The Annals of Probability
1976, Vol. 4. No. 3, 339-356

COUPLING THE SIMPLE EXCLUSION PROCESS

BY THOMAS M. LIGGETT[1]

University of California, Los Angeles

Consider the infinite particle system on the countable set S with the simple exclusion interaction and one-particle motion determined by the stochastic transition matrix $p(x, y)$. In the past, the ergodic theory of this process has been treated successfully only when $p(x, y)$ is symmetric, in which case great simplifications occur. In this paper, coupling techniques are used to give a complete description of the set of invariant measures for the system in the following three cases: (a) $p(x, y)$ is translation invariant on the integers and has mean zero, (b) $p(x, y)$ corresponds to a birth and death chain on the nonnegative integers, and (c) $p(x, y)$ corresponds to the asymmetric simple random walk on the integers.

1. Introduction. The simple exclusion process was introduced by Spitzer in [14] and has been studied extensively in [4], [9], [10], [11], [12], and [15]. It models the behavior of infinitely many identical particles on a countable set in such a way that the basic motion of each particle is that of a continuous time Markov chain. Superimposed upon this motion is the exclusion interaction, which causes transitions to occupied sites to be suppressed. The original motivating interest in this process came from the fact that when it is modified by letting the particles undergo a form of random speed change, the resulting process has the Gibbs measures of statistical mechanics as invariant measures. Thus one of the major problems concerning the exclusion process is to determine the exact structure of its set of invariant measures.

Satisfactory solutions to this problem have been obtained only when the Markov chain is symmetric ([9], [10], [15]) and when it is positive recurrent and reversible ([11]). In both cases, the solutions are based on the fact that essentially all problems concerning the infinite system can be reduced to equivalent problems concerning only finitely many interacting particles. This simplification is not available in greater generality, or even in the symmetric case if the process with speed change is considered. Therefore, techniques which deal with the infinite particle system directly are needed. In this paper, we will use the technique of coupling two copies of the process together to determine the set of invariant measures in several cases.

Coupling ideas have been used extensively and have been very effective tools in the study of systems of infinitely many interacting particles. The first and

Received July 11, 1975; revised October 20, 1975.

[1] Research supported in part by NSF Grant No. MPS 72-04591 and in part by an Alfred P. Sloan Foundation research fellowship.

AMS 1970 *subject classification*. Primary 60K35.

Key words and phrases. Infinite particle system, invariant measures, simple exclusion process, coupling.

most frequent use of these ideas has been in the area of spin-flip processes (see [1], [2], [3], [7], and [16], for example). In this context, they have been useful primarily in cases in which the system has a unique invariant measure. The simple exclusion process, on the other hand, typically has at least a one-parameter family of extremal invariant measures—a situation which requires that the technique be used somewhat differently. In the exclusion process, coupling was first used by Spitzer [15] to analyze the finite simple exclusion process in the symmetric recurrent case. It was then used indirectly in [11] and [12]. As will be seen here, while coupling is a common theme in the proofs of our theorems, it does not in general suffice to prove any one of them. Other ideas, which vary from case to case, are required to complete the results.

In order to define the simple exclusion process precisely, let S be a countable set, and $p(x, y)$ be the transition function for an irreducible discrete time Markov chain on S. The simple exclusion process determined by $p(x, y)$ is the Feller process η_t on $X = \{0, 1\}^S$ which corresponds to the semigroup $S(t)$ of contractions on $C(X)$, whose generator Ω takes the following form on functions f which depend on finitely many coordinates:

$$\Omega f(\eta) = \sum_{\eta(x)=1;\eta(y)=0} p(x, y)[f(\eta_{xy}) - f(\eta)] .$$

Here $\eta_{xy} \in X$ is the configuration obtained from $\eta \in X$ by interchanging the x and y coordinates. The existence of such a process was proved in [8] under the assumption that $\sup_y \sum_x p(x, y) < \infty$, which is always satisfied in the cases we are considering here. The set of invariant probability measures for η_t is defined as

$$\mathscr{I} = \{\mu \text{ on } X: \mu S(t) = \mu \text{ for all } t \geqq 0\} ,$$

where $\mu S(t)$ is the distribution at time t of the process when the initial distribution is μ. $\mathscr{I}$ can be described in terms of the generator in the following way:

$$\mathscr{I} = \{\mu \text{ on } X: \textstyle\int \Omega f \, d\mu = 0 \text{ for all } f \in \mathscr{D}(\Omega)\} .$$

Since $\mathscr{I}$ is convex and compact in the topology of weak convergence, it is the closed convex hull of its set $\mathscr{I}_e$ of extreme points. Thus, in order to determine $\mathscr{I}$, it suffices to study $\mathscr{I}_e$.

The verification that a given probability measure is invariant is usually a straightforward computation using the form of the generator given above. Thus in determining $\mathscr{I}_e$, one often has a collection of measures which one knows are invariant, and the difficult problem is to prove that the process has no other invariant measures. There are two cases in which one has an explicit collection of invariant measures. First, suppose that $p(x, y)$ is doubly stochastic. Then if ν_ρ is the product measure on X with $\nu_\rho\{\eta : \eta(x) = 1\} = \rho$ for all $x \in S$, it is known and easy to check that $\nu_\rho \in \mathscr{I}$ for $0 \leqq \rho \leqq 1$. Secondly, if $\pi(x)p(x, y) = \pi(y)p(y, x)$ for some positive $\pi(\cdot)$ on S (i.e., if $p(x, y)$ is reversible), then it was shown in [11] that $\nu^{(\rho)} \in \mathscr{I}$ for $0 \leqq \rho \leqq \infty$, where $\nu^{(\rho)}$ is the product measure on X with $\nu^{(\rho)}\{\eta : \eta(x) = 1\} = \rho\pi(x)/(1 + \rho\pi(x))$ for all $x \in S$. Note that these two classes of invariant measures coincide when $p(x, y)$ is symmetric.

If $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$, $\nu^{(\rho)}$ is not extremal, so some additional notation will be required to describe $\mathscr{I}_e$ in reversible cases when $\pi(\cdot)$ satisfies this property. (It should be noted in this connection that $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ is exactly the necessary and sufficient condition for the measures $\{\nu^{(\rho)}, 0 < \rho < \infty\}$ to be mutually singular.)

(a) If $\sum_x \pi(x) < \infty$, let $A = \{\eta: \sum_x \eta(x) < \infty\}$, $A_n = \{\eta: \sum_x \eta(x) = n\}$ for nonnegative integers n, and $\nu^{(\infty)} = \nu_1$, the pointmass on $\eta \equiv 1$.

(b) If $\sum_x 1/\pi(x) < \infty$, let $A = \{\eta: \sum_x [1-\eta(x)] < \infty\}$, $A_n = \{\eta: \sum_x [1-\eta(x)] = n\}$ for nonnegative integers n, and $\nu^{(\infty)} = \nu_0$, the pointmass on $\eta \equiv 0$.

(c) If $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$, $\sum_x \pi(x) = \infty$, and $\sum_x 1/\pi(x) = \infty$, there exists a $T \subset S$ for which $\sum_{x\in T} \pi(x) < \infty$ and $\sum_{x\notin T} 1/\pi(x) < \infty$. In this case, let $A = \{\eta: \sum_{x\in T} \eta(x) < \infty, \sum_{x\notin T}[1-\eta(x)] < \infty\}$, $A_n = \{\eta \in A: \sum_{x\in T}\eta(x) - \sum_{x\notin T}[1-\eta(x)] = n\}$ for all integers n, $\nu^{(-\infty)} = \nu_0$, and $\nu^{(\infty)} = \nu_1$.

In all three cases, A is countable and η_t is a Markov chain on A for which the closed irreducible classes are $\{A_n\}$. Since $\nu^{(\rho)} \in \mathscr{I}$, $\nu^{(\rho)}(A) = 1$, and $\nu^{(\rho)}(A_n) > 0$ for each $\rho \in (0, \infty)$ and finite n, it follows that this Markov chain is positive recurrent on each A_n and has unique stationary distribution

$$\nu^{(n)}(\cdot) = \nu^{(\rho)}(\cdot \mid A_n)\,,$$

which is therefore independent of ρ. Note that in case (c) above, changing T results only in a relabeling of the sequence $\{\nu^{(n)}, n \in Z^1\}$.

We now proceed to state the theorems which will be proved in this paper. In each case, the results confirm the conjectures which would be made on the basis of the comments in the previous two paragraphs. All but one of our results deal with the translation invariant case, in which $S = Z^d$ and $p(x, y) = p(0, y-x)$. In this case, $p(x, y)$ is doubly stochastic, so that $\nu_\rho \in \mathscr{I}$ for each $\rho \in [0, 1]$. The main question then is whether there can be any invariant measures which are not exchangeable. The first partial answer to this question is that if so, such a measure cannot be translation invariant. Let $\mathscr{S}$ be the set of translation invariant probability measures on X.

THEOREM 1.1. *Suppose $S = Z^d$ and $p(x, y) = p(0, y-x)$. Then*

$$(\mathscr{I} \cap \mathscr{S})_e = \{\nu_\rho, 0 \leqq \rho \leqq 1\}\,.$$

Since this result deals only with invariant measures which are also translation invariant, it is of somewhat limited interest. It is a necessary preliminary, however, to the proofs of Theorems 1.2 and 1.4 below. In the one dimensional mean zero case, the above question can be answered completely:

THEOREM 1.2. *Suppose $S = Z^1$, $p(x, y) = p(0, y-x)$, $\sum_x |x|p(0, x) < \infty$, and $\sum_x xp(0, x) = 0$. Then*

$$\mathscr{I}_e = \{\nu_\rho, 0 \leqq \rho \leqq 1\}\,.$$

These two theorems will be proved in Section 3, after some preliminary results on the coupled process are obtained in Section 2. Theorem 1.1 could also be proved by using the entropy techniques developed by Holley in [5] and [6]. However, our coupling approach appears to be somewhat simpler in this context. It should be noted that if $p(x, y)$ is symmetric, both theorems are contained in the results in [9] and [15].

THEOREM 1.3. *Suppose* $S = \{0, 1, 2, \cdots\}$, *and* $p(x, y) = 0$ *for* $|y - x| > 1$. *Note that* $p(x, y)$ *is reversible with* $\pi(0) = 1$ *and*

$$\pi(x) = \prod_{y=0}^{x-1} \frac{p(y, y+1)}{p(y+1, y)}$$

for $x \geqq 1$. *Then*

(a) $\mathscr{I}_e = \{\nu^{(\rho)}, 0 \leqq \rho \leqq \infty\}$ *if* $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$.
(b) $\mathscr{I}_e = \{\nu^{(n)}, 0 \leqq n \leqq \infty\}$ *if* $\sum_x \pi(x) < \infty$ *or* $\sum_x 1/\pi(x) < \infty$.
(c) $\mathscr{I}_e = \{\nu^{(n)}, -\infty \leqq n \leqq \infty\}$ *otherwise.*

This result will be proved in Section 4. Only parts (a) and (c) are new. Part (b) is a special case of Theorem 1.3 of [11] in case $\sum_x \pi(x) < \infty$, and a proof similar to the one used there suffices in case $\sum_x 1/\pi(x) < \infty$.

In our final case, which will be discussed in Section 5, $p(x, y)$ is both doubly stochastic and reversible, so both classes of invariant measures play a role. It illustrates the type of result which should be expected either if the zero mean assumption is eliminated from Theorem 1.2, or if Theorem 1.3 is stated with $S = Z^1$ instead of $S = \{0, 1, 2, \cdots\}$.

THEOREM 1.4. *Suppose* $S = Z^1$, $p(x, x+1) = p$, $p(x, x-1) = q$, $p + q = 1$, *and* $p \neq \frac{1}{2}$. *Then*

$$\mathscr{I}_e = \{\nu_\rho, 0 \leqq \rho \leqq 1\} \cup \{\nu^{(n)}, -\infty < n < \infty\}.$$

This result was conjectured in [12]. If $0 < p < 1$, $p(x, y)$ is reversible with $\pi(x) = (p/q)^x$, so the measures $\nu^{(n)}$ were defined above. The cases $p = 1$ and $p = 0$ are allowed here, even though the chain is not irreducible. If $p = 1$, for example, $\nu^{(n)}$ is to be interpreted as the pointmass on the configuration η_n, where $\eta_n(x) = 0$ for $x \leqq -n$ and $\eta_n(x) = 1$ for $x > -n$. These measures are clearly invariant, since no motion occurs if the process is begun in configuration η_n. One of the interesting features of this theorem is that it is just as difficult to prove in these cases as it is for $0 < p < 1$. In fact, one might expect that if $p = 1$, the process would be simple enough that one could see directly that the only probability measures μ which are solutions to the equations $\int \Omega f \, d\mu = 0$ for $f \in \mathscr{D}(\Omega)$ are the ones given in the above theorem. This does not appear to be possible.

Before proceeding to the detailed proofs, it may be helpful to give a brief indication of how the coupling idea will be used. The coupling of the two copies of the process has the property that $\eta_t(x)$ and $\zeta_t(x)$ will tend to be equal as much as is possible within the constraints provided by the fact that each process separately must have the proper initial distribution and transition probabilities. This enables one to argue along the lines of the following outline:

(a) If ν is invariant for the coupled process, then

$$\sum_{x,y}[p(x, y) + p(y, x)]\nu\{(\eta, \zeta) : \eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} < \infty .$$

(b) The above sum is actually zero.

(c) If ν is invariant for the coupled process, then

$$\nu\{(\eta, \zeta) : \eta \leqq \zeta \text{ or } \eta \geqq \zeta\} = 1 .$$

(d) If ν is extremal invariant for the coupled process, then either

$$\nu\{(\eta, \zeta) : \eta \leqq \zeta\} = 1 \qquad \text{or} \qquad \nu\{(\eta, \zeta) : \eta \geqq \zeta\} = 1 .$$

(e) If $\mu_1, \mu_2 \in \mathscr{I}_e$, then there exists an extremal invariant measure for the coupled process which has marginals μ_1 and μ_2 respectively, and hence either $\mu_1 \leqq \mu_2$ or $\mu_1 \geqq \mu_2$.

(f) This enables one to "squeeze" an arbitrary $\mu \in \mathscr{I}_e$ between two of the known invariant measures.

Not all of these steps are completely accurate in the case of all four theorems, but they do indicate the general flow of ideas in all four cases.

2. The coupled process. By the coupled process, we will mean the Markov process $\gamma_t = (\eta_t, \zeta_t)$ with state space $X \times X$ which has the following properties: (a) the marginal processes η_t and ζ_t are Markovian and have generator Ω, and (b) whenever $\eta_t(x) = \zeta_t(x) = 1$ for some $x \in S$, the two marginal processes will use the same random mechanisms to decide when the particle at x will attempt a transition, and where it will attempt to go. Of course, it may be that in only one or neither of the processes will the particle at x be allowed to move. More formally, γ_t is the Feller process on $X \times X$ whose generator $\bar{\Omega}$ takes the following form when restricted to functions $f(\eta, \zeta)$ which depend on only finitely many coordinates:

$$\begin{aligned}\bar{\Omega}f(\eta, \zeta) = & \sum_{\substack{\eta(x)=1,\eta(y)=0 \\ \text{and } \zeta(x)=\zeta(y) \\ \text{or } \zeta(x)=0,\zeta(y)=1}} p(x, y)[f(\eta_{xy}, \zeta) - f(\eta, \zeta)] \\ & + \sum_{\substack{\zeta(x)=1,\zeta(y)=0 \\ \text{and } \eta(x)=\eta(y) \\ \text{or } \eta(x)=0,\eta(y)=1}} p(x, y)[f(\eta, \zeta_{xy}) - f(\eta, \zeta)] \\ & + \sum_{\substack{\eta(x)=\zeta(x)=1 \\ \eta(y)=\zeta(y)=0}} p(x, y)[f(\eta_{xy}, \zeta_{xy}) - f(\eta, \zeta)] .\end{aligned} \tag{2.1}$$

That such a process exists and has the properties mentioned above, provided again that $\sup_y \sum_x p(x, y) < \infty$, is a consequence of the results of [8]. We will use the upper bar to denote symbols pertaining to the coupled process. Thus $\bar{\mathscr{I}}$ will be the set of invariant measures for γ_t, and $\bar{\mathscr{S}}$ will be the set of translation invariant probability measures on $X \times X$ in case $S = Z^d$.

One of the important properties of this coupling is that certain subsets of the state space are closed for the motion. It can easily be proved using the results of [8], for example, that $P^{(\eta,\zeta)}[\eta_t = \zeta_t] = 1$ if $\eta = \zeta$, $P^{(\eta,\zeta)}[\eta_t \leqq \zeta_t] = 1$ if $\eta \leqq \zeta$, and $P^{(\eta,\zeta)}[\eta_t \geqq \zeta_t] = 1$ if $\eta \geqq \zeta$, where the inequalities are to be interpreted coordinatewise. This observation leads to the following result.

LEMMA 2.2. *If* $\nu \in \bar{\mathscr{I}}_e$, *then* $\nu\{(\eta, \zeta) : \eta = \zeta\}$, $\nu\{(\eta, \zeta) : \eta \leqq \zeta\}$, *and* $\nu\{(\eta, \zeta) : \eta \geqq \zeta\}$ *is each either zero or one. The same statement holds for* $\nu \in (\bar{\mathscr{I}} \cap \bar{\mathscr{S}})_e$ *in the translation invariant case.*

PROOF. Since the proofs in the various cases are all similar, we will consider only the first one. Suppose $\nu \in \bar{\mathscr{I}}_e$ and $0 < \nu(B) < 1$ for $B = \{(\eta, \zeta) : \eta = \zeta\}$. Then $\nu = \nu(B)\alpha + [1 - \nu(B)]\beta$, where $\alpha(\cdot) = \nu(\cdot \mid B)$ and $\beta(\cdot) = \nu(\cdot \mid B^c)$. $\alpha \in \bar{\mathscr{I}}$ since $P^\gamma[\gamma_t \in B] = 1$ for $\gamma \in B$, and therefore $\beta \in \bar{\mathscr{I}}$ also. Since ν is extremal, it follows that $\nu = \alpha = \beta$, which gives a contradiction since $\nu(B) < 1 = \alpha(B)$.

In future sections, we will study the invariant measures for the coupled process, and then use information about them to draw conclusions about the invariant measures for the marginal process. In order to do this, we need to obtain relations between $\bar{\mathscr{I}}$ and $\mathscr{I}$. First suppose that $\nu \in \bar{\mathscr{I}}$ and μ_1 and μ_2 are the marginal measures of ν defined by $\mu_1(D) = \nu(D \times X)$ and $\mu_2(D) = \nu(X \times D)$. Then it is clear that μ_1 and $\mu_2 \in \mathscr{I}$. In the other direction, we have the following result.

LEMMA 2.3. (a) *If* $\mu_1, \mu_2 \in \mathscr{I}$, *there is a* $\nu \in \bar{\mathscr{I}}$ *with marginals* μ_1 *and* μ_2. (b) *If* $\mu_1, \mu_2 \in \mathscr{I}_e$, *then* ν *can be taken in* $\bar{\mathscr{I}}_e$. (c) *In the translation invariant case, if* $\mu_1, \mu_2 \in \mathscr{I} \cap \mathscr{S}$, *then* ν *can be taken in* $\bar{\mathscr{I}} \cap \bar{\mathscr{S}}$. (d) *In the translation invariant case, if* $\mu_1, \mu_2 \in (\mathscr{I} \cap \mathscr{S})_e$, *then* ν *can be taken in* $(\bar{\mathscr{I}} \cap \bar{\mathscr{S}})_e$.

PROOF. We will prove (a) and (b) only, since the proofs of (c) and (d) are similar. Take $\mu_1, \mu_2 \in \mathscr{I}$, and let ν_1 be the product measure $\mu_1 \times \mu_2$ on $X \times X$. Then $\nu_1\bar{S}(t)$ has marginals μ_1 and μ_2 for all $t > 0$, although it will not in general be a product measure. Since $X \times X$ is compact, there is a sequence $t_n \to \infty$ so that $1/t_n \int_0^{t_n} \nu_1\bar{S}(t)\,dt$ converges weakly to some probability measure ν. Since $\nu \in \bar{\mathscr{I}}$ and has marginals μ_1 and μ_2, the proof of (a) is complete. In order to prove (b), take $\mu_1, \mu_2 \in \mathscr{I}_e$ and let

$$\mathscr{A} = \{\nu \in \bar{\mathscr{I}} : \nu \text{ has marginals } \mu_1 \text{ and } \mu_2\} .$$

$\mathscr{A}$ is compact and convex, and is nonempty by part (a). Therefore $\mathscr{A}_e \neq \emptyset$ by the Krein–Millman theorem. It suffices then to prove that $\mathscr{A}_e \subset \bar{\mathscr{I}}_e$. Suppose $\nu \in \mathscr{A}_e$ and $\nu = \lambda\alpha + (1 - \lambda)\beta$ for some α and $\beta \in \bar{\mathscr{I}}$ and $0 < \lambda < 1$. The marginals of ν are the same convex combinations of the marginals of α and β, so α and $\beta \in \mathscr{A}$ since μ_1 and μ_2 are extremal. Since $\nu \in \mathscr{A}_e$, it follows that $\nu = \alpha = \beta$, and therefore that $\nu \in \bar{\mathscr{I}}_e$.

Another important property of the coupled process can be seen most clearly by thinking of it as a process in which the particles which move on S are of four types: (0, 0), (1, 1), (0, 1), (1, 0). Then some of the transitions can be thought of as resulting in the destruction of particles of the last two types and their replacement by particles of the first two types. Therefore the "number" (which is usually infinite) of particles of the first two types can only increase. This is the basis for the next result.

LEMMA 2.4. *Suppose* $\nu \in \bar{\mathscr{I}}$ *and* R *is a finite subset of* S. *Then*

$$\begin{aligned}\sum_{x,y} [p(x, y) + p(y, x)][1_R(x) + 1_R(y)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} \\ = \sum_{x \in R, y \notin R} p(x, y)[\nu\{\eta(x) = \zeta(x) = 1, \eta(y) \neq \zeta(y)\} \\ - \nu\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 0\}] \\ + \sum_{x \in R, y \notin R} p(y, x)[\nu\{\eta(x) = \zeta(x) = 0, \eta(y) \neq \zeta(y)\} \\ - \nu\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 1\}] .\end{aligned}$$

PROOF. Let $f(\eta, \zeta) = \sum_{x \in R} 1_{\{\eta(x)=\zeta(x)\}}$ be the number of sites in R at which η and ζ agree. Since f depends on only finitely many coordinates, $f \in \mathscr{D}(\bar{\Omega})$ and $\bar{\Omega} f$ is given by (2.1). Therefore $\int \bar{\Omega} f \, d\nu = 0$ for $\nu \in \bar{\mathscr{I}}$. The identity in the lemma is then obtained by a direct computation of $\int \bar{\Omega} f \, d\nu$.

LEMMA 2.5. *Suppose* $\nu \in \bar{\mathscr{I}}$ *satisfies*

$$\sum_{x,y} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} = 0 .$$

Then $\nu\{(\eta, \zeta) : \eta \leqq \zeta$ *or* $\eta \geqq \zeta\} = 1$.

PROOF. It suffices to prove that $\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} = 0$ for all $x, y \in S$. Since $p(x, y)$ is irreducible, it suffices to prove by induction on k that this holds whenever $p^{(k)}(x, y) > 0$. This is true for $k = 1$ by assumption. Assume then that it is true for $k \leqq n - 1$, and consider $x_0, x_1, \cdots, x_n$ for which $p(x_0, x_1)p(x_1, x_2) \cdots p(x_{n-1}, x_n) > 0$. Let $E = \{(\eta, \zeta) : \eta(x_0) = \zeta(x_n) = 1, \eta(x_n) = \zeta(x_0) = 0\}$. By the induction assumption,

$$\begin{aligned}\nu(E) &= \nu(E, \eta(x_i) = \zeta(x_i) \text{ for } 1 \leqq i \leqq n - 1) \\ &= \nu(E, \eta(x_i) = \zeta(x_i) = 1 \text{ for } 1 \leqq i \leqq n - 1) \\ &\quad + \sum_{i=1}^{n-1} \nu(E, \eta(x_j) = \zeta(x_j) = 1 \\ &\qquad \text{for } 1 \leqq j \leqq i - 1, \eta(x_i) = \zeta(x_i) = 0) .\end{aligned} \tag{2.6}$$

Let

$$f(\eta, \zeta) = 1_{\{\eta(x_0)=\zeta(x_{n-1})=1, \eta(x_{n-1})=\zeta(x_0)=0, \eta(x_i)=\zeta(x_i)=1 \text{ for } 1 \leqq i \leqq n-2 \text{ and } i=n\}} .$$

Then $\int \bar{\Omega} f \, d\nu = 0$ since $\nu \in \bar{\mathscr{I}}$ and $f \in \mathscr{D}(\bar{\Omega})$, and $\int f \, d\nu = 0$ by the induction assumption. From this it follows that the first term on the right side of (2.6) is zero. A similar argument, combined with an induction on i, can then be used to prove that the terms in the sum in (2.6) are also zero. Thus $\nu(E) = 0$.

If μ_1 and μ_2 are probability measures on X, we will say that $\mu_1 \leqq \mu_2$ if $\int f \, d\mu_1 \leqq \int f \, d\mu_2$ for all monotonically increasing functions f in $C(X)$. This is equivalent to saying that there is a probability measure on $X \times X$ which concentrates on $\{(\eta, \zeta) : \eta \leqq \zeta\}$ and which has marginals μ_1 and μ_2 respectively. One proof of this equivalence can be found in Theorem 53 of Chapter XI of [13]. Using this relation, Lemmas 2.2 and 2.5 can be combined in the following way.

COROLLARY 2.7. *Suppose* $\nu \in \bar{\mathscr{I}}_e$ *has marginals* μ_1 *and* μ_2 *and satisfies*

$$\sum_{x,y} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} = 0 .$$

Then either $\mu_1 \leqq \mu_2$ *or* $\mu_2 \leqq \mu_1$. *The same is true for* $\nu \in (\bar{\mathscr{I}} \cap \bar{\mathscr{S}})_e$ *in the translation invariant case.*

3. First applications. This section will be devoted to the proofs of Theorems 1.1 and 1.2. The crucial fact used in the first proof is the following.

LEMMA 3.1. *Suppose* $S = Z^d$ *and* $p(x, y) = p(0, y - x)$. *If* $\nu \in \bar{\mathscr{I}} \cap \bar{\mathscr{S}}$, *then* $\nu\{(\eta, \zeta) : \eta \leqq \zeta \text{ or } \eta \geqq \zeta\} = 1$.

PROOF. For $n \geqq 1$, let R_n be the following cube in Z^d:

$$\{x = (x_1, \cdots, x_d) \in Z^d : |x_i| \leqq n \quad \text{for} \quad 1 \leqq i \leqq d\}\,.$$

The first observation is that

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{x \in R_n, y \notin R_n} [p(x, y) + p(y, x)] = 0\,, \tag{3.2}$$

so that the right side of the equality in Lemma 2.4 is $o(n^d)$ as $n \to \infty$ for any probability measure ν. In order to see this, let $m \leqq n$ and write

$$\begin{aligned}\sum_{x \in R_n, y \notin R_n} [p(x, y) + p(y, x)] &\leqq \sum_{x \in R_m, |y-x| \geqq n-m} [p(x, y) + p(y, x)] \\ &+ \sum_{x \in R_n \backslash R_m, y \notin R_n} [p(x, y) + p(y, x)] \leqq 2(2m + 1)^d \sum_{|z| \geqq n-m} p(0, z) \\ &+ 2[(2n + 1)^d - (2m + 1)^d]\,.\end{aligned}$$

Statement (3.2) then follows by using a sequence m_n in this inequality which satisfies $m_n/n \to 1$ and $n - m_n \to \infty$. If $\nu \in \bar{\mathscr{S}}$, $\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\}$ is a function of $y - x$, and therefore

$$\begin{aligned}&\sum_{x,y \in R_n} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} \\ &\quad = (2n + 1)^d \sum_y [p(0, y) + p(y, 0)]\nu\{\eta(0) = \zeta(y) = 1, \eta(y) = \zeta(0) = 0\} \\ &\quad\quad - \sum_{x \in R_n, y \notin R_n} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\}\,.\end{aligned}$$

The last term above is $o(n^d)$ as $n \to \infty$ by (3.2), and therefore

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n^d} \sum_{x,y \in R_n} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} \\ &\quad = 2^d \sum_y [p(0, y) + p(y, 0)]\nu\{\eta(0) = \zeta(y) = 1, \eta(y) = \zeta(0) = 0\}\,.\end{aligned}$$

Thus if $\nu \in \bar{\mathscr{I}} \cap \bar{\mathscr{S}}$, Lemma 2.4 yields

$$\sum_y [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} = 0$$

for each x, and so

$$\nu\{(\eta, \zeta) : \eta \leqq \zeta \quad \text{or} \quad \eta \geqq \zeta\} = 1 \tag{3.3}$$

by Lemma 2.5.

COROLLARY 3.4. *Suppose* $S = Z^d$ *and* $p(x, y) = p(0, y - x)$. *If* $\mu_1, \mu_2 \in (\mathscr{I} \cap \mathscr{S})_e$, *then either* $\mu_1 \leqq \mu_2$ *or* $\mu_2 \leqq \mu_1$.

PROOF. Choose $\nu \in (\bar{\mathscr{I}} \cap \bar{\mathscr{S}})_e$ by Lemma 2.3 so that it has marginals μ_1 and

μ_2. By (3.3) and Lemma 2.2, either $\nu\{(\eta,\zeta):\eta \leqq \zeta\} = 1$ or $\nu\{(\eta,\zeta):\eta \geqq \zeta\} = 1$, and therefore either $\mu_1 \leqq \mu_2$ or $\mu_2 \leqq \mu_1$.

PROOF OF THEOREM 1.1. Since ν_ρ is translation invariant and ergodic, $\nu_\rho \in \mathscr{S}_e$. Also $\nu_\rho \in \mathscr{I}$, so that $\nu_\rho \in (\mathscr{I} \cap \mathscr{S})_e$. By Corollary 3.4, if $\mu \in (\mathscr{I} \cap \mathscr{S})_e$ and $\rho \in [0, 1]$, then either $\mu \leqq \nu_\rho$ or $\mu \geqq \nu_\rho$. Therefore if $\mu \in (\mathscr{I} \cap \mathscr{S})_e$, there is a $\rho_0 \in [0, 1]$ so that $\mu \geqq \nu_\rho$ for $\rho < \rho_0$ and $\mu \leqq \nu_\rho$ for $\rho > \rho_0$. Since $\int f\,d\nu_\rho$ is continuous and monotone in ρ for any monotone $f \in C(X)$, and since such functions span $C(X)$, it then follows that $\mu = \nu_{\rho_0}$. Therefore $(\mathscr{I} \cap \mathscr{S})_e = \{\nu_\rho, 0 \leqq \rho \leqq 1\}$.

The next result is the substitute for Lemma 3.1 which is needed in proving Theorem 1.2. The main difference in the proof is that the boundary terms in Lemma 2.4 require a more careful analysis.

LEMMA 3.5. *Suppose* $S = Z^1$, $p(x, y) = p(0, y - x)$, $\sum_x |x| p(0, x) < \infty$, *and* $\sum_x x p(0, x) = 0$. *If* $\nu \in \tilde{\mathscr{I}}$, *then* $\nu\{(\eta,\zeta):\eta \leqq \zeta$ or $\eta \geqq \zeta\} = 1$.

PROOF. Take $\nu \in \tilde{\mathscr{I}}$ and consider the limit of Cesàro averages on n of both sides of the identity in Lemma 2.4 with $R = R_n = [-n, n]$. The left hand side is

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \textstyle\sum_{n=1}^{N} \sum_{x,y} &[p(x, y) + p(y, x)] \\ &\times \nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\}[1_{R_n}(x) + 1_{R_n}(y)] \\ &\quad = 2 \textstyle\sum_{x,y} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\}\,.\end{aligned}$$

Since $\sum_x |x| p(0, x) < \infty$, the right hand side is equal to

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \textstyle\sum_{n=1}^{N} \{\sum_{x\leqq n<y} &p(x, y)[\nu\{\eta(x) = \zeta(x) = 1, \eta(y) \neq \zeta(y)\} \\ &- \nu\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 0\}] \\ &+ \textstyle\sum_{x\leqq n<y} p(y, x)[\nu\{\eta(x) = \zeta(x) = 0, \eta(y) \neq \zeta(y)\} \\ &- \nu\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 1\}] \\ &+ \textstyle\sum_{x\geqq -n>y} p(x, y)[\nu\{\eta(x) = \zeta(x) = 1, \eta(y) \neq \zeta(y)\} \\ &- \nu\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 0\}] \\ &+ \textstyle\sum_{x\geqq -n>y} p(y, x)[\nu\{\eta(x) = \zeta(x) = 0, \eta(y) \neq \zeta(y)\} \\ &- \nu\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 1\}]\}\,.\end{aligned} \tag{3.6}$$

By Lemma 2.5, it suffices to prove that this limit is zero. Let ν_n be the probability measure obtained from ν by shifting it in the following way:

$$\nu_n\{(\eta,\zeta):\eta(x_i) = 1, \zeta(y_j) = 1\} = \nu\{(\eta,\zeta):\eta(x_i + n) = 1, \zeta(y_j + n) = 1\}\,.$$

Then the first term in (3.6) can be written as

$$\begin{aligned}\textstyle\sum_{x\leqq 0<y} p(x, y) \frac{1}{N} \sum_{n=1}^{N} &[\nu_n\{\eta(x) = \zeta(x) = 1, \eta(y) \neq \zeta(y)\} \\ &- \nu_n\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 0\}]\,.\end{aligned} \tag{3.7}$$

Let $N_k \uparrow \infty$ be such that

$$\bar{\nu} = \lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} \nu_n$$

exists. Then $\bar{\nu} \in \bar{\mathscr{I}} \cap \bar{\mathscr{S}}$, so by Lemma 3.1, $\bar{\nu}$ can be written in the form $\bar{\nu} = \lambda\bar{\nu}_1 + (1-\lambda)\bar{\nu}_2$, where $0 \leqq \lambda \leqq 1$, $\bar{\nu}_1\{(\eta, \zeta): \eta \leqq \zeta\} = 1$, and $\bar{\nu}_2\{(\eta, \zeta): \eta \geqq \zeta, \eta \neq \zeta\} = 1$. Adding and subtracting $\bar{\nu}\{\eta(x) = \zeta(x) = 1, \eta(y) = \zeta(y) = 0\}$, we may write

$$\begin{aligned}
\bar{\nu}\{\eta(x) = \zeta(x) = 1, \eta(y) \neq \zeta(y)\} &- \bar{\nu}\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 0\} \\
&= \lambda\bar{\nu}_1\{\eta(x) = \zeta(x) = 1, \eta(y) \neq \zeta(y)\} \\
&\quad - \lambda\bar{\nu}_1\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 0\} \\
&\quad + (1-\lambda)\bar{\nu}_2\{\eta(x) = \zeta(x) = 1, \eta(y) \neq \zeta(y)\} \\
&\quad - (1-\lambda)\bar{\nu}_2\{\eta(x) \neq \zeta(x), \eta(y) = \zeta(y) = 0\} \\
&= \lambda\bar{\nu}_1\{\eta(x) = 1, \eta(y) = 0\} - \lambda\bar{\nu}_1\{\zeta(x) = 1, \zeta(y) = 0\} \\
&\quad + (1-\lambda)\bar{\nu}_2\{\zeta(x) = 1, \zeta(y) = 0\} \\
&\quad - (1-\lambda)\bar{\nu}_2\{\eta(x) = 1, \eta(y) = 0\}\,.
\end{aligned} \tag{3.8}$$

Since $\{\eta \leqq \zeta\}$, $\{\eta \geqq \zeta\}$ and $\{\eta = \zeta\}$ are translation invariant sets which are closed for the process, $\bar{\nu}_1$ and $\bar{\nu}_2 \in \bar{\mathscr{I}} \cap \bar{\mathscr{S}}$. Therefore the marginals of $\bar{\nu}_1$ and $\bar{\nu}_2$ are in $\mathscr{I} \cap \mathscr{S}$, and thus exchangeable by Theorem 1.1. From this fact and (3.8), it follows that the limit of (3.7) along the subsequence N_k is equal to

$$c \sum_{x \leqq 0 < y} p(x, y) = c \sum_{z>0} zp(0, z)\,,$$

where

$$\begin{aligned}
c = \lambda\bar{\nu}_1\{\eta(x) = 1, \eta(y) = 0\} &- \lambda\bar{\nu}_1\{\zeta(x) = 1, \zeta(y) = 0\} \\
&+ (1-\lambda)\bar{\nu}_2\{\zeta(x) = 1, \zeta(y) = 0\} - (1-\lambda)\bar{\nu}_2\{\eta(x) = 1, \eta(y) = 0\}
\end{aligned}$$

which is independent of x and y for $x \neq y$. Similarly, the limit along the sequence N_k of the second term of (3.6) is equal to $c \sum_{z<0} zp(0, z)$. Thus since $\sum_z zp(0, z) = 0$, the limit along N_k of the sum of the first two terms of (3.6) is zero. Since each subsequence has a further subsequence along which $1/N \sum_{n=1}^{N} \nu_n$ converges weakly, it follows that the limit of the sum of the first two terms of (3.6) along the full sequence is zero. A similar argument shows that the limit of the sum of the last two terms of (3.6) is also zero, thus completing the proof of the lemma.

COROLLARY 3.9. *Suppose $S = Z^1$, $p(x, y) = p(0, y-x)$, $\sum_x |x| p(0, x) < \infty$, and $\sum_x xp(0, x) = 0$. If $\mu_1, \mu_2 \in \mathscr{I}_e$, then either $\mu_1 \leqq \mu_2$ or $\mu_2 \leqq \mu_1$.*

PROOF OF THEOREM 1.2 The proof of this theorem would now proceed exactly as in the case of Theorem 1.1 if we knew that $\nu_\rho \in \mathscr{I}_e$ for $\rho \in [0, 1]$. Since all we know at this point is that $\nu_\rho \in \mathscr{I}$ for each ρ, we argue in the following way. Given $\mu \in \mathscr{I}_e$ and $\rho \in [0, 1]$, use Lemma 2.3 to choose $\nu \in \bar{\mathscr{I}}$ with marginals μ and ν_ρ respectively. By Lemma 3.5 and the fact that μ is extremal, there exist

μ_1 and $\mu_2 \in \mathscr{I}$ such that $\mu_1 \leqq \mu \leqq \mu_2$ and $\nu_\rho = \lambda\mu_1 + (1 - \lambda)\mu_2$ for $\lambda = \nu\{(\eta, \zeta): \eta \geqq \zeta\}$. If $\lambda = 0$ or 1, this is immediate. Otherwise, μ_1 and μ_2 are the second marginals of the measures obtained by conditioning ν on $\{(\eta, \zeta): \eta \geqq \zeta\}$ and its complement respectively. Suppose that $0 < \lambda < 1$. Then μ_1 and μ_2 are absolutely continuous with respect to ν_ρ, so

$$\mu_i \left\{\eta : \lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^n \eta(x) = \rho\right\} = 1$$

for $i = 1, 2$ by the strong law of large numbers. Since $\mu_1 \leqq \mu \leqq \mu_2$, it follows that

$$\mu \left\{\eta : \lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^n \eta(x) = \rho\right\} = 1$$

also. Therefore for a given $\mu \in \mathscr{I}_e$, there is at most one $\rho \in [0, 1]$ for which $0 < \lambda < 1$. Thus there is a $\rho_0 \in [0, 1]$ so that $\mu \geqq \nu_\rho$ for $\rho < \rho_0$ and $\mu \leqq \nu_\rho$ for $\rho > \rho_0$, and hence $\mu = \nu_{\rho_0}$ as in the proof of Theorem 1.1. We conclude then that

$$\mathscr{I}_e \subset \{\nu_\rho, 0 \leqq \rho \leqq 1\} \subset \mathscr{I},$$

and therefore that $\mathscr{I}_e = \{\nu_\rho, 0 \leqq \rho \leqq 1\}$.

4. The birth and death chain. Throughout this section, we will assume that $S = \{0, 1, 2, \cdots\}$ and that $p(x, y) = 0$ for $|y - x| > 1$. Our first aim is to prove that $\nu\{(\eta, \zeta): \eta \leqq \zeta$ or $\eta \geqq \zeta\} = 1$ for $\nu \in \bar{\mathscr{I}}$. This will be done via a series of lemmas.

LEMMA 4.1. *Assume that* $\inf_{x\geqq 0} p(x, x + 1) = 0$ *or* $\inf_{x\geqq 0} p(x + 1, x) = 0$. *If* $\nu \in \bar{\mathscr{I}}$, *then* $\nu\{(\eta, \zeta): \eta \leqq \zeta$ *or* $\eta \geqq \zeta\} = 1$.

PROOF. Take $\mu \in \mathscr{I}$ and let $f(\eta) = \sum_{x=0}^n \eta(x)$ for some n. Then $f \in \mathscr{D}(\Omega)$, so $\int \Omega f\, d\mu = 0$, which yields

$$\begin{aligned}(4.2) \qquad & p(n, n + 1)\mu\{\eta(n) = 1, \eta(n + 1) = 0\} \\ & \qquad = p(n + 1, n)\mu\{\eta(n + 1) = 1, \eta(n) = 0\}.\end{aligned}$$

Put $R = \{0, 1, \cdots, n\}$ in Lemma 2.4, and observe that the right side of the expression in that lemma is bounded above by

$$\begin{aligned}(4.3) \quad & p(n, n + 1)[\nu\{\eta(n) = 1, \eta(n + 1) = 0\} + \nu\{\zeta(n) = 1, \zeta(n + 1) = 0\}] \\ & \quad + p(n+1, n)[\nu\{\eta(n + 1) = 1, \eta(n) = 0\} + \nu\{\zeta(n + 1) = 1, \zeta(n) = 0\}].\end{aligned}$$

If $\nu \in \bar{\mathscr{I}}$, the marginals of ν are in $\mathscr{I}$, and therefore the two terms in (4.3) are equal by (4.2). By assumption, then, there is a subsequence along which (4.3) tends to zero, and thus by Lemma 2.4 we have

$$\sum_{x,y} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} = 0.$$

The required result then follows from Lemma 2.5.

LEMMA 4.4. *Assume that*

$$\inf_{x\geq 0} p(x, x+1) > 0 \qquad \text{and} \qquad \inf_{x\geq 0} p(x+1, x) > 0 . \tag{4.5}$$

Then

$$\sum_{x=0}^{\infty} \nu\{\eta(x) = \zeta(x+k) \neq \eta(x+k) = \zeta(x)\} < \infty \tag{4.6}$$

for $k \geqq 1$ *and* $\nu \in \bar{\mathscr{I}}$.

PROOF. Apply Lemma 2.4 with $R = \{0, 1, \cdots, n\}$ and let $n \to \infty$ to obtain

$$\sum_{x,y} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} < \infty .$$

Applying (4.5) to this yields (4.6) with $k = 1$. The general result follows by induction on k in the following way. Let $E(x, k) = \{(\eta, \zeta): \eta(x) = \zeta(x+k) \neq \eta(x+k) = \zeta(x)\}$ and write

$$\begin{aligned}\nu(E(x, k)) = \nu(E(x, k), \eta(x+1) = \zeta(x+1) = 1) \\ + \nu(E(x, k), \eta(x+1) = \zeta(x+1) = 0) \\ + \nu(E(x, k), \eta(x+1) \neq \zeta(x+1)) .\end{aligned}$$

Then $\nu(E(x, k), \eta(x+1) \neq \zeta(x+1)) \leqq \nu(E(x, 1)) + \nu(E(x+1, k-1))$. Since $\nu \in \bar{\mathscr{I}}$, $\int \bar{\Omega} 1_F \, d\nu = 0$ where $F = \{\eta(x) = \zeta(x) = 1\} \cap E(x+1, k-1)$. Evaluating $\int \bar{\Omega} 1_F \, d\nu$ and making some simple estimates yields

$$p(x+1, x)\nu(E(x, k), \eta(x+1) = \zeta(x+1) = 1) \leqq 6\nu(F) .$$

Similarly,

$$\begin{aligned}p(x, x+1)\nu(E(x, k), \eta(x+1) = \zeta(x+1) = 0) \\ \leqq 6\nu(E(x+1, k-1), \eta(x) = \zeta(x) = 0) .\end{aligned}$$

Therefore by (4.5), there is a constant L so that

$$\nu(E(x, k)) \leqq \nu(E(x, 1)) + L\nu(E(x+1, k-1)) ,$$

and the induction step is complete.

LEMMA 4.7. *Assume that* (4.5) *holds. Define* f_n *on* $X \times X$ *by*

$$f_n(\eta, \zeta) = \text{number of sign changes of} \quad \eta(x) - \zeta(x) \quad \text{in} \quad \{0, 1, \cdots, n\} .$$

If $\nu \in \bar{\mathscr{I}}$, *then* $\lim_{N\to\infty} 1/N \sum_{n=1}^{N} \int (f_{n+1} - f_n) \, d\nu = 0$.

PROOF. Fix $k \geqq 1$ and note that

$$\begin{aligned}\int (f_{n+1} - f_n) \, d\nu \leqq \nu\{\eta(n+1) \neq \zeta(n+1), \eta(l) = \zeta(l) \\ \text{for all } l, n-k \leqq l \leqq n\} \\ + \nu\{\eta(n+1) = \zeta(l) \neq \eta(l) = \zeta(n+1) \\ \text{for some } l, n-k \leqq l \leqq n\} .\end{aligned}$$

By Lemma 4.4,

$$\lim_{n\to\infty} \nu\{\eta(n+1) = \zeta(l) \neq \eta(l) = \zeta(n+1) \text{ for some } l, n-k \leqq l \leqq n\} = 0 .$$

For all m, on the other hand,

$$\sum_{n=m}^{m+k+1} \nu\{\eta(n+1) \neq \zeta(n+1), \eta(l) = \zeta(l) \text{ for all } l, n-k \leqq l \leqq n\} \leqq 1,$$

since the above sets are disjoint. Therefore

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int (f_{n+1} - f_n)\, d\nu \leqq \frac{1}{k+2},$$

and the result follows since k is arbitrary.

LEMMA 4.8. *Assume that* (4.5) *holds. If* $\nu \in \mathscr{\bar{I}}$, *then*

$$\sum_x \nu\{\eta(x) = \zeta(x+1) \neq \eta(x+1) = \zeta(x), f_x > f_{x-1}\} = 0.$$

PROOF. $f_n \in \mathscr{D}(\bar{\Omega})$, so $\int \bar{\Omega} f_n\, d\nu = 0$ for $\nu \in \mathscr{\bar{I}}$. Writing out this expression, and neglecting several nonnegative terms, we obtain the following inequality:

$$\begin{aligned} 2 \sum_{x=0}^{n-1} &[p(x, x+1) + p(x+1, x)] \\ &\times \nu\{\eta(x) = \zeta(x+1) \neq \eta(x+1) = \zeta(x), f_x > f_{x-1}\} \\ &\qquad \leqq p(n, n+1)\nu\{\eta(n) = \zeta(n) = 1, f_{n+1} > f_n\} \\ &\qquad\quad + p(n+1, n)\nu\{\eta(n) = \zeta(n) = 0, f_{n+1} > f_n\} \\ &\qquad \leqq 2 \int (f_{n+1} - f_n)\, d\nu. \end{aligned}$$

The Cesàro limit on n of the right hand side is zero by Lemma 4.7, so the result follows from (4.5).

LEMMA 4.9. *For* $\nu \in \mathscr{\bar{I}}$, $\nu\{(\eta, \zeta) : \eta \leqq \zeta$ *or* $\eta \geqq \zeta\} = 1$.

PROOF. By Lemma 4.1, we may assume that (4.5) holds. For $0 < u < v$, define

$$\begin{aligned} D_{u,v} = \{(\eta, \zeta) : \eta(u) = \zeta(v) \neq \zeta(u) = \eta(v), f_u > f_{u-1}, \eta(x) = \zeta(x) \\ \text{for } u < x < v\}. \end{aligned}$$

Since $\nu \in \mathscr{\bar{I}}$ and $1_{D_{u,v}} \in \mathscr{D}(\bar{\Omega})$, $\int \bar{\Omega} 1_{D_{u,v}}\, d\nu = 0$. Writing this out, we see that $\nu(D_{u,v}) = 0$ implies that $\nu(D_{u-1,v}) = 0$. But $\nu(D_{v-1,v}) = 0$ for $v > 1$ by Lemma 4.8, and therefore $\nu(D_{u,v}) = 0$ for all $0 < u < v$. Using a similar argument together with an induction on k, it then follows that for $\nu \in \mathscr{\bar{I}}$ and $0 < u < v$,

$$\nu\{(\eta, \zeta) : \eta(u) = \zeta(v) \neq \zeta(u) = \eta(v), f_u > f_{u-1}, \sum_{x=u}^{v-1} |\eta(x) - \zeta(x)| = k\} = 0.$$

Therefore $\nu\{(\eta, \zeta) : f_n(\eta, \zeta) \leq 1 \text{ for all } n\} = 1$. But then since $\{(\eta, \zeta) : f_n(\eta, \zeta) = 0$ for all $n\}$ is closed for the process and can be reached with positive probability from all (η, ζ) for which $\lim_{n\to\infty} f_n(\eta, \zeta) = 1$, it follows that $\nu\{(\eta, \zeta) : f_n(\eta, \zeta) = 0$ for all $n\} = 1$, which is the desired result.

COROLLARY 4.10. *If* $\mu_1, \mu_2 \in \mathscr{I}_e$, *then either* $\mu_1 \leqq \mu_2$ *or* $\mu_2 \leqq \mu_1$.

PROOF OF THEOREM 1.3. As mentioned in the introduction, part (b) of this theorem follows from the results and techniques of [11]. For part (c), note that $\nu^{(n)} \in \mathscr{I}_e$ for each n since it is the unique invariant measure which concentrates on A_n. By the convergence theorem for positive recurrent Markov chains, $\mu S(t) \to \nu^{(n)}$ for any probability measure μ which concentrates on A_n. From this, it is easily seen that $\nu^{(n)} \leqq \nu^{(n+1)}$ for all n. Now take $\mu \in \mathscr{I}_e$. By Corollary 4.10, either $\mu \leqq \nu^{(n)}$ for all finite n, or $\mu \geqq \nu^{(n)}$ for all finite n, or there exists an n for which $\nu^{(n)} \leqq \mu \leqq \nu^{(n+1)}$. In the latter case, μ concentrates on $A_n \cup A_{n+1}$, and is therefore equal to $\nu^{(n)}$ or $\nu^{(n+1)}$. If $\mu \leqq \nu^{(n)}$ for all finite n, then $\mu \leqq \nu^{(\rho)}$ for all $\rho \in (0, \infty)$, since $\nu^{(\rho)}$ is a convex combination of $\{\nu^{(n)}, -\infty < n < \infty\}$. Therefore for $x \in S$,

$$\mu\{\eta : \eta(x) = 1\} \leqq \inf_{\rho>0} \nu^{(\rho)}\{\eta : \eta(x) = 1\} = \inf_{\rho>0} \frac{\rho\pi(x)}{1 + \rho\pi(x)} = 0,$$

and hence $\mu = \nu^{(-\infty)}$. Similarly, if $\mu \geqq \nu^{(n)}$ for all finite n, $\mu = \nu^{(+\infty)}$, which completes the proof of the theorem in this case. Now consider part (a). Using the argument from the proof of Theorem 1.2 at the end of Section 3, Corollary 4.10 yields the following: if $\mu \in \mathscr{I}_e$ and $0 < \rho < \infty$, there exist $\lambda \in [0, 1]$ and $\mu_1, \mu_2 \in \mathscr{I}$ so that $\mu_1 \leqq \mu \leqq \mu_2$ and

$$\nu^{(\rho)} = \lambda\mu_1 + (1 - \lambda)\mu_2 . \tag{4.11}$$

Since $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$, there exists a sequence $x_n \to \infty$ so that one of the following happens:

(i) $\pi(x_n) \to c$ for some $c \in (0, \infty)$,
(ii) $\pi(x_n) \to 0$ and $\sum_n \pi(x_n) = \infty$, or
(iii) $\pi(x_n) \to \infty$ and $\sum_n 1/\pi(x_n) = \infty$.

Since $\nu^{(\rho)}$ is a product measure, the strong law of large numbers for independent, but not identically distributed random variables gives the following in each of these three cases:

$$\text{(i)} \quad \nu^{(\rho)}\left\{\eta : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \eta(x_n) = \frac{\rho c}{1 + \rho c}\right\} = 1,$$

$$\text{(ii)} \quad \nu^{(\rho)}\left\{\eta : \lim_{N\to\infty} \frac{\sum_{n=1}^{N} \eta(x_n)}{\sum_{n=1}^{N} \pi(x_n)} = \rho\right\} = 1, \text{ or}$$

$$\text{(iii)} \quad \nu^{(\rho)}\left\{\eta : \lim_{N\to\infty} \frac{\sum_{n=1}^{N} [1 - \eta(x_n)]}{\sum_{n=1}^{N} 1/\pi(x_n)} = \frac{1}{\rho}\right\} = 1.$$

If $0 < \lambda < 1$, then μ_1 and μ_2 are absolutely continuous with respect to $\nu^{(\rho)}$, and therefore whichever of the above three statements holds for $\nu^{(\rho)}$, holds for μ_1 and μ_2 also. Since $\mu_1 \leqq \mu \leqq \mu_2$, that statement will hold also for μ. Therefore $0 < \lambda < 1$ in (4.11) for at most one ρ, so there is a $\rho_0 \in [0, \infty]$ so that $\mu \leqq \nu^{(\rho)}$ for $\rho > \rho_0$ and $\mu \geqq \nu^{(\rho)}$ for $\rho < \rho_0$. It then follows by continuity that $\mu = \nu^{(\rho_0)}$. Thus $\mathscr{I}_e \subset \{\nu^{(\rho)}, 0 \leqq \rho \leqq \infty\}$. The reverse inclusion follows from the fact that the measures $\nu^{(\rho)}$ are all mutually singular with respect to each other, as can be seen from statement (i), (ii) or (iii) above.

5. The asymmetric simple random walk. Throughout this section, we will take $S = Z^1$, $p(x, x + 1) = p$, and $p(x, x - 1) = q$ for all $x \in Z^1$, where $p + q = 1$

and $p \neq \frac{1}{2}$. By symmetry, it suffices to consider the case $p > \frac{1}{2}$. Since both ν_ρ and $\nu^{(n)}$ are invariant in this case, it is easy to see that the direct analog of Corollaries 3.9 and 4.10 is not true. The correct result is contained in the following lemma. Let $f_n(\eta, \zeta)$ be the number of sign changes of $\eta(x) - \zeta(x)$ for $-n \leqq x \leqq n$.

LEMMA 5.1. *If* $\nu \in \bar{\mathscr{I}}$, *then*

$$\nu\{(\eta, \zeta) : f_n(\eta, \zeta) \leqq 1 \text{ for all } n \geqq 1\} = 1 .$$

PROOF. The argument parallels that in Lemmas 4.4, 4.7, 4.8 and 4.9, with the exception of the final sentence in the proof of Lemma 4.9. Therefore no details will be given.

REMARK. It is in this lemma that the requirement that $p(x, y)$ permit transitions only to nearest neighbors enters in a critical way, since it is based on the fact that an individual transition can only decrease the total number of sign changes of $\eta(x) - \zeta(x)$. If one wants to generalize the results of this section to the case of a translation invariant transition function on the integers with nonzero mean, one would probably have to show that

$$\nu\{(\eta, \zeta) : \sup_n f_n(\eta, \zeta) < \infty\} = 1$$

for $\nu \in \bar{\mathscr{I}}$, which is presumably the correct generalization of Lemma 5.1 to this case.

COROLLARY 5.2. *If* $\nu \in \bar{\mathscr{I}}_e$, *then exactly one of the following holds*:

(a) $\nu\{(\eta, \zeta) : \eta = \zeta\} = 1$,
(b) $\nu\{(\eta, \zeta) : \eta \leqq \zeta, \eta \neq \zeta\} = 1$,
(c) $\nu\{(\eta, \zeta) : \eta \geqq \zeta, \eta \neq \zeta\} = 1$,
(d) $\nu(B) = 1$,
(e) $\nu\{(\eta, \zeta) : (\zeta, \eta) \in B\} = 1$,

where $B = \{(\eta, \zeta) : \exists x \in Z^1$ *such that* $\eta(y) \leqq \zeta(y)$ *for all* $y < x$, $\eta(y) < \zeta(y)$ *for infinitely many* $y < x$, $\eta(z) \geqq \zeta(z)$ *for all* $z \geqq x$, *and* $\eta(z) > \zeta(z)$ *for infinitely many* $z \geqq x\}$.

PROOF. Each of the five sets above is closed for the motion, so that each has probability zero or one by the argument in Lemma 2.2. If in the definition of B above, we had replaced the words "infinitely many" by "some", Lemma 5.1 would say that any $\nu \in \bar{\mathscr{I}}$ must concentrate on the union of the five sets above. To complete the proof, simply use again the fact that $\nu \in \bar{\mathscr{I}}$, noting that if the number of y for which $\eta(y) > \zeta(y)$ is positive and finite, then it must decrease as t increases unless $\eta \geqq \zeta$.

COROLLARY 5.3. *Suppose* $\nu \in \bar{\mathscr{I}}_e$ *has marginals* μ_1 *and* μ_2 *respectively, and that for some* $L < \infty$,

$$|\textstyle\sum_{y=m}^{n} [\mu_1\{\eta(y) = 1\} - \mu_2\{\eta(y) = 1\}]| \leqq L \tag{5.4}$$

for all $m \leqq n$. *Then only* (a), (b) *and* (c) *can occur in Corollary* 5.2.

PROOF. Suppose that (d) in Corollary 5.2 holds. Rewriting (5.4) gives

$$(5.5)\qquad |\textstyle\sum_{y=m}^{n} [\nu\{\eta(y) = 0, \zeta(y) = 1\} - \nu\{\eta(y) = 1, \zeta(y) = 0\}]| \leqq L .$$

Since (d) holds,

$$(5.6)\qquad \lim_{y\to-\infty} \nu\{\eta(y) = 1, \zeta(y) = 0\} = \lim_{y\to+\infty} \nu\{\eta(y) = 0, \zeta(y) = 1\} = 0 .$$

Therefore by (5.5),

$$(5.7)\qquad \lim_{n\to\infty} \frac{1}{n} \textstyle\sum_{y=-n}^{0} \nu\{\eta(y) = 0, \zeta(y) = 1\}$$
$$= \lim_{n\to\infty} \frac{1}{n} \textstyle\sum_{y=0}^{n} \nu\{\eta(y) = 1, \zeta(y) = 0\} .$$

Now use Lemma 2.4 with $R_n = \{-n, \cdots, n\}$, and note that (5.6) and (5.7) imply that the Cesàro limit on n of the right side of the expression in that lemma is zero. Thus

$$\textstyle\sum_{x,y} [p(x, y) + p(y, x)]\nu\{\eta(x) = \zeta(y) = 1, \eta(y) = \zeta(x) = 0\} = 0 ,$$

and $\nu\{(\eta, \zeta): \eta \leqq \zeta \text{ or } \eta \geqq \zeta\} = 1$ by Lemma 2.5, which is a contradiction. An analogous argument shows that (e) cannot occur either if (5.4) is satisfied.

PROOF OF THEOREM 1.4. Take $\mu \in \mathscr{I}_e$, and define μ_1 and μ_2 by $\mu_1 = \mu$ and μ_2 is the following translate of μ:

$$\mu_2\{\eta: \eta(x_i) = 1 \text{ for } 1 \leqq i \leqq n\} = \mu\{\eta: \eta(x_i + 1) \text{ for } 1 \leqq i \leqq n\} .$$

Then both μ_1 and $\mu_2 \in \mathscr{I}_e$, so by Lemma 2.3, there is a $\nu \in \bar{\mathscr{I}}_e$ which has marginals μ_1 and μ_2 respectively. Since

$$\begin{aligned}\textstyle\sum_{y=m}^{n} &[\mu_1\{\eta(y) = 1\} - \mu_2\{\eta(y) = 1\}] \\ &= \textstyle\sum_{y=m}^{n} [\mu\{\eta(y) = 1\} - \mu\{\eta(y + 1) = 1\}] \\ &= \mu\{\eta(m) = 1\} - \mu\{\eta(n + 1) = 1\} ,\end{aligned}$$

(5.4) is satisfied with $L = 1$. Therefore by Corollary 5.3, ν satisfies one of (a), (b) or (c) of Corollary 5.2. If ν satisfies (a), then $\mu_1 = \mu_2$, so that μ is translation invariant, and therefore $\mu = \nu_\rho$ for some $\rho \in [0, 1]$ by Theorem 1.1. If ν satisfies (b), then $\mu_1 \leqq \mu_2$, so $\mu\{\eta(x) = 1\}$ is increasing in x, and

$$\begin{aligned}1 \leqq \textstyle\int \sum_x [\zeta(x) - \eta(x)]\, d\nu &= \textstyle\sum_x [\mu_2\{\eta(x) = 1\} - \mu_1\{\eta(x) = 1\}] \\ &= \lim_{x\to\infty} \mu\{\eta(x) = 1\} - \lim_{x\to-\infty} \mu\{\eta(x) = 1\} .\end{aligned}$$

Therefore

$$(5.8)\qquad \lim_{x\to\infty} \mu\{\eta(x) = 1\} = 1 \quad\text{and}\quad \lim_{x\to-\infty} \mu\{\eta(x) = 1\} = 0 .$$

If ν satisfies (c) instead of (b), a similar argument shows that

$$(5.9)\qquad \lim_{x\to\infty} \mu\{\eta(x) = 1\} = 0 \quad\text{and}\quad \lim_{x\to-\infty} \mu(\eta(x) = 1\} = 1 .$$

From $\int \Omega f\, d\mu = 0$ for $f(\eta) = \sum_{x=m}^{n} \eta(x)$, it follows that

$$\alpha = p\mu\{\eta(x) = 1, \eta(x + 1) = 0\} - q\mu\{\eta(x + 1) = 1, \eta(x) = 0\}$$

is independent of x for $\mu \in \mathscr{I}$. If ν satisfies either (b) or (c), (5.8) and (5.9) imply that $\alpha = 0$, and therefore that

$$p\mu\{\eta(x) = 1, \eta(x+1) = 0\} = q\mu\{\eta(x+1) = 1, \eta(x) = 0\}$$

for all $x \in Z^1$. Rewriting this as

$$(p-q)\mu\{\eta(x) = 1, \eta(x+1) = 0\} = q[\mu\{\eta(x+1) = 1\} - \mu\{\eta(x) = 1\}],$$

it follows that (5.9) is impossible and

$$\sum_x \mu\{\eta(x) = 1, \eta(x+1) = 0\} = \frac{q}{p-q} < \infty,$$

since $p > q$. Therefore

$$\mu\{\eta: \lim_{x\to\infty} \eta(x) \text{ and } \lim_{x\to-\infty} \eta(x) \text{ both exist}\} = 1.$$

Using (5.8) again, it follows that

$$\mu\{\eta: \sum_{x<0} \eta(x) < \infty, \sum_{x>0} [1 - \eta(x)] < \infty\} = 1,$$

and therefore that $\mu = \nu^{(n)}$ for some $n \in Z^1$. Thus we have shown that

$$\mathscr{I}_e \subset \{\nu_\rho, 0 \leqq \rho \leqq 1\} \cup \{\nu^{(n)}, -\infty < n < \infty\}.$$

The reverse inclusion follows from the fact that all the measures on the right side are mutually singular with respect to each other.

REFERENCES

[1] Dobrushin, R. L. (1971). Markov processes with a large number of locally interacting components: existence of a limit process and its ergodicity. *Problems of Information Transmission* **7** 149–164.

[2] Griffeath, D. (1975). Ergodic theorems for graph interactions. *Advances in Appl. Probability* **7** 179–194.

[3] Harris, T. E. (1974). Contact interactions on a lattice. *Ann. Probability* **2** 969–988.

[4] Holley, R. (1970). A class of interactions in an infinite particle system. *Advances in Math.* **5** 291–309.

[5] Holley, R. (1970). Pressure and Helmholtz free energy in a dynamic model of a lattice gas. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **3** 565–578, Univ. of California Press.

[6] Holley, R. (1971). Free energy in a Markovian model of a lattice spin system. *Comm. Math. Phys.* **23** 87–99.

[7] Holley, R. (1972). An ergodic theorem for attractive interactions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **24** 325–334.

[8] Liggett, T. M. (1972). Existence theorems for infinite particle systems. *Trans. Amer. Math. Soc.* **165** 471–481.

[9] Liggett, T. M. (1973). A characterization of the invariant measures for an infinite particle system with interactions. *Trans. Amer. Math. Soc.* **179** 433–453.

[10] Liggett, T. M. (1974). A characterization of the invariant measures for an infinite particle system with interactions, II. *Trans. Amer. Math. Soc.* **198** 201–213.

[11] Liggett, T. M. (1974). Convergence to total occupancy in an infinite particle system with interactions. *Ann. Probability* **2** 989–998.

[12] Liggett, T. M. (1975). Ergodic theorems for the asymmetric simple exclusion process. *Trans. Amer. Math. Soc.* **213** 237–261.

[13] MEYER, P. (1966). *Probability and Potentials.* Blaisdell, Waltham, Mass.
[14] SPITZER, F. (1970). Interaction of Markov processes. *Advances in Math.* **5** 246–290.
[15] SPITZER, F. (1974). Recurrent random walk of an infinite particle system. *Trans. Amer. Math. Soc.* **198** 191–199.
[16] VASERSHTEIN, L. N. (1969). Markov processes over denumerable products of spaces, describing large systems of automata. *Problems of Information Transmission* **5** 47–52.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024