (μ, θ) and nothing else, and furthermore, fixed values of the constants work for $0 < \theta < \mu$ in a compact set.

Case 1. $(1 \vee c\kappa_N) \le u \le \delta N$.

Pick an auxiliary parameter value

$$\lambda = \theta - \frac{bu}{N}. \tag{4.11}$$

We can assume $b > 0$ and $\delta > 0$ small enough so that $b\delta < \theta/2$ and then $\lambda \in (\theta/2, \theta)$. Let

$$\alpha = \exp\bigl[u\bigl(\Psi_0(\lambda) - \Psi_0(\theta)\bigr) + \delta u^2/N\bigr]. \tag{4.12}$$

Consider $0 < s < \delta$. First, split into two probabilities:

$$\mathbb{P}[Q^\omega\{\xi_x \ge u\} \ge e^{-su^2/N}] \le \mathbb{P}\left\{\prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \ge \alpha\right\} \tag{4.13}$$

$$+ \mathbb{P}\biggl(\frac{Z(\lambda)}{Z(\theta)} \ge \alpha^{-1} e^{-su^2/N}\biggr). \tag{4.14}$$

Recall that $\mathbb{E}(\log H_\theta(\eta)) = \Psi_0(\theta)$ and that overline denotes a centered random variable. Then for the second probability on line (4.13),

$$
\begin{aligned}
&\mathbb{P}\left\{\prod_{i=1}^{\lfloor u\rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \geq \alpha\right\} \\
&\quad = \mathbb{P}\left\{\sum_{i=1}^{\lfloor u\rfloor}\bigl(\overline{\log H_\lambda(\eta_i)} - \overline{\log H_\theta(\eta_i)}\bigr)\right. \\
&\qquad\qquad \left.\geq (u - \lfloor u\rfloor)\bigl(\Psi_0(\lambda) - \Psi_0(\theta)\bigr) + \delta u^2/N\right\} \qquad (4.15)\\
&\quad \leq \frac{4N^2}{\delta^2 u^3}\,\mathbb{V}\mathrm{ar}[\log H_\lambda(\eta) - \log H_\theta(\eta)] \\
&\quad \leq C\frac{N^2}{u^3}.
\end{aligned}
$$

The extra term with the integer part correction goes away because

$$
\Psi_0(\lambda) - \Psi_0(\theta) \geq -C(\theta)(\theta - \lambda) = -C(\theta)\frac{bu}{N} \geq -\frac{\delta u^2}{2N},
$$

$u \geq 1$, and we can choose b small enough.

Rewrite the probability from line (4.14) as

$$
\begin{aligned}
&\mathbb{P}\bigl(\overline{\log Z(\lambda)} - \overline{\log Z(\theta)} \geq -\mathbb{E}[\log Z(\lambda)] + \mathbb{E}[\log Z(\theta)] \\
&\qquad\qquad - \log\alpha - su^2/N\bigr). \qquad (4.16)
\end{aligned}
$$

Recall the mean from (2.5). Rewrite the right-hand side of the inequality inside the probability above as follows:

$$
\begin{aligned}
&-\mathbb{E}[\log Z(\lambda)] + \mathbb{E}[\log Z(\theta)] - \log\alpha - su^2/N \\
&\quad = \bigl(n\Psi_0(\mu-\lambda) + m\Psi_0(\lambda)\bigr) - \bigl(n\Psi_0(\mu-\theta) + m\Psi_0(\theta)\bigr) \\
&\qquad - \log\alpha - su^2/N \\
&\quad \geq \bigl(u - N\Psi_1(\mu-\theta)\bigr)\bigl(\Psi_0(\theta) - \Psi_0(\lambda)\bigr) \\
&\qquad - N\Psi_1(\theta)\bigl(\Psi_0(\mu-\theta) - \Psi_0(\mu-\lambda)\bigr) - (\delta+s)u^2/N \\
&\qquad - \kappa_N|\Psi_0(\lambda) - \Psi_0(\theta)| - \kappa_N|\Psi_0(\mu-\lambda) - \Psi_0(\mu-\theta)| \\
&\quad \geq u\Psi_1(\theta)(\theta-\lambda) + \tfrac{1}{2}N\bigl(\Psi_1(\mu-\theta)\Psi_1'(\theta) \qquad (4.17)\\
&\qquad\qquad + \Psi_1(\theta)\Psi_1'(\mu-\theta)\bigr)(\theta-\lambda)^2 \\
&\qquad - (\delta+s)u^2/N - C_1(\theta,\mu)\bigl(u(\theta-\lambda)^2 + N(\theta-\lambda)^3\bigr) \\
&\qquad - C_1(\theta,\mu)\kappa_N(\theta-\lambda)
\end{aligned}
$$

$$\geq \big(b\Psi_1(\theta) - C_2(\theta,\mu)b^2 - 2\delta - C_1(\theta,\mu)\delta(b^2+b^3)\big)\frac{u^2}{N} - C_1(\theta,\mu)\kappa_N \frac{bu}{N} \tag{4.18}$$

$$\geq \frac{c_1 u^2}{N}. \tag{4.19}$$

Inequality (4.17) with a constant $C_1(\theta,\mu) > 0$ came from the expansions

$$\Psi_0(\theta) - \Psi_0(\lambda) = \Psi_1(\theta)(\theta-\lambda) - \tfrac{1}{2}\Psi_1'(\theta)(\theta-\lambda)^2 + \tfrac{1}{6}\Psi_1''(\rho_0)(\theta-\lambda)^3$$

and

$$\Psi_0(\mu-\theta) - \Psi_0(\mu-\lambda) = -\Psi_1(\mu-\theta)(\theta-\lambda) - \tfrac{1}{2}\Psi_1'(\mu-\theta)(\theta-\lambda)^2 - \tfrac{1}{6}\Psi_1''(\rho_1)(\theta-\lambda)^3$$

for some $\rho_0, \rho_1 \in (\lambda, \theta)$. For inequality (4.18), we defined

$$C_2(\theta,\mu) = -\tfrac{1}{2}\big(\Psi_1(\mu-\theta)\Psi_1'(\theta) + \Psi_1(\theta)\Psi_1'(\mu-\theta)\big) > 0,$$

substituted in $\lambda = \theta - bu/N$ from (4.11), and recalled that $s < \delta$ and $u \leq \delta N$. To get (4.19), we fixed $b > 0$ small enough, then $\delta > 0$ small enough, defined a new constant $c_1 > 0$, and restricted u to satisfy

$$u \geq c\kappa_N \tag{4.20}$$

for another constant c. We can also restrict to $u \geq 1$ if the condition above does not enforce it.

Substitute line (4.19) on the right-hand side inside probability (4.16). This probability came from line (4.14). Apply Chebyshev, then (4.1), and finally (3.18):

$$\begin{aligned} \text{line (4.14)} &\leq \mathbb{P}\big(\overline{\log Z(\lambda)} - \overline{\log Z(\theta)} \geq c_1 u^2/N\big) \\ &\leq \frac{CN^2}{u^4}\mathbb{V}\mathrm{ar}[\log Z(\lambda) - \log Z(\theta)] \\ &\leq \frac{CN^2}{u^4}\big(\mathbb{V}\mathrm{ar}[\log Z(\lambda)] + \mathbb{V}\mathrm{ar}[\log Z(\theta)]\big) \\ &\leq \frac{CN^2}{u^4}\big(\mathbb{V}\mathrm{ar}[\log Z(\theta)] + N(\theta-\lambda)\big) \end{aligned} \tag{4.21}$$

$$\leq \frac{CN^2}{u^4} E\left[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\right] + \frac{CN^2}{u^3}. \tag{4.22}$$

Collecting (4.13), (4.14), (4.15) and (4.22) gives this intermediate result: for $0 < s < \delta$, $N \geq 1$, and $1 \vee c\kappa_N \leq u \leq \delta N$,

$$\mathbb{P}[Q^\omega\{\xi_x \geq u\} \geq e^{-su^2/N}] \leq \frac{CN^2}{u^4} E\left[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\right] + \frac{CN^2}{u^3}. \tag{4.23}$$

LEMMA 4.2. *There exists a constant* $0 < C < \infty$ *such that*

$$E\left[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\right] \le C\big(E(\xi_x) + 1\big). \tag{4.24}$$

PROOF. Write again $A_i = Y_{i,0}^{-1}$ for the Gamma$(\theta, 1)$ variables. Abbreviate $L_i = L(\theta, A_i)$, $\bar{L}_i = L_i - \mathbb{E}L_i$ and $S_k = \sum_{i=1}^{k} \bar{L}_i$

$$\begin{aligned} E\left[\sum_{i=1}^{\xi_x} L_i\right] &= \mathbb{E}(L_1)E(\xi_x) + E\left[\sum_{i=1}^{\xi_x} \bar{L}_i\right] \\ &= \mathbb{E}(L_1)E(\xi_x) + \sum_{k=1}^{m} \mathbb{E}[Q^\omega\{\xi_x = k\}S_k] \\ &\le \big(\mathbb{E}(L_1) + 1\big)E(\xi_x) + \sum_{k=1}^{m} \mathbb{E}[\mathbf{1}\{S_k \ge k\}S_k] \\ &\le CE(\xi_x) + C. \end{aligned}$$

The last bound comes from the fact that $\{\bar{L}_i\}$ are i.i.d. mean zero with all moments [recall (3.31)]:

$$\begin{aligned} \mathbb{E}[\mathbf{1}\{S_k \ge k\}S_k] &\le (k\mathbb{E}(\bar{L}^2))^{1/2}(\mathbb{P}\{S_k \ge k\})^{1/2} \\ &\le Ck^{1/2}(k^{-8}E(S_k^8))^{1/2} \\ &\le Ck^{-3/2} \end{aligned}$$

and these are summable. □

Since $u \ge 1$, we can combine (4.23) and (4.24) to give

$$\mathbb{P}[Q^\omega\{\xi_x \ge u\} \ge e^{-su^2/N}] \le \frac{CN^2}{u^4}E(\xi_x) + \frac{CN^2}{u^3} \tag{4.25}$$

still for $0 < s < \delta$ and $(1 \vee c\kappa_N) \le u \le \delta N$.

Case 2. $(1 \vee c\kappa_N \vee \delta N) \le u < \infty$.

The constant $\delta > 0$ is now fixed small enough by Case 1. Take new constants $\nu > 0$ and $\delta_1 > 0$ and set

$$\lambda = \theta - \nu$$

and

$$\alpha = \exp\big[u\big(\Psi_0(\lambda) - \Psi_0(\theta)\big) + \delta_1 u\big]. \tag{4.26}$$

Consider $0 < s < \delta_1$. First, use again (4.10) to split the probability

$$\begin{aligned}
&\mathbb{P}[Q^\omega\{\xi_x \ge u\} \ge e^{-su}] \\
&\quad \le \mathbb{P}\left\{\prod_{i=1}^{\lfloor u\rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \ge \alpha\right\} + \mathbb{P}\left(\frac{Z(\lambda)}{Z(\theta)} \ge \alpha^{-1}e^{-su}\right) \\
&\quad \le \mathbb{P}\left\{\sum_{i=1}^{\lfloor u\rfloor}\bigl(\overline{\log H_\lambda(\eta_i)} - \overline{\log H_\theta(\eta_i)}\bigr) \ge \frac{1}{2}\delta_1 u\right\} \qquad (4.27)\\
&\qquad + \mathbb{P}\bigl(\overline{\log Z(\lambda)} - \overline{\log Z(\theta)} \ge -\mathbb{E}[\log Z(\lambda)] + \mathbb{E}[\log Z(\theta)] \\
&\qquad\qquad - \log\alpha - su\bigr).
\end{aligned}$$

Logarithms of gamma variables have an exponential moment:

$$\mathbb{E}\bigl[e^{t|\log H_\theta(\eta)|}\bigr] < \infty \qquad \text{if } t < \theta.$$

Hence, standard large deviations apply and for some constant $c_4 > 0$,

$$\mathbb{P}\left\{\sum_{i=1}^{\lfloor u\rfloor}\bigl(\overline{\log H_\lambda(\eta_i)} - \overline{\log H_\theta(\eta_i)}\bigr) \ge \frac{1}{2}\delta_1 u\right\} \le e^{-c_4 u}. \qquad (4.28)$$

Following the pattern that led to (4.19), the right-hand side inside probability (4.27) is bounded as follows:

$$\begin{aligned}
&-\mathbb{E}[\log Z(\lambda)] + \mathbb{E}[\log Z(\theta)] - \log\alpha - su \\
&\quad \ge u\Psi_1(\theta)(\theta-\lambda) - NC_2(\theta)(\theta-\lambda)^2 - (\delta_1 + s)u \\
&\qquad - C_1(\theta)\bigl(u(\theta-\lambda)^2 + N(\theta-\lambda)^3\bigr) - C_1(\theta)\kappa_N(\theta-\lambda) \\
&\quad \ge u\left[\Psi_1(\theta)\nu - \frac{C_2(\theta)\nu^2}{\delta} - 2\delta_1 - C_1(\theta)(\nu^2 + \nu^3/\delta)\right] - C_1(\theta)\kappa_N\nu \\
&\quad \ge c_5 u
\end{aligned}$$

for a constant $c_5 > 0$, when we fix ν and δ_1 small enough and again also enforce (4.20) $u \ge c\kappa_N$ for a large enough c. By standard large deviations, since $\log Z(\lambda)$ and $\log Z(\theta)$ can be expressed as sums of i.i.d. random variables with an exponential moment, and for $u \ge \delta N$,

$$\text{probability (4.27)} \le \mathbb{P}\bigl(\overline{\log Z(\lambda)} - \overline{\log Z(\theta)} \ge c_5 u\bigr) \le e^{-c_6 u}. \qquad (4.29)$$

Combining (4.28) and (4.29) gives the bound

$$\mathbb{P}[Q^\omega\{\xi_x \ge u\} \ge e^{-su}] \le 2e^{-c_7 u} \qquad (4.30)$$

for $0 < s < \delta_1$ and $u \geq \delta N$. Integrate and use (4.30):

$$
\begin{aligned}
\int_{\delta N}^{\infty} P(\xi_x \geq u)\,du &= \int_{\delta N}^{\infty} du \int_0^1 dt\,\mathbb{P}[Q^{\omega}(\xi_x \geq u) \geq t] \\
&= \int_{\delta N}^{\infty} du \int_0^{\infty} ds\, u e^{-su}\mathbb{P}[Q^{\omega}(\xi_x \geq u) \geq e^{-su}] \qquad (4.31)\\
&\leq 2c_7^{-1}e^{-c_7\delta N} + \delta_1^{-1}e^{-\delta_1\delta N} \leq C.
\end{aligned}
$$

Now we combine the two cases to finish the proof of the upper bound. Let $r \geq 1$ be large enough so that $c\kappa_N \leq rN^{2/3}$ for all N for the constant c that appeared in (4.20):

$$
\begin{aligned}
E(\xi_x) &\leq rN^{2/3} + \int_{rN^{2/3}}^{\delta N} P(\xi_x \geq u)\,du + \int_{\delta N}^{\infty} P(\xi_x \geq u)\,du \\
&\leq C + rN^{2/3} + \int_{rN^{2/3}}^{\delta N} du \int_0^1 \mathbb{P}[Q^{\omega}(\xi_x \geq u) \geq t]\,dt \\
&\leq C + rN^{2/3} + \int_{rN^{2/3}}^{\delta N} du \int_0^{\delta} \mathbb{P}[Q^{\omega}\{\xi_x \geq u\} \geq e^{-su^2/N}]\frac{u^2}{N}e^{-su^2/N}\,ds
\end{aligned}
$$

[substitute in (4.25) and integrate away the s-variable]

$$
\begin{aligned}
&\leq C + rN^{2/3} + C\int_{rN^{2/3}}^{\infty}\left(\frac{N^2}{u^4}E(\xi_x) + \frac{N^2}{u^3}\right)du \\
&= C + rN^{2/3} + \frac{C}{3r^3}E(\xi_x) + \frac{CN^{2/3}}{2r^2}.
\end{aligned}
$$

If r is fixed large enough relative to C, we obtain, with a new constant C

$$E(\xi_x) \leq CN^{2/3}. \tag{4.32}$$

This is valid for all $N \geq 1$. The constant C depends on (μ, θ) and the other constants δ, δ_1, b introduced along the way. A single constant works for $0 < \theta < \mu$ that vary in a compact set.

Combining (3.18), (4.24) and (4.32) gives the upper variance bound for the free energy:

$$\mathbb{V}\mathrm{ar}[\log Z_{m,n}] \leq CN^{2/3}. \tag{4.33}$$

Combining (4.25) and (4.30) with (4.32) gives this lemma.

LEMMA 4.3. *Assume weight distributions* (2.4) *and rectangle dimensions* (4.9). *Then there are finite positive constants δ, δ_1, c, c_1 and C such that for $N \geq 1$ and $(1 \vee c\kappa_N) \leq u \leq \delta N$,*

$$\mathbb{P}[Q^{\omega}\{\xi_x \geq u\} \geq e^{-\delta u^2/N}] \leq C\left(\frac{N^{8/3}}{u^4} + \frac{N^2}{u^3}\right), \tag{4.34}$$

while for $N \geq 1$ *and* $u \geq (1 \vee c\kappa_N \vee \delta N)$,

$$\mathbb{P}[Q^\omega\{\xi_x \geq u\} \geq e^{-\delta_1 u}] \leq e^{-c_1 u}. \tag{4.35}$$

The same bounds hold for ξ_y. *The same constants work for* $0 < \theta < \mu$ *that vary in a compact set.*

Integration gives the annealed bounds in the following corollary.

COROLLARY 4.4. *There are constants* $0 < \delta, c, c_1, C < \infty$ *such that for* $N \geq 1$,

$$P\{\xi_x \geq u\} \leq \begin{cases} C\left(\dfrac{N^{8/3}}{u^4} + \dfrac{N^2}{u^3}\right), & (1 \vee c\kappa_N) \leq u \leq \delta N, \\ 2e^{-c_1 u}, & u \geq (1 \vee c\kappa_N \vee \delta N). \end{cases} \tag{4.36}$$

The same bounds hold for ξ_y.

From the upper variance bound (4.33) and Theorem 3.3, we can easily deduce the central limit theorem for off-characteristic rectangles.

PROOF OF COROLLARY 2.2. Set $m_1 = \lfloor \Psi_1(\mu - \theta)N \rfloor$. Recall that overline means centering at the mean. Since $Z_{m,n} = Z_{m_1,n} \cdot \prod_{i=m_1+1}^{m} U_{i,n}$,

$$N^{-\alpha/2}\overline{\log Z_{m,n}} = N^{-\alpha/2}\overline{\log Z_{m_1,n}} + N^{-\alpha/2} \sum_{i=m_1+1}^{m} \overline{\log U_{i,n}}.$$

Since (m_1, n) is of characteristic shape, (4.33) implies that the first term on the right is stochastically $O(N^{1/3-\alpha/2})$. Since $\alpha > 2/3$ this term converges to zero in probability. The second term is a sum of approximately $c_1 N^\alpha$ i.i.d. terms and hence satisfies a CLT. □

5. Lower bound for the model with boundaries.

In this section, we finish the proof of Theorem 2.1 by providing the lower bound. For subsets $A \subseteq \Pi_{(i,j),(k,\ell)}$ of paths, let us introduce the notation

$$Z_{(i,j),(k,\ell)}(A) = \sum_{x_\bullet \in A} \prod_{r=1}^{k-i+\ell-j} Y_{x_r} \tag{5.1}$$

for a restricted partition function. Then the quenched polymer probability can be written $Q_{m,n}(A) = Z_{m,n}(A)/Z_{m,n}$.

LEMMA 5.1. *For* $m \geq 2$ *and* $n \geq 1$ *we have this comparison of partition functions*:

$$\frac{Z_{m,n}(\xi_y > 0)}{Z_{m-1,n}(\xi_y > 0)} \leq \frac{Z_{(1,1),(m,n)}}{Z_{(1,1),(m-1,n)}} \leq \frac{Z_{m,n}(\xi_x > 0)}{Z_{m-1,n}(\xi_x > 0)}. \tag{5.2}$$

PROOF. Ignore the original boundaries given by the coordinate axes. Consider these partition functions on the positive quadrant $\mathbb{N}^2$ with boundary $\{(i,1): i \in \mathbb{N}\} \cup \{(1,j): j \in \mathbb{N}\}$. The boundary values for $Z_{(1,1),(m,n)}$ are $\{Y_{i,1}: i \ge 2\} \cup \{Y_{1,j}: j \ge 2\}$.

From the definition of $Z_{m,n}(\xi_y > 0)$,

$$Z_{1,1}(\xi_y > 0) = V_{0,1}Y_{1,1} \quad \text{and} \quad V_{1,2} = \frac{Z_{1,2}(\xi_y > 0)}{Z_{1,1}(\xi_y > 0)} = Y_{1,2}\left(1 + \frac{V_{0,2}}{Y_{1,1}}\right).$$

For $j \ge 3$, apply (3.2) inductively to compute the vertical boundary values $V_{1,j} = Y_{1,j}(1 + U_{1,j-1}^{-1}V_{0,j})$, $V_{1,j} \ge Y_{1,j}$ for all $j \ge 2$. The horizontal boundary values for $Z_{m,n}(\xi_y > 0)$ are simply $U_{i,1} = Y_{i,1}$ for $i \ge 2$. Lemma 3.1 gives

$$\frac{Z_{m,n}(\xi_y > 0)}{Z_{m-1,n}(\xi_y > 0)} \le \frac{Z_{(1,1),(m,n)}}{Z_{(1,1),(m-1,n)}} \quad \text{and} \quad \frac{Z_{m,n}(\xi_y > 0)}{Z_{m,n-1}(\xi_y > 0)} \ge \frac{Z_{(1,1),(m,n)}}{Z_{(1,1),(m,n-1)}}.$$

The second inequality of (5.2) comes by transposing the second inequality above. □

Relative to a fixed rectangle $\Lambda_{m,n} = \{0,\dots,m\} \times \{0,\dots,n\}$, define distances of entrance points on the north and east boundaries from the corner (m,n) as duals of the exit points (3.15) and (3.16):

$$\xi_x^* = \max\{k \ge 0 : x_{m+n-i} = (m-i, n) \text{ for } 0 \le i \le k\} \tag{5.3}$$

and

$$\xi_y^* = \max\{k \ge 0 : x_{m+n-j} = (m, n-j) \text{ for } 0 \le j \le k\}. \tag{5.4}$$

The next observation will not be used in the sequel, but it is curious to note the following effect of the boundary conditions: the chance that the last step of the polymer path is along the x-axis does not depend on the endpoint (m,n), but the chance that the first step is along the x-axis increases strictly with m.

PROPOSITION 5.2. *For all $m,n \ge 1$ these hold*:

$$Q_{m,n}^{\omega}\{\xi_x^* > 0\} \stackrel{d}{=} \frac{A}{A+B}, \tag{5.5}$$

where $A \sim \mathrm{Gamma}(\theta, 1)$ and $B \sim \mathrm{Gamma}(\mu - \theta, 1)$ are independent. On the other hand,

$$Q_{m,n}^{\omega}\{\xi_x > 0\} \stackrel{d}{=} Q_{m+1,n}^{\omega}\{\xi_x > 1\} < Q_{m+1,n}^{\omega}\{\xi_x > 0\}. \tag{5.6}$$

PROOF. By the definitions,

$$Q_{m,n}^{\omega}\{\xi_x^* > 0\} = \frac{Z_{m-1,n}U_{m,n}}{Z_{m,n}} = \frac{U_{m,n}^{-1}}{U_{m,n}^{-1} + V_{m,n}^{-1}}.$$

The distributional claim (5.5) follows from the Burke property Theorem 3.3.

For the distributional claim in (5.6) observe first directly from definition (3.11) that $Q^{*,\omega}_{m,n}\{\xi^*_x > 0\} = Q^{*,\omega}_{m+1,n}\{\xi^*_x > 1\}$. Note that in this equality we have dual measures defined in distinct rectangles $\Lambda_{m,n}$ and $\Lambda_{m+1,n}$. Then appeal to Lemma 3.5. The last inequality in (5.6) is immediate. □

Recall the notations $v_0(j)$ and $v_1(j)$ defined in (2.9), (2.10), and introduce their vertical counterparts:

$$w_0(i) = \min\{j \in \mathbb{Z}_+ : \exists k : x_k = (i, j)\} \tag{5.7}$$

and

$$w_1(i) = \max\{j \in \mathbb{Z}_+ : \exists k : x_k = (i, j)\}. \tag{5.8}$$

Implication $v_0(j) > k \Rightarrow w_0(k) < j$ holds, and transposition (i.e., reflection across the diagonal) interchanges v_0 and w_0. Similar properties are valid for v_1 and w_1.

PROPOSITION 5.3. *Assume weight distributions* (2.4) *and rectangle dimensions* (2.6). *Then*

$$\lim_{\delta \searrow 0} \overline{\lim_{N\to\infty}} P\{1 \le \xi_x \le \delta N^{2/3}\} = 0.$$

The same result holds for ξ_y.

PROOF. We prove the result for ξ_x, and transposition gives it for ξ_y. Take $\delta > 0$ small and abbreviate $u = \lfloor \delta N^{2/3} \rfloor$. By Fatou's lemma, it is enough to show that for all $0 < h < 1$,

$$\lim_{\delta \searrow 0} \overline{\lim_{N\to\infty}} \mathbb{P}[Q(0 < \xi_x \le u) > h] = 0. \tag{5.9}$$

Fix a small $\eta > 0$. By writing

$$\frac{Q(0 < \xi_x \le u)}{Q(\xi_x > 0)} = \frac{1}{1 + Q(\xi_x > u)/Q(0 < \xi_x \le u)}$$

we decompose the probability as follows:

$$\begin{aligned}
&\mathbb{P}[Q(0 < \xi_x \le u) > h] \\
&\quad \le \mathbb{P}\biggl[\frac{Q(0 < \xi_x \le u)}{Q(\xi_x > 0)} > h\biggr] \\
&\quad = \mathbb{P}\biggl[\frac{Q(\xi_x > u)}{Q(0 < \xi_x \le u)} < \frac{1-h}{h}\biggr] \\
&\quad = \mathbb{P}\biggl[\frac{Z_{m,n}(\xi_x > u) \cdot Z^{\square}_{(1,1),(m,n)}}{Z_{m,n}(0 < \xi_x \le u) \cdot Z^{\square}_{(1,1),(m,n)}} < \frac{1-h}{h}\biggr]
\end{aligned}$$

$$\leq \mathbb{P}\left[\frac{Z_{m,n}(\xi_x > u)}{Z^{\square}_{(1,1),(m,n)}} < e^{\eta N^{1/3}}\right] \tag{5.10}$$

$$+\,\mathbb{P}\left[\frac{Z_{m,n}(0 < \xi_x \leq u)}{Z^{\square}_{(1,1),(m,n)}} > \frac{he^{\eta N^{1/3}}}{1-h}\right]. \tag{5.11}$$

We show separately that for small δ, η can be chosen so that probabilities (5.10) and (5.11) are asymptotically small.

Step 1: *Control of probability* (5.10).

First, decompose according to the value of ξ_x:

$$\frac{Z_{m,n}(\xi_x > u)}{Z^{\square}_{(1,1),(m,n)}} = \sum_{k=u+1}^{m}\left(\prod_{i=1}^{k} U_{i,0}\right)\cdot\frac{Z^{\square}_{(k,1),(m,n)}}{Z^{\square}_{(1,1),(m,n)}}.$$

Construct a new system $\widetilde{\omega}$ in the rectangle $\Lambda_{m,n}$. Fix a parameter $a > 0$ that we will take large in the end. The interior weights of $\widetilde{\omega}$ are $Y^{\widetilde{\omega}}_{i,j} = Y_{m-i+1,n-j+1}$ for $(i,j) \in \{1,\dots,m\}\times\{1,\dots,n\}$. The boundary weights $\{U^{\widetilde{\omega}}_{i,0}, V^{\widetilde{\omega}}_{0,j}\}$ obey the standard setting (2.4) with a new parameter $\lambda = \theta - aN^{-1/3}$ (but μ stays fixed), and they are independent of the old weights ω. Define new dimensions for a rectangle by

$$(\bar{m},\bar{n}) = \big(m + \lfloor N\Psi_1(\mu-\lambda)\rfloor - \lfloor N\Psi_1(\mu-\theta)\rfloor, n + \lfloor N\Psi_1(\lambda)\rfloor - \lfloor N\Psi_1(\theta)\rfloor\big).$$

We have the bounds

$$\bar{n} - n = \lfloor N\Psi_1(\lambda)\rfloor - \lfloor N\Psi_1(\theta)\rfloor \geq a|\Psi_1'(\theta)|N^{2/3} - 1 \geq c_1 a N^{2/3}$$

for a constant $c_1 = c_1(\theta)$, and

$$\bar{u} = m - \bar{m} = \lfloor N\Psi_1(\mu-\theta)\rfloor - \lfloor N\Psi_1(\mu-\lambda)\rfloor \geq a|\Psi_1'(\mu-\lambda)|N^{2/3} - 1 \geq bN^{2/3}$$

for another constant b. By taking a large enough, we can guarantee that $b > \delta$. (It is helpful to remember here that $\Psi_1' < 0$ and $\Psi_1'' > 0$.)

By (5.2) and (3.4),

$$\begin{aligned}\frac{Z^{\square}_{(k,1),(m,n)}}{Z^{\square}_{(1,1),(m,n)}} &= \frac{Z^{\square,\widetilde{\omega}}_{(1,1),(m-k+1,n)}}{Z^{\square,\widetilde{\omega}}_{(1,1),(m,n)}} \geq \frac{Z^{\widetilde{\omega}}_{m-k+1,n}(\xi_x > 0)}{Z^{\widetilde{\omega}}_{m,n}(\xi_x > 0)}\\ &= \frac{Q^{\widetilde{\omega}}_{m-k+1,n}(\xi_x > 0)Z^{\widetilde{\omega}}_{m-k+1,n}}{Q^{\widetilde{\omega}}_{m,n}(\xi_x > 0)Z^{\widetilde{\omega}}_{m,n}}\\ &\geq Q^{\widetilde{\omega}}_{m-k+1,n}(\xi_x > 0)\left(\prod_{i=1}^{k-1} U^{\widetilde{\omega}}_{m-i+1,n}\right)^{-1}.\end{aligned}$$

After these transformations,

$$(5.10) \le \mathbb{P}\Bigg[U_{1,0}\sum_{k=u+1}^{m}\Bigg(\prod_{i=2}^{k}\frac{U_{i,0}}{U^{\widetilde{\omega}}_{m-i+2,n}}\Bigg)Q^{\widetilde{\omega}}_{m-k+1,n}(\xi_x>0)<e^{\eta N^{1/3}}\Bigg].$$

Inside this probability $\{U_{i,0}\}$ are independent of $\widetilde{\omega}$. Next, apply the distribution-preserving reversal $\widetilde{\omega}\mapsto\widetilde{\omega}^*$ and recall (3.12), to turn the probability above into

$$\mathbb{P}\Bigg[U_{1,0}\sum_{k=u+1}^{m}\Bigg(\prod_{i=2}^{k}\frac{U_{i,0}}{U^{\widetilde{\omega}^*}_{m-i+2,n}}\Bigg)Q^{*,\widetilde{\omega}}_{m-k+1,n}(\xi^*_x>0)<e^{\eta N^{1/3}}\Bigg].$$

By the definition (3.11) of the dual measure, $Q^{*,\widetilde{\omega}}_{m-k+1,n}(\xi^*_x>0)=Q^{*,\widetilde{\omega}}_{m,n}(\xi^*_x\ge k)$. Restrict the sum in the probability to $k\le\bar{u}$, and we get the bound

$$(5.10) \le \mathbb{P}\Bigg[Q^{*,\widetilde{\omega}}_{m,n}\{\xi^*_x\ge\bar{u}\}U_{1,0}\sum_{k=u+1}^{\bar{u}}\Bigg(\prod_{i=2}^{k}\frac{U_{i,0}}{U^{\widetilde{\omega}^*}_{m-i+2,n}}\Bigg)<e^{\eta N^{1/3}}\Bigg]$$

$$\le \mathbb{P}\Bigg[Q^{*,\widetilde{\omega}}_{m,n}\{\xi^*_x\ge\bar{u}\}\le\frac{1}{2}\Bigg] \tag{5.12}$$

$$+\mathbb{P}\Bigg[U_{1,0}\sum_{k=u+1}^{\bar{u}}\Bigg(\prod_{i=2}^{k}\frac{U_{i,0}}{U^{\widetilde{\omega}^*}_{m-i+2,n}}\Bigg)\le 2e^{\eta N^{1/3}}\Bigg]. \tag{5.13}$$

We treat first probability (5.12). Going over to complements,

$$(5.12)=\mathbb{P}\big[Q^{*,\widetilde{\omega}}_{m,n}\{\xi^*_x<\bar{u}\}>\tfrac{1}{2}\big].$$

We claim that

$$Q^{*,\widetilde{\omega}}_{m,n}\{\xi^*_x\le\bar{u}\}=Q^{*,\widetilde{\omega}}_{\bar{m},\bar{n}}\{\xi^*_y>\bar{n}-n\}. \tag{5.14}$$

Equality (5.14) comes from the next computation that utilizes the Markov property (3.13) of the dual measure. In the rectangle $\Lambda_{m,n}$ event $\{\xi^*_x\le\bar{u}\}$ says that the path does not touch the segment $\{0,\dots,\bar{m}-1\}\times\{n\}$. Consequently, the path uses one of the edges $((\bar{m}-1,\ell),(\bar{m},\ell))$ for $0\le\ell<n$:

$$\begin{aligned}
Q^{*,\widetilde{\omega}}_{m,n}\{\xi^*_x\le\bar{u}\}&=\sum_{\ell=0}^{n-1}Q^{*,\widetilde{\omega}}_{m,n}\{x_{\bar{m}+\ell-1}=(\bar{m}-1,\ell),x_{\bar{m}+\ell}=(\bar{m},\ell)\}\\
&=\sum_{\ell=0}^{n-1}\sum_{x_\centerdot\in\Pi_{\bar{m}-1,\ell}}\Bigg(\prod_{k=0}^{\bar{m}+\ell-1}X^{\widetilde{\omega}}_{x_k}\Bigg)\frac{1}{Z^{\widetilde{\omega}}_{\bar{m},\ell}}\\
&=\sum_{\ell=0}^{n-1}\sum_{x_\centerdot\in\Pi_{\bar{m}-1,\ell}}\Bigg(\prod_{k=0}^{\bar{m}+\ell-1}X^{\widetilde{\omega}}_{x_k}\Bigg)\Bigg(\prod_{j=\ell}^{\bar{n}-1}X^{\widetilde{\omega}}_{\bar{m},j}\Bigg)\frac{1}{Z^{\widetilde{\omega}}_{\bar{m},\bar{n}}}\\
&=Q^{*,\widetilde{\omega}}_{\bar{m},\bar{n}}\{\xi^*_y>\bar{n}-n\}.
\end{aligned}$$

The second-last equality above relies on the convention $X^{\widetilde{\omega}}_{\bar{m},j} = V^{\widetilde{\omega}}_{\bar{m},j+1}$ for the dual variables defined in the rectangle $\Lambda_{\bar{m},\bar{n}}$. This checks (5.14). Now appeal to Lemma 4.3, for $N \geq 1$ and large enough a to ensure $e^{-\delta(c_1 a)^2 N^{1/3}} \leq 1/2$:

$$\begin{aligned}(5.12) &\leq \mathbb{P}\big[Q^{*,\widetilde{\omega}}_{\bar{m},\bar{n}}\{\xi^*_y > c_1 a N^{2/3}\} \geq \tfrac{1}{2}\big] \\ &= \mathbb{P}\big[Q^{\widetilde{\omega}}_{\bar{m},\bar{n}}\{\xi_y > c_1 a N^{2/3}\} \geq \tfrac{1}{2}\big] \leq C(\theta) a^{-3}.\end{aligned} \tag{5.15}$$

To treat probability (5.13), let $A_i = U^{-1}_{i+1,0} \sim \text{Gamma}(\theta, 1)$ and $\widetilde{A}_i = (U^{\widetilde{\omega}^*}_{m-i+1,n})^{-1} \sim \text{Gamma}(\lambda, 1)$ so that we can write

$$\begin{aligned}(5.13) &= \mathbb{P}\Bigg[\sum_{k=u}^{\bar{u}-1}\Bigg(\prod_{i=1}^{k}\frac{\widetilde{A}_i}{A_i}\Bigg) \leq 2e^{\eta N^{1/3}} A_0\Bigg] \\ &\leq \mathbb{P}\Bigg[\sup_{u \leq k < \bar{u}} \exp\Bigg\{\sum_{i=1}^{k}(\log \widetilde{A}_i - \log A_i)\Bigg\} \leq 2e^{\eta N^{1/3}} A_0\Bigg].\end{aligned}$$

We approximate the sum in the exponent by a Brownian motion. Compute the mean

$$\mathbb{E}(\log \widetilde{A}_i - \log A_i) = \Psi_0(\lambda) - \Psi_0(\theta) \geq -a_1 N^{-1/3}$$

for a positive constant $a_1 \approx \Psi_1(\theta) a$. (Recall that $\Psi_1 = \Psi_0' > 0$.) Define a continuous path $\{S_N(t) : t \in \mathbb{R}_+\}$ by

$$S_N(kN^{-2/3}) = N^{-1/3} \sum_{i=1}^{k}(\log \widetilde{A}_i - \log A_i - \mathbb{E}\log \widetilde{A}_i + \mathbb{E}\log A_i), \qquad k \in \mathbb{Z}_+,$$

and by linear interpolation. Then rewrite the probability from above:

$$(5.13) \leq \mathbb{P}\Big[\sup_{\delta \leq t \leq b}\big(S_N(t) - t a_1\big) \leq \eta + N^{-1/3} \log 2A_0\Big].$$

As $N \to \infty$, S_N converges to a Brownian motion B and so

$$\varlimsup_{N \to \infty} (5.13) \leq \mathbf{P}\Big[\sup_{\delta \leq t \leq b}\big(B(t) - t a_1\big) \leq \eta\Big] \searrow 0 \qquad \text{as } \delta, \eta \searrow 0. \tag{5.16}$$

Combining (5.15) and (5.16) shows that, given $\varepsilon > 0$, we can first pick a large enough to have $\varlimsup_{N\to\infty}(5.12) \leq \varepsilon/2$. Fixing a fixes a_1, and then we fix η and δ small enough to have $\varlimsup_{N\to\infty}(5.13) \leq \varepsilon/2$. This is possible because $\sup_{0<t\leq b}(B(t) - t a_1)$ is a strictly positive random variable by the law of the iterated logarithm. Together these give $\varlimsup_{N\to\infty}(5.10) \leq \varepsilon$.

Step 2: *Control of probability* (5.11).

For later use, we prove a lemma that gives more than presently needed.

LEMMA 5.4. *Assume weight distributions* (2.4) *and rectangle dimensions* (2.6). *Let* $a, b, c > 0$.

(i) *Let* $0 < \varepsilon < 1$. *There exists a constant* $C(\theta) < \infty$ *such that, if*

$$b \ge C(\theta)\varepsilon^{-1/2}(a + \sqrt{a}), \tag{5.17}$$

then

$$\varlimsup_{N\to\infty} \mathbb{P}\biggl[\frac{Z_{m,n}(0 < \xi_x \le aN^{2/3})}{Z^{\square}_{(1,1),(m,n)}} \ge ce^{bN^{1/3}}\biggr] \le \varepsilon. \tag{5.18}$$

(ii) *There exist finite constants* $N_0(\theta, c)$ *and* $C(\theta)$ *such that, for* $N \ge N_0(\theta, c)$ *and* $b \ge 1$,

$$\mathbb{P}\biggl[\frac{Z_{m,n}(0 < \xi_x \le \sqrt{b}N^{2/3})}{Z^{\square}_{(1,1),(m,n)}} \ge ce^{bN^{1/3}}\biggr] \le C(\theta)b^{-3/2}. \tag{5.19}$$

PROOF. Let $u = \lfloor aN^{2/3} \rfloor$. First decompose

$$\frac{Z_{m,n}(0 < \xi_x \le u)}{Z^{\square}_{(1,1),(m,n)}} = \sum_{k=1}^{u} \biggl(\prod_{i=1}^{k} U_{i,0}\biggr) \frac{Z^{\square}_{(k,1),(m,n)}}{Z^{\square}_{(1,1),(m,n)}}.$$

Construct a new environment $\widetilde{\omega}$ in the rectangle $\Lambda_{m,n}$. The interior weights of $\widetilde{\omega}$ are $Y^{\widetilde{\omega}}_{i,j} = Y_{m-i+1,n-j+1}$. The boundary weights $\{U^{\widetilde{\omega}}_{i,0}, V^{\widetilde{\omega}}_{0,j}\}$ obey a new parameter $\lambda = \theta + rN^{-1/3}$ with $r > 0$. They are independent of the old weights ω. By (5.2) and (3.4),

$$\begin{aligned}
\frac{Z^{\square}_{(k,1),(m,n)}}{Z^{\square}_{(1,1),(m,n)}} &= \frac{Z^{\square,\widetilde{\omega}}_{(1,1),(m-k+1,n)}}{Z^{\square,\widetilde{\omega}}_{(1,1),(m,n)}} \le \frac{Z^{\widetilde{\omega}}_{m-k+1,n}(\xi_y > 0)}{Z^{\widetilde{\omega}}_{m,n}(\xi_y > 0)} \\
&= \frac{Q^{\widetilde{\omega}}_{m-k+1,n}\{\xi_y > 0\} Z^{\widetilde{\omega}}_{m-k+1,n}}{Q^{\widetilde{\omega}}_{m,n}\{\xi_y > 0\} Z^{\widetilde{\omega}}_{m,n}} \le \frac{1}{Q^{\widetilde{\omega}}_{m,n}\{\xi_y > 0\}} \biggl(\prod_{i=1}^{k-1} U^{\widetilde{\omega}}_{m-i+1,n}\biggr)^{-1}.
\end{aligned}$$

Write $A_i = U^{-1}_{i+1,0} \sim \text{Gamma}(\theta, 1)$ and $\widetilde{A}_i = (U^{\widetilde{\omega}}_{m-i+1,n})^{-1} \sim \text{Gamma}(\lambda, 1)$,

$$\begin{aligned}
\text{probability in (5.18)} &\le \mathbb{P}\biggl[\frac{U_{1,0}}{Q^{\widetilde{\omega}}_{m,n}\{\xi_y > 0\}} \sum_{k=1}^{u} \biggl(\prod_{i=2}^{k} \frac{U_{i,0}}{U^{\widetilde{\omega}}_{m-i+2,n}}\biggr) \ge ce^{bN^{1/3}}\biggr] \\
&\le \mathbb{P}\biggl[Q^{\widetilde{\omega}}_{m,n}\{\xi_y > 0\} < \frac{1}{2}\biggr] && (5.20) \\
&\quad + \mathbb{P}\biggl[A_0^{-1} \sum_{k=1}^{u} \biggl(\prod_{i=1}^{k-1} \frac{\widetilde{A}_i}{A_i}\biggr) \ge \frac{1}{2}ce^{bN^{1/3}}\biggr]. && (5.21)
\end{aligned}$$

To treat the probability in (5.20), define a new scaling parameter $M = N\Psi_1(\theta)/\Psi_1(\lambda)$ and new dimensions

$$(\bar{m}, \bar{n}) = \big(m + \lfloor M\Psi_1(\mu-\lambda)\rfloor - \lfloor N\Psi_1(\mu-\theta)\rfloor, n\big).$$

The deviation from characteristic shape is the same:

$$(\bar{m}, \bar{n}) - \big(\lfloor M\Psi_1(\mu-\lambda)\rfloor, \lfloor M\Psi_1(\lambda)\rfloor\big) = (m, n) - \big(\lfloor N\Psi_1(\mu-\theta)\rfloor, \lfloor N\Psi_1(\theta)\rfloor\big).$$

There exists a constant $c_2 = c_2(\theta) > 0$ such that

$$\bar{m} - m = \lfloor M\Psi_1(\mu-\lambda)\rfloor - \lfloor N\Psi_1(\mu-\theta)\rfloor \geq c_2 r M^{2/3}.$$

Consider the complement $\{\xi_x > 0\}$ of the inside event in (5.20). Apply $\widetilde{\omega} \mapsto \widetilde{\omega}^*$, and use the definition (3.11) of the dual measure to go from $\Lambda_{m,n}$ to the larger rectangle $\Lambda_{\bar{m},n} = \Lambda_{\bar{m},\bar{n}}$

$$Q_{m,n}^{\widetilde{\omega}^*}\{\xi_x > 0\} = Q_{m,n}^{*,\widetilde{\omega}}\{\xi_x^* > 0\} = Q_{\bar{m},n}^{*,\widetilde{\omega}}\{\xi_x^* > \bar{m} - m\} = Q_{\bar{m},\bar{n}}^{*,\widetilde{\omega}}\{\xi_x^* > c_2 r M^{2/3}\}.$$

By Lemmas 3.5 and 4.3, provided that

$$e^{-\delta(c_2 r)^2 M^{1/3}} \leq \tfrac{1}{2} \quad\Longleftrightarrow\quad N^{1/3} r^2 \geq c_3(\theta)\log 2, \tag{5.22}$$

we have

$$\begin{aligned}(5.20) &= \mathbb{P}\big[Q_{m,n}^{\widetilde{\omega}}\{\xi_x > 0\} > \tfrac{1}{2}\big] = \mathbb{P}\big[Q_{\bar{m},\bar{n}}^{*,\widetilde{\omega}}\{\xi_x^* > c_2 r M^{2/3}\} > \tfrac{1}{2}\big] \\ &= \mathbb{P}\big[Q_{\bar{m},\bar{n}}^{\widetilde{\omega}}\{\xi_x > c_2 r M^{2/3}\} > \tfrac{1}{2}\big] \leq r^{-3}.\end{aligned} \tag{5.23}$$

For probability (5.21), we rewrite the event in terms of mean zero i.i.d.'s. Compute the mean:

$$\mathbb{E}(\log \widetilde{A}_i - \log A_i) = \Psi_0(\lambda) - \Psi_0(\theta) \leq r_1 N^{-1/3}$$

for a positive constant $r_1 \approx \Psi_1(\theta) r$. Let

$$S_k = \sum_{i=1}^{k} (\log \widetilde{A}_i - \log A_i - \mathbb{E}\log \widetilde{A}_i + \mathbb{E}\log A_i).$$

By Kolmogorov's inequality,

$$\begin{aligned}(5.21) &\leq \mathbb{P}\bigg[\sup_{0\leq k\leq u} S_k \geq bN^{1/3} - r_1 a N^{1/3} + \log\frac{cA_0}{2aN^{2/3}}\bigg] \\ &\leq \mathbb{P}\bigg[\sup_{0\leq k\leq u} S_k \geq bN^{1/3} - r_1 a N^{1/3} + \log\frac{cb_1}{2aN^{2/3}}\bigg] + \mathbb{P}(A_0 < b_1) \\ &\leq \frac{\mathbb{E}(S_u^2)}{(bN^{1/3} - r_1 a N^{1/3} + \log(cb_1/(2aN^{2/3})))^2} + \int_0^{b_1} \frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)}\,dx \\ &\leq \frac{Ca}{(b - r_1 a + N^{-1/3}\log(cb_1/(2aN^{2/3})))^2} + Cb_1^{\theta},\end{aligned}$$

assuming that the quantity inside the parenthesis in the denominator is positive. Collecting the bounds from (5.23) and above we have, provided (5.22) holds,

$$\begin{aligned}&\mathbb{P}\biggl[\frac{Z_{m,n}(0<\xi_x\le aN^{2/3})}{Z^{\square}_{(1,1),(m,n)}}\ge ce^{bN^{1/3}}\biggr]\\ &\quad\le\frac{C}{r^3}+\frac{Ca}{(b-r_1a+N^{-1/3}\log(cb_1/(2aN^{2/3})))^2}+Cb_1^{\theta}.\end{aligned}\tag{5.24}$$

For statement (i) of the lemma choose $r=(3C\varepsilon)^{-1/3}$ and $b_1=(\varepsilon/(3C))^{1/\theta}$ for a large enough constant C. Then by assumption (5.17),

$$\varlimsup_{N\to\infty}\mathbb{P}\biggl[\frac{Z_{m,n}(0<\xi_x\le aN^{2/3})}{Z^{\square}_{(1,1),(m,n)}}\ge ce^{bN^{1/3}}\biggr]\le\frac{2\varepsilon}{3}+\frac{Ca}{b^2}\le\varepsilon.$$

For statement (ii) take $a=\sqrt{b}$, $r=\sqrt{b}/(4\Psi_1(\theta))$, and $b_1=b^{-3/(2\theta)}$. Then, since $b\ge 1$, for $N\ge N_0(\theta,c)$ the long denominator on line (5.24) is $\ge(b/2)^2$ and the entire bound becomes

$$\mathbb{P}\biggl[\frac{Z_{m,n}(0<\xi_x\le\sqrt{b}N^{2/3})}{Z^{\square}_{(1,1),(m,n)}}\ge ce^{bN^{1/3}}\biggr]\le Cb^{-3/2}.\tag{5.25}$$

With this choice of r, (5.22) also holds for $N\ge N_0(\theta,c)$. This concludes the proof of Lemma 5.4. $\square$

Now apply part (i) of Lemma 5.4 with $a=\delta$ and $b=\eta$ to show

$$\varlimsup_{N\to\infty}\mathbb{P}\biggl[\frac{Z_{m,n}(0<\xi_x\le\delta N^{2/3})}{Z^{\square}_{(1,1),(m,n)}}>\frac{he^{\eta N^{1/3}}}{1-h}\biggr]\le\varepsilon.$$

Step 1 already fixed $b=\eta>0$ small. Given $\varepsilon>0$, we can then take $a=\delta$ small enough to satisfy (5.17). Shrinking δ does not harm the conclusion from Step 1 because the bound in (5.16) becomes stronger. This concludes Step 2.

To summarize, we have shown that if δ is small enough, then

$$\varlimsup_{N\to\infty}\mathbb{P}[Q(0<\xi_x\le\delta N^{2/3})>h]\le 2\varepsilon.$$

This proves (5.9) and thereby Proposition 5.3. $\square$

From Proposition 5.3 we extract the lower bound on the variance of $\log Z_{m,n}$.

COROLLARY 5.5. *Assume weight distributions* (2.4) *and rectangle dimensions* (2.6). *Then there exists a constant c such that for large enough N, $\mathbb{V}\mathrm{ar}^{\theta}[\log Z_{m,n}]\ge cN^{2/3}$.*

PROOF. Adding equations (3.18) and (3.19) gives

$$\mathbb{V}\text{ar}[\log Z_{m,n}] = E_{m,n}\Bigg[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\Bigg] + E_{m,n}\Bigg[\sum_{j=1}^{\xi_y} L(\mu-\theta, Y_{0,j}^{-1})\Bigg].$$

Fix $\delta > 0$ so that

$$P\{0 < \xi_x < \delta N^{2/3}\} + P\{0 < \xi_y < \delta N^{2/3}\} < 1/2$$

for large N. Then for a particular N either $P\{\xi_x \geq \delta N^{2/3}\} \geq 1/4$ or $P\{\xi_y \geq \delta N^{2/3}\} \geq 1/4$. Suppose it is ξ_x. (Same argument for the other case.) Abbreviate $L_i = L(\theta, Y_{i,0}^{-1})$ and pick $a > 0$ small enough so that for some constant $b > 0$,

$$\mathbb{P}\Bigg[\sum_{i=1}^{\lfloor \delta N^{2/3}\rfloor} L_i < aN^{2/3}\Bigg] \leq e^{-bN^{2/3}} \qquad \text{for } N \geq 1.$$

This is possible because $\{L_i\}$ are strictly positive, i.i.d. random variables.

It suffices now to prove that for large N,

$$E\Bigg[\sum_{i=1}^{\xi_x} L_i\Bigg] \geq \frac{a}{8}N^{2/3}.$$

This follows now readily:

$$\begin{aligned}
E\Bigg[\sum_{i=1}^{\xi_x} L_i\Bigg] &\geq E\Bigg[\mathbf{1}\{\xi_x \geq \delta N^{2/3}\}\sum_{i=1}^{\lfloor \delta N^{2/3}\rfloor} L_i\Bigg] \\
&\geq aN^{2/3}\cdot P\Bigg\{\xi_x \geq \delta N^{2/3}, \sum_{i=1}^{\lfloor \delta N^{2/3}\rfloor} L_i \geq aN^{2/3}\Bigg\} \\
&\geq aN^{2/3}\bigg(\frac{1}{4} - e^{-bN^{2/3}}\bigg) \geq \frac{a}{8}N^{2/3}.
\end{aligned}$$

□

The corollary above concludes the proof of Theorem 2.1.

6. Fluctuations of the path in the model with boundaries.

Fix two rectangles $\Lambda_{(k,\ell),(m,n)} \subseteq \Lambda_{(k_0,\ell_0),(m,n)}$, with $0 \leq k_0 \leq k \leq m$ and $0 \leq \ell_0 \leq \ell \leq n$. As before define the partition function $Z_{(k_0,\ell_0),(m,n)}$ and quenched polymer measure $Q_{(k_0,\ell_0),(m,n)}$ in the larger rectangle. In the smaller rectangle $\Lambda_{(k,\ell),(m,n)}$, impose boundary conditions on the south and west boundaries, given by the quantities $\{U_{i,\ell}, V_{k,j} : i \in \{k+1,\dots,m\}, j \in \{\ell+1,\dots,n\}\}$ computed in the larger rectangle as in (3.4):

$$U_{i,\ell} = \frac{Z_{(k_0,\ell_0),(i,\ell)}}{Z_{(k_0,\ell_0),(i-1,\ell)}} \quad \text{and} \quad V_{k,j} = \frac{Z_{(k_0,\ell_0),(k,j)}}{Z_{(k_0,\ell_0),(k,j-1)}}. \tag{6.1}$$

Let $Z^{(k,\ell)}_{m,n}$ and $Q^{(k,\ell)}_{m,n}$ denote the partition function and quenched polymer measure in $\Lambda_{(k,\ell),(m,n)}$ under these boundary conditions. Then

$$Z^{(k,\ell)}_{m,n} = \sum_{s=k+1}^{m}\left(\prod_{i=k+1}^{s} U_{i,\ell}\right) Z^{\square}_{(s,\ell+1),(m,n)} + \sum_{t=\ell+1}^{n}\left(\prod_{j=\ell+1}^{t} V_{k,j}\right) Z^{\square}_{(k+1,t),(m,n)} = \frac{Z_{(k_0,\ell_0),(m,n)}}{Z_{(k_0,\ell_0),(k,\ell)}}. \tag{6.2}$$

For a path $x_{\centerdot} \in \Pi_{(k,\ell),(m,n)}$ with $x_1 = (k+1,\ell)$, in other words $x_{\centerdot}$ takes off horizontally,

$$Q^{(k,\ell)}_{m,n}(x_{\centerdot}) = \frac{1}{Z^{(k,\ell)}_{m,n}} \prod_{i=1}^{\xi_x^{(k,\ell)}} U_{k+i,\ell} \cdot \prod_{i=\xi_x^{(k,\ell)}+1}^{m-k+n-\ell} Y_{x_i}.$$

We wrote $\xi_x^{(k,\ell)}$ for the distance $x_{\centerdot}$ travels on the x-axis from the perspective of the new origin (k,ℓ): for $x_{\centerdot} \in \Pi_{(k,\ell),(m,n)}$

$$\xi_x^{(k,\ell)} = \max\{r \ge 0 : x_i = (k+i,\ell) \text{ for } 0 \le i \le r\}. \tag{6.3}$$

Consider the distribution of $\xi_x^{(k,\ell)}$ under $Q^{(k,\ell)}_{m,n}$: adding up all the possible path segments from $(k+r,\ell+1)$ to (m,n) and utilizing (6.1) and (6.2) gives

$$\begin{aligned} Q^{(k,\ell)}_{m,n}\{\xi_x^{(k,\ell)} = r\} &= \frac{1}{Z^{(k,\ell)}_{m,n}}\left(\prod_{i=k+1}^{k+r} U_{i,\ell}\right) Z^{\square}_{(k+r,\ell+1),(m,n)} \\ &= \frac{Z_{(k_0,\ell_0),(k+r,\ell)} Z^{\square}_{(k+r,\ell+1),(m,n)}}{Z_{(k_0,\ell_0),(m,n)}} \\ &= Q_{(k_0,\ell_0),(m,n)}\{x_{\centerdot} \text{ goes through } (k+r,\ell) \text{ and } (k+r,\ell+1)\} \\ &= Q_{(k_0,\ell_0),(m,n)}\{v_1(\ell) = k+r\}. \end{aligned} \tag{6.4}$$

Thus $\xi_x^{(k,\ell)}$ under $Q^{(k,\ell)}_{m,n}$ has the same distribution as $v_1(\ell) - k$ under $Q_{(k_0,\ell_0),(m,n)}$. We can now give the proof of Theorem 2.3.

PROOF OF THEOREM 2.3. If $\tau = 0$, then the results are already contained in Corollary 4.4 and Proposition 5.3. Let us assume $0 < \tau < 1$.

Set $u = \lfloor bN^{2/3} \rfloor$. Take $(k_0,\ell_0) = (0,0)$ and $(k,\ell) = (\lfloor \tau m \rfloor, \lfloor \tau n \rfloor)$ above. The system in the smaller rectangle $\Lambda_{(k,\ell),(m,n)}$ is a system with boundary distributions

(2.4) and dimensions $(m-k, n-\ell)$ that satisfy (2.6) for a new scaling parameter $(1-\tau)N$. By (6.4),

$$Q_{m,n}\{v_1(\lfloor \tau n\rfloor) \ge \lfloor \tau m\rfloor + u\} = Q^{(k,\ell)}_{m,n}\{\xi^{(k,\ell)}_x \ge u\} \tag{6.5}$$
$$\stackrel{d}{=} Q_{m-k,n-\ell}\{\xi_x \ge u\}.$$

Hence, bounds (4.34) and (4.35) of Lemma 4.3 are valid as they stand for the quenched probability above. The part of (2.11) that pertains to $v_1(\lfloor \tau n\rfloor)$ now follows from Corollary 4.4.

Recall definition (5.8) of w_1. To get control of the left tail of v_0, first note the implication

$$Q_{m,n}\{v_0(\lfloor \tau n\rfloor) < \lfloor \tau m\rfloor - u\} \le Q_{m,n}\{w_1(\lfloor \tau m\rfloor - u) \ge \lfloor \tau n\rfloor\}.$$

Let $k = \lfloor \tau m\rfloor - u$ and $\ell = \lfloor \tau n\rfloor - \lfloor nu/m\rfloor$. Then up to integer-part corrections, $k/\ell = m/n$. For a constant $C(\theta) > 0$, $\lfloor \tau n\rfloor \ge \ell + C(\theta)bN^{2/3}$. By (6.4), applied to the vertical counterpart w_1 of v_1,

$$Q_{m,n}\{w_1(\lfloor \tau m\rfloor - u) \ge \lfloor \tau n\rfloor\} = Q^{(k,\ell)}_{m,n}\{\xi^{(k,\ell)}_y \ge b_1 N^{2/3}\}$$
$$\stackrel{d}{=} Q_{m-k,n-\ell}\{\xi_y \ge C(\theta)bN^{2/3}\}.$$

The part of (2.11) that pertains to $v_0(\lfloor \tau n\rfloor)$ now follows from Corollary 4.4, applied to ξ_y.

Last we prove (2.12). By a calculation similar to (6.4), the event of passing through a given edge at least one of whose endpoints lies in the interior of $\Lambda_{(k,\ell),(m,n)}$ has the same probability under $Q^{(k,\ell)}_{m,n}$ and under $Q_{m,n}$. Put $(k,\ell) = (\lfloor \tau m\rfloor - 2\lfloor \delta N^{2/3}\rfloor, \lfloor \tau n\rfloor - 2\lfloor c\delta N^{2/3}\rfloor)$ where the constant c is picked so that $c > m/n$ for large enough N. If the path $x_\centerdot$ comes within distance $\delta N^{2/3}$ of $(\tau m, \tau n)$, then it necessarily enters the rectangle

$$\Lambda_{(k+1,\ell+1),(k+4\lfloor \delta N^{2/3}\rfloor, \ell+4\lfloor c\delta N^{2/3}\rfloor)}$$

through the south or the west side. This event of entering decomposes into a disjoint union according to the unique edge that is used to enter the rectangle, and consequently the probabilities under $Q^{(k,\ell)}_{m,n}$ and $Q_{m,n}$ are again the same. From the perspective of the polymer model $Q^{(k,\ell)}_{m,n}$, this event implies that either $0 < \xi^{(k,\ell)}_x \le 4\delta N^{2/3}$ or $0 < \xi^{(k,\ell)}_y \le 4c\delta N^{2/3}$. The following bound arises:

$$Q_{m,n}\{\exists k \text{ such that } |x_k - (\tau m, \tau n)| \le \delta N^{2/3}\}$$
$$\le Q^{(k,\ell)}_{m,n}\{0 < \xi^{(k,\ell)}_x \le 4\delta N^{2/3} \text{ or } 0 < \xi^{(k,\ell)}_y \le 4c\delta N^{2/3}\}$$
$$\stackrel{d}{=} Q_{m-k,n-\ell}\{0 < \xi_x \le 4\delta N^{2/3} \text{ or } 0 < \xi_y \le 4c\delta N^{2/3}\}.$$

Proposition 5.3 now gives (2.12). □

7. Polymer with fixed endpoint but without boundaries. Throughout this section, for given $0 < s, t < \infty$, let $\theta = \theta_{s,t}$ as determined by (2.15) and (m, n)

satisfy (2.21). Up to corrections from integer parts, (2.5) and definition (2.16) give

$$Nf_{s,t}(\mu) = \mathbb{E}\log Z_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}.$$

Define the scaling parameter M by

$$M = \frac{Ns}{\Psi_1(\mu-\theta)} = \frac{Nt}{\Psi_1(\theta)}. \tag{7.1}$$

Then $(Ns, Nt) = (M\Psi_1(\mu-\theta), M\Psi_1(\theta))$ is the characteristic direction for parameters M and θ.

LEMMA 7.1. *Let $\mathbb{P}$ satisfy assumption (2.4) and (m,n) satisfy (2.21). There exist finite constants N_0, C, C_0 such that, for $b \ge C_0$ and $N \ge N_0$,*

$$\mathbb{P}\big[\big|\log Z_{m,n} - \log Z^{\square}_{(1,1),(m,n)}\big| \ge bN^{1/3}\big] \le Cb^{-3/2}.$$

PROOF. Separating the paths that go through the point (1, 1) gives

$$Z_{m,n} = (U_{1,0}+V_{0,1})Z^{\square}_{(1,1),(m,n)} + Z_{m,n}(\xi_x > 1) + Z_{m,n}(\xi_y > 1). \tag{7.2}$$

Consequently,

$$\begin{aligned}\mathbb{P}\bigg[\frac{Z_{m,n}}{Z^{\square}_{(1,1),(m,n)}} \le e^{-bN^{1/3}}\bigg] &\le \mathbb{P}(U_{1,0}+V_{0,1} \le e^{-bN^{1/3}})\\ &\le C(\theta)e^{-bN^{1/3}}.\end{aligned}$$

For the other direction, abbreviate $u = \sqrt{b}(\Psi_1(\theta)/t)^{1/6}M^{2/3}$,

$$\begin{aligned}&\mathbb{P}\bigg[\frac{Z_{m,n}}{Z^{\square}_{(1,1),(m,n)}} \ge e^{bN^{1/3}}\bigg]\\ &\quad= \mathbb{P}\bigg[\frac{Z_{m,n}(\{0<\xi_x\le u\}\cup\{0<\xi_y\le u\})}{Z^{\square}_{(1,1),(m,n)}Q_{m,n}(\{0<\xi_x\le u\}\cup\{0<\xi_y\le u\})} \ge e^{bN^{1/3}}\bigg]\\ &\quad\le \mathbb{P}\bigg[\frac{Z_{m,n}(0<\xi_x\le u)}{Z^{\square}_{(1,1),(m,n)}} \ge \frac{1}{4}e^{bN^{1/3}}\bigg] &(7.3)\\ &\qquad+ \mathbb{P}\bigg[\frac{Z_{m,n}(0<\xi_y\le u)}{Z^{\square}_{(1,1),(m,n)}} \ge \frac{1}{4}e^{bN^{1/3}}\bigg] &(7.4)\\ &\qquad+ \mathbb{P}\bigg[Q_{m,n}(\{0<\xi_x\le u\}\cup\{0<\xi_y\le u\}) \le \frac{1}{2}\bigg]. &(7.5)\end{aligned}$$

By part (ii) of Lemma 5.4, (7.3)+(7.4) is bounded by $Cb^{-3/2}$. By Lemma 4.3

$$\text{line (7.5)} \le \mathbb{P}\big[Q_{m,n}\{\xi_x > u\} > \tfrac{1}{4}\big] + \mathbb{P}\big[Q_{m,n}\{\xi_y > u\} > \tfrac{1}{4}\big] \le Cb^{-3/2}$$

provided $e^{-\delta b(\Psi_1(\theta)/t)^{1/3}M^{1/3}} \le 1/4$ and $u \ge c\kappa_M$. M is now the scaling parameter and comparison of (4.9) and (2.21) shows $\kappa_M = \gamma N^{2/3}$. The requirements are satisfied with $N \geq N_0$ and $b \geq C_0$.

To summarize, we have for $b \geq C_0$ and $N \geq N_0$, and for a finite constant C,

$$\mathbb{P}\left[\frac{Z_{m,n}}{Z^{\square}_{(1,1),(m,n)}} \geq e^{bN^{1/3}}\right] \leq Cb^{-3/2}. \tag{7.6}$$

This furnishes the remaining part of the conclusion. $\square$

PROOF OF THEOREM 2.4. By Chebyshev, variance bound (4.33) and Lemma 7.1, and with a little correction to take care of the difference between $Z_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}$ and $Z^{\square}_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)}$,

$$\begin{aligned}
&\mathbb{P}\big[\big|\log Z_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} - Nf_{s,t}(\mu)\big| \geq bN^{1/3}\big] \\
&\quad\leq \mathbb{P}\big(|\log Y_{1,1}| \geq \tfrac{1}{4}bN^{1/3}\big) \\
&\qquad + \mathbb{P}\big[\big|\log Z^{\square}_{(1,1),(\lfloor Ns\rfloor,\lfloor Nt\rfloor)} - \log Z_{\lfloor Ns\rfloor,\lfloor Nt\rfloor}\big| \geq \tfrac{1}{2}bN^{1/3}\big] \\
&\qquad + \mathbb{P}\big[\big|\log Z_{\lfloor Ns\rfloor,\lfloor Nt\rfloor} - Nf_{s,t}(\mu)\big| \geq \tfrac{1}{4}bN^{1/3}\big] \\
&\quad\leq Ce^{-1/4bN^{1/3}} + Cb^{-3/2} + Cb^{-2} \leq Cb^{-3/2}.
\end{aligned}$$

This bound implies convergence in probability in (2.17). One can apply the subadditive ergodic theorem to upgrade the statement to a.s. convergence. We omit the details. $\square$

PROOF OF THEOREM 2.5. Let $(k,\ell) = (\lfloor\tau m\rfloor, \lfloor\tau n\rfloor)$ and $u = bN^{2/3} = b(\Psi_1(\theta)/t)^{2/3}M^{2/3}$. Decompose the event $\{v_1(\ell) \geq k+u\}$ according to the vertical edge $\{(i,\ell),(i,\ell+1)\}$, $k+u \leq i \leq m$, taken by the path, and utilize (7.2):

$$\begin{aligned}
&Q_{(1,1),(m,n)}\{v_1(\ell) \geq k+u\} \\
&\quad= \sum_{i\,:\,k+u\leq i\leq m} \frac{Z^{\square}_{(1,1),(i,\ell)} Z^{\square}_{(i,\ell+1),(m,n)}}{Z^{\square}_{(1,1),(m,n)}} \\
&\quad\leq \sum_{i\,:\,k+u\leq i\leq m} \frac{Z_{i,\ell} Z^{\square}_{(i,\ell+1),(m,n)}}{(U_{1,0}+V_{0,1})Z^{\square}_{(1,1),(m,n)}} \\
&\quad= \frac{Q_{m,n}\{v_1(\ell) \geq k+u\}}{U_{1,0}+V_{0,1}} \cdot \frac{Z_{m,n}}{Z^{\square}_{(1,1),(m,n)}}.
\end{aligned}$$

As explained in the paragraph of (6.5) above,

$$Q_{m,n}\{v_1(\ell) \geq k+u\} \stackrel{d}{=} Q_{m-k,n-\ell}\{\xi_x \geq u\}.$$

Let $b^{-3} < h < 1$. From above, remembering (7.1),

$$\begin{aligned}
&\mathbb{P}\big[Q_{(1,1),(m,n)}\{v_1(\ell) \ge k+u\} > h\big] \\
&\quad \le \mathbb{P}(U_{1,0} + V_{0,1} \le b^{-3}) \\
&\qquad + \mathbb{P}\biggl[\frac{Z_{m,n}}{Z^{\square}_{(1,1),(m,n)}} \ge \exp\biggl(\frac{\delta b^2 \Psi_1(\theta) N^{1/3}}{2(1-\tau)t}\biggr)\biggr] \\
&\qquad + \mathbb{P}\biggl[Q_{m-k,n-\ell}\{\xi_x \ge u\} > hb^{-3}\exp\biggl(-\frac{1}{2}\delta u^2\Big/(1-\tau)M\biggr)\biggr] \\
&\quad \le Cb^{-3}.
\end{aligned}$$

The justification for the last inequality is as follows. With a new scaling parameter $(1-\tau)M$, bound (4.34) applies to the last probability above and bounds it by Cb^{-3} for all $h > b^{-3}$ and $b \ge 1$, provided $N \ge N_0$. Apply (7.6) to the second last probability, valid if $b \ge C_0$ and $N \ge N_0$. We obtain

$$\begin{aligned}
&P_{(1,1),(m,n)}\{v_1(\ell) \ge k+u\} \\
&\quad \le b^{-3} + \int_{b^{-3}}^{1} \mathbb{P}\big[Q_{(1,1),(m,n)}\{v_1(\ell) \ge k+u\} > h\big]\,dh \\
&\quad \le Cb^{-3}.
\end{aligned}$$

The corresponding bound from below on $v_0(\ell)$ comes by reversal. If $\widetilde{Y}_{i,j} = Y_{m-i+1,n-j+1}$ for $(i,j) \in \Lambda_{(1,1),(m,n)}$, then $Q^{\widetilde{\omega}}_{(1,1),(m,n)}(x_\centerdot) = Q^{\omega}_{(1,1),(m,n)}(\widetilde{x}_\centerdot)$ where $\widetilde{x}_j = (m+1,n+1) - x_{m+n-2-j}$ for $0 \le j \le m+n-2$. This mapping of paths has the property $v_0(\ell, x_\centerdot) - k = m+1-k-v_1(n+1-\ell, \widetilde{x}_\centerdot)$, and it converts an upper bound on v_1 into a lower bound on v_0. $\square$

8. Polymer with free endpoint. In this final section, we prove Theorems 2.6 and 2.7, beginning with the three parts of Theorem 2.6.

PROOF OF LIMIT (2.23). The claimed limit is the maximum over directions in the first quadrant:

$$-\Psi_0(\mu/2) = f_{1/2,1/2}(\mu) \ge f_{s,1-s}(\mu) \qquad \text{for } 0 \le s \le 1.$$

One bound for the limit comes from $Z_N^{\text{tot}} \ge Z_{(1,1),(\lfloor N/2\rfloor, N-\lfloor N/2\rfloor)}$. To bound $\log Z_N^{\text{tot}}$ from above, fix $K \in \mathbb{N}$ and let $\delta = 1/K$. For $1 \le k \le K$ set $(s_k, t_k) = (k\delta, (K-k+1)\delta)$. Partition the indices $m \in \{1, \dots, N-1\}$ into sets

$$I_k = \big\{m \in \{1, \dots, N-1\} : (m, N-m) \in \Lambda_{\lfloor Ns_k\rfloor, \lfloor Nt_k\rfloor}\big\}.$$

The $\{I_k\}$ cover the entire set of m's because $N(k-1)\delta \le m \le Nk\delta$ implies $m \in I_k$. Overlap among the I_k's is not harmful:

$$\begin{aligned} Z_N^{\text{tot}} &\le \sum_{k=1}^{K} \sum_{m \in I_k} Z_{(1,1),(m,N-m)} \frac{Z_{(m,N-m),(\lfloor Ns_k \rfloor, \lfloor Nt_k \rfloor)}}{Z_{(m,N-m),(\lfloor Ns_k \rfloor, \lfloor Nt_k \rfloor)}} \\ &\le \Big\{ \min_{1\le k\le K, m\in I_k} Z_{(m,N-m),(\lfloor Ns_k \rfloor, \lfloor Nt_k \rfloor)} \Big\}^{-1} \\ &\quad \times \sum_{k=1}^{K} Z_{(1,1),(\lfloor Ns_k \rfloor, \lfloor Nt_k \rfloor)}. \end{aligned}$$

For each $m \in I_k$, fix a specific path $x_\bullet^{(m)} \in \Pi_{(m,N-m),(\lfloor Ns_k \rfloor, \lfloor Nt_k \rfloor)}$. Since

$$Z_{(m,N-m),(\lfloor Ns_k \rfloor, \lfloor Nt_k \rfloor)} \ge \prod_{i=1}^{\lfloor Ns_k \rfloor + \lfloor Nt_k \rfloor - N} Y_{x_i^{(m)}},$$

we get the bound

$$\begin{aligned} N^{-1} \log Z_N^{\text{tot}} &\le \max_{1\le k\le K, m\in I_k} N^{-1} \sum_i \log Y_{x_i^{(m)}}^{-1} + N^{-1}\log K \\ &\quad + \max_{1\le k\le K} N^{-1} \log Z_{(1,1),(\lfloor Ns_k \rfloor, \lfloor Nt_k \rfloor)}. \end{aligned} \tag{8.1}$$

The sum $\sum_i \log Y_{x_i^{(m)}}^{-1}$ has $\lfloor Ns_k \rfloor + \lfloor Nt_k \rfloor - N \le N\delta$ i.i.d. terms. Given $\varepsilon > 0$, we can choose $\delta = K^{-1}$ small enough to guarantee that $\mathbb{P}\{\sum_i \log Y_{x_i^{(m)}}^{-1} \ge N\varepsilon\}$ decays exponentially with N. Thus, $\mathbb{P}$-a.s. the entire first term after the inequality in (8.1) is $\le \varepsilon$ for large N. In the limit we get, utilizing law of large numbers (2.17),

$$\varlimsup_{N\to\infty} N^{-1} \log Z_N^{\text{tot}} \le \varepsilon + \max_{1\le k\le K} f_{s_k,t_k}(\mu) \le \varepsilon + \sup_{0\le s\le 1} f_{s,1-s+\delta}(\mu).$$

Let $\delta \searrow 0$ utilizing the continuity of $f_{s,t}(\mu)$ in (s,t), and then let $\varepsilon \searrow 0$. This gives $\varlimsup N^{-1} \log Z_N^{\text{tot}} \le -\Psi_0(\mu/2)$ and completes the proof of the limit (2.23). □

PROOF OF BOUND (2.24). Let

$$(m,n) = (N - \lfloor N/2 \rfloor, \lfloor N/2 \rfloor). \tag{8.2}$$

An upper bound on the left tail in (2.24) comes immediately from (2.18):

$$\begin{aligned} &\mathbb{P}\{\log Z_N^{\text{tot}} \le Nf_{1/2,1/2}(\mu) - bN^{1/3}\} \\ &\quad \le \mathbb{P}\{\log Z_{(1,1),(m,n)} \le Nf_{1/2,1/2}(\mu) - bN^{1/3}\} \\ &\quad \le Cb^{-3/2}. \end{aligned}$$

To get a bound on the right tail, start with

$$
\begin{aligned}
Z_N^{\text{tot}} &= \sum_{\ell=1}^{N-1} Z_{(1,1),(\ell,N-\ell)} \\
&\le N\biggl(Z_{(1,1),(m,n)} \cdot \max_{0\le k<n} \frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}}\biggr) \\
&\quad \vee \biggl(Z_{(1,1),(n,m)} \cdot \max_{0\le \ell<m} \frac{Z_{(1,1),(n-\ell,m+\ell)}}{Z_{(1,1),(n,m)}}\biggr).
\end{aligned} \tag{8.3}
$$

The terms in the large parentheses are transposes of each other, so we spell out the details only for the first case. In one spot below it is convenient to have $m \ge n$, hence the choice in (8.2). Thus, considering $b \ge 2$, and once N is large enough so that $\log N < N^{1/3}/3$, bounding

$$\mathbb{P}\{\log Z_N^{\text{tot}} \ge N f_{1/2,1/2}(\mu) + bN^{1/3}\}$$

boils down to bounding the sum

$$\mathbb{P}\biggl\{\log Z_{(1,1),(m,n)} \ge N f_{1/2,1/2}(\mu) + \frac{1}{3}bN^{1/3}\biggr\} \tag{8.4}$$

$$+\,\mathbb{P}\biggl\{\log \max_{0<k<n} \frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}} \ge \frac{1}{3}bN^{1/3}\biggr\}. \tag{8.5}$$

The probability on line (8.4) is again taken care of with (2.18). Utilizing both inequalities in (5.2), the first one transposed, we deduce for $1 \le k < n$,

$$
\begin{aligned}
\frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}} &= \prod_{j=1}^{k} \frac{Z_{(1,1),(m+j,n-j)}}{Z_{(1,1),(m+j-1,n-j)}} \cdot \frac{Z_{(1,1),(m+j-1,n-j)}}{Z_{(1,1),(m+j-1,n-j+1)}} \\
&\le \prod_{j=1}^{k} \frac{Z_{m+j,n-j}(\xi_x > 0)}{Z_{m+j-1,n-j}(\xi_x > 0)} \cdot \frac{Z_{m+j-1,n-j}(\xi_x > 0)}{Z_{m+j-1,n-j+1}(\xi_x > 0)} \\
&= \frac{Z_{m+k,n-k}(\xi_x > 0)}{Z_{m,n}(\xi_x > 0)} \le \frac{1}{Q_{m,n}(\xi_x > 0)} \cdot \frac{Z_{m+k,n-k}}{Z_{m,n}} \\
&= \frac{1}{Q_{m,n}(\xi_x > 0)} \cdot \prod_{j=1}^{k} \frac{U_{m+j,n-j}}{V_{m+j-1,n-j+1}}.
\end{aligned} \tag{8.6}
$$

The last equality used (3.4). In the calculation above, we switched from partition functions $Z_{(1,1),(i,j)}$ that use only bulk weights to partition functions $Z_{i,j} = Z_{(0,0),(i,j)}$ that use both bulk and boundary weights, distributed as in assumption (2.4). The parameter θ is at our disposal. We take $\theta = \mu/2 + rN^{-1/3}$ with $r > 0$ and link r to b in the next lemma. The choice $\theta > \mu/2$ makes the U/V ratios small which is good for bounding the last line of (8.6). However, this choice also makes $Q_{m,n}(\xi_x > 0)$ small which works against us. To bound $Q_{m,n}(\xi_x > 0)$ from below we switch from $\theta = \mu/2 + rN^{-1/3}$ to $\lambda = \mu/2 - rN^{-1/3}$ and pay for this by bounding the Radon–Nikodym derivative. Under parameter λ the event $\{\xi_x > 0\}$ is favored at the expense of $\{\xi_y > 0\}$, and we can get a lower bound.

Utilizing (8.6), the probability in (8.5) is bounded as follows:

$$\mathbb{P}\left\{\log \max_{1\le k\le n} \frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}} \ge \frac{1}{3}bN^{1/3}\right\}$$

$$\le \mathbb{P}\{Q_{m,n}(\xi_x > 0) \le e^{-bN^{1/3}/6}\} \tag{8.7}$$

$$+\,\mathbb{P}\left\{\max_{1\le k\le n} \sum_{j=1}^{k} (\log U_{m+j,n-j} - \log V_{m+j-1,n-j+1}) \ge bN^{1/3}/6\right\}. \tag{8.8}$$

We treat first the right-hand side probability on line (8.7).

LEMMA 8.1. *Let* $0 < \mu < \infty$ *be fixed*, $r > 0$, $b \ge 1$, $\theta = \mu/2 + rN^{-1/3}$, *weight distributions as in* (2.4) *and* (m,n) *as in* (8.2). *Then there exist finite constants* $\kappa(\mu)$, $C(\mu)$ *and* $N_0(\mu,b)$ *such that the following holds*: *if* $r = \kappa(\mu)b^{1/2}$ *and* $N \ge N_0(\mu,b)$ *then*

$$\mathbb{P}\{Q_{m,n}(\xi_x > 0) \le e^{-bN^{1/3}/6}\} \le C(\mu)b^{-3/2}. \tag{8.9}$$

PROOF. Let $U_{i,0}, V_{0,j}$ be the boundary weights with parameter $\theta = \mu/2 + rN^{-1/3}$ as specified in (2.4). Let $\widetilde{U}_{i,0}, \widetilde{V}_{0,j}$ denote boundary weights with parameter $\lambda = \mu/2 - rN^{-1/3}$ in place of θ. We ensure $\mu/4 \le \lambda < \theta \le 3\mu/4$ by considering only $N \ge N_1(\mu,r)$ for $N_1(\mu,r) = (4r/\mu)^3$. All along bulk weights have distribution $Y_{i,j}^{-1} \sim \text{Gamma}(\mu,1)$. The coupling of the boundary weights $\{U_{i,0}, V_{0,j}\}$ with $\{\widetilde{U}_{i,0}, \widetilde{V}_{0,j}\}$ is such that $U_{i,0} \le \widetilde{U}_{i,0}$. Tildes mark quantities that use $\widetilde{U}_{i,0}, \widetilde{V}_{0,j}$. Let $u = \lfloor tN^{2/3}\rfloor$ with t determined later. Recall that Ψ_0 is strictly increasing and Ψ_1 strictly decreasing,

$$Q_{m,n}(\xi_x > 0) \ge Q_{m,n}(0 < \xi_x \le u) = \frac{1}{Z_{m,n}} \sum_{k=1}^{u} \left(\prod_{i=1}^{k} U_{i,0}\right) Z^{\square}_{(k,1),(m,n)}$$

$$= \frac{1}{\widetilde{Z}_{m,n}} \sum_{k=1}^{u} \left(\prod_{i=1}^{k} \widetilde{U}_{i,0} \cdot \prod_{i=1}^{k} \frac{U_{i,0}}{\widetilde{U}_{i,0}}\right) Z^{\square}_{(k,1),(m,n)} \cdot \frac{\widetilde{Z}_{m,n}}{Z_{m,n}} \tag{8.10}$$

$$\ge \widetilde{Q}_{m,n}(0 < \xi_x \le u) \left(\prod_{i=1}^{u} \frac{U_{i,0}}{\widetilde{U}_{i,0}}\right) \frac{\widetilde{Z}_{m,n}}{Z_{m,n}}.$$

We derive tail bounds for each of the three factors on line (8.10), working our way from right to left. $C(\mu)$ denotes a constant that depends on μ and can change from one line to the next, while $C_i(\mu)$ denote constants specific to the cases.

Since $\theta > \lambda$ sit symmetrically around $\mu/2$ and $m \geq n$,

$$\mathbb{E}(\log \widetilde{Z}_{m,n}) - \mathbb{E}(\log Z_{m,n}) \\ = m\big(-\Psi_0(\lambda) + \Psi_0(\theta)\big) + n\big(-\Psi_0(\mu-\lambda) + \Psi_0(\mu-\theta)\big) \geq 0$$

and in fact vanishes for even N. By Chebyshev and the variance bound of Theorem 2.1,

$$\mathbb{P}\bigg[\frac{\widetilde{Z}_{m,n}}{Z_{m,n}} \leq e^{-bN^{1/3}/18}\bigg] = \mathbb{P}[\overline{\log \widetilde{Z}_{m,n}} - \overline{\log Z_{m,n}} \leq -bN^{1/3}/18]$$

$$\leq \frac{18^2}{N^{2/3}b^2}\big(\mathbb{V}\mathrm{ar}(\log \widetilde{Z}_{m,n}) + \mathbb{V}\mathrm{ar}(\log Z_{m,n})\big) \tag{8.11}$$

$$\leq C(\mu)(1+r)b^{-2}.$$

To understand the last inequality above for the first variance, let first a scaling parameter M be determined by $n = M\Psi_1(\lambda)$. Set $\bar{m} = \lfloor M\Psi_1(\mu-\lambda)\rfloor$ which satisfies $m - C_1(\mu)rN^{2/3} \leq \bar{m} < m$. Since $(\bar{m}, n)$ is the characteristic direction for λ,

$$\mathbb{V}\mathrm{ar}(\log \widetilde{Z}_{m,n}) = \mathbb{V}\mathrm{ar}\Bigg(\log \widetilde{Z}_{\bar{m},n} + \sum_{i=\bar{m}+1}^{m} \log \widetilde{U}_{i,n}\Bigg)$$

$$\leq 2\,\mathbb{V}\mathrm{ar}(\log \widetilde{Z}_{\bar{m},n}) + 2\,\mathbb{V}\mathrm{ar}\Bigg(\sum_{i=\bar{m}+1}^{m} \log \widetilde{U}_{i,n}\Bigg)$$

$$\leq C(\mu)(M^{2/3} + m - \bar{m}) \leq C(\mu)(1+r)N^{2/3}.$$

We used above the variance bound of Theorem 2.1 together with the feature that fixed constants work for parameters varying in a compact set. This is now valid because we have constrained λ and θ to lie in $[\mu/4, 3\mu/4]$. A similar argument works for the second variance in (8.11).

Next,

$$\mathbb{E}(\log U_{1,0} - \log \widetilde{U}_{1,0}) = -\Psi_0(\theta) + \Psi_0(\lambda) \geq -C_2(\mu)rN^{-1/3}.$$

By Chebyshev, provided we ensure $b > 36C_2(\mu)rt$,

$$\mathbb{P}\Bigg[\prod_{i=1}^{u}\frac{U_{i,0}}{\widetilde{U}_{i,0}} \leq e^{-bN^{1/3}/18}\Bigg]$$

$$= \mathbb{P}\Bigg[\sum_{i=1}^{u}(\overline{\log U_{i,0}} - \overline{\log \widetilde{U}_{i,0}}) \leq -\bigg(\frac{1}{18}b - C_2(\mu)rt\bigg)N^{1/3}\Bigg] \tag{8.12}$$

$$\leq C(\mu)tb^{-2}.$$

For the probability on line (8.10), write

$$\widetilde{Q}_{m,n}\{0 < \xi_x \leq tN^{2/3}\} = 1 - \widetilde{Q}_{m,n}\{\xi_x > tN^{2/3}\} - \widetilde{Q}_{m,n}\{\xi_y > 0\}. \tag{8.13}$$

To both probabilities on the right, we apply Lemma 4.3 after adjusting the parameters. Let M and $\bar{m}$ be as above so that $(\bar{m}, n)$ is the characteristic direction for λ. Reasoning as for the distributional equality in (5.6) and picking $t \geq 2C_1(\mu)r$,

$$\begin{aligned}\widetilde{Q}_{m,n}\{\xi_x > tN^{2/3}\} &\stackrel{d}{=} \widetilde{Q}_{\bar{m},n}\{\xi_x > tN^{2/3} - (m - \bar{m})\}\\ &\leq \widetilde{Q}_{\bar{m},n}\{\xi_x > tN^{2/3}/2\}.\end{aligned}$$

Consequently by (4.34),

$$\mathbb{P}\big[\widetilde{Q}_{\bar{m},n}\{\xi_x > tN^{2/3}/2\} \geq e^{-\delta t^2 N^{4/3}/(4M)}\big] \leq C(\mu)t^{-3}.$$

For the last probability on line (8.13), we get the same kind of bound by defining K through $m = K\Psi_1(\mu - \lambda)$, and $\bar{n} = \lfloor K\Psi_1(\lambda)\rfloor \geq n + C_4(\mu)rN^{2/3}$. Then

$$\widetilde{Q}_{m,n}\{\xi_y > 0\} \stackrel{d}{=} \widetilde{Q}_{m,\bar{n}}\{\xi_y > \bar{n} - n\} \leq \widetilde{Q}_{m,\bar{n}}\{\xi_y > C_4(\mu)rN^{2/3}\},$$

and again by (4.34)

$$\mathbb{P}\big[\widetilde{Q}_{m,\bar{n}}\{\xi_y > C_4(\mu)rN^{2/3}\} \geq e^{-\delta C_4(\mu)^2 r^2 N^{4/3}/K}\big] \leq C(\mu)r^{-3}.$$

The upshot of this paragraph is that if $N \geq N_1(\mu, r)$ and we pick $t = 2C_3(\mu)r$,

$$\mathbb{P}\big[\widetilde{Q}_{m,n}\{0 < \xi_x \leq u\} \leq \tfrac{1}{2}\big] \leq C(\mu)(t^{-3} + r^{-3}) \leq C(\mu)r^{-3}. \tag{8.14}$$

Put bounds (8.11), (8.12) and (8.14) back into (8.10). Choose $t = 2C_3(\mu)r$ as in the last paragraph. We can ensure that $b \geq 36C_2(\mu)rt$ needed for (8.12) by choosing $b = \kappa(\mu)^{-2}r^2$ for a small enough $\kappa(\mu)$. The constraint $N \geq N_1(\mu, r)$ can then be written in the form $N \geq N_0(\mu, b)$. Adding up the bounds gives

$$\begin{aligned}\mathbb{P}[Q_{m,n}\{\xi_x > 0\} \leq e^{-bN^{1/3}/6}] &\leq C(\mu)\big((1 + r)b^{-2} + tb^{-2} + r^{-3}\big)\\ &\leq C(\mu)b^{-3/2}.\end{aligned}$$

□

We turn to probability (8.8). By the Burke property Theorem 3.3 inside the probability, we have a sum of i.i.d. terms with mean

$$\begin{aligned}\mathbb{E}(\log U_{m+1,n-1} - \log V_{m,n}) &= -\Psi_0(\theta) + \Psi_0(\mu - \theta)\\ &\leq -C_5(\mu)rN^{-1/3}.\end{aligned} \tag{8.15}$$

Consequently, if we let

$$\eta_j = \log U_{m+j,n-j} - \log V_{m+j-1,n-j+1} + \Psi_0(\theta) - \Psi_0(\mu - \theta), \tag{8.16}$$

then

$$(8.8) \leq \mathbb{P}\left\{\max_{1\leq k\leq n}\sum_{j=1}^{k}(\eta_j - C_5(\mu)rN^{-1/3}) \geq bN^{1/3}/6\right\}. \tag{8.17}$$

The variables η_j have all moments. Apply part (a) of Lemma 8.2 below to the probability above with $t = N^{1/3}$, $\alpha = C_5(\mu)r$ and $\beta = b/6$. With $r = \kappa(\mu)b^{1/2}$ and p large enough, this gives

$$(8.8) \le C(\mu)b^{-3/2}. \tag{8.18}$$

Insert bounds (8.9) and (8.18) into (8.7), (8.8), and this in turn back into (8.5). This completes the proof of (2.24). □

Before the third and last part of the proof of Theorem 2.6, we state and prove the random walk lemma used to derive (8.18) above. It includes a part (b) for subsequent use.

LEMMA 8.2. *Let $Z, Z_1, Z_2, \dots$ be i.i.d. random variables that satisfy $\mathbf{E}(Z) = 0$ and $\mathbf{E}(|Z|^p) < \infty$ for some $p > 2$. Set $S_k = Z_1 + \cdots + Z_k$. Below $C = C(p)$ is a constant that depends only on p.*

(a) *For all $\alpha, \beta, t > 0$,*

$$\mathbf{P}\Big\{\sup_{k\ge 0}(S_k - k\alpha t^{-1}) \ge \beta t\Big\} \le C\mathbf{E}(|Z|^p)\alpha^{-p^2/(2(p-1))}\beta^{-p(p-2)/(2(p-1))}.$$

(b) *For all $\alpha, \beta, t > 0$ and $M \in \mathbb{N}$ such that $2\beta \le M\alpha$,*

$$\mathbf{P}\Big\{\sup_{k> Mt^2}(S_k - k\alpha t^{-1}) \ge -\beta t\Big\} \le C\mathbf{E}(|Z|^p)\alpha^{-p}M^{-(p/2)+1}.$$

PROOF. Part (a). Pick an integer $m > 0$ and split the probability:

$$\begin{aligned}
&\mathbf{P}\Big\{\sup_{k\ge 0}(S_k - k\alpha t^{-1}) \ge \beta t\Big\}\\
&\quad\le \mathbf{P}\Big\{\max_{0<k\le mt^2} S_k \ge \beta t\Big\} \\
&\qquad + \sum_{j\ge m}\mathbf{P}\Big\{\max_{jt^2<k\le(j+1)t^2}(S_k - k\alpha t^{-1}) \ge \beta t\Big\}.
\end{aligned}\tag{8.19}$$

Recall that the Burkholder–Davis–Gundy inequality ([11], Theorem 3.2) gives $\mathbf{E}|S_k|^p \le C_p\mathbf{E}|Z|^p k^{p/2}$. Doob's inequality together with BDG gives

$$\mathbf{P}\Big\{\max_{0<k\le mt^2} S_k \ge \beta t\Big\} \le C\mathbf{E}|Z|^p m^{p/2}\beta^{-p},$$

where we now write C for a constant that depends only on p. For the last probability in (8.19),

$$\begin{aligned}
\mathbf{P}\Big\{\max_{jt^2<k\le(j+1)t^2}(S_k - k\alpha t^{-1}) \ge \beta t\Big\} &\le \mathbf{P}\Big\{\max_{0<k\le(j+1)t^2} S_k \ge j\alpha t\Big\}\\
&\le C\mathbf{E}|Z|^p j^{-p/2}\alpha^{-p}.
\end{aligned}$$

Putting the bounds back into (8.19) gives

$$\mathbf{P}\Big\{\sup_{k\geq 0}(S_k - k\alpha t^{-1}) \geq \beta t\Big\} \leq C\mathbf{E}|Z|^p\bigg(\frac{m^{p/2}}{\beta^p} + \alpha^{-p}\sum_{j\geq m} j^{-p/2}\bigg)$$
$$\leq C\mathbf{E}|Z|^p\big(m^{p/2}\beta^{-p} + \alpha^{-p}m^{-(p/2)+1}\big).$$

Choosing m a constant multiple of $(\beta/\alpha)^{p/(p-1)}$ gives the conclusion for part (a).

Part (b). Proceeding as above:

$$\begin{aligned}
&\mathbf{P}\Big\{\sup_{k>Mt^2}(S_k - k\alpha t^{-1}) \geq -\beta t\Big\}\\
&\quad\leq \sum_{j\geq M}\mathbf{P}\Big\{\max_{jt^2<k\leq (j+1)t^2}(S_k - k\alpha t^{-1}) \geq -\beta t\Big\}\\
&\quad\leq \sum_{j\geq M}\mathbf{P}\Big\{\max_{0<k\leq (j+1)t^2} S_k \geq \frac{1}{2}j\alpha t\Big\} \leq C\mathbf{E}|Z|^p\alpha^{-p}\sum_{j\geq M} j^{-p/2}\\
&\quad\leq C\mathbf{E}(|Z|^p)\alpha^{-p}M^{-(p/2)+1}.
\end{aligned}$$

□

Next, the last part of the proof of Theorem 2.6.

PROOF OF BOUND (2.25). We shall show the existence of constants $c_0(\mu) > 0$ and $C(\mu)$, $N_0(\mu, s) < \infty$ such that, for $s \geq 1$ and $N \geq N_0(\mu, s)$,

$$\mathbb{P}\bigg[Q_N^{\text{tot}}\bigg\{\bigg|x_{N-2} - \bigg(\frac{N}{2}, \frac{N}{2}\bigg)\bigg| \geq 2sN^{2/3}\bigg\} \geq e^{-c_0(\mu)s^2N^{1/3}}\bigg] \leq C(\mu)s^{-3}. \tag{8.20}$$

Abbreviating $A_N = \{|x_{N-2} - (\frac{N}{2}, \frac{N}{2})| \geq 2sN^{2/3}\}$, then (2.25) follows from

$$\begin{aligned}
P_N^{\text{tot}}(A_N) &= \mathbb{E}Q_N^{\text{tot}}(A_N)\\
&\leq e^{-c_0(\mu)s^2N^{1/3}} + \mathbb{P}\big[Q_N^{\text{tot}}(A_N) \geq e^{-c_0(\mu)s^2N^{1/3}}\big]\\
&\leq C(\mu)s^{-3}.
\end{aligned}$$

To show (8.20), we control sums of ratios of partition functions:

$$\begin{aligned}
&Q_N^{\text{tot}}\bigg\{\bigg|x_{N-2} - \bigg(\frac{N}{2}, \frac{N}{2}\bigg)\bigg| \geq 2sN^{2/3}\bigg\}\\
&\quad\leq \sum_{0<\ell<N/2-sN^{2/3}} \frac{Z_{(1,1),(\ell,N-\ell)}}{Z_N^{\text{tot}}}\\
&\qquad + \sum_{N/2+sN^{2/3}<\ell<N} \frac{Z_{(1,1),(\ell,N-\ell)}}{Z_N^{\text{tot}}}.
\end{aligned}$$

We treat the second sum from above. The first one develops the same way. With (m,n) as in (8.2) and utilizing (8.6), write

$$\begin{aligned}
&\sum_{N/2+sN^{2/3}<\ell<N} \frac{Z_{(1,1),(\ell,N-\ell)}}{Z_N^{\text{tot}}} \\
&\quad\le \sum_{sN^{2/3}\le k<N/2} \frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}} \\
&\quad\le \frac{1}{Q_{m,n}(\xi_x>0)} \sum_{sN^{2/3}\le k<N/2} \prod_{j=1}^{k} \frac{U_{m+j,n-j}}{V_{m+j-1,n-j+1}} \\
&\quad\le \frac{N}{Q_{m,n}(\xi_x>0)} \cdot \max_{sN^{2/3}\le k<N/2} \prod_{j=1}^{k} \frac{U_{m+j,n-j}}{V_{m+j-1,n-j+1}}.
\end{aligned}$$

As in (8.2), we introduced again boundary weights with parameter $\theta=\mu/2+rN^{-1/3}$. Let $c_0=c_0(\mu)$ be a small constant whose value will be determined below. Consider N large enough so that $N\le e^{c_0N^{1/3}}$ and take $s\ge 1$. Define η_j as in (8.16) and let $C_5(\mu)$ be as in (8.15). Then

$$\mathbb{P}\biggl[\sum_{N/2+sN^{2/3}<\ell<N} \frac{Z_{(1,1),(\ell,N-\ell)}}{Z_N^{\text{tot}}} \ge e^{-c_0s^2N^{1/3}}\biggr]$$

$$\le \mathbb{P}[Q_{m,n}(\xi_x>0)\le e^{-c_0s^2N^{1/3}}] \tag{8.21}$$

$$+\,\mathbb{P}\biggl[\max_{sN^{2/3}\le k\le N/2} \sum_{j=1}^{k}(\eta_j - C_5(\mu)rN^{-1/3}) \ge -3c_0s^2N^{1/3}\biggr] \tag{8.22}$$

$$\le C(\mu)s^{-3}.$$

The justification for the last inequality is in the previous lemmas. Apply Lemma 8.1 with $b=6c_0s^2$ to the probability on line (8.21) to bound it by $C(\mu)s^{-3}$. For this purpose, set $r=\kappa(\mu)b^{1/2}=\kappa(\mu)s\sqrt{6c_0}$. Then apply Lemma 8.2(b) to the probability on line (8.22) to bound it also by $C(\mu)s^{-3}$. The condition $2\beta\le M\alpha$ of that lemma is equivalent to $\sqrt{6c_0}\le C_5(\mu)\kappa(\mu)$, and we can fix c_0 small enough to satisfy this. This completes the proof of (8.20) and thereby the proof of Theorem 2.6. □

PROOF OF THEOREM 2.7. *Case* 1: $\theta\ne\mu/2$. We do the subcase $0<\theta<\mu/2$. By (3.4),

$$\log Z_N^{\text{tot}}(\theta,\mu) = \log Z_{N,0} + \log\Biggl(1+\sum_{k=1}^{N}\prod_{i=1}^{k}\frac{V_{N-i+1,i}}{U_{N-i+1,i}}\Biggr). \tag{8.23}$$

Since

$$\mathbb{E}(\log V_{N-i+1,i} - \log U_{N-i+1,i}) = -\Psi_0(\mu-\theta) + \Psi_0(\theta) < 0$$

the random variable

$$\log\left(1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{V_{N-i+1,i}}{U_{N-i+1,i}}\right)$$

is positive and finite. Since $\log Z_{N,0}$ is a sum of i.i.d. variables $\log U_{i,0}$ with $U_{i,0}^{-1} \sim \text{Gamma}(\theta, 1)$, the conclusions follow for the case $0 < \theta < \mu/2$.

Case 2: $\theta = \mu/2$. Let $(m,n) = (N - \lfloor N/2 \rfloor, \lfloor N/2 \rfloor)$. Separate the partition function in the characteristic direction and use (3.4):

$$\begin{aligned} &\log Z_N^{\text{tot}}(\mu/2, \mu) \\ &\quad = \log Z_{m,n} + \log\left(\sum_{k=0}^{m} \prod_{i=1}^{k} \frac{V_{m-i+1,n+i}}{U_{m-i+1,n+i}} + \sum_{k=1}^{n} \prod_{i=1}^{k} \frac{U_{m+i,n-i+1}}{V_{m+i,n-i+1}}\right). \end{aligned} \tag{8.24}$$

By the Burke property, the mean zero random variables $\eta_i = \log U_{m+i,n-i+1} - \log V_{m+i,n-i+1}$ for $i \in \mathbb{Z}$ are i.i.d. For $k \geq 1$, define sums

$$S_k = \sum_{i=1}^{k} \eta_i, \qquad S_0 = 0 \quad \text{and} \quad S_{-k} = -\sum_{i=1}^{k} \eta_{-i+1}.$$

At $\theta = \mu/2$, $\mathbb{E}(\log Z_{m,n}) = Ng(\mu/2, \mu)$. Consequently, (8.24) gives

$$\begin{aligned} &\log Z_N^{\text{tot}}(\mu/2, \mu) - Ng(\mu/2, \mu) \\ &\quad = \overline{\log Z_{m,n}} + O(\log N) + \max_{-m \leq k \leq n} S_k. \end{aligned} \tag{8.25}$$

By the usual strong law of large numbers, $N^{-1} \max_{-m \leq k \leq n} S_k \to 0$ a.s. and so together with (2.7), (8.25) gives the law of large numbers (2.27) in the case $\theta = \mu/2$. Second, since $\overline{\log Z_{m,n}}$ is stochastically of order $O(N^{1/3})$ by Theorem 2.1 and since $N^{-1/2} \max_{-m \leq k \leq n} S_k$ converges weakly to $\zeta(\mu/2, \mu)$ defined in (2.26), (8.25) implies also the weak limit (2.28). □

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