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DE FINETTI THEOREMS FOR EASY QUANTUM GROUPS

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We study sequences of noncommutative random variables which are invariant under "quantum transformations" coming from an orthogonal quantum group satisfying the "easiness" condition axiomatized in our previous paper. For 10 easy quantum groups, we obtain de Finetti type theorems characterizing the joint distribution of any infinite quantum invariant sequence. In particular, we give a new and unified proof of the classical results of de Finetti and Freedman for the easy groups S_n, O_n, which is based on the combinatorial theory of cumulants. We also recover the free de Finetti theorem of Köstler and Speicher, and the characterization of operator-valued free semicircular families due to Curran. We consider also finite sequences, and prove an approximation result in the spirit of Diaconis and Freedman.

Introduction. In the study of probabilistic symmetries, the classical groups S_n and O_n play central roles. De Finetti's fundamental theorem states that an infinite sequence of random variables whose joint distribution is invariant under finite permutations must be conditionally independent and identically distributed. In [20], Freedman considered sequences of real-valued random variables whose joint distribution is invariant under orthogonal transformations, and proved that any infinite sequence with this property must form a conditionally independent Gaussian family with mean zero and common variance. Although these results fail for finite sequences, approximation results may still be obtained (see [17, 18]). For a thorough treatment of probabilistic symmetries, the reader is referred to the recent text of Kallenberg [23].

The free analogues S_n^+ and O_n^+ of the permutation and orthogonal groups were constructed by Wang in [31, 32]. These are compact quantum groups in the sense of Woronowicz [34]. In [24], Köstler and Speicher discovered that de Finetti's theorem has a natural free analogue: an infinite sequence of noncommutative random variables has a joint distribution which is invariant under "quantum permutations" coming from S_n^+ if and only if the variables are freely independent and identically distributed with amalgamation, that is, with respect to a conditional expectation. This was further studied in [13], where this result was extended to more general

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sequences and an approximation result was given for finite sequences. The free analogue of Freedman's result was obtained in [14], where it was shown that an infinite sequence of self-adjoint noncommutative random variables has a joint distribution which is invariant under "quantum orthogonal transformations" if and only if the variables form an operator-valued free semicircular family with mean zero and common variance.

In this paper, we present a unified approach to de Finetti theorems by using the "easiness" formalism from [7]. Stated roughly, a quantum group $S_n \subset G \subset O_n^+$ is called *easy* if its tensor category is spanned by certain partitions coming from the tensor category of S_n. This might look, of course, to be a quite technical condition. However, we feel that this provides a good framework for understanding certain probabilistic and representation theory aspects of orthogonal quantum groups. There are 14 natural examples of easy quantum groups, listed as follows:

(1) Groups: $O_n, S_n, H_n, B_n, S_n', B_n'$.
(2) Free versions: $O_n^+, S_n^+, H_n^+, B_n^+, S_n'^+, B_n'^+$.
(3) Half-liberations: O_n^*, H_n^*.

Except for H_n^*, which was found in [5], these are all described in [7]. The four "primed" versions above are rather trivial modifications of their "unprimed" versions, corresponding to taking a product with a copy of $\mathbb{Z}_2$. We will focus then on the remaining ten examples in this paper, from which similar results for the "primed" versions may be easily deduced.

As explained in [5, 7], our motivating belief is that "any result which holds for S_n, O_n should have a suitable extension to all easy quantum groups." This is, of course, a quite vague statement, whose target is formed by several results at the borderline of representation theory and probability. This paper represents the first application of this philosophy.

If G is an easy quantum group, there is a natural notion of G-invariance for a sequence of noncommutative random variables, which agrees with the usual definition when G is a classical group. Our main result is the following de Finetti type theorem, which characterizes the joint distributions of infinite G-invariant sequences for the 10 natural easy quantum groups discussed above.

THEOREM 1. *Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of self-adjoint random variables in a W^*-probability space (M, φ), and suppose that the sequence is G-invariant, where G is one of $O, S, H, B, O^*, H^*, O^+, S^+, H^+, B^+$. Assume that M is generated as a von Neumann algebra by $\{x_i : i \in \mathbb{N}\}$. Then there is a W^*-subalgebra $1 \subset \mathcal{B} \subset M$ and a φ-preserving conditional expectation $E : M \to \mathcal{B}$ such that the following hold*:

(1) *Free case*:

(a) *If $G = S^+$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$.*

(b) *If $G = H^+$, then $(x_i)_{i\in\mathbb{N}}$ are freely independent, and have even and identical distributions, with amalgamation over $\mathcal{B}$.*

(c) *If $G = O^+$, then $(x_i)_{i\in\mathbb{N}}$ form a $\mathcal{B}$-valued free semicircular family with mean zero and common variance.*

(d) *If $G = B^+$, then $(x_i)_{i\in\mathbb{N}}$ form a $\mathcal{B}$-valued free semicircular family with common mean and variance.*

(2) *Half-liberated case: Suppose that $x_i x_j x_k = x_k x_j x_i$ for any $i, j, k \in \mathbb{N}$.*

(a) *If $G = H^*$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally half-independent and identically distributed given $\mathcal{B}$.*

(b) *If $G = O^*$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally half-independent, and have symmetrized Rayleigh distributions with common variance, given $\mathcal{B}$.*

(3) *Classical case: Suppose that $(x_i)_{i\in\mathbb{N}}$ commute.*

(a) *If $G = S$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent and identically distributed given $\mathcal{B}$.*

(b) *If $G = H$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent, and have even and identical distributions, given $\mathcal{B}$.*

(c) *If $G = O$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent, and have Gaussian distributions with mean zero and common variance, given $\mathcal{B}$.*

(d) *If $G = B$, then $(x_i)_{i\in\mathbb{N}}$ are conditionally independent, and have Gaussian distributions with common mean and variance, given $\mathcal{B}$.*

The notion of half-independence, appearing in (2) above, will be introduced in Section 2. The basic example of a half-independent family of noncommutative random variables is $(x_i)_{i\in I}$,

$$x_i = \begin{pmatrix} 0 & \xi_i \\ \overline{\xi_i} & 0 \end{pmatrix},$$

where $(\xi_i)_{i\in I}$ are independent, complex-valued random variables and $\mathbb{E}[\xi_i^n \overline{\xi_i}^m] = 0$ unless $n = m$ (see Example 2.4 and Proposition 2.8). Note that in particular, if $(\xi_i)_{i\in\mathbb{N}}$ are independent and identically distributed complex Gaussian random variables, then x_i has a symmetrized Rayleigh distribution $(\xi_i\overline{\xi_i})^{1/2}$ and we obtain the joint distribution in (2) corresponding to the half-liberated orthogonal group O^*. Since the complex Gaussian distribution is known to be characterized by unitary invariance, this appears to be closely related to the connection between U_n and O_n^* observed in [5, 8].

Let us briefly outline the proof of Theorem 1, to be presented in Section 5. We define von Neumann subalgebras $\mathcal{B}_n \subset M$ consisting of "functions" of the variables $(x_i)_{i\in\mathbb{N}}$ which are invariant under "quantum transformations" of $x_1, \dots, x_n$ coming from the quantum group G_n. The *G-invariant* subalgebra $\mathcal{B}$ is defined as the intersection of the nested sequence $\mathcal{B}_n$ (note that if $G = S$, then $\mathcal{B}$ corresponds to the classical *exchangeable subalgebra*). There are natural conditional expectations onto $\mathcal{B}_n$ given by "averaging" over G_n, that is, integrating with respect to the *Haar state* on the compact quantum group G_n. By using an explicit formula for the Haar states on easy quantum groups from [7], and a noncommutative reversed martingale convergence argument, we obtain a simple combinatorial formula for computing joint moments with respect to the conditional expectation onto the G-invariant subalgebra. What emerges from these computations is a *moment-cumulant formula*, and Theorem 1 follows from the characterizations of these joint distributions by the structure of their cumulants. Note that, in particular, we obtain a new proof of de Finetti's classical result for S_n which is based on cumulants. This method also allows us to give certain approximation results for finite sequences, which will be explained in Section 4.

The paper is organized as follows. Section 1 contains preliminaries. Here we collect the basic notions from the combinatorial theory of classical and free probability. We also recall some basic notions and results from [7] about the class of "easy" quantum groups. In Section 2, we introduce half-independence and develop its basic combinatorial theory. In Section 3, we recall the Weingarten formula from [7] for computing integrals on easy quantum groups, and give a new estimate on the asymptotic behavior of these integrals. This will be essential to the proofs of our main results, and we believe that this estimate will also find applications to other problems involving easy quantum groups. In Section 4, we define quantum invariance for finite sequences, prove a converse to Theorem 1, and give approximate de Finetti type results. Section 5 contains the proof of Theorem 1, and a discussion of the situation for unbounded random variables in the classical and half-liberated cases. Section 6 contains concluding remarks.

1. Background and notation.

Noncommutative probability. We begin by recalling the basic notions of noncommutative probability spaces and distributions of random variables. For further details, see the texts [25, 30].

DEFINITION 1.1.

(1) A *noncommutative probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$, and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$. Elements in a noncommutative probability space $(\mathcal{A}, \varphi)$ will be called *noncommutative random variables*, or simply *random variables*.

(2) A W*-probability space (M, φ) is a von Neumann algebra M together with a faithful normal state φ. We will not assume that φ is a trace.

EXAMPLE 1.2. Let (Ω, Σ, μ) be a (classical) probability space.

(1) The pair $(L^\infty(\mu), \mathbb{E})$ is a W*-probability space, where $L^\infty(\mu)$ is the algebra of bounded Σ-measurable random variables, and $\mathbb{E}$ is the expectation functional $\mathbb{E}(f) = \int f \, d\mu$.

(2) Let

$$L(\mu) = \bigcap_{1 \le p < \infty} L^p(\mu)$$

be the algebra of random variables with finite moments of all orders. Then $(L(\mu), \mathbb{E})$ is a noncommutative probability space.

The joint distribution of a sequence $(X_1, \ldots, X_n)$ of (classical) random variables can be defined as the linear functional on $C_b(\mathbb{R}^n)$ determined by

$$f \mapsto \mathbb{E}[f(X_1, \ldots, X_n)].$$

In the noncommutative context, it is generally not possible to make sense of $f(x_1, \ldots, x_n)$ for $f \in C_b(\mathbb{R}^n)$ if the random variables $x_1, \ldots, x_n$ do not commute. Instead, we work with an algebra of noncommutative polynomials.

NOTATION 1.3. Let I be a nonempty set. We let $\mathscr{P}_I$ denote the algebra $\mathbb{C}\langle t_i : i \in I\rangle$ of noncommutative polynomials, with generators indexed by the set I. Note that $\mathscr{P}_I$ is spanned by 1 and monomials of the form $t_{i_1} \cdots t_{i_k}$, for $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in I$. If $I = \{1, \ldots, n\}$, we set $\mathscr{P}_n = \mathscr{P}_I$, and if $I = \mathbb{N}$ we denote $\mathscr{P}_\infty = \mathscr{P}_I$.

Given a family $(x_i)_{i \in I}$ of noncommutative random variables in a noncommutative probability space $(\mathcal{A}, \varphi)$, there is a unique unital homomorphism $\mathrm{ev}_x : \mathscr{P}_I \to \mathcal{A}$ which sends t_i to x_i for each $i \in I$. We also denote this map by $p \mapsto p(x)$.

DEFINITION 1.4. Let $(x_i)_{i \in I}$ be a family of random variables in the noncommutative probability space $(\mathcal{A}, \varphi)$. The *joint distribution* of $(x_i)_{i \in I}$ is the linear functional $\varphi_x : \mathscr{P}_I \to \mathbb{C}$ defined by

$$\varphi_x(p) = \varphi(p(x)).$$

Note that the joint distribution of $(x_i)_{i \in I}$ is determined by the collection of *joint moments*

$$\varphi_x(t_{i_1} \cdots t_{i_k}) = \varphi(x_{i_1} \cdots x_{i_k})$$

for $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in I$.

REMARK 1.5. In the classical de Finetti's theorem, the independence which occurs is only after conditioning. Likewise the free de Finetti's theorem is a statement about freeness with amalgamation. Both of these concepts may be expressed in terms of operator-valued probability spaces, which we now recall.

DEFINITION 1.6. An *operator-valued probability* space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ consists of a unital algebra $\mathcal{A}$, a subalgebra $1 \in \mathcal{B} \subset \mathcal{A}$, and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$, that is, E is a linear map such that $E[1] = 1$ and

$$E[b_1 a b_2] = b_1 E[a] b_2$$

for all $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$.

EXAMPLE 1.7. Let (Ω, Σ, μ) be a probability space, and let $\mathcal{F} \subset \Sigma$ be a σ-subalgebra. Let $\mathcal{A} = L^\infty(\mu)$, and let $\mathcal{B} = L^\infty(\mu|_{\mathcal{F}})$ be the subalgebra of bounded, $\mathcal{F}$-measurable functions on Ω. Then $(\mathcal{A}, \mathbb{E}[\cdot|\mathcal{F}])$ is an operator-valued probability space.

To define the joint distribution of a family $(x_i)_{i \in I}$ in an operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$, we will use the algebra $\mathcal{B}\langle t_i : i \in I\rangle$ of noncommutative polynomials with coefficients in $\mathcal{B}$. This algebra is spanned by monomials of the form $b_0 t_{i_1} \cdots t_{i_k} b_k$, for $k \in \mathbb{N}$, $b_0, \dots, b_k \in \mathcal{B}$ and $i_1, \dots, i_k \in I$. There is a unique homomorphism from $\mathcal{B}\langle t_i : i \in I\rangle$ into $\mathcal{A}$ which acts as the identity on $\mathcal{B}$ and sends t_i to x_i, which we denote by $p \mapsto p(x)$.

DEFINITION 1.8. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and let $(x_i)_{i \in I}$ be a family in $\mathcal{A}$. The *B-valued joint distribution* of the family $(x_i)_{i \in I}$ is the linear map $E_x : \mathcal{B}\langle t_i : i \in I\rangle \to \mathcal{B}$ defined by

$$E_x(p) = E[p(x)].$$

Observe that the joint distribution is determined by the *$\mathcal{B}$-valued joint moments*

$$E_x[b_0 t_{i_1} \cdots t_{i_k} b_k] = E[b_0 x_{i_1} \cdots x_{i_k} b_k]$$

for $b_0, \dots, b_k \in \mathcal{B}$ and $i_1, \dots, i_k \in I$. Observe that if $\mathcal{B}$ commutes with the variables $(x_i)_{i \in I}$, then

$$E[b_0 x_{i_1} \cdots x_{i_k} b_k] = b_0 \cdots b_k E[x_{i_1} \cdots x_{i_k}],$$

so that the $\mathcal{B}$-valued joint distribution is determined simply by the collection of moments $E[x_{i_1} \cdots x_{i_k}]$ for $i_1, \dots, i_k \in I$.

DEFINITION 1.9. Let $(x_i)_{i \in I}$ be a family in the operator-valued probability space $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$.

(1) If the algebra generated by $\mathcal{B}$ and $\{x_i : i \in I\}$ is commutative, then the variables are called *conditionally independent given B* if

$$E[p_1(x_{i_1}) \cdots p_k(x_{i_k})] = E[p_1(x_{i_1})] \cdots E[p_k(x_{i_k})],$$

whenever $i_1, \dots, i_k$ are distinct and $p_1, \dots, p_k$ are polynomials in $\mathcal{B}\langle t\rangle$.

(2) The variables $(x_i)_{i\in I}$ are called *free with amalgamation over $\mathcal{B}$*, or *free with respect to E*, if

$$E[p_1(x_{i_1})\cdots p_k(x_{i_k})]=0,$$

whenever $i_1,\dots,i_k\in I$ are such that $i_l\neq i_{l+1}$ for $1\leq l<k$, and $p_1,\dots,p_k\in\mathcal{B}\langle t\rangle$ are such that $E[p_l(x_{i_l})]=0$ for $1\leq l\leq k$.

REMARK 1.10. Voiculescu first defined freeness with amalgamation, and developed its basic theory in [29]. Conditional independence and freeness with amalgamation also have rich combinatorial theories, which we now recall. In the free case this is due to Speicher [28]; see also [25].

DEFINITION 1.11.

(1) A *partition* π of a set S is a collection of disjoint, nonempty sets $V_1,\dots,V_r$ such that $V_1\cup\cdots\cup V_r=S$. $V_1,\dots,V_r$ are called the *blocks* of π, and we set $|\pi|=r$. The collection of partitions of S will be denoted $P(S)$, or in the case that $S=\{1,\dots,k\}$ by $P(k)$.

(2) Given $\pi,\sigma\in P(S)$, we say that $\pi\leq\sigma$ if each block of π is contained in a block of σ. There is a least element of $P(S)$ which is larger than both π and σ, which we denote by $\pi\vee\sigma$.

(3) If S is ordered, we say that $\pi\in P(S)$ is *noncrossing* if whenever V,W are blocks of π and $s_1<t_1<s_2<t_2$ are such that $s_1,s_2\in V$ and $t_1,t_2\in W$, then $V=W$. The set of noncrossing partitions of S is denoted by $NC(S)$, or by $NC(k)$ in the case that $S=\{1,\dots,k\}$.

(4) The noncrossing partitions can also be defined recursively, a partition $\pi\in P(S)$ is noncrossing if and only if it has a block V which is an interval, such that $\pi\setminus V$ is a noncrossing partition of $S\setminus V$.

(5) Given $i_1,\dots,i_k$ in some index set I, we denote by $\ker\mathbf{i}$ the element of $P(k)$ whose blocks are the equivalence classes of the relation

$$s\sim t\quad\Leftrightarrow\quad i_s=i_t.$$

Note that if $\pi\in P(k)$, then $\pi\leq\ker\mathbf{i}$ is equivalent to the condition that whenever s and t are in the same block of π, i_s must equal i_t.

DEFINITION 1.12. Let $(\mathcal{A},E:\mathcal{A}\to\mathcal{B})$ be an operator-valued probability space.

(1) A *$\mathcal{B}$-functional* is a n-linear map $\rho:\mathcal{A}^n\to\mathcal{B}$ such that

$$\rho(b_0a_1b_1,a_2b_2,\dots,a_nb_n)=b_0\rho(a_1,b_1a_2,\dots,b_{n-1}a_n)b_n$$

for all $b_0,\dots,b_n\in\mathcal{B}$ and $a_1,\dots,a_n$. Equivalently, ρ is a linear map from $\mathcal{A}^{\otimes_{\mathcal{B}}n}$ to $\mathcal{B}$, where the tensor product is taken with respect to the natural $\mathcal{B}-\mathcal{B}$-bimodule structure on $\mathcal{A}$.

(2) Suppose that $\mathcal{B}$ is commutative. For $k\in\mathbb{N}$ let $\rho^{(k)}$ be a $\mathcal{B}$-functional. Given $\pi\in P(n)$, we define a $\mathcal{B}$-functional $\rho^{(\pi)}:\mathcal{A}^n\to\mathcal{B}$ by the formula

$$\rho^{(\pi)}[a_1,\dots,a_n]=\prod_{V\in\pi}\rho(V)[a_1,\dots,a_n],$$

where if $V = (i_1 < \cdots < i_s)$ is a block of π then

$$\rho(V)[a_1, \ldots, a_n] = \rho_s(a_{i_1}, \ldots, a_{i_s}).$$

If $\mathcal{B}$ is noncommutative, there is no natural order in which to compute the product appearing in the above formula for $\rho^{(\pi)}$. However, the nesting property of noncrossing partitions allows for a natural definition of $\rho^{(\pi)}$ for $\pi \in NC(n)$, which we now recall from [28].

DEFINITION 1.13. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and for $k \in \mathbb{N}$ let $\rho^{(k)} : \mathcal{A}^k \to \mathcal{B}$ be a $\mathcal{B}$-functional. Given $\pi \in NC(n)$, define a $\mathcal{B}$-functional $\rho^{(n)} : \mathcal{A}^n \to \mathcal{B}$ recursively as follows:

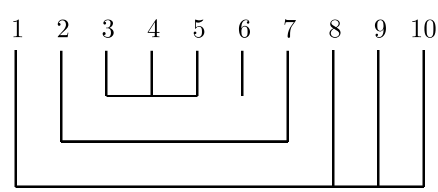
(1) If $\pi = 1_n$ is the partition containing only one block, define $\rho^{(\pi)} = \rho^{(n)}$.
(2) Otherwise, let $V = \{l+1, \ldots, l+s\}$ be an interval of π and define

$$\rho^{(\pi)}[a_1, \ldots, a_n] = \rho^{(\pi \setminus V)}\big[a_1, \ldots, a_l \cdot \rho^{(s)}(a_{l+1}, \ldots, a_{l+s}), a_{l+s+1}, \ldots, a_n\big]$$

for $a_1, \ldots, a_n \in \mathcal{A}$.

EXAMPLE 1.14. Let

$$\pi = \{\{1, 8, 9, 10\}, \{2, 7\}, \{3, 4, 5\}, \{6\}\} \in NC(10),$$



then the corresponding $\rho^{(\pi)}$ is given by

$$\rho^{(\pi)}[a_1, \ldots, a_{10}] = \rho^{(4)}\big(a_1 \cdot \rho^{(2)}\big(a_2 \cdot \rho^{(3)}(a_3, a_4, a_5), \rho^{(1)}(a_6) \cdot a_7\big), a_8, a_9, a_{10}\big).$$

DEFINITION 1.15. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and let $(x_i)_{i \in I}$ be a family of random variables in $\mathcal{A}$.

(1) The *operator-valued classical cumulants* $c_E^{(k)} : \mathcal{A}^k \to \mathcal{B}$ are the $\mathcal{B}$-functionals defined by the *classical moment-cumulant formula*

$$E[a_1 \cdots a_n] = \sum_{\pi \in P(n)} c_E^{(\pi)}[a_1, \ldots, a_n].$$

Note that the right-hand side of the equation is equal to $c_E^{(n)}[a_1, \ldots, a_n]$ plus lower order terms, and hence $c_E^{(n)}$ can be solved for recursively.

(2) The *operator-valued free cumulants* $\kappa_E^{(k)} : \mathcal{A}^k \to \mathcal{B}$ are the $\mathcal{B}$-functionals defined by the *free moment-cumulant formula*

$$E[a_1, \dots, a_n] = \sum_{\pi \in NC(n)} \kappa_E^{(\pi)}[a_1, \dots, a_n].$$

As above, this equation can be solved recursively for $\kappa_E^{(n)}$.

While the definitions of conditional independence and freeness with amalgamation given above appear at first to be quite different, they have very similar expressions in terms of cumulants. In the free case, the following theorem is due to Speicher [28].

THEOREM 1.16. *Let* $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ *be an operator-valued probability space, and* $(x_i)_{i \in I}$ *a family of random variables in* $\mathcal{A}$.

(1) *If the algebra generated by* $\mathcal{B}$ *and* $(x_i)_{i \in I}$ *is commutative, then the variables are conditionally independent given* B *if and only if*

$$c_E^{(n)}[b_0 x_{i_1} b_1, \dots, x_{i_n} b_n] = 0,$$

whenever there are $1 \leq k, l \leq n$ *such that* $i_k \neq i_l$.

(2) *The variables are free with amalgamation over* $\mathcal{B}$ *if and only if*

$$\kappa_E^{(n)}[b_0 x_{i_1} b_1, \dots, x_{i_n} b_n] = 0,$$

whenever there are $1 \leq k, l \leq n$ *such that* $i_k \neq i_l$.

Note that the condition in (1) is equivalent to the statement that if $\pi \in P(n)$, then

$$c_E^{(\pi)}[b_0 x_{i_1} b_1, \dots, x_{i_n} b_n] = 0$$

unless $\pi \leq \ker \mathbf{i}$, and likewise in (2) for $\pi \in NC(n)$. Stronger characterizations of the joint distribution of $(x_i)_{i \in I}$ can be given by specifying what types of partitions may contribute nonzero cumulants.

THEOREM 1.17. *Let* $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ *be an operator-valued probability space, and let* $(x_i)_{i \in I}$ *be a family of random variables in* $\mathcal{A}$.

(1) *Suppose that* $\mathcal{B}$ *and* $(x_i)_{i \in I}$ *generate a commutative algebra. The* $\mathcal{B}$*-valued joint distribution of* $(x_i)_{i \in I}$ *has the property corresponding to* D *in the table below if and only if for any* $\pi \in P(n)$

$$c_E^{(\pi)}[b_0 x_{i_1} b_1, \dots, x_{i_n} b_n] = 0$$

unless $\pi \in D(n)$ *and* $\pi \leq \ker \mathbf{i}$.

Partitions D	***Joint distribution***
P: All partitions	*Independent*
P_h: Partitions with even block sizes	*Independent and even*
P_b: Partitions with block size ≤ 2	*Independent Gaussian*
P_2: Pair partitions	*Independent centered Gaussian*

(2) *The $\mathcal{B}$-valued joint distribution of $(x_i)_{i\in I}$ has the property corresponding to D in the the table below if and only if for any $\pi \in NC(n)$*

$$\kappa_E^{(\pi)}[b_0 x_{i_1} b_1, \dots, x_{i_n} b_n] = 0$$

unless $\pi \in D(n)$ and $\pi \leq \ker \mathbf{i}$.

Noncrossing partitions D	***Joint distribution***
NC: Noncrossing partitions	*Freely independent*
NC_h: Noncrossing partitions with even block sizes	*Freely independent and even*
NC_b: Noncrossing partitions with block size ≤ 2	*Freely independent semicircular*
NC_2: Noncrossing pair partitions	*Freely independent centered semicircular*

PROOF. These results are well known. In the classical case, note that the results for P_2, P_b are equivalent to the *Wick formula* for computing moments of independent Gaussian families. In the free case, see [25, 28]. □

REMARK 1.18. It is clear from the definitions that the classical and free cumulants can be solved for from the joint moments. In fact, a combinatorial formula for the cumulants in terms of the moments can be given. First we recall the definition of the Möbius function on a partially ordered set.

DEFINITION 1.19. Let $(P, <)$ be a finite partially ordered set. The *Möbius function* $\mu_P : P \times P \to \mathbb{Z}$ is defined by

$$\mu_P(p,q) = \begin{cases} 0, & p \not\leq q, \\ 1, & p = q, \\ -1 + \displaystyle\sum_{l\geq 1} (-1)^{l+1} \#\{(p_1, \dots, p_l) \in P^l : p < p_1 < \cdots < p_l < q\}, \\ \qquad p < q. \end{cases}$$

THEOREM 1.20. *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and let $(x_i)_{i\in I}$ be a family of random variables. Define the $\mathcal{B}$-valued moment functionals $E^{(n)}$ by*

$$E^{(n)}[a_1, \dots, a_n] = E[a_1 \cdots a_n].$$

(1) *Suppose that $\mathcal{B}$ is commutative. Then for any $\sigma \in P(n)$ and $a_1, \dots, a_n \in \mathcal{A}$, we have*

$$c_E^{(\sigma)}[a_1, \dots, a_n] = \sum_{\substack{\pi \in P(n) \\ \pi \leq \sigma}} \mu_{P(n)}(\pi, \sigma) E^{(\pi)}[a_1, \dots, a_n].$$

(2) *For any $\sigma \in NC(n)$ and $a_1, \dots, a_n \in \mathcal{A}$, we have*

$$\kappa_E^{(\sigma)}[a_1, \dots, a_n] = \sum_{\substack{\pi \in NC(n) \\ \pi \leq \sigma}} \mu_{NC(n)}(\pi, \sigma) E^{(\pi)}[a_1, \dots, a_n].$$

PROOF. This follows from the *Möbius inversion formula*; see [25, 28]. $\square$

Easy quantum groups. We will now briefly recall some notions and results from [7].

Consider a compact group $G \subset O_n$. By the Stone–Weierstrauss theorem, $C(G)$ is generated by the n^2 coordinate functions u_{ij} sending a matrix in G to its (i, j) entry. The structure of G as a compact group is captured by the commutative Hopf C*-algebra $C(G)$ together with comultiplication, counit and antipode determined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},$$

$$\varepsilon(u_{ij}) = \delta_{ij},$$

$$S(u_{ij}) = u_{ji}.$$

Dropping the condition of commutativity, we obtain the following definition, adapted from the fundamental paper of Woronowicz [34].

DEFINITION 1.21. An *orthogonal Hopf algebra* is a unital C*-algebra A generated by n^2 self-adjoint elements u_{ij}, such that the following conditions hold:

(1) The inverse of $u = (u_{ij}) \in M_n(A)$ is the transpose $u^t = (u_{ji})$.
(2) $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ determines a morphism $\Delta : A \to A \otimes A$.
(3) $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
(4) $S(u_{ij}) = u_{ji}$ defines a morphism $S : A \to A^{op}$.

It follows from the definitions that Δ, ε, S satisfy the usual Hopf algebra axioms. If A is an orthogonal Hopf algebra, we use the heuristic formula "$A = C(G)$," where G is an *compact orthogonal quantum group*. Of course if A is noncommutative, then G cannot exist as a concrete object, and all statements about G must be interpreted in terms of the Hopf algebra A.

The following two examples, constructed by Wang in [31, 32], are fundamental to our considerations.

DEFINITION 1.22.

(1) $A_o(n)$ is the universal C*-algebra generated by n^2 self-adjoint elements u_{ij}, such that $u = (u_{ij}) \in M_n(A_o(n))$ is orthogonal.

(2) $A_s(n)$ is the universal C*-algebra generated by n^2 projections u_{ij}, such that the sum along any row or column of $u = (u_{ij}) \in M_n(A_s(n))$ is the identity.

As discussed above, we use the notation $A_o(n) = C(O_n^+)$, $A_s(n) = C(S_n^+)$, and call O_n^+ and S_n^+ the *free orthogonal group* and *free permutation group*, respectively.

We now recall the "easiness" condition from [7] for a compact orthogonal quantum group $S_n \subset G \subset O_n^+$. Let u, v be the fundamental representations of G, S_n on $\mathbb{C}^n$, respectively. By functoriality, the space $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ of intertwining operators is contained in $\mathrm{Hom}(v^{\otimes k}, v^{\otimes l})$ for any k, l. But the Hom-spaces for v are well known: they are spanned by operators T_π with π belonging to the set $P(k,l)$ of partitions between k upper and l lower points. Explicitly, if $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{C}^n$, then the formula for T_π is given by

$$T_\pi(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1,\dots,j_l} \delta_\pi \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_l \end{pmatrix} e_{j_1} \otimes \cdots \otimes e_{j_l}.$$

Here the δ symbol appearing on the right-hand side is 1 when the indices "fit," that is, if each block of π contains equal indices, and 0 otherwise.

It follows from the above discussion that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ consists of certain linear combinations of the operators T_π, with $\pi \in P(k,l)$. We call G "easy" if these spaces are spanned by partitions.

DEFINITION 1.23. A compact orthogonal quantum group $S_n \subset G \subset O_n^+$ is called *easy* if for each $k, l \in \mathbb{N}$, there exist sets $D(k,l) \subset P(k,l)$ such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_\pi : \pi \in D(k,l))$. If we have $D(k,l) \subset NC(k,l)$ for each $k, l \in \mathbb{N}$, we say that G is a *free quantum group*.

There are four natural examples of classical groups which are easy:

Group	**Partitions**
Permutation group S_n	P: All partitions
Orthogonal group O_n	P_2: Pair partitions
Hyperoctahedral group H_n	P_h: Partitions with even block sizes
Bistochastic group B_n	P_b: Partitions with block size ≤ 2

There are also the two trivial modifications $S_n' = S_n \times \mathbb{Z}_2$ and $B_n' = B_n \times \mathbb{Z}_2$, and it was shown in [7] that these six examples are the only ones.

There is a one-to-one correspondence between classical easy groups and free quantum groups, which on a combinatorial level corresponds to restricting to noncrossing partitions:

Quantum group	**Partitions**
S_n^+	NC: All noncrossing partitions
O_n^+	NC_2: Noncrossing pair partitions
H_n^+	NC_h: Noncrossing partitions with even block sizes
B_n^+	NC_b: Noncrossing partitions with block size ≤ 2

There are also free versions of S_n', B_n', constructed in [7].

In general, the class of easy quantum groups appears to be quite rigid (see [5] for a discussion here). However, two more examples can be obtained as "half-liberations." The idea is that instead of removing the commutativity relations from the generators u_{ij} of $C(G)$ for a classical easy group G, which would produce $C(G^+)$, we instead require that the the generators "half-commute," that is, $abc = cba$ for $a, b, c \in \{u_{ij}\}$. More precisely, we define $C(G^*) = C(G^+)/I$, where I is the ideal generated by the relations $abc = cba$ for $a, b, c \in \{u_{ij}\}$. For $G = S_n, S_n', B_n, B_n'$ we have $G^* = G$, however for O_n, H_n, we obtain new quantum groups O_n^*, H_n^*. The corresponding partition categories E_2, E_h consist of all pair partitions, respectively all partitions, which are *balanced* in the sense that each block contains as many odd as even legs.

2. Half independence. In this section, we introduce a new kind of independence which appears in the de Finetti theorems for the half-liberated quantum groups H^* and O^*. To define this notion, we require that the variables have a certain degree of commutativity.

DEFINITION 2.1. Let $(x_i)_{i\in I}$ be a family of noncommutative random variables. We say that the variables *half-commute* if

$$x_i x_j x_k = x_k x_j x_i$$

for all $i, j, k \in I$.

Observe that if $(x_i)_{i\in I}$ half-commute, then in particular x_i^2 commutes with x_j for any $i, j \in I$.

DEFINITION 2.2. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and suppose that $\mathcal{B}$ is contained in the center of $\mathcal{A}$. Let $(x_i)_{i\in I}$ be a family of random variables in $\mathcal{A}$ which half-commute. We say that $(x_i)_{i\in I}$ are *conditionally half-independent given* $\mathcal{B}$, or *half-independent with respect to* E, if the following conditions are satisfied:

(1) The variables $(x_i^2)_{i\in I}$ are conditionally independent given $\mathcal{B}$.
(2) For any $i_1, \ldots, i_k \in I$, we have

$$E[x_{i_1} \cdots x_{i_k}] = 0$$

unless for each $i \in I$ the set of $1 \leq j \leq k$ such that $i_j = i$ contains as many odd as even numbers, that is, unless $\ker \mathbf{i}$ is balanced.

If $B = \mathbb{C}$, then the variables are simply called *half-independent*.

REMARK 2.3. As a first remark, we note that half-independence is defined only between random variables and not at the level of algebras, in contrast with classical and free independence. In fact, it is known from [27] there are no other good notions of independence between unital algebras other than classical and free.

The conditions may appear at first to be somewhat artificial, but are motivated by the following natural example.

EXAMPLE 2.4. Let (Ω, Σ, μ) be a (classical) probability space, and let $L(\mu)$ denote the algebra of complex-valued random variables on Ω with finite moments of all orders.

(1) Let $(\xi_i)_{i\in I}$ be a family of independent random variables in $L(\mu)$. Suppose that for each $i \in I$, the distribution of ξ_i is such that

$$\mathbb{E}[\xi_i^n \overline{\xi_i^m}] = 0$$

unless $n = m$. Define random variables x_i in $(M_2(L(\mu)), \mathbb{E} \circ \mathrm{tr})$ by

$$x_i = \begin{pmatrix} 0 & \xi_i \\ \overline{\xi}_i & 0 \end{pmatrix}.$$

A simple computation shows that the variables $(x_i)_{i\in I}$ half-commute. Since

$$x_i^2 = |\xi_i|^2 I_2,$$

it is clear that $(x_i^2)_{i\in I}$ are independent with respect to $\mathbb{E} \circ \mathrm{tr}$. Moreover, the assumption on the distributions of the ξ_i clearly implies that $\mathbb{E}[\mathrm{tr}[x_{i_1} \cdots x_{i_k}]] = 0$ unless k is even and $\ker \mathbf{i}$ is balanced. So $(x_i)_{i\in I}$ are half-independent.

Observe also that the distribution of x_i is equal to that of $(\xi_i \overline{\xi_i})^{1/2}$, where the square root is chosen such that the distribution is even. We call this the *squeezed version* of the complex distribution ξ_i (cf. [7]).

(2) Of particular interest is the case that the $(\xi_i)_{i\in I}$ have complex Gaussian distributions. Here the distribution of x_i is the squeezed version of the complex Gaussian ξ_i, which is a symmetrized Rayleigh distribution.

REMARK 2.5. We will show in Proposition 2.8 below that any half-independent family can be modeled as in the example above. First, we will show that, as for classical and free independence, the joint distribution of a family of half-independent random variables $(x_i)_{i\in I}$ is determined by the distributions of x_i for $i \in I$. It is convenient to first introduce the following family of permutations which are related to the half-commutation relation.

DEFINITION 2.6. We say that a permutation $\omega \in S_n$ *preserves parity* if $\omega(i) \equiv i \pmod 2$ for $1 \leq i \leq n$.

The collection of parity preserving partitions in S_n clearly form a subgroup, which is simply $S(\{1,3,\dots\}) \times S(\{2,4,\dots\})$. Moreover, this subgroup is generated by the transpositions $(i\ i+2)$ for $1 \leq i \leq n-2$. It follows that if $(x_i)_{i\in I}$ half-commute, then

$$x_{i_1} \cdots x_{i_n} = x_{i_{\omega(1)}} \cdots x_{i_{\omega(n)}},$$

whenever $\omega \in S_n$ preserves parity.

LEMMA 2.7. *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space such that $\mathcal{B}$ is contained in the center of $\mathcal{A}$. Suppose that $(x_i)_{i\in I}$ is a family of random variables in $\mathcal{A}$ which are conditionally half-independent given $\mathcal{B}$. Then the $\mathcal{B}$-valued joint distribution of $(x_i)_{i\in I}$ is uniquely determined by the $\mathcal{B}$-valued distributions of x_i for $i \in I$.*

PROOF. Let $i_1, \dots, i_k \in I$. We know that

$$E[x_{i_1} \cdots x_{i_k}] = 0$$

unless we have that for each $i \in I$, the set of $1 \leq j \leq k$ such that $i_j = i$ has as many odd as even elements. So suppose that this the case. By the remark above, we know that $x_{i_1} \cdots x_{i_k} = x_{i_{\omega(1)}} \cdots x_{i_{\omega(k)}}$ whenever $\omega \in S_k$ is parity preserving. With an appropriate choose of ω, it follows that

$$x_{i_1} \cdots x_{i_k} = x_{j_1}^{2(k_1)} \cdots x_{j_m}^{2(k_m)}$$

for some $j_1, \dots, j_m \in I$ and $k_1, \dots, k_m \in \mathbb{N}$ such that $k = 2(k_1 + \cdots + k_m)$. Since the joint distribution of $(x_i^2)_{i\in I}$ is clearly determined by the distributions of x_i for $i \in I$, the result follows. $\square$

PROPOSITION 2.8. *Let $(x_i)_{i\in I}$ be a half-commuting family of random variables in a W^*-probability space (M, φ) which are half-independent. Then there are independent complex-valued random variables $(\xi_i)_{i\in I}$ such that $\mathbb{E}[\xi_i^n \overline{\xi_i^m}] = 0$ unless $n = m$, and such that $(x_i)_{i\in I}$ has the same joint distribution as the family $(y_i)_{i\in I}$,*

$$y_i = \begin{pmatrix} 0 & \xi_i \\ \overline{\xi}_i & 0 \end{pmatrix}.$$

PROOF. Let $(X_i)_{i\in I}$ be a family of independent random variables such that X_i has the same distribution as x_i. Let $(U_i)_{i\in I}$ be a family of independent Haar unitary random variables which are independent from $(X_i)_{i\in I}$, and let $\xi_i = U_i X_i$. Then $(\xi_i)_{i\in I}$ are independent and

$$\mathbb{E}[\xi_i^n \overline{\xi_i^m}] = \mathbb{E}[X_i^{n+m}]\mathbb{E}[U_i^n \overline{U_i}^m] = \delta_{nm}\varphi(x_i^{2n}).$$

From Example 2.4, the variables $(y_i)_{i\in I}$ defined by

$$y_i = \begin{pmatrix} 0 & \xi_i \\ \overline{\xi}_i & 0 \end{pmatrix}$$

are half-independent, and y_i has the same distribution as x_i for each $i \in I$. By Lemma 2.7, $(y_i)_{i\in I}$ has the same joint distribution as $(x_i)_{i\in I}$. □

REMARK 2.9. We have stated our results in the scalar case $\mathcal{B} = \mathbb{C}$ for simplicity, but note that with suitable modifications, Example 2.4 and Proposition 2.8 apply equally well to conditionally half-independent families.

We will now develop a combinatorial theory for half-independence, based on the family E_h of balanced partitions.

DEFINITION 2.10. Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and suppose that $\mathcal{B}$ is contained in the center of $\mathcal{A}$. Let $(x_i)_{i\in I}$ be a family of random-variables in $\mathcal{A}$, and suppose that

$$E[x_{i_1} \cdots x_{i_k}] = 0$$

for any odd k and $i_1, \dots, i_k \in I$. Define the *half-liberated cumulants* $\gamma_E^{(n)}$ by the *half-liberated moment-cumulant formula*

$$E[x_{i_1} \cdots x_{i_k}] = \sum_{\substack{\pi \in E_h(k) \\ \pi \leq \ker \mathbf{i}}} \gamma_E^{(\pi)}[x_{i_1}, \dots, x_{i_k}],$$

where $\gamma_E^{(\pi)}[x_{i_1}, \dots, x_{i_k}]$ is defined, as in the classical case, by the formula

$$\gamma_E^{(\pi)}[x_{i_1}, \dots, x_{i_k}] = \prod_{V\in\pi} \gamma_E^{(|V|)}(V)[x_{i_1}, \dots, x_{i_k}].$$

Observe that both sides of the moment-cumulant formula above are equal to zero for odd values of k, and for even values the right hand side is equal to $\gamma_E^{(k)}[x_{i_1}, \dots, x_{i_k}]$ plus products of lower ordered terms and hence $\gamma_E^{(k)}$ may be solved for recursively. As in the free and classical cases, we may apply the Möbius inversion formula to obtain the following equation for $\gamma_E^{(\pi)}$, $\pi \in E_h(k)$:

$$\gamma_E^{(\pi)}(x_{i_1}, \dots, x_{i_k}) = \sum_{\substack{\sigma \in E_h(k) \\ \sigma \leq \pi}} \mu_{E_h(k)}(\sigma, \pi) E^{(\pi)}[x_{i_1}, \dots, x_{i_k}].$$

THEOREM 2.11. *Let $(\mathcal{A}, E : \mathcal{A} \to \mathcal{B})$ be an operator-valued probability space, and suppose that $\mathcal{B}$ is contained in the center of $\mathcal{A}$. Suppose $(x_i)_{i\in\mathbb{N}}$ is a family of variables in $\mathcal{A}$ which half-commute. Then the following conditions are equivalent*:

(1) $(x_i)_{i\in\mathbb{N}}$ *are half-independent with respect to* E.
(2) $E[x_{i_1}\cdots x_{i_k}]=0$ *whenever* k *is odd, and*

$$\gamma_E^{(\pi)}[x_{i_1},\dots,x_{i_k}]=0$$

for any $\pi\in E_h(k)$ *such that* $\pi\not\leq\ker\mathbf{i}$.

PROOF. First, suppose that condition (2) holds. From the moment-cumulant formula, we have

$$E[x_{i_1}\cdots x_{i_k}]=\sum_{\substack{\pi\in E_h(k)\\ \pi\leq\ker\mathbf{i}}}\gamma_E^{(\pi)}[x_{i_1},\dots,x_{i_k}]$$

for any $k\in\mathbb{N}$ and $i_1,\dots,i_k\in I$. Observe that if $\ker\mathbf{i}$ is not balanced then there is no $\pi\in E_h(k)$ such that $\pi\leq\ker\mathbf{i}$, so it follows that $E[x_{i_1}\cdots x_{i_k}]=0$. It remains to show that $(x_i^2)_{i\in I}$ are independent. Choose $k_1,\dots,k_m\in\mathbb{N}$, distinct $i_1,\dots,i_m\in I$ and let $k=2(k_1+\cdots+k_m)$. Let $\tau\in E_h(k)$ be the partition with blocks $\{1,\dots,2k_1\},\dots,\{2(k_1+\cdots+k_{m-1})+1,\dots,2k\}$. Then

$$\begin{aligned}E\bigl[x_{i_1}^{(2k_1)}\cdots x_{i_m}^{(2k_m)}\bigr]&=\sum_{\substack{\pi\in E_h(k)\\ \pi\leq\tau}}\gamma_E^{(\pi)}[x_{i_1},\dots,x_{i_1},x_{i_2},\dots,x_{i_m},\dots,x_{i_m}]\\&=\prod_{1\leq j\leq m}\sum_{\pi\in E_h(2k_j)}\gamma_E^{(\pi)}[x_{i_j},\dots,x_{i_j}]\\&=\prod_{1\leq j\leq m}E\bigl[x_{i_j}^{(2k_j)}\bigr],\end{aligned}$$

so that $(x_i^2)_{i\in I}$ are independent and hence $(x_i)_{i\in I}$ are half-independent.

The implication (1) $\Rightarrow$ (2) actually follows from (2) $\Rightarrow$ (1). Indeed, suppose that $(x_i)_{i\in I}$ are half-independent. Consider the algebra $\mathcal{A}'=\mathcal{B}\langle y_i:i\in I\rangle/\langle y_iy_jy_k=y_ky_jy_i\rangle$ of polynomials in half-commuting indeterminates $(y_i)_{i\in I}$ and coefficients in $\mathcal{B}$. Define a conditional expectation $E':\mathcal{A}'\to\mathcal{B}$ by

$$E'[y_{i_1}\cdots y_{i_k}]=\sum_{\substack{\pi\in E_h(k)\\ \pi\leq\ker\mathbf{i}}}\gamma_E^{(\pi)}[x_{i_1},\dots,x_{i_k}].$$

(It is easy to see that E' is well defined, that is, compatible with the half-commutation relations.) Since the half-liberated cumulants are uniquely determined by the moment-cumulant formula, it follows that

$$\gamma_{E'}^{(\pi)}[y_{i_1},\dots,y_{i_k}]=\begin{cases}\gamma_E^{(\pi)}[x_{i_1},\dots,x_{i_k}], & \pi\leq\ker\mathbf{i},\\ 0, & \text{otherwise}.\end{cases}$$

By the first part, it follows that $(y_i)_{i\in I}$ are half-independent with respect to E'. Since y_i has the same $\mathcal{B}$-valued distribution as x_i, it follows from Lemma 2.7

that $(y_i)_{i\in I}$ have the same joint distribution as $(x_i)_{i\in I}$. It then follows from the moment-cumulant formula that these families have the same half-liberated cumulants, and hence $\gamma_E^{(\pi)}[x_{i_1},\dots,x_{i_k}]=0$ unless $\pi\le\ker\mathbf{i}$. $\square$

Recall that (centered) Gaussian and semicircular distributions are characterized by the property that their nonvanishing cumulants are those corresponding to pair and noncrossing pair partitions, respectively. We will now show that for half-independence, it is the symmetrized Rayleigh distribution which has this property. This follows from the considerations in [7], but we include here a direct proof.

PROPOSITION 2.12. *Let x be a random variable in $(\mathcal{A},\varphi)$ which has an even distribution. Then x has a symmetrized Rayleigh distribution if and only if*

$$\gamma_E^{(\pi)}[x,\dots,x]=0$$

for any $\pi\in E_h(k)$ such that $\pi\notin E_2(k)$.

PROOF. Since the distribution of x is determined uniquely by its half-liberated cumulants, it suffices to show that if the cumulants have the stated property then x has a symmetrized Rayleigh distribution. Suppose that this is the case, then

$$\begin{aligned}\varphi(x^k)&=\sum_{\pi\in E_2(k)}\gamma^{(\pi)}[x,\dots,x]\\&=\gamma^{(2)}[x,x]\#\{\pi\in E_2(k)\}.\end{aligned}$$

It is easy to see that the number of partitions in $E_2(k)$ is $m!$ if $k=2m$ is even and is zero if k is odd. Since these agree with the moments of a symmetrized Rayleigh distribution, the result follows. $\square$

3. Weingarten estimate. It is a fundamental result of Woronowicz [34] that if G is a compact orthogonal quantum group, then there is a unique state $\int : C(G)\to\mathbb{C}$, called the *Haar state*, which is left and right invariant in the sense that

$$\left(\int\otimes\mathrm{id}\right)\Delta(f)=\int(f)\cdot 1_{C(G)}=\left(\mathrm{id}\otimes\int\right)\Delta(f)\qquad (f\in C(G)).$$

If $G\subset O_n$ is a compact group, then the Haar state on $C(G)$ is given by integrating against the Haar measure on G.

One quite useful aspect of the easiness condition for a compact orthogonal quantum group is that it leads to a combinatorial *Weingarten formula* for computing the Haar state, which we now recall from [7].

DEFINITION 3.1. Let $D(k)\subset P(k)$ be a collection of partitions. For $n\in\mathbb{N}$, define the *Gram matrix* $(G_{kn}(\pi,\sigma))_{\pi,\sigma\in D(k)}$ by the formula

$$G_{kn}(\pi,\sigma)=n^{|\pi\vee\sigma|}.$$

G_{kn} is invertible for n sufficiently large (see Proposition 3.4), define the *Weingarten matrix* W_{kn} to be its inverse.

THEOREM 3.2. *Let $G \subset O_n^+$ be an easy quantum group and let $D(k) \subset P(0,k)$ be the corresponding collection of partitions having no upper points. If G_{kn} is invertible, then*

$$\int u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{\substack{\pi,\sigma \in D(k) \\ \pi \le \ker \mathbf{i} \\ \sigma \le \ker \mathbf{j}}} W_{kn}(\pi,\sigma).$$

REMARK 3.3. The statement of the theorem above is from [7], but goes back to work of Weingarten [33] and was developed in a series of papers [2, 3, 11, 12]. Note that this reduces the problem of evaluating integrals over G to computing the entries of the Weingarten matrix. We will now give an estimate on the asymptotic behavior of W_{kn} as $n \to \infty$. This unifies and extends the estimates given in [2] and [15] for O^+, S^+.

PROPOSITION 3.4. *Let $k \in \mathbb{N}$ and $D(k) \subset P(k)$. For n sufficiently large, the Gram matrix G_{kn} is invertible. Moreover, the entries of the Weingarten matrix $W_{kn} = G_{kn}^{-1}$ satisfy the following*:

(1) $W_{kn}(\pi,\sigma) = O(n^{|\pi \vee \sigma| - |\pi| - |\sigma|})$.
(2) *If $\pi \le \sigma$, then*

$$n^{|\pi|} W_{kn}(\pi,\sigma) = \mu_{D(k)}(\pi,\sigma) + O(n^{-1}),$$

where $\mu_{D(k)}$ is the Möbius function on the partially ordered set $D(k)$ under the restriction of the order on $P(k)$.

PROOF. We use a standard method from [11, 12], further developed in [2, 3, 13].

First, note that

$$G_{kn} = \Theta_{kn}^{1/2}(1 + B_{kn})\Theta_{kn}^{1/2},$$

where

$$\Theta_{kn}(\pi,\sigma) = \begin{cases} n^{|\pi|}, & \pi = \sigma, \\ 0, & \pi \ne \sigma, \end{cases}$$

$$B_{kn}(\pi,\sigma) = \begin{cases} 0, & \pi = \sigma, \\ n^{|\pi \vee \sigma| - (|\pi| + |\sigma|)/2}, & \pi \ne \sigma. \end{cases}$$

Note that the entries of B_{kn} are $O(n^{-1/2})$, it follows that for n sufficiently large $1 + B_{kn}$ is invertible and

$$(1 + B_{kn})^{-1} = 1 - B_{kn} + \sum_{l \ge 1} (-1)^{l+1} B_{kn}^{l+1}.$$

G_{kn} is then invertible, and

$$W_{kn}(\pi,\sigma)=\sum_{l\geq 1}(-1)^{l+1}(\Theta_{kn}^{-1/2}B_{kn}^{l+1}\Theta_{kn}^{-1/2})(\pi,\sigma) + \begin{cases} n^{-|\pi|}, & \pi=\sigma, \\ -n^{|\pi\vee\sigma|-|\pi|-|\sigma|}, & \pi\neq\sigma. \end{cases}$$

Now for $l \geq 1$ we have

$$(\Theta_{kn}^{-1/2}B_{kn}^{l+1}\Theta_{kn}^{-1/2})(\pi,\sigma) = \sum_{\substack{\nu_1,\dots,\nu_l\in D(k)\\ \pi\neq\nu_1\neq\cdots\neq\nu_l\neq\sigma}} n^{|\pi\vee\nu_1|+|\nu_1\vee\nu_2|+\cdots+|\nu_l\vee\sigma|-|\nu_1|-\cdots-|\nu_l|-|\pi|-|\sigma|}.$$

So to prove (1), it suffices to show that if $\nu_1,\dots,\nu_l\in D(k)$, then

$$|\pi\vee\nu_1|+|\nu_1\vee\nu_2|+\cdots+|\nu_l\vee\sigma|\leq|\pi\vee\sigma|+|\nu_1|+\cdots+|\nu_l|.$$

We will use the fact that $P(k)$ is a *semi-modular lattice* ([10], Section I.8, Example 9): If $\nu,\tau\in\mathcal{P}(k)$, then

$$|\nu|+|\tau|\leq|\nu\vee\tau|+|\nu\wedge\tau|.$$

We will now prove the claim by induction on l, for $l=1$ we may apply the formula above to find

$$\begin{aligned}|\pi\vee\nu|+|\nu\vee\sigma| &\leq |(\pi\vee\nu)\vee(\nu\vee\sigma)|+|(\pi\vee\nu)\wedge(\nu\vee\sigma)| \\ &\leq |\pi\vee\sigma|+|\nu|.\end{aligned}$$

Now let $l>1$, by induction we have

$$|\pi\vee\nu_1|+|\nu_1\vee\nu_2|+\cdots+|\nu_{l-1}\vee\nu_l|\leq|\pi\vee\nu_l|+|\nu_1|+\cdots+|\nu_{l-1}|.$$

Also $|\nu_l\vee\sigma|\leq|\pi\vee\sigma|+|\nu_l|-|\pi\vee\nu_l|$, and the result follows.

To prove (2), suppose $\pi,\sigma\in D(k)$ and $\pi\leq\sigma$. The terms which contribute to order $n^{-|\pi|}$ in the expansion come from sequences $\nu_1,\dots,\nu_l\in D(k)$ such that $\pi\neq\nu_1\neq\cdots\neq\nu_l\neq\sigma$ and

$$|\pi\vee\nu_1|+\cdots+|\nu_l\vee\sigma|=|\sigma|+|\nu_1|+\cdots+|\nu_l|.$$

Since $|\pi\vee\nu_1|\leq|\nu_1|$, $|\nu_1\vee\nu_2|\leq|\nu_2|,\dots,|\nu_l\vee\sigma|\leq\sigma$, it follows that each of these must be an equality, which implies $\pi<\nu_1<\cdots<\nu_l<\sigma$. Conversely, any $\nu_1,\dots,\nu_l\in D(k)$ such that $\pi<\nu_1<\cdots<\nu_l<\sigma$ clearly satisfy this equation. Therefore, the coefficient of $n^{-|\pi|}$ in $W_{kn}(\pi,\sigma)$ is

$$\begin{cases} 1, & \pi=\sigma, \\ -1+\displaystyle\sum_{l=1}^{\infty}(-1)^{l+1}\#\{(\nu_1,\dots,\nu_l)\in D(k)^l:\pi<\nu_1<\cdots<\nu_l<\sigma\}, & \pi<\sigma, \end{cases}$$

which is precisely $\mu_{D(k)}(\pi, \sigma)$. $\square$

Recall that the free, half-liberated and classical cumulants are obtained from moment functionals by using the Möbius functions on NC, E_h and P, respectively. To show that this is compatible with Proposition 3.4, we will need the following result.

PROPOSITION 3.5.
(1) *If* $D = NC, NC_2, NC_b, NC_h$, *then*

$$\mu_{D(k)}(\pi, \sigma) = \mu_{NC(k)}(\pi, \sigma)$$

for all $\pi, \sigma \in D(k)$.
(2) *If* $D = E_h, E_2$, *then*

$$\mu_{D(k)}(\pi, \sigma) = \mu_{E_h(k)}(\pi, \sigma)$$

for all $\pi, \sigma \in D(k)$.
(3) *If* $D = P, P_2, P_b, P_h$, *then*

$$\mu_{D(k)}(\pi, \sigma) = \mu_{P(k)}(\pi, \sigma)$$

for all $\pi, \sigma \in D(k)$.

PROOF. Let $Q = NC, E_h, P$ according to cases (1), (2), (3). It is easy to see in each case that $D(k)$ is closed under taking intervals in $Q(k)$, that is, if $\pi_1, \pi_2 \in D(k)$, $\sigma \in Q(k)$ and $\pi_1 < \sigma < \pi_2$ then $\sigma \in D(k)$. The result now follows immediately from the definition of the Möbius function. $\square$

4. Finite quantum invariant sequences. We begin this section by defining the notion of quantum invariance for a sequence of noncommutative random variables under "transformations" coming from an orthogonal quantum group $G_n \subset O_n^+$.

Let $\mathscr{P}_n = \mathbb{C}\langle t_1, \dots, t_n\rangle$, and let $\alpha_n : \mathscr{P}_n \to \mathscr{P}_n \otimes C(G_n)$ be the unique unital homomorphism such that

$$\alpha_n(t_j) = \sum_{i=1}^{n} t_i \otimes u_{ij}.$$

It is easily verified that α_n is an action of G_n, that is,

$$(\mathrm{id} \otimes \Delta) \circ \alpha_n = (\alpha_n \otimes \mathrm{id}) \circ \alpha_n$$

and

$$(\mathrm{id} \otimes \varepsilon) \circ \alpha_n = \mathrm{id}.$$

DEFINITION 4.1. Let $(x_1, \dots, x_n)$ be a sequence of random variables in a noncommutative probability space $(\mathcal{B}, \varphi)$. We say that the joint distribution of this sequence is *invariant under* G_n, or that the sequence is G_n*-invariant*, if the distribution functional $\varphi_x : \mathscr{P}_n \to \mathbb{C}$ is invariant under the coaction α_n, that is,

$$(\varphi_x \otimes \mathrm{id})\alpha_n(p) = \varphi_x(p)$$

for all $p \in \mathscr{P}_n$. More explicitly, the sequence $(x_1, \dots, x_n)$ is G_n-invariant if

$$\varphi(x_{j_1} \cdots x_{j_k}) 1_{C(G_n)} = \sum_{1 \le i_1, \dots, i_k \le n} \varphi(x_{i_1} \cdots x_{i_k}) u_{i_1 j_1} \cdots u_{i_k j_k}$$

as an equality in $C(G_n)$, for all $k \in \mathbb{N}$ and $1 \le j_1, \dots, j_k \le n$.

REMARK 4.2. Suppose that $G_n \subset O_n$ is a compact group. By evaluating both sides of the above equation at $g \in G_n$, we see that a sequence $(x_1, \dots, x_n)$ is G_n-invariant if and only if

$$\varphi(x_{j_1} \cdots x_{j_k}) = \sum_{1 \le i_1, \dots, i_k \le n} g_{i_1 j_1} \cdots g_{i_k j_k} \varphi(x_{i_1} \cdots x_{i_k})$$

for each $k \in \mathbb{N}$, $1 \le j_1, \dots, j_k \le n$ and $g = (g_{ij}) \in G_n$, which coincides with the usual notion of G_n-invariance for a sequence of classical random variables.

We will now prove a converse to Theorem 1, which holds for finite sequences and in a purely algebraic context. The proof is adapted from the method of [24], Proposition 3.1.

PROPOSITION 4.3. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, $1 \in \mathcal{B} \subset \mathcal{A}$ a unital subalgebra and $E : \mathcal{A} \to \mathcal{B}$ a conditional expectation which preserves φ. Let $(x_1, \dots, x_n)$ be a sequence in $\mathcal{A}$.*

(1) *Free case*:

(a) *If $x_1, \dots, x_n$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$, then the sequence is S_n^+-invariant.*

(b) *If $x_1, \dots, x_n$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$, and have even distributions with respect to E, then the sequence is H_n^+-invariant.*

(c) *If $x_1, \dots, x_n$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$, and have semicircular distributions with respect to E, then the sequence is B_n^+-invariant.*

(d) *If $x_1, \dots, x_n$ are freely independent and identically distributed with amalgamation over $\mathcal{B}$, and have centered semicircular distributions with respect to E, then the sequence is O_n^+-invariant.*

(2) *Half-liberated case*: *Suppose that $(x_1, \dots, x_n)$ half-commute, and that $\mathcal{B}$ is central in $\mathcal{A}$.*

(a) *If* $x_1, \dots, x_n$ *are half-independent and identically distributed given* $\mathcal{B}$, *then the sequence is* H_n^**-invariant.*

(b) *If* $x_1, \dots, x_n$ *are half-independent and identically distributed given* $\mathcal{B}$, *and have symmetrized Rayleigh distributions with respect to* E, *then the sequence is* O_n^**-invariant.*

(3) *Suppose that* $\mathcal{B}$ *and* $x_1, \dots, x_n$ *generate a commutative algebra.*

(a) *If* $x_1, \dots, x_n$ *are conditionally independent and identically distributed given* $\mathcal{B}$, *then the sequence is* S_n*-invariant.*

(b) *If* $x_1, \dots, x_n$ *are conditionally independent and identically distributed given* $\mathcal{B}$, *and have even distributions with respect to* E, *then the sequence is* H_n*-invariant.*

(c) *If* $x_1, \dots, x_n$ *are conditionally independent and identically distributed given* $\mathcal{B}$, *and have Gaussian distributions with respect to* E, *then the sequence is* B_n*-invariant.*

(d) *If* $x_1, \dots, x_n$ *are conditionally independent and identically distributed given* $\mathcal{B}$, *and have centered Gaussian distributions with respect to* E, *then the sequence is* O_n*-invariant.*

PROOF. Suppose that the joint distribution of $(x_1, \dots, x_n)$ satisfies one of the conditions specified in the statement of the proposition, and let D be the partition family associated to the corresponding easy quantum group. By Propositions 1.17 and 2.11, and the moment-cumulant formulae, for any $k \in \mathbb{N}$ and $1 \le j_1, \dots, j_k \le n$ we have

$$\begin{aligned}
&\sum_{1 \le i_1, \dots, i_k \le n} \varphi(x_{i_1} \cdots x_{i_k}) u_{i_1 j_1} \cdots u_{i_k j_k} \\
&\quad = \sum_{1 \le i_1, \dots, i_k \le n} \varphi(E[x_{j_1} \cdots x_{j_k}]) u_{i_1 j_1} \cdots u_{i_k j_k} \\
&\quad = \sum_{1 \le i_1, \dots, i_k \le n} \sum_{\substack{\pi \in D(k) \\ \pi \le \ker \mathbf{i}}} \varphi\big(\xi_E^{(\pi)}[x_1, \dots, x_1]\big) u_{i_1 j_1} \cdots u_{i_k j_k} \\
&\quad = \sum_{\pi \in D(k)} \varphi\big(\xi_E^{(\pi)}[x_1, \dots, x_1]\big) \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \pi \le \ker \mathbf{i}}} u_{i_1 j_1} \cdots u_{i_k j_k},
\end{aligned}$$

where ξ denotes the free, half-liberated or classical cumulants in cases (1), (2) and (3), respectively. It follows from the considerations in [7], or by direct computation, that if $\pi \in D(k)$ then

$$\sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \pi \le \ker \mathbf{i}}} u_{i_1 j_1} \cdots u_{i_k j_k} = \begin{cases} 1_{C(G_n)}, & \pi \le \ker \mathbf{j}, \\ 0, & \text{otherwise.} \end{cases}$$

Applying this above, we find

$$\sum_{1\le i_1,\dots,i_k\le n} \varphi(x_{i_1}\cdots x_{i_k})u_{i_1j_1}\cdots u_{i_kj_k} = \sum_{\substack{\pi\in D(k)\\ \pi\le\ker\mathbf{j}}} \varphi\big(\xi_E^{(\pi)}[x_1,\dots,x_1]\big)1_{C(G_n)}$$
$$= \varphi(x_{j_1}\cdots x_{j_k})1_{C(G_n)},$$

which completes the proof. $\square$

REMARK 4.4. To prove the approximation result for finite sequences, we will require more analytic structure. Throughout the rest of the section, we will assume that $G_n \subset O_n^+$ is a compact quantum group, (M,φ) is a W*-probability space and $(x_1,\dots,x_n)$ is a sequence of self-adjoint random variables in M. We denote the von Neumann algebra generated by $(x_1,\dots,x_n)$ by M_n, and define the G_n-*invariant subalgebra* by

$$\mathcal{B}_n = \mathrm{W}^*\big(\{p(x) : p\in\mathscr{P}_n^{\alpha_n}\}\big),$$

where $\mathscr{P}_n^{\alpha_n}$ denotes the fixed point algebra of the action α_n, that is,

$$\mathscr{P}_n^{\alpha_n} = \big\{p\in\mathscr{P}_n : \alpha_n(p) = p\otimes 1_{C(G_n)}\big\}.$$

We now begin the technical preparations for our approximation result. First, we will need to extend the action α_n to the von Neumann algebra context. $L^\infty(G_n)$ will denote the von Neumann algebra obtained by taking the weak closure of $\pi_n(C(G_n))$, where π_n is the GNS representation of $C(G_n)$ on the GNS Hilbert space $L^2(G_n)$ for the Haar state. $L^\infty(G_n)$ is a *Hopf von Neumann algebra*, with the natural structure induced from $C(G_n)$.

PROPOSITION 4.5. *Suppose that $(x_1,\dots,x_n)$ is G_n-invariant. Then there is a right coaction $\widetilde{\alpha}_n : M_n \to M_n\otimes L^\infty(G_n)$ determined by*

$$\widetilde{\alpha}_n(p(x)) = (\mathrm{ev}_x\otimes\pi_n)\alpha_n(p)$$

for $p\in\mathscr{P}_n$. Moreover, the fixed point algebra of $\widetilde{\alpha}_n$ is precisely the G_n-invariant subalgebra $\mathcal{B}_n$.

PROOF. This follows from [13], Theorem 3.3, after identifying the GNS representation of $\mathscr{P}_n$ for the state φ_x with the homomorphism $\mathrm{ev}_x : \mathscr{P}_n \to M_n$. $\square$

There is a natural conditional expectation $E_n : M_n \to \mathcal{B}_n$ given by integrating the coaction $\widetilde{\alpha}_n$ with respect to the Haar state, that is,

$$E_n[m] = \left(\mathrm{id}\otimes\int\right)\widetilde{\alpha}_n(m).$$

By using the Weingarten calculus, we can give a simple combinatorial formula for the moment functionals with respect to E_n if G_n is one of the easy quantum groups under consideration. In the half-liberated case, we must first show that $\mathcal{B}_n$ is central in M_n.

LEMMA 4.6. *Suppose that $(x_1, \dots, x_n)$ half-commute. If $H_n^* \subset G_n$, then the G_n-invariant subalgebra $\mathcal{B}_n$ is contained in the center of M_n.*

PROOF. Since the G_n-invariant subalgebra is clearly contained in the H_n^*-invariant subalgebra, it suffices to prove the result for $G_n = H_n^*$. Observe that the representation of G_n on the subspace of $\mathscr{P}_n$ consisting of homogeneous noncommutative polynomials of degree k, given by the restriction of α_n, is naturally identified with $u^{\otimes k}$, where u is the fundamental representation of G_n. As discussed in Section 1, $\mathrm{Fix}(u^{\otimes k})$ is spanned by the operators T_π for $\pi \in E_h(k)$. It follows that the fixed point algebra of α_n is spanned by

$$p_\pi = \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \pi \le \ker \mathbf{i}}} t_{i_1} \cdots t_{i_k}$$

for $k \in \mathbb{N}$ and $\pi \in E_h(k)$. Therefore, $\mathcal{B}_n$ is generated by $p_\pi(x)$, for $k \in \mathbb{N}$ and $\pi \in E_h(k)$. Recall from Section 2 that if $\omega \in S_k$ is a parity preserving permutation, then $x_{i_1} \cdots x_{i_k} = x_{i_{\omega(1)}} \cdots x_{i_{\omega(k)}}$ for any $1 \le i_1, \dots, i_k \le n$. It follows that $p_\pi(x) = p_{\omega(\pi)}(x)$, where $\omega(\pi)$ is given by the usual action of permutations on set partitions. Now if $\pi \in E_h(k)$, it is easy to see that there is a parity preserving permutation $\omega \in S_k$ such that

$$\omega(\pi) = \{(1, \dots, 2k_1), \dots, (2(k_1 + \cdots + k_{l-1}) + 1, \dots, 2(k_1 + \ldots + k_l))\}$$

is an interval partition. We then have

$$p_\pi(x) = p_{\omega(\pi)}(x) = \left(\sum_{i_1=1}^{n} x_{i_1}^{2k_1}\right) \cdots \left(\sum_{i_l=1}^{n} x_{i_l}^{2k_l}\right).$$

Since x_i^2 is central in M_n for $1 \le i \le n$, the result follows. □

PROPOSITION 4.7. *Suppose that $(x_1, \dots, x_n)$ is G_n-invariant, and that one of the following conditions is satisfied*:

(1) *G_n is a free quantum group O_n^+, S_n^+, H_n^+ or B_n^+.*
(2) *G_n is a half-liberated quantum group O_n^* or H_n^* and $(x_1, \dots, x_n)$ half-commute.*
(3) *G_n is an easy group O_n, S_n, H_n or B_n and $(x_1, \dots, x_n)$ commute.*

Then for any π in the partition category $D(k)$ for the easy quantum group G_n, and any $b_0, \dots, b_k \in \mathcal{B}_n$, we have

$$E_n^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k] = \frac{1}{n^{|\pi|}} \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \pi \le \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k,$$

which holds if n is sufficiently large that the Gram matrix G_{kn} is invertible.

PROOF. We prove this by induction on the number of blocks of π. First, suppose that $\pi = 1_k$ is the partition with only one block. Then

$$E_n^{(1_k)}[b_0x_1b_1, \dots, x_1b_k] = E_n[b_0x_1 \cdots x_1b_k]$$
$$= \sum_{1 \le i_1, \dots, i_k \le n} b_0x_{i_1} \cdots x_{i_k}b_k \int u_{i_1 1} \cdots u_{i_k 1},$$

where we have used the fact that $b_0, \dots, b_k$ are fixed by the coaction $\widetilde{\alpha}_n$. Applying the Weingarten integration formula in Proposition 3.2, we have

$$E_n[b_0x_1 \cdots x_1b_k] = \sum_{1 \le i_1, \dots, i_k \le n} b_0x_{i_1} \cdots x_{i_k}b_k \sum_{\substack{\sigma, \pi \in D(k) \\ \pi \le \ker \mathbf{i}}} W_{kn}(\pi, \sigma)$$
$$= \sum_{\pi \in D(k)} \left(\sum_{\sigma \in D(k)} W_{kn}(\pi, \sigma) \right) \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \pi \le \ker \mathbf{i}}} b_0x_{i_1} \cdots x_{i_k}b_k.$$

Observe that $G_{kn}(\sigma, 1_k) = n^{|\sigma \vee 1_k|} = n$ for any $\sigma \in D(k)$. It follows that for any $\pi \in D(k)$, we have

$$n \cdot \sum_{\sigma \in D(k)} W_{kn}(\pi, \sigma) = \sum_{\sigma \in D(k)} W_{kn}(\pi, \sigma) G_{kn}(\sigma, 1_k)$$
$$= \delta_{\pi 1_k}.$$

Applying this above, we find

$$E_n[b_0x_1 \cdots x_1b_k] = \sum_{\pi \in D(k)} n^{-1} \delta_{\pi 1_k} \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \pi \le \ker \mathbf{i}}} b_0x_{i_1} \cdots x_{i_k}b_k$$
$$= \frac{1}{n} \sum_{i=1}^{n} b_0x_i \cdots x_ib_k,$$

as desired.

If condition (2) or (3) are satisfied, then the general case follows from the formula

$$E_n^{(\pi)}[b_0x_1b_1, \dots, x_1b_k] = b_1 \cdots b_k \prod_{V \in \pi} E_n(V)[x_1, \dots, x_1],$$

where in the half-liberated case we are applying the previous lemma. The one thing we must check here is that if $\pi \in D(k)$ and V is a block of π with s elements, then $1_s \in D(s)$. This is easily verified, in each case, for $D = P, P_2, P_h, P_b, E_h, E_2$.

Suppose now that condition (1) is satisfied. Let $\pi \in D(k)$. Since π is noncrossing, π contains an interval $V = \{l+1, \dots, l+s+1\}$. We then have

$$E_n^{(\pi)}[b_0x_1b_1, \dots, x_1b_k]$$
$$= E_n^{(\pi \setminus V)}[b_0x_1b_1, \dots, E_n[x_1b_{l+1} \cdots x_1b_{l+s}]x_1, \dots, x_1b_k].$$

To apply induction, we must check that $\pi \setminus V \in D(k-s)$ and $1_s \in D(s)$. Indeed, this is easily verified for NC, NC_2, NC_h and NC_b. Applying induction, we have

$$\begin{aligned}
&E_n^{(\pi)}[b_0x_1b_1, \dots, x_1b_k] \\
&\quad = \frac{1}{n^{|\pi|-1}} \sum_{\substack{1\le i_1,\dots,i_l,\\ i_{l+s+1},\dots,i_k\le n\\ (\pi\setminus V)\le \ker \mathbf{i}}} b_0x_{i_1}\cdots b_l(E_n[x_1b_{l+1}\cdots x_1b_{l+s}])x_{i_{l+s}}\cdots x_{i_k}b_k \\
&\quad = \frac{1}{n^{|\pi|-1}} \sum_{\substack{1\le i_1,\dots,i_l,\\ i_{l+s+1},\dots,i_k\le n\\ (\pi\setminus V)\le \ker \mathbf{i}}} b_0x_{i_1}\cdots b_l\left(\frac{1}{n}\sum_{i=1}^n x_ib_{l+1}\cdots x_ib_{l+s}\right)x_{i_{l+s}}\cdots x_{i_k}b_k \\
&\quad = \frac{1}{n^{|\pi|}} \sum_{\substack{1\le i_1,\dots,i_k\le n\\ \pi\le \ker \mathbf{i}}} b_0x_{i_1}\cdots x_{i_k}b_k,
\end{aligned}$$

which completes the proof. $\square$

We are now prepared to prove the approximation result for finite sequences.

THEOREM 4.8. *Suppose that $(x_1, \dots, x_n)$ is G_n-invariant, and that one of the following conditions is satisfied*:

(1) *G_n is a free quantum group O_n^+, S_n^+, H_n^+ or B_n^+.*

(2) *G_n is a half-liberated quantum group O_n^* or H_n^* and $(x_1, \dots, x_n)$ half-commute.*

(3) *G_n is an easy group O_n, S_n, H_n or B_n and $(x_1, \dots, x_n)$ commute.*

Let $(y_1, \dots, y_n)$ be a sequence of $\mathcal{B}_n$-valued random variables with $\mathcal{B}_n$-valued joint distribution determined as follows:

- *$G = O^+$: Free semicircular, centered with same variance as x_1.*
- *$G = S^+$: Freely independent, y_i has same distribution as x_1.*
- *$G = H^+$: Freely independent, y_i has same distribution as x_1.*
- *$G = B^+$: Free semicircular, same mean and variance as x_1.*
- *$G = O^*$: Half-liberated Gaussian, same variance as x_1.*
- *$G = H^*$: Half-independent, y_i has same distribution as x_1.*
- *$G = O$: Independent Gaussian, centered with same variance as x_1.*
- *$G = S$: Independent, y_i has same distribution as x_1.*
- *$G = H$: Independent, y_i has same distribution as x_1.*
- *$G = B$: Independent Gaussian, same mean and variance as x_1.*

If $1 \le j_1, \dots, j_k \le n$ and $b_0, \dots, b_k \in \mathcal{B}_n$, then

$$\|E_n[b_0x_{j_1}\cdots x_{j_k}b_k] - E[b_0y_{j_1}\cdots y_{j_k}b_k]\| \le \frac{C_k(G)}{n}\|x_1\|^k\|b_0\|\cdots\|b_k\|,$$

where $C_k(G)$ is a universal constant which depends only on k and the easy quantum group G.

PROOF. First, we note that it suffices to prove the statement for n sufficiently large, in particular we will assume throughout that n is sufficiently large for the Gram matrix G_{kn} to be invertible.

Let $1 \leq j_1, \dots, j_k \leq n$ and $b_0, \dots, b_k \in \mathcal{B}_n$. We have

$$\begin{aligned} E_n[b_0 x_{j_1} \cdots x_{j_k} b_k] &= \sum_{1 \leq i_1, \dots, i_k \leq n} b_0 x_{i_1} \cdots x_{i_k} b_k \int u_{i_1 j_1} \cdots u_{i_k j_k} \\ &= \sum_{1 \leq i_1, \dots, i_k \leq n} b_0 x_{i_1} \cdots x_{i_k} b_k \sum_{\substack{\pi, \sigma \in D(k) \\ \pi \leq \ker \mathbf{i} \\ \sigma \leq \ker \mathbf{j}}} W_{kn}(\pi, \sigma) \\ &= \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\pi \in D(k)} W_{kn}(\pi, \sigma) \sum_{\substack{1 \leq i_1, \dots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k. \end{aligned}$$

On the other hand, it follows from the assumptions on $(y_1, \dots, y_n)$ and the various moment-cumulant formulae that

$$E[b_0 y_{j_1} \cdots y_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \xi_{E_n}^{(\sigma)}[b_0 x_1 b_1, \dots, x_1 b_k],$$

where ξ denotes the relevant free, classical or half-liberated cumulants. The right-hand side can be expanded, via Möbius inversion, in terms of expectation functionals $E_n^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k]$ where π is a partition in NC, E_h, P according to cases (1), (2), (3), and $\pi \leq \sigma$ for some $\sigma \in D(k)$. Now if $\pi \notin D(k)$ then we claim that this expectation functional is zero. Indeed this is only possible if $D = NC_2, NC_h, P_2, P_h$ and π has a block with an odd number of legs. But it is easy to see that in these cases x_1 has an even distribution with respect to E_n, and therefore $E_n^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k] = 0$ as claimed. This observation, together with Proposition 3.5, allows to to rewrite the above equation as

$$E[b_0 y_{j_1} \cdots y_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\substack{\pi \in D(k) \\ \pi \leq \sigma}} \mu_{D(k)}(\pi, \sigma) E_n^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k].$$

Applying Lemma 4.7, we have

$$E[b_0 y_{j_1} \cdots y_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} \sum_{\substack{\pi \in D(k) \\ \pi \leq \sigma}} \mu_{D(k)}(\pi, \sigma) n^{-|\pi|} \sum_{\substack{1 \leq i_1, \dots, i_k \leq n \\ \pi \leq \ker \mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k.$$

Comparing these two equations, we find that

$$E_n[b_0x_{j_1}\cdots x_{j_k}b_k]-E[b_0y_{j_1}\cdots y_{j_k}b_k] = \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}}\sum_{\pi\in D(k)}\big(W_{kn}(\pi,\sigma)-\mu_{D(k)}(\pi,\sigma)n^{-|\pi|}\big)\sum_{\substack{1\le i_1,\dots,i_k\le n\\ \pi\le\ker\mathbf{i}}} b_0x_{i_1}\cdots x_{i_k}b_k.$$

Now since $x_1,\dots,x_n$ are identically distributed with respect to the faithful state φ, it follows that these variables have the same norm. Therefore,

$$\Big\|\sum_{\substack{1\le i_1,\dots,i_k\le n\\ \pi\le\ker\mathbf{i}}} b_0x_{i_1}\cdots x_{i_k}b_k\Big\|\le n^{|\pi|}\|x_1\|^k\|b_0\|\cdots\|b_k\|$$

for any $\pi\in D(k)$. Combining this with former equation, we have

$$\|E_n[b_0x_{j_1}\cdots x_{j_k}b_k]-E[b_0y_{j_1}\cdots y_{j_k}b_k]\| \le\sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}}\sum_{\pi\in D(k)}\big|W_{kn}(\pi,\sigma)n^{|\pi|}-\mu_{D(k)}(\pi,\sigma)\big|\|x_1\|^k\|b_0\|\cdots\|b_k\|.$$

Setting

$$C_k(G)=\sup_{n\in\mathbb{N}} n\cdot\sum_{\sigma,\pi\in D(k)}\big|W_{kn}(\pi,\sigma)n^{|\pi|}-\mu_{D(k)}(\pi,\sigma)\big|,$$

which is finite by Proposition 3.4, completes the proof. □

5. Infinite quantum invariant sequences. In this section, we will prove Theorem 1. Throughout this section, we will assume that G is one of the easy quantum groups $O, S, H, B, O^*, H^*, O^+, S^+, H^+$ or B^+. We will make use of the inclusions $G_n\hookrightarrow G_m$ for $n<m$, which correspond to the Hopf algebra morphisms $\omega_{n,m}: C(G_m)\to C(G_n)$ determined by

$$\omega_{n,m}(u_{ij})=\begin{cases}u_{ij}, & 1\le i,j\le n,\\ \delta_{ij}1_{C(G_n)}, & \max\{i,j\}>n.\end{cases}$$

The existence of $\omega_{n,m}$ may be verified in each case by using the universal relations of $C(G_n)$.

We begin by extending the notion of G_n-invariance to infinite sequences.

DEFINITION 5.1. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in a noncommutative probability space $(\mathcal{A},\varphi)$. We say that the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is *invariant under* G, or that the sequence is G*-invariant*, if $(x_1,\dots,x_n)$ is G_n-invariant for each $n\in\mathbb{N}$.

This means that the joint distribution functional of $(x_1,\dots,x_n)$ is invariant under the action $\alpha_n:\mathscr{P}_n\to\mathscr{P}_n\otimes C(G_n)$ for each $n\in\mathbb{N}$. It will be convenient to extend these actions to $\mathscr{P}_\infty = \mathbb{C}\langle t_i : i \in \mathbb{N}\rangle$, by defining $\beta_n : \mathscr{P}_\infty \to \mathscr{P}_\infty \otimes C(G_n)$ to be the unique unital homomorphism such that

$$\beta_n(t_j) = \begin{cases} \sum_{i=1}^{n} t_i \otimes u_{ij}, & 1 \le j \le n, \\ t_j \otimes 1_{C(G_n)}, & j > n. \end{cases}$$

It is clear that β_n is an action of G_n, moreover we have the relations

$$(\mathrm{id} \otimes \omega_{n,m}) \circ \beta_m = \beta_n$$

and

$$(\iota_n \otimes \mathrm{id}) \circ \alpha_n = \beta_n \circ \iota_n,$$

where $\iota_n : \mathscr{P}_n \to \mathscr{P}_\infty$ is the natural inclusion. Using these compatibilities, it is not hard to see that a sequence $(x_i)_{i\in\mathbb{N}}$ is G-invariant if and only if the joint distribution functional $\varphi_x : \mathscr{P}_\infty \to \mathbb{C}$ is invariant under β_n for each $n \in \mathbb{N}$.

Throughout the rest of the section, (M, φ) will be a W*-probability space and $(x_i)_{i\in\mathbb{N}}$ a sequence of self-adjoint random variables in (M, φ). We will assume that M is generated as a von Neumann algebra by $\{x_i : i \in \mathbb{N}\}$. $L^2(M, \varphi)$ will denote the GNS Hilbert space, with inner product $\langle m_1, m_2\rangle = \varphi(m_1^* m_2)$. The strong topology on M will be taken with respect to the faithful representation on $L^2(M, \varphi)$. We set

$$\mathcal{B}_n = \mathrm{W}^*\big(\{p(x) : p \in \mathscr{P}_\infty^{\beta_n}\}\big),$$

where $\mathscr{P}_\infty^{\beta_n}$ is the fixed point algebra of the action β_n. Since

$$(\mathrm{id} \otimes \omega_{n,n+1}) \circ \beta_{n+1} = \beta_n,$$

it follows that $\mathcal{B}_{n+1} \subset \mathcal{B}_n$ for all $n \ge 1$. We then define the *G-invariant subalgebra* by

$$\mathcal{B} = \bigcap_{n\ge 1} \mathcal{B}_n.$$

REMARK 5.2. If $(x_i)_{i\in\mathbb{N}}$ is G-invariant, then as in Proposition 4.5, for each $n \in \mathbb{N}$ there is a right coaction $\widetilde{\beta}_n : M \to M \otimes L^\infty(G_n)$ determined by

$$\widetilde{\beta}_n(p(x)) = (\mathrm{ev}_x \otimes \pi_n)\beta_n(p)$$

for $p \in \mathscr{P}_\infty$, and moreover the fixed point algebra of $\widetilde{\beta}_n$ is $\mathcal{B}_n$. For each $n \in \mathbb{N}$, there is then a φ-preserving conditional expectation $E_n : M \to \mathcal{B}_n$ given by integrating the action $\widetilde{\beta}_n$, that is,

$$E_n[m] = \Big(\mathrm{id} \otimes \int\Big)\widetilde{\beta}_n(m)$$

for $m \in M$. By taking the limit as $n \to \infty$, we obtain a φ-preserving conditional expectation onto the G-invariant subalgebra.

PROPOSITION 5.3. *Suppose that $(x_i)_{i\in\mathbb{N}}$ is G-invariant. Then*:

(1) *For any $m \in M$, the sequence $E_n[m]$ converges in $|\cdot|_2$ and the strong topology to a limit $E[m]$ in $\mathcal{B}$. Moreover, E is a φ-preserving conditional expectation of M onto $\mathcal{B}$.*

(2) *Fix $\pi \in NC(k)$ and $m_1, \dots, m_k \in M$, then*

$$E^{(\pi)}[m_1 \otimes \cdots \otimes m_k] = \lim_{n\to\infty} E_n^{(\pi)}[m_1 \otimes \cdots \otimes m_k],$$

with convergence in the strong topology.

PROOF. The proof follows from [14], Proposition 4.7. Note that (1) is just a simple noncommutative reversed martingale convergence theorem. More sophisticated convergence theorems for noncommutative martingales have been obtained; see, for example, [21, 22]. □

We are now prepared to prove Theorem 1.

PROOF OF THEOREM 1. Let $j_1, \dots, j_k \in \mathbb{N}$ and $b_0, \dots, b_k \in B$. As in the proof of Theorem 4.8, we have

$$\begin{aligned}
&E[b_0 x_{j_1} \cdots x_{j_k} b_k] \\
&\quad = \lim_{n\to\infty} E_n[b_0 x_{j_1} \cdots x_{j_k} b_k] \\
&\quad = \lim_{n\to\infty} \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} \sum_{\pi\in D(k)} W_{kn}(\pi,\sigma) \sum_{\substack{1\le i_1,\dots,i_k\le n\\ \pi\le\ker\mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k \\
&\quad = \lim_{n\to\infty} \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} \sum_{\substack{\pi\in D(k)\\ \pi\le\sigma}} \mu_{D(k)}(\pi,\sigma) n^{-|\pi|} \sum_{\substack{1\le i_1,\dots,i_k\le n\\ \pi\le\ker\mathbf{i}}} b_0 x_{i_1} \cdots x_{i_k} b_k.
\end{aligned}$$

By Proposition 4.7, and using the compatibility

$$(\widetilde{\iota}_n \otimes \mathrm{id}) \circ \widetilde{\alpha}_n = \widetilde{\beta}_n \circ \widetilde{\iota}_n,$$

where $\widetilde{\iota}_n : W^*(x_1, \dots, x_n) \to M$ is the obvious inclusion and $\widetilde{\alpha}_n$ is as in the previous section, we have

$$E[b_0 x_{j_1} \cdots x_{j_k} b_k] = \lim_{n\to\infty} \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} \sum_{\substack{\pi\in D(k)\\ \pi\le\sigma}} \mu_{D(k)}(\pi,\sigma) E_n^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k].$$

By (2) of Proposition 5.3, we obtain

$$E[b_0 x_{j_1} \cdots x_{j_k} b_k] = \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} \sum_{\substack{\pi\in D(k)\\ \pi\le\sigma}} \mu_{D(k)}(\pi,\sigma) E^{(\pi)}[b_0 x_1 b_1, \dots, x_1 b_k].$$

As discussed in the proof of Theorem 4.8, we can replace the sum of expectation functionals by cumulants to obtain

$$E[b_0 x_{j_1} \cdots x_{j_k} b_k] = \sum_{\substack{\sigma \in D(k) \\ \sigma \le \ker \mathbf{j}}} \xi_E^{(\sigma)}[b_0 x_1 b_1, \dots, x_1 b_k],$$

where ξ denotes the relevant free, half-liberated or classical cumulants. Since the cumulants are determined by the moment-cumulant formulae, we find that

$$\xi_E^{(\sigma)}[b_0 x_{j_1} b_1, \dots, x_{j_k} b_k] = \begin{cases} \xi_E^{(\sigma)}[b_0 x_1 b_1, \dots, x_1 b_k], & \sigma \in D(k) \text{ and } \sigma \le \ker \mathbf{j}, \\ 0, & \text{otherwise.} \end{cases}$$

The result then follows from the characterizations of these joint distributions in terms of cumulants given in Theorem 1.17 and Propositions 2.11 and 2.12. $\square$

REMARK 5.4. For simplicity, we have restricted to elements of a von Neumann algebra, that is, bounded random variables, in the statement of Theorem 1. However, for the easy quantum groups O, B and O^* the result implies that the variables must have unbounded distributions. In the classical setting, the boundedness assumption can be easily replaced by the condition that x_1 has finite moments of all orders. The key differences are as follows:

First, in the classical case one can replace the uniform bound in Theorem 4.8 by the L^p estimate

$$|E_n[x_{j_1} \cdots x_{j_k}] - E[y_{j_1} \cdots y_{j_k}]|_p \le \frac{C_k(G)}{n} |x_1|_{pk}^k,$$

where $|\cdot|_p$ denotes the L^p-norm. The proof is identical, except that one uses Hölder's identity $|x_{i_1} \cdots x_{i_k}|_p \le |x_1|_{pk}^k$ for any $1 \le i_1, \dots, i_p \le n$.

Second, Proposition 5.3 is replaced by a standard L^p reversed martingale convergence theorem (the statement for expectation functionals requiring another application of Hölder).

With these technical modifications, the proof of Theorem 1 shows that any infinite B (resp., O) invariant sequence of classical random variables with finite moments of all orders has the same joint moments with respect to $\mathcal{B}$ as a conditionally i.i.d. (centered) Gaussian family. But this is sufficient to determine the joint distribution with respect to $\mathcal{B}$, since the Gaussian distribution is characterized by its moments.

Likewise, the result for O^* still holds if $(x_i)_{i \in \mathbb{N}}$ are of the form in Example 2.4, where $|\xi_i|$ has finite moments of all orders. The details are left to the reader.

6. Concluding remarks.

We have seen in this paper that the "easiness" condition from [7] provides a good framework for the study of de Finetti type theorems for orthogonal quantum groups.

A first natural question is what happens in the unitary case. For the classical unitary group U_n, it is well known that an infinite sequence of complex-valued random variables is unitarily invariant if and only if they are conditionally i.i.d. centered complex Gaussians. For the free unitary group U_n^+ this is considered in [14], where it is shown that an infinite sequence of noncommutative random variables is quantum unitarily invariant if and only if they form an operator-valued free circular family with mean zero and common variance. However, the study and classification of easy quantum groups seems to be a quite difficult combinatorial problem in the unitary case, we refer to the concluding section of [7] for a discussion here.

In addition to the 14 easy quantum groups discussed in this paper, there are also two infinite series $H_n^{(s)}$ and $H_n^{[s]}$, $s = 2, 3, \dots, \infty$, which are related to the complex reflection groups $H_n^s = \mathbb{Z}_s \wr S_n$. These are described in [5], with the conjectural conclusion that the class of easy quantum groups consists of the 14 examples discussed in this paper, and a multi-parameter "hyperoctahedral series" unifying $H_n^{(s)}$ and $H_n^{[s]}$. It is a natural question whether there are de Finetti type results for this series, with corresponding notions of "independence," and we plan to return to this question after completing the construction.

A third question is whether the approximation result in Theorem 4.8 can be strengthened. The main tool that we have available at this time, namely the Weingarten formula, is only suitable for estimates on the joint moments. In [17], Diaconis and Freedman give refined estimates on the variation norm between the distribution of the coordinates $(u_{11}, \dots, u_{1k})$ on S_n (resp., O_n) and an independent Bernoulli (resp., Gaussian) distribution. This is used to prove finite de Finetti type results, where the approximations hold in variation norm. It is known from [2, 3] that the coordinates $(u_{11}, \dots, u_{1k})$ on S_n^+ and O_n^+ converge in moments to freely independent Bernoulli and semicircular distributions, and it is a natural question whether these converge in a stronger sense. For $k = 1$, it is known from [4] that the distribution of $n^{1/2}u_{11}$ in $C(O_n^+)$ "superconverges" (in the sense of [9]) to the semicircle law, but nothing is currently known for $k > 1$.

Another question is whether the results of Aldous [1] for invariant arrays of random variables have suitable extensions to easy quantum groups. We will consider this problem first for free quantum groups in a forthcoming paper [16].

Another basic symmetry for a sequence of classical random variables is *spreadability*, that is, invariance under taking subsequences. Ryll-Nardzewski proved in [26] that de Finetti's theorem in fact holds under this apparently weaker condition. A free analogue of this condition, and of Ryll-Nardzewski's theorem, has been obtained in [15].

Finally, there is the general question of applying our "S_n, O_n philosophy" to other situations. In [6], we have developed a global approach, using the "easiness" formalism, to the fundamental stochastic eigenvalue computations of Diaconis and Shahshahani [19].

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