The Annals of Probability
1976, Vol. 4, No. 1. 89–90

SIMPLE PROOF OF A RESULT ON THINNED POINT PROCESSES

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A simple proof of a result that a point process subject to independent thinning converges to a Poisson process is given.

Consider a point process $N(\cdot)$ on the real line R. It undergoes independent thinning; a point is deleted with probability p, $0 < p < 1$, and retained with probability $q = 1 - p$, independently for each point of $N(\cdot)$. Define $N_q(\cdot)$ to be the thinned process after a scale contraction by a factor q, that is $N_q(A)$ corresponds to a thinning of $N(q^{-1}A)$, where $cA = \{cx : x \in A\}$ and $A \in \mathscr{B}$, the σ-field of Borel sets of R.

A natural question is whether $N_q(\cdot)$ converges weakly as $q \to 0$, weak convergence meaning here convergence of all finite-dimensional distributions. We give a simple proof of the following result.

THEOREM. *$N_q(\cdot)$ converges weakly to a stationary Poisson process with intensity λ, $0 < \lambda < \infty$, if and only if $N(\cdot)$ satisfies*

$$qN(q^{-1}I) \to_p \lambda|I| \,, \qquad q \to 0 \,, \tag{1}$$

for all intervals I with one end point at zero. Here, $|\cdot|$ denotes Lebesgue measure.

The result is not essentially new. Belyaev [1] showed that (1), with an additional uniformity condition, was sufficient. Recently, Kallenberg [4] has given a comprehensive result on independent thinning with convergence to a doubly stochastic Poisson process; our theorem is partly a special case of his in Belyaev's framework, with a quite elementary proof.

PROOF. Define $\phi_q(s; A) = E[\exp\{-sN_q(A)\}]$, $\psi_q(s; A) = E[\exp\{-sqN(q^{-1}A)\}]$, $s \geqq 0$. The thinning mechanism implies that

$$\phi_q(s; A) = E(\exp[N(q^{-1}A) \log \{1 - q(1 - e^{-s})\}]) \,. \tag{2}$$

But for any ε in $(0, \frac{1}{2})$,

$$q(1 - e^{-s}) < -\log \{1 - q(1 - e^{-s})\} < q(1 - e^{-s})(1 + \varepsilon) \tag{3}$$

if $q < \varepsilon$, using standard inequalities. Since $q \to 0$ we may choose ε arbitrarily small and deduce from (2), (3) that $\phi_q(s; A)$ and $\psi_q(1 - e^{-s}; A)$ have the same limit, if any, as $q \to 0$.

To show necessity, note that $\psi_q(s'; I)$ converges to $e^{-\lambda s'|I|}$ for all s' in $[0, 1)$.

Received May 7, 1974; revised May 20, 1975.
AMS 1970 *subject classifications.* 60F05, 60K99.
Key words and phrases. Thinning of point processes.

Now apply the standard continuity theorem for Laplace transforms (Feller [2], page 431), remembering that such functions are analytic and hence determined, in their domain of definition, by values in an interval. For sufficiency, (1) implies that $qN(q^{-1}A) \to_p \lambda|A|$ for all $A \in \mathscr{R}$, the ring generated by the intervals I, so for such A,

$$\phi_q(s; A) \to \exp\{-\lambda|A|(1 - e^{-s})\} \qquad q \to 0 . \tag{4}$$

But, from a result of Rényi [5], the Poisson process is uniquely determined by having a Poisson distribution of counts for all $A \in \mathscr{R}$, and (4) implies all convergent subsequences of finite-dimensional distributions possess this property. Thus they all have the same limit, hence the weak convergence of $N_q(\cdot)$.

It is clear from the proof that the result holds for point processes in more general spaces provided (1) is taken for a class of sets generating the Borel algebra in the space (see, for example, Kallenberg [3]). Also, if λ is a random variable then (1) is still sufficient for convergence to a mixed Poisson process but it is necessary only that (1) holds in the sense of convergence in distribution. The latter will not be sufficient in this case unless we insist directly that (1) is true for all sets in $\mathscr{R}$; see [4].

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