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LEVEL CROSSINGS FOR RANDOM FIELDS[1]

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For an n-dimensional random field $X(\mathbf{t})$ we define the excursion set A of $X(\mathbf{t})$ by $A = \{\mathbf{t} \in I_0 : X(\mathbf{t}) \geqq u\}$, where I_0 is the unit cube in R^n. It is shown that the natural generalisation of the number of upcrossings of a one-dimensional stochastic process to random fields is via the characteristic of the set A introduced by Hadwiger (1959). This characteristic is related to the number of connected components of A. A formula is obtained for the mean value of this characteristic when $n = 2, 3$. This mean value is calculated explicitly when $X(\mathbf{t})$ is a homogeneous Gaussian field satisfying certain regularity conditions.

1. Introduction. The generalisation of level crossings of a one-dimensional stochastic process to an n-dimensional field $X(\mathbf{t})$, $\mathbf{t} \in R^n$, involves random point sets of the form $S = \{\mathbf{t} \in R^n : X(\mathbf{t}) = u\}$, which, for $n = 2$, form a family of contour lines in the plane. As is noted by Belyaev (1972a, 1972b) no technique has yet been developed to handle the distributional properties of these sets.

It has only been possible to obtain results for high levels u by considering the rather complex set of "A_α crossings" (Belyaev (1972a)) or the associated point process of local maxima. In both cases the results obtained are only partial. For example, for a homogeneous Gaussian field satisfying regularity conditions similar to those in Section 3, it has only proved possible to obtain an asymptotic formula for the mean number of maxima, or A_α upcrossings, above a level u for high levels u (Belyaev (1972a, 1972b), Nosko (1969), Hasofer (1976)). The results obtained in this paper are exact for all values of the level u, and can be used to explain how the known asymptotic formulae arise.

Our technique is based on considering the point set S indirectly, via the set $A = \{\mathbf{t} \in R^n : X(\mathbf{t}) \geqq u\}$ which it bounds. If X is a one-dimensional process satisfying the conditions appearing in Chapter 10 of Cramér and Leadbetter (1967) it is immediate that A is made up of the union of disjoint closed intervals, so that the number of upcrossings of X in the interval $[0, 1]$ is the number of closed intervals in the set $A \cap (0, 1]$. We shall generalise this approach to upcrossings to the multi-dimensional situation.

In Section 2 we shall describe some results from the theory of Integral Geometry which will be shown in Section 3 to provide the theoretical foundation for solving the upcrossing problem in R^n. Section 3 also shows how our technique

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reduces to considering a particular point process for the case $n = 2$, and Section 4 is devoted to calculating certain mean values in the Gaussian case. In Section 5 we sketch the results for $n = 3$, and Section 6 is devoted to some further results and conjectures.

2. Some integral geometry. We shall now summarise definitions and results relating to a particular type of point set in R^n, which we shall call a normal body. A full treatment is available in Hadwiger (1959). Let $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ be a set of n vectors which form a basis for R^n. Call an i-dimensional hyperplane $R^i (i = 0, 1, \cdots, n)$ of R^n **V**-*associated* if it is parallel to an i-dimensional subspace of R^n generated by i of the vectors in **V**. Furthermore, call a compact set of points $P \subset R^n$ a **V**-*parcel* if the intersections $P \cap R^*$ are connected or empty for every **V**-associated hyperplane R^* of R^n, including the case $R^* = R^n$ itself.

Define now a body (point set) $A \subset R^n$ to be a **V**-*package* if it can be represented as the union of **V**-parcels, such that the intersection of any subset of these parcels is again a **V**-parcel. The set of these parcels gives a **V**-*partition* of A, and their number m is called the **V**-*order* of the partition.

DEFINITION 2.1. A compact set $A \subset R^n$ will be called a *normal body* in R^n if for each arbitrary choice of basis **V** A is a **V**-package, and there exists a **V**-partition of A with finite order. Denote by $\mathscr{N}$ the set of all normal bodies.

Note that this definition is slightly broader than the corresponding one in Hadwiger (1959), but it is clear on checking through his paper that the properties of $\mathscr{N}$ we shall require continue to hold.

It is immediate that $\mathscr{N}$ is invariant under either a rotation of the axes, or change of origin. Furthermore, $\mathscr{N}$ is additive in the sense that $A \in \mathscr{N}$, $B \in \mathscr{N}$, $A \cap B = \emptyset$ (the empty set) implies $A \cup B \in \mathscr{N}$. Further properties of $\mathscr{N}$ can be found in Hadwiger (1959).

In Section 4 we shall show that under certain conditions on $X(\mathbf{t})$ the set $A = \{\mathbf{t} \in I_0 : X(\mathbf{t}) \geqq u\}$ belongs almost surely to $\mathscr{N}$. With this application in mind, it is now clear that we wish to find a functional on $\mathscr{N}$, φ say, which will have the following properties.

(a) For all **V**-parcels P

$$\varphi(P) = 0 \quad P = \emptyset , \qquad (2.1)$$
$$= 1 \quad P \neq \emptyset .$$

(b) When each of A, B, $A \cup B$, $A \cap B$ is in $\mathscr{N}$

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) . \qquad (2.2)$$

Such a functional does in fact exist, and, moreover, is *uniquely* determined by (2.1) and (2.2). It can be defined in at least two distinct ways, but we shall give only a definition via a recurrence relation which is more suited to our purposes.

Choose for $\mathbf{V}$ a set of n orthogonal vectors $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$, and consider the corresponding set of axes which give rise to coordinates $(t_1, \cdots, t_n)$. Let E_x denote the $(n-1)$-dimensional hyperplane orthogonal to $\mathbf{v}_n$, so that all points on E_x have their nth coordinate equal to x.

Noting that if $A \in \mathscr{N}$ and $A \subset R^1$, A is composed of the union of a finite number of disjoint closed (and possibly degenerate) intervals we have

DEFINITION 2.2. Define the *characteristic functional* $\varphi : \mathscr{N} \to \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, and $\mathscr{N}$ is the set of normal bodies in R^n, by

(a) $n = 1$

$$\varphi(A) = \text{number of disjoint closed intervals in } A, \tag{2.3}$$

(b) $n > 1$

$$\varphi(A) = \textstyle\sum_x \{\varphi(A \cap E_x) - \varphi(A \cap E_{x^-})\} \tag{2.4}$$

where

$$\varphi(A \cap E_{x^-}) = \lim_{y \to 0} \varphi(A \cap E_{x-y}), \quad y \geqq 0 \tag{2.5}$$

and the summation is over all real x for which the summand is nonzero.

Hadwiger (1959) has shown that the limit in (2.5) exists, and that the summation (2.4) is over a finite number of values of x. Furthermore the following is true.

LEMMA 2.1. *If $A \subset R^n$ is a normal body, then its characteristic $\varphi(A)$ is independent of the basis $\mathbf{V}$ used for its definition.*

Figure 1 shows an example of this concept in R^2. Note in particular the set with the hole "in the middle." It is on sets like this, and their analogues in higher dimensions, that the characteristic φ and the number of connected components of the set differ. In this example, they are respectively zero and one.

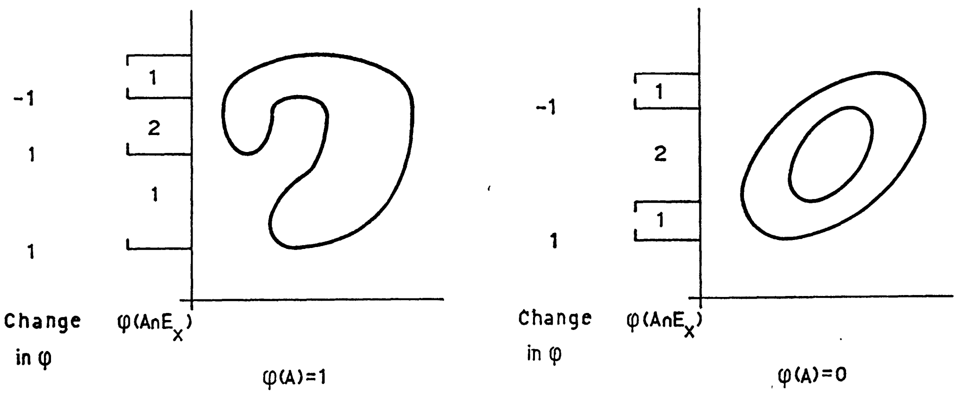

FIG. 1. The characteristic φ.

3. Level crossings in R^n. Let us now restrict our attention to the values of $X(\mathbf{t})$ in I_0, the unit cube in R^n, i.e. $I_0 = \{\mathbf{t} \in R^n, \mathbf{t} = (t_1, \cdots, t_n), 0 \leqq t_j \leqq 1,$

$j = 1, \cdots, n\}$. Furthermore, let us assume that the set $A = \{\mathbf{t} \in I_0 : X(\mathbf{t}) \geqq u\}$ is with probability one a normal body. Sufficient conditions for this will be given in Lemma 3.2. Then, when $n = 1$, the number of upcrossings in $[0, 1]$ is $\varphi(A)$ if $X(0) < u$ or $\varphi(A) - 1$ if $X(0) \geqq u$. In fact, by (2.3), the number of upcrossings equals $\varphi(A \cap [0, 1]) - \varphi(A \cap [0])$, where $[0]$ denotes the set whose only element is the origin. This way of counting upcrossings generalises to random fields.

For an n-dimensional random field $X(\mathbf{t})$ and arbitrary real u we shall call the set $A = \{\mathbf{t} \in I_0 : X(\mathbf{t}) \geqq u\}$ the *excursion set* of $X(t)$ above the level u. Then if $\hat{I}_0$ represents all those faces of I_0 which contain the origin, a natural generalisation of the number of upcrossings to random fields is the quantity $\chi(A)$, defined as follows.

DEFINITION 3.1. The *characteristic* of the excursion set A of a random field is defined to be the number

$$\chi(A) = \varphi(A \cap I_0) - \varphi(A \cap \hat{I}_0) .$$

It is not hard to see that $\chi(A)$ also has the following representation

$$\chi(A) = \textstyle\sum_x \{\chi(A \cap E_x) - \chi(A \cap E_{x^-})\} \tag{3.1}$$

where the summation is, as in (2.4), over all $x \in (0, 1]$ for which the summand is nonzero.

Let us now write X_j for the first order partial derivatives $\partial X/\partial t_j$ and X_{ij} for the second order derivatives $\partial^2 X/\partial t_i \, \partial t_j$, $i, j = 1, 2, \cdots, n$, of the field X.

For the remainder of this and the following section we shall restrict ourselves to the simple but representative case $n = 2$. As one would expect, the following analysis is only valid when the sample paths of the field X and its partial derivatives satisfy certain regularity conditions which, not surprisingly, are similar to those required by Belyaev (1972 b).

DEFINITION 3.2. We shall call a two-dimensional random field X *suitably regular* if its realisations satisfy, with probability one, the following conditions for arbitrary real u.

(3.2) The sample functions have continuous partial derivatives of up to second order, with finite variance, in I_0.

(3.3) The number of points $\mathbf{t} \in I_0$, where $X(\mathbf{t}) = u$ and either $X_1(\mathbf{t}) = 0$ or $X_2(\mathbf{t}) = 0$ is finite.

(3.4) There is no point $\mathbf{t}$ on the boundary of I_0 for which $X(\mathbf{t}) = u$ and either $X_1(\mathbf{t}) = 0$ or $X_2(\mathbf{t}) = 0$.

(3.5) There is no point $\mathbf{t} \in I_0$ for which $X(\mathbf{t}) = u$, $X_1(\mathbf{t}) = 0$ and $X_2(\mathbf{t}) = 0$.

(3.6) There is no point $\mathbf{t} \in I_0$ for which $X(\mathbf{t}) = u$, and either $X_1(\mathbf{t}) = X_{11}(\mathbf{t}) = 0$ or $X_2(\mathbf{t}) = X_{22}(\mathbf{t}) = 0$.

REMARK. It is not hard to find conditions under which a homogeneous random field is suitably regular. The proofs are straightforward, although messy, and examples of the type of argument involved can be found in Belyaev (1972b). Indeed if X is a homogeneous random field over I_0 satisfying (3.2), and if each of the joint probability densities of (X, X_1, X_2), (X, X_1, X_{11}) and (X, X_2, X_{22}) is bounded by a finite constant, then X is suitably regular. In the case when X is also Gaussian these last conditions are automatically fulfilled. Conditions on the covariance function of the field so that it satisfies (3.2) can be found in Garsia (1972).

We shall now show how the characteristic of an excursion set, as defined by (3.1), can be calculated by considering a particular point process, and from this prove that excursion sets are normal bodies with probability one.

Except where explicitly stated otherwise we shall, for the remainder of this section, assume that we are dealing with a particular realisation $X(\mathbf{t})$ of the random field that satisfies (3.2)—(3.6). Consider the summation in (3.1). For given x, E_x is a straight line parallel to the t_1 axis, and $A \cap E_x$ is composed of a sequence of n_x disjoint closed intervals not containing the point $(0, x)$, and one more such interval containing this point if $X(0, x) \geqq u$. Thus $\chi(A \cap E_x) = n_x$ and so values of x contributing to the sum clearly correspond to values of x where n_x changes. It is immediately clear from continuity considerations that contributions to $\chi(A)$ can only occur when E_x is tangential to the boundary of A (Type I contributions) or when $X(0, x) = u$ or $X(1, x) = u$, (Type II contributions). Consider the former first.

Since all the points on the boundary of A satisfy $X(t_1, t_2) = u$, t_2 can be defined locally for such points as an implicit function of t_1. The standard rules for differentiating implicit functions give us that $dt_2/dt_1 = -X_1/X_2$, so that applying what we have just noted about the tangency of E_x to the boundary of A we have that for each unit contribution of Type I to $\chi(A)$ there must be a point $\mathbf{t} \in I_0$ satisfying

$$X(\mathbf{t}) = u\,, \tag{3.7}$$

and

$$X_1(\mathbf{t}) = 0\,, \tag{3.8}$$

since there are no points with $X(\mathbf{t}) - u = X_1(\mathbf{t}) = X_2(\mathbf{t}) = 0$ by (3.5). Furthermore, since the limit in (3.1) (see (2.5)) is one-sided, continuity considerations imply that contributing points must also satisfy

$$X_2(\mathbf{t}) > 0\,. \tag{3.9}$$

Conversely, for each point satisfying (3.7)—(3.9) there is a unit contribution of Type I to $\chi(A)$. Note that there is no contribution of Type I to $\chi(A)$ from points on the boundary of I_0 when (3.4) holds. Thus we have set up a one-one correspondence between unit contributions of Type I to $\chi(A)$ and points in the interior of I_0 satisfying (3.7)—(3.9). It is easily seen that contributions of $+1$ will correspond to points for which $X_{11}(\mathbf{t}) < 0$ and contributions of -1 to points for which $X_{11}(\mathbf{t}) > 0$. By (3.6) there are no contributing points for which $X_{11}(\mathbf{t}) = 0$.

Consider now Type II contributions to $\chi(A)$. Using similar arguments it can be seen that we obtain a contribution of $+1$ for every point $\mathbf{t} = (1, x)$, $x \in (0, 1]$, where $X(\mathbf{t}) = u$, $X_1(\mathbf{t}) > 0$, $X_2(\mathbf{t}) > 0$, and a contribution of -1 for each point $\mathbf{t} = (0, x)$, $x \in (0, 1]$ satisfying the same conditions. Thus if we define

(3.10) $\chi_1^+(\chi_1^-)$ = number of points in I_0 satisfying (3.7)—(3.9) and $X_{11}(\mathbf{t}) < 0$ $(X_{11}(\mathbf{t}) > 0)$,

(3.11) $\chi_2^+(\chi_2^-)$ = number of values of x, $x \in (0, 1]$, for which if $\mathbf{t} = (1, x)$ $(\mathbf{t} = (0, x))$ $X(\mathbf{t}) = u$, $X_1(\mathbf{t}) > 0$, $X_2(\mathbf{t}) > 0$,

we have the following result.

LEMMA 3.1. *If X is a suitably regular two-dimensional random field on I_0 the characteristic of its excursion set is, with probability one, given by*

$$\chi(A) = \chi_1^+ + \chi_2^+ - (\chi_1^- + \chi_2^-) \,.$$

We now wish to establish that excursion sets are normal bodies, so that the sum (3.1) defining $\chi(A)$ has the properties discussed in Section 2.

A is clearly closed and bounded, and thus, by the Heine–Borel theorem, is compact. Thus we need only show that A has, for any basis of R^2, a partition into a finite number of parcels, whose intersections are again parcels. Consider the points $\mathbf{t} \in I_0$ where $X(\mathbf{t}) - u = X_1(\mathbf{t}) = 0$, or $X(\mathbf{t}) - u = X_2(\mathbf{t}) = 0$. For each of these points draw a line parallel to the t_2, or respectively, t_1 axis, containing the point. These lines form a grid over I_0, and, from the characterisation of the points contributing to $\chi(A)$, the connected regions of A within each cell of this grid are parcels. See Figure 2 for some examples.

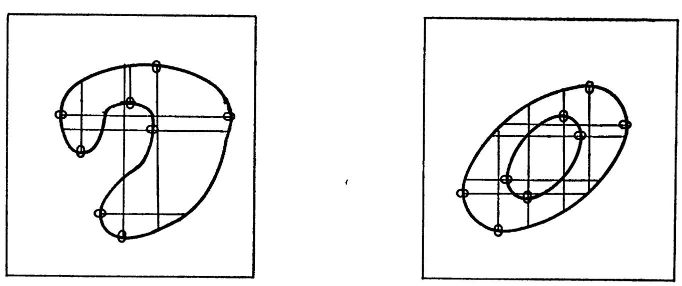

FIG. 2. Partitioning excursion sets. (Partitioning points are marked with 0 or o.)

By (3.3) there is only a finite number of cells, and so we have exhibited A as the union of a finite number of parcels, whose intersections are either closed intervals, points, or empty, and thus also parcels. A similar argument holds for any choice of basis, and so we have

LEMMA 3.2. *If X is a suitably regular two-dimensional random field on I_0, then its excursion set is a normal body with probability one.*

The measurability of $\chi(A)$ follows from its representation as the sum of integrals in the following section.

4. The mean value of $\chi(A)$.

THEOREM 4.1. *Let X be a zero mean, homogeneous, two dimensional Gaussian field on I_0. Assume $X(\mathbf{t})$ satisfies (3.2), has finite spectral moments of up to and including order 6, and has a spectrum with a continuous component. Then the mean value of the characteristic of its excursion set is given by*

$$E\{\chi(A)\} = -\int_{x_2>0}\int_{x_{11}=-\infty}^{\infty} x_2 x_{11}\varphi(u, 0, x_1, x_{11})\, dx_2\, dx_{11} \tag{4.1}$$

where $\varphi(x, x_1, x_2, x_{11})$ is the joint density function of (X, X_1, X_2, X_{11}).

PROOF. This is essentially a generalisation to higher dimensions of a counting technique in one dimension used by Kac (1943).

For a realisation $X(\mathbf{t})$ satisfying (3.2)—(3.6) consider the mapping $f: I_0 \to R^2$, given by $f(\mathbf{t}) = (X(\mathbf{t}) - u, X_1(\mathbf{t})) = \mathbf{X}$. Let K be the set of points in I_0 for which $X_2(\mathbf{t}) > 0$ and $X_{11}(\mathbf{t}) < 0$. Then K is clearly open, and if we denote by K^* the interior of the complement of K in I_0 we have under (3.5) and (3.6) that the points in I_0 with $X(\mathbf{t}) - u = X_1(\mathbf{t}) = 0$ lie in K or K^*.

Since from (3.3) these points are isolated, we can surround each one by an open sphere $\sigma(\eta)$ in such a way that the spheres neither overlap nor cross the boundaries of K or I_0.

Now let $\sigma(\varepsilon)$ denote the disc $|\mathbf{X}| < \varepsilon$ in the image space. We can show, using (3.4), that we can choose ε so small that the inverse image of $\sigma(\varepsilon)$ in I_0 is contained in the union of spheres $\sigma(\eta)$.

Furthermore, since $X_2(\mathbf{t}) > 0$ and $X_{11}(\mathbf{t}) < 0$ in K, we can choose ε, η so small that for each $\sigma(\eta)$ in K, $\sigma(\varepsilon) \subset f(\sigma(\eta))$ and so the restriction of f to each $\sigma(\eta)$ is one-one. Its Jacobian will be given by $|X_1X_{12} - X_2X_{11}| = |J|$ say.

Defining a function $\delta_\varepsilon(\mathbf{X})$ on R^2 which is constant over $\sigma(\varepsilon)$ and zero elsewhere, and for which $\int_{\sigma(\varepsilon)} \delta_\varepsilon(\mathbf{X})\, d\mathbf{X} = 1$, it follows that we can choose ε so small that

$$\chi_1^+ = \chi_1^+(\varepsilon),$$

where $\chi_1^+(\varepsilon)$ is given by

$$\chi_1^+(\varepsilon) = \int_{I_0} \delta_\varepsilon(\mathbf{X}(\mathbf{t}))I(K)|J|\, d\mathbf{t},$$

and $I(K)$ is the indicator function of the set K.

Since χ_1^+ does not depend on ε, we also have

$$\chi_1^+ = \lim_{\varepsilon\to 0} \chi_1^+(\varepsilon). \tag{4.2}$$

Now let $\varphi(x, x_1, x_2, x_{11}, x_{12})$ denote the joint density function of $(X, X_1, X_2, X_{11}, X_{12})$. Then, for fixed ε we have

$$E\{\chi_1^+(\varepsilon)\} = \int_{I_0} d\mathbf{t} \int \delta_\varepsilon(x - u, x_1)|J|\varphi(x, x_1, x_2, x_{11}, x_{12})\, dx\, dx_1\, dx_2\, dx_{11}\, dx_{12}$$

where the second integral is over $x_2 > 0$, $x_{11} < 0$, and all x, x_1, x_{12}. The integral interchange is justified by Fubini's theorem.

Because of homogeneity, the second integral does not depend on $\mathbf{t}$ and so the above expression equals

$$\int_{x_{11}} \int_{x_2} \int_{x_{12}} dx_{11}\, dx_2\, dx_{12} \int_x \int_{x_1} \delta_\varepsilon(x - u, x_1)|x_1 x_{12} - x_2 x_{11}|\varphi(x, x_1, x_2, x_{11}, x_{12})\, dx\, dx_1 \,.$$

The inner integral clearly converges to $-x_2 x_{11}\varphi(u, 0, x_2, x_{11}, x_{12})$ as $\varepsilon \to 0$. Furthermore, as we shall now show, this inner integral is bounded by an integrable function, so that applying dominated convergence and integrating out x_{12} we have

$$\lim_{\varepsilon\to 0} E\{\chi_1^+(\varepsilon)\} = -\int_{x_{11}<0} \int_{x_2>0} x_2 x_{11}\varphi(u, 0, x_2, x_{11})\, dx_2 dx_{11} \,.$$

To obtain the bounding function we note that the inner integral is not greater than

$$(\pi\varepsilon^2)^{-1}\varphi_2(x_2, x_{11}, x_{12}) \int_{x=u-\varepsilon}^{u+\varepsilon} \int_{x_1=-\varepsilon}^{\varepsilon} \{\varepsilon|x_{12}| + |x_2 x_{11}|\}\varphi_1(x, x_1 \,|\, x_2, x_{11}, x_{12})\, dx\, dx_1 \,,$$

where φ_1 and φ_2 are respectively the joint probability densities of $(X(\mathbf{t}), X_1(\mathbf{t}) \,|\, X_2(\mathbf{t}), X_{11}(\mathbf{t}), X_{12}(\mathbf{t}))$ and $(X_2(\mathbf{t}), X_{11}(\mathbf{t}), X_{12}(\mathbf{t}))$. Since under the conditions of the theorem the determinants of the covariance matrices of these densities are bounded away from zero it follows that if $\varepsilon < 1$ the above expression is bounded by

$$K\pi^{-1}\varphi_2(x_2, x_{11}, x_{12})\{|x_{12}| + |x_2 x_{11}|\}$$

for some $K < \infty$. This expression is clearly integrable since under the conditions of the theorem each of X_{12}, X_2, X_{11} has finite variance. If we can show that $\sup_{\varepsilon\to 0} E\{[\chi_1^+(\varepsilon)]^2\} < \infty$ we can then apply a result of Feller (1971) (pages 251—252) to obtain

$$\lim_{\varepsilon\to 0} E\{\chi_1^+(\varepsilon)\} = E\{\chi_1^+\} \,,$$

so that

$$E(\chi_1^+) = -\int_{x_2>0} \int_{x_{11}<0} x_2 x_{11}\varphi(u, 0, x_2, x_{11})\, dx_2 dx_{11} \,. \tag{4.3}$$

That this in fact holds under to conditions given in the statement of the theorem is a consequence of Lemmata 4.1 and 4.2 which follow this proof. Using completely analogous arguments we can also show

$$E(\chi_1^-) = \int_{x_2>0} \int_{x_{11}>0} x_2 x_{11}\varphi(u, 0, x_2, x_{11})\, dx_2\, dx_{11} \,. \tag{4.4}$$

Furthermore, it follows simply from the homogeneity of the random field that $E(\chi_2^+) = E(\chi_2^-)$ so that

$$E(\chi) = -\int_{x_2>0} \int_{x_{11}=-\infty}^{\infty} x_2 x_{11}\varphi(u, 0, x_2, x_{11})\, dx_2\, dx_{11}$$

and the theorem is established.

We shall state the following two results without proof. The proofs follow exactly the pattern of similar results for the kth factorial moment of the number of upcrossings by a one-dimensional process due to Cramér and Leadbetter

(1965) and Belyaev (1966). Both proofs are tedious, but not difficult. As in the one-dimensional analogue, the condition that the spectrum of $X(\mathbf{t})$ has a continuous component plays an important role in establishing (4.5).

LEMMA 4.1. *If $X(\mathbf{t})$ satisfies all the conditions of Theorem 4.1 then*

$$\begin{aligned} E\{\chi_1^+[\chi_1^+ - 1]\} &\leqq \lim_{\varepsilon\to 0} E\{\chi_1^+(\varepsilon)[\chi_1^+(\varepsilon) - 1]\} \\ &= \int_{I_0}\int_{I_0} E\{(X_2(\mathbf{t}))^+(X_2(\mathbf{s}))^+(X_{11}(\mathbf{t}))^-(X_{11}(\mathbf{s}))^- \mid B(\mathbf{t}, \mathbf{s})\} \\ &\quad\times p(u, u, 0, 0, \mathbf{t}, \mathbf{s})\, d\mathbf{t}\, d\mathbf{s}\,, \end{aligned} \tag{4.5}$$

where $(y)^+ = \max(y, 0)$, $(y)^- = \min(y, 0)$, $p(x, y, x_1, y_1, \mathbf{t}, \mathbf{s})$ is the joint density of $[X(\mathbf{t}), X(\mathbf{s}), X_1(\mathbf{t}), X_1(\mathbf{s})]$ and $B(\mathbf{t}, \mathbf{s})$ is the event that this vector valued variable equals $[u, u, 0, 0]$.

LEMMA 4.2. *If a two-dimensional homogeneous Gaussian field has partial derivatives of order 3 which are mean square continuous, then the integral on the right hand side of (4.5) is finite.*

The requirement that the field has continuous partial derivatives of order 3, rather than of order 2 as would at first seem to be appropriate, arises from the fact that we are considering not only zeroes of $X(\mathbf{t}) - u$, but also zeroes of the derivative $X_1(\mathbf{t})$. In the one-dimensional case for example, it follows from Theorem 3 of Belyaev (1966) that the variance of the number of stationary points of a process $X(t)$, $t \in R^1$, is finite if $X(t)$ is 3 times differentiable in mean square. Note that the conditions of the lemma correspond to the condition that $\partial^6 R(\mathbf{t})/\partial t_1^{k_1}\,\partial t_2^{k_2}$, $k_1 + k_2 = 6$, is continuous at the origin, where $R(\mathbf{t})$ is the covariance function of $X(\mathbf{t})$, or equivalently to the condition that the spectral moments of order 6 are finite.

The last two results complete the proof of Theorem 4.1. We now give an explicit form for (4.1). It follows from considering the spectral representation of X that we can write

$$\varphi(x, x_1, x_2, x_{11}) = \varphi_1(x)\varphi_2(x_1, x_2)\varphi_3(x_{11} \mid x)$$

where the φ_i are normal densities. Now let $\sigma^2 = E(X^2)$, and let $\mathbf{A}$ be the covariance matrix of the first order derivatives of X. Computing the integral we obtain

THEOREM 4.2. *Let X be a zero mean homogeneous Gaussian field on R^2 satisfying the conditions of Theorem 4.1. Then the mean value of the characteristic of its excursion set is given by*

$$E\{\chi(A)\} = (2\pi\sigma^2)^{-\frac{3}{2}}|\mathbf{A}|^{\frac{1}{2}}u\exp(-\tfrac{1}{2}u^2/\sigma^2)\,.$$

5. Three-dimensional fields. We shall now briefly consider the mean value of $\chi(A)$ when X is a Gaussian field on R^3. The analysis is very similar to the two-dimensional case. With a similar set of conditions for suitable regularity to those in Definition 3.2 we can show that there is, for suitably regular processes, a positive contribution to $\chi(A)$ in the interior of I_0 whenever the implicit function defining t_3 as a function of t_1 and t_2 on the boundary of A has a maximum or minimum and $X_3(\mathbf{t}) > 0$. Negative contributions correspond to saddle points. Writing $\mathbf{J}(\mathbf{t}) = (J_{ij}(\mathbf{t}))$ as the $n \times n$ matrix with $J_{ij}(\mathbf{t}) = X_{ij}(\mathbf{t})$, and denoting by $\mathbf{J}_k$ the matrix $\mathbf{J}$ with its kth row and column removed, we find that $E\{\chi(A)\}$ is equal to the mean number of points $\mathbf{t} \in I_0$ with $X(\mathbf{t}) - u = X_1(\mathbf{t}) = X_2(\mathbf{t}) = 0$, $X_3(\mathbf{t}) > 0$, and $\det(\mathbf{J}_3(\mathbf{t})) > 0$, minus the mean number of points satisfying these conditions but with $\det(\mathbf{J}_3(\mathbf{t})) < 0$. As in the two-dimensional case the mean contribution from the boundaries vanishes. It then follows that

$$E\{\chi(A)\} = \int x_3(x_{11}x_{22} - x_{12}^2)\varphi(u, 0, 0, x_3, x_{11}, x_{22}, x_{12})\, dx_3\, dx_{11}\, dx_{22}\, dx_{12}$$

where the integral is over all values of x_{11}, x_{22}, x_{12} and $x_3 > 0$, and φ is the joint density function of $(X, X_1, X_2, X_3, X_{11}, X_{22}, X_{12})$. Computing this integral gives

THEOREM 5.1. *Let X be a zero mean homogeneous Gaussian field on R^3 whose sample functions have* a.s. *continuous partial derivatives of up to second order with finite variance in I_0. Furthermore, let X have finite spectral moments of up to and including order* 6, *and a spectrum with a continuous component. Finally, suppose that for each $k = 1, 2, 3$ and any real numbers x_j, the following determinant nondegeneracy condition is satisfied.*

$$(5.1) \qquad P\{\det(\mathbf{J}_k(\mathbf{t})) = 0 \mid X(\mathbf{t}) = u,\ X_j(\mathbf{t}) = x_j,\ 1 \leqq j \leqq 3,\ j \neq k\} = 0\,.$$

Then

$$(5.2) \qquad E\{\chi(A)\} = (2\pi)^{-2}|\mathbf{A}|^{\frac{1}{2}}\sigma^{-5}(u^2 - \sigma^2)\exp(-\tfrac{1}{2}u^2/\sigma^2)$$

where $\mathbf{A}$ *is the covariance matrix of the X_j and $\sigma^2 = E\{X^2(\mathbf{t})\}$.*

An explanation of how (5.1) arises can be gleaned from Belyaev (1972b).

6. Remarks. 1. As we noted in Section 3, $\chi(A)$ is a natural generalisation to random fields of the notion of the number of upcrossings in one dimension and it would thus be of interest to know its mean value for $n > 3$ as well. We conjecture, but can as yet only prove heuristically, that for an n-dimensional, homogeneous, zero mean Gaussian field satisfying similar conditions to those in Theorem 5.1

$$(6.1) \qquad E\{\chi(A)\} = (2\pi)^{-\frac{1}{2}(n+1)}|\mathbf{A}|^{\frac{1}{2}}\sigma^{-2(n-\frac{1}{2})}u^{n-1}\exp(-\tfrac{1}{2}u^2/\sigma^2)(1 - O(u^{-2}))\,.$$

2. It is of considerable interest that (6.1) leads to the asymptotic formula obtained by Nosko (1969) and Hasofer (1976) for the mean number of maxima above a high level. This follows from Theorem 2 of Nosko (1969), where it is shown under more restrictive conditions than ours that excursions above a high level, u, can be approximated within $O(u^{-1})$ by the segment of a particular second order surface lying above the hyperplane $X(\mathbf{t}) = u$, such that each such excursion has one associated local maximum. From the point of view of this paper, this means that each excursion has a surface which is approximately a hyper-ellipse,

and so has characteristic one. Thus for large values of u the number of maxima above u, and $\chi(A)$, are essentially the same. The obvious advantage of our approach is that our results are exact for all values of u, at least for $n = 2, 3$.

3. Since none of our results explicitly involve the second order derivatives, it seems possible that they will continue to hold in the absence of the conditions we have placed on them. This is certainly the case with the corresponding one-dimensional result.

4. The recent work by Nosko (1973) represents the only serious attempt other than our own to obtain a generalisation of level crossings via the topological properties of the excursion set. Nosko considers only excursions by two-dimensional fields over rather special subsets of R^2, using the number of connected components of the excursion set to generalise the number of excursions of a one-dimensional process. This random variable is not readily amenable to statistical investigation however. For example, it is not possible to obtain an exact expression for its mean value but only upper and lower bounds. It is perhaps worth mentioning here that exact expressions for $E\{\chi(A)\}$ such as those given in Theorems 4.2 and 5.1 open up the possibility of using the characteristic of an excursion set to estimate $|\mathbf{A}|$ in the same way that one uses the number of upcrossings of a one-dimensional process to estimate its second spectral moment. (See, for example, Lindgren (1974).) The parameter, $|\mathbf{A}|$, plays an important role in the distribution of the maximum of $X(\mathbf{t})$ (Belyaev (1972b)).

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