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QUENCHED SCALING LIMITS OF TRAP MODELS

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In this paper, we study Bouchaud's trap model on the discrete d-dimensional torus $\mathbb{T}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$. In this process, a particle performs a symmetric simple random walk, which waits at the site $x \in \mathbb{T}_n^d$ an exponential time with mean ξ_x, where $\{\xi_x, x \in \mathbb{T}_n^d\}$ is a realization of an i.i.d. sequence of positive random variables with an α-stable law. Intuitively speaking, the value of ξ_x gives the depth of the trap at x. In dimension $d = 1$, we prove that a system of independent particles with the dynamics described above has a hydrodynamic limit, which is given by the degenerate diffusion equation introduced in [*Ann. Probab.* **30** (2002) 579–604]. In dimensions $d > 1$, we prove that the evolution of a single particle is metastable in the sense of Beltrán and Landim [Tunneling and Metastability of continuous time Markov chains (2009) Preprint]. Moreover, we prove that in the ergodic scaling, the limiting process is given by the K-process, introduced by Fontes and Mathieu in [*Ann. Probab.* **36** (2008) 1322–1358].

1. Introduction. Scaling limits of random walks in random trap environments have been examined recently [5, 6, 11] as stochastic models which exhibit aging [3, 7, 11, 18], a phenomenon of considerable interest in physics and mathematics.

To describe the dynamics, fix an unoriented graph $G = (V, E)$ with finite degree and consider a sequence of i.i.d. strictly positive random variables $\{\xi_z : z \in V\}$ indexed by the vertices. Let $\{X_t : t \geq 0\}$ be a continuous-time random walk on V which waits a mean ξ_z exponential time at site z, at the end of which it jumps to one of its neighbors with uniform probability.

The time spent by the random walk at a vertex z is proportional to the value of ξ_z. It is thus natural to regard the environment as a landscape of valleys or traps with depth given by the value of the random variables $\{\xi_z : z \in V\}$. As the random walk evolves, it explores the random landscape, finding deeper and deeper traps, and aging appears as a consequence of the longer and longer times the process remains at the same vertex.

It is clear from the description that random walks on random trap environments should present a very rich scaling fractal structure if one chooses appropriate graphs and random environments. For each given time scale, only traps at a

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certain depth matter. The deeper valleys are too sparse to influence the evolution and the shallower wells are not deep enough to retain the process.

We are concerned in this article with the lattice case: $\{\xi_z : z \in \mathbb{Z}^d\}$ is a sequence of i.i.d. strictly positive random variables and $\{X_t : t \geq 0\}$ a continuous-time random walk on $\mathbb{Z}^d$ which waits a mean ξ_z exponential time at site z, at the end of which it jumps to one of its neighbors with probability $1/2d$.

When ξ_0 has finite mean, for almost all environments $\{\xi_z : z \in \mathbb{Z}^d\}$, the rescaled random walk $\epsilon X_{t\epsilon^{-2}}$ converges in distribution to a Brownian motion. In dimension 1, we can use the method of random time change to study the problem explicitly and a simple computation establishes the result [20]. In this case, the diffusion coefficient is equal to $E[\xi_0]^{-1}$, the harmonic mean of the random rates $\{\xi_z^{-1} : z \in \mathbb{Z}^d\}$. Observing that the random walk is a martingale, in higher dimensions, by examining the evolution of the environment as seen from the position of the random walk, the proof of the invariance principle is reduced to the proof of an ergodic theorem for the dynamics of the environment [19]. An explicit formula for the variance is, however, no longer available.

To investigate the case where the environment has an infinite mean, a natural assumption is to suppose that the distribution of ξ_0 belongs to the domain of attraction of an α-stable law, $0 < \alpha < 1$. The variables $\{\xi_z : z \in \mathbb{Z}^d\}$ now take large values in certain sites, forcing the random walk to stay still for a long time when it reaches one of them, causing a macroscopic subdiffusive behavior.

In dimension 1, Fontes, Isopi and Newman [11] proved, under these hypotheses, that for almost all environments, the random walk converges, in the time scale $t^{1+(1/\alpha)}$, to a singular diffusion with a random discrete speed measure. In dimension $d \geq 2$, Ben Arous and Černý [5] proved that for almost all environments, the Bouchaud trap model converges, in a proper time scale $t^{2/\alpha}$ in dimension $d \geq 3$ and a logarithmic scale smaller than $t^{2/\alpha}$ in dimension 2, to the fractional-kinetic process, a self-similar, non-Markovian, continuous process, obtained as the time change of a Brownian motion by the inverse of an independent α-stable subordinator. In fact, they proved, under quite general conditions on the environment, that the clock process converges to an α-stable subordinator for a large range of time scales [6]. In these time scales, the random walk does not visit the deepest traps, but exhibits an aging behavior. During the exploration of the random scenery, the process discovers deeper and deeper traps which slow down its evolution, the mechanism responsible for the aging phenomenon. We refer to [4, 9] for recent reviews.

In this article, we present two results. The first establishes the hydrodynamic behavior, almost sure with respect to the environment, of a superposition of independent random walks evolving on the one-dimensional torus with a trap environment of α-stable i.i.d. random variables. The hydrodynamic equation, describing the macroscopic evolution of the density, is given by the generalized second order linear equation

$$\frac{d}{dt}\rho(t,x) = \frac{d}{dW}\frac{d}{dx}\rho(t,x),$$

where W is an α-stable subordinator deriving from the realization of the environment. The Krein–Feller operator $(d/dW)(d/dx)$ is the generator of the singular diffusion obtained by Fontes, Isopi and Newman [11] as scaling limit of the random walk in the trap environment.

The striking feature of this result is that the random environment survives entirely in the limit since even the differential operator, which describes the macroscopic evolution of the density, depends on the specific realization of the environment. A similar phenomenon was observed in [10, 14, 21, 22] for exclusion processes with α-stable random conductances.

The second result describes the evolution of the random walk in the random environment, produced by α-stable i.i.d. random variables in dimension $d \geq 2$, in the time scale needed to visit the deepest traps. Using the notation of Theorem 4.1 in [6], this corresponds to the case $\gamma = 0$.

In dimension 2, on the time scale $N^{2/\alpha} \log N$, we prove that the random walk, evolving on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^2$, converges to the Markov K-process introduced by Fontes and Mathieu [13], which, in the present context, can be informally described as follows. The state space is formed by the countable and dense subset of deepest traps. The process stays at one of these sites an exponential time, with expectation proportional to the depth of the trap, at the end of which it jumps to a new location, chosen with uniform probability among the deepest traps. The scaling limit is similar in dimension $d \geq 3$, but the time scale is now $N^{d/\alpha}$. In the terminology of [2], these results establish the tunneling behavior of the random walk in dimension $d \geq 2$.

Convergence to the K-process has been proven by Fontes and Mathieu [13] for the trap model in the complete graph and by Fontes and Lima [12] for the trap model in the hypercube. We believe that this is a universal behavior of random walks on graphs with heavy-tailed random trap environments in the ergodic time scale, the scale proportional to the time needed to jump from one very deep trap to another (at least in sufficiently high dimension).

It is, in fact, quite surprising that, even in low dimensions, the geometry of the torus is completely wiped out in the scaling limit of the random walk in a random trap environment, as proved below.

We conclude this introduction by specifying the random environment which we consider in this article. Although we shall work on the torus, we present the construction on $\mathbb{R}^d$. Let λ be the measure on $\mathbb{R}^d \times (0, \infty)$ given by $\lambda = \alpha w^{-(1+\alpha)}\, dx\, dw$, $0 < \alpha < 1$, where dx (resp. dw) denotes the Lebesgue measure on $\mathbb{R}^d$ [resp. $(0, \infty)$]. Denote by $\{(\mathbf{x}_i, w_i) \in \mathbb{R}^d \times (0, \infty) : i \geq 1\}$ the marks of a Poisson point process of intensity λ and define the measure W on $\mathbb{R}^d$ by

$$W = \sum_{i \geq 1} w_i \delta_{\mathbf{x}_i}.$$

For $z = (z_1, \dots, z_d)$ in $\mathbb{Z}^d$, let $[z/N, [z+\mathbf{1}]/N)$ be the d-dimensional cube $\prod_{1 \le i \le d} [z_i/N, [z_i + 1]/N)$ and let

$$\xi_z^N = N^{d/\alpha} \sum_{i \ge 1} w_i \mathbf{1}\{\mathbf{x}_i \in [z/N, (z+\mathbf{1})/N)\},$$

where $\mathbf{1}\{A\}$ stands for the indicator of the set A. In the next section, we show that for each $N \ge 1$, $\{\xi_z^N : z \in \mathbb{Z}^d\}$ are i.i.d. random variables with a common α-stable distribution, independent of N. Following [11], Section 3, we may refine this construction to obtain i.i.d. random variables distributed according to any law in the domain of attraction of an α-stable law.

Taking the array $\{\xi_z^N : z \in \mathbb{Z}^d\}$, $N \ge 1$, as our environment, instead of a sequence $\{\xi_z : z \in \mathbb{Z}^d\}$ of i.i.d. random variables in the domain of attraction of an α-stable law, as it is usually done, produces noticeable differences in the scaling limit, the main one being the survival of the measure W.

2. Notation and results. Fix a finite, strictly positive measure W on the d-dimensional torus $\mathbb{T}^d$:

$$W(A) > 0 \qquad \text{for any open set } A. \tag{2.1}$$

Denote by $\mathbb{T}_N^d$ the d-dimensional, discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$. Let W_x^N, $x \in \mathbb{T}_N^d$, be the W-measure of the d-dimensional cube $[x/N, (x+\mathbf{1})/N)$, where $\mathbf{1}$ is the vector with all components equal to 1: $\mathbf{1} = (1, \dots, 1)$:

$$W_x^N = W\{[x/N, (x+\mathbf{1})/N)\}. \tag{2.2}$$

In this article, we examine the evolution of a continuous-time, nearest-neighbor, symmetric random walk on $\mathbb{T}_N^d$ which waits a mean W_x^N exponential time at site x. Its generator $\mathcal{L}_N$ is given by

$$(\mathcal{L}_N f)(x) = \frac{1}{2d} \frac{1}{W_x^N} \sum_{y \sim x} [f(y) - f(x)] \tag{2.3}$$

for every $f : \mathbb{T}_N^d \to \mathbb{R}$, where $(y_1, \dots, y_d) = y \sim x = (x_1, \dots, x_d)$ if $|y - x| = \sum_{1 \le i \le d} |x_i - y_i| = 1$.

2.1. *Hydrodynamic limit in dimension* 1. Consider a finite number of random walks evolving independently on $\mathbb{T}_N$ according to the dynamics defined by the generator $\mathcal{L}_N$. Let $\mathbb{N}_0$ be the nonnegative integers: $\mathbb{N}_0 = \{0, 1, \dots\}$. Denote by $\Omega_N = \mathbb{N}_0^{\mathbb{T}_N}$ the state space of the process and by η the configurations of Ω_N so that $\eta(x)$, $x \in \mathbb{T}_N$, represents the number of particles at site x for the configuration η.

This evolution corresponds to a Markov process on Ω_N whose generator L_N is given by

$$(L_N f)(\eta) = \frac{1}{2} \sum_{x \in \mathbb{T}_N} \sum_{y \sim x} \frac{\eta(x)}{N W_x^N} [f(\eta^{x,y}) - f(\eta)],$$

where $f : \Omega_N \to \mathbb{R}$ is a bounded function and $\eta^{x,y}$ stands for the configuration obtained from η by moving a particle from site x to site y:

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1, & z = x, \\ \eta(y) + 1, & z = y, \\ \eta(z), & z \neq x, y. \end{cases}$$

Note that we have slowed down the dynamics by a factor N. We did that in order to have a jump rate NW_x^N of order one if the measure W is absolutely continuous with respect to the Lebesgue measure in a neighborhood of x/N. Indeed, in this case, if we denote by w the Radon–Nikodym derivative of W, $NW_x^N = N \int_{[x/N,(x+1)/N)} w(y)\,dy$ is of order one. In contrast, if W has a point mass at x/N, then NW_x^N is of order N, which means that particles wait exponential times of order N at sites where W has point masses. Particles are thus trapped at these sites.

Denote by $\{\eta_t : t \geq 0\}$ the Markov process with generator L_N *speeded up* by N^2. Let $D(\mathbb{R}_+, \Omega_N)$ be the space of right-continuous trajectories $\xi : \mathbb{R}_+ \to \Omega_N$ with left limits, endowed with the Skorokhod topology. For a measure μ on Ω_N, let $\mathbb{P}_\mu$ be the probability measure on $D(\mathbb{R}_+, \Omega_N)$ induced by the Markov process $\{\eta_t : t \geq 0\}$ starting from μ.

For $\rho \geq 0$, let $\mathfrak{P}_\rho$ be the Poisson probability distribution with parameter ρ in $\mathbb{N}_0$: $\mathfrak{P}_\rho\{k\} = e^{-\rho}\rho^k/k!$, $k \geq 0$. Denote by ν_ρ^N the product measure on Ω_N with marginals defined by

$$\nu_\rho^N\{\eta : \eta(x) = k\} = \mathfrak{P}_{\rho W_x^N}\{k\}, \qquad x \in \mathbb{T}_N,\ k \geq 0. \tag{2.4}$$

It is not hard to see that the measures ν_ρ^N are invariant and reversible for the generator L_N.

Let $\mathcal{M}(\mathbb{T})$ be the space of finite positive measures on the torus $\mathbb{T}$, endowed with the weak topology. Fix $\gamma > 0$ and denote by $\pi^N = \pi^N(\eta) \in \mathcal{M}(\mathbb{T})$ the measure obtained from a configuration η by assigning mass $N^{-\gamma}$ to each particle:

$$\pi^N = \frac{1}{N^\gamma} \sum_{x \in \mathbb{T}_N} \eta(x) \delta_{x/N}, \tag{2.5}$$

where $\delta_{x/N}$ stands for the Dirac measure at x/N. For a continuous function $H : \mathbb{T} \to \mathbb{R}$, denote by $\langle \pi^N, H \rangle$ the integral of H with respect to π^N so that

$$\langle \pi^N, H \rangle = \frac{1}{N^\gamma} \sum_{x \in \mathbb{T}_N} H(x/N)\eta(x).$$

Fix a continuous function $u_0 : \mathbb{T} \to \mathbb{R}_+$ and denote by $\mu_{u_0(\cdot)}^N$ the product measure on Ω_N with marginals given by

$$\mu_{u_0(\cdot)}^N\{\eta : \eta(x) = k\} = \mathfrak{P}_{u_0(x/N)N^\gamma W_x^N}\{k\}, \qquad x \in \mathbb{T}_N,\ k \geq 0. \tag{2.6}$$

When u_0 is a constant function equal to ρ, we write $\mu^N_{u_0(\cdot)}$ simply as $\mu^N_\rho = \mu_\rho$. Thus, under $\mu^N_{u_0(\cdot)}$, $\eta(x)$ has a Poisson distribution with parameter $u_0(x/N)W^N_x N^\gamma$.

An elementary computation shows that $\langle \pi^N, H\rangle$ converges to $\int H(x)u_0(x) \times W(dx)$ in $L^2(\mu^N_{u_0(\cdot)})$ for every continuous function H:

$$\lim_{N\to\infty} E_{\mu^N_{u_0(\cdot)}}\left[\left(\langle \pi^N, H\rangle - \int_{\mathbb{T}} H(x)u_0(x)W(dx)\right)^2\right] = 0.$$

The hydrodynamic equation. Let $\mathcal{H}_1$ be the Sobolev space of all functions in $L^2(\mathbb{T})$ with generalized derivative in $L^2(\mathbb{T})$ endowed with the scalar product $\langle \cdot, \cdot\rangle_{1,2}$ defined by

$$\langle f, g\rangle_{1,2} = \langle f, g\rangle + \int_{\mathbb{T}} (\partial_x f)(x)(\partial_x g)(x)\, dx,$$

where $\langle \cdot, \cdot\rangle$ stands for the usual scalar product of $L^2(\mathbb{T})$. It is well known that the space of functions with continuous derivatives of all orders is dense in $\mathcal{H}_1$. Moreover, any function in $\mathcal{H}_1$ has a continuous version.

Denote by $L^2(dW)$ the Hilbert space associated to the measure $W(dx)$ and by $\langle f, g\rangle_W$ the corresponding inner product.

DEFINITION 2.1. A bounded measurable function $u:[0,T]\times\mathbb{T}\to\mathbb{R}$ is a weak solution of

$$\begin{cases} \dfrac{d}{dt}u = \dfrac{1}{2}\dfrac{d}{dW}\dfrac{d}{dx}u, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{2.7}$$

if:

(i) it has finite energy, that is,

$$\int_0^T \langle u_t, u_t\rangle_{1,2}\, dt < \infty;$$

(ii) for any smooth function $G:[0,T]\times\mathbb{T}\to\mathbb{R}$ vanishing at T, $G_T = 0$,

$$\langle G_0, u_0\rangle_W + \int_0^T \langle \partial_t G_t, u_t\rangle_W\, dt = \frac{1}{2}\int_0^T \langle \partial_x G_t, \partial_x u_t\rangle\, dt.$$

At the end of Section 5, we prove that there is at most one weak solution of (2.7). Denote by π^N_t, $t \ge 0$, the empirical measure associated to the state of the process at time t,

$$\pi^N_t = \frac{1}{N^\gamma}\sum_{x\in\mathbb{T}_N} \eta_t(x)\delta_{x/N},$$

and recall that time has been speeded up by N^2.

THEOREM 2.2. *Let W be a finite, positive measure on $\mathbb{T}$ satisfying* (2.1). *Assume that there exists $\gamma_0 > 0$ such that*

$$\lim_{N\to\infty} \frac{1}{N^{2+\gamma_0}} \sum_{x\in\mathbb{T}_N} \frac{1}{W_x^N} = 0. \tag{H1}$$

Fix $\gamma \geq \gamma_0$. Then, for every $t \geq 0$, every continuous function $H : \mathbb{T} \to \mathbb{R}$ and every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N_{u_0(\cdot)}} \left[\left| \langle \pi_t^N, H\rangle - \int_{\mathbb{T}} H(x) u(t,x) W(dx) \right| > \delta \right] = 0,$$

where u is the unique weak solution of (2.7).

If the measure W is absolutely continuous with respect to the Lebesgue measure, and its Radon–Nikodym derivative, denoted by $w(x)$, is strictly positive, $w > 0$ a.s., then the previous theorem states that the empirical measure π_t^N converges to the measure $\pi(t, dx) = u(t,x)w(x)\,dx$, whose density u is a solution of

$$\begin{cases} \partial_t u = (1/2) w^{-1} \Delta u, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

The proof of the hydrodynamic behavior of the empirical measure differs substantially from the usual ones due to the spatial irregularity of the environment. The lack of smoothness is reflected in the dynamics by an erratic time evolution. To overcome this issue, we average not only in space, but also in time, investigating the asymptotic behavior of the measure $\mathfrak{M}^N$ on $[0, T] \times \mathbb{T}$, defined by

$$\mathfrak{M}^N = \frac{1}{N^{1+\gamma}} \sum_{x\in\mathbb{T}_N} \frac{\eta_t(x)}{W_x^N} \delta_{x/N}\, dt,$$

which does not capture space and time discontinuities.

2.2. *Metastable behavior of the trap model in dimension $d \geq 2$.* Fix a finite, strictly positive, atomic measure W on the d-dimensional torus $\mathbb{T}^d$:

$$W = \sum_{i\geq 1} w_i \delta_{x_i},$$

where $\{x_i : i \geq 1\}$ is a dense subset of $\mathbb{T}^d$, $\sum_{i\geq 1} w_i < \infty$ and $w_i \neq w_j$ for $i \neq j$.

Denote by $\{\hat{w}_i : i \geq 1\}$ the weights of W in decreasing order so that $\{\hat{w}_i : i \geq 1\} = \{w_i : i \geq 1\}$ and $\hat{w}_1 > \hat{w}_2 > \cdots$. Let $\{\hat{x}_i : i \geq 1\}$ be the positions of the atoms of W corresponding to the weights $\{\hat{w}_i : i \geq 1\}$:

$$W = \sum_{i\geq 1} w_i \delta_{x_i} = \sum_{i\geq 1} \hat{w}_i \delta_{\hat{x}_i}.$$

Recall the definition of W_x^N given in (2.2). Denote by $\{X_t^N : t \geq 0\}$ the random walk on $\mathbb{T}_N^d$ with generator $\mathcal{L}_N$. Let $D(\mathbb{R}_+, \mathbb{T}_N^d)$ be the path space of right-continuous trajectories $\omega : \mathbb{R}_+ \to \mathbb{T}_N^d$ with left limits endowed with the Skorokhod topology. Denote by $\mathbb{P}_x^N$, $x \in \mathbb{T}_N^d$, the probability measure on $D(\mathbb{R}_+, \mathbb{T}_N^d)$ induced by the Markov process $\{X_t^N\}$ starting from x. Expectation with respect to $\mathbb{P}_x^N$ is denoted by $\mathbb{E}_x^N$.

Denote by ν^N the unique stationary state of the process $\{X_t^N : t \geq 0\}$. An elementary computation shows that ν^N is, in fact, reversible and given by

$$\nu^N(x) = \frac{1}{W(\mathbb{T}^d)} W_x^N.$$

Enumerate $\mathbb{T}_N^d$ according to the weights $\{W_x^N\}$ in decreasing order:

$$\mathbb{T}_N^d = \{x_1^N, x_2^N, \dots, x_{N^d}^N\}, \qquad W_{x_1^N}^N \geq W_{x_2^N}^N \geq \cdots \geq W_{x_{N^d}^N}^N.$$

In the case of ties, we choose the smallest site according to some pre-established order. Following [5], we call the sites x_j^N, j fixed, the *very deep* traps. These are the relevant states of the trap random walk on the scale observed here.

Since $W(\mathbb{T}^d)$ is finite, we may assume that for every $M > 0$, there exists N_0 such that $\hat{x}_j \in [x_j^N/N - (1/2N)\mathbf{1}, x_j^N/N + (1/2N)\mathbf{1}]$, $1 \leq j \leq M$, for all $N \geq N_0$.

To define the trace of the process $\{X_t^N : t \geq 0\}$ on a subset F of $\mathbb{T}_N^d$, let $\mathcal{T}_N^F(t)$, $t \geq 0$, $F \subset \mathbb{T}_N^d$, be the time the process remains in the set F in the interval $[0, t]$,

$$\mathcal{T}_N^F(t) := \int_0^t \mathbf{1}\{X_s^N \in F\}\, ds,$$

and let $\mathcal{S}_N^F(t)$ be the generalized inverse of $\mathcal{T}_N^F(t)$,

$$\mathcal{S}_N^F(t) := \sup\{s \geq 0 : \mathcal{T}_N^F(s) \leq t\}.$$

It is well known that the process $\{X_t^{N,F} : t \geq 0\}$ defined by

$$X_t^{N,F} = X^N(\mathcal{S}_N^F(t))$$

is a Markov process with state space F, called the *trace* of $\{X_t^N\}$ on F.

Let $\{\mathbb{Y}_k : k \geq 0\}$ be the d-dimensional, nearest-neighbor, symmetric, discrete-time random walk on $\mathbb{Z}^d$ starting from the origin. For $d \geq 3$, denote by v_d the probability that $\{\mathbb{Y}_k\}$ never returns to the origin:

$$v_d = \mathbb{P}_0[\mathbb{Y}_k \neq 0 \text{ for all } k \geq 1].$$

Let $A_M^N = \{x_1^N, \dots, x_M^N\}$, $1 \leq M \leq N^d$, and denote by $\{\hat{X}_t^{N,M} : t \geq 0\}$ the trace of the process $\{X_t^N : t \geq 0\}$ on the set A_M^N. The second main result of this article states that in dimension $d \geq 3$, the trace process $\{\hat{X}_t^{N,M}\}$ converges, as $N \uparrow \infty$, to the random walk on $\{\hat{x}_1, \dots, \hat{x}_M\}$ which waits a mean $\hat{w}_j / v_d$ exponential time at

$\hat{x}_j$ and then jumps to $\{\hat{x}_i : 1 \leq i \leq M\}$ with uniform probability. Note that we do not rule out the possibility that the process jumps back to the site where it was. To state the result, let $\Psi_N : \mathbb{T}_N^d \to \mathbb{N}$ be defined by $\Psi_N(x_j^N) = j$ and let $X_t^{N,M} = \Psi_N(\hat{X}_t^{N,M})$. Clearly, $\{X_t^{N,M} : t \geq 0\}$ is a Markov process on $\{1, \dots, M\}$.

THEOREM 2.3. *Fix $T > 0$ and assume that $d \geq 3$. As $N \uparrow \infty$, the law of $\{X_t^{N,M} : 0 \leq t \leq T\}$ converges in distribution to a random walk in $\{1, \dots, M\}$ with generator $\mathfrak{L}_M$ given by*

$$(\mathfrak{L}_M f)(i) = \frac{v_d}{M \hat{w}_i} \sum_{j=1}^{M} [f(j) - f(i)].$$

Moreover,

$$\lim_{M \to \infty} \limsup_{N \to \infty} \max_{1 \leq j \leq M} \mathbb{E}_{x_j^N}^N [\mathcal{T}_N^{\Delta_{N,M}}(T)] = 0,$$

where $\Delta_{N,M} = \mathbb{T}_N^d \setminus A_M^N$.

In dimension 2, the picture is similar, but the process needs to be *speeded up* by $\log N$. Denote by $\{\mathfrak{X}_t^N : t \geq 0\}$ the random walk on $\mathbb{T}_N^2$ with generator $(\log N)\mathcal{L}_N$, where $\mathcal{L}_N$ has been introduced in (2.3). Hence, $\{\mathfrak{X}_t^N : t \geq 0\}$ has the same distribution as $\{X_{t \log N}^N : t \geq 0\}$.

Denote by $\mathbf{P}_x^N$, $x \in \mathbb{T}_N^2$, the probability measure on $D(\mathbb{R}_+, \mathbb{T}_N^2)$ induced by the Markov process $\{\mathfrak{X}_t^N : t \geq 0\}$ starting from x. Expectation with respect to $\mathbf{P}_x^N$ is denoted by $\mathbf{E}_x^N$. Denote by $\{\hat{\mathfrak{X}}_t^{N,M} : t \geq 0\}$ the trace of the process $\{\mathfrak{X}_t^N : t \geq 0\}$ on the set A_M^N and let $\mathfrak{X}_t^{N,M} = \Psi_N(\hat{\mathfrak{X}}_t^{N,M})$.

THEOREM 2.4. *Fix $T > 0$ and assume that $d = 2$. As $M \uparrow \infty$, the law of $\{\mathfrak{X}_t^{N,M} : 0 \leq t \leq T\}$ converges in distribution to a random walk in $\{1, \dots, M\}$ with generator $\mathfrak{L}_M^\star$ given by*

$$(\mathfrak{L}_M^\star f)(i) = \frac{\pi}{2} \frac{1}{M \hat{w}_i} \sum_{j=1}^{M} [f(j) - f(i)].$$

Moreover, if we denote by $\mathfrak{T}_N^{\Delta_{N,M}}(T)$ the time spent by the process $\mathfrak{X}_t^N$ in the set $\Delta_{N,M}$ on the time interval $[0, t]$, then

$$\lim_{M \to \infty} \limsup_{N \to \infty} \max_{1 \leq j \leq M} \mathbf{E}_{x_j^N}^N [\mathfrak{T}_N^{\Delta_{N,M}}(T)] = 0.$$

We prove in Proposition 6.19 that in dimension 2, the random walk $\{X_t^N : t \geq 0\}$ with generator $\mathcal{L}_N$ does not leave a very deep trap, staying there indefinitely.

Therefore, on time scales of order 1, the random walk does not move and on scales of order $\log N$, the geometry is wiped out and the random walk jumps from a very deep trap to another one, chosen with uniform probability.

We now give a characterization of the K-process which is a Markov process on $\overline{\mathbb{N}}$, the one-point compactification of $\mathbb{N}$. Following [13], Theorem 4.1, the K-process with parameters $\{\gamma_i > 0 : i \geq 1\}$ such that $\sum_{i\geq 1} \gamma_i < +\infty$ and $c = 0$ is the unique strong Markov process Z_t in $D(\mathbb{R}_+, \overline{\mathbb{N}})$ such that:

(i) Z_t stays in $i \in \mathbb{N}$ an exponential time with mean γ_i;
(ii) starting from ∞, the hitting time of any nonempty finite set $A \subset \mathbb{N}$ is almost surely finite and Z_t enters A at a point which is chosen uniformly at random;
(iii) the total time spent at ∞ by Z_t is almost surely zero.

Denote by $\{Z_t^M : t \geq 0\}$ the Markov process with generator $\mathfrak{L}_M$. Fontes and Mathieu [13], Lemma 3.11, proved that the process Z_t^M converges, as $M \uparrow \infty$, to the K-process with parameters $c = 0$ and $\{\hat{w}_i / v_d : i \geq 1\}$. The next result follows from this fact and from Theorem 2.3.

THEOREM 2.5. *Fix $T > 0$ and assume that $d \geq 3$. There exists a sequence $\{\ell_N^* : N \geq 1\}$, $\ell_N^* \uparrow \infty$, such that for any sequence $\{\ell_N : N \geq 1\}$, $\ell_N \leq \ell_N^*$, $\ell_N \uparrow \infty$, the law of $\{X_t^{N,\ell_N} : 0 \leq t \leq T\}$ converges in distribution to the K-process with parameters $\{\hat{w}_i / v_d : i \geq 1\}$ and $c = 0$. Moreover,*

$$\lim_{N\to\infty} \max_{1\leq j\leq \ell_N} \mathbb{E}^N_{x_j^N}[\mathcal{T}_N^{\Delta_{N,\ell_N}}(T)] = 0.$$

Of course, a similar statement holds in dimension 2, that is, for $\mathfrak{L}_M^\star$ in place of $\mathfrak{L}_M$. In the terminology of Definition 2.2 in [2], Theorem 2.5 states that in dimension $d \geq 3$, the trap random walk $\{X_t^N : t \geq 0\}$ exhibits a tunneling behavior with metastates $\{\hat{x}_1, \hat{x}_2, \ldots\}$ and limit given by the K-process with parameters $\{\hat{w}_i / v_d : i \geq 1\}$ and $c = 0$. Analogously, in dimension 2, the trap random walk $\{\mathfrak{X}_t^N : t \geq 0\}$ exhibits a tunneling behavior with metastates $\{\hat{x}_1, \hat{x}_2, \ldots\}$ and limit given by the K-process with parameters $\{2\hat{w}_i / \pi : i \geq 1\}$ and $c = 0$.

2.3. *Bouchaud's trap model.* In this subsection, we present an example of a random measure W which satisfies almost surely assumption (H1). Fix $0 < \alpha < 1$ and let λ be the measure on $\mathbb{T}^d \times (0, \infty)$ given by $\lambda = \alpha w^{-(1+\alpha)}\, dx\, dw$. Since λ is a positive Radon measure, the Poisson point process Γ of intensity λ is well defined. Let $\{(x_i, w_i) : i \geq 1\}$ be the Poisson marks and define the measure W by

$$W = \sum_{i\geq 1} w_i \delta_{x_i}.$$

Note that $W(\mathbb{T}^d)$ is a.s. finite. On one hand, the random variable $\Gamma(\mathbb{T}^d \times (1, \infty))$ has finite mean which implies that there are only a finite number of Poisson marks on $\mathbb{T}^d \times [1, \infty)$. On the other hand, $\sum_{i \geq 1} w_i \mathbf{1}\{w_i \leq 1\}$ has finite expectation. Note, also, that $W(A)$, $W(B)$ are independent if A and B are disjoint.

Denote by $|A|$ the Lebesgue measure of a measurable set $A \subseteq \mathbb{T}^d$. A simple computation shows that the random variable $W(A)$ has an α-stable distribution for any A with $|A| > 0$. In particular, the random measure W is self-similar with index α/d, in the sense that the distributions of $W(\beta A)$ and $\beta^{d/\alpha} W(A)$ are the same for any $\beta \in (0, 1)$ and any measurable set $A \subseteq \mathbb{T}^d$.

We call the random measure $W(dx)$ a *d-dimensional subordinator of index α*. For $x \in \mathbb{T}_N^d$, we define

$$\tau_x^N = N^{d/\alpha} W_x^N.$$

Since $W(dx)$ is self-similar with index α/d, $\{\tau_x^N; x \in \mathbb{T}_N^d\}$ is a sequence of i.i.d. random variables with common α-stable distribution $\zeta = W(\mathbb{T}^d)$ which does not depend on N.

The Fontes–Isopi–Newman version of the Bouchaud trap model is the symmetric, nearest-neighbor, continuous-time random walk on $\mathbb{T}_N^d$ with generator $\mathcal{L}_N$, in which W_x^N is replaced by $\tau_x^N = N^{d/\alpha} W_x^N$:

$$(\mathcal{L}_N^\tau f)(x) = \frac{1}{2d} \frac{1}{\tau_x^N} \sum_{y \sim x} [f(y) - f(x)].$$

In dimension 1, the generator on Ω_N corresponding to the superposition of independent random walks is given by

$$(L_N^\tau f)(\eta) = \frac{1}{2} \sum_{x \in \mathbb{T}_N} \sum_{y \sim x} \frac{\eta(x)}{\tau_x^N} [f(\eta^{x,y}) - f(\eta)].$$

Denote by $\{\eta_t^\tau : t \geq 0\}$ the Markov process with generator L_N^τ *speeded up* by $N^{1+(1/\alpha)}$ and denote by $\pi_t^{N,\tau}$ the empirical measure associated to the configuration η_t^τ by formula (2.5). Observe that the time scaling is subdiffusive.

We show below in (2.8) that assumption (H1) is in force almost surely. Moreover, if the Markov process starts from $\mu_{u_0(\cdot)}^N$ for some continuous function $u_0 : \mathbb{T} \to \mathbb{R}_+$, by Theorem 2.2, for almost all measures W and all $t \geq 0$, the random measure $\pi_t^{N,\tau}$ converges in probability to the measure $u(t, x) W(dx)$, where u is the unique weak solution of (2.7). Note that the noise W survives entirely in the limit; even the differential equation depends on W.

We conclude this section by showing that assumption (H1) is in force for $\gamma_0 > (1/\alpha) - 1$. Indeed, with the notation introduced above, assumption (H1) can be restated as

$$\frac{1}{N^{2+\gamma-(1/\alpha)}} \sum_{x \in \mathbb{T}_N} \frac{1}{\tau_x^N} \longrightarrow 0 \qquad \text{a.s.} \tag{2.8}$$

It is well known that $1/\tau_x^N$ has finite moments of any order. Denote by m_1 the expectation of $1/\tau_x^N$. The variance of the previous sum is equal to

$N^{-\{3+2\gamma-(2/\alpha)\}}\sigma^2$ for some constant $\sigma^2 > 0$. Therefore, by Chebyshev's inequality, for every $\epsilon > 0$,

$$P\left[\left|\frac{1}{N^{\{2+\gamma-(1/\alpha)\}}}\sum_{x\in\mathbb{T}_N}\left\{\frac{1}{\tau_x^N} - m_1\right\}\right| \geq \epsilon\right] \leq \frac{\sigma^2}{\epsilon^2 N^{3+2\gamma-(2/\alpha)}}.$$

Taking $\epsilon = N^{-\delta}$ for $\delta > 0$ small enough, it follows from the Borel–Cantelli lemma that the sum in (2.8) vanishes a.s., provided $\gamma > (1/\alpha) - 1$.

3. Proof of the hydrodynamic limit. In this section, we prove Theorem 2.2. Fix $T > 0$ and denote by $\mathcal{M}([0, T] \times \mathbb{T})$ the space of finite, positive measures on $[0, T] \times \mathbb{T}$, endowed with the weak topology. For each $N \geq 1$, consider the measure $\mathfrak{M}^N$ on $[0, T] \times \mathbb{T}$ defined by

$$\mathfrak{M}^N = \frac{1}{N^{1+\gamma}}\sum_{x\in\mathbb{T}_N}\frac{\eta_t(x)}{W_x^N}\delta_{x/N}\,dt.$$

Hence, if we denote by $\langle\langle \mathfrak{M}^N, H\rangle\rangle$ the integral of a continuous function $H : [0, T] \times \mathbb{T} \to \mathbb{R}$ with respect to $\mathfrak{M}^N$, then we have that

$$\langle\langle \mathfrak{M}^N, H\rangle\rangle = \int_0^T \frac{1}{N^{1+\gamma}}\sum_{x\in\mathbb{T}_N} H(t, x/N)\frac{\eta_t(x)}{W_x^N}\,dt.$$

Let $D([0, T], \mathcal{M})$ be the space of right-continuous trajectories $\pi : [0, T] \to \mathcal{M}$ with left limits, endowed with the Skorokhod topology. Fix a continuous function $u_0 : \mathbb{T} \to \mathbb{R}_+$. Let $\mathcal{Q}_N$, $N \geq 1$, be the probability measure on $D([0, T], \mathcal{M}) \times \mathcal{M}([0, T] \times \mathbb{T})$ induced by the initial distribution $\mu^N_{u_0(\cdot)}$ and the pair $(\{\pi_t^N : 0 \leq t \leq T\}, \mathfrak{M}^N)$: $\mathcal{Q}_N = \mathbb{P}_{\mu^N_{u_0(\cdot)}} \circ (\{\pi_t^N : 0 \leq t \leq T\}, \mathfrak{M}^N)^{-1}$. We prove, in Lemma 3.5 below, that the sequence $\{\mathcal{Q}_N : N \geq 1\}$ is tight for the uniform topology in the first variable and, in Section 3.3, that all limit points of the sequence $\{\mathcal{Q}_N : N \geq 1\}$ are concentrated on measures $(\{\pi_t : 0 \leq t \leq T\}, \mathfrak{M})$ whose first coordinate is absolutely continuous with respect to W, $\pi(t, dx) = v(t, x)W(dx)$ and whose density v_t is a weak solution of the hydrodynamic equation (2.7). Since, by Theorem 5.1, there is at most one weak solution, for each $0 \leq t \leq T$, π_t^N converges weakly to $v(t, x)W(dx)$, where v_t is the unique weak solution of (2.7), as claimed in Theorem 2.2.

3.1. *Entropy estimates.* Recall from [15], Section A1.8, the definition of the relative entropy $H(\lambda|\mu)$ of a probability measure λ with respect to another probability measure μ defined on the same space, as well as its explicit formula presented in [15], Theorem A1.8.3. An elementary computation shows that there exists a constant $K_0 > 0$ such that

$$H\big(\mu^N_{u_0(\cdot)}|\mu_\rho\big) \leq K_0 N^\gamma \tag{3.1}$$

for all $N \geq 1$. In fact, $N^{-\gamma} H(\mu^N_{u_0(\cdot)}|\mu_\rho)$ converges to $\int\{u_0(x)\log[u_0(x)/\rho] - [u_0(x)-\rho]\}W(dx)$ as $N \uparrow \infty$.

Denote by $\langle \cdot, \cdot \rangle_{\mu_\rho}$ the scalar product of $L^2(\mu_\rho)$ and denote by I_N^W the convex and lower semicontinuous [15], Corollary A1.10.3, functional defined by

$$I_N^W(f) = \langle -L_N \sqrt{f}, \sqrt{f} \rangle_{\mu_\rho}$$

for all probability densities f with respect to μ_ρ (i.e., $f \geq 0$ and $\int f\, d\mu_\rho = 1$). An elementary computation shows that

$$I_N^W(f) = \sum_{x \in \mathbb{T}_N} I^W_{x,x+1}(f),$$

where

$$I^W_{x,x+1}(f) = \frac{1}{2N} \int \frac{\eta(x)}{W^N_x} \bigl\{\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)}\bigr\}^2 \, d\mu_\rho.$$

By [15], Theorem A1.9.2, if $\{S^N_t : t \geq 0\}$ stands for the semigroup associated to the generator $N^2 L_N$, then, for all $t \geq 0$,

$$H_N\bigl(\mu^N_{u_0(\cdot)} S^N_t | \mu_\rho\bigr) + N^2 \int_0^t I^W_N(f^N_s)\, ds \leq H_N\bigl(\mu^N_{u_0(\cdot)}|\mu_\rho\bigr), \tag{3.2}$$

provided f^N_s stands for the Radon–Nikodym derivative of $\mu^N_{u_0(\cdot)} S^N_s$ with respect to μ_ρ.

3.2. *Attractiveness and coupling estimates.* Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\preceq$ be a partial order in Ω. We say that a function $f : \Omega \to \mathbb{R}$ is *increasing* if $f(\eta) \leq f(\xi)$ whenever $\eta \preceq \xi$. Let λ, μ be two probability measures in Ω. We say that λ is *stochastically dominated* by μ if $\int f\, d\lambda \leq \int f\, d\mu$ for any increasing bounded function $f : \Omega \to \mathbb{R}$. An equivalent definition is the following. We say that a probability measure Λ defined in $\Omega \times \Omega$ is a *coupling* of λ and μ if $\Lambda(A \times \Omega) = \lambda(A)$, $\Lambda(\Omega \times A) = \mu(A)$ for any $A \in \mathcal{F}$. The measure λ is stochastically dominated by μ if there is a coupling Λ of λ and μ such that $\Lambda((\eta, \xi); \eta \preceq \xi) = 1$.

We say that a stochastic process η_t defined in Ω is *attractive* if, for any two probability measures $\lambda_1 \preceq \lambda_2$, there is a process (η^1_t, η^2_t) in $\Omega \times \Omega$ such that η^i_t is distributed as the process η_t with initial distribution μ_i for $i = 1, 2$, and such that $P(\eta^1_t \preceq \eta^2_t) = 1$ for any $t \geq 0$. We call the process (η^1_t, η^2_t) a *coupling*.

In Ω_N, we say that $\eta \preceq \xi$ if $\eta(x) \leq \xi(x)$ for any $x \in \mathbb{T}_N$. For this partial order, it is easy to see that η_t is attractive. Indeed, since the state space is finite, it is enough to show the existence of a coupling for measures μ_i concentrated on fixed configurations η^i, with $\eta^1 \preceq \eta^2$. Define η^1_t as the process η_t with initial configuration η^1. Then, define the process $\bar{\eta}_t$ as a copy of the process η_t, independent of η^1_t and starting from $\bar{\eta}$, where $\bar{\eta}(x) = \eta^2(x) - \eta^1(x)$. Now, define η^2_t by taking

$\eta_t^2(x) = \eta_t^1(x) + \bar{\eta}_t(x)$. Since the motion of different particles is independent, it is clear that (η_t^1, η_t^2) is the desired coupling as, by construction, $\eta_t^1 \preceq \eta_t^2$ for any $t \geq 0$.

In terms of stochastic domination, the definition of attractiveness reads as follows. If λ^1 is stochastically dominated by λ^2, then λ_t^1 is stochastically dominated by λ_t^2 for any time $t \geq 0$, where λ_t^i denotes the distribution in Ω_N of the process η_t with initial distribution λ^i. In particular, we obtain the following inequality, which we call the *coupling estimate*.

PROPOSITION 3.1. *Let λ^1, λ^2 be two probability measures on Ω_N. If λ^1 is stochastically dominated by λ^2, then*

$$\mathbb{E}_{\lambda^1}[F(\eta_t)] \leq \mathbb{E}_{\lambda^2}[F(\eta_t)]$$

for any $t \geq 0$ and any bounded increasing function $F : \Omega_N \to \mathbb{R}$.

We now need a criterion to decide whether an initial distribution is stochastically dominated by another one. In $\mathbb{N}_0$, consider the canonical ordering. It is easy to show that $\mathfrak{P}_{\rho_1}$ is stochastically dominated by $\mathfrak{P}_{\rho_2}$ whenever $\rho_1 \leq \rho_2$. Since the measures μ_ρ are of product form, μ_{ρ_1} is stochastically dominated by μ_{ρ_2} each time $\rho_1 \leq \rho_2$. More interesting for us, we have the following.

PROPOSITION 3.2. *Fix an initial bounded non-negative profile $u_0 : \mathbb{T} \to \mathbb{R}_+$. Define $\bar{\rho} = \|u_0\|_\infty$. Then, $\mu^N_{u_0(\cdot)}$ is stochastically dominated by $\mu_{\bar{\rho}}$ for any $N > 0$. In particular,*

$$\mathbb{E}_{\mu^N_{u_0(\cdot)}}[F(\eta_t)] \leq \mathbb{E}_{\mu_{\bar{\rho}}}[F(\eta_t)]$$

for any $t \geq 0$ and for any increasing bounded function $F : \Omega_N \to \mathbb{R}$.

The coupling estimate shows that π_t^N, $\mathfrak{M}^N$ converge to measures which are absolutely continuous with respect to W, the Lebesgue measure, respectively.

LEMMA 3.3. *Every limit point $\mathcal{Q}^*$ of the sequence $\mathcal{Q}_N$ is concentrated on measures $\pi(t, dx) = u(t, x)W(dx)$ [resp. $\mathfrak{M}(dt, dx) = v(t, x)\,dt\,dx$] which are absolutely continuous with respect to W (resp. the Lebesgue measure) and whose density $u(t, x)$ [resp. $v(t, x)$] is positive and bounded by $\|u_0\|_\infty$.*

PROOF. Fix a limit point $\mathcal{Q}^*$ of the sequence $\mathcal{Q}_N$ and assume, without loss of generality, that $\mathcal{Q}_N$ converges to $\mathcal{Q}^*$ (in the uniform topology on the first coordinate). Fix a continuous, positive function $G : [0, T] \times \mathbb{T} \to \mathbb{R}$, $\varepsilon > 0$ and recall that $\bar{\rho} = \|u_0\|_\infty$. By the previous proposition,

$$\begin{aligned}
&\mathbb{P}_{\mu^N_{u_0(\cdot)}}\left[\langle\!\langle \mathfrak{M}, G \rangle\!\rangle \geq \bar{\rho} \int_0^T dt \int_{\mathbb{T}} G(t, x)\,dx + \varepsilon\right] \\
&\quad \leq \mathbb{P}_{\mu_{\bar{\rho}}}\left[\langle\!\langle \mathfrak{M}, G \rangle\!\rangle \geq \bar{\rho} \int_0^T dt \int_{\mathbb{T}} G(t, x)\,dx + \varepsilon\right]
\end{aligned}$$

for every $N \geq 1$. We may replace the integral $\int_{\mathbb{T}} G(t,x)\,du$ by Riemann sum because G is continuous. Thus, for N large enough, the previous expression is bounded above by

$$\mathbb{P}_{\mu_{\bar\rho}}\biggl[\int_0^T \frac{1}{N^{1+\gamma}} \sum_{x\in\mathbb{T}_N} G(t,x/N)\biggl\{\frac{\eta_t(x)}{W_x^N} - \bar\rho N^\gamma\biggr\}\,dt \geq \varepsilon/2\biggr].$$

By the Chebyshev and Schwarz inequalities, since $\mu_{\bar\rho}$ is a stationary state given by a product of Poisson measures, this expression is less than or equal to

$$\frac{4T}{\varepsilon^2}\int_0^T \frac{\bar\rho}{N^{2+\gamma}} \sum_{x\in\mathbb{T}_N} G(t,x/N)^2 \frac{1}{W_x^N}\,dt.$$

In view of assumption (H1), this expression vanishes as $N \uparrow \infty$ because G is a continuous bounded function.

Since $\mathcal{Q}_N$ converges to $\mathcal{Q}^*$, for every $\varepsilon > 0$,

$$\mathcal{Q}^*\biggl[\langle\!\langle \mathfrak{M}, G\rangle\!\rangle \geq \bar\rho\int_0^T dt\int_{\mathbb{T}} G(t,x)\,dx + \varepsilon\biggr] = 0.$$

Letting $\varepsilon \downarrow 0$, we conclude that $\mathcal{Q}^*$ is concentrated on measures $\mathfrak{M}$ such that $\langle\!\langle \mathfrak{M}, G\rangle\!\rangle \leq \bar\rho\int_0^T dt\int_{\mathbb{T}} G(t,x)\,dx$. Taking a set $\{G_k : k \geq 1\}$ of positive, bounded, continuous functions dense for the uniform topology, we conclude that $\mathcal{Q}^*$ is concentrated on absolutely continuous measures $\mathfrak{M}(dt,dx) = v(t,x)\,dt\,dx$, whose density $v(t,x)$ is bounded by $\bar\rho$.

A similar coupling argument shows that for every $0 \leq t \leq T$ and every continuous, positive function $H : \mathbb{T} \to \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N_{u_0(\cdot)}}\biggl[\langle \pi_t^N, H\rangle \geq \bar\rho\int_{\mathbb{T}} H(x)W(dx) + \varepsilon\biggr] = 0.$$

Since we assumed compactness in the uniform topology, we deduce from this formula that

$$\mathcal{Q}^*\biggl[\langle \pi_t, H\rangle \geq \bar\rho\int_{\mathbb{T}} H(x)W(dx) + \varepsilon\biggr] = 0.$$

It remains to recall the arguments presented for $\mathfrak{M}$ to complete the proof. $\square$

3.3. *Hydrodynamic limit.* In this subsection, we prove Theorem 2.2.

THEOREM 3.4. *The sequence of probability measures* $\{\mathcal{Q}_N : N \geq 1\}$ *converges to the measure* $\mathcal{Q}^*$ *concentrated on the absolutely continuous pair* $(\{\pi_t : 0 \leq t \leq T\}, \mathfrak{M})$, $\pi_t = v(t,x)W(dx)$, $\mathfrak{M} = v(t,x)\,dt\,dx$, *whose density* $v(t,x)$ *is the weak solution of the equation* (2.7).

PROOF. By Lemma 3.5 below, the sequence $\{\mathcal{Q}_N : N \geq 1\}$ is tight. Fix a limit point $\mathcal{Q}^*$ and assume, without loss of generality, that $\mathcal{Q}_N$ converges to $\mathcal{Q}^*$.

Fix a smooth function $H : [0, T] \times \mathbb{T} \to \mathbb{R}$ such that $H(T, \cdot) = 0$. Consider the martingale $M_N^H(t)$ defined by

$$\begin{aligned} M_N^H(t) &= \langle \pi_t^N, H_t \rangle - \langle \pi_0^N, H_0 \rangle \\ &\quad - \int_0^t \langle \pi_s^N, \partial_s H_s \rangle \, ds - N^2 \int_0^t L_N \langle \pi_s^N, H \rangle \, ds. \end{aligned} \tag{3.3}$$

The variance of this martingale is equal to

$$\frac{N}{2N^{2\gamma}} \sum_{x \in \mathbb{T}_N} \sum_{y : |y-x|=1} \int_0^t \frac{\eta_s(x)}{W_x^N} \{H(s, y/N) - H(s, x/N)\}^2 \, ds.$$

The coupling estimate shows that the expectation of this expression with respect to $\mathbb{P}_{\mu^N_{u_0(\cdot)}}$ is bounded by $C_0 N^{-\gamma}$ for some constant $C_0 > 0$ which depends on H and $\bar{\rho}$. On the other hand, an elementary computation shows that

$$N^2 \int_0^T L_N \langle \pi_s^N, H \rangle \, ds = \frac{1}{2} \langle\!\langle \mathfrak{M}^N, \Delta_N H \rangle\!\rangle,$$

where Δ_N stands for the discrete Laplacian. In particular, in view of (3.3) and since $H(T, \cdot)$ vanishes, for every $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}_{\mu^N_{u_0(\cdot)}} \left[\left| \langle \pi_0^N, H_0 \rangle + \int_0^T \langle \pi_s^N, \partial_s H_s \rangle \, ds + (1/2) \langle\!\langle \mathfrak{M}^N, \Delta_N H \rangle\!\rangle \right| > \delta \right] = 0.$$

The first term of this sum converges to $\int_{\mathbb{T}} H_0(x) u_0(x) W(dx)$ in $\mathbb{P}_{\mu^N_{u_0(\cdot)}}$-probability as $N \uparrow \infty$. The last expression can be written, up to smaller order terms, as $(1/2) \langle\!\langle \mathfrak{M}^N, \Delta H \rangle\!\rangle$. Hence, since $\mathcal{Q}^N$ converges to $\mathcal{Q}^*$, for every $\delta > 0$ and every smooth function H,

$$\mathcal{Q}^* \left[\left| \int_{\mathbb{T}} H_0(x) u_0(x) W(dx) + \int_0^T \langle \pi_s, \partial_s H_s \rangle \, ds + (1/2) \langle\!\langle \mathfrak{M}, \Delta H \rangle\!\rangle \right| > \delta \right] = 0.$$

Letting $\delta \downarrow 0$, by Lemma 3.3, $\mathcal{Q}^*$-a.s.,

$$\begin{aligned} &\int_{\mathbb{T}} H_0(x) u_0(x) W(dx) + \int_0^T ds \int_{\mathbb{T}} (\partial_s H)(s, x) u(s, x) W(dx) \\ &\quad + (1/2) \int_0^T ds \int_{\mathbb{T}} (\Delta H)(s, x) v(s, x) \, dx = 0. \end{aligned}$$

According to Lemma 4.6, we may replace u by v in the second term. By Proposition 4.3, we may integrate by parts the last term to obtain that

$$\langle H_0, u_0 \rangle_W + \int_0^T \langle \partial_s H_s, u_s \rangle_W \, ds - (1/2) \int_0^T \langle \partial_x H_s, \partial_x v_s \rangle \, ds = 0.$$

This proves that $\mathcal{Q}^*$ is concentrated on weak solutions of (2.7). By Proposition 4.3, $\partial_x v$ belongs to $L^2([0,T]\times\mathbb{T})$ and by Lemma 3.3, v is positive and bounded. Since the previous identity holds for all smooth functions H, v is a weak solution of (2.7). □

Theorem 2.2 follows from this result and the tightness in the uniform topology of the sequence $\{\mathcal{Q}_N : N \ge 1\}$ proved in Lemma 3.5 below.

LEMMA 3.5. *The sequence* $\{\mathcal{Q}_N : N \ge 1\}$ *is tight in the uniform topology in the first coordinate.*

PROOF. To prove tightness of the sequence $\{\mathcal{Q}_N : N \ge 1\}$, we need to examine the two coordinates separately.

Clearly, the sequence of random measures $\mathfrak{M}^N$ is tight if and only if the sequence of random variables $\langle\langle \mathfrak{M}^N, G\rangle\rangle$ is tight for every continuous function $G : [0,T]\times\mathbb{T}\to\mathbb{R}$. Tightness of the sequence $\langle\langle \mathfrak{M}^N, G\rangle\rangle$ follows from a coupling argument similar to the one used in the proof of Lemma 3.3.

To prove tightness of the sequence of processes $\{\pi_t^N : 0 \le t \le T\}$ in the uniform topology, it is enough to examine the process $\langle \pi_t^N, H\rangle$ for some fixed smooth function H. Recall the definition of the martingale $M_N^H(t)$ introduced in (3.3). Tightness of $\langle \pi_t^N, H\rangle$ follows from tightness of the martingale $M_N^H(t)$ and tightness of the additive functional $\int_0^t N^2 L_N \langle \pi_s^N, H\rangle\, ds$.

The martingale is tight in the uniform topology because, by Doob's inequality and the explicit computation of the quadratic variation of $M_N^H(t)$, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N_{u_0(\cdot)}}\Big[\sup_{0\le t\le T} |M_N^H(t)| > \delta\Big] = 0.$$

On the other hand, computing $N^2 L_N\langle \pi_r^N, H\rangle$, by the Chebyshev and Schwarz inequalities, for every $\delta > 0$,

$$\mathbb{P}_{\mu^N_{u_0(\cdot)}}\Big[\sup_{0\le |t-s|\le\epsilon}\Big|\int_s^t N^2 L_N\langle \pi_r^N, H\rangle\, dr\Big| > \delta\Big]$$
$$\le \frac{\epsilon C_0}{\delta^2}\mathbb{E}_{\mu^N_{u_0(\cdot)}}\Big[\int_0^T \Big\{\frac{1}{N^{1+\gamma}}\sum_{x\in\mathbb{T}_N}\frac{\eta_s(x)}{W_x^N}\Big\}^2 ds\Big]$$

for some constant $C_0 > 0$ which depends only on H. By the coupling estimate, we may replace the measure $\mu^N_{u_0(\cdot)}$ by the stationary measure $\mu^N_{\bar\rho}$, estimating the expectation by

$$C(\bar\rho)T\Big\{1 + \frac{1}{N^{2+\gamma}}\sum_{x\in\mathbb{T}_N}\frac{1}{W_x^N}\Big\}.$$

By assumption (H1), this expression is bounded uniformly in N, which concludes the proof of tightness. □

4. Entropy estimates. We prove in this section the main estimates needed in the proof of the hydrodynamic limit. For $\ell \geq 1$, let Λ_ℓ be a cube of length ℓ, $\Lambda_\ell = \{1, \dots, \ell\}$, and let $\Lambda_{x,\ell} = x + \Lambda_\ell$. Denote by $M^\ell(x)$ the number of particles on $\Lambda_{x,\ell}$ and by $W_N(x, \ell)$ the W-measure of the cube $\Lambda_{x,\ell}$ rescaled by N:

$$M^\ell(x) = \sum_{y \in \Lambda_\ell} \eta(x+y), \qquad W_N(x,\ell) = \sum_{y \in \Lambda_\ell} W^N_{x+y}.$$

Note that $W_N(x, \epsilon N) \sim W_\epsilon(x) := W([x, x+\epsilon))$.

4.1. *Two-blocks estimate.* We prove in this subsection the so-called two-blocks estimate.

LEMMA 4.1. *Fix a bounded function $G : [0,T] \times \mathbb{T} \to \mathbb{R}$. Then,*

$$\lim_{\epsilon \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu^N_{u_0(\cdot)}} \left[\left| \int_0^T \frac{1}{N^{1+\gamma}} \sum_{x \in \mathbb{T}_N} G(s, x/N) \left\{ \frac{\eta_s(x)}{W^N_x} - \frac{M^{\epsilon N}_s(x)}{W_N(x, \epsilon N)} \right\} ds \right| \right] = 0.$$

PROOF. Fix a bounded function $G : [0,T] \times \mathbb{T} \to \mathbb{R}$, $\delta > 0$ and a positive constant $C_1 = C_1(\delta)$ to be specified later. Let

$$V^0_\epsilon(s, \eta) = \frac{1}{N^{1+\gamma}} \sum_{x \in \mathbb{T}_N} G(s, x/N) \left\{ \frac{\eta(x)}{W^N_x} - \frac{M^{\epsilon N}(x)}{W_N(x, \epsilon N)} \right\},$$

$$R_\epsilon(s, \eta) = \frac{C_1 \epsilon}{N^{2+\gamma}} \sum_{x \in \mathbb{T}_N} G(s, x/N)^2 \sum_{y=0}^{\epsilon N} \frac{\eta(x+y)}{W^N_{x+y}}.$$

By the coupling estimate,

$$\mathbb{E}_{\mu^N_{u_0(\cdot)}} \left[\int_0^T R_\epsilon(s, \eta_s)\, ds \right] \leq C_0 \epsilon^2 T \tag{4.1}$$

for some constant $C_0 > 0$ depending only on C_1, G, ρ. It is therefore enough to prove that

$$\lim_{\epsilon \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu^N_{u_0(\cdot)}} \left[\left| \int_0^T V^0_\epsilon(s, \eta_s)\, ds \right| - \int_0^T R_\epsilon(s, \eta_s)\, ds \right] = 0.$$

By the entropy inequality, Jensen's inequality and the entropy estimate (3.1), the previous expectation is bounded above by

$$\frac{K_0}{A} + \frac{1}{AN^\gamma} \log \mathbb{E}_{\mu_\rho} \left[\exp AN^\gamma \left\{ \left| \int_0^T V^0_\epsilon(s, \eta_s)\, ds \right| - \int_0^T R_\epsilon(s, \eta_s)\, ds \right\} \right]$$

for every positive $A > 0$.

Let $A = K_0\delta^{-1}$. Since $e^{|x|} \le e^x + e^{-x}$ and $\limsup_{n\to\infty} N^{-\gamma}\log\{a_N^1 + a_N^2\} = \max_{i=1,2} \limsup_{n\to\infty} N^{-\gamma}\log a_N^i$, to prove the lemma, it is enough to show that for every $\delta > 0$,

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{1}{AN^\gamma}\log \mathbb{E}_{\mu_\rho}\left[\exp\left\{AN^\gamma \int_0^T V_\epsilon(s,\eta_s)\,ds\right\}\right] \le 0,$$

where $V_\epsilon = V_\epsilon^0 - R_\epsilon$.

By classical arguments relying on the Feynman–Kac formula (see [15], page 267), the previous expectation is bounded above by

$$\int_0^T \sup_f \left\{\int V_\epsilon(s,\eta) f(\eta)\mu_\rho(d\eta) - \frac{N^2}{AN^\gamma} I_N^W(f)\right\} ds,$$

where the supremum is carried over all densities f with respect to μ_ρ. Hence, to conclude the proof of the lemma, it is enough to show that

$$\int V_\epsilon^0(s,\eta) f(\eta)\mu_\rho(d\eta) \le \int R_\epsilon(s,\eta) f(\eta)\mu_\rho(d\eta) + \frac{\delta N^2}{K_0 N^\gamma} I_N^W(f) \tag{4.2}$$

for every density function f and every $\delta > 0$.

Recall the definition of $W_N(x,\epsilon N)$ to rewrite $V_\epsilon^0(s,\eta)$ as

$$\frac{1}{N^{1+\gamma}}\sum_{x\in\mathbb{T}_N} G(s,x/N)\sum_{y=1}^{\epsilon N}\frac{W_{x+y}^N}{W_N(x,\epsilon N)}\left\{\frac{\eta(x)}{W_x^N} - \frac{\eta(x+y)}{W_{x+y}^N}\right\}.$$

Fix a density f with respect to μ_ρ. Performing a simple change of variables, we see that

$$\begin{aligned}\int\left\{\frac{\eta(x)}{W_x^N} - \frac{\eta(x+y)}{W_{x+y}^N}\right\} f\,d\mu_\rho &= \sum_{z=0}^{y-1}\int\left\{\frac{\eta(x+z)}{W_{x+z}^N} - \frac{\eta(x+z+1)}{W_{x+z+1}^N}\right\} f\,d\mu_\rho\\ &= \sum_{z=0}^{y-1}\int \frac{\eta(x+z)}{W_{x+z}^N}\{f(\eta) - f(\sigma^{x+z,x+z+1}\eta)\}\,d\mu_\rho.\end{aligned}$$

Since $(a-b) = (\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$, by the Schwarz inequality, the previous expression is less than or equal to

$$\begin{aligned}&\frac{N}{\beta}\sum_{z=0}^{y-1} I_{x+z,x+z+1}^W(f)\\ &\qquad + \frac{\beta}{2}\sum_{z=0}^{y-1}\int \frac{\eta(x+z)}{W_{x+z}^N}\left\{\sqrt{f(\sigma^{x+z,x+z+1}\eta)} + \sqrt{f(\eta)}\right\}^2 d\mu_\rho\end{aligned}$$

for all $\beta > 0$. The same change of variables permits the second term to be estimated as

$$\beta \sum_{z=0}^{y-1} \int \left\{ \frac{\eta(x+z)}{W^N_{x+z}} + \frac{\eta(x+z+1)}{W^N_{x+z+1}} \right\} f(\eta)\, d\mu_\rho.$$

It follows from the previous estimates that for any density f with respect to μ_ρ, and all $\beta > 0$,

$$\begin{aligned}
&\int V^0_\epsilon(s,\eta) f(\eta) \mu_\rho(d\eta) \\
&\quad \le \frac{1}{\beta N^\gamma} \sum_{x \in \mathbb{T}_N} \sum_{y=1}^{\epsilon N} \frac{W^N_{x+y}}{W_N(x,\epsilon N)} \sum_{z=0}^{y-1} I^W_{x+z,x+z+1}(f) \\
&\qquad + \frac{2\beta}{N^{1+\gamma}} \sum_{x \in \mathbb{T}_N} G(s,x/N)^2 \sum_{y=1}^{\epsilon N} \frac{W^N_{x+y}}{W_N(x,\epsilon N)} \sum_{z=0}^{y} \int \frac{\eta(x+z)}{W^N_{x+z}} f(\eta)\, d\mu_\rho.
\end{aligned} \tag{4.3}$$

We examine each term on the right-hand side separately. Set $\beta = 2\epsilon N^{-1} A$. Changing the order of summation, we obtain that the second term is less than or equal to

$$\frac{4\epsilon A}{N^{2+\gamma}} \sum_{x \in \mathbb{T}_N} G(s,x/N)^2 \int \sum_{y=0}^{\epsilon N} \frac{\eta(x+y)}{W^N_{x+y}} f(\eta)\, d\mu_\rho.$$

This expression is bounded by the expectation of R_ϵ with respect to $f(\eta)\, d\mu_\rho$, provided we choose $C_1 \ge 4A = 4K_0\delta^{-1}$. By similar reasoning, the first term on the right-hand side of (4.3) is bounded above by

$$\frac{N(\epsilon N + 1)}{2A\epsilon N^\gamma} \sum_{z \in \mathbb{T}_N} I^W_{z,z+1}(f).$$

Hence, (4.2) holds and the lemma is proved. □

Consider a sequence $\{G_{N,\epsilon} : N \ge 1, \epsilon > 0\}$ of functions $G_{N,\epsilon} : [0,T] \times \mathbb{T}_N \to \mathbb{R}$. In the proof of the two-blocks estimate, the boundedness assumption on G was used only in (4.1). In particular, the proof presented above shows that

$$\begin{aligned}
&\lim_{\epsilon \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu^N_{u_0(\cdot)}} \left[\left| \int_0^T \frac{1}{N^{1+\gamma}} \sum_{x \in \mathbb{T}_N} G_{N,\epsilon}(s,x) \left\{ \frac{M^{\epsilon N}_s(x)}{W_N(x,\epsilon N)} - \frac{\eta_s(x)}{W^N_x} \right\} ds \right| \right] \\
&\quad = 0,
\end{aligned} \tag{4.4}$$

provided that

$$\lim_{\epsilon \to 0} \limsup_{N \to \infty} \int_0^T \frac{\epsilon^2}{N} \sum_{x \in \mathbb{T}_N} G_{N,\epsilon}(s,x)^2\, ds = 0.$$

Recall that $W_\epsilon : \mathbb{T} \to \mathbb{R}$ is defined by $W_\epsilon(x) = W([x, x+\epsilon))$.

COROLLARY 4.2. *Let $J : [0,T] \times \mathbb{T} \to \mathbb{R}$ be a continuous function. Then*,

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \mathbb{E}_{\mu^N_{u_0(\cdot)}}\left[\left|\int_0^T \langle \pi^N_s, J_s\rangle\, ds - \langle\!\langle \mathfrak{M}^N, J\epsilon^{-1} W_\epsilon \rangle\!\rangle\right|\right] = 0.$$

PROOF. Since J is a continuous function,

$$\langle \pi^N_s, J_s\rangle - \frac{1}{N^\gamma} \sum_{x\in\mathbb{T}_N} \eta_s(x) \frac{1}{(\epsilon N)} \sum_{-y\in\Lambda_{-x,\epsilon N}} J(s, y/N)$$

is absolutely bounded by $C(\epsilon)N^{-\gamma}\sum_{x\in\mathbb{T}_N}\eta(x)$ for some constant $C(\epsilon) > 0$ which vanishes as $\epsilon \downarrow 0$. In particular, by the coupling estimate and changing the order of summation, we get that

$$\lim_{\epsilon\to 0} \sup_{N\ge 1} \mathbb{E}_{\mu^N_{u_0(\cdot)}}\Bigg[\Bigg|\int_0^T \langle \pi^N_s, J_s\rangle\, ds - \int_0^T \frac{\epsilon^{-1}}{N^{1+\gamma}} \sum_{x\in\mathbb{T}_N} J(s, x/N) M^{\epsilon N}_s(x)\, ds\Bigg|\Bigg] = 0.$$

The second term inside the absolute value can be rewritten as

$$\frac{1}{N^{1+\gamma}} \sum_{x\in\mathbb{T}_N} J(s, x/N)\epsilon^{-1} W_N(x, \epsilon N) \frac{M^{\epsilon N}_s(x)}{W_N(x,\epsilon N)}.$$

Let $G_{N,\epsilon}(s, x/N) = J(s, x/N)\epsilon^{-1}W_N(x, \epsilon N)$. Since J is a bounded function, by the definition of $W_N(x, \epsilon N)$, we have

$$\int_0^T \frac{\epsilon^2}{N} \sum_{x\in\mathbb{T}_N} G_{N,\epsilon}(s, x)^2\, ds \le \frac{C_0 T}{N} \sum_{x\in\mathbb{T}_N} W_{2\epsilon}(x/N)^2$$

for some constant $C_0 > 0$ which depends only on J. As $N \uparrow \infty$, this expression converges to

$$C_0 T \int_{\mathbb{T}} W_{2\epsilon}(x)^2\, dx.$$

For Lebesgue almost all x, $W_{2\epsilon}(x)$ vanishes as $\epsilon \downarrow 0$. Therefore, by (4.4),

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \mathbb{E}_{\mu^N_{u_0(\cdot)}}\Bigg[\Bigg|\int_0^T \frac{\epsilon^{-1}}{N^{1+\gamma}} \sum_{x\in\mathbb{T}_N} J(s, x/N) W_N(x, \epsilon N) \times \left\{\frac{M^{\epsilon N}_s(x)}{W_N(x,\epsilon N)} - \frac{\eta_s(x)}{W^N_x}\right\} ds\Bigg|\Bigg] = 0.$$

Finally, since

$$\frac{1}{N^d}\sum_{x\in\mathbb{T}_N}|W_N(x,\epsilon N)-W_\epsilon(x/N)|$$

vanishes as $N\uparrow\infty$, we may replace $W_N(x,\epsilon N)$ by $W_\epsilon(x/N)$ to complete the proof of the corollary. $\square$

4.2. *Energy estimate.* We proved in Lemma 3.3 that any limit point $\mathcal{Q}^*$ of the sequence $\{\mathcal{Q}_N : N\geq 1\}$ is concentrated on measures $\mathfrak{M}$ which are absolutely continuous with respect to the Lebesgue measures: $\mathfrak{M}=v(t,x)\,dt\,dx$. We show in this section that the density v has a generalized space derivative in $L^2([0,T]\times\mathbb{T})$.

PROPOSITION 4.3. *Any limit point $\mathcal{Q}^*$ of the sequence $\{\mathcal{Q}_N : N\geq 1\}$ is concentrated on measures $\mathfrak{M}=v(t,x)\,dt\,dx$ with the property that there exists a function F in $L^2([0,T]\times\mathbb{T})$ such that*

$$\int_0^T ds\int_{\mathbb{T}}(\partial_x H)(s,x)v(s,x)\,dx=-\int_0^T ds\int_{\mathbb{T}}H(s,x)F(s,x)\,dx$$

for all smooth functions H. We denote the generalized derivative F of v by $\partial_x v$.

The proof of this proposition relies on the following estimate.

LEMMA 4.4. *Fix a set of smooth functions $H_i:[0,T]\times\mathbb{T}\to\mathbb{R}$, $1\leq i\leq\ell$. Let*

$$V_i(s,\eta)=\frac{1}{N^\gamma}\sum_{x\in\mathbb{T}_N}H_i(s,x/N)\left\{\frac{\eta(x)}{W_x^N}-\frac{\eta(x+1)}{W_{x+1}^N}\right\}.$$

Then, for any $\beta>0$,

$$\limsup_{N\to\infty}\mathbb{E}_{\mu^N_{u_0(\cdot)}}\left[\max_{1\leq i\leq l}\int_0^T\left\{V_i(s,\eta)-\frac{2\beta}{N^{1+\gamma}}\sum_{x\in\mathbb{T}_N}H_i(s,x/N)^2\frac{\eta_s(x)}{W_x^N}\right\}ds\right]\leq\frac{K_0}{\beta},$$

where K_0 is the constant given by (3.1).

PROOF. Fix $\beta>0$ and let

$$X_i(s,\eta)=V_i(s,\eta)-\frac{\beta}{N^{1+\gamma}}\sum_{x\in\mathbb{T}_N}H_i(s,x/N)^2\left\{\frac{\eta(x)}{W_x^N}+\frac{\eta(x+1)}{W_{x+1}^N}\right\}.$$

A summation by parts and the coupling estimate similar to the one used in the proof of Lemma 3.3 show that it is enough to prove that

$$\limsup_{N\to\infty}\mathbb{E}_{\mu^N_{u_0(\cdot)}}\left[\max_{1\leq i\leq l}\int_0^T X_i(s,\eta)\,ds\right]\leq\frac{K_0}{\beta}.$$

By the entropy inequality, Jensen's inequality and the entropy estimate (3.1),

$$\begin{aligned}
&\mathbb{E}_{\mu^N_{u_0(\cdot)}}\left[\max_{1\le i\le \ell}\int_0^T X_i(s,\eta_s)\,ds\right] \\
&\qquad \le \frac{K_0}{\beta}+\frac{1}{\beta N^\gamma}\log \mathbb{E}_{\mu_\rho}\left[\exp\left\{\max_{1\le i\le\ell}\beta N^\gamma\int_0^T X_i(s,\eta_s)\,ds\right\}\right]
\end{aligned}$$

for every $\beta > 0$.

Since, on one hand, $\exp\{\max_{1\le i\le\ell} a^i_N\}\le\sum_{1\le i\le\ell}\exp\{a^i_N\}$ and, on the other hand, $\limsup_{n\to\infty} N^{-\gamma}\log\{\sum_{1\le i\le\ell} b^i_N\}=\max_{1\le i\le\ell}\limsup_{n\to\infty}N^{-\gamma}\log b^i_N$, to prove the lemma, it is enough to show that

$$\limsup_{n\to\infty}\frac{1}{\beta N^\gamma}\log\mathbb{E}_{\mu_\rho}\left[\exp\left\{\beta N^\gamma\int_0^T X_i(s,\eta_s)\,ds\right\}\right]\le 0 \tag{4.5}$$

for $1\le i\le\ell$ and any $\beta>0$.

By classical arguments relying on the Feynman–Kac formula (see [15], page 267), the previous expectation is bounded above by

$$\int_0^T\sup_f\left\{\int X_i(s,\eta)f(\eta)\mu_\rho(d\eta)-\frac{N^2}{\beta N^\gamma}I^W_N(f)\right\}ds,$$

where the supremum is carried over all densities f with respect to μ_ρ.

Therefore, to conclude the proof of the lemma, it is enough to show that

$$\begin{aligned}
&\int V_i(s,\eta)f(\eta)\mu_\rho(d\eta)\\
&\qquad\le\int\frac{\beta}{N^{1+\gamma}}\sum_{x\in\mathbb{T}_N}H_i(s,x/N)^2\left\{\frac{\eta(x)}{W^N_x}+\frac{\eta(x+1)}{W^N_{x+1}}\right\}f(\eta)\mu_\rho(d\eta)\\
&\qquad\quad+\frac{N^2}{\beta N^\gamma}I^W_N(f)
\end{aligned}$$

for all densities f and $\beta>0$.

Recall the definition of V_i. Performing a simple change of variables, we see that

$$\int\left\{\frac{\eta(x)}{W^N_x}-\frac{\eta(x+1)}{W^N_{x+1}}\right\}f\,d\mu_\rho=\int\frac{\eta(x)}{W^N_x}\{f(\eta)-f(\sigma^{x,x+1}\eta)\}\,d\mu_\rho.$$

Since $(a-b)=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$, by the Schwarz inequality, the previous expression is less than or equal to

$$\frac{N}{A}I^W_{x,x+1}(f)+\frac{A}{2}\int\frac{\eta(x)}{W^N_x}\{\sqrt{f(\sigma^{x,x+1}\eta)}+\sqrt{f(\eta)}\}^2\,d\mu_\rho$$

for all $A>0$. The same change of variables permit the second term to be estimated as

$$A\int\left\{\frac{\eta(x)}{W^N_x}+\frac{\eta(x+1)}{W^N_{x+1}}\right\}f(\eta)\,d\mu_\rho.$$

Choosing $A = \beta N^{-1}|H(s, x/N)|$, we obtain that for any density f with respect to μ_ρ,

$$\int V_i(s,\eta)(s,\eta) f(\eta)\mu_\rho(d\eta)$$

$$\leq \frac{N^2}{\beta N^\gamma}\sum_{x\in\mathbb{T}_N} I^W_{x,x+1}(f)$$

$$+ \frac{\beta}{N^{1+\gamma}}\sum_{x\in\mathbb{T}_N} H_i(s,x/N)^2 \int\left\{\frac{\eta(x)}{W^N_x} + \frac{\eta(x+1)}{W^N_{x+1}}\right\} f(\eta)\,d\mu_\rho,$$

which proves the lemma. □

Recall that, by Lemma 3.3, $\mathcal{Q}^*$ is concentrated on absolutely continuous measures $\mathfrak{M} = v(t,x)\,dt\,dx$.

COROLLARY 4.5. *Any limit point $\mathcal{Q}^*$ of the sequence $\{\mathcal{Q}_N : N \geq 1\}$ is concentrated on measures $\mathfrak{M} = v(t,x)\,dt\,dx$ such that*

$$E_{\mathcal{Q}^*}\left[\sup_H\left\{\int_0^T ds\int_{\mathbb{T}}(\partial_x H)(s,x)v(s,x)\,dx - 2\int_0^T ds\int_{\mathbb{T}} H(s,x)^2 v(s,x)\,dx\right\}\right] \leq K_0.$$

In this formula, the supremum is taken over all functions H in $C^{0,1}([0,T]\times\mathbb{T})$.

PROOF. Fix a limit point $\mathcal{Q}^*$ of the sequence $\mathcal{Q}_N$ and assume, without loss of generality, that $\mathcal{Q}_N$ converges to $\mathcal{Q}^*$. Consider a sequence $\{H_j : j \geq 1\}$ of functions in $C^{0,1}([0,T]\times\mathbb{T})$, dense with respect to the uniform topology. It follows from Lemma 4.4 with $\beta = 1$, a summation by parts and a coupling estimate similar to the one used in the proof of Lemma 3.3, to replace the discrete derivative $N\{H(s,(x+1)/N) - H(s,x/N)\}$ by the continuous one $(\partial_x H)(s,x/N)$, that

$$E_{\mathcal{Q}^*}\left[\max_{1\leq i\leq \ell}\left\{\int_0^T ds\int_{\mathbb{T}}(\partial_x H_i)(s,x)v(s,x)\,dx - 2\int_0^T ds\int_{\mathbb{T}} H_i(s,x)^2 v(s,x)\,dx\right\}\right] \leq K_0.$$

Letting $\ell \uparrow \infty$, we complete the proof of the lemma by applying the monotone convergence theorem. □

PROOF OF PROPOSITION 4.3. The proof is similar to that of [15], Theorem 5.7.1, and is left to the reader. Note that we in fact have

$$\int_0^T ds\int_{\mathbb{T}}\frac{(\partial_x v)(s,x)^2}{v(s,x)}\,dx < \infty.$$

□

4.3. $\mathfrak{M}_t = \pi_t$, *W almost surely.* We prove in this section that $\mathfrak{M}_t = \pi_t$, W almost surely.

LEMMA 4.6. *Every limit point $\mathcal{Q}^*$ of the sequence $\mathcal{Q}_N$ is concentrated on measures $\mathfrak{M}(dt, dx) = v(t,x)\,dt\,dx$ and $\pi(t,dx) = u(t,x)W(dx)$ such that $u = v$ $(dt \times W(dx))$ almost surely on* $[0,T] \times \mathbb{T}$.

PROOF. Fix a limit point $\mathcal{Q}^*$ of the sequence $\mathcal{Q}_N$ and assume, without loss of generality, that $\mathcal{Q}_N$ converges to $\mathcal{Q}^*$. Fix a continuous function $J : [0,T] \times \mathbb{T} \to \mathbb{R}$. By Corollary 4.2 and Lemma 3.3,

$$\lim_{\epsilon\to 0} E_{\mathcal{Q}^*}\biggl[\biggl|\int_0^T ds \int_{\mathbb{T}} J(s,x)u(s,x)W(dx) - \int_0^T ds \int_{\mathbb{T}} J(s,x)\epsilon^{-1}W_\epsilon(x)v(s,x)\,dx\biggr|\biggr] = 0.$$

It follows from the energy estimate stated in Proposition 4.3 that

$$\lim_{\epsilon\to 0}\int_0^T ds \int_{\mathbb{T}} dx\, J(s,x)\frac{1}{\epsilon}\int_{[x,x+\epsilon)}\{v(s,x) - v(s,y)\}W(dy) = 0$$

$\mathcal{Q}^*$ almost surely. Changing the order of summation, it follows from the continuity of J that

$$\lim_{\epsilon\to 0}\int_0^T ds \int_{\mathbb{T}} dx\, J(s,x)\frac{1}{\epsilon}\int_{[x,x+\epsilon)} v(s,y)W(dy) = \int_0^T ds \int_{\mathbb{T}} J(s,x)v(s,x)W(dx).$$

Hence, for all continuous functions J, $\mathcal{Q}^*$ almost surely

$$\int_0^T ds \int_{\mathbb{T}} J(s,x)\{u(s,x) - v(s,x)\}W(dx) = 0,$$

which proves the lemma. $\square$

5. Uniqueness of weak solutions.

THEOREM 5.1. *There exists at most one weak solution of* (2.7).

PROOF. We use a method due to Oleinik (see page 90 in [22]). Due to the linearity of problem (2.7), it is enough to show that the constant function equal to 0 is the unique weak solution of equation (2.7) with initial condition $u_0 \equiv 0$.

Fix such a solution u. By condition (i), u belongs to $L^2([0,T];\mathcal{H}_1)$. Since $u(t,\cdot)$ is continuous for almost all t, it is not difficult to show that there exists a sequence of smooth functions $u_\epsilon : [0,T] \times \mathbb{T} \to \mathbb{R}$, $\epsilon > 0$, such that $\|u_\epsilon\|_\infty \le \|u\|_\infty$ and

$$\lim_{\epsilon\to 0}\int_0^T dt\,\{\|u_\epsilon(t,\cdot) - u(t,\cdot)\|_{2,W}^2 + \|(\partial_x u_\epsilon)(t,\cdot) - (\partial_x u)(t,\cdot)\|_2^2\} = 0.$$

Consider the test function $G_\epsilon : [0, T] \times \mathbb{T} \to \mathbb{R}$ defined by

$$G_\epsilon(t, x) = -\int_t^T u_\epsilon(s, x)\, ds.$$

Since $\partial_t G_\epsilon = u_\epsilon$ and since $u_\epsilon(t, \cdot)$ converges to $u(t, \cdot)$ in $L^2(dW)$ for almost all t, by the dominated convergence theorem, we have

$$\lim_{\epsilon \to 0} \int_0^T \langle \partial_t G_\epsilon(t, \cdot), u(t, \cdot)\rangle_W \, dt = \int_0^T \langle u_t, u_t\rangle_W \, dt.$$

On the other hand, since

$$\int_0^T \langle (\partial_x G_\epsilon)(t, \cdot), (\partial_x u)(t, \cdot)\rangle \, dt = -\int_0^T dt \int_t^T \langle (\partial_x u_\epsilon)(s, \cdot), (\partial_x u)(t, \cdot)\rangle \, ds$$

and since $\partial_x u_\epsilon$ converges to $\partial_x u$ in $L^2([0, T] \times \mathbb{T})$, we have

$$\lim_{\epsilon \to 0} \int_0^T \langle (\partial_x G_\epsilon)(t, \cdot), (\partial_x u)(t, \cdot)\rangle \, dt = -\frac{1}{2} \int_{\mathbb{T}} \left(\int_0^T (\partial_x u)(t, x)\, dt \right)^2 dx.$$

Hence, by condition (ii), since $u_0 = 0$, we have

$$\int_0^T \langle u_t, u_t\rangle_W \, dt = -\frac{1}{4} \int_{\mathbb{T}} \left(\int_0^T (\partial_x u)(t, x)\, dt \right)^2 dx.$$

This show that $u_t \equiv 0$ for almost every t and uniqueness follows. □

6. Atomic trap models in dimension $d \geq 2$. We prove in this section Theorems 2.3, 2.4 and 2.5.

6.1. *Capacity and trace process*. To help the reader to follow the arguments of this section, we summarize below known results on capacity and trace processes used later. Consider a reversible, ergodic Markov chain $\{X_t : t \geq 0\}$ on a countable set E. Fix a nonempty subset F of E and denote by $\{X_t^F : t \geq 0\}$ the trace process of $\{X_t : t \geq 0\}$ on F, as defined in Section 2.2.

Denote by ν the unique invariant probability measure of $\{X_t : t \geq 0\}$ and by ν^F the invariant probability measure of the trace process $\{X_t^F : t \geq 0\}$. By Proposition 6.3 of [2], ν^F coincides with the measure ν conditioned to F and ν^F is reversible.

For $x \in E$ (resp. $x \in F$), let $\mathbb{P}_x$ (resp. $\mathbb{P}_x^F$) be the distribution on the path space $D(\mathbb{R}_+, E)$ [resp. $D(\mathbb{R}_+, F)$] induced by the process $\{X_t : t \geq 0\}$ (resp. $\{X_t^F : t \geq 0\}$) starting from x.

For a subset B of E (or F), denote by $H(B)$ the entry time in B, defined as

$$H(B) = \inf\{t \geq 0 : Z_t \in B\},$$

where Z_t stands either for X_t or for X_t^F. The context will always clarify to which process we are referring. Denote by $\tau(B)$ the time of first return of $\{X_t : t \geq 0\}$ to B:

$$\tau(B) = \inf\{t > T_1 : X_t \in B\},$$

where T_1 stands for the time of the first jump of $\{X_t : t \geq 0\}$. When the set B is a singleton $\{x\}$, we denote $H(\{x\})$, $\tau(\{x\})$ by $H(x)$, $\tau(x)$, respectively.

Denote by $\lambda : E \to \mathbb{R}_+$ the holding times of the Markov process $\{X_t : t \geq 0\}$. By Proposition 6.1 in [2], the rate $r^F(x, y)$ at which the trace process $\{X_t^F : t \geq 0\}$ jumps from a site $x \in F$ to a site $y \in F$, $y \neq x$, is given by

$$r^F(x, y) = \lambda(x)\mathbb{P}_x[H(y) < \tau(F \setminus \{y\})]. \tag{6.1}$$

The expectation of an entry time has a simple expression in terms of the capacities associated to the process $\{X_t : t \geq 0\}$. Denote by L the generator of the process $\{X_t : t \geq 0\}$. Let $A, B \subseteq E$ be two disjoint sets. Define

$$\mathcal{B}(A, B) = \{f : E \to \mathbb{R} : f(x) = 1 \text{ for } x \in A \text{ and } f(x) = 0 \text{ for } x \in B\}.$$

Let D be the Dirichlet form associated to $\{X_t : t \geq 0\}$: $D(f) = -\int f L f \, d\nu$ for any $f : E \to \mathbb{R}$. The capacity $\operatorname{cap}(A, B)$ between A and B is defined as

$$\operatorname{cap}(A, B) = \inf\{D(f) : f \in \mathcal{B}(A, B)\}. \tag{6.2}$$

Notice that $\operatorname{cap}(A, B) = \operatorname{cap}(B, A)$; it is enough to consider $\tilde{f} = 1 - f$. An elementary computation shows that $\operatorname{cap}(A, B) = D(f_{A,B})$, where $f_{A,B}$ is the unique solution of

$$\begin{cases} (Lf)(x) = 0, & \text{if } x \in E \setminus (A \cup B), \\ f(x) = 1, & \text{if } x \in A, \\ f(x) = 0, & \text{if } x \in B. \end{cases}$$

It is easy to see, by the strong Markov property, that

$$f_{A,B}(x) = \mathbb{P}_x[H(A) < H(B)]. \tag{6.3}$$

Let A, B be two disjoint subsets of F. Define $\operatorname{cap}_F(A, B)$ as the capacity between A and B with respect to the trace process $\{X_t^F : t \geq 0\}$. By Lemma 6.9 in [2],

$$\operatorname{cap}_F(A, B) = \frac{\operatorname{cap}(A, B)}{\nu(F)}. \tag{6.4}$$

The first result of this section establishes the relation between capacity and expectation of hitting times.

LEMMA 6.1. *For any subset A of F and any y in $F \setminus A$,*

$$\mathbb{E}_y^F[H(A)] = \frac{1}{\operatorname{cap}(y, A)} \sum_{z \in F} \nu(z)\mathbb{P}_z[H(y) < H(A)].$$

PROOF. Fix $A \subset F$, y in $F \setminus A$ and note that

$$\mathbb{E}_y^F[H(A)] = \mathbb{E}_y\left[\int_0^{H(A)} \mathbf{1}\{X_s \in F\} \, ds\right].$$

The result follows from Proposition 6.10 in [2] with $A = \{y\}$ and $B = A$. $\square$

Note that $\mathbb{P}_z[H(y) < H(A)]$ vanishes for z in A. In particular,

$$\mathbb{E}_y^F[H(A)] \le \frac{\nu(F \setminus A)}{\mathrm{cap}(y, A)}. \tag{6.5}$$

Capacities are also related to return times. The next result follows from Lemma 6.6 in [2] with $A = \{y\}$, $B = A$.

LEMMA 6.2. *Let A be a finite subset of E and let $y \in E \setminus A$. Then,*

$$\mathbb{P}_y[H(A) < \tau(y)] = \frac{\mathrm{cap}(A, \{y\})}{\lambda(y)\nu(y)}.$$

When the Markov chain $\{X_t : t \ge 0\}$ is not ergodic, the definition of the capacity can be generalized in a natural way. Assume that there exists a positive measure ν, reversible and invariant for $\{X_t : t \ge 0\}$. In this case, of course, $\nu(E) = +\infty$.

To define the capacity between a finite set A and infinity, consider an increasing sequence of finite sets $B_n \subseteq E$ such that $\bigcup_n B_n = E$. Since A is finite, $\mathrm{cap}(A, B_n^c)$, given by the variational formula (6.2), is well defined for n large enough. The sequence of functions f_{A,B_n^c}, introduced in (6.3) is increasing and bounded. Therefore, we can define $f_A(x) = \lim_n f_{A,B_n^c}(x)$. It is not difficult to check that

$$f_A(x) = \mathbb{P}_x[H(A) < \infty], \qquad D(f_A) = \inf\{D(f) : f \in \mathcal{B}(A)\},$$

where $\mathcal{B}(A)$ is the set of finitely supported functions $f : E \to \mathbb{R}$ such that $f(x) = 1$ for $x \in A$. Let $\mathrm{cap}(A) := D(f_A)$ be the capacity of A with respect to infinity.

By the dominated convergence theorem and Lemma 6.2, the following result holds.

LEMMA 6.3. *For any y in E,*

$$\mathbb{P}_y[\tau(y) = +\infty] = \frac{\mathrm{cap}(y)}{\lambda(y)\nu(y)}.$$

We conclude this subsection with two estimates for the simple symmetric random walk on the torus $\mathbb{T}_N^d$ or on the lattice $\mathbb{Z}^d$. Taking advantage of the commuting time identity, which relates expectations of hitting times with capacities, Proposition 10.13 of [17] establishes the following bounds for the hitting times of the simple symmetric random walk on $\mathbb{T}_N^d$.

LEMMA 6.4. *Let x, y be two points at distance k on the torus $\mathbb{T}_N^d$. There exist constants $0 < c_d < C_d < +\infty$ such that, in dimension $d \ge 3$,*

$$c_d N^d \le \mathbb{E}_x[H(y)] \le C_d N^d \qquad \textit{uniformly in } k$$

and in dimension 2,

$$c_2 N^2 \log k \le \mathbb{E}_x[H(y)] \le C_2 N^2 \log(k+1).$$

For $N > 0$, let us denote by Λ_N^* the cube of length $2N+1$ centered at the origin: $\Lambda_N^* = \{-N, \dots, N\}^d$. Denote by $\partial\Lambda_N$ its inner boundary: $\partial\Lambda_N = \{x \in \Lambda_N^* : \exists y \notin \Lambda_N^*$ with $|y-x| = 1\}$. The following lemma is Proposition 2.2.2 of [16].

LEMMA 6.5. *Let $X(t)$ be the simple symmetric random walk in $\mathbb{Z}^d$, $d \geq 3$. Then, there exist constants $0 < c_d < C_d < +\infty$ such that for any set $A \subseteq \Lambda_N^*$ and any $x \in \partial\Lambda_{2N}^*$, we have*

$$c_d N^{2-d} \operatorname{cap}(A) \leq \mathbb{P}_x[H_A < +\infty] \leq C_d N^{2-d} \operatorname{cap}(A).$$

Note that $\operatorname{cap}(A)$ is finite because the random walk is transient.

6.2. *Random walks in $\mathbb{T}_N^d$, $d \geq 3$.* In this subsection, we prove some properties of the simple random walk on $\mathbb{T}_N^d$ which will be used to establish its metastable behavior. Denote by $\{Y_k^N : k \geq 0\}$ the discrete-time, nearest-neighbor, symmetric random walk on $\mathbb{T}_N^d$ and let $\mathbb{Q}_x^N$, $x \in \mathbb{T}_N^d$, be the measure on $D(\mathbb{Z}_+, \mathbb{T}_N^d)$ induced by the random walk Y^N starting from x. Expectation with respect to $\mathbb{Q}_x^N$ is denoted by the same symbol.

For a subset B of $\mathbb{T}_N^d$, denote by $H(B)$ the entry time in B, defined as

$$H(B) = \inf\{k \geq 0 : Y_k^N \in B\}.$$

In Lemma 6.6 below, we prove that whenever the simple random walk starts from a point isolated from the very deep traps, asymptotically, the next very deep trap to be visited is uniformly chosen. In Corollary 6.7, we obtain the limiting distribution of the next very deep trap to be visited starting from another very deep trap. Corollary 6.8 presents the limit of the capacity between two points of $\mathbb{T}_N^d$ which are far apart.

Let $d(x, y)$ be the distance induced by the graph $\mathbb{T}_N^d$. For a subset Γ of $\mathbb{T}^d$ and $r > 0$, denote by $B(\Gamma, r)$ the set of sites in $\mathbb{T}_N^d$ at distance less than or equal to r from Γ: $B(\Gamma, r) = \{x \in \mathbb{T}_N^d : d(x, \Gamma) \leq r\}$. Denote by ∂G the sites not in Γ which are at distance one from Γ: $\partial\Gamma = \{x \notin \Gamma : d(x, \Gamma) = 1\}$. In these definitions, we have identified Γ with its immersion in $\mathbb{T}_N^d$: $\Gamma = \Gamma \cap \mathbb{T}_N^d$.

LEMMA 6.6. *Suppose that $l_N \uparrow \infty$ and $A_M^N = \{x_1^N, \dots, x_M^N\} \subset \mathbb{T}_N^d$ are such that, if $i \neq j$, $d(x_i^N, x_j^N) \geq l_N$. Then, defining $A_{M,i}^N$ to be $A_M^N \setminus \{x_i^N\}$, we have*

$$\lim_{N\to\infty} \sup_{y \in B(A_M^N, l_N)^c} \left| \mathbb{Q}_y^N[H(x_1^N) < H(A_{M,1}^N)] - \frac{1}{M} \right| = 0.$$

PROOF. The proof is based on the fact that a site is reached on the scale N^d and equilibrium is reached on the scale N^2. Hence, on an intermediate scale, the process has not reached A_M^N and is in equilibrium. In particular, it has a probability $1/M$ to attain x_1^N before the set $A_{M,1}^N$.

First, we prove that the process does not reach a site on the scale $N^{5/2}$:

$$\lim_{N\to\infty} \sup_{y;d(y,0)\ge l_N} \mathbb{Q}_y^N[H(0) < N^{5/2}] = 0. \tag{6.6}$$

By the strong Markov property,

$$\mathbb{Q}_y^N[H(0) < N^{5/2}] \le \mathbb{Q}_y^N\bigl[H(0) < H\bigl(\partial B(0, N/8)\bigr)\bigr] + \sup_{z\in\partial B(0,N/8)} \mathbb{Q}_z^N[H(0) < N^{5/2}].$$

Since $d(y,0) \ge l_N$, by Lemma 6.5, the first term in the sum above is bounded by C_0/l_N^{d-2} for some constant $C_0 > 0$ independent of N. We can therefore suppose in (6.6) that $l_N = N/8$.

Denote by $\{R_k : k \ge 1\}$ and $\{D_k : k \ge 1\}$ the successive return and departure times between $B_1 = B(0, N/8)$ and $B_2 = B(0, N/4)$:

$$R_1 = H_{B_1}, \qquad D_1 = R_1 + H_{B_2^c} \circ \theta_{R_1},$$
$$R_n = D_{n-1} + H_{B_1} \circ \theta_{D_{n-1}}, \qquad D_n = R_n + H_{B_2^c} \circ \theta_{R_n}, \qquad n \ge 2.$$

Here, $\theta_s : D(\mathbb{Z}_+, \mathbb{T}_N^d) \to D(\mathbb{Z}_+, \mathbb{T}_N^d)$ is the shift map given by $Y_t \circ \theta_s = Y_{s+t}$.

For y such that $d(y,0) \ge N/8$, by the strong Markov property, we have

$$\begin{aligned} \mathbb{Q}_y^N[H(0) < N^{5/2}] &\le \mathbb{Q}_y^N[R_{N^{d-9/4}} \le N^{5/2}] + \mathbb{Q}_y^N[H(0) < R_{N^{d-9/4}}] \\ &\le \sup_{z;d(z,0)\ge N/8} \mathbb{Q}_z^N[R_{N^{d-9/4}} \le N^{5/2}] \\ &\quad + N^{d-9/4} \sup_{z\in\partial B(0,N/8)} \mathbb{Q}_z^N\bigl[H(0) < H\bigl(\partial B(0, N/4)\bigr)\bigr]. \end{aligned} \tag{6.7}$$

The right-hand side does not depend on the choice of y and tends to zero as $N \uparrow \infty$, by [8], Proposition 1.1, with $u = N^{-1/2}$, and Lemma 6.5. This proves (6.6).

In the time scale $N^{5/2}$, the process reaches equilibrium. More precisely, denote by π^N the uniform probability measure on $\mathbb{T}_N^d$ and by $\|\mu - \nu\|$ the total variation distance between two measures μ, ν on $\mathbb{T}_N^d$. By Corollary 5.3 and equation (5.9) in [17], for an arbitrary sequence $y^N \in B(A_M^N, l_N)^c$,

$$\lim_{N\to\infty} \|\mathbb{Q}_{y^N}^N[Y_{N^{5/2}} = \cdot] - \pi^N(\cdot)\| = 0. \tag{6.8}$$

In particular, for $1 \le i \le M$ and an arbitrary sequence y^N in $B(A_M^N, l_N)^c$,

$$\lim_{N\to\infty} \frac{1}{N^d} |\mathbb{Q}_{y^N}^N[H(x_i^N)] - \mathbb{Q}_{\pi^N}^N[H(x_i^N)]| = 0. \tag{6.9}$$

To prove this claim, fix $1 \le i \le M$ and introduce the indicators of the sets $H(x_i^N) < N^{5/2}$, $H(x_i^N) \ge N^{5/2}$, to obtain that

$$\mathbb{Q}_{y^N}^N[H(x_i^N)] = \mathbb{Q}_{y^N}^N[\mathbb{Q}_{Y_{N^{5/2}}}^N[H(x_i^N)]] + R_N,$$

where the remainder R_N is absolutely bounded by

$$N^{5/2} + \sup_{z \in \mathbb{T}_N^d} \mathbb{Q}_z^N[H(x_i^N)]\mathbb{Q}_{y^N}^N[H(x_i^N) < N^{5/2}].$$

Hence,

$$\mathbb{Q}_{y^N}^N[H(x_i^N)] - \mathbb{Q}_{\pi^N}^N[H(x_i^N)] = \mathbb{Q}_{y^N}^N\big[\mathbb{Q}_{Y_{N^{5/2}}}^N[H(x_i^N)] - \mathbb{Q}_{\pi^N}^N[H(x_i^N)]\big] + R_N$$

is absolutely bounded by

$$N^{5/2} + \sup_{z \in \mathbb{T}_N^d} \mathbb{Q}_z^N[H(x_i^N)]\{\mathbb{Q}_{y^N}^N[H(x_i^N) < N^{5/2}] + \|\mathbb{Q}_{y^N}^N[Y_{N^{5/2}} = \cdot] - \pi^N(\cdot)\|\}.$$

By (6.6), (6.8) and Lemma 6.4, this expression divided by N^d vanishes as $N \uparrow \infty$. This proves (6.9).

Recall that v_d stands for the probability that a symmetric, nearest-neighbor random walk on $\mathbb{Z}^d$ never returns to its starting point. The escape probability v_d equals the capacity between the origin and infinity in $\mathbb{Z}^d$; see Lemma 6.3. Note that $\lambda(0) = 1$ and that the capacity is computed with respect to the counting measure.

In [23], a variational characterization is used to estimate the expected hitting time of certain sets when the random walk starts from the uniform measure; see (3.2) of [23]. In particular, it gives that

$$\lim_{N\to\infty} \frac{N^d}{\mathbb{Q}_{\pi^N}^N[H(C_N)]} = K \cdot v_d,$$

where $C_N \subset A_M^N$ has K points.

By the equation above and (6.9), it follows that

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N^d}\mathbb{Q}_{y^N}^N[H(x_i^N)] = \frac{1}{v_d} \qquad \text{and, analogously, for } j \neq 1, \\ &\lim_{N\to\infty} \frac{1}{N^d}\mathbb{Q}_{x_j^N}^N[H(x_1^N)] = \frac{1}{v_d} \quad \text{and} \qquad (6.10) \\ &\lim_{N\to\infty} \frac{1}{N^d}\mathbb{Q}_{x_1^N}^N[H(A_{M,1}^N)] = \frac{1}{v_d(M-1)}. \end{aligned}$$

Define S to be the stopping time given by the first visit to y^N after the first visit to x_1^N, after visiting $A_{M,1}^N$. By the strong Markov property, $\mathbb{Q}_{y^N}^N[S]$ is equal to

$$\mathbb{Q}_{y^N}^N[H(A_{M,1}^N)] + \sum_{j\neq 1} \mathbb{Q}_{y^N}^N\big[Y_{H(A_{M,1}^N)} = x_j^N\big]\mathbb{Q}_{x_j^N}^N[H(x_1^N)] + \mathbb{Q}_{x_1^N}^N[H(y^N)].$$

Hence, by (6.10),

$$\lim_{N\to\infty} \frac{1}{N^d} \mathbb{Q}^N_{y^N}[S] = \frac{1}{v_d}\Big(\frac{1}{M-1} + 2\Big). \tag{6.11}$$

We are now in a position to prove the lemma. Fix an arbitrary sequence y^N in $B(A^N_M, l_N)^c$. Since $\mathbb{Q}^N_{y^N}[\#\{\text{visits to } x^N_1 \text{ before } H(A^N_{M,1})\}] = \mathbb{Q}^N_{y^N}[\#\{\text{visits to } x^N_1 \text{ before } H(A^N_{M,1})\}\mathbf{1}\{H(x^N_1) < H(A^N_{M,1})\}]$, by the strong Markov property, we have

$$\mathbb{Q}^N_{y^N}[H(x^N_1) < H(A^N_{M,1})] = \frac{\mathbb{Q}^N_{y^N}[\#\{\text{visits to } x^N_1 \text{ before } H(A^N_{M,1})\}]}{\mathbb{Q}^N_{x^N_1}[\#\{\text{visits to } x^N_1 \text{ before } H(A^N_{M,1})\}]}. \tag{6.12}$$

To estimate the numerator, observe that, by [1], Chapter 2, Proposition 3 and Lemma 7,

$$\begin{aligned}
\frac{1}{N^d}\mathbb{Q}^N_{y^N}[S] &= \mathbb{Q}^N_{y^N}[\#\{\text{visits to } x^N_1 \text{ before } S\}] \\
&= \mathbb{Q}^N_{y^N}[\#\{\text{visits to } x^N_1 \text{ before } H(A^N_{M,1})\}] \\
&\quad + \mathbb{Q}^N_{x^N_1}[\#\{\text{visits to } x^N_1 \text{ before } H(y^N)\}] \\
&= \mathbb{Q}^N_{y^N}[\#\{\text{visits to } x^N_1 \text{ before } H(A^N_{M,1})\}] \\
&\quad + \frac{1}{N^d}\{\mathbb{Q}^N_{x^N_1}[H(y^N)] + \mathbb{Q}^N_{y^N}[H(x^N_1)]\}.
\end{aligned}$$

Hence, the right-hand side of (6.12) can be written as

$$\frac{N^{-d}(\mathbb{Q}^N_{y^N}[S] - \mathbb{Q}^N_{x^N_1}[H(y^N)] - \mathbb{Q}^N_{y^N}[H(x^N_1)])}{\mathbb{Q}^N_{x^N_1}[\#\{\text{visits to } x^N_1 \text{ before } H(x^N_1)\circ\theta_{H(A^N_{M,1})} + H(A^N_{M,1})\}]}.$$

Let S' be the stopping time $H(A^N_{M,1}) + H(x^N_1)\circ\theta_{H(A^N_{M,1})}$. By Lemma 7, Chapter 2 in [1], the denominator is equal to $N^{-d}\mathbb{Q}^N_{x^N_1}[S']$. Hence, by the strong Markov property on $H(A^N_{M,1})$, the previous ratio is equal to

$$\frac{N^{-d}(\mathbb{Q}^N_{y^N}[S] - \mathbb{Q}^N_{x^N_1}[H(y^N)] - \mathbb{Q}^N_{y^N}[H(x^N_1)])}{N^{-d}(\mathbb{Q}^N_{x^N_1}[H(A^N_{M,1})] + \sum_{j\neq 1}\mathbb{Q}^N_{x^N_1}[Y_{H(A^N_{M,1})} = x^N_j]\mathbb{Q}^N_{x^N_j}[H(x^N_1)])}.$$

By (6.10), (6.11), this expression converges to M^{-1} as $N \uparrow \infty$. Since the sequence $y^N \in B(A^N, l_N)^c$ is arbitrary, we are done. □

For a subset F of $\mathbb{T}^d_N$, let $\hat\tau(F)$ be the return time to F for the discrete-time random walk $\{Y^N_k : k \geq 0\}$:

$$\hat\tau(F) = \inf\{k \geq 1 : Y^N_k \in F\}.$$

For subsets A, B of $\mathbb{T}_N^d$ such that $A \cap B = \phi$, let $\mathrm{cap}_{Y^N}(A, B)$ be the capacity between A and B induced by the process Y^N:

$$\mathrm{cap}_{Y^N}(A, B) = \inf_f \frac{1}{4d} \sum_{x \in \mathbb{T}_N^d} \sum_{y \sim x} [f(y) - f(x)]^2,$$

where the infimum is carried over all functions $f : \mathbb{T}_N^d \to \mathbb{R}$ such that $f(x) = 1$ for all x in A, $f(x) = 0$ for all x in B.

COROLLARY 6.7. *Under the same conditions of Lemma 6.6, for $j \neq 1$,*

$$\lim_{N\to\infty} \mathbb{Q}^N_{x_j^N}[H(x_1^N) < \hat{\tau}(A^N_{M,1})] = \frac{v_d}{M}.$$

PROOF. Since

$$\lim_{N\to\infty} \mathbb{Q}^N_{x_j^N}[H(\partial B(x_j^N, l_N)) < \hat{\tau}(x_j^N)] = v_d,$$

the result follows from the strong Markov property and Lemma 6.6. □

COROLLARY 6.8. *If $l_N \uparrow \infty$ and $x^N, y^N \in \mathbb{T}_N^d$ are such that $d(x^N, y^N) \geq l_N$, then*

$$\lim_{N\to\infty} \mathrm{cap}_{Y^N}(x^N, y^N) = \frac{v_d}{2}. \tag{6.13}$$

PROOF. The corollary is a direct application of Lemma 6.2 and Corollary 6.7. □

6.3. *Random walk on $\mathbb{T}_N^d$ for $d \geq 2$.* We present similar results to the ones stated in the previous subsection, but which also hold in dimension 2. Here, however, we need to impose that the distances l_N between the very deep traps and the starting point of the walk grow close to linearly.

Although Lemma 6.9 below also holds for $d \geq 3$ and could be used in place of Lemma 6.6, we keep both results since they are based on different arguments. Let l_N be an increasing sequence such that $l_N/N \to 0$ and $l_N/N^\alpha \to \infty$ for every $\alpha < 1$. In this way,

$$\lim_{N\to\infty} \frac{l_N}{N} = 0, \qquad \lim_{N\to\infty} \frac{\log l_N}{\log N} = 1. \tag{6.14}$$

In this subsection, it will be more convenient to work with the distance $d_2(x, y)$ in $\mathbb{T}_N^d$ given by N times the Euclidean distance between x and y in $\mathbb{T}^d$.

LEMMA 6.9. *Consider a sequence of sets $A_M^N = \{x_1^N, \ldots, x_M^N\} \subset \mathbb{T}_N^d$ such that $d_2(x_i, x_j) \geq l_N$ $(0 \leq i < j \leq M)$ for some sequence $\{l_N : N \geq 1\}$ satisfying (6.14). Then,*

$$\lim_{N\to\infty} \sup_{y \in B(A_M^N, l_N)^c} \left| \mathbb{Q}_y^N[H(x_1^N) < H(A^N_{M,1})] - \frac{1}{M} \right| = 0.$$

Since the result only concerns the first point in A_M^N to be visited, we can suppose that the process $\{Y_k : k \geq 0\}$ is a lazy random walk in $\mathbb{T}_N^d$, that is, with probability $1/2$, Y does not move, otherwise it jumps uniformly to one of its neighbors. Before proceeding to the proof of Lemma 6.9, we collect some properties of the hitting and mixing times of Y.

Recall that $\|\cdot\|$ denotes the total variation distance between two probability measures. The following bound on the mixing time on the torus follows, for instance, from Corollary 5.3 and equation (5.9) in [17]:

$$\lim_{\beta\to\infty} \limsup_{N\to\infty} \|\mathbb{Q}_y^N[Y_{\beta N^2} = \cdot] - \pi^N(\cdot)\| = 0. \tag{6.15}$$

Of course, the same result holds for any sequence $\{t_N : N \geq 1\}$ which increases to ∞ faster than N^2.

We claim that for every $\beta > 0$ and $x^N, y^N \in \mathbb{T}_N^d$ such that $d_2(x^N, y^N) \geq l_N$,

$$\lim_{N\to\infty} \mathbb{Q}_{y^N}^N[H(x^N) \leq \beta N^2] = 0. \tag{6.16}$$

Since, for $d \geq 3$, this statement follows from (6.6), we concentrate on the case $d = 2$. Recall that $\mathbb{Y}$ stands for the simple random walk on $\mathbb{Z}^2$ (with law $\mathbb{P}$) and denote by $\phi_N : \mathbb{Z}^2 \to \mathbb{T}_N^2$ the canonical projection. In view of the invariance principle in $\mathbb{Z}^2$, it is enough to prove that for every $R \geq 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\bar{y}^N}[\phi_N(\mathbb{Y}) \text{ visits } x^N \text{ before } \mathbb{Y} \text{ exits } B(\bar{y}^N, RN)] = 0,$$

where $\bar{y}^N \in \mathbb{Z}^2$ is such that $\phi_N(\bar{y}^N) = y^N \in \mathbb{T}_N^2$. This follows, for instance, from [7], (225), since for every point $\bar{x}^N \in B(\bar{y}^N, RN)$ such that $\phi(\bar{x}^N) = x^N$, we have $|\bar{y}^N - \bar{x}^N| \geq l_N$ and, moreover, the number of possible choices for $\bar{x}^N$ is bounded uniformly on $N \geq 1$. This establishes (6.16) for $d = 2$.

A straightforward consequence of (6.16) is that

$$\lim_{N\to\infty} \mathbb{Q}_{\pi^N}^N[H(x^N) \leq \beta N^2] = 0 \tag{6.17}$$

for any sequence $\{x^N : N \geq 1\}$ in $\mathbb{T}_N^d$.

Now, we can state the hitting time estimate which we will use in the proof. Consider the scales

$$h_N^d = \begin{cases} N^2 \log N, & \text{if } d = 2, \\ N^d, & \text{if } d \geq 3. \end{cases}$$

We claim that for every $d \geq 2$,

$$\lim_{\gamma\to 0} \limsup_{N\to\infty} \sup_{x,y; d_2(x,y) > l_N} \mathbb{Q}_y^N[H(x) < \gamma h_N^d] = 0. \tag{6.18}$$

Indeed, fix $\delta > 0$. By (6.16), for any $\beta > 0$, any N sufficiently large and any x, y such that $d_2(x, y) > l_N$, we have

$$\mathbb{Q}_y^N[H(x) < \gamma h_N^d] \leq \delta + \mathbb{Q}_y^N[H(x) \geq \beta N^2, H(x) < \gamma h_N^d].$$

On the set $H(x) \ge \beta N^2$, $H(x) = \beta N^2 + H(x) \circ \theta_{\beta N^2}$. Therefore, by the Markov property at βN^2, the second term is less than or equal to

$$E_{\mathbb{Q}_y^N}[\mathbb{Q}^N_{Y_{\beta N^2}}[H(x) < \gamma h_N^d]].$$

By (6.15), for β large enough, this expression is bounded by

$$\delta + \mathbb{Q}^N_{\pi^N}[H(x) < \gamma h_N^d].$$

The result now follows from Theorem 2.1 of [23] and Lemma 6.4.

Finally, we claim that for every positive γ, the probability to hit a very deep trap before γh_N^d is bounded away from zero in N. More precisely, for every $\gamma > 0$,

$$\limsup_{N\to\infty} \mathbb{Q}^N_{\pi^N}[H(x) \ge \gamma h_N^d] < 1. \tag{6.19}$$

The claim above also follows from Theorem 2.1 of [23] and Lemma 6.4.

PROOF OF LEMMA 6.9. Our strategy is to consider several consecutive attempts to hit one of the points in A_M^N. For $\gamma > 0$ and a positive integer L, define the times

$$a_i = i(LN^2 + \gamma h_N^d), \qquad b_i = i(LN^2 + \gamma h_N^d) + LN^2$$

for $i \ge 0$. Intuitively, for each i, we use the intervals $[a_i, b_i]$ to approach equilibrium measure π^N and the intervals $[b_i, a_{i+1}]$ to attempt to hit the set A_M^N.

Let

$$\begin{aligned} R_{L,N} &:= \max_{y\in\mathbb{T}_N^d} \|\mathbb{Q}_y^N[Y^N_{LN^2} = \cdot] - \pi^N(\cdot)\|, \\ S_{L,N} &:= \sup_{y\in B(A_M^N, l_N)^c} \mathbb{Q}_y^N[H(A_M^N) \le LN^2], \\ T_{L,N} &:= \mathbb{Q}^N_{\pi^N}[H(A_M^N) \le LN^2]. \end{aligned}$$

By symmetry, the maximum is irrelevant in the definition of $R_{L,N}$ and, by (6.15), $R_{L,N}$ vanishes as $N \uparrow \infty$ and then $L \uparrow \infty$. By (6.16) and (6.17), $S_{L,N}$ and $T_{L,N}$ vanish as $N \uparrow \infty$ for every $L \ge 1$.

For $0 \le s < t$, define the random variable $J_{s,t}$, which takes the value 0 if the set $A_M^N = \{x_1^N, \dots, x_M^N\}$ is not visited between the times s and t, and otherwise $J_{s,t} = 1, \dots, M$, according to the index of the first point in A_M^N visited in this interval.

The proof of the lemma is divided into two parts. We first claim that for every $\gamma > 0$,

$$\limsup_{N\to\infty} \sup_{y\in B(A_M^N, l_N)^c} \left| \mathbb{Q}_y^N[H(x_1^N) < H(A^N_{M,1})] - \frac{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} = 1]}{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} \ne 0]} \right| = 0. \tag{6.20}$$

Note that this expression does not depend on L, but only on γ and N. We then prove that

$$\lim_{\gamma\to 0}\limsup_{N\to\infty}\left|\frac{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=1]}{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}\neq 0]}-\frac{1}{M}\right|=0. \tag{6.21}$$

Clearly, Lemma 6.9 follows from (6.20) and (6.21).

The proof of (6.20) relies on three estimates. Consider the event $\mathfrak{D}_F=[J_{a_i,b_i}\neq 0 \text{ for some } i=0,\dots,F-1]$, $F\geq 1$. This event indicates that some very deep trap is visited in one of the "mixing" intervals $[a_j,b_j]$. We claim that

$$\sup_{y\in B(A^N_M,l_N)^c}\mathbb{Q}^N_y[\mathfrak{D}_F]\leq FT_{L,N}+S_{L,N}+R_{L,N}. \tag{6.22}$$

Fix a site y in $B(A^N_M,l_N)^c$ and decompose the event $\mathfrak{D}_F$ according to whether the set A^N_M has been attained before time LN^2 or not to get that

$$\mathbb{Q}^N_y[\mathfrak{D}_F]\leq\mathbb{Q}^N_y[H(A^N_M)\leq LN^2]+\mathbb{Q}^N_y[\mathfrak{D}_F,H(A^N_M)>LN^2].$$

The first term is bounded by $S_{L,N}$. On the set $H(A^N_M)>LN^2$, the event $\mathfrak{D}_F$ is equal to $\bigcup_{1\leq i\leq F-1}\{J_{a_i,b_i}\neq 0\}$. Hence, by the Markov property at time b_0, the second term on the right-hand side of the previous inequality is bounded by

$$E_{\mathbb{Q}^N_y}\left[\mathbb{Q}^N_{Y^N_{b_0}}\left[\bigcup_{i=1}^{F-1}\{J_{a_i-b_0,b_i-b_0}\neq 0\}\right]\right]\leq R_{L,N}+\mathbb{Q}^N_{\pi^N}\left[\bigcup_{i=0}^{F-2}\{J_{a_i,b_i}\neq 0\}\right].$$

Since π^N is the stationary state, the second term is less than or equal to $(F-1)$ $\mathbb{Q}^N_{\pi^N}[H(A^N_N)\leq LN^2]$. In conclusion, we have proven that

$$\mathbb{Q}^N_y[\mathfrak{D}_F]\leq S_{L,N}+F\mathbb{Q}^N_{\pi^N}[H(A^N_M)\leq LN^2]+R_{L,N},$$

which is exactly (6.22).

Let $\mathfrak{E}_F$, $F\geq 1$, be the event that no site in A^N_M has been visited in the "hitting" intervals $[b_i,a_{i+1}]$, $0\leq i\leq F-1$: $\mathfrak{E}_F=[J_{b_i,a_{i+1}}=0 \text{ for all } i=0,\dots,F-1]$. We claim that

$$\sup_{y\in B(A^N_M,l_N)^c}|\mathbb{Q}^N_y[\mathfrak{E}_F]-\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]^F|\leq FR_{L,N}. \tag{6.23}$$

Fix a site y in $B(A^N_M,l_N)^c$. By the Markov property,

$$\mathbb{Q}^N_y[\mathfrak{E}_F]=\mathbb{Q}^N_y\left[\bigcap_{i=0}^{F-2}\{J_{b_i,a_{i+1}}=0\}\mathbb{Q}^N_{Y^N_{a_{F-1}}}[J_{b_0,a_1}=0]\right].$$

Applying the Markov property at time $b_0=LN^2$, this expression can be written as

$$\mathbb{Q}^N_y\left[\bigcap_{i=0}^{F-2}\{J_{b_i,a_{i+1}}=0\}\right]\mathbb{Q}^N_{\pi^N}[J_{0,a_1-b_0}=0]+R_N,$$

where the remainder R_N is absolutely bounded by $R_{L,N}$. Since π^N is the stationary state, $\mathbb{Q}^N_{\pi^N}[J_{0,a_1-b_0}=0]=\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]$. We proceed by induction to derive (6.23).

Let $\mathfrak{H}_F$, $F\ge 1$, be the event $\{H(x_1)<H(A^N_{M,1})\}\cap\mathfrak{D}^c_F\cap\mathfrak{E}^c_F$. We claim that

$$\begin{aligned}&\sup_{y\in B(A^N_M,l_N)^c}\left|\mathbb{Q}^N_y[\mathfrak{H}_F]-\frac{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=1]}{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}\neq 0]}\right|\\ &\qquad\le F^2R_{L,N}+\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]^F+\mathbb{Q}^N_y[\mathfrak{D}_F].\end{aligned}\tag{6.24}$$

Clearly,

$$\mathfrak{H}_F=\mathfrak{D}^c_F\cap\bigcup_{j=0}^{F-1}\left\{\bigcap_{k=0}^{j-1}\{J_{b_k,a_{k+1}}=0\}\cap\{J_{b_j,a_{j+1}}=1\}\right\}.$$

Repeating the arguments which led to (6.23), we get that, for each $0\le j\le F-1$ and any site y in $B(A^N_M,l_N)^c$,

$$\begin{aligned}&\mathbb{Q}^N_y\left[\bigcap_{k=0}^{j-1}\{J_{b_k,a_{k+1}}=0\}\cap\{J_{b_j,a_{j+1}}=1\}\right]\\ &\qquad=\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]^j\,\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=1]+R_N,\end{aligned}$$

where the remainder R_N is absolutely bounded by $(j+1)R_{L,N}$. The value of the remainder R_N may change from line to line below. Summing over j, we get that

$$\mathbb{Q}^N_y[\mathfrak{H}_F]=\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=1]\sum_{j=0}^{F-1}\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]^j+R_N,$$

where the remainder R_N is now absolutely bounded by $F^2R_{L,N}+\mathbb{Q}^N_y[\mathfrak{D}_F]$. This expression can be written as

$$\frac{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=1]}{1-\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]}+R_N$$

for a remainder R_N absolutely bounded by $F^2R_{L,N}+\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]^F+\mathbb{Q}^N_y[\mathfrak{D}_F]$, which is precisely (6.24).

By the estimates (6.22), (6.23), (6.24), the supremum in (6.20) is bounded by

$$2FT_{L,N}+2S_{L,N}+4F^2R_{L,N}+2\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]^F$$

for every $F\ge 1$, $L\ge 1$, $\gamma>0$. By (6.19), $\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1}=0]$, which does not depend on L, is bounded above by a constant strictly less than one. Hence, as $N\uparrow\infty$ and then $L\uparrow\infty$, $R_{L,N}$, $S_{L,N}$ and $T_{L,N}$ vanish. It remains to let $F\uparrow\infty$ to complete the proof of (6.20).

It remains to prove (6.21). Decompose the event $\{J_{b_0,a_1} = 1\}$ according to the event that at least two sites in A_M^N have been visited in the time interval $[b_0, a_1]$ to get that

$$\big|\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} = 1] - \mathbb{Q}^N_{\pi^N}\big[Y_{[b_0,a_1]} \cap A_M^N = \{x_1\}\big]\big| \le \mathbb{Q}^N_{\pi^N}\big[\#\big(Y_{[b_0,a_1]} \cap A_M^N\big) > 1\big].$$

In this formula, $Y_{[b_0,a_1]}$ stands for the sites visited by the random walk in the interval $[b_0, a_1]$, $Y_{[b_0,a_1]} = \{Y_t : b_0 \le t \le a_1\}$ and $\#A$ denotes the cardinality of A. By similar reasoning,

$$\big|\mathbb{Q}^N_{\pi^N}\big[Y_{[b_0,a_1]} \cap A_M^N = \{x_1\}\big] - \mathbb{Q}^N_{\pi^N}\big[x_1 \in Y_{[b_0,a_1]}\big]\big| \le \mathbb{Q}^N_{\pi^N}\big[\#\big(Y_{[b_0,a_1]} \cap A_M^N\big) > 1\big].$$

Therefore,

$$\big|\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} = 1] - \mathbb{Q}^N_{\pi^N}\big[x_1 \in Y_{[b_0,a_1]}\big]\big| \le 2\mathbb{Q}^N_{\pi^N}\big[\#\big(Y_{[b_0,a_1]} \cap A_M^N\big) > 1\big].$$

Note that the probability $\mathbb{Q}^N_{\pi^N}[y \in Y_{[b_0,a_1]}]$ does not depend on y, by symmetry. In particular, it also follows from the previous arguments that

$$\big|\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} \ne 0] - M\mathbb{Q}^N_{\pi^N}\big[x_1 \in Y_{[b_0,a_1]}\big]\big| \le (M+1)\mathbb{Q}^N_{\pi^N}\big[\#\big(Y_{[b_0,a_1]} \cap A_M^N\big) > 1\big]$$

so that

$$|\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} = 1] - (1/M)\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} \ne 0]| \le 4\mathbb{Q}^N_{\pi^N}\big[\#\big(Y_{[b_0,a_1]} \cap A_M^N\big) > 1\big].$$

Equivalently,

$$\left|\frac{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} = 1]}{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} \ne 0]} - \frac{1}{M}\right| \le \frac{4\mathbb{Q}^N_{\pi^N}[\#(Y_{[b_0,a_1]} \cap A_M^N) > 1]}{\mathbb{Q}^N_{\pi^N}[J_{b_0,a_1} \ne 0]}\cdot$$

By the strong Markov property, the right-hand side is bounded above by

$$\max_{1 \le i \le M} \mathbb{Q}^N_{x_i}[H(A^N_{M,i}) < \gamma h_N^d].$$

By (6.18), this expression vanishes as $N \uparrow \infty$ and then $\gamma \downarrow 0$. This completes the proof of the lemma. $\square$

We also need to estimate in dimension 2 the probability that the random walk escapes from a deep trap. More precisely,

LEMMA 6.10. *Assume that $d = 2$ and let $\{l_N : N \ge 1\}$ be a sequence satisfying (6.14). Then, for any sequence $\{y^N \in \mathbb{T}_N^2 : N \ge 1\}$,*

$$\lim_{N \to \infty} \log N \mathbb{Q}^N_{y^N}[H(B(y^N, l_N)^c) < \hat{\tau}(y^N)] = \frac{\pi}{2}\cdot$$

PROOF. The number of visits that the random walk performs to y^N before exiting $B(y^N, l_N)$ is a geometric random variable with failure probability given by $\mathbb{Q}^N_{y^N}[H(B(y^N, l_N)^c) < \hat{\tau}(y^N)]$. The inverse of this probability is equal to the expected number of visits to y^N before exiting $B(y^N, l_N)$. By [16], Theorem 1.6.6, the expected number of visits, denoted by $G_{B(y^N,l_N)}(y^N, y^N)$ in the notation of Green's functions, is given by

$$G_{B(y^N,l_N)}(y^N, y^N) = \frac{2}{\pi} \log N + K + O(N^{-1})$$

for some constant K in $\mathbb{R}$. The result follows from this estimate. □

COROLLARY 6.11. *Assume that* $d = 2$. *Under the hypotheses of Lemma* 6.9, *for* $j \neq 1$,

$$\lim_{N\to\infty} \log N \mathbb{Q}^N_{x_j^N}[H(x_1^N) < \hat{\tau}(A^N_{M,1})] = \frac{\pi}{2M}\cdot$$

PROOF. This result follows from the strong Markov property and Lemmas 6.9 and 6.10. □

COROLLARY 6.12. *Assume that* $d = 2$ *and let* $\{R_N : N \geq 1\}$ *be a sequence such that* $R_N \uparrow \infty$. *Let* $x^N, y^N \in \mathbb{T}^2_N$ *be such that* $d(x^N, y^N) \geq R_N$. *Then,*

$$\lim_{N\to\infty} \log N \operatorname{cap}_{Y^N}(x^N, y^N) = \frac{\pi}{4}\cdot \tag{6.25}$$

PROOF. The corollary is a direct application of Lemma 6.2 and Corollary 6.11. □

6.4. *Metastability of the trap model in dimension* $d \geq 3$. Recall that we have denoted by v_d the probability that a nearest-neighbor, symmetric random walk on $\mathbb{Z}^d$ never returns to its starting point. As in the previous subsections, we denote by Y^N_k the discrete-time random walk on the torus $\mathbb{T}^d_N$, inducing the law $\mathbb{Q}^N_x$ on $D(\mathbb{Z}_+, \mathbb{T}^d_N)$. The proof of Theorem 2.3 is divided into two parts. In Proposition 6.13 below, we show that the trace process converges, and in Corollary 6.15, that the time spent outside A^N_M is negligible.

PROPOSITION 6.13. *Fix* $M > 1$ *and* $T > 0$. *As* $N \uparrow \infty$, *the process* $\{X^{N,M}_t : 0 \leq t \leq T\}$ *converges in distribution to the Markov process on* $\{1, \dots, M\}$ *with generator* L_M *given by*

$$(\mathfrak{L}_M f)(i) = \frac{v_d}{M \hat{w}_i} \sum_{j=1}^{M} [f(j) - f(i)].$$

PROOF. Fix $M \ge 1$ and denote by $r_{N,M} : \{1, \dots, M\} \times \{1, \dots, M\} \to \mathbb{R}_+$ the jump rates of the trace process $\{X_t^{N,M} : 0 \le t\}$. By (6.1), for $j \ne i$,

$$r_{N,M}(i, j) = \frac{1}{W_{x_i^N}^N} \mathbb{P}_{x_i^N}^N[H(x_j^N) < \tau(A_{M,j}^N)],$$

where, again, $A_{M,j}^N = \{x_1^N, \dots, x_{j-1}^N, x_{j+1}^N, x_M^N\}$, $1 \le j \le M \le N^d$. To compute this probability, we need only to examine the discrete skeleton Markov chain,

$$\mathbb{P}_{x_i^N}^N[H(x_j^N) < \tau(A_{M,j}^N)] = \mathbb{Q}_{x_i^N}^N[H(x_j^N) < \hat{\tau}(A_{M,j}^N)].$$

Since x_j^N converges, as $N \uparrow \infty$, to $\hat{x}_j$, $1 \le j \le M$, and since $\min_{1 \le i \ne j \le M} \|\hat{x}_i - \hat{x}_j\| > 0$, by Corollary 6.7, we have

$$\lim_{N \to \infty} \mathbb{Q}_{x_i^N}^N[H(x_j^N) < \hat{\tau}(A_{M,j}^N)] = \frac{v_d}{M}\cdot$$

Hence, for $j \ne i$, $r_{N,M}(i, j)$ converges, as $N \uparrow \infty$, to $(v_d / M\hat{w}_i)$ because $W_{x_i^N}^N$ converges to $\hat{w}_i$. This completes the proof of the proposition. $\square$

To examine the time spent by the random walk $\{X_t^N : 0 \le t \le T\}$ on $\mathbb{T}_N^d \setminus A_M^N$, denote by cap_N the capacity associated to the process X^N. Of course, for any two disjoint subsets A, B of $\mathbb{T}_N^d$,

$$\mathrm{cap}_N(A, B) = \frac{1}{W(\mathbb{T}^d)} \mathrm{cap}_{Y^N}(A, B). \tag{6.26}$$

For $x \ne y$ in $\mathbb{T}_N^d$, denote by $\mathbb{P}_y^{N,x}$ the probability measure on the path space $D(\mathbb{R}_+, \mathbb{T}_N^d \setminus \{x\})$ induced by the trace of $\{X_t^N : t \ge 0\}$ on $\mathbb{T}_N^d \setminus \{x\}$ starting from y. Expectation with respect to $\mathbb{P}_y^{N,x}$ is denoted by $\mathbb{E}_y^{N,x}$.

LEMMA 6.14. *We have that*

$$\lim_{M \to \infty} \limsup_{N \to \infty} \max_{1 \le j \le M} \max_{y : |y - x_j^N| = 1} M \mathbb{E}_y^{N, x_j^N}[H(A_{M,j}^N)] = 0.$$

PROOF. Fix $1 \le j \le M$ and $y \sim x_j^N$. By Lemma 6.1, the expectation appearing in the statement of the lemma is equal to

$$\frac{1}{\mathrm{cap}_N(y, A_{M,j}^N)} \sum_{z \ne x_j^N} \nu^N(z) \mathbb{P}_z^N[H(y) < H(A_{M,j}^N)]. \tag{6.27}$$

By (6.26), the denominator is equal to

$$\frac{1}{W(\mathbb{T}^d)} \mathrm{cap}_{Y^N}(y, A_{M,j}^N) \ge \frac{1}{W(\mathbb{T}^d)} \mathrm{cap}_{Y^N}(y, x_1^N).$$

In view of Corollary 6.8, this latter expression is bounded below, uniformly in N, by a strictly positive constant.

To estimate the numerator in (6.27), we need only examine the discrete skeleton Markov chain because $\mathbb{P}_z^N[H(y) < H(A_{M,j}^N)] = \mathbb{Q}_z^N[H(y) < H(A_{M,j}^N)]$. Fix a sequence $\{\ell_N : N \geq 1\}$ such that $1 \ll \ell_N \ll N$ and let $B_N = \{z \in \mathbb{T}_N^d : d(z, A_M^N) \leq \ell_N\} \setminus A_M^N$, $C_N = \{z \in \mathbb{T}_N^d : d(z, A_M^N) > \ell_N\}$. Since $\mathbb{Q}_z^N[H(y) < H(A_{M,j}^N)]$ vanishes on the set $A_{M,j}^N$, we have

$$\sum_{z \neq x_j^N} \nu^N(z) \mathbb{Q}_z^N[H(y) < H(A_{M,j}^N)] = \sum_{z \in B_N} \nu^N(z) \mathbb{Q}_z^N[H(y) < H(A_{M,j}^N)]$$

$$+ \sum_{z \in C_N} \nu^N(z) \mathbb{Q}_z^N[H(y) < H(A_{M,j}^N)].$$

The first term on the right-hand side is bounded by $\nu^N(B_N)$, which vanishes as $N \uparrow \infty$ because $\ell_N \ll N$. Since $\ell_N \gg 1$, by Lemma 6.6, as $N \uparrow \infty$, the second term converges to $M^{-1}[1 - W(\mathbb{T}^d)^{-1} \sum_{1 \leq j \leq M} \hat{w}_j]$. The expression inside brackets vanishes as $M \uparrow \infty$, by definition of the sequence $\{\hat{w}_i\}$. This proves the lemma. $\square$

Recall that $\Delta_{N,M} = \mathbb{T}_N^d \setminus A_M^N$.

COROLLARY 6.15. *For every $t \geq 0$,*

$$\lim_{M \to \infty} \limsup_{N \to \infty} \max_{1 \leq j \leq M} \mathbb{E}_{x_j^N}^N [\mathcal{T}_t^{\Delta_{N,M}}] = 0.$$

PROOF. Fix $M \geq 1$ and $1 \leq j \leq M$. Consider the stochastic process $\hat{Z}_t^{N,M}$ with state space A_M^N defined as

$$\hat{Z}_t^{N,M} = X^N(\sigma(t)),$$

where $\sigma(t) := \sup\{s \leq t : X_s^N \in A_M^N\}$. Hence, during an excursion in $\Delta_{N,M}$ by X^N, the process $\hat{Z}_t^{N,M}$ stays at the last-visited site in A_M^N.

For a path $\omega \in D(\mathbb{R}_+, \mathbb{T}_N^d)$ performing infinitely many jumps, denote by $\tau_n(\omega)$, $n \geq 0$, the jumping times of ω: $\tau_0(\omega) = 0$ and

$$\tau_n(\omega) := \inf\{t > \tau_{n-1}(\omega) : \omega(t) \neq \omega(\tau_{n-1}(\omega))\}.$$

Let

$$T_n(\omega) := \tau_n(\omega) - \tau_{n-1}(\omega), \qquad n \geq 1,$$

and let N_t be the number of jumps up to time t:

$$N_t(\omega) := \sup\{j \geq 0 : \tau_j(\omega) \leq t\}.$$

The process $\hat{X}_t^{N,M}$, defined on the path space $D(\mathbb{R}_+, A_M^N)$, can be thought of as a process on $D(\mathbb{R}_+, \mathbb{T}_N^d)$. Couple the processes $\hat{X}_t^{N,M}$, $\hat{Z}_t^{N,M}$ forcing them to visit the same sequence of sites. By Lemma 6.14 and the proof of Proposition 4.4 in [2], for every $K \geq 1$, we have

$$\lim_{M\to\infty} \limsup_{N\to\infty} \max_{1\le j\le M} \mathbb{E}_{x_j^N}^N[\tau_{KM}(\hat{Z}^{N,M}) - \tau_{KM}(\hat{X}^{N,M})] = 0. \tag{6.28}$$

Set $\hat{N}_t := N_t(\hat{Z}^{N,M})$, $\hat{T}_n := T_n(\hat{Z}^{N,M})$ and $T_n := T_n(\hat{X}^{N,M})$. Fix $1 \le j \le M$. Under $\mathbb{P}_{x_j^N}^N$,

$$\mathcal{T}_t^{\Delta_{N,M}} \le t \wedge \sum_{n=1}^{\hat{N}_t+1} (\hat{T}_n - T_n) \le \mathbf{1}\{\hat{N}_t \ge KM\}t + \sum_{n=1}^{KM} (\hat{T}_n - T_n)$$

for any positive integer K. Therefore,

$$\mathbb{E}_{x_j^N}^N[\mathcal{T}_t^{\Delta_{N,M}}] \le t\mathbb{P}_{x_j^N}^N[\hat{N}_t \ge KM] + \mathbb{E}_{x_j^N}^N[\tau_{KM}(\hat{Z}^{N,M}) - \tau_{KM}(\hat{X}^{N,M})].$$

By (6.28), the second term vanishes as $N \uparrow \infty$ and then $M \uparrow \infty$. It remains to prove that

$$\lim_{K\to\infty} \limsup_{M\to\infty} \limsup_{N\to\infty} \max_{1\le j\le M} \mathbb{P}_{x_j^N}^N[N_t(\hat{Z}^{N,M}) \ge KM] = 0. \tag{6.29}$$

Since $N_t(\hat{Z}^{N,M}) \le N_t(\hat{X}^{N,M})$, $\mathbb{P}_{x_j^N}^N$-a.s., we have

$$\mathbb{P}_{x_j^N}^N[N_t(\hat{Z}^{N,M}) \ge KM] \le \mathbb{P}_{x_j^N}^N[N_t(\hat{X}^{N,M}) \ge KM].$$

Fix $M \ge 1$ and $1 \le j \le M$ such that

$$\limsup_{N\to\infty} \max_{1\le k\le M} \mathbb{P}_{x_k^N}^N[N_t(\hat{X}^{N,M}) \ge KM] = \limsup_{N\to\infty} \mathbb{P}_{x_j^N}^N[N_t(\hat{X}^{N,M}) \ge KM].$$

Since $[N_t \ge KM]$ is a closed set for the Skorokhod topology on $D(\mathbb{R}_+, A_M^N)$, since $N_t(\hat{X}^{N,M})$ has the same distribution as $N_t(X^{N,M})$ and since, by Proposition 6.13, $X^{N,M}$ converges in distribution to Z^M, we have

$$\limsup_{N\to\infty} \mathbb{P}_{x_j^N}^N[N_t(\hat{X}^{N,M}) \ge KM] \le \max_{1\le k\le M} P_k[N_t(Z^M) \ge KM],$$

where P_k is the distribution of the process Z^M starting from k.

To estimate the right-hand side, we compare $N_t(Z^M)$ with a counting process C_t in which we replace the holding times T_n by 0 if $Z^M(\tau_{n-1}) \ne 1$. In other words, let $G_0 := C_0$ be the number of times the process Z^M jumped before hitting 1 for the first time. Since Z^M jumps from any site uniformly to all others, G_0 is a random variable with geometric distribution: $P[G_0 = n] = (1/M)[(M-1)/M]^{n-1}$, $n \ge 1$.

When hitting 1, as Z^M, the process C_t stays there for a mean $\hat{w}_1/v_d$ exponential time. At the end of this exponential time, C_t jumps from G_0 to $G_0 + G_1$, where G_1 stands for the number of jumps performed by Z^M before hitting 1 again.

By construction, $N_t(Z^M) \le C_t$ for all $t \ge 0$ and $C_t = \sum_{0\le j\le \hat{N}_t} G_j$, where $\{G_j : j \ge 0\}$ are i.i.d. random variables with geometric distribution: $P[G_1 = n] = (1/M)[(M-1)/M]^{n-1}$, $n \ge 1$. $\hat{N}_t$ is a Poisson process with rate $v_d/\hat{w}_1$, independent of the sequence $\{G_i\}$. In particular, $E_{P_k}[N_t(Z^M)] \le M[1 + (t\hat{w}_1/v_d)]$. This proves (6.29) and the corollary. □

PROOF OF THEOREM 2.3. Theorem 2.3 follows from Proposition 6.13 and Corollary 6.15. □

PROOF OF THEOREM 2.5. Denote by $\{Z_t : t \ge 0\}$ the K-process with parameters $\{\hat{w}_i/v_d : i \ge 1\}$ and $c = 0$. Using independent exponential and Poisson random variables, we may define in the same probability space $(\Omega, \mathcal{F}, P)$ processes $\{\mathbb{X}_t^{N,M} : t \ge 0\}$, $\{\mathbb{Z}_t^M : t \ge 0\}$ and $\{\mathbb{Z}_t : t \ge 0\}$ which have the same distribution as $\{X_t^{N,M} : t \ge 0\}$, $\{Z_t^M : t \ge 0\}$ and $\{Z_t : t \ge 0\}$, respectively. Fix a common starting point j for all processes and $T > 0$. By [13], Lemma 3.11, $\{\mathbb{Z}_t^M : 0 \le t \le T\}$ converges a.s., as $M \uparrow \infty$, to $\{\mathbb{Z}_t : t \ge 0\}$ in the Skorokhod metric. On the other hand, by Proposition 6.13, if d_S stands for the Skorokhod metric on $D([0,T], \overline{\mathbb{N}})$, for every $M \ge 1$ and $\epsilon > 0$,

$$\lim_{N\to\infty} P[d_S(\mathbb{X}^{N,M}, \mathbb{Z}^M) > \epsilon] = 0.$$

In particular, there exists a strictly increasing sequence $\{N_M^* : M \ge 1\}$ such that

$$P[d_S(\mathbb{X}^{N,M}, \mathbb{Z}^M) > M^{-1}] \le \frac{1}{M}$$

for all $N \ge N_M^*$. Hence, by the triangular inequality, for any sequence $\{N_M : M \ge 1\}$ such that $N_M \ge N_M^*$, we have

$$\lim_{M\to\infty} P[d_S(\mathbb{X}^{N_M,M}, \mathbb{Z}) > \epsilon] = 0$$

for any $\epsilon > 0$. The sequence $\{\ell_N^* : N \ge 1\}$, defined as the inverse of $\{N_M^* : M \ge 1\}$, fulfills the requirements of the first part of Theorem 2.5.

To prove the second statement of Theorem 2.5, fix $M \ge 2$ and observe that $\mathcal{T}_t^{\Delta_{N,\ell_N}} \le H(A_M^N) + \mathcal{T}_t^{\Delta_{N,M}} \circ \theta(H(A_M^N))$ provided $\ell_N \ge M$. In this formula, $\theta(s) : D(\mathbb{R}_+, \mathbb{T}_N^d) \to D(\mathbb{R}_+, \mathbb{T}_N^d)$, $s \ge 0$, stands for the time shift by s of a path ω: $(\theta(s)\omega)(t) = \omega(s+t)$, $t \ge 0$. Therefore,

$$\max_{1\le j\le \ell_N} \mathbb{E}_{x_j^N}^N[\mathcal{T}_t^{\Delta_{N,\ell_N}}] \le \max_{1\le j\le \ell_N} \mathbb{E}_{x_j^N}^N[H(A_M^N)] + \max_{1\le j\le M} \mathbb{E}_{x_j^N}^N[\mathcal{T}_t^{\Delta_{N,M}}],$$

provided $M \le \ell_N$. By Corollary 6.15, the second expression converges to 0 as $N \uparrow \infty$, $M \uparrow \infty$. By the definition of $H(A^N_M)$, the first expectation on the right-hand side vanishes for $1 \le j \le M$. For $M < j \le \ell_N$, by (6.5) with $F = E = \mathbb{T}^d_N$,

$$\mathbb{E}^N_{x^N_j}[H(A^N_M)] \le \frac{\nu^N(\Delta_{N,M})}{\mathrm{cap}(x^N_j, A^N_M)}.$$

The denominator is bounded below by $\min_{z \in \mathbb{T}^d_N} \max_{1 \le k \le M} \mathrm{cap}(z, x^N_k)$. Since $\|\hat{x}_1 - \hat{x}_2\| > 0$, by Corollary 6.8, this expression is bounded below by a positive constant, uniformly in M and N. This proves the second statement of Theorem 2.5 because $\nu^N(\Delta_{N,M})$ vanishes as $N \uparrow \infty$, $M \uparrow \infty$. $\square$

6.5. *Metastability for $d = 2$.* In this subsection, we adapt to dimension 2 the results presented in the previous subsection. Most of the proofs are similar to the case $d \ge 3$.

The main difference with respect to dimension 3 is that the process is speeded up by $\log N$. Recall from Section 2 that we denote by $\{\mathfrak{X}^N_t : t \ge 0\}$ the random walk with generator $\mathcal{L}_N$, defined in (2.3), speeded up by $\log N$. Moreover, $\mathbf{P}^N_x$, $\mathbb{P}^N_x$, $x \in \mathbb{T}^2_N$, stand for the probability measures on $\mathbb{D}(\mathbb{R}_+, \mathbb{T}^2_N)$ induced by the processes $\{\mathfrak{X}^N_t : t \ge 0\}$, $\{X^N_t : t \ge 0\}$, respectively, starting from x.

PROPOSITION 6.16. *For a fixed $M > 1$, and $T > 0$, the process $\{\mathfrak{X}^{N,M}_t : 0 \le t \le T\}$ converges in distribution, as $N \uparrow \infty$, to the Markov process in $\{1, \dots, M\}$ given by the following generator*:

$$(\mathfrak{L}^\star_M f)(i) = \frac{\pi}{2} \frac{1}{M \hat{w}_i} \sum_{j=1}^{M} [f(j) - f(i)].$$

PROOF. As in the proof of Proposition 6.13, we use (6.1) to write the jump rates of $\mathfrak{X}^{N,M}_t$ in terms of the excursion probabilities between the very deep traps:

$$r_{N,M}(i, j) = \frac{\log N}{W^N_{x^N_i}} \mathbb{Q}^N_{x^N_i}[H(x^N_j) < \hat{\tau}(A^N_{M,j})].$$

Note the factor $\log N$ which appears because the generator $\mathcal{L}_M$ is multiplied by this constant.

Consider a sequence $\{l_N : N \ge 1\}$ satisfying (6.14). Use the strong Markov property on $H(B(x^N_i, l_N)^c)$ to obtain

$$\begin{aligned}
&\mathbb{Q}^N_{x^N_i}[H(x^N_j) < \hat{\tau}(A^N_{M,j})] \\
&\quad = \mathbb{Q}^N_{x^N_i}\big[\mathbf{1}\{H(B(x^N_i, l_N)^c) < \hat{\tau}(x^N_i)\} \mathbb{Q}^N_{Y(H(B(x^N_i, l_N)^c))}[H(x^N_j) < H(A^N_{M,j})]\big].
\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\log N\mathbb{Q}^N_{x_i^N}[H(x_j^N) < \hat{\tau}(A^N_{M,j})] - \frac{\pi}{2M}\right| \\ \leq \left|\log N\mathbb{Q}^N_{x_i^N}[H(B(x_i^N, l_N)^c) < \hat{\tau}(x_i^N)] - \frac{\pi}{2}\right| \\ + \frac{\pi}{2}\sup_{z\in\partial B(x_i^N, l_N)}\left|\mathbb{Q}^N_z[H(x_j^N) < H(A^N_{M,j})] - \frac{1}{M}\right|.\end{aligned}$$

By Lemmas 6.9 and 6.10, these expressions vanish as $N \uparrow \infty$. Since $W^N_{x_i^N}$ converges to $\hat{w}_i$, $1 \leq i \leq M$, we are done. $\square$

Recall the definition of the measures $\mathbb{P}^{N,x}_y$, $x \neq y \in \mathbb{T}^2_N$, introduced in the previous subsection. This corresponds to the trace on $\mathbb{T}^2_N \setminus \{x\}$ of the process $\{X^N_t : t \geq 0\}$, which has not been speeded up.

LEMMA 6.17. *In dimension* 2,

$$\lim_{M\to\infty}\limsup_{N\to\infty}\max_{1\leq j\leq M}\max_{y:|y-x_j^N|=1}\frac{M}{\log N}\mathbb{E}^{N,x_j^N}_y[H(A^N_{M,j})] = 0.$$

PROOF. The proof of this result follows the same argument as in Lemma 6.14. One must simply note that the denominator of (6.27) is now multiplied by $\log N$, which forces us to use Corollary 6.12 in place of Corollary 6.8. The argument to bound the numerator is also the same. However, one should choose a sequence $\{\ell_N : N \geq 1\}$ satisfying (6.14) in order to apply Lemma 6.9. $\square$

For $x \neq y$, in $\mathbb{T}^2_N$, denote by $\mathbf{P}^{N,x}_y$ the probability measure on the path space $D(\mathbb{R}_+, \mathbb{T}^d_N \setminus \{x\})$ induced by the trace of $\{\mathfrak{X}^N_t : t \geq 0\}$ on $\mathbb{T}^d_N \setminus \{x\}$ starting from y. Expectation with respect to $\mathbf{P}^{N,x}_y$ is denoted by $\mathbf{E}^{N,x}_y$. The difference between $\mathbf{P}^{N,x}_y$ and $\mathbb{P}^{N,x}_y$ is that the first probability measure is associated to the random walk speeded up by $\log N$. Therefore, for every subset A of $\mathbb{T}^2_N \setminus \{x_j^N\}$,

$$\mathbf{E}^{N,x_j^N}_y[H(A)] = \frac{1}{\log N}\mathbb{E}^{N,x_j^N}_y[H(A)].$$

In particular, it follows from the previous lemma that

$$\lim_{M\to\infty}\limsup_{N\to\infty}\max_{1\leq j\leq M}\max_{y:|y-x_j^N|=1} M\mathbf{E}^{N,x_j^N}_y[H(A^N_{M,j})] = 0. \tag{6.30}$$

COROLLARY 6.18. *In dimension* 2, *for every* $t \geq 0$,

$$\lim_{M\to\infty}\limsup_{N\to\infty}\max_{1\leq j\leq M}\mathbf{E}^N_{x_j^N}[\mathcal{T}^{\Delta_{N,M}}_t] = 0.$$

PROOF. The argument is identical to the one in $d \geq 3$ presented in Corollary 6.15. We just use (6.30) and Proposition 6.16 instead of Lemma 6.14 and Proposition 6.13. At the end of the proof, the rate of the process $N_t(Z^M)$ is replaced by $\pi/(2\hat{w}_1)$, but its exact value is superfluous. □

PROOF OF THEOREM 2.4. The proof is a direct consequence of Proposition 6.16 and Corollary 6.18. □

6.6. *Dimension* 2 *with no acceleration*. We prove in this subsection that in dimension 2, the trap model with generator (2.3) starting from a very deep trap does not move. Hence, on the order 1 scale, the random walk does not move, and on the scale $\log N$, it converges to the K-process in which all the geometry is wiped out.

PROPOSITION 6.19. *For every* $j \geq 1$, *every* $t > 0$ *and every sequence* $\{\ell_N : N \geq 1\}$ *such that* $\ell_N \uparrow \infty$,

$$\lim_{N\to\infty} \mathbb{P}^N_{x_j^N}[|X^N(t) - x_j^N| \geq \ell_N] = 0.$$

PROOF. Fix $j \geq 1$ and a sequence $\{\ell_N : N \geq 1\}$ such that $\ell_N \uparrow \infty$. Following [5], we denote by $S_N : \mathbb{Z}_+ \to \mathbb{R}$ the clock process: $S_N(0) = 0$,

$$S_N(k) = \sum_{i=0}^{k-1} \mathfrak{e}_i W^N(Y^N(i)), \qquad k \geq 1,$$

where $\{Y^N(i) : i \geq 0\}$ is a nearest-neighbor, symmetric, discrete-time random walk on $\mathbb{T}^2_N$ starting from x_j^N; $\{\mathfrak{e}_i : i \geq 0\}$ is a sequence of i.i.d., mean-one, exponential random variables, independent of the Markov chain $\{Y^N(i)\}$; and $W^N(x) = W^N_x$, $x \in \mathbb{T}^2_N$. Denote by $T_N : \mathbb{R}_+ \to \mathbb{R}_+$ the inverse of S_N:

$$T_N(t) = \sup\{k : S_N(k) \leq t\}.$$

Clearly, $\{X^N(t) : t \geq 0\}$ has the same distribution as $\{Y^N(T_N(t)) : t \geq 0\}$. Hence,

$$\begin{aligned}
&\mathbb{P}^N_{x_j^N}[|X^N(t) - x_j^N| \geq \ell_N] \\
&\quad = P^N[|Y^N(T_N(t)) - x_j^N| \geq \ell_N] \qquad (6.31)\\
&\quad \leq P^N\Big[\max_{0\leq k\leq r_N} |Y^N(k) - x_j^N| \geq \ell_N\Big] + P^N[T_N(t) \geq r_N]
\end{aligned}$$

for a sequence r_N such that $1 \ll r_N \ll \ell_N^2$.

We estimate separately the expressions on the right-hand side of (6.31). Since T_N is the inverse of S_N, $\{T_N(t) \geq r_N\} = \{S_N(r_N) \leq t\}$. In particular,

$$P^N[T_N(t) \geq r_N] \leq P^N\left[\hat{w}_j \sum_{i=0}^{r_N-1} \mathfrak{e}_i \mathbf{1}\{Y^N(i) = x_j^N\} \leq t\right]$$

because $S_N(k) \geq W^N_{x^N_j} \sum_{0 \leq i \leq k-1} \mathfrak{e}_i \mathbf{1}\{Y^N(i) = x^N_j\}$, $W^N_{x^N_j} \geq \hat{w}_j$. Since $\{Y^N(k): k \geq 0\}$ starts from x^N_j and the two-dimensional random walk is recurrent, the previous probability vanishes as $N \uparrow \infty$ because $r_N \uparrow \infty$.

On the other hand, since $Y^N(k) - x^N_j$ is a bidimensional martingale, by Doob's inequality,

$$P^N\Big[\max_{0 \leq k \leq r_N} |Y^N(k) - x^N_j| \geq \ell_N\Big] \leq \frac{4r_N}{\ell^2_N},$$

which vanishes as $N \uparrow \infty$. This completes the proof of the proposition. $\square$

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