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RECURRENCE SETS OF NORMED RANDOM WALK IN R^d [1]

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In this paper we give examples of the sets of recurrent points (or accumulation points) of random walks in R^d normed by nice sequences of constants. These examples, interesting in their own right, give rise to some very interesting conjectures concerning the general structure of such sets. Of particular interest are the recurrent points of the ordinary averages or sample means. It turns out that any closed subset of R^d can be the finite points of recurrence of a sequence of averages: $(X_1 + \cdots + X_n)/n$, X_i i.i.d. random vectors. This seems to be a property not shared by most other normalizing sequences. We also give some results on rates of escape of random walks in a domain of attraction. In looking for rates of escape we are looking for normalizing constants which give rise to no finite recurrent points of the normalized walk.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with values in R^d = real Euclidean space of d-dimensions, and let F denote the common distribution. Put $S_n = \sum_1^n X_K$ and let $\{\gamma_n\}$ be a nondecreasing sequence of positive constants. The recurrence set of $\{S_n/\gamma_n\}$ is the (random) set of accumulation points of $\{S_n/\gamma_n\}$, i.e.:

$$A(S_n; \gamma_n) = \bigcap_{n\geq 1} \overline{\{S_k/\gamma_k : k \geqq n\}} .$$

The bar in the right-hand side denotes closure in the extended, compactified space $\bar{R}^d = R^d \cup R_\infty{}^d$ obtained by adjoining $R_\infty{}^d$, the sphere at ∞.[2] For any set $B \subset \bar{R}^d$ we write B^∞ for $B \cap R_\infty{}^d$ the infinite points of B and B^f for $B \cap R^d$ the finite points of B. An extension of the argument of [5], proof of Theorem 1, page 1174, shows that there exist nonrandom sets $B^f(F, \{\gamma_n\})$, $B^\infty(F, \{\gamma_n\})$ depending only on F and $\{\gamma_n\}$ such that

$$P\{A^i(S_n, \gamma_n) = B^i(F, \{\gamma_n\}); i = f, \infty\} = 1 ,$$

and in fact

$$B^f(F, \{\gamma_n\}) = \{b : P\{\liminf |\gamma_n{}^{-1}S_n - b| = 0\} = 1\} ,$$

$$B^\infty(F, \{\gamma_n\}) = \{\infty_b : P\{\liminf [|b - |S_n|^{-1}S_n| + |S_n|^{-1}\gamma_n] = 0\} = 1\} .$$

Clearly $B^f(B^\infty)$ is a closed possibly empty subset of $R^d(R_\infty{}^d)$ respectively, however $B = B^f \cup B^\infty \neq \emptyset$. One may also define $B_s(F, \{\gamma_n\})$ the strong accumulation

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[2] The points of $R_\infty{}^d$ will be written ∞_b or $\infty \cdot b$ where $b = (b(1), \cdots, b(d))$ is a unit vector $|b| \equiv (\sum_1^d b^2(k))^{\frac{1}{2}} = 1$. Thus a sequence $\{c_n\}$ in R^d converges to $\infty_b \in R^d$ iff $|c_n| \to \infty$ and $c_n/|c_n| \to b$ as $n \to \infty$.

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points of $\{S_n/\gamma_n\}$: $b \in B_s$ if and only if there exist nonrandom sequences $\{n_k\}$ of integers and $\{c_k\}$ of vectors in R^d such that $\lim c_k = b$ and $\lim |\gamma_{n_k}^{-1} S_{n_k} - c_k| = 0$ w.p. 1.[3] $B_s(F, \{\gamma_n\})$ is a closed possibly empty subset of $B(F, \{\gamma_n\})$. Except for Theorems 6 and 7 below and Theorem 8 in Section 4, the main results are examples of $B(F, \{\gamma_n\})$ and $B_s(F, \{\gamma_n\})$ for well-behaved sequences $\{\gamma_n\}$ (primarily $\gamma_n = n^\alpha$). These examples illustrate and extend some results of [1] and [5] and raise some new and tantalizing questions and conjectures. As Hilbert said on another occasion if we could understand the examples we would not need the theorems. In what follows $B(F, \alpha)$ stands for $B(F, \{n^\alpha\})$. Our first result extends the remarkable Theorem 7 of Kesten [5].

THEOREM 1. *If C is any closed subset of real Euclidean space R^d there is a distribution F such that $B^f(F, 1) = C$, moreover, F may be chosen so that $B^\infty(F, 1) = R_\infty{}^d$.*

The strong accumulation points are somewhat easier to control. The next theorem is a straightforward (almost trivial) extension of Theorem 2 of [1] but it seems worth recording.

THEOREM 2. *Given any closed subset C of $\bar{R}^d$ there is a distribution F such that $B_s(F, 1) = C$.*

One interesting aspect of Theorem 2 is that the distribution F constructed in the proof is a product of one-dimensional distributions, that is, if $\{S_n \equiv (S_n(1), \cdots, S_n(d))\}$, $n \geqq 1$, is the induced random walk then $S_n(1), \cdots, S_n(d)$ is, for each n, a set of d independent random variables; yet, $B_s(F, 1)$ could be, for example, the surface of the unit sphere in R^d, a highly "correlated" set.

As another instance of the manipulability of $B_s(F, \{\gamma_n\})$ we have:

THEOREM 3. *Let $d = 1$. If α_1 and α_2 are any two numbers satisfying $\frac{1}{2} < \alpha_1 < \alpha_2 < 1$ then there is a distribution F such that simultaneously $B_s{}^f(F, \alpha_1) = (-\infty, 0]$ and $B_s{}^f(F, \alpha_2) = [0, +\infty)$.*

Consider the following problem: if $\{S_n/n^\alpha\}$ is dense in R^1 (or R^d) w.p. 1 for $\alpha = \alpha_1$ and $\alpha = \alpha_2 > \alpha_1$, does it follow that $\{S_n/n^\alpha\}$ is dense somewhere for $\alpha_1 < \alpha < \alpha_2$? The answer appears to be no, as the following theorem implies; however, it should be noted that the values $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = 1$ play a special role in the theory of $B(F; \alpha)$ as one can easily see from a quick reading of [1] and [5]; thus the answer may be yes if one requires $\alpha_1 > \frac{1}{2}$, $\alpha_2 \neq 1$.

THEOREM 4. *There exists a distribution on R^1 such that $B(F, 1) = B(F, \frac{1}{2}) = \bar{R}^1$ (in fact $B(F, \alpha) = \bar{R}^1$ for all $\alpha \leqq \frac{1}{2}$ as well) but $B(F, \alpha)$ consists of just 3 points $\{+\infty, 0, -\infty\}$ for $\frac{1}{2} < \alpha < 1$. Thus in going from $\alpha = 0$ to $\alpha = 1$ $A^f(S_n; n^\alpha)$ opens to R^1, then closes to a single point and finally opens up again.*

When $\alpha \neq 1$ the possibilities for $B(F, \alpha)$ and $B_s(F, \alpha)$ are quite limited. For

[3] For a finite b this definition of strong accumulation point is equivalent to the definition given in [2]: $\lim_k S_{n_k}/\gamma_{n_k} = b$ a.s. for some deterministic sequence of indices n_k. However, for infinite b our definition is slightly stronger, see Remark 4 in [2], page 565.

example if F on R^1 is not concentrated at 0 then for any $0 < \alpha < \frac{1}{2}$ the set $B^f(F; \alpha)$ is either empty or is all of R^1, there are no other possibilities; $B^f(F, \frac{1}{2})$ is either empty or contains a half line ([5], Theorem 4, page 1173). For any $\alpha \neq 1$ $B_s{}^f(F, \alpha)$ must be one of the five sets $\varnothing$, the single point $\{0\}$, $[0, \infty)$, $(-\infty, 0]$, R^1; and, for $\alpha \leqq \frac{1}{2}$, $B_s{}^f(F, \alpha)$ is always empty ([1], Theorem 1 and remarks preceding Theorem 1, page 554). It was conjectured in [1], page 572 that if $\alpha \neq 1$ and $b \in B(F, \alpha)$ for some $0 < |b| < \infty$ then sign (b) $[0, \infty] \subset B(F, \alpha)$. This conjecture is not true. M. Klass [6] has constructed a distribution F such that for an $\alpha \neq 1$ $\limsup n^{-\alpha}S_n = b$ a.s., where $0 < b < \infty$. Consequently $b \in B(F, \alpha) \subset [-\infty, b]$. (He did not discuss the other limit points of $n^{-\alpha}S_n$). Here is another counterexample.

THEOREM 5. *Let* $\{X_i\}$ *be* i.i.d. *nonnegative random variables with*

$$P\{X_i > t\} \sim ct^{-1/\alpha}(\log\log t)^{1-(1/\alpha)} \quad \text{as} \quad t \to \infty\,, \tag{1.1}$$

where c and α *are constants and* $\alpha > 1$. *Then there is a constant b with* $0 < b < \infty$ *such that*

$$B(F, \alpha) = [b, \infty]\,. \tag{1.2}$$

The modified conjecture may yet be true: If $\alpha \neq 1$ and $E(X_1^+)^{1/\alpha} + E(X_1^-)^{1/\alpha} = \infty$, then $B^f(F, \alpha)$ is one of the sets R^1, $\varnothing$, $\{b\}$, $(-\infty, b]$ or $[b, \infty)$ for some real b. Here is another problem: Characterize those distributions for which (1.2) holds for some $0 < b < \infty$. Does (1.1) have to hold or nearly hold? For more discussion of this problem see Section 4.

For our last results we suppose there is a nondecreasing sequence of constants $\{b_n\}$ such that $b_n \to \infty$ and the distribution of $\{b_n{}^{-1}S_n\}$ converges to a proper d-dimensional stable distribution G_β of exponent β. (This entails, of course, that $\{S_n\}$ be genuinely d-dimensional.[4]) See [9], [2], Chapter XVII or [8] for basic notions. It is known that $0 < \beta \leqq 2$ and G_β has a bounded continuous density g_β (with respect to Lebesgue measure in R^d). Thus we are assuming

$$\lim_n P\{b_n{}^{-1}S_n \in I\} = \textstyle\int_I g_\beta(x)\, dx \tag{1.3}$$

for every Borel I whose boundary ∂I has Lebesgue measure 0. It is also known that the constants b_n are given by

$$b_n = n^{1/\beta}H(n)\,, \tag{1.4}$$

where $H(t)$ is a slowly varying function ($H(tx)/H(t) \to 1$ as $t \to \infty$ for every $x > 0$) which is asymptotically uniquely determined by the tail $P\{|X_1| > t\} = q(t)$ of F (see Remark 4 below).

THEOREM 6. *Under the above assumptions if either* (i) $\beta < \min(2, d)$ *or* (ii) $\beta = 2 < d$ *and F is either arithmetic or nonlattice, then*

$$\liminf n^{-\alpha}|S_n| = \lim n^{-\alpha}|S_n| = \infty \quad \text{a.s.} \tag{1.5}$$

[4] A distribution F in R^d and by extension the corresponding random walk $\{S_n\}$ is said to be genuinely d-dimensional if the support of F is not contained in a $d-1$ dimensional hyperplane.

for every $\alpha < 1/\beta$ (*so* $B^f(F, \alpha) = \emptyset$). *If* $\alpha > 1/\beta$, $\lim n^{-\alpha}|S_n| = 0$ a.s. *In the case* $d = 1$ *and* $\beta > 1$, $B(F, \alpha) = \bar{R}^1$ *for all* $0 < \alpha < 1/\beta$.

NOTE. F on R^d is nonlattice if $|\hat{F}(\theta)| < 1$ for all $\theta \in R^d$, $\theta \neq 0$, where $\hat{F}(\theta) = \int_{R^d} \exp(ix \cdot \theta)F\{dx\}$, $x \cdot \theta$ denotes the usual dot product. F is arithmetic if F is concentrated on a set of the form $\{(n_1 a_1, n_2 a_2, \cdots, n_d a_d) : n_i = 0, \pm 1, \pm 2, \cdots, i = 1, \cdots, d\}$ where $a_1, \cdots, a_d$ are given positive numbers. There are distributions in R^d that are neither nonlattice nor arithmetic. The condition in (iii) that F be arithmetic or nonlattice can almost be dispensed with; see Section 5.

If u and v are functions on a set I we write $u(t) \asymp v(t)$ on $I_0 \subset I$ to mean the ratio $|u(t)/v(t)|$ is bounded away from 0 and ∞ for all $t \in I_0$. The subset I_0 is usually clear from the context and is omitted.

THEOREM 7. *In addition to* (1.3) *suppose further that* $g_\beta(0) \neq 0$, *that* H *of* (1.4) *also satisfies as* $t \to \infty$

$$H(t^\theta) \asymp H(t) \quad \text{uniformly for} \quad \theta \in [p^{-1}, p] \tag{1.6}$$

for some $p > 1$, *and that in the case* $\beta = 2$ F *is either arithmetic or nonlattice* (*again this can be weakened*). *If* $\beta < d$ *and if* $\psi(t)$ *is nonincreasing and slowly varying at* ∞, *then* $P\{|S_n| \leqq b_n \psi(n) \text{ i.o.}\} = 0$ *or* 1 *according as*

$$\sum_{n=1}^{\infty} \frac{\psi^{d-\beta}(n)}{n} \tag{1.7}$$

converges or diverges.

COROLLARY 1 (to the proof of Theorem 7). *In case* (1.7) *diverges we have in fact* $B(F, \{\gamma_n\}) = \bar{R}^d$ *and, in case of convergence, we have* $\liminf \gamma_n^{-1}|S_n| = \lim \gamma_n^{-1}|S_n| = \infty$ a.s., $\gamma_n = b_n \psi(n)$.

COROLLARY 2. *If* $d \geqq 3$, $E(X_1) = 0$ *and the matrix* $(EX_1(i)X_1(j))$ $1 \leqq i, j \leqq d$ *is finite and nonsingular* (*and* F *is arithmetic or nonlattice*), *then, with* ψ *as above*, $P\{|S_n| \leqq n^{\frac{1}{2}}\psi(n) \text{ i.o.}\} = 0$ *or* 1 *according as* $\sum_n \psi^{d-2}(2^n)$ *converges or diverges.*

REMARKS. 1. Although we will not prove it here, in the case $\beta = d = 1$ of Theorem 6 it can be shown that for any $\alpha \leqq 1$ either $B(F, \alpha) = \bar{R}^1$ or else $B(F, \alpha) \subset \{+\infty, -\infty\}$; and if H satisfies (1.6) then $B(F, \alpha) = \bar{R}^1$.

2. Theorem 7 is a generalization of well-known result on rates of escape of isotropic stable processes in R^d; see [3], page 361. Corollary 2 is of course a generalization of the Dvoretsky–Erdös test for simple random walk or Brownian motion in R^d, $d \geqq 3$. As may easily be shown (1.7) converges (diverges) if and only if $\sum \psi^{d-\beta}(2^n)$ or $\int^\infty x^{-1}\psi^{d-\beta}(x)\,dx$ converges (diverges).

3. Condition (1.6) seems stringent, but it vastly simplifies certain formulas. One may see from the proof of Theorem 7 how to formulate a more general theorem allowing arbitrary slowly varying H to appear in (1.4) (provided $n^{1/\beta}H(n)$ is nondecreasing). The resulting criterion is in terms of a series which involves both H and ψ in an unpleasant manner. We leave the details to regularly varying

function enthusiasts. It should be noted that (1.8) is satisfied whenever H satisfies

(1.8) $$H(t) \asymp \prod_{k=1}^{r} (\log_k t)^{a_k}, \qquad t \geqq t_0$$

for some constants $a_1, \cdots, a_r$ ($\log_k$ denotes the kth iterated log function). Finally note that if (1.6) holds for some $p = p_0 > 1$, then (1.6) holds for all $p > 1$.

4. We take H as given in Theorems 6 and 7; however it can be shown that when $E|X_1|^2 = \infty$ the constants b_n may be defined by

$$b_n = \inf \left\{ t : P\{|X_1| > t\} \leqq \frac{1}{n} \leqq P\{|X_1| \geqq t\} \right\}$$

(when $E|X_1|^2 < \infty$, $\beta = 2$, $b_n = cn^{\frac{1}{2}}$ so $H(n) = c$). From this we see that H, being slowly varying, is asymptotically uniquely determined from the formulas

$$\lim L(t^{1/\beta}H(t))/H^\beta(t) = 1 ,$$

where $L(t) = t^\beta P\{|X_1| > t\}$. (See [9] or [2], XVII.) A simple argument (use [2], page 277) now shows that H satisfies (1.6) (or (1.8)) if and only if L satisfies (1.6) (or (1.8)).

5. Conjecture: If $d \geqq 3$ and $\{S_n\}$ is any genuinely d-dimensional random walk then

(1.9) $$\lim_{n\to\infty} n^{-\alpha}|S_n| = \infty \quad \text{a.s.} \qquad \text{for all} \quad \alpha < \tfrac{1}{2} .$$

Thus Theorem 6 says, roughly, that (1.9) holds whenever (1.3) holds. For more discussion see Section 5.

While preparing this paper for publication, Professor Kesten informed me that Mr. Steven Kalikow of Cornell has also obtained, independently, a proof of Theorem 1. Elsewhere, one of us (hopefully Mr. Kalikow) will give a thorough discussion of the structure of $B^\infty(F, 1)$. The proof of Theorem 1 is sketched in Section 6.

I would like to thank Professor Kesten for pointing out to me some useful facts in connection with Theorem 4.

2. Proof of Theorems 2 and 3. Let us write for any real random variable X with distribution F

$$q(t) = P\{|X| > t\} , \qquad v(t) = E(X^2; |X| \leqq t) = \int_{-t}^{t} x^2 F\{dx\}$$
$$\mu(t) = E(X; |X| \leqq t) = \int_{-t}^{t} xF\{dx\} .$$

Let $X = (X(1), \cdots, X(d))$ be a random vector and let $q_i, v_i, \mu_i, i = 1, \cdots, d$ be the quantities above defined for $X(i)$, $i = 1, \cdots, d$. Let $X_1, X_2, \cdots$ be independent copies of X, $S_t = \sum_{k=1}^{[t]} X_k$.

LEMMA 1. *In order that $b \in \bar{R}^d$ be a strong accumulation point of $\{S_n/n^\alpha\}$ it is necessary and sufficient that there be a sequence $t_k \to \infty$ as $k \to \infty$ such that*

$$t_k q_i(t_k^\alpha) \to 0 \quad \textit{and} \quad t_k^{1-2\alpha} v_i(t_k^\alpha) \to 0 \quad \textit{as} \quad k \to \infty \quad \textit{for each}$$
$$i = 1, \cdots, d \quad \textit{and}$$

(2.1) $$t_k^{1-\alpha}\mu(t_k^\alpha) \to b \qquad \textit{as} \quad k \to \infty \quad \text{(see footnote 2)}$$

where $\mu(t) = (\mu_1(t), \cdots, \mu_d(t))$. *If* (2.1) *holds then, whether or not* $\lim t_k^{1-\alpha}\mu(t_k^{\alpha})$ *exists,*

$$|t_k^{-\alpha}S_{t_k} - t_k^{1-\alpha}\mu(t_k^{\alpha})| \to_p 0 \qquad as \quad k \to \infty .$$

PROOF. This is a straightforward generalization of Lemma 5 in [1], page 564. Note again that when b is infinite our definition of strong accumulation point is stronger than that given in [1].

PROOF OF THEOREM 2. We proceed as in [1], page 567. Let $\{c_k = (c_k(1), \cdots, c_k(d))\}_{k=0}^{\infty}$ be points in R^d so that $c_0 = 0$ and

$$\bigcap_{n=1}^{\infty} \overline{\{c_n, c_{n+1}, \cdots\}} = C = \text{the given closed set.}$$

Assume also that $c_k \neq c_{k-1}$. Define the positive nondecreasing sequences $\{b_k\}$ and $\{a_k\}$ as in (3.10) and (3.11), respectively or page 567 of [1]; recall that $|c| = (c^2(1) + \cdots + c^2(d))^{\frac{1}{2}}$ for $c = (c(1), \cdots, c(d))$. Next define the distribution F of a random vector $X = (X(1), \cdots, X(d))$ by

$$P\{X(i) = a_k\} = (2b_k + c_k(i) - c_{k-1}(i))/2a_k$$
$$P\{X(i) = -a_k\} = (2b_k + c_{k-1}(i) - c_k(i))/2a_k , \qquad i = 1, \cdots, d$$

and

$$P\{X = (z_1, \cdots, z_d)\} = \prod_{j=1}^{d} P\{X(j) = z_j\} ,$$

where $z_1, \cdots, z_d$ take on (independently) any of the values $\{\pm a_1, \pm a_2, \cdots\}$. Thus the components of X are independent. Note that for $i = 1, \cdots, d$

$$\mu_i(t) = \sum_{r=1}^{k} (c_r(i) - c_{r-1}(i)) = c_k(i) \qquad \text{for} \quad a_k \leqq t < a_{k+1} .$$

Hence by Lemma 1 ($\alpha = 1$)

$$B_s(F, 1) \subset \bigcap_{n\geqq 1} \overline{\{c_k : k \geqq n\}} = C . \tag{2.2}$$

Defining $t_k = (a_k a_{k+1})^{\frac{1}{2}} \in (a_k, a_{k+1})$ and proceeding as on page 568 of [1] we find that

$$\lim_{k\to\infty} (t_k^{-1}v_i(t_k) + t_k q_i(t_k)) = 0 , \qquad i = 1, \cdots, d .$$

Hence, by Lemma 1, as $k \to \infty$

$$|t_k^{-1}S_{t_k} - \mu(t_k)| = |t_k^{-1}S_{t_k} - c_k| \to 0 \quad \text{in probability.}$$

On considering a.s. convergent subsequences, we conclude $\bigcap_{n\geqq 1} \overline{\{c_k : k \geqq n\}} = C \subset B_s(F, 1)$. This and (2.2) give $B_s(F, 1) = C$.

PROOF OF THEOREM 3. Let $\frac{1}{2} < \alpha_1 < \alpha_2 < 1$. To simplify some of the notation let us write $\alpha = \alpha_1$ and $\beta = \alpha_2$ for this proof only. We are going to construct an F so that $B_s^f(F, \alpha) = (-\infty, 0]$ and $B_s^f(F, \beta) = [0, \infty)$. It will follow incidentally that $B(F, \gamma) = \bar{R}$ for $\alpha \leqq \gamma \leqq \beta$ by Theorem 4 of [1]. Put

$$\lambda = \frac{2\beta(1 - \alpha)}{(1 - \beta)(2\alpha - 1)} - 1 .$$

Then

$$\lambda = \left(1 + \frac{2(\beta - \alpha)}{1 - \beta}\right)\left(\frac{1}{2\alpha - 1}\right) > \frac{1}{2\alpha - 1} > 1 .$$

Let $\{b_n\}$ be any sequence of numbers such that $2^{1/(1-\beta)} < b_1 < b_2 < \cdots \to \infty$ and

(2.3) $$\sum_{n=1}^{\infty} b_n^{-1} = 1 ,$$

(2.4) $$(b_n b_{n+1})^{1/2\alpha} \sum_{j=n+1}^{\infty} b_j^{-1} \to 0 \quad \text{and}$$
$$(b_n b_{n+1})^{(1/2\alpha)-1} \sum_{j=1}^{n} b_j \to 0 \quad \text{as} \quad n \to \infty ,$$

and for some positive finite constant A

(2.5) $$b_{n+1} < A b_n^{\lambda} \qquad \text{for all} \quad n \geqq 1 ,$$

where λ is given above. The choice $b_n = c_1 \exp(c_2 \exp(\theta n))$ for some positive constants c_1, c_2 and $\theta = \log \lambda$ will work as the reader may verify. (The fact $\lambda > 1/(2\alpha - 1)$ is needed for this.) Since $\frac{1}{2} < 1/2\beta < 1/2\alpha < 1$ it follows that also

(2.6) $$(b_n b_{n+1})^{1/2\beta} \sum_{j=n+1}^{\infty} b_j^{-1} \to 0 \quad \text{and}$$
$$(b_n b_{n+1})^{(1/2\beta)-1} \sum_{j=1}^{n} b_j \to 0 \quad \text{as} \quad n \to \infty .$$

Define

$$s_n = (b_n b_{n+1})^{1/2\alpha} \qquad \text{if} \quad n \quad \text{is even,}$$
$$= (b_n b_{n+1})^{1/2\beta} \qquad \text{if} \quad n \quad \text{is odd.}$$

Then

(2.7) $$b_{2j} < s_{2j}^{\alpha} < b_{2j+1} \quad \text{and} \quad b_{2j-1} < s_{2j-1}^{\beta} < b_{2j} , \qquad j \geqq 1 .$$

Next define p_n, q_n by $b_1(p_1 - q_1) = s_1^{\beta-1}$ and

(2.8) $$b_n(p_n + q_n) = 1 , \qquad n \geqq 1 ,$$

(2.9) $$b_n(p_n - q_n) = s_n^{\beta-1} + s_{n-1}^{\alpha-1} \qquad \text{for} \quad n \quad \text{odd;}$$
$$= -(s_n^{\alpha-1} + s_{n-1}^{\beta-1}) \qquad \text{for} \quad n \quad \text{even.}$$

Since $b_n > 2^{1/(1-\beta)}$, $\frac{1}{2} < \alpha < \beta < 1$ and $s_n \to \infty$ we have $\frac{1}{2} > s_n^{\beta-1} \geqq s_n^{\alpha-1}$ and $\lim s_n^{\beta-1} = \lim s_n^{\alpha-1} = 0$ so that on solving for p_n and q_n we find that

$$p_n > 0 , \qquad q_n > 0 \qquad \text{and} \qquad p_n/q_n \to 1 \quad \text{as} \quad n \to \infty .$$

We now define F to be the distribution with weights p_n, q_n at b_n, $-b_n$ respectively, i.e.,

$$P\{X_1 = b_n\} = p_n \qquad P\{X_1 = -b_n\} = q_n , \qquad n \geqq 1 .$$

From (2.3), (2.8) and the preceding remark it follows that this defines a genuine probability distribution concentrated on $\{\pm b_1, \pm b_2, \cdots\}$. Now $\mu(z) = \int_{-z}^{z} xF\{dx\} = \sum_{b_k \leqq z} b_k(p_k - q_k)$ so by (2.9)

(2.10) $$\mu(z) = -s_{2k}^{\alpha-1} \qquad \text{when} \quad b_{2k} \leqq z < b_{2k+1} ,$$
$$= s_{2k-1}^{\beta-1} \qquad \text{when} \quad b_{2k-1} \leqq z < b_{2k} .$$

Similarly by (2.8)

$$(2.11)\qquad q(z) = P\{|X_1| > z\} = \textstyle\sum_{j=n+1}^{\infty} b_j^{-1} \quad \text{and}$$
$$v(z) = \textstyle\int_{-z}^{z} x^2 F\{dx\} = \sum_{j=1}^{n} b_j \quad \text{for} \quad b_n \leqq z < b_{n+1}\,.$$

Put $t_k = s_{2k}$ and $r_k = s_{2k-1}$. Then $t_k^{\alpha} \in (b_{2k}, b_{2k+1})$ and $r_k^{\beta} \in (b_{2k-1}, b_{2k})$ by (2.7). Applying (in order) (2.10), (2.11), the definition of s_n, (2.4) and finally (2.6), we conclude $t_k^{1-\alpha}\mu(t_k^{\alpha}) = -1$, $r_k^{1-\beta}(r_k^{\beta}) = +1$, and as $k \to \infty$ $t_k q(t_k^{\alpha}) \to 0$, $t_k^{1-2\alpha}v(t_k^{\alpha}) \to 0$, $r_k q(r_k^{\beta}) \to 0$ and $r_k^{1-2\beta}v(r_k^{\beta}) \to 0$. Hence, by Lemma 1 ($d = 1$), as $k \to \infty$ $t_k^{-\alpha}S_{t_k} \to -1$ and $r_k^{-\beta}S_{r_k} \to +1$ in probability. Consequently (by going over to a.s. convergent subsequences), $-1 \in B_s(F, \alpha)$ and $+1 \in B_s(F, \beta)$. But this fact and Theorem 1 of [1], page 554 enable us to conclude

$$(2.12)\qquad (-\infty, 0] \subset B_s^f(F, \alpha) \qquad \text{and} \qquad [0, +\infty) \subset B_s^f(F, \beta)\,.$$

To get the reverse inclusions we must show that $B_s^f(F, \alpha) \cap (0, \infty) = \emptyset$ and $B_s^f(F, \beta) \cap (-\infty, 0) = \emptyset$. Suppose to the contrary. Then, by Theorem 1 of [1] again, every positive number is in $B_s(F, \alpha)$ or every negative number is in $B_s(F, \beta)$. In particular by Lemma 1 it follows that there exist $n_1 < n_2 < \cdots \to \infty$, $m_1 < m_2 < \cdots \to \infty$ such that (2.13) or (2.14) hold where

$$(2.13)\qquad \lim_{j\to\infty} n_j^{1-\alpha}\mu(n_j^{\alpha}) = +1\,,$$

$$(2.14)\qquad \lim_{j\to\infty} m_j^{1-\beta}\mu(m_j^{\beta}) = -1\,.$$

We show that (2.13) and (2.14) each lead to a contradiction.

Consider (2.13) first. We may suppose that $\mu(n_j^{\alpha}) > 0$ for all j. Consideration of (2.10) leads us to conclude that there exists $k_j \to \infty$ such that

$$b_{2k_j-1} \leqq n_j^{\alpha} < b_{2k_j}$$

and $\mu(n_j^{\alpha}) = s_{2k_j-1}^{\beta-1}$ for all j. Returning to (2.13) we see that this implies

$$(2.15)\qquad n_j \sim (s_{2k_j-1})^{(1-\beta)/(1-\alpha)} \qquad \text{as} \quad j \to \infty$$

($x_n \sim y_n$ means $x_n/y_n \to 1$ as $n \to \infty$). Now by Lemma 1 we must also have

$$(2.16)\qquad n_j^{1-2\alpha}v(n_j^{\alpha}) \to 0 \qquad \text{as} \quad j \to \infty\,.$$

But from (2.11) and (2.15) we have

$$n_j^{1-2\alpha}v(n_j^{\alpha}) = (s_{2k_j-1})^{(1-\beta)(1-2\alpha)/(1-\alpha)}\left(\textstyle\sum_{i=1}^{2k_j-1} b_i\right)(1 + o(1))$$
$$> \tfrac{1}{2} b_{2k_j-1}(s_{2k_j-1})^{-(1-\beta)(2\alpha-1)/(1-\alpha)}$$

eventually. Write $N = 2k_j - 1$. Then N is odd so $s_N = (b_N b_{N+1})^{1/2\beta}$. On recalling (2.5) (for the first time) we get for all j sufficiently large

$$\log [n_j^{1-2\alpha}v(n_j^{\alpha})] > -(2\alpha - 1)\left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{1}{2\beta}\right)\log (b_N b_{N+1}) + \log b_N - \log 2$$
$$> \left[1 - (1 + \lambda)\frac{(1-\beta)(2\alpha-1)}{2\beta(1-\alpha)}\right]\log b_N + \delta$$

(note that $2\alpha - 1 > 0$), where $\delta = -((1-\beta)(2\alpha-1)/2\beta(1-\alpha))\log A - \log 2$.

But $1 + \lambda = 2\beta(1-\alpha)/[(1-\beta)(2\alpha-1)]$. Hence,

$$\liminf_{j\to\infty} n_j^{1-2\alpha} v(n_j^{\alpha}) \geqq e^{\delta} > 0$$

which contradicts (2.16). Thus $+1 \notin B_s(F, \alpha)$, so $B_s^f(F, \alpha) \cap (0, \infty) = \emptyset$ and it follows from (2.12) that $B_s^f(F, \alpha) = (-\infty, 0]$.

Consider now (2.14). Arguing as above we see that (2.14) gives for some $k_j \to \infty$ $b_{2k_j} \leqq m_j^{\beta} < b_{2k_j+1}$ and then

$$m_j \sim (s_{2k_j})^{(1-\alpha)/(1-\beta)} \quad \text{as} \quad j \to \infty .$$

Lemma 1 says that we must have

$$m_j q(m_j^{\beta}) \to 0 \quad \text{as} \quad j \to \infty . \tag{2.17}$$

Write $M = 2k_j$. Then M is even so $s_M = (b_M b_{M+1})^{1/2\alpha}$ and recalling (2.5) again we have for j sufficiently large

$$\begin{aligned} \log[m_j q(m_j^{\beta})] &= \log[(s_M)^{(1-\alpha)/(1-\beta)}(\textstyle\sum_{i=M+1}^{\infty} b_i^{-1})(1+o(1))] \\ &> \frac{(1-\alpha)}{2\alpha(1-\beta)} \log(b_{M+1}b_M) - \log b_{M+1} - \log 2 \\ &> \left[\frac{1}{2\alpha}\left(\frac{1}{\lambda}+1\right)\left(\frac{1-\alpha}{1-\beta}\right) - 1\right]\log b_{M+1} - \delta_1 \\ &\equiv \Delta \log b_{M+1} - \delta_1 , \end{aligned} \tag{2.18}$$

where $\delta_1 = \log 2 + (1-\alpha)/[2\alpha\lambda(1-\beta)]\log A$. Now $\lambda + 1 = 2\beta(1-\alpha)(1-\beta)^{-1}(2\alpha-1)^{-1}$ and we have

$$\begin{aligned} \lambda\Delta &= \frac{1}{2\alpha}\left(\frac{1-\alpha}{1-\beta}\right)(\lambda+1) - \lambda \\ &= \tfrac{1}{2}(\lambda+1)\left[\frac{1-\alpha}{\alpha(1-\beta)} + \frac{(1-\beta)(2\alpha-1)}{\beta(1-\alpha)} - 2\right] \\ &= \frac{(\lambda+1)(\beta-\alpha)}{2\alpha\beta(1-\beta)(1-\alpha)}[1-\alpha(2-\beta)] , \end{aligned}$$

as the reader may check. But, $\alpha < \beta < 2$ so $1 - \alpha(2-\beta) > 1 - \beta(2-\beta) = (1-\beta)^2 > 0$. Hence $\Delta > 0$ and returning to (2.18) we get

$$\lim_{j\to\infty} m_j q(m_j^{\beta}) \geqq e^{-\delta_1} \lim_{M\to\infty} b_{M+1}^{\Delta} = \infty$$

contradicting (2.17). Thus $-1 \notin B_s(F, \beta)$, $B_s(F, \beta) \cap (-\infty, 0) = \emptyset$ and $B_s^f(F, \beta) = [0, \infty)$ by (2.12). This concludes the proof of Theorem 3 except for the remark $B(F, \gamma) = \bar{R}$ for $\alpha \leqq \gamma \leqq \beta$. But clearly for any $a > 0$, $\gamma \in [\alpha, \beta] \subset (\frac{1}{2}, 1)$,

$$\limsup_{n\to\infty} P\{|n^{-\gamma}S_n| \leqq a\} = 1 .$$

Also since $p_n/q_n \to 1$ as $n \to \infty$, see (2.8)—(2.9), it follows that $\min(p_n, q_n) > \frac{1}{4}(p_n + q_n)$ for $n \geqq n_0$, hence, for $\gamma \leqq \beta < 1$,

$$\min\{E(X_1^+)^{1/\gamma}, E(X_1^-)^{1/\gamma}\} \geqq \tfrac{1}{4}\textstyle\sum_{n=n_0}^{\infty} b_n^{1/\beta}(p_n + q_n) = \tfrac{1}{4}\textstyle\sum_{n=n_0}^{\infty} b_n^{(1/\beta)-1} = \infty .$$

Thus Theorem 4 of [1] applies and we get $B(F, \gamma) = \bar{R}$ as asserted.

3. Proof of Theorem 4. To prove Theorem 4 we will use the construction in the proof of (as well as the conclusion of) Theorem 7 in Section 5 of Kesten [5], pages 1196–1204. All notation will be more or less the same as found there and many steps will be omitted. Numbers (3.x) refer to displays in this paper, numbers (5.x) to displays in [5].

Let $\{c_k\}$ be a sequence of real numbers such that

$$c_k = 0 \quad \text{for infinitely many} \quad k\,; \tag{3.1}$$

$$\text{if} \quad c_k \neq 0\,, \quad \text{then} \quad 1/k \leqq |c_k| \leqq k\,; \tag{3.2}$$

$$\bigcap_{m=1}^{\infty} \overline{\{c_k : k \geqq m\}} = \bar{R}\,. \tag{3.3}$$

Next let $\delta_k \in (0, \frac{1}{2}]$ decrease slowly to 0, the choice

$$\delta_1 = \tfrac{1}{2}\,, \qquad \delta_k = (4 \log k)^{-1}\,, \qquad k \geqq 2 \tag{3.4}$$

will suffice, and put $\alpha_k = (1 + \delta_k)/2$. Next define inductively the parameters a_k, b_k, p_k, r_k, λ_k and ν_k so that they satisfy the relations (5.26)—(5.32), page 1199, and in addition we require, for all k,

$$p_{k+1}^{\delta_{k+1}} \leqq \min\{[k^6 \textstyle\sum_{j=1}^{k-1} p_j(a_j^2 r_j^2 + b_j^2)]^{-2}, \delta_{k+1}^2\} \qquad \text{and} \tag{3.5}$$
$$p_{k+1} \leqq \min\{e^{-(k+1)}, p_k\}\,;$$

$$a_k \geqq p_{k+1}^{-3}(512 k^6 p_k p_{k-1}(a_{k-1}^2 r_{k-1}^2 + b_{k-1}^2))^{\frac{1}{2}}\,; \tag{3.6}$$

$$\lambda_k^{\delta_k} \geqq \max\{[16 k^2 p_{k-1}(a_{k-1}^2 r_{k-1}^2 + b_{k-1}^2)\delta_k^{-1}], k^3/\delta_k p_k\}\,. \tag{3.7}$$

Now the right-hand sides of (3.5)—(3.7) involve subsets of the same parameters which occur in the corresponding right-hand sides of (5.30), (5.32) and (5.27); moreover the direction of the inequalities in each of (3.5)—(3.7) is the same as the directions of the corresponding inequalities in (5.30), (5.32) and (5.27). It follows that (3.5)—(3.7) is compatible with the inductive construction on page 1199 and therefore such a set can be chosen. Let F be the distribution (5.6) with these parameters:

$$F(x) = p_0 \varepsilon_0(x) + \sum_{k=1}^{\infty} p_k G_k\left(\frac{x - b_k}{a_k}\right)$$

with G_k uniform on the integers in $[-r_k, r_k]$ and ε_0 the distribution with unit mass concentrated at the origin. By (3.3) and the proof of Theorem 7 of [5] we have

$$A(S_n; n) = B(F, 1) = \bar{R}\,. \tag{3.8}$$

Moreover, on account of (3.1) $\{S_n\}$ is recurrent (last paragraph page 1204) so $\liminf |S_n| < \infty$ a.s.; and then, of course, $\liminf n^{-\alpha}|S_n| = 0$ a.s. or

$$0 \in B(F, \alpha) \qquad \text{for all} \quad \alpha > 0\,. \tag{3.9}$$

By [5] (Theorem 4, page 1173, and the remark at the end of 4 on page 1190), (3.9) implies

$$B(F, \alpha) = \bar{R} \qquad \text{for} \quad 0 < \alpha \leqq \tfrac{1}{2}\,. \tag{3.10}$$

From (3.8) and $n^\alpha < n$ for $\alpha < 1$ we have $\limsup n^{-\alpha}S_n \geqq \limsup n^{-1}S_n = +\infty$ and $\liminf n^{-\alpha}S_n \leqq \liminf n^{-1}S_n = -\infty$, a.s. This and (3.9) give

$$\{+\infty, 0, -\infty\} \subset B(F, \alpha) \qquad \text{for } \tfrac{1}{2} < \alpha < 1 . \tag{3.11}$$

The hard part is to demonstrate the reverse inclusion:

$$B(F, \alpha) \subset \{+\infty, 0, -\infty\} \qquad \text{for } \tfrac{1}{2} < \alpha < 1 . \tag{3.12}$$

Put $j_0 = \min\{j: \text{(5.36) holds for all } k \geqq j\}$; of course j_0 is random but, as shown on page 1200, $P\{j_0 < \infty\} = 1$ and for all $k \geqq j_0$ we have the decomposition (5.14), namely,

$$S_n = V_n{}^k + U_n{}^{k-1} + U_n{}^k \qquad \text{on } N_k \leqq n < N_{k+1} . \tag{5.14}$$

Write

$$\Delta_j = c_j - c_{j-1}$$
$$\alpha_j = \tfrac{1}{2} + \tfrac{1}{2}\delta_j .$$

CLAIM. There is a sure event Ω_0 such that on Ω_0 for every $\varepsilon > 0$ there is a finite random integer $j_1 \geqq j_0$ so that $k \geqq j_1$ implies

$$n^{-\alpha_k}|V_n{}^k - nc_{k-2}| \leqq \varepsilon \qquad \text{and} \tag{3.13a}$$

$$n^{-\alpha_k}|U_n{}^{k-1} - n\Delta_{k-1}| \leqq \tfrac{1}{8}a_k p_{k+1} k^{-1} \qquad \text{for all } n \geqq N_k ; \tag{3.13b}$$

$$n^{-\alpha_k}|U_n{}^{k-1} - n\Delta_{k-1}| \leqq \varepsilon \qquad \text{and} \tag{3.14a}$$

$$n^{-\alpha_k}|U_n{}^k - n\Delta_k| \leqq \varepsilon \qquad \text{for any } n \in [N_k, N_{k+1}) \tag{3.14b}$$

satisfying

$$\textstyle\sum_{i=1}^n Y_i{}^k I(\eta_i = k) = 0 ; \tag{3.15}$$

$$n^{-1}|U_n{}^k - n\Delta_k| \geqq \tfrac{3}{8}a_k p_{k+1} k^{-1} \quad \text{whenever } n \in [N_k, N_{k+1}) \text{ and (3.15) fails.} \tag{3.16}$$

Again I remind the reader that notation and basic setup are as on pages 1197–1204, 1184(b) of [5].

PROOF OF (3.12) FROM THE CLAIM. Fix ε, α with $0 < \varepsilon < 1$ and $\frac{1}{2} < \alpha < 1$. Let j_1 be as in the claim, $k \geqq j_1 + 3$ and $n \in [N_k, N_{k+1})$. Then by (5.14)

$$S_n - nc_k = U_n{}^k - n\Delta_k + U_n{}^{k-1} - n\Delta_{k-1} + V_n{}^k - nc_{k-2} .$$

(A) Suppose (3.13) and (3.14) hold. Then

$$||n|c_k| - |S_n|| \leqq |S_n - nc_k| \leqq 3\,\varepsilon\, n^{\alpha_k} \qquad \text{and thus}$$
$$n^{-\alpha}|S_n| \leqq 3n^{\alpha_k - \alpha} \quad \text{if } c_k = 0 \qquad \text{and}$$
$$n^{-\alpha}|S_n| \geqq \frac{1}{k} n^{1-\alpha} - 3n^{\alpha_k - \alpha} \quad \text{if } c_k \neq 0 ;$$

see (3.2).

(B) Suppose (3.13) and (3.16) hold. Then

$$|S_n - nc_k| \geqq n(\tfrac{3}{8}a_k p_{k+1} k^{-1}) - n^{\alpha_k}(\tfrac{1}{8}a_k p_{k+1} k^{-1}) - n^{\alpha_k}\varepsilon$$
$$\geqq \tfrac{8}{3}k^2(\tfrac{3}{8}n - \tfrac{1}{8}n^{\alpha_k}) - n \geqq (\tfrac{2}{3}k^2 - 1)n$$

since $a_k p_{k+1} \geqq 8k^3/3$ by (5.32). Now $|c_k| \leqq k$ for all k by (3.1)—(3.2) and it follows that

$$n^{-\alpha}|S_n| \geqq (\tfrac{2}{3}k^2 - k - 1)n^{1-\alpha} \geqq \tfrac{1}{9}k^2 n^{1-\alpha} > \frac{1}{k} n^{1-\alpha}$$

in this case. Now $p_k \leqq e^{-k}$ by (3.5) so when (5.36) holds we have

$$N_k \geqq (k^2 p_k)^{-1} \geqq k^{-2}e^k .$$

Also for $k > K_\alpha$ sufficiently large ($K_\alpha = \exp(4\alpha - 2)^{-1}$ if (3.4) holds) we will have

$$\alpha_k = \tfrac{1}{2}(1 + \delta_k) < \tfrac{1}{2}(\alpha + \tfrac{1}{2}) < \alpha .$$

Combining these facts with (A) and (B) we find that (on Ω_0) if $k \geqq j_2 = j_1 + 3 + K_\alpha$ and $n \in [N_k, N_{k+1})$

(3.17) $$n^{-\alpha}|S_n| \geqq k^{-(3-2\alpha)}e^{(1-\alpha)k} - 3 \quad \text{if either (3.14) holds and } c_k \neq 0 \quad \text{or else if (3.16) holds; and}$$

(3.18) $$n^{-\alpha}|S_n| \leqq 3k^{\alpha-\frac{1}{2}}e^{-(\alpha-\frac{1}{2})k/2} \quad \text{if} \quad c_k = 0 \quad \text{and (3.14) holds.}$$

Let Z denote the positive integers and let

$$A_1 = \bigcup\nolimits_{k \geqq j_2, c_k \neq 0} [N_k, N_{k+1}) \cap Z ,$$
$$A_2 = \bigcup\nolimits_{k \geqq j_2, c_k = 0} \{n : n \in [N_k, N_{k+1}) \text{ and (3.15) fails (so (3.16) holds)}\} ,$$
$$A_3 = \bigcup\nolimits_{k \geqq j_2, c_k = 0} \{n : n \in [N_k, N_{k+1}) \text{ and (3.15) holds (so (3.14) holds)}\} .$$

Then A_1, A_2, A_3 are random disjoint sets of positive integers and, more importantly, on account of (3.1) and (3.3) and

$$P\{(3.15)_{k,n} \text{ fails for all } n \in [N_k, L_k) \text{ for all } k \text{ suff. large}\} = 1 ,$$
$$P\{(3.15)_{k,n} \text{ holds for some } n \in [L_k, N_{k+1}) \text{ for all } k \text{ suff. large}\} = 1$$

(see pages 1203–1204; $L_k \in (N_k, N_{k+1})$ is defined on page 1201), it follows that A_1, A_2, A_3 are each infinite and with probability 1

(3.19) $$Z \backslash \{1, \cdots, n-1\} = \{n, n+1, \cdots\} \subset A_1 \cup A_2 \cup A_3 \quad \text{eventually.}$$

For each n define k_n to be the unique (if it exists) random solution to $N_k \leqq n < N_k$. Then k_n is defined for all n sufficiently large and $k_n \to \infty$ w.p. 1. Since A_1, A_2, A_3 are infinite we have from (3.17)—(3.18)

(3.20) $$\lim\nolimits_{n\to\infty; n \in A_1 \cup A_2} n^{-\alpha}|S_n| \geqq \lim\nolimits'_{n\to\infty} k_n^{-(3-2\alpha)}e^{(1-\alpha)k_n} - 3 = \infty \quad \text{and}$$
$$\lim\nolimits_{n\to\infty; n \in A_3} n^{-\alpha}|S_n| \leqq 3 \lim\nolimits_{n\to\infty} k_n^{\alpha-\frac{1}{2}} \exp(-\tfrac{1}{2}(\alpha - \tfrac{1}{2})k_n) = 0$$

w.p. 1. Clearly (3.12) follows immediately from (3.19) and (3.20).

Proof of the claim. Let us suppose for definiteness that δ_k is given by (3.4). Next note that the assertion of (3.16) is already established: pages 1201–1203, in particular the last few lines on page 1203, note that (3.15) is (5.56). So we need only prove (3.13) and (3.14).

LEMMA. *If* $\{\gamma_i\}$ *are* i.i.d. *random variables with* $E\gamma_i = 0$, $E\gamma_i^2 = \sigma^2 < \infty$ *and* $\Gamma_n = \gamma_1 + \cdots + \gamma_n$. *Then for* $\alpha > \frac{1}{2}$, $\varepsilon > 0$.

$$P\{|\Gamma_n| \geqq n^\alpha \varepsilon \text{ for some } n \geqq A\} \leqq 24\sigma^2[(2\alpha - 1)A^{2\alpha-1}\varepsilon^2]^{-1}. \tag{3.21}$$

PROOF. This is proved as (5.37) is proved. Use the fact that $2^{2\alpha-1} - 1 > (2\alpha - 1)\log 2$ for $\alpha > \frac{1}{2}$.

PROOF OF (3.13a). Apply (3.21) with $A = (k^2p_k)^{-1}$, $\alpha = \alpha_k$, $\Gamma_n = V_n^k - nc_{k-2}$ and (see page 1201)

$$\sigma^2 \leqq 2k \textstyle\sum_{j=1}^{k-2} p_j(a_j^2r_j^2 + b_j^2) \leqq 2k((k-1)^6 p_k^{\delta_k/2})^{-1};$$

see (3.5). Thus as in (5.38),

$$\begin{aligned} P\{|V_n^k - nc_{k-2}| &\geqq n^{\alpha_k}\varepsilon \text{ for some } n \geqq N_k\} \\ &= O(k^{-2}) + O(k^{-5}p_k^{-\delta_k/2}(k^2p_k)^{\delta_k}\delta_k^{-1}) \\ &= O(k^{-2}) + O(k^{-3}p_k) = O(k^{-2}), \end{aligned}$$

calculation and Borel–Cantelli show that (3.13a) holds w.p. 1 for all $n \geqq N_k$ and all k sufficiently large.

PROOF OF (3.13b). Apply (3.21) to $\Gamma_n = U_n^{k-1} - n\Delta_{k-1}$, $\alpha = \alpha_k$, $A = (k^2p_k)^{-1}$, $\varepsilon = \frac{1}{8}a_kp_{k+1}k^{-1}$ and $\sigma^2 = \sigma_{k-1}^2$, where

$$\begin{aligned} \sigma_j^2 &= E(U_1^j - \Delta_{j-1})^2 = E(X_1^jI(\eta_1 = j))^2 - b_j^2p_{j_2}^2 \\ &\leqq p_jE(a_jY_1^j + b_j)^2 \leqq 2p_j(a_j^2r_j^2 + b_j^2). \end{aligned} \tag{3.22}$$

Thus

$$\begin{aligned} P\{|U_n^{k-1} - n\Delta_{k-1}| &\geqq n^{\alpha_k}(\tfrac{1}{8}a_kp_{k+1}k^{-1}) \text{ for some } n \geqq N_k\} \\ &= O(k^{-2}) + O(\delta_k^{-1}(k^2p_k)^{\delta_k}(a_kp_{k+1}k^{-1})^{-2}p_{k-1}(a_{k-1}^2r_{k-1}^2 + b_{k-1}^2)) \\ &= O(k^{-2}) + O(\delta_k^{-1}p_k^{\delta_k}k^2a_k^{-2}p_{k+1}^{-2}k^2a_k^2p_{k+1}^6k^{-6}p_k^{-1}) \\ &= O(k^{-2}) + O(\delta_kk^{-2}p_{k+1}^4p_k^{-1}) = O(k^{-2}) \end{aligned}$$

by (3.6) and (3.5). As before this and the Borel–Cantelli lemma gives (3.13b) for $n \geqq N_k$ all k sufficiently large.

PROOF OF (3.14a). Proceed as on page 1203, applying (3.21) with $A = \lambda_k/2p_k$, $\Gamma_n = U_n^{k-1} - n\Delta_{k-1}$. As above we have

$$\begin{aligned} P\{n^{-\alpha_k}|U_n^{k-1} &- n\Delta_{k-1}| > \varepsilon \text{ for some } n \geqq L_k\} \\ &= P\{L_k < \lambda_k/2p_k\} + O[(p_k\lambda_k^{-1})^{\delta_k}\delta_k^{-1}p_{k-1}(a_{k-1}^2r_{k-1}^2 + b_{k-1}^2)] \\ &= O(k^{-2}) + O(p_k^{\delta_k}k^{-2}) = O(k^{-2}); \end{aligned}$$

see (3.7) and (5.41). This gives (3.14a) for all $n \geqq L_k$, k sufficiently large w.p. 1 regardless of (3.15). We finish the argument below.

PROOF OF (3.14b). Referring to page 1203, we have as in (5.55) but using

(3.21)

$$\begin{aligned} P\{|b_k|\,|\textstyle\sum_{i=1}^n [I(\eta_i = k) - p_k]| \geqq n^{\alpha_k \varepsilon} \text{ for some } n \geqq L_k\} &= O(k^{-2}) + O((p_k/\lambda_k)^{\delta_k}\delta_k^{-1}p_k b_k^2) \\ &= O(k^{-2}) + O(p_k^2 b_k^2 k^{-3}) = O(k^{-2} + |\Delta_k|k^{-3}) = O(k^{-2}); \end{aligned}$$

see (3.2), (3.7) and (5.26). Thus w.p. 1 for all k sufficiently large and all $n \geqq L_k$, in particular for all $n \in [L_k, N_{k+1})$, we have

$$n^{-\alpha_k}|\textstyle\sum_{i=1}^n b_k[I(\eta_i = k) - p_k]| < \varepsilon .$$

Hence since $U_n^k - n\Delta_k = a_k \sum_{i=1}^n Y_i^k I(\eta_i = k) + \sum_{i=1}^n b_k(I(\eta_i = k) - p_k)$, (5.51), we must have w.p. 1 (3.14b) for all k sufficiently large whenever $n \in [L_k, N_{k+1})$ satisfies (3.15).

We have established (3.14) whenever $n \in [L_k, N_{k+1})$ satisfies (3.15). But this does it since w.p. 1 (3.15) fails for all $n \in [N_k, L_k]$ for all k sufficiently large by (5.49), page 1202. This finishes the proof of the claim and the theorem.

4. Proofs of Theorems 5 and related facts. Theorem 8. Let F be a distribution on $[0, \infty)$ whose tail $1 - F(t)$ satisfies

$$1 - F(t) = t^{-\beta}L(t) \qquad t > 0, \tag{4.1}$$

where $L > 0$ is slowly varying at ∞ and $0 < \beta < 1$. Define

$$v(\lambda) = -\log \textstyle\int_0^\infty e^{-\lambda x}F\{dx\} \qquad \lambda \geqq 0 .$$

Then v is strictly increasing (since F is not concentrated at the origin) and maps $[0, \infty)$ onto $[0, x_0)$ where $x_0 = -\log F\{0\} > 0$ ($F\{0\}$ = mass at 0, $0 \leqq F\{0\} < 1$ so $0 < x_0 \leqq \infty$). A standard Abelian argument (use [2], Theorem 4, page 446, $u(x) = 1 - F(x)$) gives

$$v(\lambda) \sim \Gamma(1-\beta)\lambda^\beta L\left(\frac{1}{\lambda}\right) \qquad \text{as} \quad \lambda \to 0+ , \tag{4.2}$$

where the symbol $\sim$ means the ratio of both sides has limit 1. We will denote by v_1 the continuous strictly increasing inverse of v: v_1 maps $[0, x_0)$ onto $[0, \infty)$, $v(0) = v_1(0) = 0$ and

$$v_1(v(\lambda)) = v(v_1(\lambda)) = \lambda \qquad \lambda < x_0 . \tag{4.3}$$

PROPOSITION 1. *Write $\alpha = 1/\beta$ and define*

$$w(\lambda) = \lambda^\alpha L^{-\alpha}(1/\lambda), \qquad \lambda > 0,\ w(0) = 0 . \tag{4.4}$$

If for each fixed $p > 1$, L also satisfies

$$L(t^\theta) \asymp L(t) \qquad \textit{as} \quad t \to \infty \tag{4.5}$$

uniformly for $p^{-1} \leqq \theta \leqq p$ (see Section 1 for meaning of $\asymp$), *then*

$$v_1(\lambda) \asymp w(\lambda) \qquad \textit{as} \quad \lambda \to 0^+ . \tag{4.6}$$

If L satisfies

$$L(t^\theta) \sim r_\theta L(t) \qquad as \quad t \to \infty \tag{4.7}$$

for every $\theta > 0$, *where* $0 < r_\theta < \infty$, *then*

$$v_1(\lambda) \sim B_0 w(\lambda) \qquad as \quad \lambda \to 0 , \tag{4.8}$$

where $B_0 = [r_\alpha \Gamma(1-\beta)]^{-\alpha}$, $\alpha = 1/\beta$.

LEMMA 1. *If* (4.7) *holds for every* $\theta > 0$ *then* (4.7) *holds uniformly on* $p^{-1} \leqq \theta \leqq p$ *for each* $p > 1$.

PROOF. (4.7) says that for some constant ρ the function $L(e^t)$ varies regularly with exponent ρ, i.e.,

$$L(e^t) = t^\rho z(t) \tag{4.9}$$

with z slowly varying. Now $0 < r_\theta < \infty$ implies that $-\infty < \rho < \infty$ and then $r_\theta = \theta^\rho$. A well-known property of slowly varying functions ([2], Lemma 2, page 277) is that for each $p > 1$, $z(t\theta)/z(t) \to 1$ as $t \to \infty$ uniformly for $p^{-1} \leqq \theta \leqq p$. This fact and (4.9) give the desired conclusion.

LEMMA 2. *Under* (4.5)

$$L(t/L(t)) \asymp L(t) \qquad as \quad t \to \infty . \tag{4.10}$$

If (4.7) *holds then* (4.10) *holds with the symbol* $\sim$ *in place of* $\asymp$.

PROOF. By Lemma 2 of [2], page 277, we have for every $\varepsilon > 0$ fixed, $t^{-\varepsilon} < L(t) < t^\varepsilon$, $t \geqq t_\varepsilon$, t_ε sufficiently large. Thus $t/L(t) = t^\theta$ for some $\theta = \theta(t)$ in $[1-\varepsilon, 1+\varepsilon]$. (4.10) follows from the uniformity of (4.5). That $L(t/L(t)) \sim L(t)$ under (4.7) follows from Lemma 1 and the fact that $r_\theta = \theta^\rho$.

PROOF OF PROPOSITION 1. Assume (4.5). As $\lambda \to 0$ we have $w(\lambda) \to 0$ and

$$\begin{aligned} v(w(\lambda)) &\sim w^\beta(\lambda) L(1/w(\lambda)) \Gamma(1-\beta) \\ &= \lambda L^{-1}(\lambda^{-1}) L(\lambda^{-\alpha} L^\alpha(\lambda^{-1})) \Gamma(1-\beta) \\ &\asymp \lambda L^{-1}(\lambda^{-1}) L(\lambda^{-1} L(\lambda^{-1})) \asymp \lambda \end{aligned}$$

by, in order of application, (4.2), (4.4), (4.5) and (4.10). In other words for some $c > 1$, $\lambda_0 > 0$

$$c^{-1} \leqq \frac{v(w(\lambda))}{v(v_1(\lambda))} \leqq c , \qquad 0 < \lambda \leqq \lambda_0 . \tag{4.11}$$

Suppose (4.6) were not true: Let $\lambda_n \to 0^+$ as $n \to \infty$ so that either $x_n/y_n \to \infty$ or $x_n/y_n \to 0$ as $n \to \infty$ where $x_n = w(\lambda_n)$, $y_n = v_1(\lambda_n)$. Suppose in fact

$$1 \leqq \frac{x_n}{y_n} \to \infty \qquad \text{as} \quad n \to 0 . \tag{4.12}$$

Note that $x_n \to 0$ and $y_n \to 0$. An application of Lemma 2 of [2], page 277, shows that for $\varepsilon < \beta$

$$\left(\frac{x_n}{y_n}\right)^{\beta-\varepsilon} < \frac{v(x_n)}{v(y_n)} < \left(\frac{x_n}{y_n}\right)^{\beta+\varepsilon} \tag{4.13}$$

eventually; see (4.2). (4.13) and (4.11) contradict (4.12). This proves (4.6). If we assume (4.7), then similar asymptotic calculations lead to (4.8).

PROOF OF THEOREM 5. Suppose (1.1) holds:

$$L(t) = c(\log_2 t)^{1-\beta}, \qquad \beta = 1/\alpha,\ \alpha > 1$$

($\log_2 = \log\log$). Define the random functions

$$H_n(t) = S_{[nt]}/a_n, \qquad t \geqq 0,\ n \geqq 1,$$

where $[\cdot]$ is the greatest integer and, with v_1 given by (4.3),

$$a_n = \log_2 n/v_1(\log_2 n/n). \tag{4.14}$$

According to Wichura ([11], Theorem 3.1, page 1116), with respect to a certain metric w.p. 1, $\{H_n\}$ is relatively compact in a certain function space K_β and $\{H_n\}$ has K_β for its set of limit points (our β is Wichura's α). In particular this implies

$$\text{acc. pts. of } \{H_n(1)\} = A(S_n; a_n) = [b_0, \infty] \tag{4.15}$$

w.p. 1 for some positive finite constant b_0 (see [11], (i) page 1118). But $L(t^\theta) \sim L(t)$ as $t \to \infty$ for all $\theta > 0$; consequently by (4.8) $v_1(\lambda) \sim dw(\lambda) = c^{-\alpha} d\lambda^{-\alpha}(\log_2 \lambda^{-1})^{1-\alpha}$ as $\lambda \to 0$, where c is the constant of (1.1) and $d = \Gamma(1-\beta)^{-\alpha}$. Therefore

$$a_n \sim b_1 n^\alpha \quad \text{as} \quad n \to \infty$$

for some constant b_1. From this and (4.15) we get (1.2) with $b = b_0 b_1$.

EXAMPLE 1. As noted in the introduction it seems reasonable to hope for some sort of converse to Theorem 5, that is, if one knows that (1.2) holds then it ought to be the case that (1.1) holds or nearly holds. Here we give an example which shows that the weaker

$$0 < \liminf n^{-\alpha} S_n < \infty \quad \text{a.s.} \tag{4.16}$$

does not imply $P(X_1 > t) \asymp t^{-\beta}(\log_2 t)^{1-\beta}$ as $t \to \infty$, where $\beta = 1/\alpha$. Let $\alpha > 1$, $\beta = 1/\alpha < 1$ and put

$$h(t) = (\log_2 t)^{1-\beta} \exp\{(1-\beta)(\log_3 t)(1 - \cos(\log_4 t))\}$$

and define a distribution F on $[0, \infty)$ by

$$P(X_1 > t) = 1 - F(t) = t^{-\beta} h(t), \qquad t \geqq t_0.$$

Choose t_0 so that $t^{-\beta}h(t) < 1$ and is decreasing for $t > t_0$. For this distribution we have $\liminf_{t\to\infty} (\limsup_{t\to\infty})(\log_2 t)^{\beta-1} t^\beta P(X_1 > t) = 1\ (= \infty)$ so in fact (1.1) does hold if $t \to \infty$ through some rapidly increasing sequence $\{t_n\}$. We now indicate briefly how one may show that (4.16) holds for this example. The lengthy details are not difficult and are omitted.

1°. The function h is strictly increasing on (t_0, ∞), is slowly varying at ∞ and satisfies (1.6). Put $b_n = n^\alpha h^\alpha(n)$ then $\lim \text{Law}\,(b_n^{-1}S_n) = G = a$ stable law of index $\beta\,(= 1/\alpha)$ concentrated on $(0, \infty)$.

2°. Let $a_n = \log_2(n)/v_1(n^{-1}\log_2 n)$ where v_1 is the inverse function of $v(s) = -\log \int_0^\infty \exp(-sx)\,dF(x)$, then $\liminf a_n^{-1}S_n = \text{const.} > 0$ a.s., see [4], page 181. From 1° and (4.8) of Proposition 1 we get for some $0 < c < \infty$ $a_n \sim cn^\alpha \exp[(\alpha - 1)(\log_3 n)(1 - \cos\log_4 n)] \geqq cn^\alpha$. Consequently $\liminf_{n\to\infty} n^{-\alpha}S_n > 0$ a.s.

3°. Put $\psi(t) = h(t)/k$ where k is a number to be chosen later. The distribution F and the function ψ satisfy the assumptions of Theorem 1, page 138, of Lipschutz [7] (the most tedious assumption to check is condition (7) on page 136). Applying that theorem we find, after some simplifications, that

$$P\{S_n \leqq b_n/\psi^\alpha(b_n) \text{ infinitely often}\} = 1 \tag{4.17}$$

will hold if we can show that $\int^\infty x^{\frac{1}{2}} \exp[x(1 - qu(x))]\,dx = \infty$ where $u(x) = \exp[(\log x)(1 - \cos\log_2 x)]$ and $q = k_0/k^{(1-\beta)}$ (k is the constant in the definition of ψ, k_0 is another constant). Since $\cos(\log_2 x)$ is near 1 sufficiently often, we can indeed show that the integral diverges if k is sufficiently large.

4°. From the properties of h we have that $b_n/\psi^\alpha(b_n) \sim n^\alpha k^\alpha$, as $n \to \infty$, so (4.17) implies $\liminf_{n\to\infty} n^{-\alpha}S_n \leqq k^\alpha < \infty$ a.s. This fact and 2° give us (4.16).

Conditions which guarantee $B(F, \alpha) = [0, \infty]$. The following result can be proved by the methods used in examples 4 and 5, pages 576–579 of Erickson–Kesten [1]. We omit the proof.

THEOREM 8. *Let* $\alpha = 1/\beta > 1$. *Let* $\{S_n\}$ *be a random walk associated with a distribution F such that*

$$P\{\liminf n^{-\alpha}S_n \geqq 0\} = 1 \quad \textit{and} \tag{4.18}$$

$$1 - F(t) = t^{-\beta}L(t)\,, \qquad t > 0\,, \tag{4.19}$$

where L is slowly varying and $\beta = 1/\alpha < 1$. *Suppose also that either set of conditions* (4.20) *or* (4.21) *below hold. Then* $B(F, \alpha) = [0, \infty]$.

$$\text{(i)}\ \ \lim_{t\to\infty} L(t) = 0 \qquad \text{(ii)}\ \ \int^\infty \frac{L(x)}{x}\,dx = \infty\,. \tag{4.20}$$

$$\text{(i)}\ \ \lim_{t\to\infty} L(t) = \infty \qquad \text{(ii)}\ \ L \ \textit{ satisfies } (1.6) \tag{4.21}$$

$$\text{(iii)}\ \ \int^\infty \frac{L(x)}{x} e^{-kL^{q}(x)}\,dx = \infty \qquad \textit{for every}\ \ k > 0\,, \quad q = \frac{\alpha}{\alpha - 1}\,.$$

Note that (4.18) is satisfied if, for example, $E(X_1^-)^\beta < \infty$ or if $P\{S_n \leqq 0 \text{ i.o.}\} = 0$. Note also that (4.20ii) is equivalent to $E(X_1^+)^\beta = \infty$. If $E(X_1^+)^\beta < \infty$ then we will have $\limsup n^{-\alpha}S_n \leqq 0$ a.s.

REMARK 6. There is no random walk on R^1 which has

$$0 < \liminf n^{-\alpha}S_n < \infty \quad \text{a.s.} \tag{4.22}$$

for some $0 < \alpha < 1$. For (4.22) implies $P(S_n \leqq 0 \text{ i.o.}) = 0$ and then $0 < \alpha < 1$ implies $\liminf n^{-1}S_n = 0$ a.s. It then would follow from [5], page 1195 that $E|X_1| < \infty$ and $EX_1 = 0$. But then $\{S_n\}$ is recurrent and $P(S_n \leqq 0 \text{ i.o.}) = 1$ contradicting (4.22).

5. Proofs of Theorems 6, 7 and corollaries. For any vector $v = (v(1), \cdots, v(d)) \in R^d$ $d \geqq 2$ we write

$$||v|| = \max_{1 \leqq i \leqq d} |v(i)| .$$

LEMMA 1. *Let* $\gamma_n = \gamma(n)$ *where* $\gamma(t)$ *is regularly varying with positive exponent*:

$$\lim_{t\to\infty} \gamma(tx)/\gamma(t) = x^\alpha \qquad \textit{for all } x > 0 \tag{5.1}$$

for some $\alpha > 0$. *Then*

$$b \in B(F, \{\gamma_n\}) \qquad b \in R^d$$

if and only if

$$\textstyle\sum_{n=1}^\infty \Delta_n^{-1} P\{||\gamma_n^{-1}S_n - b|| \leqq \varepsilon\} = \infty \qquad \textit{for all } \varepsilon > 0 \tag{5.2}$$

where $\Delta_1 = 1$,

$$\Delta_n = \Delta_n(\gamma_n) = 1 + \textstyle\sum_{k=1}^{n-1} P\{||S_k|| \leqq \gamma_n\} \qquad n \geqq 2 .$$

This is a generalization of Theorem 3 of [5]. The proof of Lemma 1 follows almost exactly the same lines as the proof in [5] so we omit the details.

PROOF OF THEOREM 6. I. Let us suppose first that F is nonlattice, that is, we suppose $|F(\theta)| < 1$ for all $\theta \neq 0$ where $F(\theta) = \int_{R^d} \exp(i\theta \cdot x) F\{dx\}$, $\theta \in R^d$. Later we show how to remove this restriction.

LEMMA 2. *Assume* (1.3) *and that* F *is nonlattice. For* $h > 0$ *put*

$$I_h = [0, h]^d = \{x \in R^d : 0 \leqq x(i) \leqq h, i = 1, \cdots, d\} .$$

Let a be any fixed finite positive constant. Then

$$P\{b_n^{-1}S_n \in I_h + x\} = O(h^d) \qquad \textit{as} \quad n \to \infty , \tag{5.3}$$

uniformly for $x \in R^d$ *and* $ab_n^{-1} \leqq h < \infty$. *If* $g_\beta(0) > 0$ *then given finite positive* a, h_0 *and* r_0

$$P\{b_n^{-1}S_n \in I_h + x\} \asymp h^d \qquad \textit{as} \quad n \to \infty , \tag{5.4a}$$

uniformly for $||x|| \leqq r_0$ *and* $ab_n^{-1} \leqq h \leqq h_0$. *If* $g_\beta(0) > 0$, *and if* $x_n \to 0$ $(x_n \in R^d)$ *and* $h_n \to 0$, *but* $b_n^{-1} = O(h_n)$ *as* $n \to \infty$, *then*

$$P\{b_n^{-1}S_n \in I_{h_n} + x_n\} \sim g_\beta(0) h_n^d \qquad \textit{as} \quad n \to \infty . \tag{5.4b}$$

Lemma 2 follows immediately from Theorem 1 of Stone [10] (see also his Lemma 2 on page 550) and the fact that g_β is continuous and bounded on R^d.

From Lemma 1 and the inequality $||v|| \leqq |v| \leqq d||v||$ it follows that to get (1.5) it suffices to show

$$\textstyle\sum_{n=1}^\infty \Delta_n^{-1}(n^\alpha) P\{||S_n|| \leqq cn^\alpha\} < \infty \tag{5.5}$$

for every $c > 0$, $\alpha < 1/\beta$, $\beta < d$. Let $\varepsilon > 0$ and $\varepsilon < 1/\beta - \alpha$. Then

$$(5.6) \qquad n^{-\varepsilon} < H(n) < n^{\varepsilon} \quad \text{for all} \quad n \geqq m_0 \quad \text{(suff. large)}$$

([2], page 277) so $n^\alpha b_n^{-1} = (n^{-(\beta^{-1}-\alpha)})/H(n) \to 0$ as $n \to \infty$ and $2cn^\alpha b_n^{-1} \in [b_n^{-1}, 1]$ for all n sufficiently large. Applying (5.3) and then (5.6) we get

$$(5.7) \qquad P\{||S_n|| \leqq cn^\alpha\} = O((n^\alpha b_n^{-1})^d) = O(n^{-d(\beta^{-1}-\alpha-\varepsilon)}) .$$

Let the integer k_n satisfy $k_n \leqq (A_0^{-1}n^\alpha)^{\beta/(\varepsilon\beta+1)} \leqq k_n + 1$, then for $m_0 \leqq k \leqq k_n$, $b_k^{-1}n^\alpha \geqq k^{-(\beta^{-1}+\varepsilon)}n^\alpha \geqq A_0$ so

$$(5.8) \qquad \sum_{k=1}^{n-1} P\{||S_k|| \leqq n^\alpha\} \geqq \sum_{k=m_0}^{k_n} P\{||b_k^{-1}S_k|| \leqq A_0\} .$$

But $P\{||b_k^{-1}S_k|| \leqq A_0\} > 1/2G_\beta\{||x|| \leqq A_0\} = a_0 > 0$ for an A_0 sufficiently large and all $k \geqq m_1$ by (1.3). Since $k_n \to \infty$ it follows that

$$(5.9) \qquad \begin{aligned} &\Delta_n(n^\alpha) > \text{RHS}\,(5.8) > (k_n - m_0 - m_1)a_0 , \qquad \text{or} \\ &\Delta_n^{-1} = O(k_n^{-1}) = O(n^{-\alpha\beta/(\varepsilon\beta+1)}) \quad \text{as} \quad n \to \infty . \end{aligned}$$

Combining (5.7) and (5.9) shows that the nth term of the series in (5.5) is $O(n^{-\theta})$ where $\theta = d(\beta^{-1} - \alpha - \varepsilon) + \alpha\beta(\varepsilon\beta + 1)^{-1}$. But $\beta < d$ and $\alpha < 1/\beta$ so, as the reader may easily check, $\theta > 1$ for $\varepsilon > 0$ sufficiently small. It follows that (5.5) does indeed hold for all $c > 0$. This establishes (1.5) for all $\alpha < 1/\beta$ under (i) or (ii) when F is nonlattice. In the case $\beta = 2$, $d \geqq 3$ and F arithmetic a lattice version of Lemma 2 is available (apply Theorem (6.1), page 202 of [9] to the process $S_n^0 = S_n \operatorname{diag}(a_1^{-1}, \cdots, a_j^{-1})$ where $a_1, \cdots, a_d$ are the spans). Essentially the same calculations again give us (5.5) for all $\alpha < \frac{1}{2} = 1/\beta$.

II. We now show how to drop the nonlattice assumption when $\beta < 2$. Fix an α_0 with

$$\tfrac{1}{2} < \alpha_0 < 1/\beta$$

and let $T_n = Y_1 + \cdots + Y_n$ be a d-dimensional Brownian random walk independent of $\{S_n\}$: the Y_i are i.i.d. with a common standard normal distribution N having density $n(y_1, \cdots, y_d) = (2\pi)^{-d/2}\exp\{-\frac{1}{2}(y_1^2 + \cdots + y_d^2)\}$. Then $E||Y_1||^k < \infty$ for all $k > 0$ and $E(Y_1) = 0$, so

$$(5.10) \qquad \lim_{n\to\infty} n^{-\alpha}||T_n|| = 0 \quad \text{a.s.}$$

for $\alpha > \frac{1}{2}$ (see part III below). Write $S_n' = S_n + T_n$. Then $||S_n|| - ||T_n|| \leqq ||S_n'|| \leqq ||S_n|| + ||T_n||$ so (5.10) tells us

$$(5.11) \qquad \lim_n n^{-\alpha_0}||S_n|| = \infty \quad \text{iff} \quad \lim_n n^{-\alpha_0}||S_n'|| = \infty .$$

Now $b_n > n^{\frac{1}{2}+\varepsilon}$ eventually by (5.6) and $\beta < 2$ so again by (5.10) $b_n^{-1}T_n \to 0$ as $n \to \infty$ and it follows that (1.3) holds with S_n' in place of S_n. The increment distribution F_1 of the steps $X_i + Y_i$ of the random walk $\{S_n'\}$ is the convolution of F and N and thus F_1 has a bounded density since N does. This implies that F_1 is nonlattice (in fact $|\hat{F}_1(\theta)| \leqq \exp\{-\frac{1}{2}|\theta|^2\}$). We may therefore apply part I to $\{S_n'\}$ and then conclude from (5.11) that $n^{-\alpha_0}||S_n|| \to \infty$ a.s. as $n \to \infty$ for any $\frac{1}{2} < \alpha_0 < 1/\beta$. But clearly this gives (1.5) for any $\alpha < 1/\beta$.

III. Let us show $\lim n^{-\alpha}||S_n|| = 0$ if $\alpha > 1/\beta$. As is well known, and is easy to verify, any distribution in a domain of attraction of index β must have $E||X_1||^k < \infty$ for all $k < \beta$ (use [2], page 578), so $\alpha > 1/\beta$ implies $E||X_1||^{1/\alpha} < \infty$.

CASE 1. $\beta \leqq 1$. Then $\alpha > 1/\beta \geqq 1$ and it follows from a Marcinkiewicz–Zygmund theorem ([8], page 243) applied to each of the coordinate processes that $n^{-\alpha}||S_n|| \to 0$ as $n \to \infty$.

CASE 2. $\beta > 1$. Then $1/\beta < 1$ and $\alpha > 1/\beta$ and by the same theorem $n^{-\alpha}||S_n - n\mu|| \to 0$ as $n \to \infty$ where $\mu = E(X_1)$ is the mean vector. But with $\beta > 1$ we have $b_n^{-1} > n^{-1+\varepsilon}$ eventually for $\varepsilon > 0$ sufficiently small by (5.6), so $||b_n^{-1}S_n|| \geqq n^\varepsilon||S_n/n||$. If $\mu \neq 0$ then $||S_n/n|| \to ||\mu|| > 0$ hence $||b_n^{-1}S_n|| \to \infty$ as $n \to \infty$ w.p. 1 and we could not possibly have (1.3). Hence $\mu = 0$ and $n^{-\alpha}||S_n|| \to 0$ as $n \to \infty$ w.p. 1.

IV. We now want to show that in case $d = 1 < \beta$

$$B(F, \alpha) = \bar{R}^1 \tag{5.12}$$

for all $0 < \alpha < 1/\beta$. Note first that (as in III above) since $\beta > 1$ we must have $E|X_1| < \infty$ and $E(X_1) = 0$ and then, as is well known, $\{S_n\}$ is a recurrent random walk. This means in particular that $\liminf |S_n| < \infty$ and a fortiori $\liminf n^{-\alpha}|S_n| = 0$ a.s. or $B^f(F, \alpha) \neq \emptyset$ for $\alpha > 0$. Theorem 4, page 1173, and the remarks at the end of Section 3, page 1190 of [5], now show that (5.12) holds for $0 < \alpha \leqq \frac{1}{2}$. If $\beta = 2$ we are done (and have obtained a slight improvement). If $\beta < 2$ we need only establish (5.12) for $\frac{1}{2} < \alpha \leqq 1/\beta$. Using the argument of part II we find

$$B(F, \alpha) = B(F * N, \alpha) , \qquad \alpha > \tfrac{1}{2} ,$$

where N is the standard normal distribution with mean 0 $\sigma^2 = 1$ and $*$ denotes convolution. So we may assume without loss of generality that F is nonlattice.

As in part I we will estimate the series (5.2) using Lemma 2. Now $g_\beta(0) \neq 0$ for $\beta \geqq 1$ (this follows from [2], page 448–449, (6.9) page 583, in particular the remarks before proof of Theorem 2, page 449); so by Lemma 2 for $a > 0$ and any fixed x

$$P\{n^{-\alpha}S_n \in (x, x + a)\} \sim a g_\beta(0) b_n^{-1} n^\alpha \qquad \text{as} \quad n \to \infty . \tag{5.13}$$

Recall that $b_n^{-1}n^\alpha = n^{-(\beta^{-1}-\alpha)}/H(n) \to 0$ as $n \to \infty$ and g_β is continuous. The symbol $\sim$ means, as usual, the ratio of both sides has limit $= 1$. The denominator Δ_n in (5.2) requires more care. Choose integers $k = k_n$ to satisfy

$$b_k^{-1} n^\alpha \asymp 1 \qquad \text{as} \quad n \to \infty . \tag{5.14}$$

Then for any $0 < \varepsilon < 1/\beta$ and constants c_1, c_2 we will have (see (5.6))

$$c_1 n^{\alpha\beta/(1+\beta\varepsilon)} < k_n < c_2 n^{\alpha\beta/(1-\beta\varepsilon)} \tag{5.15}$$

eventually. Now $\alpha\beta < \alpha + 1 - 1/\beta$ for $\alpha < 1/\beta$, $\beta > 1$ so a glance at (5.6) and (5.15) should convince us that

$$k_n \to \infty \qquad \text{and} \qquad k_n = o(n^{\alpha+1-1/\beta}/H(n)) = o(n) \tag{5.16}$$

as $n \to \infty$. Now $h = b_j^{-1}n^\alpha \geqq b_j^{-1}$, so by (5.3) $P\{|S_j| \leqq n^\alpha\} \leqq Mb_j^{-1}n^\alpha$ for all j sufficiently large and $n \geqq 1$. The constant M is independent of j and n. Consequently,

$$\Delta_n = 1 + \sum_{j=1}^{n-1} P\{|S_j| \leqq n^\alpha\} \leqq k_n + \sum_{j=k_n}^{n} P\{|S_j| \leqq n^\alpha\}$$
$$\leqq k_n + 2Mn^\alpha \sum_{j=k_n}^{n} b_j^{-1} .$$

Also $1/\beta < 1$ so one of the Karamata theorems on regular variation gives us

$$\sum_{j=j_0}^{n} b_j^{-1} = \sum_{j=j_0}^{n} j^{-1/\beta}/H(j) \sim \frac{\beta}{\beta - 1}\frac{n^{1-\beta^{-1}}}{H(n)}$$

as $n \to \infty$ for each fixed j_0 (see [2], page 281). Applying these facts and (5.16) we get

$$\Delta_n \leqq k_n + 2Mn^\alpha \sum_{j=j_0}^{n} b_j^{-1} \leqq c_3 n^{\alpha+1-\beta^{-1}}/H(n) = c_3 n^{\alpha+1} b_n^{-1}$$

for some constant $c_3 > 0$ and n sufficiently large. This estimate and (5.13) give

$$\Delta_n^{-1} P\{n^{-\alpha}S_n \in (b - \varepsilon, b + \varepsilon)\} \geqq c_3^{-1}\varepsilon g(0)\frac{1}{n}$$

for n sufficiently large. So (5.2) holds for all $b \in R^1$ and $\varepsilon > 0$ since $g_\beta(0) > 0$ and $\sum 1/n$ diverges. Therefore (5.12) does indeed hold for all $\alpha < 1/\beta$.

PROOFS OF THEOREM 7 AND COROLLARIES 1 AND 2. Much of the proof is like that of Theorem 6 so we omit some detail. We will suppose throughout that F is nonlattice. To get around this see parts I and II of the preceding proof. Clearly, on account of (1.3) and $\psi(t)$ nonincreasing, we may assume without loss of generality that

$$1 > \psi(t) \to 0 \quad \text{as} \quad t \to \infty . \tag{5.17}$$

Write

$$\gamma(t) = t^{1/\beta}H(t)\psi(t) , \qquad \gamma_n = \gamma(n) = b_n\psi(n) .$$

For each fixed d-cube $J = I_a + x$ we have

$$P\{\gamma_n^{-1}S_n \in J\} \sim g_\beta(0)a^d\psi^d(n) \quad \text{as} \quad n \to \infty , \tag{5.18}$$

by Lemma 2, (5.17), continuity of g_β and the fact $g_\beta(0) \neq 0$.

To estimate $\Delta_n = \Delta_n(\gamma_n)$ define integers $k_n \to \infty$ so that

$$k_n - 1 \leqq n\psi^\beta(n) \leqq k_n < n . \tag{5.19}$$

Then, $k_n \sim n\psi^\beta(n)$ and, since ψ is slowly varying, for any $\varepsilon > 0$

$$n^{1-\varepsilon} < k_n < n \tag{5.20}$$

eventually. Thus $H(k_n) \asymp H(n)$ by (1.6) and since $\{b_k\}$ is nondecreasing and $\gamma_n \to \infty$ it follows that for some constant $h_0 < \infty$ and $k \geqq k_n$,

$$b_k^{-1} \leqq b_k^{-1}\gamma_n \leqq b_{k_n}^{-1}b_n\psi(n) \leqq h_0$$

for all n sufficiently large. Therefore by Lemma 2

$$(5.21) \qquad \sum_{k=k_n}^{n-1} P\{||S_k|| \leqq \gamma_n\} \asymp \gamma_n^{\,d} \sum_{k=k_n}^{n-1} b_k^{-d} \qquad \text{as} \quad n \to \infty .$$

Put $V(t) = \sum_{k=t}^{\infty} b_k^{-d} = \sum_{k=t}^{\infty} k^{-d/\beta} H^{-d}(k)$. This series converges since H^{-d} is slowly varying and $d/\beta > 1$. Moreover ([2], pages 279–282)

$$(5.22) \qquad V(t) \sim \frac{\beta}{d-\beta} t^{(1-d\beta^{-1})} H^{-d}(t) \sim \frac{\beta}{d-\beta} t b_t^{-d}$$

as $t \to \infty$. From (1.6), (5.20), (5.19) and (5.22) we get $b_{k_n} \asymp b_n \psi(n)$ and

$$(5.23) \qquad V(k_n) \sim \frac{\beta}{d-\beta} n\psi^{\beta}(n) b_{k_n}^{-d}$$

$$\asymp \frac{\beta}{d-\beta} n b_n^{-d} \psi^{\beta-d}(n) \sim V(n)\psi^{\beta-d}(n)$$

as $n \to \infty$. (Recall that $y_n \asymp z_n$ as $n \to \infty$ means $|y_n/z_n|$ is bounded away from 0 and ∞ for all n large.) But $\psi(n) \to 0$ and $\beta < d$ so $V_n(n) = o(V(k_n))$ and going back to (5.21) we get as $n \to \infty$

$$(5.24) \qquad \sum_{k=k_n}^{n-1} P\{||S_k|| \leqq \gamma_n\} \asymp \gamma_n^{\,d}(V(k_n) - V(n))$$

$$\asymp V(k_n) b_n^{\,d} \psi^d(n) \asymp n\psi^{\beta}(n) .$$

From (5.19) we also have

$$1 \leqq 1 + \sum_{k=1}^{k_n} P\{||S_k|| \leqq \gamma_n\} = O(k_n) = O(n\psi^{\beta}(n)) .$$

This and (5.24) clearly imply

$$\Delta_n = \sum_{k=1}^{n-1} P\{||S_k|| \leqq \gamma_n\} \asymp n\psi^{\beta}(n)$$

as $n \to \infty$. Going back to (5.18) we obtain finally

$$(5.25) \qquad \Delta_n^{-1} P\{\gamma^{-1} S_n \in J\} \asymp \frac{1}{n} \psi^{d-\beta}(n) \qquad \text{as} \quad n \to \infty$$

for every d-cube J of positive volume. The conclusions of Theorem 7 and Corollary 1 follow immediately from (5.25) and Lemma 1. Corollary 2 follows from Theorem 7, Remark 2 and the central limit theorem: $S_n/n^{\frac{1}{2}}$ converges in law to the normal law $N(0, V)$ where V is the nonsingular covariance matrix $E(X(i)X(j))$. $N(0, V)$, consequently, is nonsingular in R^d.

REMARK 7. Some comments on the conjecture of Remark 5, Section 1. As noted in Section 1 the conjecture is true whenever the assumption of Theorem 6 holds (and $d \geqq 3$). Suppose we could imbed a random walk $\{S_n\}$ in a d-dimensional stable process $\{Z(t)\}_{t \geqq 0}$ of index β (see [3] for definitions, when $\beta = 2$ Z is a Brownian motion). That is, suppose we could find i.i.d. positive random variables $t_1, t_2, \cdots$, (on the sample space of $\{Z(t)\}$) such that the processes $\{S_n\}_{n \geqq 1}$ and $\{Z(V_n)\}_{n \geqq 1}$ are equivalent, where $V_n = t_1 + \cdots + t_n$. Then clearly $\lim n^{-\alpha}|S_n| = \infty$ a.s. if and only if $\lim n^{-\alpha}|Z(V_n)| = \infty$ a.s. Now by Theorem

11.5, page 365 of [3], $\lim t^{-\alpha}|Z(t)| = \infty$ a.s. for $\alpha < \frac{1}{2}$ ($d \geqq 3$); therefore,

$$\lim n^{-\alpha}|Z(V_n)| = \lim \left(\frac{V_n}{n}\right)^{\alpha} \frac{|Z(V_n)|}{V_n^{\alpha}} = \infty$$

since $\lim V_n/n = E(t_1) > 0$ by the strong law. If $E(t_1) = \infty$ so much the better. Thus the conjecture is true in this case.

The following lemma is of some interest.

Lemma. *If F on R^d is genuinely d-dimensional, if $d > 2$ and if $\alpha < \frac{1}{2} - 1/d$, then*

$$\lim n^{-\alpha}|S_n - u_n| = \infty \quad \text{w.p. } 1$$

for any sequence of constants $\{u_n\}$ in R^d.

Proof. An inequality of Esseen (*Z. Wahr.* **9** 306 (1968), Corollary to Theorem (6.2)) says we can find a constant $c > 0$ such that for all $t > 0$, $n = 1, 2, \cdots$, we have

$$\sup_{x \text{ in } R^d} P\{|x + S_n| \leqq t\} \leqq ct^d n^{-\frac{1}{2}d} .$$

Setting $t = an^{\alpha}$ we get

$$P\{|S_n - u_n| \leqq an^{\alpha}\} \leqq ca^d n^{d(\alpha - \frac{1}{2})} .$$

Hence if $\alpha < \frac{1}{2} - 1/d$ the series $\sum P\{|S_n - u_n| \leqq an^{\alpha}\}$ converges for every $a > 0$ and the conclusion follows from the Borel–Cantelli lemma.

As a complement to the preceding lemma the following interesting observation of H. Kesten should be noted. If $\frac{1}{2} - 1/d \leqq \alpha < \frac{1}{2}$, then under various conditions on F, the assumptions of Corollary 2 in Section 1, for example, one in fact can find a sequence of constants $u_n \in R^d$ such that w.p. 1 $\liminf n^{-\alpha}|S_n - u_n| = 0$. (Of course, $\limsup |u_n| = \infty$ by Corollary 1.) We shall not prove this here.

6. Proof of Theorem 1. Since we are mainly interested in getting any closed (but not necessarily bounded) C of R^d as a $B^f(F, 1)$ set we will assume that C has at least one point. (A brief description is given in Remark 8 at the end of this section of the possible structure of $B^{\infty}(F, 1)$ for the distribution F constructed in the proof.) Let $\{c_k = (c_k(1), \cdots, c_k(d))\}$ be a sequence of vectors in R^d such that

(6.1) $$||c_k|| \equiv \max\{|c_k(1)|, \cdots, |c_k(d)|\} \leqq k ,$$

(6.2) $c_k \in C$ for every k and for every $m \geqq 1$ $\{c_k\}_{k \geqq m}$ is dense in C.

Let $\{Y_i^k, \eta_m\}$ $i, k, m \geqq 1$ be a bunch of independent random variables such that, for each k, Y_i^k is uniformly distributed on the integers in $[-r_k, r_k]$ and

$$P(\eta_i = k) = p_k \qquad k = 0, 1, 2, \cdots$$

(p_k, r_k defined below). Put $T_n^k = Y_1^k + \cdots + Y_n^k$ then $\{T_n^k\}$ is a recurrent random walk on the integers. We now choose inductively parameters a_k, b_k, λ_k, ν_k, r_k, p_k to satisfy $a_0 = b_0 = 0$, $\lambda_0 = \nu_0 = r_0 = 1$, $p_1 = \frac{1}{16}$ and for $k \geqq 1$

$$b_k = p_k^{-1}[c_k - \textstyle\sum_{i=1}^{k-1} p_i b_i] = p_k^{-1}(c_k - c_{k-1}) \tag{6.3}$$

(the b_k are thus vectors in R^d),

$$\lambda_k \geqq \max\{k^7, a_{k-1}^2 r_{k-1}^2 + ||b_{k-1}||^2\} \tag{6.4}$$

$$2r_k + 1 \geqq 3k^2 \lambda_k^{\frac{1}{2}} \tag{6.5}$$

$$P\{T_n^k = 0 \text{ for some } \lambda_k \leqq n \leqq \nu_k\} \geqq 1 - k^{-2} \tag{6.6}$$

$$0 < p_{k+1} \leqq \min\{p_k k^{-4} \nu_k^{-1}, (k^6 \textstyle\sum_{j=1}^{k-1} p_j[a_j^2 r_j^2 + ||b_j||^2])^{-1}\} \tag{6.7}$$

$$a_k \geqq p_{k+1}^{-1} \max\{k^7(a_{k-1}^2 r_{k-1}^2 + ||b_{k-1}||^2)^{\frac{1}{2}}, k^4 p_k^{-1}\}\,. \tag{6.8}$$

r_k, ν_k, λ_k are to be integers and we choose p_0 so that

$$\textstyle\sum_{k=0}^{\infty} p_k = 1\,.$$

The reader may verify as in Kesten ([5], pages 1196–1200) that such a set of parameters can indeed be chosen. Moreover our parameters also satisfy for $k \geqq 16$ the relations (5.26)—(5.30) and (5.32), page 1199 of [5] (with $|b_k|$ replaced by $|b_k(\gamma)|$ or $||b_k||$). For example (6.1) and (6.3) imply $||b_k||p_k \leqq 2k$ so $\lambda_k \geqq k^7 \geqq 8k^2 p_k^2 ||b_k||^2$. (Note that we can forget about $f(k)$, page 1199, since (5.31) is used only for the recurrence assertion of Theorem 7, page 1196; see page 1204.) From now on numbers (5.x) will refer to displays in [5], Section 5, pages 1196–1204.

Define $X_i^k(\gamma)$ by

$$X_i^k(1) = a_k Y_i^k + b_k(1)$$
$$X_i^k(\gamma) = b_k(\gamma) \qquad \gamma = 2, 3, \cdots, d$$

where $b_k(1), \cdots, b_k(d)$ are the components of b_k and put

$$\begin{aligned} X_i(\gamma) &= \textstyle\sum_{k=0}^{\infty} X_i^k(\gamma) I(\eta_i = k)\,, & \gamma = 1, \cdots, d\,,\\ X_i &= (X_i(1), X_i(2), \cdots, X_i(d))\,, & i \geqq 1\,,\\ S_n &= (S_n(1), \cdots, S_n(d)) = \textstyle\sum_{i=1}^{n} X_i\,, \end{aligned} \tag{6.9}$$

where $I(\eta \in A)$ denotes the random variable which is 1 when $\eta \in A$ and 0 when $\eta \notin A$. The X_i are i.i.d. with distribution

$$F = \textstyle\sum_{k=0}^{\infty} p_k F_k \times \mu_k^2 \times \cdots \times \mu_k^d$$

where F_k is the uniform distribution on the set of points $\{b_k(1) + ma_k : m = -r_k, -r_k + 1, \cdots, r_k\}$ and μ_k^γ is the probability measure which assigns all its mass to the point $b_k(\gamma)$. We claim that

$$B^f(F, 1) = \text{finite acc. pts of } \{S_n/n\} = C\,. \tag{6.10}$$

To prove (6.10) bring in the random variables N_k as on page 1197:

$$N_k = \min\{i : \eta_i = k\}$$

and put

$$W_n^k = \sum_{i=1}^n X_i I(\eta_i \leqq k) = \sum_{i=1}^n \sum_{j=0}^k X_1^j I(\eta_i = j) ;$$

as shown on pages 1197–1200 we have w.p. 1

(6.11) $N_k < N_{k+1}$ for all k sufficiently large;

in fact $k^{-2}p_k^{-1} \leqq N_k \leqq kp_k^{-1}$ which implies (6.11) and

(6.12) $S_n = W_n^k$ on $N_k \leqq n < N_{k+1}$.

(Obviously we also have the decomposition analogous to (5.14), page 1197, but we do not need it.) By now the discerning reader will have noticed that the first coordinate $\{S_n(1)\}$ of our random walk is equivalent to the random walk $\{S_n\}$ constructed for the proof of Theorem 7 of [5]. According to that proof, see pages 1201–1203, there exists a random variable L_k such that

(6.13) $P\{\frac{1}{2}\lambda_k/p_k \leqq L_k \leqq 2\lambda_k/p_k\} \geqq 1 - k^{-2}$,

and then w.p. 1 for all k sufficiently large

(6.14) $N_k < L_k < N_{k+1}$:

and w.p. 1 for infinitely many values of k

(6.15) $\sum_{i=0}^n Y_i^k I(\eta_i = k) = 0$ for some $n \in [L_k, N_{k+1})$.

As shown on pages 1202–1203 we have

(6.16) $|n^{-1}S_n(1) - c_k(1)| \geqq k^2$

whenever $N_k \leqq n < N_{k+1}$ and

(6.17) $\sum_{i=0}^n Y_i^k I(\eta_i = k) \neq 0$.

Moreover w.p. 1 for all k sufficiently large (6.17) holds for all $n \in [N_k, L_k)$. Now, see pages 1203–1204, for any $\varepsilon > 0$ there is a k_ε, $P\{k_\varepsilon < \infty\} = 1$ such that on $\{k_\varepsilon < \infty\}$ we have for all $k \geqq k_\varepsilon$

(6.18) $|n^{-1}S_n(1) - c_k(1)| < \varepsilon$

for any $n \in [L_k, N_{k+1})$ which satisfies (6.15).

Clearly (6.16) and (6.1) imply that for the full walk $S_n = (S_n(1), \cdots, S_n(d))$ w.p. 1

(6.19) $||n^{-1}S_n|| \geqq \frac{1}{2}k^2$ whenever (6.17) holds

and $n \in [N_k, N_{k+1})$, in particular (6.19) holds for all $n \in [N_k, L_k)$ for all k sufficiently large. We are now going to demonstrate that w.p. 1 for every $\varepsilon > 0$

(6.20) $|n^{-1}S_n(\gamma) - c_k(\gamma)| \leqq \varepsilon$ for all $n \in [L_k, N_{k+1})$

for each $\gamma = 2, 3, \cdots, d$ and all sufficiently large k. Clearly (6.18)—(6.20) give (6.10). For suppose $n_j \to \infty$ so that w.p. 1

$$\lim_{j\to\infty} n_j^{-1}S_{n_j} = b \in R^d .$$

Then eventually $n_j \in [N_{k_j}, N_{k_j+1})$ where $k_j \to \infty$, and then for every $\varepsilon > 0$ either

$$||n_j^{-1}S_{n_j}|| \geqq \tfrac{1}{2}k_j^2 \qquad \text{or else} \qquad ||n_j^{-1}S_{n_j} - c_{k_j}|| < \varepsilon$$

for all j sufficiently large. The first relation cannot hold forever, hence $||n_j^{-1}S_{n_j} - c_{k_j}|| \to 0$ and therefore $c_{k_j} \to b$ as $j \to \infty$ or $b \in C$ by (6.2) or

$$B^f(F, 1) \subset C .$$

But (6.18) and (6.20) hold for some $n \in [L_k, N_{k+1})$ for all k sufficiently large w.p. 1, or, since $\varepsilon > 0$ is arbitrary,

$$\lim_{k\to\infty} \inf_{n \in [N_k, N_{k+1})} ||n^{-1}S_n - c_k|| = 0 \quad \text{a.s.}$$

But $\{c_k, k \geqq m\}$ is dense in C for every m so it follows that every c in C is the a.s. limit of some (random) subsequence of $\{n^{-1}S_n\}$. Thus

$$C \subset B^f(F, 1)$$

and (6.10) follows.

Here is the proof of (6.20). Clearly since $d < \infty$ we need only verify (6.20) for each $\gamma = 2, 3, \cdots, d$ separately. By (6.12)

$$(6.21) \qquad \begin{aligned} &P\{|n^{-1}S_n(\gamma) - c_k(\gamma)| > \varepsilon \text{ for some } n \in [L_k, N_{k+1})\} \\ &\qquad \leqq P\{|W_n^k(\gamma) - nc_k(\gamma)| > n\varepsilon \text{ for some } n \geqq L_k\} . \end{aligned}$$

Now $\{W_n^k(\gamma)\}n \geqq 1$ is a random walk in R^1 whose increments have mean

$$EW_1^k(\gamma) = \textstyle\sum_{j=1}^k b_j(\gamma)p_j = c_k(\gamma) ,$$

see (6.3), and variance

$$\begin{aligned} E(W_1^k(\gamma) - c_k(\gamma))^2 &\leqq E(W_1^k(\gamma))^2 = E[\textstyle\sum_{j=1}^k b_j(\gamma)I(\eta_1 = j)]^2 \\ &= \textstyle\sum_{j=1}^k b_j^2(\gamma)p_j \leqq p_k^{-1} \sum_{j=1}^k b_j^2(\gamma)p_j^2 \\ &\leqq p_k^{-1} \textstyle\sum_{j=1}^k 4j^2 = O\left(\frac{k^3}{p_k}\right), \qquad \gamma \geqq 2 . \end{aligned}$$

See (6.1), (6.3) and note that the p_j are decreasing by (6.7). Applying (5.37), page 1200, to the RHS (6.21) (see also (5.54), page 1203) gives us

$$\begin{aligned} &P\{|W_n^k(\gamma) - nc_k(\gamma)| \geqq n\varepsilon \text{ for some } n \geqq L_k\} \\ &\quad \leqq P\{L_k < \tfrac{1}{2}(\lambda_k/p_k)\} + P\left\{|W_n^k(\gamma) - nc_k(\gamma)| \geqq n\varepsilon \text{ some } n \geqq \frac{\lambda_k}{2p_k}\right\} \\ &\quad = O\left(\frac{1}{k^2}\right) + O\left(\frac{p_k^{-1}k^3}{(\varepsilon^2/2)\lambda_k p_k^{-1}}\right) = O\left(\frac{1}{k^2}\right) + O\left(\frac{k^3}{\lambda_k}\right). \end{aligned}$$

But $\lambda_k \geqq k^7$ by (6.4) so finally

$$P\{|n^{-1}S_n(\gamma) - c_k(\gamma)| > \varepsilon \text{ for some } n \in [L_k, N_{k+1})\} \leqq \frac{c}{k^2}$$

for some constant c independent of k (but not ε). This estimate and a Borel–Cantelli lemma give (6.20).

REMARK 8. If $\{c_k\}$ has infinite limit points $A^\infty\{c_k\} \subset R_\infty{}^d$, then the preceding construction clearly shows that $\{n^{-1}S_n\}$ will also have $A^\infty\{c_k\}$ as infinite limit points. In addition (6.2) shows that

$$\frac{S_n(\gamma)}{|S_n|} = O\left(\frac{1}{k}\right), \qquad 2 \leqq \gamma \leqq d,$$

where $|S_n| = (\sum_{\gamma=1}^d S^2(\gamma))^{\frac{1}{2}}$, whenever (6.19) and (6.20) hold for the same $n \in [N_k, N_{k+1})$, k sufficiently large. It can be shown that this is the case infinitely often, so in addition to $A^\infty\{c_k\}$ we also get $\infty \cdot (1, 0, \cdots, 0) \equiv \infty \cdot e_1$ or $\infty \cdot (-e_1)$ or both in $B(F, 1)$. Reexamination of [5], pages 1198–1203, shows that in fact we get both. Thus

$$A^\infty\{c_k\} \cup \{\infty \cdot e_1\} \cup \{\infty \cdot (-e_1)\} \subset B^\infty(F, 1)\,. \tag{6.22}$$

By taking $\{c_k\}$ dense in $R_\infty{}^d$ as well as dense in C we get $B^\infty(F, 1) = R_\infty{}^d$. If one controls $S_n(\gamma)$, $\gamma = 2, 3, \cdots, d$ for $n \in [N_k, L_k)$ more carefully, one may get equality in (6.22).

Added in proof. Dr. Joop Mijnheer pointed out to me that one actually gets $B(F, \alpha) = [b, \infty]$ for some $0 < b < \infty$ in the Example 1 of Section 4. His proof uses a modification of the arguments of Wichura [11]. He is preparing a general theorem which will cover a large class of distributions (including that of Example 1) in a domain of attraction of a positive stable law.

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