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LIMIT LAWS FOR MAXIMA AND SECOND MAXIMA FROM STRONG-MIXING PROCESSES

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In 1972, R. E. Welsch obtained a class of two-dimensional laws which includes the class of all possible limit laws for a sequence of normalized pairs (M_n, S_n), where M_n and S_n are the maximum and the second maximum of the first n terms from a stationary strong-mixing sequence of random variables. In this note an example is given to show that these two classes are identical.

1. Introduction. Let $\{X_n\}_{n\geqq 1}$ be a strictly stationary strong-mixing sequence of random variables, with M_n and S_n being the maximum and the second maximum of $X_1, \cdots, X_n$ respectively. Loynes (1965) proved that the only possible nondegenerate limit laws G for suitably normalized M_n are the same three types that occur in the independent case, i.e., G is one of the following three types (except for scale and location parameters)

$$(1)\qquad \begin{aligned} G_1(x) &= 0 && x \leqq 0 \\ &= \exp(-x^{-\alpha}) && x > 0\,, \quad \alpha > 0 \\ G_2(x) &= \exp(-(-x)^{-\alpha}) && x < 0\,, \quad \alpha > 0 \\ &= 1 && x \geqq 0 \\ G_3(x) &= \exp(-e^{-x}) && -\infty < x < \infty\,. \end{aligned}$$

Welsch (1972) has extended the work of Loynes by considering the possible limit laws of order statistics of fixed rank other than the maximum. In this direction he has obtained the following result.

THEOREM (Welsch). *Let $\{X_n\}_{n\geqq 1}$ be a stationary strong-mixing sequence. If there exist two sequences of constants $a_n > 0$ and b_n so that $P\{M_n \leqq a_n x + b_n, S_n \leqq a_n y + b_n\}$ has a limit distribution $H(x, y)$, with the limiting distribution $G(x)$ of $P\{M_n \leqq a_n x + b_n\}$ nondegenerate, then*

$$(2)\qquad \begin{aligned} H(x, y) &= G(y)\{1 - \rho[(\log G(x))/\log G(y)] \log G(y)\} && y < x \\ &= G(x) && y \geqq x \end{aligned}$$

where $\rho(s)$, $0 \leqq s \leqq 1$, is a concave nonincreasing function which satisfies $0 \leqq \rho(s) \leqq 1 - s$ and G is one of the three types given in (1).

The purpose of this note is to give two examples. The first example shows that every H satisfying the conditions in Welsch's theorem is a limiting joint distribution of normalized M_n and S_n from a suitable strong-mixing sequence.

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This answers a question posed in Welsch (1972). The second example shows that the joint distribution of $(M_n - b_n)/a_n$ and $(S_n - b_n)/a_n$ does not necessarily converge even if $(M_n - b_n)/a_n$ has a nondegenerate limiting distribution. Both of these examples are one-dependent and constructed by a method similar to that of Newell (1964).

2. Examples. At first we note that any function satisfying the conditions in Welsch's theorem is represented as

$$\rho(s) = \int_s^1 [1 - F(-\log u)]\, du\,, \qquad 0 \leqq s \leqq 1\,, \tag{3}$$

where F is a possibly defective distribution function such that $F(-0) = 0$. In fact it follows from the concavity of ρ and $\rho(1) = 0$, that ρ is represented as

$$\rho(s) = \int_s^1 h(u)\, du\,, \qquad 0 \leqq s \leqq 1\,,$$

with a nondecreasing h. One may choose as $-h$ the right (or left) Dini derivative of ρ. Since ρ is nonincreasing and $0 \leqq \rho(s) \leqq 1 - s$ one has $0 \leqq h(s) \leqq 1$ for $s \in (0, 1)$. Obviously one may assume that h is left continuous. Let $F(x) = 1 - h(e^{-x})$ for $x \geqq 0$ and $F(x) = 0$ for $x < 0$. Then F is a possibly defective distribution function and satisfies (3).

EXAMPLE 1. Let $\{Z_n\}_{n \geqq 0}$ be i.i.d. random variables having a common exponential distribution with mean one. Let ρ be a function satisfying the conditions in Welsch's theorem and let F be a possibly defective distribution function determined by (3). Let $\{\zeta_n\}_{n \geqq 1}$ be i.i.d. nonnegative random variables independent of $\{Z_n\}$ and having common distribution function F. Note that $\zeta_n = \infty$ with probability $1 - F(\infty)$.

Let

$$X_n = \max(Z_{n-1}, Z_n - \zeta_n)\,, \qquad n \geqq 1\,. \tag{4}$$

It is obvious that $\{X_n\}_{n \geqq 1}$ is a stationary one-dependent sequence. We shall prove that in this case the sequence $\{(M_n - \log n, S_n - \log n)\}_{n \geqq 1}$ converges in distribution to $H(x, y)$ given by (2). Since for each x

$$\begin{aligned} 0 &\leqq P\{\max_{0 \leqq j \leqq n-1} Z_j \leqq \log n + x\} - P\{M_n \leqq \log n + x\} \\ &\leqq P\{Z_n - \zeta_n > \log n + x\} \to 0 \end{aligned}$$

as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} P\{M_n \leqq \log n + x\} &= \lim_{n\to\infty} [P\{Z_j \leqq \log n + x\}]^n \\ &= \exp(-e^{-x}) = G_3(x)\,. \end{aligned} \tag{5}$$

Hence if $x \leqq y$ then

$$\begin{aligned} &\lim_{n\to\infty} P\{M_n \leqq \log n + x, S_n \leqq \log n + y\} \\ &\qquad = \lim_{n\to\infty} P\{M_n \leqq \log n + x\} = G_3(x)\,. \end{aligned} \tag{6}$$

If $x > y$ then we can write

$$P\{M_n \leqq \log n + x, S_n \leqq \log n + y\} = P\{M_n \leqq \log n + y\} + P\{A_n\}\,, \tag{7}$$

where

$$A_n = \{\log n + y < M_n \leqq \log n + x,\ S_n \leqq \log n + y\} .$$

Let

$$A_n' = \bigcup_{k=1}^{n-1} \{\log n + y < Z_k \leqq \log n + x,\ Z_k - \zeta_k \leqq \log n + y, \text{ and } Z_j \leqq \log n + y \text{ for } j \neq k, 1 \leqq j \leqq n-1\}$$

and

$$B_n = \{Z_0 \leqq \log n + y,\ Z_n - \zeta_n \leqq \log n + y\} .$$

Then we have $A_n \cap B_n = A_n' \cap B_n$ and $\lim_{n\to\infty} P\{B_n\} = 1$. Therefore

$$\lim_{n\to\infty} (P\{A_n\} - P\{A_n'\}) = 0 . \tag{8}$$

On the other hand we have

$$\begin{aligned} P\{A_n'\} &= (n-1)[P\{Z_1 \leqq \log n + y\}]^{n-2} \\ &\quad \times P\{\log n + y < Z_1 \leqq \log n + \min(x, y + \zeta_1)\} \\ &= (n-1)(1 - n^{-1}e^{-y})^{n-2} n^{-1} e^{-y} \\ &\quad \times [\textstyle\int_0^\infty [1 - \exp\{-\min(x-y, u)\}]F(du) \\ &\quad + (1 - e^{-x+y})(1 - F(\infty))] . \end{aligned} \tag{9}$$

Since integration by parts shows that

$$\int_0^\infty [1 - \exp\{-\min(x-y, u)\}]F(du) = \int_{\exp(-x+y)}^1 [F(\infty) - F(-\log u)]\, du ,$$

from (8), (9) and (3) we have

$$\begin{aligned} \lim_{n\to\infty} P\{A_n\} &= \exp(-e^{-y})e^{-y} \int_{\exp(-x+y)}^1 [1 - F(-\log u)]\, du \\ &= \exp(-e^{-y})e^{-y}\rho(e^{-x+y}) . \end{aligned} \tag{10}$$

It follows from (5), (7) and (10) that if $x > y$ then

$$\begin{aligned} &\lim_{n\to\infty} P\{M_n \leqq \log n + x,\ S_n \leqq \log n + y\} \\ &\quad = \lim_{n\to\infty} P\{M_n \leqq \log n + y\} + \lim_{n\to\infty} P\{A_n\} \\ &\quad = G_3(y)[1 - \rho(\log G_3(x)/\log G_3(y)) \log G_3(y)] . \end{aligned} \tag{11}$$

The relations (6) and (11) show that the limiting law of $(M_n - \log n, S_n - \log n)$ is given by (2) with $G = G_3$.

In order to obtain examples with $G = G_1$ and $G = G_2$ one needs only to replace the variables X_n constructed above by $\exp(\alpha^{-1}X_n)$ and $-\exp(-\alpha^{-1}X_n)$ respectively.

EXAMPLE 2. Let $\{Z_n\}_{n\geqq 0}$ be as in Example 1. Define X_n by $X_n = \max(Z_{n-1}, Z_n - f(Z_n))$, $n \geqq 1$, where $f(x) = 1$ if $x \in \bigcup_{j=0}^\infty [2^{2j}, 2^{2j+1})$ and $f(x) = 0$ otherwise. Then the sequence $\{X_n\}_{n\geqq 1}$ is stationary one-dependent, and as in Example 1 we have

$$\lim_{n\to\infty} P\{M_n \leqq \log n + x\} = G_3(x) .$$

We shall now prove that the sequence of pairs $(M_n - \log n, S_n - \log n)$ does not converge in law. Let $\{n_j\}_{j\geqq 1}$ be a sequence of positive integers such that

$[\log n_j] = 3 \cdot 2^{2j-1}$. Let $X_n' = \max(Z_n, Z_n - 1)$, $n \geqq 1$, M_n' and S_n' be the maximum and the second maximum of $X_1', \cdots, X_n'$ respectively.

We can show that $M_{n_j} = M'_{n_j}$ and $S_{n_j} = S'_{n_j}$ on C_j, where C_j is the event specified by the following three conditions:

$$(12) \qquad \max(Z_1, \cdots, Z_{n_j-1}) - \log n_j \in [-2^{2j-1} + 1, 2^{2j-1} - 1)\,,$$
$$Z_0 \leqq \log n_j - 2^{2j-1} - 1\,, \qquad Z_{n_j} \leqq \log n_j - 2^{2j-1} - 1\,.$$

In fact assume (12) and let Z_k and Z_l be the maximum and the second maximum of $Z_1, \cdots, Z_{n_j-1}$ respectively. Then $M_{n_j} = \max(Z_0, \cdots, Z_{n_j-1}, Z_{n_j} - f(Z_{n_j})) = \max(Z_1, \cdots, Z_{n_j-1}) = Z_k\ (= X_{k+1})$. Similarly we have $M'_{n_j} = Z_k$. Furthermore

$$\begin{aligned} S_{n_j} &= \max(X_1, \cdots, X_k, X_{k+2}, \cdots, X_{n_j}) \\ &= \max(Z_0, \cdots, Z_{k-1}, Z_{k+1}, \cdots, Z_{n_j-1}, Z_1 - f(Z_1), \cdots, Z_k - f(Z_k), \\ &\qquad Z_{k+2} - f(Z_{k+2}), \cdots, Z_{n_j} - f(Z_{n_j})) \\ &= \max(Z_0, Z_l, Z_k - f(Z_k), Z_{n_j} - f(Z_{n_j}))\,. \end{aligned}$$

Since $Z_k \in [2^{2j} + 1, 2^{2j+1})$ we have $f(Z_k) = 1$ and $S_{n_j} \geqq Z_k - f(Z_k) \geqq 2^{2j} > \max(Z_0, Z_{n_j} - f(Z_{n_j}))$. Hence $S_{n_j} = \max(Z_l, Z_k - 1)$. Similarly we have $S'_{n_j} = \max(Z_l, Z_k - 1)$. This proves our assertion. Since $P\{C_j\}$ converges to one we have

$$(13) \qquad \lim_{j\to\infty} P\{M_{n_j} = M'_{n_j}, S_{n_j} = S'_{n_j}\} = 1\,.$$

The sequence $\{X_n'\}$ is the one constructed by the method of Example 1 with $\rho = \rho_0$, where $\rho_0(s) = 1 - e^{-1}$ for $s \in [0, e^{-1}]$ and $\rho_0(s) = 1 - s$ for $s \in [e^{-1}, 1]$. Hence as was shown in Example 1

$$(14) \qquad \lim_{n\to\infty} P\{M_n' \leqq \log n + x, S_n' \leqq \log n + y\} = H_0(x, y)\,,$$

where H_0 is given by (2) with $G = G_3$ and $\rho = \rho_0$. It follows from (13) and (14) that

$$(15) \qquad \lim_{j\to\infty} P\{M_{n_j} \leqq \log n_j + x, S_{n_j} \leqq \log n_j + y\} = H_0(x, y)\,.$$

Next choose a sequence $\{m_j\}$ such that $[\log m_j] = 3 \cdot 2^{2j}$. Applying an argument similar to the one above with $X_n' = \max(Z_{n-1}, Z_n)$, we can conclude that if $x > y$ then

$$(16) \qquad \lim_{j\to\infty} P\{M_{m_j} \leqq \log m_j + x, S_{m_j} \leqq \log m_j + y\} = G_3(y) \neq H_0(x, y)\,.$$

The relations (15) and (16) show that the sequence $\{(M_n - \log n, S_n - \log n)\}$ does not converge in law, while $\{M_n - \log n\}$ has a nondegenerate limit distribution.

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