The Annals of Probability
2011, Vol. 39, No. 4, 1286–1304
DOI: 10.1214/10-AOP581

ON MONOCHROMATIC ARM EXPONENTS FOR 2D CRITICAL PERCOLATION

BY VINCENT BEFFARA AND PIERRE NOLIN[1]

ENS Lyon and the Courant Institute

We investigate the so-called *monochromatic arm exponents* for critical percolation in two dimensions. These exponents, describing the probability of observing j disjoint macroscopic paths, are shown to exist and to form a different family from the (now well understood) polychromatic exponents. More specifically, our main result is that the monochromatic j-arm exponent is strictly between the polychromatic j-arm and $(j+1)$-arm exponents.

1. Introduction. Percolation is one of the most-studied discrete models in statistical physics. The usual setup is that of *bond percolation* on the square lattice $\mathbb{Z}^2$, where each bond is open (resp., closed) with probability $p \in (0, 1)$ (resp., $1 - p$), independently of the others. This model exhibits a phase transition at a critical point $p_c \in (0, 1)$ (in this particular case, $p_c = 1/2$): for $p < p_c$, almost surely all connected components are finite, while for $p > p_c$ there exists a unique infinite component with density $\theta(p) > 0$. Site percolation is defined in a similar fashion, the difference being that the vertices are open or closed, instead of the edges; one can then see it as a random coloring of the lattice, and use the terms *black* and *white* in place of *open* and *closed*.

The behavior of percolation away from the critical point is well understood; however it is only recently that precise results have been obtained at and near criticality. For critical site percolation on the regular triangular lattice, the proof of conformal invariance in the scaling limit was obtained by Smirnov [12], and SLE processes, as introduced by Schramm [11] and further studied by Lawler, Schramm and Werner [7, 8], provide an explicit description of the interfaces (in the scaling limit) in terms of SLE(6) (see, e.g., [16]).

This description allows for the derivation of the so-called *polychromatic arm exponents* [13], which describe the probability of observing connections across annuli of large modulus by disjoint connected paths of specified colors (with at least one arm of each color), and also the derivation of the one-arm exponent [9]. Combined with Kesten's scaling relations [6], these exponents then provide the existence and the values of most of the other critical exponents, like, for example, the exponent $\beta = 5/36$ associated with the density of the infinite cluster, as $p \downarrow p_c$,

$$\theta(p) = (p - p_c)^{5/36+o(1)}.$$

Received June 2009; revised July 2010.
[1]Supported by NSF Grant OISE-07-30136.
MSC2010 subject classifications. 82B43, 60K35.
Key words and phrases. Percolation, critical exponent, scaling limit.

On the other hand, very little is known concerning the *monochromatic* arm exponents (i.e., with all the connections of the same color—see below for a formal definition) with more than one arm. Here, the SLE approach does not seem to work, and, correspondingly, there is no universally established conjecture for the values of those exponents. One notable exception, however, is the two-arm monochromatic exponent, for which an interpretation in terms of SLE(6) is proposed at the end of [9]—but again no explicit value has been computed. That particular exponent is actually of physical interest: known as the *backbone exponent*, it describes the "skeleton" of a percolation cluster. Even the existence of these exponents is not clear, as there does not seem to be any direct sub-additivity argument.

In this paper, we prove that the monochromatic exponents do exist, and investigate how they are related to the polychromatic exponents. We show that they have different values than their polychromatic counterparts. As an illustration, our result implies that the backbone of a typical large percolation cluster at criticality is much "thinner" than its boundary.

2. Background.

2.1. *The setting.* We restrict ourselves here to site percolation on the triangular lattice, at criticality ($p = p_c = 1/2$). Recall that it can be obtained by coloring the faces of the honeycomb lattice randomly, each cell being black or white with probability $1/2$ independently of the others. In the following, we denote by $\mathbb{P} = \mathbb{P}_{1/2}$ the corresponding probability measure on the set of configurations. Let us mention, however, that many of the results of combinatorial nature based on Russo–Seymour–Welsh-type estimates should also hold for bond percolation on $\mathbb{Z}^2$, due to the self-duality property of this lattice.

Let S_n denote the ball of radius n in the triangular lattice (i.e., the intersection of the triangular lattice with the Euclidean disc of radius n, though the specifics of the definition are of little relevance), seen as a set of vertices. We will denote by $\partial^i S_n$ (resp., $\partial^e S_n$) its internal (resp., external) boundary, that is, the set of vertices in (resp., outside) S_n that have at least one neighbor outside (resp., in) S_n, and, for $n < N$, by

$$S_{n,N} := S_N \setminus S_n$$

the annulus of radii n and N. To describe critical and near-critical percolation, certain exceptional events play a central role: the arm events, referring to the existence of a number of crossings ("arms") of $S_{n,N}$, the color of each crossing (black or white) being prescribed.

DEFINITION 1. Let $j \geq 1$ be an integer and $\sigma = (\sigma_1, \dots, \sigma_j)$ be a sequence of colors (black or white). For any two positive integers $n < N$, a (j, σ)*-arm configuration* in the annulus $S_{n,N}$ is the data of j disjoint monochromatic, nonself-intersecting paths $(r_i)_{1 \leq i \leq j}$—the *arms*—connecting the inner boundary $\partial^e S_n$ and the outer boundary $\partial^i S_N$ of the annulus, ordered counterclockwise in a cyclic way, where the color of the arm r_i is given by σ_i. We denote by

$$A_{j,\sigma}(n,N) := \{\partial^e S_n \underset{j,\sigma}{\rightsquigarrow} \partial^i S_N\} \tag{2.1}$$

the corresponding event. It depends only on the state of the vertices in $S_{n,N}$.

We will write down color sequences by abbreviating colors, using B and W for black and white, respectively. To avoid the obvious combinatorial obstructions, we will also use the notation $n_0 = n_0(j)$ for the smallest integer such that j arms can possibly arrive on $\partial^e S_{n_0}$ [$n_0(j)$ is of the order of j] and only consider annuli of inner radius larger than n_0. This restriction will be done implicitly in what follows.

The so-called *color exchange trick* (noticed in [1, 13]) shows that for a fixed number j of arms, prescribing the color sequence σ changes the probability only by at most a constant factor, *as long as both colors are present in* σ (because an interface is needed to proceed). The asymptotic behavior of that probability can be described precisely using SLE(6): it is possible to prove the existence of the (polychromatic) arm exponents and to derive their values [13], which had been predicted in the physics literature (see, e.g., [1] and the references therein).

THEOREM 2. *Fix $j \geq 2$. Then for any color sequence σ containing both colors,*

$$\mathbb{P}(A_{j,\sigma}(n,N)) = N^{-\alpha_j + o(1)} \tag{2.2}$$

as $N \to \infty$ for any fixed $n \geq n_0(j)$, where $\alpha_j = (j^2-1)/12$.

The value of the exponent for $j = 1$ (corresponding to the probability of observing one arm crossing the annulus) has also been established [9] and it is equal to $5/48$ (oddly enough formally corresponding to $j = 3/2$ in the above formula).

For future reference, let us mention the following facts about critical percolation that we will use.

1. *A priori bound for arm configurations*: there exist constants $C, \varepsilon > 0$ such that for all $n < N$,

$$\mathbb{P}(A_{1,B}(n,N)) = \mathbb{P}(A_{1,W}(n,N)) \leq C\left(\frac{n}{N}\right)^{\varepsilon}. \tag{2.3}$$

For all $j \geq 1$, there exist constants $c_j, \beta_j > 0$ such that for all $n < N$,

$$\mathbb{P}(A_{j,B\ldots BB}(n,N)) \geq c_j\left(\frac{n}{N}\right)^{\beta_j}. \tag{2.4}$$

2. *Quasi-multiplicativity property*: For any $j \geq 1$ and any sequence σ, there exist constants $C_1, C_2 > 0$ such that for all $n_1 < n_2 < n_3$,

$$\begin{aligned} C_1 \mathbb{P}(A_{j,\sigma}(n_1, n_2))\mathbb{P}(A_{j,\sigma}(n_2, n_3)) \\ \leq \mathbb{P}(A_{j,\sigma}(n_1, n_3)) \\ \leq C_2 \mathbb{P}(A_{j,\sigma}(n_1, n_2))\mathbb{P}(A_{j,\sigma}(n_2, n_3)). \end{aligned}$$

The first of these two properties actually relies on the so-called Russo–Seymour–Welsh (RSW) lower bounds, that we will use extensively in various situations: roughly speaking, these bounds state that the probability of crossing a given shape of fixed aspect ratio is bounded below independently of the scale. For instance, the probability of crossing a $3n \times n$ rectangle in its longer direction is bounded below, uniformly as $n \to \infty$. We refer the reader to [5] for more details.

In the second one, the independence of arm events in disjoint annuli implies that one can actually take $C_2 = 1$, which we will do from now on. The lower bound is obtained using a so-called *separation lemma*, as first proved by Kesten [6], Lemma 6 (technically, he does the proof in the case of four arms, but the argument extends easily to the general case). The monochromatic case is rather easier, as is follows directly from RSW estimates and Harris's lemma.

2.2. *A correlation inequality.* For two increasing events, the probability of their disjoint occurrence can be bounded below by the classic van den Berg–Kesten (BK) inequality [15]; Reimer's inequality, conjectured in [15] and proved in [10], extends it to the case of arbitrary events. A key ingredient in our proof will be a not-that-classic correlation inequality which is an intermediate step in the proof of Reimer's inequality. Note that actually, in the simpler case of increasing events (which is the only one we need here), this inequality was obtained earlier by Talagrand [14]; but we choose to follow the more established name of the general result even for this particular case.

Instead of using the terminology in Reimer's original paper [10] we follow the rephrasing (with more "probabilistic" notation) of his proof in the review paper [2]. Consider an integer n, and $\Omega = \{0, 1\}^n$. For any configuration $\omega \in \Omega$ and any set of indices $S \subseteq \{1, \dots, n\}$, we introduce the cylinder

$$[\omega]_S := \{\tilde{\omega} : \forall i \in S, \tilde{\omega}_i = \omega_i\},$$

and more generally for any $X \subseteq \Omega$, any $S : X \to \mathcal{P}(\{1, \dots, n\})$,

$$[X]_S := \bigcup_{\omega \in X} [\omega]_{S(\omega)}.$$

For any two $A, B \subseteq \Omega$, we denote by $A \circ B$ the disjoint occurrence of A and B (the notation $A \square B$ is also often used to denote this event)

$$A \circ B := \{\omega : \text{for some } S(\omega) \subseteq \{1, \dots, n\}, [\omega]_S \subseteq A \text{ and } [\omega]_{S^c} \subseteq B\}.$$

Recall that Reimer's inequality states that

$$\mathbb{P}(A \circ B) \le \mathbb{P}(A)\mathbb{P}(B). \tag{2.5}$$

We also denote by $\bar{\omega} = 1 - \omega$ the configuration obtained by "flipping" every bit of the configuration $\omega \in \Omega$, and if $X \subseteq \Omega$, we define $\bar{X} := \{\bar{\omega} : \omega \in X\}$. We are now in a position to state the correlation inequality that will be a key ingredient in the following:

THEOREM 3 ([10], Theorem 1.2). *For any $A, B \subseteq \Omega$, we have*

$$|A \circ B| \le |A \cap \bar{B}| = |\bar{A} \cap B|. \tag{2.6}$$

For the sake of completeness, let us just mention that this inequality is not stated explicitly in that form in [10]. It can be deduced from Theorem 1.2 by applying it to the *flock of butterflies* $\mathcal{B} = \{(\omega, f(\omega)) : \omega \in A \circ B\}$, where $f(\omega)$ coincides with ω exactly in the coordinates in $S(\omega)$. Here, $S(\omega) \subseteq \{1, \dots, n\}$ is the subset of indices associated with A and B by the definition of disjoint occurrence, that is, so as to satisfy $[\omega]_{S(\omega)} \subseteq A$ and $[\omega]_{S(\omega)^c} \subseteq B$ for all $\omega \in A \circ B$. For this particular $\mathcal{B}$, we indeed have $\mathrm{Red}(\mathcal{B}) \subseteq A$ and $\mathrm{Yellow}(\mathcal{B}) \subseteq B$.

Equivalently, it can be obtained from Lemma 4.1 in [2] by taking $X = A \circ B$, and $S : X \to \mathcal{P}(\{1, \dots, n\})$ associated with $A \circ B$ (so that $[X]_S \subseteq A$ and $[X]_{S^c} \subseteq B$).

2.3. *Statement of the results.* In this paper, we will be interested in the asymptotic behavior of the probability of the event $A_{j,\sigma}(n_0(j), N)$ as $N \to \infty$ for a monochromatic σ, say $\sigma = B \dots B$, so that $A_{j,\sigma}$ simply refers to the existence of j disjoint black arms. Our first result shows that this probability follows a power law, as in the case of a polychromatic choice of σ.

THEOREM 4. *For any $j \ge 2$, there exists an exponent $\alpha'_j > 0$ such that*

$$\mathbb{P}(A_{j,B\dots B}(n, N)) = N^{-\alpha'_j + o(1)} \tag{2.7}$$

as $N \to \infty$ for any fixed $n \ge n_0(j)$.

These exponents α'_j are known as the *monochromatic arm exponents*, and it is natural to try to relate them to the previously mentioned polychromatic exponents α_j.

Consider any $j \ge 2$; we start with a few easy remarks. On the one hand, one can apply Harris's lemma (more often referred to in the statistical mechanics community as the FKG inequality, which is its generalization to certain nonproduct measures; we will follow that convention in what follows): it implies that

$$\begin{aligned}\mathbb{P}(A_{j+1,B\dots BW}(n_0, N)) &= \mathbb{P}\big(A_{j,B\dots B}(n_0, N) \cap A_{1,W}(n_0, N)\big) \\ &\le \mathbb{P}(A_{j,B\dots B}(n_0, N))\mathbb{P}(A_{1,W}(n_0, N)),\end{aligned}$$

and by using item 1 above, we get that, for some constants $C, \varepsilon > 0$,

$$\mathbb{P}(A_{j+1,B\dots BW}(n_0, N)) \le CN^{-\varepsilon}\mathbb{P}(A_{j,B\dots B}(n_0, N)) \tag{2.8}$$

or, in other words, that $\alpha'_j < \alpha_{j+1}$. On the other hand, inequality (2.6) directly implies that

$$\begin{aligned}\mathbb{P}(A_{j,B\dots BB}(n_0, N)) &= \mathbb{P}\big(A_{j-1,B\dots B}(n_0, N) \circ A_{1,B}(n_0, N)\big)\\ &\le \mathbb{P}\big(A_{j-1,B\dots B}(n_0, N) \cap A_{1,W}(n_0, N)\big)\\ &= \mathbb{P}(A_{j,B\dots BW}(n_0, N)),\end{aligned}$$

hence $\alpha'_j \ge \alpha_j$. We will actually prove the following, stronger result.

THEOREM 5. *For any $j \ge 2$, we have*

$$\alpha_j < \alpha'_j < \alpha_{j+1}. \tag{2.9}$$

The monochromatic exponents α'_j thus form a family of exponents different from the polychromatic exponents.

We would like to stress the fact that the case of half-plane exponents (or more generally, boundary exponents in any planar domain) is considerably different: indeed, whenever a boundary is present, the color-exchange trick implies that the probability of observing j arms of prescribed colors is *exactly* the same for all color prescriptions, whether mono- or poly-chromatic. In particular there is no difference between the monochromatic and polychromatic boundary exponents. (For the reader's peace of mind, he can notice that the presence of the boundary provides for a canonical choice of a leftmost arm; the lack of which is precisely the core idea of the proof of our main result in the whole plane.)

We will first prove the inequality $\alpha_j < \alpha'_j$ (which is the main statement in the above theorem, the other strict inequality being the simple consequence of the FKG inequality we mentioned earlier), since its proof only requires combinatorial arguments, and postpone the proof of the existence of the exponents to the end of the paper.

In order not to refer to the α'_j's, we adopt the following equivalent formulation of the inequality: what we formally prove is that, for any $j \ge 2$, there exists $\varepsilon > 0$ such that for any N large enough,

$$\mathbb{P}(A_{j,B\dots BB}(n_0, N)) \le N^{-\varepsilon}\mathbb{P}(A_{j,B\dots BW}(n_0, N)). \tag{2.10}$$

The proof of this inequality only relies on discrete features such as self-duality and RSW-type estimates, and hence it is possible that our proof could be extended to the case of bond percolation on $\mathbb{Z}^2$, where the existence of the exponents, which strongly relies on the knowledge of the scaling limit, is still unproved; however, the statement of duality in that case is different enough (the color exchange trick for instance has no exact counterpart) that some of our arguments do not seem to extend directly.

3. The set of winding angles.

3.1. *Strict inequalities between the exponents*. Our proof is based on an energy versus entropy consideration. The difference between the monochromatic and the polychromatic j-arm exponents can be written in terms of the expected number of "really different" choices of j arms out of a percolation configuration with j arms: for a polychromatic configuration, this number is equal to 1, whereas for a monochromatic configuration, it grows at least like a positive power of the modulus, and the ratio between these two numbers behaves exactly like $(N/n)^{\alpha_j-\alpha'_j}$ because, for fixed disjoint arms $(r_1, \dots, r_j)$ with respective lengths $(\ell_1, \dots, \ell_j)$, the probability that they are present in the configuration with a prescribed coloring does not depend on that coloring (it is equal to $2^{-(\ell_1+\cdots+\ell_j)}$).

More precisely, but still roughly speaking, the proof relies on the following observation: given a configuration where j black arms are present, there are many ways to choose them, since by RSW there is a positive density of circuits around the origin [allowing "surgery" on the arms (see Figure 1)], while if we consider a configuration with arms of both colors, then there is essentially only one way to select them. Of course the geometry of an arm is quite intricate and many local modifications, on every scale, are always possible, both in the monochromatic and polychromatic setups; what we mean here is that this choice is unique from a macroscopic point of view. To formalize this intuition, we thus have to find a way of distinguishing two macroscopic choices of arms, and for this we will use the *set of winding angles* associated with a configuration.

DEFINITION 6. For any configuration of arms, one can choose a continuous determination of the argument along one of the arms; we call *winding angle of the arm* (or simply *angle* for short) the overall (algebraic) variation of the argument along that arm.

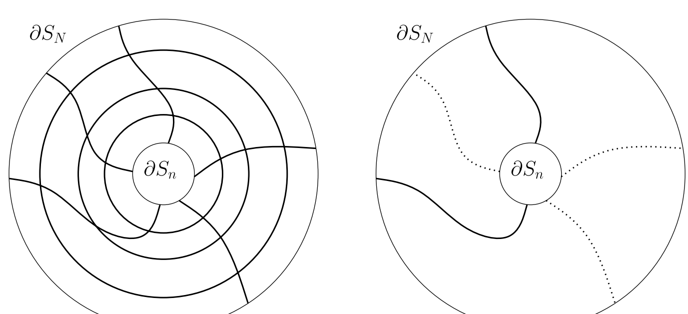

FIG. 1. *To a given monochromatic configuration correspond many different "macroscopic" ways to choose the arms, contrary to the polychromatic case. The left-hand picture shows the sort of realization in which many macroscopically different* 5*-arm configurations can be found—the actual topology needed is a little bit more involved*; *see Figure* 3.

Clearly, the winding angles of the arms corresponding to a given (j,σ)-arm configuration differ by at most 2π. However, for the same percolation configuration, there might exist many different choices of a (j,σ)-arm configuration, corresponding to different winding angles: we denote by $I_{j,\sigma}(n,N)$ the set of all the winding angles which can be obtained from such a configuration; we omit the subscript from the notation whenever j and σ are clear from the context (notice though that whenever there are enough arms for two different arm events with the same j to be realized simultaneously, the corresponding sets of winding angles are essentially the same). For the sake of completeness, we also declare $I_{j,\sigma}(n,N)$ to be empty if the configuration does not contain j arms of the prescribed colors.

We will actually rather use $\bar{I}_{j,\sigma}(n,N)$, the set of angles obtained by "completing" $I_{j,\sigma}(n,N)$

$$\bar{I}_{j,\sigma}(n,N) := \bigcup_{\alpha\in I_{j,\sigma}(n,N)} (\alpha-\pi,\alpha+\pi].$$

It is an easy remark that in the polychromatic case (σ nonconstant), we have for any $\alpha \in I_{j,\sigma}(n,N)$

$$I_{j,\sigma}(n,N) \subseteq (\alpha-2\pi,\alpha+2\pi)$$

(because two arms of different colors cannot cross), so that $\bar{I}_{j,\sigma}(n,N)$ is an interval of length at most 4π. In the monochromatic case (σ constant), no such bound applies [and actually it is not obvious that $\bar{I}_{j,\sigma}(n,N)$ is an interval; this is proved as Proposition 7 below].

In the case of a polychromatic arm configuration, considering successive annuli of a given modulus as independent, one would expect a central limit theorem to hold on the angles, or at least fluctuations of order $\sqrt{\log N}$. On the other hand, for a monochromatic configuration, performing surgery using circuits in successive annuli should imply that every time one multiplies the outer radius by a constant, the expected largest available angle would increase by a constant, so that one would guess that, by a careful choice of arms, the total angle can be made of order $\pm\log N$.

Fix $\varepsilon > 0$, and let $A^{\varepsilon}_{1,B}$ (resp., $A^{\varepsilon}_{1,W}$, resp., $A^{\varepsilon}_{j,\sigma}$) be the event that there exists a black arm (resp., a white arm, resp., j arms with colors given by σ) with angle larger than $\varepsilon\log N$ between radii n_0 and N. Applying inequality (2.6) with $A = A_{j-1,B\dots B}$ and $B = A^{\varepsilon}_{1,B}$, if the above intuition was correct, this would imply

$$\begin{aligned}\mathbb{P}(A_{j,B\dots BB}) &\asymp \mathbb{P}(A\circ B)\\ &\le \mathbb{P}(A_{j-1,B\dots B}\cap A^{\varepsilon}_{1,W})\\ &= \mathbb{P}(A^{\varepsilon}_{j,B\dots BW}),\end{aligned}$$

and we could expect

$$\mathbb{P}(A^{\varepsilon}_{j,B\dots BW}) \le N^{-\varepsilon'}\mathbb{P}(A_{j,B\dots BW})$$

by a large-deviation principle. However, proving this LDP seems to be difficult, and we propose here an alternative proof that relies on the same ideas, but bypasses some of the difficulties.

PROOF OF THEOREM 5. We shall in fact prove (2.10). As noted earlier, assuming Theorem 4 (which will be proved in Section 4), Theorem 5 follows immediately.

Step 1. First, note that it suffices to prove that the ratio

$$\frac{\mathbb{P}(A_{j,B\dots BB}(n,N))}{\mathbb{P}(A_{j,B\dots BW}(n,N))}$$

can be made arbitrarily small as $N/n \to \infty$, uniformly in n: indeed, assuming that this is the case, then for any $\delta > 0$, there exists $R > 0$ such that this ratio is less than δ as soon as $N/n \geq R$. Then, as a direct consequence of the quasi-multiplicativity property (item 2 above), we have

$$\begin{aligned}
&\mathbb{P}(A_{j,B\dots BB}(n, R^k n)) \\
&\qquad \leq C_2^{k-1}\mathbb{P}(A_{j,B\dots BB}(n, Rn))\cdots\mathbb{P}(A_{j,B\dots BB}(R^{k-1}n, R^k n)) \\
&\qquad \leq C_2^{k-1}\delta^k\mathbb{P}(A_{j,B\dots BW}(n, Rn))\cdots\mathbb{P}(A_{j,B\dots BW}(R^{k-1}n, R^k n)) \\
&\qquad \leq C_2^{k-1}\delta^k(C_1^{-1})^{k-1}\mathbb{P}(A_{j,B\dots BW}(n, R^k n)),
\end{aligned}$$

and for $\delta = 1/(2C_2C_1^{-1})$ this gives

$$\mathbb{P}(A_{j,B\dots BB}(n, R^k n)) \leq 2^{-k}\mathbb{P}(A_{j,B\dots BW}(n, R^k n)), \tag{3.1}$$

which immediately implies that for some $C, \varepsilon > 0$,

$$\mathbb{P}(A_{j,B\dots BB}(n,N)) \leq C\left(\frac{N}{n}\right)^{-\varepsilon}\mathbb{P}(A_{j,B\dots BW}(n,N)).$$

In particular, applying this for $n = n_0$ (and N large enough) leads to the inequality that we need.

Step 2. The key step of the proof is as follows. Given a configuration with j arms in an annulus of large modulus, we use RSW-type estimates to prove the existence of a large number of disjoint sub-annuli of it, in each of which one can find black paths topologically equivalent to those in Figure 2 (in the case $j = 2$) or its reflection. Every time this configuration appears, one has the possibility to replace the original arms (in solid lines on the figure) with modified, and still disjoint, arms, obtained by using one of the dashed spirals in each of them. The new arms then land at the same points on the outer circle, but with a winding angle differing by 2π. This allows us to show that, with high probability, the set of angles $\bar{I}(n,N)$ contains an interval of length at least $\varepsilon \log(N/n)$, for some $\varepsilon > 0$ (which can be written in terms of the RSW estimates). We now proceed to make the construction in detail.

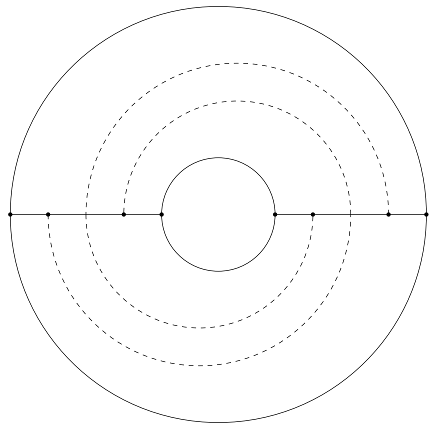

FIG. 2. *When they encounter this configuration, the arms (here in solid lines) can be modified, detouring via the dashed lines, to make an extra turn.*

Let $j \geq 2$, and let m be a positive integer. Define a *j-spiral* between radii m and $4m$ as the configuration pictured in Figure 3. More precisely, a j-spiral is the union of 4 families of j black paths in a percolation configuration, namely:

- j disjoint rays between radii m and $4m$;

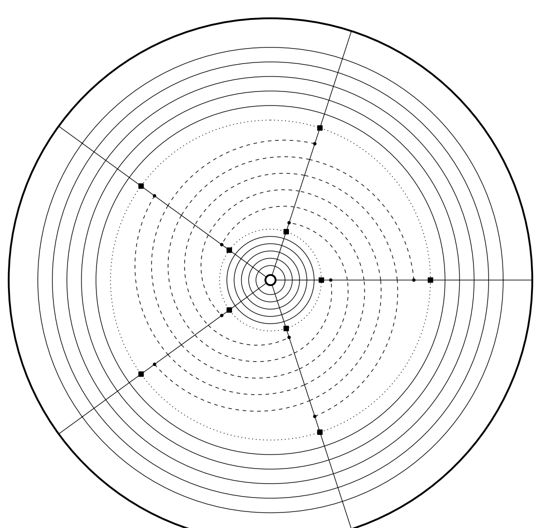

FIG. 3. *Generalization of Figure 2 in the case of $j \geq 3$ arms. The additional circuits (in solid lines) are needed to apply Menger's theorem; the circles of radii m and $4m$ (resp., $2m$ and $3m$) are drawn in heavy (resp., dotted) lines, the spiraling paths in dashed lines and the active points are marked with a black square.*

- j disjoint "spiraling paths" contained in the annulus $S_{2m,3m}$, each connecting two points of one of the rays and making one additional turn around the origin;
- j disjoint circuits around the origin, contained in the annulus $S_{m,2m}$;
- j disjoint circuits around the origin, contained in the annulus $S_{3m,4m}$.

RSW-type estimates directly show that, uniformly as $m \to \infty$, the probability of observing a j-spiral between radii m and $4m$ is bounded below by a positive constant (depending only on j). In addition, with each such spiral we associate two families of j *active points*: for each of the j rays, oriented starting at radius m, its last intersection with the circle of radius $2m$ is called an *inner* active point, and its first intersection with the circle of radius $3m$ after it, an *outer* active point; in particular, the section of the ray between its two active points remains within the annulus $S_{2m,3m}$.

The same RSW arguments show that with positive probability, such a j-spiral actually satisfies a few additional properties (which we will consider part of the definition from now on): the different pieces remain well separated whenever they are not forced to intersect by topological constraints; whenever two pieces intersect, they can do so multiple times, but all intersection points remain close to each other, and it is easy to check that this is not a problem in the proof. In fact, the only thing that might cause trouble is that the rays may cross each of the circles of radii $2m$ and $3m$ several times; one cannot use RSW to avoid it, but it is taken care of by our choice of the active points.

The presence of j-spirals in disjoint annuli are independent events, each with positive probability, so that, for some $\varepsilon > 0$, the probability of the event $E_j^{(\varepsilon)}(n, N)$ of having at least $\varepsilon \log(N/n) + 1$ disjoint j-spirals between radii n and N goes to 1 as N/n goes to infinity. The presence of j-spirals being an increasing event, the FKG inequality ensures the conditional probability of $E_j^{(\varepsilon)}(n, N)$, given the existence of j black arms between radii n and N, still goes to 1 as N/n goes to infinity.

We now explain how to use j-spirals to perform surgery on black arms. Assume that between radii n and N, there are a certain number $s \geq 2$ of disjoint spiral configurations; let $(m_i)_{1 \leq i \leq s}$ be the corresponding values of m. Assume in addition that there are j disjoint arms between radii n and N. The first remark is the following: for every $i \in \{1, \dots, s-1\}$, the event $A_{j,B\dots BB}(3m_i, 2m_{i+1})$ is realized, and one can choose j disjoint arms $(\gamma_{i,k})_{1 \leq k \leq j}$ in the annulus $S_{3m_i, 2m_{i+1}}$ accordingly. The main part of the argument then consists of proving that, within the union of all those arms together with the spirals, it is always possible to find 2^{s-1} j-arm configurations between radii n and N, spanning winding angles in an interval of length $2(s-2)\pi$.

Each of the constructed arms will consist of sections of two types, connecting at the active points (and disjoint otherwise, by our choice of the active points): the "movable" ones, formed out of the rays and spiraling parts of the j-spirals and contained in the union of the annuli $S_{2m_i,3m_i}$; and the "intermediate" ones, made from the arms $\gamma_{i,k}$ (which we have chosen to stay in the annuli $S_{3m_i,2m_{i+1}}$) and from pieces of the j-spirals lying outside the active points. Within every j-spiral there are two choices for the corresponding movable section, which is how arms of very different winding angles will be produced. For that, all we need to prove is the existence of one family of j disjoint arms following this decomposition.

The key argument is as follows. Define Γ_1 as the union of the $(\gamma_{1,k})_{1\le k\le j}$ (which cross the annulus $S_{3m_1,2m_2}$), together with the parts of the j-spiral Σ_1 (resp., Σ_2) outside its outer active points (resp., inside its inner active points). It is easy to check that, whenever one marks $(j-1)$ vertices on Γ_1, there still exists a path completely contained in Γ_1 and avoiding the marked points, and connecting one of the outer active points of Σ_1 to one of the inner active points of Σ_2. Indeed, marking $(j-1)$ vertices leaves untouched at least one arm, one inner (resp., outer) circle and one ray of Σ_1 (resp., Σ_2). Menger's theorem (see [4]) then ensures that Γ_1 contains j disjoint arms, each connecting one of the outer active points of Σ_1 to one of the inner active points of Σ_2.

The same construction can be performed between each pair of successive j-spirals, to produce the desired "intermediate" sections. An obvious variation of the construction can also be applied in the annuli $S_{n,2m_1}$ and $S_{3m_s,N}$, leading to the following fact: whenever there are j arms between radii n and N, and the event $E_j^{(\varepsilon)}(n,N)$ is realized, the set $\bar{I}(n,N)$ contains an interval of length at least $2\pi\varepsilon\log(N/n)$ (obtained by playing with the appropriate movable sections), and this occurs with conditional probability going to 1 as N/n goes to infinity. Notice that the winding angles of the original j arms need not be within the interval we just constructed; however, this has no bearing on what follows.

Step 3. We now use the BK inequality to control the probability, given the presence of j black arms between radii n and N, that there is a choice of arms with a very large winding angle. In fact, the argument shows a little more: it is very unlikely, even without the conditioning, that there is a single arm with large winding.

Let $R_{m,m'}$ be the intersection of the annulus $S_{m,m'}$ with the cone $\mathcal{C} := \{z \in \mathbb{C} : |\arg(z)| < \pi/10\}$ (using the standard identification of the plane $\mathbb{R}^2$ with the complex plane). We will consider two families of "rectangles": the $R_{e^k,e^{k+2}}$ (which we call the *long* ones), and the $R_{e^k,e^{k+1}}$ (the *wide* ones). It is easy to see that any curve connecting two points of the plane of arguments $-\pi/10$ and $+\pi/10$ while staying within the cone $\mathcal{C}$ has to cross at least one of these rectangles between two opposite sides. More precisely, it must either cross a long rectangle from one straight side to the other, or a wide rectangle from one curved side to the other.

Now, assume that there exist j arms between radii n and N, and that their winding angle is at least equal to $2\pi K\log(N/n)+4\pi$ (where K is a positive constant which will be chosen later). Each of these arms has to cross the cone $\mathcal{C}$ at least (the integer part of) $K\log(N/n)$ times, so the configuration contains $jK\log(N/n)$ disjoint paths (at least), each one crossing one of the rectangles. On the other hand, the probability that a rectangle of a given shape is crossed by a path is bounded above by $1-\delta$ for some $\delta > 0$, as provided by RSW estimates.

Combined with the BK inequality, this implies that the probability that there are j arms winding of at least an angle of $2\pi K \log(N/n) + 4\pi$ is bounded above by

$$\mathcal{S} := \sum_{(l_k, l'_k)} \prod_k (1-\delta)^{l_k + l'_k},$$

where the sum is taken over all $\log(N/n)$-tuples of (l_k, l'_k) having a sum equal to $jK\log(N/n)$. The number of such tuples is the same as the number of choices of $2\log(N/n) - 1$ disjoint elements out of $(jK+2)\log(N/n) - 1$, so we obtain that

$$\mathcal{S} \le (1-\delta)^{jK\log(N/n)} \binom{(jK+2)\log(N/n)-1}{2\log(N/n)-1}.$$

It is then a straightforward application of Stirling's formula to obtain that

$$\mathcal{S} \le C\exp[(-cK + C\log K)\log(N/n)],$$

where the constants c and C do not depend on the value of K. Choosing K large enough, one then obtains that

$$\mathcal{S} \le C\left(\frac{n}{N}\right)^{\beta_j + 1}$$

[where β_j is the same as in (2.4)].

Hence, the conditional probability, given $A_{j,B\ldots BB}(n,N)$, that $\bar{I}(n,N)$ is contained in the interval of length $4\pi K\log(N/n) + 8\pi$ centered at 0 goes to 1 as N/n goes to infinity. Dividing that interval into sub-intervals of length $\frac{\varepsilon}{2}\log(N/n)$, and using the previous step, we get that for one of them, say $i_\varepsilon(n,N)$,

$$\mathbb{P}\big(i_\varepsilon(n,N) \subseteq \bar{I}(n,N) | A_{j,B\ldots BB}(n,N)\big) \ge C',$$

where $C' > 0$ is a universal constant.

Step 4. We are now in a position to conclude. If we take $\alpha_{\min}$ such that

$$\mathbb{P}\big(A_{j,B\ldots BW}(n,N) \cap \{\alpha_{\min} \in \bar{I}_{j,B\ldots BW}(n,N)\}\big)$$

is minimal among all $\alpha_{\min} \in i_\varepsilon(n,N) \cap (4\pi\mathbb{Z})$, then

$$\begin{aligned}
&\mathbb{P}\big(A_{j,B\ldots BW}(n,N) \cap \{\alpha_{\min} \in \bar{I}_{j,B\ldots BW}(n,N)\}\big) \\
&\quad \le \frac{4\pi}{\varepsilon/2\log(N/n)} \mathbb{P}(A_{j,B\ldots BW}(n,N))
\end{aligned}$$

since, as we noted earlier, whenever there are arms of different colors, $4\pi\mathbb{Z}$ cannot contain more than one element of $\bar{I}(n,N)$. On the other hand, we know from the previous step that

$$\begin{aligned}
&\mathbb{P}\big(A_{j,B\ldots BB}(n,N) \cap \{\alpha_{\min} \in \bar{I}_{j,B\ldots BB}(n,N)\}\big) \\
&\quad \ge \mathbb{P}\big(A_{j,B\ldots BB}(n,N) \cap \{i_\varepsilon(n,N) \subseteq \bar{I}_{j,B\ldots BB}(n,N)\}\big) \\
&\quad \ge C'\mathbb{P}(A_{j,B\ldots BB}(n,N)).
\end{aligned}$$

If we apply (2.6) to $A = A_{j-1,B\dots B}(n,N) \cap \{\alpha_{\min} \in \bar{I}_{j-1,B\dots B}(n,N)\}$ and $B = A_{1,B}(n,N)$, we obtain that

$$\begin{aligned}
C'\mathbb{P}(A_{j,B\dots BB}(n,N)) &\le \mathbb{P}\bigl(A_{j,B\dots BB}(n,N) \cap \{\alpha_{\min} \in \bar{I}_{j,B\dots B}(n,N)\}\bigr) \\
&= \mathbb{P}(A \circ B) \\
&\le \mathbb{P}(A \cap \bar{B}) \\
&= \mathbb{P}\bigl(A_{j,B\dots BW}(n,N) \cap \{\alpha_{\min} \in \bar{I}_{j-1,B\dots B}(n,N)\}\bigr) \\
&\le \mathbb{P}\bigl(A_{j,B\dots BW}(n,N) \cap \{\alpha_{\min} \in \bar{I}_{j,B\dots BW}(n,N)\}\bigr) \\
&\le \frac{4\pi}{\varepsilon/2\log(N/n)}\mathbb{P}(A_{j,B\dots BW}(n,N)),
\end{aligned}$$

which completes the proof. $\square$

3.2. *The density of the set of winding angles.* In this section, we further describe the set of winding angles $I(n,N)$, which happened to be a key tool in the previous proof, in the monochromatic case. We prove that (conditionally on the existence of j disjoint black arms) $\bar{I}(n,N)$ is always an interval, as in the polychromatic case. For that, we use the following deterministic statement that $I(n,N)$ does not have large "holes":

PROPOSITION 7. *Let $j \ge 1$ and $\sigma = B\dots BB$ of length j. Let $\alpha, \alpha' \in I_{j,\sigma}(n,N)$ with $\alpha < \alpha'$; then there exists a sequence $(\alpha_i)_{0\le i\le r}$ of elements of $I_{j,\sigma}(n,N)$, satisfying the following two properties*:

- $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_r = \alpha'$;
- *for every $i \in \{0,\dots,r-1\}$, $\alpha_{i+1} - \alpha_i < 2\pi$.*

This result directly implies that $\bar{I}(n,N)$ is an interval, and the construction of the previous subsection, creating extra turns (step 2 of the proof), gives a lower bound on the diameter of $\bar{I}(n,N)$: we hence get that for σ constant, there exists some $\varepsilon > 0$ (depending only on j) such that $\bar{I}(n,N)$ is an interval of length at least $\varepsilon\log(N/n)$ with probability tending to 1 as N/n gets large.

The main step in the proof of the density result is the following topological lemma:

LEMMA 8. *Let $j \ge 1$, and let $\gamma_1,\dots,\gamma_j$ be j disjoint Jordan curves contained in the* (*closed*) *annulus $\{n \le |z| \le N\}$, ordered cyclically and each having its starting point on the circle of radius n and its endpoint on the circle of radius N. For each $k \in \{1,\dots,j\}$, let α_k be the winding angle of γ_k* (*as defined above*) *and let δ_k be the ray $[ne^{2i\pi k/j}, Ne^{2i\pi k/j}]$. Assume that, for each pair (k,k'), the intersection of γ_k and $\delta_{k'}$ is finite. Then, provided all the α_k are larger than $2\pi(1+2/j)$, the union of all the paths γ_k and δ_k contains j disjoint paths $\tilde{\delta}_1,\dots,\tilde{\delta}_j$, all having winding angle $2\pi/j$ and sharing the same endpoints as the δ_k.*

In other words: starting from two collections of paths, if their angles differ enough, one can "correct" the one with the smaller angle in such a way as to make it turn a little bit more.

PROOF OF LEMMA 8. We shall construct the paths $\tilde{\delta}_k$ explicitly. The first step is to reduce the situation to one of lower combinatorial complexity, namely to the case where the starting points of the γ_k are separated by those of the δ_k. For each $k \le j$, let $\tau_k = \inf\{t : \gamma_k(t) \in [ne^{i\pi(2k-1)/j}, Ne^{i\pi(2k-1)/j}]\}$ (which is always finite by our hypotheses), and let

$$\Gamma := \bigcup_{k=1}^{j} \{\gamma_k(t) : 0 \le t \le \tau_k\}.$$

Γ intersects each of the δ_k finitely many times, so each of the $\delta_k \setminus \Gamma$ has finitely many connected components: let Δ be the union of those components that do not intersect the circle of radius N, and let

$$\Omega_0 := \{n \le |z| \le N\} \setminus (\Gamma \cup \Delta).$$

Let Ω be the connected component of Ω_0 having the circle of radius N as a boundary component. Ω is homeomorphic to an annulus, and for each k, the point $\gamma_k(\tau_k)$ is on its boundary; by construction, the $\gamma_k(\tau_k)$ are intertwined with the (remaining portions of the) rays of angles $2\pi k/j$. We will perform our construction of the $\tilde{\delta}_k$ inside Ω; continuing them with the δ_k outside Ω then produces j disjoint paths satisfying the conditions we need.

Up to homeomorphism, we can now assume without loss of generality that for each k, $\gamma_k(0) = ne^{i\pi(2k-1)/j}$. The only thing we lose in the above reduction is the assumption on the angles of the γ_k; but since it takes at most one turn for each of the γ_k to reach the appropriate argument, we can still assume that the remaining angles are all larger than $4\pi/j$. In particular, each of the γ_k will cross the wedge between angles $2\pi k/j$ and $2\pi(k+1)/j$ in the positive direction before hitting the circle of radius N.

For every $k \le j$, let $\theta_k(t)$ be the continuous determination of the argument of $\gamma_k(t)$ satisfying $\theta_k(0) = (2k-1)\pi/j$, and let

$$\mathcal{T}_k := \left\{t > 0 : \frac{2\pi k}{j} < \theta_k(t) < \frac{2\pi(k+1)}{j}\right\} \quad \text{and} \quad \tilde{\Gamma}_k = \overline{\{\gamma_k(t) : t \in \mathcal{T}_k\}}.$$

We now describe informally the construction of $\tilde{\delta}_k$. Start from the point $ne^{2\pi ik/j}$, and start following δ_k outward, until the first intersection of δ_k with $\tilde{\Gamma}_k$. Then, follow the corresponding connected component of $\tilde{\Gamma}_k$, until intersecting either δ_k or δ_{k+1}; follow that one outward until it intersects either $\tilde{\Gamma}_k$ or the circle of radius N; iterating the construction, one finally obtains a Jordan path joining $ne^{2\pi ik/j}$ to $Ne^{2\pi i(k+1)/j}$, and contained in the union of δ_k, δ_{k+1} and $\tilde{\Gamma}_k$ (see Figure 4).

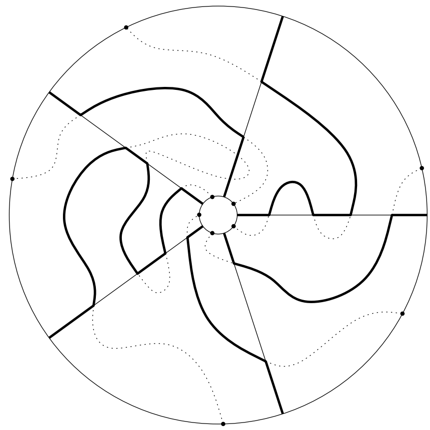

FIG. 4. *The construction of the* $\tilde{\delta}_k$ (*in the case* $j = 5$). *The dotted lines are the paths* γ_k, *and the heavy lines are the* $\tilde{\delta}_k$ *obtained at the end of the construction.*

All that remains is to prove that the $\tilde{\delta}_k$ are indeed disjoint; by symmetry, it is enough to do so for $\tilde{\delta}_1$ and $\tilde{\delta}_2$. Besides, because the γ_k are themselves disjoint, any intersection point between $\tilde{\delta}_1$ and $\tilde{\delta}_2$ has to occur on δ_2 (at least in the case $j > 2$, but the case $j = 2$, where they could also intersect along δ_1, again follows by symmetry).

The intersection of $\tilde{\delta}_1$ with δ_2 consists of a finite collection (I_m) of compact intervals; besides, the points of the intersection are visited by $\tilde{\delta}_1$ in order of increasing distance to the origin. Similarly, the intersection of $\tilde{\delta}_2$ with δ_2 consists of a finite collection (J_l) of compact intervals, which are also visited in order of increasing distance to the origin.

Suppose that $\bigcup I_p$ and $\bigcup J_p$ have a nonempty intersection, and let z_0 be the intersection point lying closest to the origin. Let p_0 and q_0 be such that $z_0 \in I_{p_0} \cap J_{q_0}$; notice that z_0 is the endpoint closest to the origin of either I_{p_0} or J_{q_0}. According to the order in which γ_1 (resp., γ_2) visits the endpoints of I_{p_0} (resp., J_{q_0}), this gives rise to eight possible configurations; it is straightforward in all cases to apply Jordan's theorem to prove that γ_1 and γ_2 then have to intersect, thus leading to a contradiction. $\square$

For the purpose of the proof of Proposition 7, we will need a slight variation of the lemma, where the hypothesis of finiteness of the intersections between paths is replaced with the assumption that the paths considered are all polygonal lines. The proof is exactly the same though, and does not even require any additional notation: whenever two paths, say γ_k and $\delta_{k'}$, coincide along a line segment, the definition of Γ_k amounts to considering some of the endpoints of this segment as intersections, which in other words is equivalent to shifting γ_k by an infinitesimal amount toward the exterior of the wedge used to define $\mathcal{T}_k$ in order to recover finiteness.

PROOF OF PROPOSITION 7. The previous lemma is stated with particular curves on which a surgery can be done, but it can obviously be applied to more general cases through a homeomorphism of the annulus. The general statement is then the following (roughly speaking): assuming the existence of two families of j arms with different enough winding angles, it is possible to produce a third family using the same endpoints as the first one but with a slightly larger winding angle.

We are now ready to prove Proposition 7. Consider a configuration in which one can find two families of crossings, say (λ_k) and (λ'_k), in such a way that for every k, the difference between the winding angles of λ_k and λ'_k is at least 2π. Let α_0 be the minimal angle in the first family, and apply the topological lemma with $\delta_k = \lambda_k$ and $\gamma_k = \lambda'_k$: one obtains a new family of pairwise disjoint paths (λ^1_k), which share the same family of endpoints as the (λ_k), the endpoint of λ^1_k being that of λ_{k+1} (with the obvious convention that $j+1=1$).

One can then iterate the procedure, applying the topological lemma with this time $\delta_k = \lambda^1_k$, and still letting $\gamma_k = \lambda'_k$; one gets a new family (λ^2_k) with the endpoints again shifted amongst the paths in the same direction. Continuing as long as the winding angle difference is at least 2π, this construction produces a sequence (λ^i_k) of j-tuples of disjoint paths, the winding angles of which vary by less than 2π at each step. Besides, the construction ends in finitely many steps, for after j steps, each of the winding angles has increased by exactly 2π. This readily implies our claim. $\square$

REMARK 9. Notice that, as early as the second step of the procedure, (λ^n_k) and (λ'_k) will always coincide on a positive fraction of their length, which is why we needed the above extension of the lemma.

4. Existence of the monochromatic exponents. We now prove Theorem 4, stating the existence of the monochromatic exponents α'_j. For that, we use a rather common argument, presented, for example, in [13]: since the quasi-multiplicativity property holds (item 2 above), it is actually enough to check that there exists a function f_j (which will automatically be sub-multiplicative itself; one can take $C_2 = 1$ in the quasi-multiplicativity property) such that, for every $R > 1$,

$$\mathbb{P}(A_{j,B\dots BB}(n, Rn)) \to f_j(R) \tag{4.1}$$

as $n \to \infty$. Notice that RSW-type estimates provide both the fact that the left-hand term in bounded above and below by constants for fixed R as $n \to \infty$, and a priori estimates on any (potentially subsequential) limit, of the form

$$R^{-1/\varepsilon_j} \le f_j(R) \le R^{-\varepsilon_j},$$

where ε_j depends only on j.

By Menger's theorem (see, e.g., [4]), the complement of the event $A_{j,B\dots BB}(n, N)$ can be written as

$$D_j(n, N) = \{\text{There exists a circuit in } S_{n,N} \text{ that surrounds } \partial^i S_n \\ \text{and contains at most } j-1 \text{ black sites}\}.$$

This makes it possible to express the event $A_{j,B\dots BB}(n, N)$ in terms of the collection of all cluster interfaces (or "loops"): it is just the event that there does not exist a "necklace" of at most $(j-1)$ loops, with white vertices on their inner boundary and black ones on their outer boundary, forming a chain around $\partial^i S_n$ and such that two consecutive loops are separated by only one black site.

Standard arguments show that the probability that two interfaces touch in the scaling limit is exactly the asymptotic probability that they "almost touch" (in the sense that they are separated by exactly one vertex) on discrete lattices; it is, for example, a simple consequence of the fact that the polychromatic six-arm exponent is strictly larger than 2, which in turn is a consequence of RSW-type estimates (the fact that the polychromatic five-arm exponent is equal to 2 being true on any lattice on which RSW holds, at least for colors $BWBWW$).

What this means, is that to show convergence of the probability in (4.1), it is enough to know the probability of the corresponding continuous event. While we do not know the exact value of the limit, it is nevertheless easy to check that the event itself is measurable with respect to the *full scaling limit* of percolation, as constructed by Camia and Newman in [3], and that is enough for our purpose. Notice that the measurability of the event in terms of the full scaling limit is ensured by the exploration procedure described in that paper: it is proved there that for every $\varepsilon > 0$, all loops of diameter at least ε (with the proper orientation) are discovered after finitely many steps of the exploration procedure.

REMARK 10. Proving the existence of the exponents only requires the existence of the function f_j, but it is easily seen that in fact the value of α'_j describes the power law decay of $f_j(R)$ as R goes to infinity. However, deriving the value of the exponent directly from the full scaling limit seems to be difficult, and we were not able to do it.

Acknowledgments. We are indebted to J. van den Berg for pointing out inequality (2.6) to us. We are also very grateful to W. Werner for many stimulating discussions, and to an anonymous referee for many insightful remarks about the first version of this paper. Part of this research was realized during a semester (resp., year) spent by Pierre Nolin (resp., Vincent Beffara) at Université de Genève, and both authors would like to thank the mathematics department there for its hospitality, in particular Stanislav Smirnov.

REFERENCES

[1] AIZENMAN, M., DUPLANTIER, B. and AHARONY, A. (1999). Path crossing exponents and the external perimeter in 2D percolation. *Phys. Rev. Lett.* **83** 1359–1362.

[2] BORGS, C., CHAYES, J. and RANDALL, D. (1999). The van den Berg–Kesten–Reimer inequality: A review. In *Perpexing Problems in Probability*: *Festschrifft in honor of Harry Kesten*. Birkhäuser, Boston, MA. MR1703130

[3] CAMIA, F. and NEWMAN, C. M. (2005). Two-dimensional critical percolation: The full scaling limit. *Comm. Math. Phys.* **268** 1–38. MR2249794

[4] DIESTEL, R. (2000). *Graph Theory*, 2nd ed. Springer, New York. MR1743598

[5] GRIMMETT, G. R. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339

[6] KESTEN, H. (1987). Scaling relations for 2D-percolation. *Comm. Math. Phys.* **109** 109–156. MR879034

[7] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2001). Values of Brownian intersection exponents I: Half-plane exponents. *Acta Math.* **187** 237–273. MR1879850

[8] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2001). Values of Brownian intersection exponents II: Plane exponents. *Acta Math.* **187** 275–308. MR1879851

[9] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2002). One-arm exponent for critical 2D percolation. *Electron. J. Probab.* **7** 13 pp. (electronic). MR1887622

[10] REIMER, D. (2000). Proof of the van den Berg–Kesten conjecture. *Combin. Probab. Comput.* **9** 27–32. MR1751301

[11] SCHRAMM, O. (2000). Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118** 221–288. MR1776084

[12] SMIRNOV, S. (2001). Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.* **333** 239–244. MR1851632

[13] SMIRNOV, S. and WERNER, W. (2001). Critical exponents for two-dimensional percolation. *Math. Res. Lett.* **8** 729–744. MR1879816

[14] TALAGRAND, M. (1994). Some remarks on the Berg–Kesten inequality. In *Probability in Banach Spaces*, 9 (*Sandjberg*, 1993). *Progress in Probability* **35** 293–297. Birkhäuser, Boston, MA. MR1308525

[15] VAN DEN BERG, J. and KESTEN, H. (1985). Inequalities with applications to percolation and reliability. *J. Appl. Probab.* **22** 556–569. MR799280

[16] WERNER, W. (2009). Lectures on two-dimensional critical percolation. In *Statistical Mechanics*. *IAS/Park City Math. Ser.* **16** 297–360. Amer. Math. Soc., Providence, RI. MR2523462

UMPA—ENS LYON
46, ALLÉE D'ITALIE
F-69364 LYON CEDEX 07
FRANCE
E-MAIL: Vincent.Beffara@ens-lyon.fr

COURANT INSTITUTE OF
MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
251 MERCER STREET
NEW YORK, NEW YORK 10012
USA
E-MAIL: nolin@cims.nyu.edu