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# THE SKOROHOD OBLIQUE REFLECTION PROBLEM IN TIME-DEPENDENT DOMAINS

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The deterministic Skorohod problem plays an important role in the construction and analysis of diffusion processes with reflection. In the form studied here, the multidimensional Skorohod problem was introduced, in time-independent domains, by H. Tanaka [61] and further investigated by P.-L. Lions and A.-S. Sznitman [42] in their celebrated article. Subsequent results of several researchers have resulted in a large literature on the Skorohod problem in time-independent domains. In this article we conduct a thorough study of the multidimensional Skorohod problem in time-dependent domains. In particular, we prove the existence of càdlàg solutions $(x, \lambda)$ to the Skorohod problem, with oblique reflection, for $(D, \Gamma, w)$ assuming, in particular, that $D$ is a time-dependent domain (Theorem 1.2). In addition, we prove that if $w$ is continuous, then $x$ is continuous as well (Theorem 1.3). Subsequently, we use the established existence results to construct solutions to stochastic differential equations with oblique reflection (Theorem 1.9) in time-dependent domains. In the process of proving these results we establish a number of estimates for solutions to the Skorohod problem with bounded jumps and, in addition, several results concerning the convergence of sequences of solutions to Skorohod problems in the setting of time-dependent domains.

**1. Introduction.** In time-independent domains the Skorohod problem, in the form studied in this article, goes back to Tanaka [61], who established existence and uniqueness of solutions to the Skorohod problem in convex domains with normal reflection. These results were subsequently generalized to wider classes of time-independent domains by, in particular, Lions and Sznitman [42] and Saisho [53]. By imposing an admissibility condition on the domain, Lions and Sznitman [42] proved existence and uniqueness of solutions to the Skorohod problem in two different cases. The first of the two cases considered normal reflection on domains satisfying a uniform exterior sphere condition, meaning that the domain is smooth except for "convex corners." Moreover, the second case considered smoothly varying (possibly oblique) directions of reflection on smooth domains. In addition, for smoothly varying directions of reflection on domains satisfying a uniform exterior

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sphere condition, existence and uniqueness results were obtained in the special case when the oblique reflection cone can be transformed into the normal cone by multiplication by a smooth matrix function. Saisho [53] later showed that in the first case considered in [42], that is, for normal reflection, the admissibility condition is not necessary and can be removed. Moreover, concerning oblique reflection, that is, when the cone of reflection differs from the cone of inward normals, we note that in the case of an orthant with constant directions of reflection on the sides, Harrison and Reiman [33] found sufficient conditions for the existence and uniqueness of solutions to the Skorohod problem as well as for continuity of the reflection map. In this context we also mention that Bernard and El Kharroubi [6] provided necessary and sufficient conditions for the existence of solutions to the Skorohod problem in an orthant with constant directions of reflection on each face. The most general results so far concerning the existence of solutions to the Skorohod problem with oblique reflection in time-independent domains were derived by Costantini [15]. Costantini [15] proved existence of solutions to the Skorohod problem for domains satisfying a uniform exterior sphere condition with a nontangential reflection cone given as a continuous transformation of the normal cone. Note that this allowed for discontinuous directions of reflection at the corners. The question of uniqueness of solutions to the Skorohod problem with oblique reflection is, in general, still an open question and has been settled only in some specific cases. For example, Dupuis and Ishii [17] obtained uniqueness for a convex polyhedron with constant directions of reflection on the faces assuming the existence of a certain convex set, defined in terms of the normal directions and the directions of reflection. Dupuis and Ishii [18, 19] later extended this result to piecewise smooth domains with smoothly varying directions of reflection on each face. In addition, we here also mention the work of Dupuis and Ramanan [21, 22] based on convex duality techniques. In particular, in [22] convex duality is used to transform the condition of Dupuis and Ishii [17] into one that is much easier to verify. Before we proceed, we here note that the outline above is an attempt to briefly discuss relevant previous developments concerning the Skorohod problem in the form studied in this article. In particular, the study of reflected diffusion based on Skorohod problems was first introduced by Skorohod [57] and this approach has, as briefly described, subsequently been developed in many articles, including [15, 17, 33, 42, 53] and [61]. However, we emphasize that the literature devoted to Skorohod problems, their extensions and applications is much larger than what is conveyed above and, in fact, many more researcher have contributed to this rich field. In particular, applied areas where Skorohod problems occur include heavy traffic analysis of queueing networks (see, e.g, [2, 20, 24, 40, 47, 48, 51, 52]), control theory, game theory and mathematical economics (see, e.g., [3, 37, 49, 50, 58]), image processing (see, e.g., [8]) and molecular dynamics (see, e.g., [54–56]). For further results concerning Skorohod problems, as well as applications of Skorohod problems, we also refer to [1, 4, 5, 14, 23, 25, 29, 31, 35, 38, 44, 46] and [60].

An important novelty of this article is that we conduct a thorough study of the Skorohod problem, and the subsequent applications to stochastic differential equations reflected at the boundary, in the setting of time-dependent domains. To our knowledge, the Skorohod problem is indeed less developed in time-dependent domains. In particular, a first treatment of the Skorohod problem in time-dependent domains was given by Costantini, Gobet and El Karoui [16], who proved existence and uniqueness of solutions to the Skorohod problem with normal reflection in smooth time-dependent domains. Moreover, existence and uniqueness for deterministic problems of Skorohod type in time-dependent intervals have recently also been established by Burdzy, Chen and Sylvester [9], Burdzy, Kang and Ramanan [12]. The main contribution of this article is that we are able to generalize the results in [15], concerning càdlàg solutions to the Skorohod problem with oblique reflection, to time-dependent domains assuming less regularity on the domains compared to [16]. Note also that in [45] we use the results of this article to construct a numerical method for weak approximation of stochastic differential equations with oblique reflection in time-dependent domains. Finally, as in [12], we note, in particular, that reflecting Brownian motions in time-dependent domains arise in queueing theory (see, e.g., [36, 43]), statistical physics, (see, e.g., [13, 59]), control theory (see, e.g., [27, 28]) and finance (see, e.g., [26]). In particular, in future articles we hope to be able to further explore the results and techniques developed in this article in several applications.

To properly formulate the multidimensional Skorohod problem considered in this article, and our results, we in the following first have to introduce some notation. Given $d \geq 1$, we let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^d$ and we let $|z| = \langle z, z\rangle^{1/2}$ be the Euclidean norm of $z$. Whenever $z \in \mathbb{R}^d, r > 0$, we let $B_r(z) = \{y \in \mathbb{R}^d : |z - y| < r\}$ and $S_r(z) = \{y \in \mathbb{R}^d : |z - y| = r\}$. Moreover, given $D \subset \mathbb{R}^{d+1}$, $E \subset \mathbb{R}^d$, we let $\bar{D}$, $\bar{E}$ be the closure of $D$ and $E$, respectively, and we let $d(y, E)$ denote the Euclidean distance from $y \in \mathbb{R}^d$ to $E$. Given $d \geq 1$, $T > 0$ and an open, connected set $D' \subset \mathbb{R}^{d+1}$, we will refer to

$$D = D' \cap ([0, T] \times \mathbb{R}^d), \tag{1.1}$$

as a time-dependent domain. Given $D$ and $t \in [0, T]$, we define the time sections of $D$ as $D_t = \{z : (t, z) \in D\}$, and we assume that

$$D_t \neq \varnothing \text{ and that } D_t \text{ is bounded and connected for every } t \in [0, T]. \tag{1.2}$$

We let $\partial D$ and $\partial D_t$, for $t \in [0, T]$, denote the boundaries of $D$ and $D_t$, respectively. A convex cone of vectors in $\mathbb{R}^d$ is a subset $\Gamma \subset \mathbb{R}^d$ such that $\alpha u + \beta v \in \Gamma$ for all $\alpha, \beta \in \mathbb{R}_+$ and all $u, v \in \Gamma$. We let $\Gamma = \Gamma_t(z) = \Gamma(t, z)$ be a function defined on $\mathbb{R}^{d+1}$ such that $\Gamma_t(z)$ is a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_t$, $t \in [0, T]$. To give an example of a closed convex cone, we consider the set $C = C_\Omega = \{\lambda\gamma : \lambda > 0, \gamma \in \Omega\}$, where $\Omega$ is a closed, connected subset of $S_1(0)$ satisfying $\gamma_1 \cdot \gamma_2 > -1$ for all $\gamma_1, \gamma_2 \in \Omega$. Given $C$, we define

$C^* = \{\alpha u + \beta v : \alpha, \beta \in \mathbb{R}_+, u, v \in C\}$. Then $C^*$ is an example of a closed convex cone and we note that $C^* = C^*_{\Omega^*}$, where $\Omega^*$ can be viewed as the "convex hull" of $\Omega$ on $S_1(0)$. Given $\Gamma = \Gamma_t(z)$, we let $\Gamma^1_t(z) := \Gamma_t(z) \cap S_1(0)$. Given $T > 0$, we let $\mathcal{D}([0, T], \mathbb{R}^d)$ denote the set of càdlàg functions $w = w_t : [0, T] \to \mathbb{R}^d$, that is, functions which are right continuous with left limits. For $w \in \mathcal{D}([0, T], \mathbb{R}^d)$ we introduce the norm

$$\|w\|_{t_1,t_2} = \sup_{t_1 \le r \le s \le t_2} |w_s - w_r| \tag{1.3}$$

for $0 \le t_1 \le t_2 \le T$ and, given $\delta > 0$, we let

$$\mathcal{D}^\delta([0, T], \mathbb{R}^d) = \Big\{ w \in \mathcal{D}([0, T], \mathbb{R}^d) : \sup_t |w_t - w_{t^-}| < \delta \Big\} \tag{1.4}$$

denote the set of càdlàg functions with jumps bounded by $\delta$. We denote the set of functions $\lambda = \lambda_t : [0, T] \to \mathbb{R}^d$ with bounded variation by $\mathcal{BV}([0, T], \mathbb{R}^d)$ and we let $|\lambda|$ denote the total variation of $\lambda \in \mathcal{BV}([0, T], \mathbb{R}^d)$.

In this article we consider the Skorohod problem in the following form.

DEFINITION 1.1. Let $d \ge 1$ and $T > 0$. Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2) and let $\Gamma = \Gamma_t(z)$ be, for every $z \in \partial D_t$, $t \in [0, T]$, a closed convex cone of vectors in $\mathbb{R}^d$. Given $w \in \mathcal{D}([0, T], \mathbb{R}^d)$, with $w_0 \in \overline{D_0}$, we say that the pair $(x, \lambda)$ is a solution to the Skorohod problem for $(D, \Gamma, w)$, on $[0, T]$, if $(x, \lambda) \in \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{BV}([0, T], \mathbb{R}^d)$ and if $(w, x, \lambda)$ satisfies, for all $t \in [0, T]$,

$$x_t = w_t + \lambda_t, \qquad x_t \in \overline{D_t}, \tag{1.5}$$

$$\lambda_t = \int_0^{t^+} \gamma_s \, d|\lambda|_s, \qquad \gamma_s \in \Gamma^1_s(x_s)\, d|\lambda|\text{-a.e on } \bigcup_{s \in [0,t]} \partial D_s \tag{1.6}$$

and

$$d|\lambda|\big(\{t \in [0, T] : (t, x_t) \in D\}\big) = 0. \tag{1.7}$$

The main results of this article will be proved for time-dependent domains $D \subset \mathbb{R}^{d+1}$ satisfying (1.2). However, several additional restrictions will be imposed on $D$, on the cones of reflection $\Gamma$ as well as on the interaction between $D$ and $\Gamma$. In the following we will outline these assumptions in order to be able to properly state our existence result concerning the Skorohod problem with oblique reflection. However, while these assumptions are introduced quite briefly here, the intuition behind the assumptions, as well as the implications of the assumptions, is explained in more detail in Section 3.2 below.

*Geometry of the time-slice $D_t$.* We let $N_t(z)$ denote the cone of inward normal vectors at $z \in \partial D_t$, $t \in [0, T]$; see (3.11) below for a definition. In particular, we assume that $N_t(z) \neq \emptyset$ whenever $z \in \partial D_t$, $t \in [0, T]$. Note that we allow for the possibility of several inward normal vectors at the same boundary point. Given $N_t(z)$, we let $N_t^1(z) := N_t(z) \cap S_1(0)$. Then the spatial domain $D_t$ is said to verify the uniform exterior sphere condition if there exists a radius $r_0 > 0$ such that

$$B_{r_0}(z - r_0 n) \subseteq ([0, T] \times \mathbb{R}^d \setminus D_t) \cap (\mathbb{R}^{d+1} \setminus D), \tag{1.8}$$

whenever $z \in \partial D_t$, $n \in N_t^1(z)$. Note that $B_{r_0}(z - r_0 n)$ is the open Euclidean ball with center $z - r_0 n$ and radius $r_0$. We say that a time-dependent domain $D$ satisfies a uniform exterior sphere condition in time if the uniform exterior sphere condition in (1.8) holds, with the same radius $r_0$, for all spatial domains $D_t$, $t \in [0, T]$.

*Temporal variation of the domain.* Following [16], we let

$$l(r) = \sup_{\substack{s,t \in [0,T] \\ |s-t| \leq r}} \sup_{z \in \overline{D_s}} d(z, D_t) \tag{1.9}$$

be the modulus of continuity of the variation of $D$ in time. In particular, in several of our estimates related to the Skorohod problem we will assume that

$$\lim_{r \to 0^+} l(r) = 0. \tag{1.10}$$

*Cones of reflection.* Following [15], we assume that

$$\begin{aligned} &\gamma_1 \cdot \gamma_2 > -1 \text{ holds whenever } \gamma_1, \gamma_2 \in \Gamma_t^1(z) \text{ and} \\ &\qquad \text{for all } z \in \partial D_t, t \in [0, T]. \end{aligned} \tag{1.11}$$

The assumption in (1.11) eliminates the possibility of $\Gamma$ containing vectors in opposite directions. We also assume that the set

$$G^\Gamma = \{(t, z, \gamma) : \gamma \in \Gamma_t(z), z \in \partial D_t, t \in [0, T]\} \text{ is closed.} \tag{1.12}$$

The interpretation of the condition in (1.12) is discussed in Section 3.2. In addition, we need the following assumption concerning the variation of the cones $\Gamma_t(z)$. Let

$$h(E, F) = \max(\sup\{d(z, E) : z \in F\}, \sup\{d(z, F) : z \in E\}) \tag{1.13}$$

denote the Hausdorff distance between the sets $E, F \subset \mathbb{R}^d$. Moreover, let $\{(s_n, z_n)\}$ be a sequence of points in $\mathbb{R}^{d+1}$, $s_n \in [0, T]$, $z_n \in \partial D_{s_n}$, such that $\lim_{n\to\infty} s_n = s \in [0, T]$, $\lim_{n\to\infty} z_n = z \in \partial D_s$. We assume, for any such sequence of points $\{(s_n, z_n)\}$, that

$$\lim_{n \to \infty} h(\Gamma_{s_n}(z_n), \Gamma_s(z)) = 0. \tag{1.14}$$

*Interaction between the geometry and the cones of reflection.* For $z \in \partial D_s$, $s \in [0, T]$, and $\rho, \eta > 0$ we define

$$a_{s,z}(\rho, \eta) = \max_{u \in S_1(0)} \min_{s \le t \le s+\eta} \min_{y \in \partial D_t \cap \overline{B_\rho(z)}} \min_{\gamma \in \Gamma_t^1(y)} \langle \gamma, u \rangle \tag{1.15}$$

and

$$c_{s,z}(\rho, \eta) = \max_{s \le t \le s+\eta} \max_{y \in \partial D_t \cap \overline{B_\rho(z)}} \max_{\hat{z} \in \overline{D_t} \cap \overline{B_\rho(z)}, \hat{z} \neq y} \max_{\gamma \in \Gamma_t^1(y)} \left( \frac{\langle \gamma, y - \hat{z} \rangle}{|y - \hat{z}|} \vee 0 \right). \tag{1.16}$$

For technical reasons we also introduce the quantity

$$e_{s,z}(\rho, \eta) = \frac{c_{s,z}(\rho, \eta)}{(a_{s,z}(\rho, \eta))^2 \vee a_{s,z}(\rho, \eta)/2}. \tag{1.17}$$

In the proof of certain a priori estimates for the Skorohod problem, established in the bulk of the article, we will consider time-dependent domains satisfying (1.2) and the uniform exterior sphere condition in time, with radius $r_0$. In addition, we will assume that there exist $0 < \rho_0 < r_0$ and $\eta_0 > 0$, such that

$$\inf_{s \in [0,T]} \inf_{z \in \partial D_s} a_{s,z}(\rho_0, \eta_0) = a > 0, \tag{1.18}$$

$$\sup_{s \in [0,T]} \sup_{z \in \partial D_s} e_{s,z}(\rho_0, \eta_0) = e < 1. \tag{1.19}$$

Interpretations of (1.15), (1.16), (1.18) and (1.19) are given in Section 3.2.

*Existence of good projections.* Let $0 < \delta_0 < r_0$, $h_0 > 1$ and let $\Gamma = \Gamma_t(z) = \Gamma(t, z)$ be given for all $z \in \partial D_t$, $t \in [0, T]$. We say that $([0, T] \times \mathbb{R}^d) \setminus \overline{D}$ has the $(\delta_0, h_0)$-property of good projections along $\Gamma$ if there exists, for any $y \in \mathbb{R}^d \setminus \overline{D_t}$, $t \in [0, T]$, such that

$$d(y, D_t) < \delta_0, \tag{1.20}$$

at least one projection of $y$ onto $\partial D_t$ along $\Gamma_t$, denoted $\pi_{\partial D_t}^{\Gamma_t}(y)$, which satisfies

$$|y - \pi_{\partial D_t}^{\Gamma_t}(y)| \le h_0 d(y, D_t). \tag{1.21}$$

Concerning the existence and continuity of solutions to the Skorohod problem, as defined in Definition 1.1, we prove the following two theorems.

THEOREM 1.2. *Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2), (1.10) *and a uniform exterior sphere condition in time with radius $r_0$ in the sense of* (1.8). *Let $\Gamma = \Gamma_t(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_t$, $t \in [0, T]$, and assume that $\Gamma$ satisfies* (1.11), (1.12) *and* (1.14). *Assume that* (1.18) *and* (1.19) *hold for some $0 < \rho_0 < r_0$, $\eta_0 > 0$, $a$ and $e$. Finally, assume that $([0, T] \times \mathbb{R}^d) \setminus \overline{D}$ has the $(\delta_0, h_0)$-property of good projections along $\Gamma$, for some $0 < \delta_0 < \rho_0$, $h_0 > 1$, as defined in* (1.20) *and* (1.21). *Then, given $w \in \mathcal{D}^{(\delta_0/4 \wedge \rho_0/(4h_0))}([0, T], \mathbb{R}^d)$, with $w_0 \in \overline{D_0}$, there exists a solution $(x, \lambda)$ to the Skorohod problem for $(D, \Gamma, w)$, in the sense of Definition 1.1, with $x \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R})$.*

THEOREM 1.3. *Assume that the assumptions stated in Theorem 1.2 are satisfied and let $\rho_0$ be as in the statement of Theorem 1.2. Let $w : [0, T] \to \mathbb{R}^d$ be a continuous function and let $(x, \lambda)$ be any solution to the Skorohod problem for $(D, \Gamma, w)$ in the sense of Definition 1.1. If $x \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$, then $x$ is continuous.*

In the following remarks we have gathered comments concerning the importance of the assumptions imposed in Theorems 1.2 and 1.3, as well as comments concerning situations when these assumptions are fulfilled.

REMARK 1.4. Our proofs of Theorems 1.2 and 1.3 rely, as outlined below, on certain a priori estimates proved in Section 4. These estimates are proved assuming that $D \subset \mathbb{R}^{d+1}$ is a time-dependent domain satisfying (1.2), (1.10) and a uniform exterior sphere condition in time with radius $r_0$ in the sense of (1.8). Furthermore, to derive these estimates, we also assume that (1.18) and (1.19) hold for some $0 < \rho_0 < r_0$, $\eta_0 > 0$, $a$ and $e$ and that $([0, T] \times \mathbb{R}^d) \setminus \overline{D}$ has the $(\delta_0, h_0)$-property of good projections along $\Gamma$, for some $0 < \delta_0 < \rho_0$, $h_0 > 1$, as defined in (1.20) and (1.21). In particular, we do not have to assume that $\Gamma = \Gamma_t(z)$ satisfies (1.11), (1.12) and (1.14) in order to derive the results in Section 4.

REMARK 1.5. In Section 5 we proceed toward the final proof of Theorem 1.2. In particular, we use the a priori estimates of Section 4 to derive general results concerning the convergence of solutions to Skorohod problems in time-dependent domains. We note that our assumptions on $D$ do not exclude the possibility of holes in $D$ and $D_t$, for some $t \in [0, T]$. Nevertheless, the assumptions on $D$ ensure that the number of holes in $D_t$ stays the same for all $t \in [0, T]$ and that these holes cannot shrink too much as time changes. This observation, Lemma 3.1 below and its proof allow us to conclude the validity of the conclusion in Remark 3.2, which, in turn, is used to complete the proofs in Section 5. Simple examples show that the conclusion in Remark 3.2 would not hold if we, for instance, allowed the number of holes in $D_t$ to change as a function of $t$ and if we, in particular, allowed the holes to vanish.

REMARK 1.6. The assumption that $\Gamma = \Gamma_t(z)$ satisfies (1.11), (1.12) and (1.14) is used to complete the proofs in Section 5. In particular, focusing on Theorem 5.3, which is the convergence result actually used in the proof of Theorem 1.2, we note that we need to assume (1.14) in order to be able derive (5.69). We then use (1.12) to complete the argument in the proof of Theorem 5.3. Note also the difference between (1.12) and (1.14). Assumption (1.12) simply states that if $(s_n, z_n, \gamma_n)$ is a sequence such that $\gamma_n \in \Gamma_{s_n}(z_n)$, $z_n \in \partial D_{s_n}$, $s_n \in [0, T]$, and if $(s_n, z_n, \gamma_n) \to (t, z, \gamma)$ in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, for some $(s, z, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, then $\gamma \in \Gamma_s(z)$, $z \in \partial D_s$, $s \in [0, T]$. Assumption (1.14), on the other hand, is a statement concerning the convergence, in the Hausdorff distance sense, of the cones

$\{\Gamma_{s_n}(z_n)\}$. Finally, to comment on assumption (1.11), which was also imposed in [15], we note that (1.11) is only used in the proofs of Theorems 5.1 and 5.3 and, in particular, in the verification of (5.51) and (5.52). Assumption (1.11) eliminates the possibility of $\Gamma$ containing vectors in opposite directions and we have not been able to complete our argument without this assumption. However, there are articles dealing with Skorohod type lemmas and reflected Brownian motion; see [14], in particular, where this assumption is not required. As noted in [14], the inclusion of vectors in opposite directions can be viewed as a critical case and [14] considers a related problem in a particular setting in the plane. In our general case we leave this question as a subject for future research.

REMARK 1.7. For examples of cases when the geometric assumptions imposed in Theorems 1.2 and 1.3 are fulfilled, we refer to Appendix. However, we here briefly discuss Theorems 1.2 and 1.3 in the context of convex domains. In particular, let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2) and (1.10). Assume, in addition, that $D_t$ is convex whenever $t \in [0, T]$. Let $\Gamma = \Gamma_t(z)$ be as in the statement of Theorem 1.2. Assume that

$$\lim_{\eta\to 0}\lim_{\rho\to 0}\inf_{s\in[0,T]}\inf_{z\in\partial D_s} a_{s,z}(\rho,\eta) = a > 0, \tag{1.22}$$

$$\lim_{\eta\to 0}\lim_{\rho\to 0}\sup_{s\in[0,T]}\sup_{z\in\partial D_s} e_{s,z}(\rho,\eta) = e < 1. \tag{1.23}$$

If $D_t$ is convex whenever $t \in [0, T]$, then there exists, for every $0 < \delta_0$ given, $h_0 > 1$ such that $([0, T] \times \mathbb{R}^d) \setminus \overline{D}$ has the $(\delta_0, h_0)$-property of good projections along $\Gamma$ as defined in (1.20) and (1.21). In this case the conclusion of Theorem 1.2 is, as can be seen from the proofs below, that given $w \in \mathcal{D}([0, T], \mathbb{R}^d)$, with $w_0 \in \overline{D_0}$, there exists a solution $(x, \lambda)$ to the Skorohod problem for $(D, \Gamma, w)$, in the sense of Definition 1.1, with $x \in \mathcal{D}([0, T], \mathbb{R})$. Moreover, if $w$ is a continuous function, then $x$ is continuous. In particular, if the time-slices $\{D_t\}$ are convex, then the restrictions, in Theorems 1.2 and 1.3, on the jump-sizes in terms of $\delta_0$, $\rho_0$ can be removed. Moreover, this is consistent with the results in [15] valid in time-independent domains; see Theorem 4.1 and Proposition 2.3 in [15].

We next formulate a subsequent application of Theorems 1.2 and 1.3 to the problem of constructing weak solutions to stochastic differential equations in $\overline{D}$ with reflection along $\Gamma_t$ on $\partial D_t$ for all $t \in [0, T]$. Given $T > 0$, we let $\mathcal{C}([0, T], \mathbb{R}^d)$ denote the class of continuous functions from $[0, T]$ to $\mathbb{R}^d$. In the following, we let $m$ be a positive integer and we let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be given functions which are bounded and continuous.

DEFINITION 1.8. Let $d \geq 1$ and $T > 0$. Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2), let $\Gamma = \Gamma_t(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_t$, $t \in [0, T]$, and let $\hat{z} \in \overline{D_0}$. A weak solution to the stochastic differential equation in $\overline{D}$ with coefficients $b$ and $\sigma$, reflection along $\Gamma_t$ on $\partial D_t$, $t \in [0, T]$, and with initial condition $\hat{z}$ at $t = 0$, is a stochastic process $(X^{0,\hat{z}}, \Lambda^{0,\hat{z}})$ with paths in $\mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{BV}([0, T], \mathbb{R}^d)$, which is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and satisfies, $P$-almost surely, whenever $t \in [0, T]$,

$$X_t^{0,\hat{z}} = \hat{z} + \int_0^t b(s, X_s^{0,\hat{z}})\,ds + \int_0^t \sigma(s, X_s^{0,\hat{z}})\,dW_s + \Lambda_t^{0,\hat{z}}, \tag{1.24}$$

$$\Lambda_t^{0,\hat{z}} = \int_0^t \gamma_s\,d|\Lambda^{0,\hat{z}}|_s, \qquad \gamma_s \in \Gamma_s(X_s^{0,\hat{z}}) \cap S_1(0),\ d|\Lambda^{0,\hat{z}}|\text{-a.e.}, \tag{1.25}$$

$$X_t^{0,\hat{z}} \in \overline{D_t}, \qquad d|\Lambda^{0,\hat{z}}|(\{t \in [0, T] : X_t^{0,\hat{z}} \in D_t\}) = 0. \tag{1.26}$$

Here $W$ is a $m$-dimensional Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and $(X^{0,\hat{z}}, \Lambda^{0,\hat{z}})$ is $\{\mathcal{F}_t\}$-adapted.

Concerning weak solutions to stochastic differential equations in $\overline{D}$ with oblique reflection along $\partial D$, we prove the following theorem.

THEOREM 1.9. *Let $T > 0$, $D \subset \mathbb{R}^{d+1}$ and $\Gamma = \Gamma_t(z)$ be as in the statement of Theorem 1.2. Let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be given, bounded and continuous functions on $\overline{D}$ and let $\hat{z} \in \overline{D_0}$. Then there exists a weak solution, in the sense of Definition 1.8, to the stochastic differential equation in $\overline{D}$ with coefficients $b$ and $\sigma$, reflection along $\Gamma_t$ on $\partial D_t$, $t \in [0, T]$, and with initial condition $\hat{z}$ at $t = 0$.*

We note that Theorem 1.9 generalizes the corresponding results in [15, 16] and [53]. Furthermore, we note that there has recently been considerable activity in the study of reflected diffusions in time-dependent intervals. In particular, in this context we mention [9–11] and [12] and we refer the interested reader to these articles for more information as well as for references to other related articles.

The rest of the article is organized as follows. In Section 2 we first briefly outline two general and important themes present in the proofs of the results in this article. The first theme concerns a priori estimates and compactness for solutions to Skorohod problems and the second theme concerns convergence results for sequences of solutions to Skorohod problems. Second, we discuss the proofs of Theorems 1.2, 1.3 and 1.9 and we try to point out the new difficulties occurring due to the time-dependent character of the domain. This section is included for further reference and, in particular, to convey some of the ideas to the reader. In Section 3 we introduce additional notation, outline the restrictions imposed on $D$ and $\Gamma$ and collect a few notions and facts from the Skorohod topology. There is also an appendix attached to Section 3, Appendix. In Appendix we state sufficient conditions for the $(\delta_0, h_0)$-property of good projections along $\Gamma$ and we give examples of time-dependent domains satisfying the assumptions stated in Theorems 1.2, 1.3 and 1.9. Section 4 is devoted to estimates for solutions to the Skorohod problem, with oblique reflection, which have bounded jumps and also to the corresponding estimates for certain approximations of the Skorohod problem. In Section 5 we first prove Theorem 5.1, containing a general result concerning convergence of solutions to Skorohod problems in time-dependent domains. Furthermore, we establish the somewhat similar result for certain approximations of the Skorohod problem. The latter estimates are then used in the proof of Theorem 1.2. The final proofs of Theorems 1.2, 1.3 and 1.9 are given in Section 6. The article ends with the Appendix, discussed above.

**2. A brief outline of proofs and our contribution.** Concerning proofs, we note that the arguments in this article follow two general and important themes which we here, to start with, briefly outline.

*A priori estimates and compactness.* To explain the a priori estimates, we let $T > 0$, $D \subset \mathbb{R}^{d+1}$ and $\Gamma = \Gamma_t(z)$ be as in the statement of Theorem 1.2, and we let $w \in \mathcal{D}([0, T], \mathbb{R}^d)$ with $w_0 \in \overline{D_0}$. Assume that $(x, \lambda)$ is a solution to the Skorohod problem for $(D, \Gamma, w)$ such that $x \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$. Under these assumptions, we prove (see Theorem 4.2 below) that there exist positive constants $L_1(w, T)$, $L_2(w, T)$, $L_3(w, T)$ and $L_4(w, T)$ such that

$$\begin{aligned} \|x\|_{t_1,t_2} &\leq L_1(w, T)\|w\|_{t_1,t_2} + L_2(w, T)l(t_2 - t_1), \\ |\lambda|_{t_2} - |\lambda|_{t_1} &\leq L_3(w, T)\|w\|_{t_1,t_2} + L_4(w, T)l(t_2 - t_1), \end{aligned} \tag{2.1}$$

whenever $0 \leq t_1 \leq t_2 \leq T$. Furthermore, we prove that if $\mathcal{W} \subset \mathcal{D}([0, T], \mathbb{R}^d)$ is relatively compact in the Skorohod topology and $w_0 \in \overline{D_0}$, whenever $w \in \mathcal{W}$, then there exist positive constants $L_1^T$, $L_2^T$, $L_3^T$ and $L_4^T$, such that

$$\sup_{w \in \mathcal{W}} L_i(w, T) \leq L_i^T < \infty \qquad \text{for } i = 1, 2, 3, 4. \tag{2.2}$$

*Convergence results for sequences of solutions to Skorohod problems.* The a priori estimates and compactness result in (2.1) and (2.2) are useful for proving convergence of solutions to Skorohod problems. To explain this further, let $\{D^n\}_{n=1}^\infty$ be a sequence of time-dependent domains $D^n \subset \mathbb{R}^{d+1}$ satisfying (1.2) and let $\{\Gamma^n\}_{n=1}^\infty = \{\Gamma_t^n(z)\}_{n=1}^\infty$ be a sequence of closed convex cones of vectors in $\mathbb{R}^d$. Assume that $\{D^n\}_{n=1}^\infty$ and $\{\Gamma^n\}_{n=1}^\infty$ satisfy the conditions stated in Theorem 1.2 with constants that are "uniform with respect to $n$" in a sense made precise in Section 5. Let $\{w^n\}$, with $w_0^n \in \overline{D_0^n}$, be a sequence in $\mathcal{D}([0, T], \mathbb{R}^d)$ which is relatively compact in the Skorohod topology and which converges to $w \in \mathcal{D}([0, T], \mathbb{R}^d)$ with $w_0 \in \overline{D_0}$. Furthermore, let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2), let $\Gamma = \Gamma_t(z)$ be, for every $z \in \partial D_t$, $t \in [0, T]$, a closed convex cone of vectors in $\mathbb{R}^d$ satisfying (1.11) and (1.12). Assume that the sequences $\{D^n\}_{n=1}^\infty$ and $\{\Gamma^n\}_{n=1}^\infty$ converge to $D$ and $\Gamma$, respectively, in a sense specified in Theorem 5.1. If there exists, for all $n \geq 1$, a solution $(x^n, \lambda^n)$ to the Skorohod problem for $(D^n, \Gamma^n, w^n)$ such that $x_t^n \in \overline{D_t^n}$, for all $t \in [0, T]$, and $x^n \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$, then it follows, using (2.1) and (2.2), that $\{(w^n, x^n, \lambda^n, |\lambda^n|)\}$ is relatively compact in $\mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}_+)$. Hence, we are able to conclude that $\{(x^n, \lambda^n)\}$ converges to some $(x, \lambda) \in \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}^d)$ with $x \in \overline{D}$ and we can, in addition, prove that $(x, \lambda)$ is indeed a solution to the Skorohod problem for $(D, \Gamma, w)$. This result, found in Theorem 5.1 below, constitutes a general convergence result for sequences of solutions to Skorohod problems based on the a priori estimates and compactness result in (2.1) and (2.2).

Although Theorem 1.2 does not follow directly from the results outlined above, we claim that (2.1), (2.2) and Theorem 5.1, stated below, are of independent interest and may be useful in other applications involving the Skorohod problem. To start an outline of the actual proofs of Theorems 1.2, 1.3 and 1.9, we note that to prove Theorem 1.2 we use arguments similar to those outlined above, but in this case we have to construct, given $(D, \Gamma, w)$, an approximating sequence $\{(D^n, \Gamma^n, w^n)\}$ such that a solution $(x^n, \lambda^n)$ to the Skorohod problem for $(D^n, \Gamma^n, w^n)$ can be found explicitly.

*Proof of Theorems* 1.2, 1.3 *and* 1.9. To discuss the construction of $\{(D^n, \Gamma^n, w^n)\}$ and $\{(x^n, \lambda^n)\}$ used in the proof of Theorem 1.2, we consider $w \in \mathcal{D}([0, T], \mathbb{R}^d)$, with $w_0 \in \overline{D_0}$ and with jumps bounded by some constant, and we now let $\{\tau_k\}_{k=0}^N$ define a partition $\Delta$ of the interval $[0, T]$, that is, $0 = \tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = T$. Given $\Delta$, we let

$$\Delta^* := \max_{k \in \{0, \dots, N-1\}} \tau_{k+1} - \tau_k, \tag{2.3}$$

and, given $\Delta$ and $w$, we define

$$w_t^\Delta = w_{\tau_{k-1}} \qquad \text{whenever } t \in [\tau_{k-1}, \tau_k), k \in \{1, \dots, N\}, \tag{2.4}$$

and $w_T^\Delta = w_T$. Then $w^\Delta \in \mathcal{D}([0, T], \mathbb{R}^d)$ is a step function approximation of $w$. Furthermore, assume that $\Delta$ and $w^\Delta$ are such that

$$\|w^\Delta\|_{\tau_{k-1}, \tau_k} + l(\Delta^*) < \delta_0, \tag{2.5}$$

whenever $k \in \{1, \dots, N\}$. Recall that $\delta_0$ is the constant appearing in the notion of the $(\delta_0, h_0)$-property of good projections. We next define

$$\begin{aligned} D_t^\Delta &= D_{\tau_{k-1}}, \\ \Gamma_t^\Delta &= \Gamma_{\tau_{k-1}} \qquad \text{whenever } t \in [\tau_{k-1}, \tau_k), k \in \{1, \dots, N\}, \end{aligned} \tag{2.6}$$

and $D_T^\Delta = D_T, \Gamma_T^\Delta = \Gamma_T$. Given $w^\Delta$, $D^\Delta$ and $\Gamma^\Delta$ as above, we define a pair of processes $(x^\Delta, \lambda^\Delta)$ as follows. We let

$$x_t^\Delta = w_0, \qquad \lambda_t^\Delta = 0 \qquad \text{for } t \in [0, \tau_1). \tag{2.7}$$

If $x^{\Delta}_{\tau_{k-1}} \in \overline{D^{\Delta}_{\tau_{k-1}}}$ for some $k \in \{1, \dots, N\}$, then, by the triangle inequality and (2.5),

$$d(x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}}, \overline{D^{\Delta}_{\tau_k}}) \leq \|w^n\|_{\tau_{k-1},\tau_k} + l(\Delta^*) < \delta_0. \tag{2.8}$$

Hence, by the $(\delta_0, h_0)$-property of good projections, it follows that if $x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}} \notin \overline{D^{\Delta}_{\tau_k}}$, then there exists a point

$$\pi^{\Gamma^{\Delta}_{\tau_k}}_{\partial D^{\Delta}_{\tau_k}}(x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}}) \in \partial D^{\Delta}_{\tau_k}, \tag{2.9}$$

which is the projection of $x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}}$ onto $\partial D^{\Delta}_{\tau_k}$ along $\Gamma^{\Delta}_{\tau_k}$. Furthermore, if $x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}} \in \overline{D^{\Delta}_{\tau_k}}$, then we let

$$\pi^{\Gamma^{\Delta}_{\tau_k}}_{\partial D^{\Delta}_{\tau_k}}(x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}}) = x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}}. \tag{2.10}$$

Based on this argument, we define, whenever $t \in [\tau_k, \tau_{k+1})$, $k \in \{1, \dots, N-1\}$,

$$\begin{aligned} x^{\Delta}_t &= \pi^{\Gamma^{\Delta}_{\tau_k}}_{\partial D^{\Delta}_{\tau_k}}(x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}}), \\ \lambda^{\Delta}_t &= \lambda^{\Delta}_{\tau_{k-1}} + \big(x^{\Delta}_t - (x^{\Delta}_{\tau_{k-1}} + w^{\Delta}_{\tau_k} - w^{\Delta}_{\tau_{k-1}})\big). \end{aligned} \tag{2.11}$$

Finally, we define $x^{\Delta}_T$ and $\lambda^{\Delta}_T$ as in (2.11) by putting $k = N$ in (2.11). By construction, the pair $(x^{\Delta}, \lambda^{\Delta})$ is a solution to the Skorohod problem for $(D^{\Delta}, \Gamma^{\Delta}, w^{\Delta})$. Moreover, using the assumption on the size of the jumps of $w$ stated in Theorem 1.2, we will be able to make the construction so that we can conclude that $x^{\Delta} \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$. As the next step we then apply Theorem 4.6 stated below, showing the existence of positive constants $\hat{L}_1(w, T)$, $\hat{L}_2(w, T)$, $\hat{L}_3(w, T)$ and $\hat{L}_4(w, T)$ such that

$$\begin{aligned} \|x^{\Delta}\|_{t_1,t_2} &\leq \hat{L}_1(w, T)\|w\|_{t_1,t_2} + \hat{L}_2(w, T)\big(l(t_2 - t_1) + l(\Delta^*)\big), \\ |\lambda^{\Delta}|_{t_2} - |\lambda^{\Delta}|_{t_1} &\leq \hat{L}_3(w, T)\|w\|_{t_1,t_2} + \hat{L}_4(w, T)\big(l(t_2 - t_1) + l(\Delta^*)\big), \end{aligned} \tag{2.12}$$

whenever $0 \leq t_1 \leq t_2 \leq T$. Provided with the estimates in (2.12), we are then able to prove Theorem 1.2 by means of compactness arguments similar to those outlined above. Indeed, we construct an appropriate sequence of partitions $\{\Delta_n\}_{n=1}^{\infty}$, based on $w$, such that $x^{\Delta_n} \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$ for $n$ larger than some $n_0$ and such that $(x^{\Delta_n}, \lambda^{\Delta_n})$ is a solution to the Skorohod problem for $(D^{\Delta_n}, \Gamma^{\Delta_n}, w^{\Delta_n})$ for $n \geq n_0$. Then, using (2.12), we conclude that $\{(w^{\Delta_n}, x^{\Delta_n}, \lambda^{\Delta_n}, |\lambda^{\Delta_n}|)\}$ is a relatively compact sequence in the Skorohod topology and that $\{(x^{\Delta_n}, \lambda^{\Delta_n})\}$ converges in the sense of the Skorohod topology to a pair of functions $(x, \lambda)$. Note that an important difference here, compared to the situation outlined above, is that $D^{\Delta}$ and $\Gamma^{\Delta}$ as defined in (2.6) are discontinuous in time. To be able to handle this situation, we employ some additional arguments, similar to the ones in the proof of

Theorem 5.1, in order to prove that $(x, \lambda)$ is a solution to the Skorohod problem for $(D, \Gamma, w)$ on $[0, T]$. This completes the proof of Theorem 1.2. Concerning Theorem 1.3, we see that this theorem follows immediately from the continuity of $w$ and (1.10) using the estimates in (2.1). To prove Theorem 1.9, we argue somewhat similarly as in the proof of Theorem 1.2 and we refer to the bulk of the article for details.

To conclude, we note that the proof of Theorem 1.2 is more involved compared to the proof of the corresponding result for time-independent domains established in [15] and that new difficulties occur, naturally, due to the fact that we are considering time-dependent domains. In the time-independent case a solution $(x, \lambda)$ to the Skorohod problem for $(D, \Gamma, w)$ is constructed as the limit of a sequence $\{(x^{\Delta_n}, \lambda^{\Delta_n})\}$, where $(x^{\Delta_n}, \lambda^{\Delta_n})$ is a solution to a Skorohod problem based on $w^{\Delta_n}$. In this case $(x^{\Delta_n}, \lambda^{\Delta_n})$ is a solution to a Skorohod problem for $(D, \Gamma, w^{\Delta_n})$, while in our case $(x^{\Delta_n}, \lambda^{\Delta_n})$ is a solution to a Skorohod problem for $(D^{\Delta_n}, \Gamma^{\Delta_n}, w^{\Delta_n})$. Hence, in the time-dependent case we, at each step, also have to discretize and approximate $D$ and $\Gamma$ due to the time-dependent character of the domain. In particular, the fact that $D^{\Delta_n}$ and $\Gamma^{\Delta_n}$, as defined in (2.6), are discontinuous in time induces several new difficulties which we have to overcome in order to complete the proof of Theorem 1.2.

**3. Preliminaries.** In this section we introduce notation, collect a number of preliminary results concerning the geometry of time-dependent domains and recall a few notions and facts from the Skorohod topology.

3.1. *Notation.* Points in Euclidean $(d+1)$-space $\mathbb{R}^{d+1}$ are denoted by $(t, z) = (t, z_1, \dots, z_d)$. Given a differentiable function $f = f(t, z)$ defined on $\mathbb{R} \times \mathbb{R}^d$, we let $\partial_{z_i} f(t, z)$ denote the partial derivative of $f$ at $(t, z)$ with respect to $z_i$ and we let $\nabla_z f$ denote the gradient $(\partial_{z_1} f, \dots, \partial_{z_d} f)$. Higher order derivatives of $f$ with respect to the space variables will often be denoted by $\partial_{z_i z_j} f(t, z)$, $\partial_{z_i z_j z_k} f(t, z)$ and so on. Furthermore, given a multi-index $\beta = (\beta_1, \dots, \beta_d)$, $\beta_i \in \mathbb{Z}_+$, we define $|\beta| = \beta_1 + \cdots + \beta_d$ and we let $\partial_z^\beta f(t, z)$ denote the associated partial derivative of $f(t, z)$ with respect to the space variables. Time derivatives of $f$ will be denoted by $\partial_t^j f(t, z)$ where $j \in \mathbb{Z}_+$. As in the Introduction, we let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^d$ and we let $|z| = \langle z, z \rangle^{1/2}$ be the Euclidean norm of $z$. Whenever $z \in \mathbb{R}^d$, $r > 0$, we let $B_r(z) = \{y \in \mathbb{R}^d : |z - y| < r\}$ and $S_r(z) = \{y \in \mathbb{R}^d : |z - y| = r\}$. In addition, $dz$ denotes the Lebesgue $d$-measure on $\mathbb{R}^d$. Moreover, given $E \subset \mathbb{R}^d$, we let $\bar{E}$ and $\partial E$ be the closure and boundary of $E$, respectively, and we let $d(z, E)$ denote the Euclidean distance from $z \in \mathbb{R}^d$ to $E$. Given $(t, z), (s, y) \in \mathbb{R}^{d+1}$, we let $d_p((t, z), (s, y)) = \max\{|z - y|, |t - s|^{1/2}\}$ denote the parabolic distance between $(t, z)$ and $(s, y)$ and for $F \subset \mathbb{R}^{d+1}$, we let $d_p((t, z), F)$ denote the parabolic distance from $(t, z) \in \mathbb{R}^{d+1}$ to $F$. Moreover, for $(t, z) \in \mathbb{R}^{d+1}$ and $r > 0$, we introduce the parabolic cylinder $C_r(t, z) =$

$\{(s, y) \in \mathbb{R}^{d+1} : |y - z| < r, |t - s| < r^2\}$. Given two real numbers $a$ and $b$, we let $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Finally, given a Borel set $E \subset \mathbb{R}^{d+1}$, we let $\chi_E$ denote the characteristic function associated to $E$.

Given a time-dependent domain $D'$, a function $f$ defined on $D'$ and a constant $\alpha \in (0, 1]$, we adopt the definition on page 46 in [41] and introduce

$$
\begin{aligned}
|f|_{1+\alpha,D'} = {} & \sum_{|\beta|\leq 1} \sup_{D'} |\partial_z^\beta f| + \sup_{(t,z)\in D'} \sup_{(s,z)\in D'\setminus\{(t,z)\}} \frac{|f(t,z) - f(s,z)|}{|t-s|^{(\alpha+1)/2}} \\
& + \sum_{|\beta|=1} \sup_{(t,z)\in D'} \sup_{(s,y)\in D'\setminus\{(t,z)\}} \frac{|\partial_z^\beta f(t,z) - \partial_z^\beta f(s,y)|}{[d_p((t,z),(s,y))]^\alpha}.
\end{aligned}
\tag{3.1}
$$

The third term on the right-hand side of $|f|_{1+\alpha,D'}$ is superfluous for our purposes, but we include it here for agreement with the theory of partial differential equations in time-dependent domains (see [45]). Using the norm $|f|_{1+\alpha,D'}$, we let $\mathcal{H}_{1+\alpha}(D')$ denote the Banach space of functions $f$ on $D'$ with finite $|f|_{1+\alpha,D'}$-norm.

3.2. *Geometry of time-dependent domains.* We here outline the geometric restrictions which we impose on the time-dependent domains and cones of reflections. Concerning $D$, we first prove the following auxiliary lemma.

LEMMA 3.1. *Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2) *and* (1.10) *and assume that $D$ satisfies a uniform exterior sphere condition in time with radius $r_0$ in the sense of* (1.8). *Let*

$$
\hat{l}(r) := \sup_{\substack{s,t\in[0,T] \\ |s-t|\leq r}} \sup_{z\in\partial D_s} d(z, \partial D_t). \tag{3.2}
$$

*Then, $l(r) = \hat{l}(r)$ for all $r > 0$ such that $l(r) < r_0$.*

PROOF. In the following we let $\epsilon > 0$ be arbitrary and we consider $r$ small enough to ensure that $l(r) < r_0$. With $\epsilon$ and $r$ fixed, we let $s, t \in [0, T]$, $|s - t| \leq r$, be such that

$$
\tilde{l}_1 \leq \hat{l}(r) \leq \tilde{l}_1 + \epsilon \qquad \text{where } \tilde{l}_1 = \sup_{z\in\partial D_s} d(z, \partial D_t). \tag{3.3}
$$

Naturally,

$$
\tilde{l}_1 = \max\Big\{ \sup_{z\in\partial D_s\cap(\mathbb{R}^d\setminus D_t)} d(z, \partial D_t), \sup_{z\in\partial D_s\cap D_t} d(z, \partial D_t) \Big\}. \tag{3.4}
$$

Assume $z \in \partial D_s \cap (\mathbb{R}^d \setminus D_t)$. Then we immediately obtain

$$
\sup_{z\in\partial D_s\cap(\mathbb{R}^d\setminus D_t)} d(z, \partial D_t) = \sup_{z\in\partial D_s\cap(\mathbb{R}^d\setminus D_t)} d(z, D_t) \leq \sup_{z\in\overline{D_s}} d(z, D_t) \leq l(r). \tag{3.5}
$$

Assume, on the contrary, that $z \in \partial D_s \cap D_t$. In this case, as $l(r) < r_0$ and $D_s$ satisfies the uniform exterior sphere condition with radius $r_0$, we can conclude that there exists at least one point $y_z \in \partial D_t \cap \{z + n_\lambda \in \mathbb{R}^d : n_\lambda \in N_s(z) \cap S_\lambda(0), 0 < \lambda < r_0\}$ and obviously $d(z, \partial D_t) \le |z - y_z|$. Furthermore, again applying the uniform exterior sphere condition, we see that $z$ minimizes the distance from $y_z \in \partial D_t$ to $D_s$. Hence,

$$\begin{aligned}\sup_{z\in\partial D_s\cap D_t} d(z, \partial D_t) &\le \sup_{z\in\partial D_s\cap D_t} |z - y_z| \le \sup_{z\in\partial D_s\cap D_t} d(y_z, D_s)\\ &\le \sup_{y_z\in\partial D_t} d(y_z, D_s) \le \sup_{y\in\overline{D_t}} d(y, D_s) \le l(r).\end{aligned} \tag{3.6}$$

Combining (3.4)–(3.6), we conclude that $\tilde{l}_1 \le l(r)$. Using (3.3), it is clear that $\hat{l}(r) \le l(r) + \epsilon$ and then, as $\epsilon$ is arbitrary, $\hat{l}(r) \le l(r)$. We next consider the opposite inequality. With $\epsilon$ and $r$ fixed, we let $s, t \in [0, T]$, $|s - t| \le r$, be such that

$$\tilde{l}_2 \le l(r) \le \tilde{l}_2 + \epsilon \qquad \text{where } \tilde{l}_2 = \sup_{z\in\overline{D_s}} d(z, D_t). \tag{3.7}$$

In this case we have

$$\tilde{l}_2 = \max\Big\{\sup_{z\in\overline{D_s}\cap(\mathbb{R}^d\setminus D_t)} d(z, D_t), 0\Big\} = \max\Big\{\sup_{z\in\overline{D_s}\cap(\mathbb{R}^d\setminus D_t)} d(z, \partial D_t), 0\Big\} \tag{3.8}$$

and in the following we can assume, without loss of generality, that $\tilde{l}_2 > 0$. Then, by the uniform exterior sphere condition, and the fact that $l(r) < r_0$, we see that every point $z \in \overline{D_s} \cap (\mathbb{R}^d \setminus D_t)$ can be written as $z = y_z + n_\lambda$ for some $y_z \in \partial D_t$ and some $n_\lambda \in N_t(y_z) \cap S_\lambda(0)$, $0 < \lambda < r_0$. Furthermore, there exists a point $\tilde{z} = y_z + n_{\tilde{\lambda}} \in \partial D_s$, with $0 < \lambda < \tilde{\lambda} < r_0$. Once again applying the uniform exterior sphere condition, we see that $y_z$ minimizes the distance from $\tilde{z}$ to $\partial D_t$ and we obtain

$$\begin{aligned}\sup_{z\in\overline{D_s}\cap(\mathbb{R}^d\setminus D_t)} d(z, \partial D_t) &\le \sup_{z\in\overline{D_s}\cap(\mathbb{R}^d\setminus D_t)} |z - y_z|\\ &\le \sup_{z\in\overline{D_s}\cap(\mathbb{R}^d\setminus D_t)} |\tilde{z} - y_z| \le \sup_{\tilde{z}\in\partial D_s} d(\tilde{z}, \partial D_t) \le \hat{l}(r).\end{aligned} \tag{3.9}$$

Combining (3.8)–(3.9), we conclude that $\tilde{l}_2 \le \hat{l}(r)$. Using (3.7), it is clear that $l(r) \le \hat{l}(r) + \epsilon$ and then, as $\epsilon$ is arbitrary, $l(r) \le \hat{l}(r)$. This completes the proof of the lemma. $\square$

REMARK 3.2. Note also that the prerequisites of Lemma 3.1 ensure that the number of holes in $D_t$ stays the same for all $t \in [0, T]$ and, in particular, that these holes cannot shrink too much as time changes. Furthermore, Lemma 3.1 and its proof allow us to conclude that

$$h(\overline{D_s}, \overline{D_t}) = h(D_s, D_t) = h(\partial D_s, \partial D_t) \tag{3.10}$$

whenever $s, t \in [0, T]$, $|s - t| \le r$, $l(r) < r_0$.

Concerning $\Gamma$, we let $\Gamma = \Gamma_t(z) = \Gamma(t, z)$ be a function defined on $\mathbb{R}^{d+1}$ such that $\Gamma_t(z)$ is a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_t$, $t \in [0, T]$ and we assume that $\Gamma$ satisfies (1.11) and (1.12). To understand the condition in (1.12), that is, the assumption that the graph $G^\Gamma$ is closed, we observe that one motivation for using a cone of reflection, rather than a single-valued direction of reflection, is to be able to deal with discontinuities in the direction of reflection. Such discontinuities arise, for instance, in the normal direction for a convex polygon. At a point of discontinuity of the direction of reflection one can use the cone generated by all the limit vectors (if they exist) of the direction of reflection. For a cone of reflection, the assumption that the graph $G^\Gamma$ is closed provides a form of continuity of the cone. In fact, for a cone of the form $\Gamma_t(z) = \{\lambda\gamma_t(z), \lambda \geq 0\}$, for some $\mathbb{R}^d$-valued function $\gamma_t(z)$, the assumption that $G^\Gamma$ is closed is equivalent to the assumption that the function $\gamma_t(z)$ is continuous as a function of $(t, z)$.

The cone $N_t(z)$ of inward normal vectors at $z \in \partial D_t$, $t \in [0, T]$, is defined as being equal to the set consisting of the union of the set $\{0\}$ and the set

$$\{v \in \mathbb{R}^d : v \neq 0, \exists \rho > 0 \text{ such that } B_\rho(z - \rho v/|v|) \subset ([0, T] \times \mathbb{R}^d) \setminus D\}. \tag{3.11}$$

Note that this definition does not rule out the possibility of several unit inward normal vectors at the same boundary point. Given $N_t(z)$, we let $N_t^1(z) := N_t(z) \cap S_1(0)$, so that $N_t^1(z)$ contains the set of vectors in $N_t(z)$ with unit length. Moreover, based on $N_t(z)$, we introduce the set

$$G^N = \{(t, z, n) : n \in N_t(z), z \in \partial D_t, t \in [0, T]\}. \tag{3.12}$$

The spatial domain $D_t$ is said to verify the uniform exterior sphere condition if there exists a radius $r_0 > 0$ such that (1.8) holds. It is easy to see that (1.8) is equivalent to the statement that

$$\langle n, y - z\rangle + \frac{1}{2r_0}|y - z|^2 \geq 0 \tag{3.13}$$

for all $y \in \overline{D_t}$, $n \in N_t^1(z)$ and $z \in \partial D_t$. Moreover, as deduced from Remark 2.1 in [15], the uniform exterior sphere condition in time asserts that $N_t(z)$ is a closed convex cone for all $z \in \partial D_t$, $t \in [0, T]$ and that $G^N$ is closed.

For $z \in \partial D_s$, $s \in [0, T]$, and $\rho, \eta > 0$, recall the definition of the quantity $a_{s,z}(\rho, \eta)$ introduced in (1.15),

$$a_{s,z}(\rho, \eta) = \max_{u \in S_1(0)} \min_{s \leq t \leq s+\eta} \min_{y \in \partial D_t \cap \overline{B_\rho(z)}} \min_{\gamma \in \Gamma_t^1(y)} \langle \gamma, u\rangle.$$

The vector $u$ that maximizes the minimum of $\langle\gamma, u\rangle$ over all vectors $\gamma \in \Gamma_t^1(y)$ in a time–space neighborhood of a point $(s, z)$, $z \in \partial D_s$, $s \in [0, T]$, can be regarded as the best approximation of the $\Gamma_t^1(y)$-vectors in that neighborhood. With this interpretation $a_{s,z}(\rho, \eta)$ represents the cosine of the largest angle between the best approximation and a $\Gamma_t^1(y)$-vector in the neighborhood. Hence, in a sense,

$a_{s,z}(\rho,\eta)$ quantifies the variation of $\Gamma$ in a space–time neighborhood of $(s,z)$. For $z\in\partial D_s$, $s\in[0,T]$ and $\rho,\eta>0$, recall the definition of the quantity $c_{s,z}(\rho,\eta)$ introduced in (1.16),

$$c_{s,z}(\rho,\eta)=\max_{s\le t\le s+\eta}\max_{y\in\partial D_t\cap\overline{B_\rho(z)}}\max_{\hat z\in\overline{D_t}\cap\overline{B_\rho(z)},\hat z\ne y}\max_{\gamma\in\Gamma_t^1(y)}\left(\frac{\langle\gamma,y-\hat z\rangle}{|y-\hat z|}\vee 0\right).$$

This quantity is close to one if the vectors $\gamma\in\Gamma_t^1(y)$, in a time–space neighborhood, deviate much from the normal vectors and/or the domain is very concave. Hence, in a sense, $c_{s,z}(\rho,\eta)$ quantifies the skewness of $\Gamma$ and the concavity of $D$. Note that (1.16) implies

$$\langle\gamma,\hat z-y\rangle+c_{s,z}(\rho,\eta)|y-\hat z|\ge 0 \tag{3.14}$$

for all $y\in\partial D_t\cap\overline{B_\rho(z)}$, $\hat z\in\overline{D_t}\cap\overline{B_\rho(z)}$, $\hat z\ne y$ and $\gamma\in\Gamma_t^1(y)$ with $z\in\partial D_t$, $t\in[s,s+\eta]\subset[0,T]$. This condition exhibits some similarity with the uniform exterior sphere property (3.13). Finally, recall the definition of the quantity $e_{s,z}(\rho,\eta)$ introduced in (1.17),

$$e_{s,z}(\rho,\eta)=\frac{c_{s,z}(\rho,\eta)}{(a_{s,z}(\rho,\eta))^2\vee a_{s,z}(\rho,\eta)/2}.$$

Furthermore, as stated in the Introduction, in the subsequent section we prove estimates related to the Skorohod problem in time-dependent domains satisfying (1.2) and the uniform exterior sphere condition in time, with radius $r_0$. Moreover, to derive these estimates, we also assume that there exist $0<\rho_0<r_0$ and $\eta_0>0$, such that the assumptions in (1.18) and (1.19) hold, that is,

$$\inf_{s\in[0,T]}\inf_{z\in\partial D_s}a_{s,z}(\rho_0,\eta_0)=a>0,$$
$$\sup_{s\in[0,T]}\sup_{z\in\partial D_s}e_{s,z}(\rho_0,\eta_0)=e<1.$$

REMARK 3.3. The function $a_{s,z}(\rho,\eta)$ is a straightforward generalization of the function

$$\alpha_z(\rho)=\max_{u\in S_1(0)}\min_{y\in\partial\Omega\cap\overline{B_\rho(z)}}\min_{n\in N^1(y)}\langle n,u\rangle, \tag{3.15}$$

introduced by Tanaka [61] in his treatment of the Skorohod problem. Here $\Omega\subset\mathbb{R}^d$ is a bounded spatial domain and $N^1(y)$ is the set of unit inward normals at $y\in\partial\Omega$. In [15, 42, 53] and [61] the condition

$$\lim_{\rho\to 0}\inf_{z\in\partial\Omega}\alpha_z(\rho)=\alpha>0 \tag{3.16}$$

is used to rule out the case of tangential normal directions (see [15] for equivalent characterizations of domains satisfying this criterion).

REMARK 3.4. The functions $c_{s,z}(\rho,\eta)$ and $e_{s,z}(\rho,\eta)$, introduced in (1.16) and (1.17), are straightforward generalizations of the functions $\widetilde{c}$ and $\widetilde{e}$, respectively, introduced in [15]. Moreover, the related functions $c$ and $e$, also introduced in [15], are useful only in the context of convex domains. Hence, as we here consider general (possibly nonconvex) domains, only generalizations of the functions $\widetilde{c}$ and $\widetilde{e}$ are useful. For notational simplicity, we have removed the tilde in our definition of the generalized versions of $\widetilde{c}$ and $\widetilde{e}$.

Given $T > 0$, let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2) and a uniform exterior sphere condition in time with radius $r_0$ in the sense of (1.8). Given a point $(t,z) \in ([0,T] \times \mathbb{R}^d) \setminus \overline{D}$ in a neighborhood of $D$, in this article we heavily use the projection of $(t,z)$ onto $\partial D$ along the vectors in the cone $\Gamma$. While such projections can be defined in several ways, in this article we here only consider projections in space along vectors $\gamma \in \Gamma_t(y)$, $y \in \partial D_t$, onto $\partial D_t$. With this restriction, the analysis of Section 4 in [15] can be used to derive sufficient conditions for the existence of a projection of a point $z \in \mathbb{R}^d \setminus \overline{D_t}$, onto $\partial D_t$, along $\Gamma_t$. In other words, we can determine whether or not there exist, for a given point $z \in \mathbb{R}^d \setminus \overline{D_t}$, a point $y \in \partial D_t$ and a vector $\gamma \in \Gamma_t(y)$ such that $y - z \parallel \gamma$. In particular, it can be understood when, for $0 < \delta_0 < r_0$, $h_0 > 1$ and $\Gamma = \Gamma_t(z) = \Gamma(t,z)$ given, $([0,T] \times \mathbb{R}^d) \setminus \overline{D}$ has the $(\delta_0, h_0)$-property of good projections along $\Gamma$ in the sense defined in the introduction; see (1.20) and (1.21). We refer to the Appendix, for more on this, as well as for a discussion of examples of time-dependent domains satisfying the restrictions imposed in Theorems 1.2, 1.3 and 1.9.

3.3. *Càdlàg functions and the Skorohod topology.* Let $T > 0$ and let $x \in \mathcal{D}([0,T], \mathbb{R}^d)$. Given a bounded set $I \subset [0,T]$, we let

$$\widehat{w}(x, I) = \sup_{u,r \in I} |x_u - x_r|. \tag{3.17}$$

Then, using Lemma 1 on page 122 in [7], we see that there exists, for $\epsilon > 0$ given, a sequence of points $t_0, \dots, t_\nu$, such that

$$0 = t_0 < t_1 < \cdots < t_\nu = T, \qquad \widehat{w}(x, [t_{i-1}, t_i)) < \epsilon. \tag{3.18}$$

In particular, there can only be finitely many points $t \in [0,T]$ at which the jump $|x_t - x_{t-}|$ exceeds a given positive number. To proceed, in the following we use the notation and exposition of Chapter 3 in [30]. We let $q(x,y) = |x-y| \wedge 1$ whenever $x, y \in \mathbb{R}^d$ and we let $d_{\mathcal{D}}([0,T], x, y)$ be the metric on the space $\mathcal{D}([0,T], \mathbb{R}^d)$ introduced, for the interval $[0,T]$, as in display (5.2) in [30]. Then, by Theorem 5.6 in [30], we see that $(\mathcal{D}([0,T], \mathbb{R}^d), d_{\mathcal{D}}([0,T], \cdot, \cdot))$ is a complete metric space and the topology on $\mathcal{D}([0,T], \mathbb{R}^d)$, induced by the metric $d_{\mathcal{D}}([0,T], \cdot, \cdot)$, is known as the Skorohod topology on $\mathcal{D}([0,T], \mathbb{R}^d)$. Recall that if $x, y \in \mathcal{D}([0,T], \mathbb{R}^d)$, then $d_{\mathcal{D}}([0,T], x, y) = 0$ implies that $x_t = y_t$ for every $t$. Furthermore, if $\{x^n\}$ is a sequence in $\mathcal{D}([0,T], \mathbb{R}^d)$ and $x \in \mathcal{D}([0,T], \mathbb{R}^d)$, then the statement that

$d_{\mathcal{D}}([0,T],x^n,x) \to 0$ as $n \to \infty$ is equivalent to the statement that there exists $\{\lambda_n\} \subset \Lambda$ (see [30] for the definition of the space $\Lambda$) such that (5.6) in [30] holds and such that

$$\lim_{n\to\infty} \sup_{0\le t\le T} |x_t^n - x_{\lambda_n(t)}| = 0. \tag{3.19}$$

For a proof of this result we refer to Proposition 5.3 in [30]. Furthermore, to understand the relatively compact sets in $\mathcal{D}([0,T],\mathbb{R}^d)$, we introduce and use a modulus of continuity. In particular, for $x \in \mathcal{D}([0,T],\mathbb{R}^d)$ and $\delta > 0$ we define the quantity

$$w'(x,\delta,T) = \inf_{\{t_i\}} \max_i \sup_{u,r\in[t_i,t_{i+1})} |x_u - x_r|, \tag{3.20}$$

where the infimum is taken with respect to all partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \le t_n$, with $\min_i |t_i - t_{i-1}| > \delta$. Furthermore, given $\mathcal{W} \subset \mathcal{D}([0,T],\mathbb{R}^d)$, we let

$$\mu(\mathcal{W},\delta,T) = \sup_{w\in\mathcal{W}} w'(w,\delta,T). \tag{3.21}$$

Using this notation, we first quote Theorem 6.3 in [30] which states that $\mathcal{W} \subset \mathcal{D}([0,T],\mathbb{R}^d)$ is relatively compact in the Skorohod topology if and only if for every rational $t \in [0,T]$ there exists a relatively compact set $A_t \subset \mathbb{R}^d$ such that $w_t \in A_t$ for all $w \in \mathcal{W}$ and such that

$$\lim_{\delta\to 0} \mu(\mathcal{W},\delta,T) = 0. \tag{3.22}$$

Finally, we also note the following. Given $\delta' > \delta$, let

$$\tilde{w}'(x,\delta,\delta',T) = \inf_{\{t_i\}} \max_i \sup_{u,r\in[t_i,t_{i+1})} |x_u - x_r|, \tag{3.23}$$

where the infimum is taken with respect to all partitions as above but with the additional restriction that $\max_i |t_i - t_{i-1}| < \delta'$. Furthermore, given $\mathcal{W} \subset \mathcal{D}([0,T],\mathbb{R}^d)$, we let

$$\tilde{\mu}(\mathcal{W},\delta,\delta',T) = \sup_{w\in\mathcal{W}} \tilde{w}'(w,\delta,\delta',T). \tag{3.24}$$

Then

$$\tilde{w}'(x,\delta,\delta',T) = w'(x,\delta,T) \quad \text{and} \quad \tilde{\mu}(\mathcal{W},\delta,\delta',T) = \mu(\mathcal{W},\delta,T). \tag{3.25}$$

**4. Estimates for solutions and approximations to Skorohod problems.** In this section we first prove certain estimates for solutions to the Skorohod problem for $(D,\Gamma,w)$, assuming that $D$ satisfies the assumptions stated in Theorem 1.2 and that $w \in \mathcal{D}([0,T],\mathbb{R}^d)$ with $w_0 \in \overline{D_0}$. In particular, we prove that the modulus of continuity of càdlàg solutions to the Skorohod problem for $(D,\Gamma,w)$, with bounded jumps, can be estimated from above by the modulus of continuity of $w$ and the modulus of continuity $l$. This result is derived in two steps. In the first step we prove (see Lemma 4.1 below) a local compactness result which is valid in a spatial neighborhood of a given boundary point and on a constructed time interval. In the second step we then prove that corresponding global estimates (see Theorem 4.2 below) can be derived based on the local compactness result. In particular, Theorem 4.2 is the main result we establish in this context. In Section 4.1 we derive these estimates for solutions to the Skorohod problem and in Section 4.2 we establish the corresponding results for approximations to the Skorohod problem. In particular, given $w \in \mathcal{D}([0,T],\mathbb{R}^d)$ with $w_0 \in \overline{D_0}$ and a partition $\{\tau_k\}_{k=0}^N$, which we denote by $\Delta$, of the interval $[0,T]$, we define $w^\Delta$, $D^\Delta$, $\Gamma^\Delta$, $x^\Delta$ and $\lambda^\Delta$ as in (2.4), (2.6), (2.7) and (2.11). Then, by construction, the pair $(x^\Delta,\lambda^\Delta)$ is a solution to the Skorohod problem for $(D^\Delta,\Gamma^\Delta,w^\Delta)$. In Lemma 4.5 and Theorem 4.6 we prove estimates for solutions to the Skorohod problem for $(D^\Delta,\Gamma^\Delta,w^\Delta)$, which are similar to the ones established in Lemma 4.1 and Theorem 4.2 for the Skorohod problem for $(D,\Gamma,w)$. We note that the reason for this twofold approach is that since we are considering time-dependent domains, the condition in (1.10) will in general not hold for $D^\Delta$.

4.1. *Estimates for solutions to Skorohod problems.* Given $a>0$ and $e\in(0,1)$, we define the positive functions $K_1$, $K_2$, $K_3$ and $K_4$ as follows:

$$\begin{aligned} K_1(a,e) &= \frac{a+2a^2e+2+ae}{a(1-e)}, \qquad & K_2(a,e) &= \frac{2a^2e+2+ae}{a(1-e)}, \\ K_3(a,e) &= \frac{1+K_1(a,e)}{a}, \qquad & K_4(a,e) &= \frac{1+K_2(a,e)}{a}. \end{aligned} \tag{4.1}$$

In this section we first prove the following two general results for solutions to the Skorohod problem.

LEMMA 4.1. *Let $T>0$ and let $D\subset\mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2), (1.10) *and a uniform exterior sphere condition in time with radius $r_0$ in the sense of* (1.8). *Let $\Gamma=\Gamma_t(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z\in\partial D_t$, $t\in[0,T]$. Assume that* (1.18) *and* (1.19) *hold for some $0<\rho_0<r_0$, $\eta_0>0$, $a$ and $e$. Finally, assume that $([0,T]\times\mathbb{R}^d)\setminus\overline{D}$ has the $(\delta_0,h_0)$-property of good projections along $\Gamma$, for some $0<\delta_0<r_0$, $h_0>1$ as defined in* (1.20) *and* (1.21). *Let $w\in\mathcal{D}([0,T],\mathbb{R}^d)$ with $w_0\in\overline{D_0}$ and let $(x,\lambda)$ be a solution to the Skorohod problem for $(D,\Gamma,w)$. Consider a fixed but arbitrary $s\in[0,T]$, such that $x_s\in\partial D_s$, and note that it follows from* (1.18) *and* (1.19) *that there exist $0<\rho<r_0$ and $\eta>0$ such that*

$$a_{s,x_s}(\rho,\eta)>0, \qquad e_{s,x_s}(\rho,\eta)<1. \tag{4.2}$$

*Then, for $0\le s\le t_1\le t_2<\tau_{\rho,\eta}$,*

$$\|x\|_{t_1,t_2} \le K_1(a,e)\|w\|_{t_1,t_2} + K_2(a,e)l(t_2-t_1), \tag{4.3}$$

$$|\lambda|_{t_2}-|\lambda|_{t_1} \le K_3(a,e)\|w\|_{t_1,t_2} + K_4(a,e)l(t_2-t_1), \tag{4.4}$$

*where* $a = a_{s,x_s}(\rho, \eta)$, $e = e_{s,x_s}(\rho, \eta)$. *Here* $\tau_{\rho,\eta}$ *is defined as follows. If there exists some* $t$ *such that* $s \leq t < (s+\eta) \wedge T$ *and* $|x_t - x_s| + l(t-s) \geq \rho$, *then*

$$\tau_{\rho,\eta} = \inf\{t : s \leq t < (s+\eta) \wedge T, |x_t - x_s| + l(t-s) \geq \rho\}, \tag{4.5}$$

*whereas if* $|x_t - x_s| + l(t-s) < \rho$ *for all* $s \leq t < (s+\eta) \wedge T$, *then*

$$\tau_{\rho,\eta} = (s+\eta) \wedge T. \tag{4.6}$$

THEOREM 4.2. *Let* $T > 0$, $D \subset \mathbb{R}^{d+1}$, $r_0$, $\Gamma = \Gamma_t(z)$, $0 < \rho_0 < r_0$, $\eta_0 > 0$, $a$, $e$, $\delta_0$ *and* $h_0$ *be as in the statement of Lemma* 4.1. *Let* $w \in \mathcal{D}([0, T], \mathbb{R}^d)$ *with* $w_0 \in \overline{D_0}$ *and let* $(x, \lambda)$ *be a solution to the Skorohod problem for* $(D, \Gamma, w)$. *Moreover, assume in addition that* $x \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$. *Then there exist positive constants* $L_1(w, T)$, $L_2(w, T)$, $L_3(w, T)$ *and* $L_4(w, T)$ *such that*

$$\|x\|_{t_1,t_2} \leq L_1(w, T)\|w\|_{t_1,t_2} + L_2(w, T)l(t_2 - t_1), \tag{4.7}$$

$$|\lambda|_{t_2} - |\lambda|_{t_1} \leq L_3(w, T)\|w\|_{t_1,t_2} + L_4(w, T)l(t_2 - t_1), \tag{4.8}$$

*whenever* $0 \leq t_1 \leq t_2 \leq T$. *Furthermore, if* $\mathcal{W} \subset \mathcal{D}([0, T], \mathbb{R}^d)$ *is relatively compact in the Skorohod topology and satisfies* $w_0 \in \overline{D_0}$, *whenever* $w \in \mathcal{W}$, *then there exist positive constants* $L_1^T$, $L_2^T$, $L_3^T$ *and* $L_4^T$, *such that*

$$\sup_{w \in \mathcal{W}} L_i(w, T) \leq L_i^T < \infty \qquad \text{for } i = 1, 2, 3, 4. \tag{4.9}$$

REMARK 4.3. Versions of Lemma 4.1 and Theorem 4.2, valid only in the setting of time-independent domains, are proved in Lemma 2.1, Theorems 2.2 and 2.4 in [15]. Our contribution is that we are able to establish similar results when $D \subset \mathbb{R}^{d+1}$ is a time-dependent domain. Furthermore, concerning related results in the setting of time-dependent domains, we note that if $D$ is an $\mathcal{H}_2$-domain and if $\Gamma_t(z) = \{\lambda\gamma_t(z), \lambda \geq 0\}$, for some $S_1(0)$-valued continuous function $\gamma_t(z)$ such that

$$\inf_{z \in \partial D_t, t \in [0,T]} \langle \gamma_t(z), n_t(z)\rangle > \frac{\sqrt{3}}{2}, \tag{4.10}$$

then a version of Theorem 4.2 is proved in Theorem C.3 in [16]. Note also that if $D$ is an $\mathcal{H}_2$-domain, then there exists a unique unit inward normal, $n_t(z)$, at $z \in \partial D_t$, $t \in [0, T]$.

REMARK 4.4. Unlike in the statements of Theorems 1.2, 1.3 and 1.9, we need not assume that $\Gamma$ satisfies (1.11), (1.12) and (1.14) in the prerequisites of Lemma 4.1 and Theorem 4.2. This remark also applies to Lemma 4.5 and Theorem 4.6 stated below.

PROOF OF LEMMA 4.1. To simplify the notation, we in the following let $a = a_{s,x_s}(\rho,\eta)$, $c = c_{s,x_s}(\rho)$ and $e = e_{s,x_s}(\rho,\eta)$. Moreover, we let $u$ be a unit vector such that

$$\langle \gamma_r, u\rangle \geq a \tag{4.11}$$

for all $\gamma_r \in \Gamma_r^1(y)$, $y \in \partial D_r \cap \overline{B_\rho(x_s)}$ and $r \in [s, \tau_{\rho,\eta}] \subset [s,(s+\eta)\wedge T]$. The existence of such a vector follows from the definition of $a_{s,x_s}(\rho,\eta)$. Using properties (1.5)–(1.6) in Definition 1.1, we see that

$$\langle x_{t_2} - x_{t_1}, u\rangle = \langle w_{t_2} - w_{t_1}, u\rangle + \int_{t_1^+}^{t_2^+} \underbrace{\langle \gamma_r, u\rangle}_{\geq a} \, d|\lambda|_r \tag{4.12}$$

for any $0 \leq s \leq t_1 \leq t_2 < \tau_{\rho,\eta}$. Based on (4.12), we deduce that

$$|\lambda|_{t_2} - |\lambda|_{t_1} \leq \frac{1}{a}(|w_{t_2} - w_{t_1}| + |x_{t_2} - x_{t_1}|). \tag{4.13}$$

Furthermore, again using properties (1.5)–(1.6) in Definition 1.1, we also see that

$$\begin{aligned} |x_{t_2} - x_{t_1}|^2 &= \left(w_{t_2} - w_{t_1} + \int_{t_1^+}^{t_2^+} \gamma_r \, d|\lambda|_r\right)^2 \\ &= |w_{t_2} - w_{t_1}|^2 + \left(\int_{t_1^+}^{t_2^+} \gamma_r \, d|\lambda|_r\right)^2 + 2\int_{t_1}^{t_2^+} \langle w_{t_2} - w_{t_1}, \gamma_r\rangle \, d|\lambda|_r, \end{aligned} \tag{4.14}$$

whenever $0 \leq s \leq t_1 \leq t_2 < \tau_{\rho,\eta}$. Note that the integrand in the last term in this display can be rewritten as

$$\begin{aligned} \langle w_{t_2} - w_{t_1}, \gamma_r\rangle &= \langle w_{t_2} - w_r, \gamma_r\rangle + \langle w_r - w_{t_1}, \gamma_r\rangle \\ &= \langle w_{t_2} - w_r, \gamma_r\rangle + \langle x_r - x_{t_1}, \gamma_r\rangle - \left\langle \left(\int_{t_1^+}^{r^+} \gamma_u \, d|\lambda|_u\right), \gamma_r\right\rangle. \end{aligned} \tag{4.15}$$

In particular, combining (4.14) and (4.15), we see that

$$\begin{aligned} |x_{t_2} - x_{t_1}|^2 = |w_{t_2} - w_{t_1}|^2 &+ 2\int_{t_1^+}^{t_2^+} \langle w_{t_2} - w_r, \gamma_r\rangle \, d|\lambda|_r \\ &+ 2\int_{t_1^+}^{t_2^+} \langle x_r - x_{t_1}, \gamma_r\rangle \, d|\lambda|_r + \left(\int_{t_1^+}^{t_2^+} \gamma_r \, d|\lambda|_r\right)^2 \\ &- 2\int_{t_1^+}^{t_2^+} \left\langle \left(\int_{t_1^+}^{r^+} \gamma_u(x_u) \, d|\lambda|_u\right), \gamma_r\right\rangle d|\lambda|_r. \end{aligned} \tag{4.16}$$

We now intend to derive bounds from above for all integrals in (4.16). To do this, we first note that the first integral on the right-hand side of (4.16) is bounded from above by

$$2\int_{t_1^+}^{t_2^+} \langle w_{t_2} - w_r, \gamma_r\rangle \, d|\lambda|_r \leq 2\int_{t_1^+}^{t_2^+} |w_{t_2} - w_r| \, d|\lambda|_r. \tag{4.17}$$

To find an upper bound of the second integral in (4.16), we must take into account that, due to the fact that our domain is time-dependent, $x_{t_1}$ might not belong to $\overline{D_r}$. Recall that we are assuming that $N_r(y) \neq \emptyset$ for all $y \in \partial D_r$, $r \in [0, T]$, and, given $r \in [0, T]$, $y \in \mathbb{R}^d \setminus \overline{D_r}$, in the following we denote a projection of $y$ onto $\partial D_r$ along $N_r$ by $\pi_{\partial D_r}^{N_r}(y)$. Furthermore, whenever $y \in \overline{D_r}$ we let $\pi_{\partial D_r}^{N_r}(y) = y$. Using this notation, and the definition of $\tau_{\rho,\eta}$, we see that

$$|\pi_{\partial D_r}^{N_r}(x_{t_1}) - x_s| \leq |x_{t_1} - x_s| + l(r - t_1) \leq \rho. \tag{4.18}$$

Equation (4.18) implies that $\pi_{\partial D_r}^{N_r}(x_{t_1}) \in \overline{B_\rho(x_s)} \cap \partial D_r \subset \overline{B_\rho(x_s)} \cap \overline{D_r}$. Next, writing

$$\langle x_r - x_{t_1}, \gamma_r \rangle = \langle x_r - \pi_{\partial D_r}^{N_r}(x_{t_1}), \gamma_r \rangle + \langle \pi_{\partial D_r}^{N_r}(x_{t_1}) - x_{t_1}, \gamma_r \rangle, \tag{4.19}$$

and using the fact that $x_r \in \overline{B_\rho(x_s)} \cap \partial D_r$ a.e. when $d|\lambda|_r \neq 0$, together with a version of (3.14), we deduce that

$$\langle x_r - \pi_{\partial D_r}^{N_r}(x_{t_1}), \gamma_r \rangle \leq c|x_r - \pi_{\partial D_r}^{N_r}(x_{t_1})| \leq c|x_r - x_{t_1}| + cl(r - t_1). \tag{4.20}$$

Furthermore,

$$\langle \pi_{\partial D_r}^{N_r}(x_{t_1}) - x_{t_1}, \gamma_r \rangle \leq |\pi_{\partial D_r}^{N_r}(x_{t_1}) - x_{t_1}| \leq l(r - t_1). \tag{4.21}$$

Using the estimates derived above, we conclude that the second integral in (4.16) has the upper bound

$$\begin{aligned} 2\int_{t_1^+}^{t_2^+} \langle x_r - x_{t_1}, \gamma_r \rangle \, d|\lambda|_r \leq{}& 2c\int_{t_1^+}^{t_2^+} |x_r - x_{t_1}| \, d|\lambda|_r \\ &+ 2(c+1)l(t_2 - t_1)(|\lambda|_{t_2} - |\lambda|_{t_1}). \end{aligned} \tag{4.22}$$

Next, we use Lemma 2.1(ii) in [53] and rewrite the third integral in (4.16) as

$$\begin{aligned} \left(\int_{t_1^+}^{t_2^+} \gamma_r \, d|\lambda|_r\right)^2 ={}& 2\int_{t_1^+}^{t_2^+} \left\langle \left(\int_{t_1^+}^{r^+} \gamma_u(x_u) \, d|\lambda|_u\right), \gamma_r \right\rangle d|\lambda|_r \\ &- \sum_{t_1 < r \leq t_2} \underbrace{|\gamma_r|^2}_{=1} (|\lambda|_r - |\lambda|_{r-})^2. \end{aligned} \tag{4.23}$$

Based on the last display, it is clear that and the third and fourth integral in (4.16) reduce to the term

$$-\sum_{t_1 < r \leq t_2} (|\lambda|_r - |\lambda|_{r-})^2. \tag{4.24}$$

Putting the relations (4.16)–(4.24) together, we obtain

$$\begin{aligned} |x_{t_2} - x_{t_1}|^2 \leq{}& |w_{t_2} - w_{t_1}|^2 + 2\int_{t_1^+}^{t_2^+} |w_{t_2} - w_r| \, d|\lambda|_r + 2c\int_{t_1^+}^{t_2^+} |x_r - x_{t_1}| \, d|\lambda|_r \\ &- \sum_{t_1 < r \leq t_2} (|\lambda|_r - |\lambda|_{r-})^2 + 2(c+1)l(t_2 - t_1)(|\lambda|_{t_2} - |\lambda|_{t_1}). \end{aligned} \tag{4.25}$$

If we now combine (4.25) and the properties (1.5)–(1.6) in Definition 1.1, we first get

$$
\begin{aligned}
|x_{t_2}-x_{t_1}|^2 \le |w_{t_2}-w_{t_1}|^2 &+ 2\int_{t_1^+}^{t_2^+} |w_{t_2}-w_r|\,d|\lambda|_r + 2c\int_{t_1^+}^{t_2^+} |w_r-w_{t_1}|\,d|\lambda|_r \\
&+ 2c\int_{t_1^+}^{t_2^+} (|\lambda|_r-|\lambda|_{t_1})\,d|\lambda|_r - \sum_{t_1<r\le t_2} (|\lambda|_r-|\lambda|_{r-})^2 \\
&+ 2(c+1)l(t_2-t_1)(|\lambda|_{t_2}-|\lambda|_{t_1}),
\end{aligned} \tag{4.26}
$$

and then, again using Lemma 2.1(ii) in [53] as well as the fact that $0\le c\le 1$, we conclude that

$$
\begin{aligned}
|x_{t_2}-x_{t_1}|^2 \le |w_{t_2}-w_{t_1}|^2 &+ 2\int_{t_1^+}^{t_2^+} |w_{t_2}-w_r|\,d|\lambda|_r + 2c\int_{t_1^+}^{t_2^+} |w_r-w_{t_1}|\,d|\lambda|_r \\
&+ c(|\lambda|_{t_2}-|\lambda|_{t_1})^2 + 2(c+1)l(t_2-t_1)(|\lambda|_{t_2}-|\lambda|_{t_1}).
\end{aligned} \tag{4.27}
$$

Relation (4.13) and the inequality in the last display yield

$$
\begin{aligned}
\|x\|_{t_1,t_2}^2 \le \left(1+\frac{2(c+1)}{a}+\frac{c}{a^2}\right)\|w\|_{t_1,t_2}^2 &+ 2\left(\frac{c+1}{a}+\frac{c}{a^2}\right)\|x\|_{t_1,t_2}\|w\|_{t_1,t_2} \\
&+ \frac{c}{a^2}\|x\|_{t_1,t_2}^2 + \frac{2(c+1)}{a}l(t_2-t_1)\|x\|_{t_1,t_2} \\
&+ \frac{2(c+1)}{a}l(t_2-t_1)\|w\|_{t_1,t_2}.
\end{aligned} \tag{4.28}
$$

In addition, combining (4.13) and (4.25), we obtain

$$
\begin{aligned}
\|x\|_{t_1,t_2}^2 \le \left(1+\frac{2}{a}\right)\|w\|_{t_1,t_2}^2 &+ \frac{2(c+1)}{a}\|x\|_{t_1,t_2}\|w\|_{t_1,t_2} + \frac{2c}{a}\|x\|_{t_1,t_2}^2 \\
&+ \frac{2(c+1)}{a}l(t_2-t_1)\|x\|_{t_1,t_2} + \frac{2(c+1)}{a}l(t_2-t_1)\|w\|_{t_1,t_2}.
\end{aligned} \tag{4.29}
$$

The inequalities (4.28) and (4.29) can both be written on the form

$$
A\|x\|_{t_1,t_2}^2 - B\|x\|_{t_1,t_2}\|w\|_{t_1,t_2} - C\|w\|_{t_1,t_2}^2 - D\|x\|_{t_1,t_2} - D\|w\|_{t_1,t_2} \le 0, \tag{4.30}
$$

where the positive constants $A$, $B$ and $C$ are easily shown to satisfy the condition $A+B=C$ in both cases. We claim that

$$
\|x\|_{t_1,t_2} \le \frac{C}{A}\|w\|_{t_1,t_2} + \frac{D}{A}. \tag{4.31}
$$

Indeed, suppose, on the contrary, that

$$
0 \le \|w\|_{t_1,t_2} < \frac{A}{C}\|x\|_{t_1,t_2} - \frac{D}{C}. \tag{4.32}
$$

Then, by (4.32),

$$
\begin{aligned}
&A\|x\|_{t_1,t_2}^2 - B\|x\|_{t_1,t_2}\|w\|_{t_1,t_2} - C\|w\|_{t_1,t_2}^2 - D\|x\|_{t_1,t_2} - D\|w\|_{t_1,t_2} \\
&\quad > A\|x\|_{t_1,t_2}^2 + B\|x\|_{t_1,t_2}\left(-\frac{A}{C}\|x\|_{t_1,t_2} + \frac{D}{C}\right) - C\left(\frac{A}{C}\|x\|_{t_1,t_2} - \frac{D}{C}\right)^2 \\
&\qquad - D\|x\|_{t_1,t_2} + D\left(-\frac{A}{C}\|x\|_{t_1,t_2} + \frac{D}{C}\right) \\
&\quad = A\|x\|_{t_1,t_2}^2 \underbrace{\left(1 - \frac{B}{C} - \frac{A}{C}\right)}_{=\frac{C-B-A}{C}=0} + D\|x\|_{t_1,t_2} \underbrace{\left(\frac{B}{C} + \frac{2A}{C} - \frac{A}{C} - 1\right)}_{=\frac{A+B-C}{C}=0} = 0.
\end{aligned} \tag{4.33}
$$

Obviously (4.33) contradicts (4.30) and, hence, the claim in (4.31) is proved. To complete the proof of Lemma 4.1, we first note that (4.28) implies

$$
\begin{aligned}
\|x\|_{t_1,t_2} &\le \frac{C}{A}\|w\|_{t_1,t_2} + \frac{D}{A} \\
&= \frac{1 + 2(c+1)/a + c/a^2}{1 - c/a^2}\|w\|_{t_1,t_2} + \frac{2(c+1)/a}{1 - c/a^2} l(t_2 - t_1) \\
&= \frac{a^2 + 2ac + 2a + c}{a^2 - c}\|w\|_{t_1,t_2} + \frac{2a(c+1)}{a^2 - c} l(t_2 - t_1),
\end{aligned} \tag{4.34}
$$

and that (4.29) implies

$$
\begin{aligned}
\|x\|_{t_1,t_2} &\le \frac{C}{A}\|w\|_{t_1,t_2} + \frac{D}{A} = \frac{1 + 2/a}{1 - 2c/a}\|w\|_{t_1,t_2} + \frac{2(c+1)/a}{1 - 2c/a} l(t_2 - t_1) \\
&= \frac{a+2}{a - 2c}\|w\|_{t_1,t_2} + \frac{2(c+1)}{a - 2c} l(t_2 - t_1).
\end{aligned} \tag{4.35}
$$

From the definition $e = \frac{c}{a^2 \vee a/2}$ we know that if $a/2 \le a^2$, then we can set $c = a^2 e$ in (4.34) and obtain

$$
\begin{aligned}
\|x\|_{t_1,t_2} &\le \frac{a^2 + 2a^3 e + 2a + a^2 e}{a^2(1-e)}\|w\|_{t_1,t_2} + \frac{2a^3 e + 2a}{a^2(1-e)} l(t_2 - t_1) \\
&= \frac{a + 2a^2 e + 2 + ae}{a(1-e)}\|w\|_{t_1,t_2} + \frac{2a^2 e + 2}{a(1-e)} l(t_2 - t_1),
\end{aligned} \tag{4.36}
$$

whereas if $a/2 \ge a^2$, then we can set $2c = ae$ in (4.35) and obtain

$$
\|x\|_{t_1,t_2} \le \frac{a+2}{a(1-e)}\|w\|_{t_1,t_2} + \frac{ae + 2}{a(1-e)} l(t_2 - t_1). \tag{4.37}
$$

Hence, in either case, we arrive at

$$
\|x\|_{t_1,t_2} \le \frac{a + 2a^2 e + 2 + ae}{a(1-e)}\|w\|_{t_1,t_2} + \frac{2a^2 e + 2 + ae}{a(1-e)} l(t_2 - t_1), \tag{4.38}
$$

and the proof of estimate (4.3) is complete. Finally, we note that estimate (4.4) now follows directly from (4.3) and (4.13). This completes the proof of Lemma 4.1. □

PROOF OF THEOREM 4.2. We first note that the assumptions stated in Theorem 4.2 ensure that there exist some $a > 0$ and $0 < e < 1$ such that $a_{s,x_s}(\rho_0, \eta_0) \geq a$ and $e_{s,x_s}(\rho_0, \eta_0) \leq e$ for all $x_s \in \partial D_s$, $s \in [0, T]$. Next we recursively define two sets of time-points $\{\hat{T}_i\}$ and $\{T_i\}$. In particular, we let $\hat{T}_0 = T_0 = 0$ and define, for $i \geq 0$, $T_{i+1} = T$ if $x_t \in D_t$ for all $t \in [0, T]$ and

$$T_{i+1} = \inf\{t : \hat{T}_i \leq t \leq T, x_t \in \partial D_t\}, \tag{4.39}$$

otherwise. Similarly, for $i \geq 0$, we let $\hat{T}_{i+1} = (T_{i+1} + \eta_0) \wedge T$, if $|x_t - x_{T_{i+1}}| + l(t - T_{i+1}) < \rho_0$ for all $t$ such that $T_{i+1} \leq t \leq (T_{i+1} + \eta_0) \wedge T$, and

$$\hat{T}_{i+1} = \inf\{T_{i+1} \leq t < (T_{i+1} + \eta_0) \wedge T : |x_t - x_{T_{i+1}}| + l(t - T_{i+1}) \geq \rho_0\}, \tag{4.40}$$

otherwise. Using (1.10) and the fact that $x$ is a right continuous function, it follows that $T_{i+1} < \hat{T}_{i+1}$ for all $i \geq 0$. Moreover, using (4.39)–(4.40), we can apply Lemma 4.1 to any pair of time points $(t_1, t_2)$ such that $T_i \leq t_1 \leq t_2 < \hat{T}_i$ and obtain

$$\begin{aligned} \|x\|_{t_1,t_2} &\leq K_1(a, e)\|w\|_{t_1,t_2} + K_2(a, e)l(t_2 - t_1), \\ |\lambda|_{t_2} - |\lambda|_{t_1} &\leq K_3(a, e)\|w\|_{t_1,t_2} + K_4(a, e)l(t_2 - t_1), \end{aligned} \tag{4.41}$$

whenever $T_i \leq t_1 \leq t_2 < \hat{T}_i$ where $K_1$, $K_2$, $K_3$ and $K_4$ are defined as in Lemma 4.1 based on $a$ and $e$ introduced above. Next, we want to find a similar estimate whenever $\hat{T}_i \leq t_1 \leq t_2 < T_{i+1}$. If $\hat{T}_i = T_{i+1}$, we are done and, hence, we assume that $\hat{T}_i < T_{i+1}$. In that case $x_t \in D_t$ for all $\hat{T}_i \leq t < T_{i+1}$ and, as a consequence, the changes in $x$ and $w$ coincide on this time interval. Finally, considering the case $T_i \leq t_1 < \hat{T}_i \leq t_2 < T_{i+1}$, we have

$$\begin{aligned} |x_{t_2} - x_{t_1}| &\leq |w_{t_2} - w_{\hat{T}_i}| + |x_{\hat{T}_i} - x_{\hat{T}_i^-}| + |x_{\hat{T}_i^-} - x_{t_1}|, \\ |\lambda|_{t_2} - |\lambda|_{t_1} &\leq (|\lambda|_{\hat{T}_i} - |\lambda|_{\hat{T}_i^-}) + (|\lambda|_{\hat{T}_i^-} - |\lambda|_{t_1}). \end{aligned} \tag{4.42}$$

The terms $|x_{\hat{T}_i^-} - x_{t_1}|$ and $|\lambda|_{\hat{T}_i^-} - |\lambda|_{t_1}$ in (4.42) can be handled using (4.41). Regarding the terms $|x_{\hat{T}_i} - x_{\hat{T}_i^-}|$ and $|\lambda|_{\hat{T}_i} - |\lambda|_{\hat{T}_i^-}$ in (4.42), we can, since $|x_{\hat{T}_i} - x_{\hat{T}_i^-}| \leq \rho_0$, use (3.14) and the definition of the Skorohod problem to first conclude that

$$\begin{aligned} |w_{\hat{T}_i} - w_{\hat{T}_i^-}|^2 &= |x_{\hat{T}_i} - x_{\hat{T}_i^-}|^2 + \big|\gamma_{\hat{T}_i}(|\lambda|_{\hat{T}_i} - |\lambda|_{\hat{T}_i^-})\big|^2 \\ &\quad - 2(x_{\hat{T}_i} - x_{\hat{T}_i^-}) \cdot \gamma_{\hat{T}_i}(|\lambda|_{\hat{T}_i} - |\lambda|_{\hat{T}_i^-}) \\ &\geq |x_{\hat{T}_i} - x_{\hat{T}_i^-}|^2 + (|\lambda|_{\hat{T}_i} - |\lambda|_{\hat{T}_i^-})^2 \\ &\quad - 2c_{x_{\hat{T}_i}, \hat{T}_i}(\rho_0, \eta_0)|x_{\hat{T}_i} - x_{\hat{T}_i^-}|(|\lambda|_{\hat{T}_i} - |\lambda|_{\hat{T}_i^-}). \end{aligned} \tag{4.43}$$

Then

$$
\begin{aligned}
|w_{\hat{T}_i} - w_{\hat{T}_i^-}|^2 &\geq \big(1 - c_{x_{\hat{T}_i},\hat{T}_i}(\rho_0,\eta_0)\big)|x_{\hat{T}_i} - x_{\hat{T}_i^-}|^2 \\
&\quad + \big(1 - c_{x_{\hat{T}_i},\hat{T}_i}(\rho_0,\eta_0)\big)(|\lambda|_{\hat{T}_i} - |\lambda|_{\hat{T}_i^-})^2,
\end{aligned} \tag{4.44}
$$

and, as $(a_{s,y}(\rho_0,\eta_0))^2 \vee a_{s,y}(\rho_0,\eta_0)/2 \leq 1$, for all $y \in \partial D_s$, $s \in [0,T]$, we obtain

$$
\begin{aligned}
c_{x_{\hat{T}_i},\hat{T}_i}(\rho_0,\eta_0) &\leq \frac{c_{x_{\hat{T}_i},\hat{T}_i}(\rho_0,\eta_0)}{(a_{x_{\hat{T}_i},\hat{T}_i}(\rho_0,\eta_0))^2 \vee a_{x_{\hat{T}_i},\hat{T}_i}(\rho_0,\eta_0)/2} \\
&= e_{x_{\hat{T}_i},\hat{T}_i}(\rho_0,\eta_0) \leq e.
\end{aligned} \tag{4.45}
$$

Combining the estimates in (4.44) and (4.45), we arrive at

$$
\begin{aligned}
&|x_{\hat{T}_i} - x_{\hat{T}_i^-}| \leq \frac{1}{\sqrt{1 - c_{x_{\hat{T}_i},\hat{T}_i}(\rho_0)}}|w_{\hat{T}_i} - w_{\hat{T}_i^-}| \leq \frac{1}{\sqrt{1-e}}|w_{\hat{T}_i} - w_{\hat{T}_i^-}|, \\
&|\lambda|_{\hat{T}_i} - |\lambda|_{\hat{T}_i^-} \leq \frac{1}{\sqrt{1-e}}|w_{\hat{T}_i} - w_{\hat{T}_i^-}|.
\end{aligned} \tag{4.46}
$$

Introducing the notation

$$
\begin{aligned}
K_1 &= K_1(a,e) + 1 + \frac{1}{\sqrt{1-e}}, \qquad K_2 = K_2(a,e), \\
K_3 &= K_3(a,e) + \frac{1}{\sqrt{1-e}}, \qquad K_4 = K_4(a,e),
\end{aligned} \tag{4.47}
$$

we can use the deductions in (4.41)–(4.46) to conclude that

$$
\begin{aligned}
\|x\|_{t_1,t_2} &\leq K_1\|w\|_{t_1,t_2} + K_2 l(t_2 - t_1), \\
|\lambda|_{t_2} - |\lambda|_{t_1} &\leq K_3\|w\|_{t_1,t_2} + K_4 l(t_2 - t_1),
\end{aligned} \tag{4.48}
$$

whenever $T_i \leq t_1 \leq t_2 < T_{i+1}$. We now intend to make use of the estimates in (4.48) to complete the proof of Theorem 4.2. Note that above we have constructed a set of time-points $\{T_i\}_{i=0}^{M+1}$, where $M$ is so far undetermined, and

$$
0 = T_0 < T_1 < \cdots < T_M < T = T_{M+1}. \tag{4.49}
$$

If $M \geq 1$, let

$$
0 \leq t_1 \leq u \leq r \leq t_2 \leq T, \qquad T_{h-1} \leq u < T_h,\ T_v \leq r < T_{v+1},\ h - 1 \leq v. \tag{4.50}
$$

Then, using (4.48), we have

$$
\begin{aligned}
|x_r - x_u| &\leq (M+1)\big(K_1\|w\|_{t_1,t_2} + K_2 l(t_2 - t_1)\big) + \sum_{i=h}^{v}|x_{T_i} - x_{T_i^-}|, \\
|\lambda|_r - |\lambda|_u &\leq (M+1)\big(K_3\|w\|_{t_1,t_2} + K_4 l(t_2 - t_1)\big) + \sum_{i=h}^{v}|\lambda|_{T_i} - |\lambda|_{T_i^-}.
\end{aligned} \tag{4.51}
$$

Moreover, arguing exactly as in the deduction of (4.46), we obtain

$$
\begin{aligned}
|x_{T_i} - x_{T_i^-}| &\le \frac{1}{\sqrt{1-e}}|w_{T_i} - w_{T_i^-}|,\\
|\lambda|_{T_i} - |\lambda|_{T_i^-} &\le \frac{1}{\sqrt{1-e}}|w_{T_i} - w_{T_i^-}|,
\end{aligned} \tag{4.52}
$$

whenever $1 \le i \le M+1$. Hence, to complete the proof of Theorem 4.2, we have to estimate $M$. To do this, we consider $\mathcal{W} \subset \mathcal{D}([0,T],\mathbb{R}^d)$, which is assumed to be relatively compact in the Skorohod topology and for which $w_0 \in \overline{D_0}$ whenever $w \in \mathcal{W}$. We shall prove that the $M$ introduced above is bounded for every such set $\mathcal{W}$. To do this, we use the notation introduced in Section 3.3 concerning the Skorohod topology. In the following let $\delta'$ be a fixed number such that

$$
\delta' = \min\{\eta_0, \hat{\delta}'\} \qquad \text{where } \hat{\delta}' \text{ is such that } l(\hat{\delta}') \le \rho_0/\bigl(2(K_2+1)\bigr). \tag{4.53}
$$

Note that the existence of $\hat{\delta}'$ follows immediately from (1.10). Using the definition of $\delta'$ and the fact that $\mathcal{W} \subset \mathcal{D}([0,T],\mathbb{R}^d)$ is relatively compact in the Skorohod topology, we see, by (3.22) and (3.25), that

$$
\lim_{\delta \to 0} \tilde{\mu}(\mathcal{W}, \delta, \delta', T) = 0. \tag{4.54}
$$

In particular, using (4.54), we can find a $0 < \delta < \delta'$ such that for every $w \in \mathcal{W}$ there exists a partition $\{t_j\}_{j=0}^{M}$, in general depending on $w$, such that

$$
\delta < |t_{j+1} - t_j| < \delta' \qquad \text{for } j \in \{0, \dots, M-1\} \tag{4.55}
$$

and

$$
\max_{0 \le j \le M-1} \sup_{u,r \in [t_j, t_{j+1})} |w_u - w_r| < \frac{\rho_0}{2K_1}. \tag{4.56}
$$

We claim that none of the intervals $\{[t_j, t_{j+1})\}$ in this partition can contain more than one point from the sequence $\{T_i\}$. To prove this, we suppose, on the contrary, that there exist $i$ and $j$ such that $t_j \le T_i < T_{i+1} < t_{j+1}$. Then, by construction,

$$
t_j \le T_i < \hat{T}_i \le T_{i+1} < t_{j+1}. \tag{4.57}
$$

We intend to estimate $|x_{\hat{T}_i} - x_{T_i}| + l(\hat{T}_i - T_i)$. We first note that if $|x_t - x_{T_i}| + l(t - T_i) < \rho_0$ for all $t$ such that $T_i \le t \le (T_i + \eta_0) \wedge T$, then $\hat{T}_i = (T_i + \eta_0) \wedge T$. However, using (4.53) and (4.55), it is clear that neither $\hat{T}_i = (T_i + \eta_0)$ nor $\hat{T}_i = T$ can occur. Hence, we can assume that $\hat{T}_i$ is given by (4.40) and, as a consequence, that

$$
|x_{\hat{T}_i} - x_{T_i}| + l(\hat{T}_i - T_i) \ge \rho_0. \tag{4.58}
$$

But on the other hand, using (4.48), we first see that

$$
\begin{aligned}
|x_{\hat{T}_i} - x_{T_i}| + l(\hat{T}_i - T_i) &\leq \|x\|_{T_i,\hat{T}_i} + l(\hat{T}_i - T_i) \\
&\leq K_1\|w\|_{T_i,\hat{T}_i} + (K_2+1)l(\hat{T}_i - T_i),
\end{aligned} \tag{4.59}
$$

and then, using (4.53), (4.55) and (4.56), we deduce

$$
|x_{\hat{T}_i} - x_{T_i}| + l(\hat{T}_i - T_i) < K_1\frac{\rho_0}{2K_1} + (K_2+1)l(\delta') < \rho_0, \tag{4.60}
$$

which contradicts the assumption $t_j \leq T_i < T_{i+1} < t_{j+1}$. Hence, none of the intervals $\{[t_j, t_{j+1})\}$ in the partition can contain more than one point from the sequence $\{T_i\}$ and, in particular, we conclude that

$$
M \leq \frac{T}{\delta} + 1. \tag{4.61}
$$

Combining (4.51), (4.52) and (4.61), we see that

$$
\begin{aligned}
\|x\|_{t_1,t_2} &\leq (M+1)\big(K_1\|w\|_{t_1,t_2} + K_2 l(t_2-t_1)\big) + M|x_{T_i} - x_{T_i^-}| \\
&\leq \left(\frac{T}{\delta}+2\right)\big(K_1\|w\|_{t_1,t_2} + K_2 l(t_2-t_1)\big) + \left(\frac{T}{\delta}+1\right)\frac{1}{\sqrt{1-e}}\|w\|_{t_1,t_2} \\
&\leq \left(K_1\left(\frac{T}{\delta}+2\right) + \frac{1}{\sqrt{1-e}}\left(\frac{T}{\delta}+1\right)\right)\|w\|_{t_1,t_2} \\
&\quad + K_2\left(\frac{T}{\delta}+2\right)l(t_2-t_1),
\end{aligned} \tag{4.62}
$$

and, similarly, that

$$
\begin{aligned}
|\lambda|_{t_2} - |\lambda|_{t_1} &\leq \left(K_3\left(\frac{T}{\delta}+2\right) + \frac{1}{\sqrt{1-e}}\left(\frac{T}{\delta}+1\right)\right)\|w\|_{t_1,t_2} \\
&\quad + K_4\left(\frac{T}{\delta}+2\right)l(t_2-t_1).
\end{aligned} \tag{4.63}
$$

The deductions in the last two displays complete the proof of Theorem 4.2. □

4.2. *Estimates for approximations to Skorohod problems*. Let $T > 0$, $D \subset \mathbb{R}^{d+1}$ and $\Gamma = \Gamma_t(z)$ satisfy the assumptions stated in Theorem 4.2. In this section we derive estimates for approximations to the Skorohod problem for $(D, \Gamma, w)$. In particular, given $w \in \mathcal{D}([0,T], \mathbb{R}^d)$ with $w_0 \in \overline{D_0}$ and a partition $\{\tau_k\}_{k=0}^N$ of the interval $[0,T]$, which we denote by $\Delta$, we define $w^\Delta$ as in (2.4). Recall that $\Delta^*$ was defined in (2.3). Furthermore, in the following, we will assume that (2.5) holds whenever $k \in \{1, \dots, N\}$. Based on the assumption in (2.5), we define $D^\Delta$, $\Gamma^\Delta$, $x^\Delta$ and $\lambda^\Delta$ as in (2.6), (2.7) and (2.11). Then, by construction, the pair $(x^\Delta, \lambda^\Delta)$ is a solution to the Skorohod problem for $(D^\Delta, \Gamma^\Delta, w^\Delta)$. In this section we prove the following results.

LEMMA 4.5. *Let $T > 0$, $D \subset \mathbb{R}^{d+1}$, $r_0$, $\Gamma = \Gamma_t(z)$, $0 < \rho_0 < r_0$, $\eta_0 > 0$, $a$, $e$, $\delta_0$ and $h_0$ be as in the statement of Theorem 4.2. Given $a > 0$ and $e \in (0, 1)$, let the functions $K_1$, $K_2$, $K_3$ and $K_4$ be defined as in (4.1) and let $w \in \mathcal{D}([0, T], \mathbb{R}^d)$ with $w_0 \in \overline{D_0}$. Let $\Delta = \{\tau_k\}_{k=0}^N$ be a partition of the interval $[0, T]$, let $w^\Delta$ be defined as in (2.4) and assume that (2.5) holds. Given $\Delta$ and $w^\Delta$, let $D^\Delta$, $\Gamma^\Delta$, $x^\Delta$ and $\lambda^\Delta$ be defined as in (2.6), (2.7) and (2.11). Consider a fixed but arbitrary $s \in [0, T]$, such that $x_s^\Delta \in \partial D_s^\Delta$. Then, for $0 \le s \le t_1 \le t_2 < \tau^\Delta_{\rho_0,\eta_0}$, $(w^\Delta, x^\Delta, \lambda^\Delta)$ satisfies the estimates*

$$\|x^\Delta\|_{t_1,t_2} \le K_1(a, e)\|w\|_{t_1,t_2} + K_2(a, e)\big(l(t_2 - t_1) + l(\Delta^*)\big), \tag{4.64}$$

$$|\lambda^\Delta|_{t_2} - |\lambda^\Delta|_{t_1} \le K_3(a, e)\|w\|_{t_1,t_2} + K_4(a, e)\big(l(t_2 - t_1) + l(\Delta^*)\big). \tag{4.65}$$

*Here $\tau^\Delta_{\rho_0,\eta_0}$ is defined as follows. If there exists some $t$ such that $s \le t < (s + \eta_0) \wedge T$ and $|x_t^\Delta - x_s^\Delta| + l(t - s) + l(\Delta^*) \ge \rho_0$, then*

$$\tau^\Delta_{\rho_0,\eta_0} = \inf\{t : s \le t < (s + \eta_0) \wedge T, |x_t^\Delta - x_s^\Delta| + l(t - s) + l(\Delta^*) \ge \rho_0\}, \tag{4.66}$$

*whereas if $|x_t^\Delta - x_s^\Delta| + l(t - s) + l(\Delta^*) < \rho_0$ for all $s \le t < (s + \eta_0) \wedge T$, then*

$$\tau^\Delta_{\rho_0,\eta_0} = (s + \eta_0) \wedge T. \tag{4.67}$$

THEOREM 4.6. *Let $T > 0$, $D \subset \mathbb{R}^{d+1}$, $r_0$, $\Gamma = \Gamma_t(z)$, $0 < \rho_0 < r_0$, $\eta_0 > 0$, $a$, $e$, $\delta_0$ and $h_0$ be as in the statement of Theorem 4.2. Given $a > 0$ and $e \in (0, 1)$, let the functions $K_1$, $K_2$, $K_3$ and $K_4$ be defined as in (4.1) and let $w \in \mathcal{D}([0, T], \mathbb{R}^d)$ with $w_0 \in \overline{D_0}$. Let $\Delta = \{\tau_k\}_{k=0}^N$ be a partition of the interval $[0, T]$, let $w^\Delta$ be defined as in (2.4) and assume that (2.5) holds. Let $\Delta$ be such that $l(\Delta^*) \le \rho_0/(4(K_2(a, e) + 1))$ and let*

$$\begin{aligned} \delta' = \min\{\eta_0, \hat{\delta}'\} \qquad &\textit{where } \hat{\delta}' \textit{ is such that} \\ l(\hat{\delta}') + l(\Delta^*) \le \rho_0/\big(2\big(K_2(a, e) + 1\big)\big).& \end{aligned} \tag{4.68}$$

*Given $\Delta$ and $w^\Delta$, let $D^\Delta$, $\Gamma^\Delta$, $x^\Delta$ and $\lambda^\Delta$ be defined as in (2.6), (2.7) and (2.11). Moreover, assume that $x^\Delta \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$. Then there exist positive constants $\hat{L}_1(w, T)$ , $\hat{L}_2(w, T)$, $\hat{L}_3(w, T)$ and $\hat{L}_4(w, T)$, independent of $\Delta$, such that*

$$\begin{aligned} \|x^\Delta\|_{t_1,t_2} &\le \hat{L}_1(w, T)\|w\|_{t_1,t_2} + \hat{L}_2(w, T)\big(l(t_2 - t_1) + l(\Delta^*)\big), \\ |\lambda^\Delta|_{t_2} - |\lambda^\Delta|_{t_1} &\le \hat{L}_3(w, T)\|w\|_{t_1,t_2} + \hat{L}_4(w, T)\big(l(t_2 - t_1) + l(\Delta^*)\big), \end{aligned} \tag{4.69}$$

*whenever $0 \le t_1 \le t_2 \le T$.*

PROOF OF LEMMA 4.5. Naturally, the proof of this lemma is similar to the proof of Lemma 4.1 and, thus, we only describe the main differences compared to the proof of Lemma 4.1. First we note, by the assumptions on $D$ and the construction of $D^\Delta$ based on $D$, that there exists a unit vector $u$ such that

$$\langle \gamma_r^\Delta, u\rangle \ge a \tag{4.70}$$

for all $\gamma_r^\Delta \in \Gamma_r^{\Delta,1}(y)$, $y \in \partial D_r^\Delta \cap \overline{B_{\rho_0}(x_s^\Delta)}$ and $r \in [s, \tau^\Delta_{\rho_0,\eta_0}] \subset [s, (s+\eta_0) \wedge T]$. We also note that if $t_1 \in [\tau_j, \tau_{j+1})$ and $t_2 \in [\tau_k, \tau_{k+1})$, for some $j, k \in \{0, \dots, N-1\}$, then $|x^\Delta_{t_2} - x^\Delta_{t_1}| = 0$ if $j = k$ and otherwise $|x^\Delta_{t_2} - x^\Delta_{t_1}| = |x^\Delta_{\tau_k} - x^\Delta_{\tau_j}|$. Now, using the fact that $(x^\Delta, \lambda^\Delta)$ solves the Skorohod problem for $(D^\Delta, \Gamma^\Delta, w^\Delta)$, we conclude, in analogy with (4.12), that

$$\langle x^\Delta_{t_2} - x^\Delta_{t_1}, u\rangle = \langle w^\Delta_{t_2} - w^\Delta_{t_1}, u\rangle + \int_{t_1^+}^{t_2^+} \underbrace{\langle \gamma_r^\Delta, u\rangle}_{\geq a} \, d|\lambda^\Delta|_r \tag{4.71}$$

for any $0 \leq s \leq t_1 \leq t_2 < \tau^\Delta_{\rho_0,\eta_0}$, where $\gamma_r^\Delta \in \Gamma_r^{\Delta,1}(y)$ for some $y \in \partial D_r^\Delta$. Based on (4.71), we obtain

$$|\lambda^\Delta|_{t_2} - |\lambda^\Delta|_{t_1} \leq \frac{1}{a}(|w^\Delta_{t_2} - w^\Delta_{t_1}| + |x^\Delta_{t_2} - x^\Delta_{t_1}|). \tag{4.72}$$

Furthermore, arguing as in the proof of Lemma 4.1, we derive

$$\begin{aligned} |x^\Delta_{t_2} - x^\Delta_{t_1}|^2 &= |w^\Delta_{t_2} - w^\Delta_{t_1}|^2 + 2\int_{t_1^+}^{t_2^+} \langle w^\Delta_{t_2} - w^\Delta_r, \gamma_r^\Delta\rangle \, d|\lambda^\Delta|_r \\ &\quad + 2\int_{t_1^+}^{t_2^+} \langle x^\Delta_r - x^\Delta_{t_1}, \gamma^\Delta_r\rangle \, d|\lambda^\Delta|_r + \left(\int_{t_1^+}^{t_2^+} \gamma_r^\Delta \, d|\lambda^\Delta|_r\right)^2 \\ &\quad - 2\int_{t_1^+}^{t_2^+} \left\langle \left(\int_{t_1+}^{r^+} \gamma_u^\Delta \, d|\lambda^\Delta|_u\right), \gamma_r^\Delta\right\rangle d|\lambda^\Delta|_r. \end{aligned} \tag{4.73}$$

As in the proof of Lemma 4.1, we have to find upper bounds of all integrals in (4.73) and, naturally, particular attention has to be paid to the second integral, as $x^\Delta_{t_1}$ might not belong to $\overline{D_r^\Delta}$. For $y \in \partial D_t^\Delta$, $t \in [0, T]$, we let $N_t^\Delta(y)$ denote the set of inward normals at $y \in \partial D_t^\Delta$ and given $r \in [0, T]$, $y \in \mathbb{R}^d \setminus \overline{D_r^\Delta}$ in the following we denote a projection of $y$ onto $\partial D_r^\Delta$ along $N_r^\Delta$ by $\pi^{N_r^\Delta}_{\partial D_r^\Delta}(y)$. Furthermore, if $y \in \overline{D_r^\Delta}$, then we let $\pi^{N_r^\Delta}_{\partial D_r^\Delta}(y) = y$. Using this notation, and the definition of $\tau^\Delta_{\rho_0,\eta_0}$, we see that

$$|\pi^{N_r^\Delta}_{\partial D_r^\Delta}(x^\Delta_{t_1}) - x^\Delta_s| \leq |x^\Delta_{t_1} - x^\Delta_s| + l(r - t_1) + l(\Delta^*) \leq \rho_0. \tag{4.74}$$

Equation (4.74) implies that $\pi^{N_r^\Delta}_{\partial D_r^\Delta}(x^\Delta_{t_1}) \in \overline{B_{\rho_0}(x_s^\Delta)} \cap \partial D_r^\Delta \subset \overline{B_{\rho_0}(x_s^\Delta)} \cap \overline{D_r^\Delta}$. Arguing as in (4.19)–(4.21), we then deduce that

$$\begin{aligned} \langle x_r^\Delta - \pi^{N_r^\Delta}_{\partial D_r^\Delta}(x^\Delta_{t_1}), \gamma_r^\Delta\rangle &\leq c|x_r^\Delta - \pi^{N_r^\Delta}_{\partial D_r^\Delta}(x^\Delta_{t_1})| \\ &\leq c|x_r^\Delta - x^\Delta_{t_1}| + cl(r - t_1) + cl(\Delta^*), \end{aligned} \tag{4.75}$$

and that

$$\langle \pi^{N_r^\Delta}_{\partial D_r^\Delta}(x^\Delta_{t_1}) - x^\Delta_{t_1}, \gamma_r^\Delta\rangle \leq |\pi^{N_r^\Delta}_{\partial D_r^\Delta}(x^\Delta_{t_1}) - x^\Delta_{t_1}| \leq l(r - t_1) + l(\Delta^*). \tag{4.76}$$

Using the estimates derived above, we conclude that the second integral in (4.73) has the upper bound

$$2c\int_{t_1^+}^{t_2^+} |x_r^\Delta - x_{t_1}^\Delta|\, d|\lambda^\Delta|_r + 2(c+1)\bigl(l(t_2-t_1)+l(\Delta^*)\bigr)(|\lambda^\Delta|_{t_2} - |\lambda^\Delta|_{t_1}). \tag{4.77}$$

Equipped with (4.77), the proof of Lemma 4.5 can now be completed following the lines of the proof of Lemma 4.1. □

PROOF OF THEOREM 4.6. Proceeding as in the proof of Theorem 4.2, we first recursively define two sets of time-points $\{\hat{T}_i^\Delta\}$ and $\{T_i^\Delta\}$ in order to use Lemma 4.5. In particular, we let $\hat{T}_0^\Delta = T_0^\Delta = 0$ and define, for $i \geq 0$, $T_{i+1}^\Delta = T$ if $x_t^\Delta \in D_t^\Delta$ for all $t \in [0, T]$ and

$$T_{i+1}^\Delta = \inf\{t : \hat{T}_i^\Delta \leq t \leq T, x_t^\Delta \in \partial D_t^\Delta\}, \tag{4.78}$$

otherwise. Similarly, for $i \geq 0$ we let $\hat{T}_{i+1}^\Delta = (T_{i+1}^\Delta + \eta_0) \wedge T$, if $|x_t^\Delta - x_{T_{i+1}}^\Delta| + l(t - T_{i+1}^\Delta) + l(\Delta^*) < \rho_0$ for all $t$ such that $T_{i+1}^\Delta \leq t \leq (T_{i+1}^\Delta + \eta_0) \wedge T$. Moreover, if the latter is not the case, we then define $\hat{T}_{i+1}^\Delta$ to equal

$$\inf\{T_{i+1}^\Delta \leq t < (T_{i+1}^\Delta + \eta_0) \wedge T : |x_t^\Delta - x_{T_{i+1}}^\Delta| + l(t - T_{i+1}^\Delta) + l(\Delta^*) \geq \rho_0\}. \tag{4.79}$$

We can then repeat the argument in (4.41)–(4.48) to conclude that

$$\begin{aligned} \|x^\Delta\|_{t_1,t_2} &\leq K_1\|w\|_{t_1,t_2} + K_2\bigl(l(t_2-t_1)+l(\Delta^*)\bigr), \\ |\lambda^\Delta|_{t_2} - |\lambda^\Delta|_{t_1} &\leq K_3\|w\|_{t_1,t_2} + K_4\bigl(l(t_2-t_1)+l(\Delta^*)\bigr), \end{aligned} \tag{4.80}$$

whenever $T_i^\Delta \leq t_1 \leq t_2 < T_{i+1}^\Delta$. Furthermore, we note that the $M$ in $\{T_i\}_{i=0}^{M+1}$ is so far undetermined and, as in (4.51)–(4.52), we derive

$$\begin{aligned} \|x^\Delta\|_{t_1,t_2} &\leq (M+1)\bigl(K_1\|w\|_{t_1,t_2} + K_2\bigl(l(t_2-t_1)+l(\Delta^*)\bigr)\bigr) \\ &\quad + \sqrt{\frac{1}{1-e}}\sum_{i=1}^{M}|w_{T_i^\Delta} - w_{T_i^{\Delta,-}}|, \\ |\lambda^\Delta|_{t_2} - |\lambda^\Delta|_{t_1} &\leq (M+1)\bigl(K_3\|w\|_{t_1,t_2} + K_4\bigl(l(t_2-t_1)+l(\Delta^*)\bigr)\bigr) \\ &\quad + \sqrt{\frac{1}{1-e}}\sum_{i=1}^{M}|w_{T_i^\Delta} - w_{T_i^{\Delta,-}}|, \end{aligned} \tag{4.81}$$

whenever $0 \leq t_1 \leq t_2 \leq T$. To complete the proof of Theorem 4.6, we can now proceed as in the proof of Theorem 4.2 and conclude that $M \leq T/\delta + 1$, where $\delta$ is given in the proof of Theorem 4.2. This completes the proof of Theorem 4.6. □

**5. Convergence and approximation of Skorohod problems.** In the first subsection of this section we prove the general convergence result for sequences of Skorohod problems (see Theorem 5.1 stated below) referred to in Section 2. Then, in the second subsection we explicitly construct, given $(D, \Gamma, w)$, an approximating sequence $\{(D^n, \Gamma^n, w^n)\}$ and, for each $n$, an explicit solution $(x^n, \lambda^n)$ to the Skorohod problem for $(D^n, \Gamma^n, w^n)$. We then prove that the constructed sequence $\{(x^n, \lambda^n)\}$ of solutions converges to a solution to the Skorohod problem for $(D, \Gamma, w)$.

5.1. *Convergence of a sequence of solutions to Skorohod problems.* Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2). Let $\Gamma = \Gamma_t(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_t$, $t \in [0, T]$ and assume that $\Gamma$ satisfies (1.11) and (1.12). Let $\{D^n\}_{n=1}^{\infty}$ be a sequence of time-dependent domains $D^n \subset \mathbb{R}^{d+1}$ and let $\{\Gamma^n\}_{n=1}^{\infty} = \{\Gamma_t^n(z)\}_{n=1}^{\infty}$ be a sequence of closed convex cones of vectors in $\mathbb{R}^d$. Let $w \in \mathcal{D}([0, T], \mathbb{R}^d)$ with $w_0 \in \overline{D_0}$ and let $\{w^n\}$ with $w_0^n \in \overline{D_0^n}$ be a sequence of càdlàg functions converging to $w$ in the Skorohod topology. Assume that there exists a solution $(x^n, \lambda^n)$ to the Skorohod problem for $(D^n, \Gamma^n, w^n)$. Then in Theorem 5.1 we prove, by making appropriate assumptions on $D$, $\Gamma$, $\{D^n\}_{n=1}^{\infty}$ and $\{\Gamma^n\}_{n=1}^{\infty}$, that if $D^n \to D$ and $\Gamma^n \to \Gamma$ in the sense defined in Theorem 5.1, then $(x^n, \lambda^n)$ converges to $(x, \lambda)$ and $(x, \lambda)$ is a solution to the Skorohod problem for $(D, \Gamma, w)$. However, to state Theorem 5.1, we need to introduce some additional notions and notation. In particular, in the following we let $a_{s,z}^n$ and $e_{s,z}^n$ be defined as in (1.15) and (1.17) but with respect to $(D^n, \Gamma^n)$. We assume that $D^n$, for $n \geq 1$, satisfies the uniform exterior sphere condition in time with radius $r_0$, independent of $n$, and that there exist $0 < \rho_0 < r_0$ and $\eta_0 > 0$ such that, for all $n \geq 1$,

$$\inf_{s \in [0,T]} \inf_{z \in \partial D_s^n} a_{s,z}^n(\rho_0, \eta_0) = a_n > 0, \tag{5.1}$$

$$\sup_{s \in [0,T]} \sup_{z \in \partial D_s^n} e_{s,z}^n(\rho_0, \eta_0) = e_n < 1. \tag{5.2}$$

Furthermore, we let

$$l_n(r) = \sup_{\substack{s,t \in [0,T] \\ |s-t| \leq r}} \sup_{z \in \overline{D_s^n}} d(z, D_t^n), \tag{5.3}$$

and we assume that

$$\lim_{r \to 0^+} \sup_{n \geq 1} l_n(r) = 0. \tag{5.4}$$

Note also that if we define $\hat{l}_n(r)$ as in (3.2) but with $D$ replaced by $D^n$, then Lemma 3.1 and (5.4) imply that

$$\lim_{r \to 0^+} \sup_{n \geq 1} \hat{l}_n(r) = 0. \tag{5.5}$$

Moreover, we assume that there exists $\hat{R} > 0$ such that $D_t^n \subset B(0, \hat{R})$ and $D_t \subset B(0, \hat{R})$, for all $n \geq 1$ and $t \in [0, T]$, and we let

$$R = 2 \sup_{t\in[0,T]} \sup_n \max\{\operatorname{diam}(D_t^n), \operatorname{diam}(D_t)\}, \tag{5.6}$$

where $\operatorname{diam}(D_t^n)$, $\operatorname{diam}(D_t)$ are the Euclidean diameters of the spatial regions $D_t^n$ and $D_t$, respectively. Recalling that the set $G^\Gamma$ was introduced in (1.12), we here also introduce, for $n \geq 1$,

$$G^{\Gamma^n} = \{(s, z, \gamma) : \gamma \in \Gamma_t^n(z), z \in \partial D_s^n, s \in [0, T]\}, \tag{5.7}$$

and we let, whenever $t \in [0, T]$,

$$\begin{aligned} G_t &= G^\Gamma \cap \big([0, t] \times B_R(0) \times S_1(0)\big), \\ G_t^n &= G^{\Gamma^n} \cap \big([0, t] \times B_R(0) \times S_1(0)\big). \end{aligned} \tag{5.8}$$

In the following we need to measure the distance between the sets $G_T$ and $G_T^n$ and, hence, we introduce an appropriate Hausdorff distance for subsets of $[0, T] \times B_R(0) \times S_1(0)$. In particular, we let, given $(s, z, \gamma) \in [0, T] \times B_R(0) \times S_1(0)$ and $(\hat{s}, \hat{z}, \hat{\gamma}) \in [0, T] \times B_R(0) \times S_1(0)$,

$$E((s, z, \gamma), (\hat{s}, \hat{z}, \hat{\gamma})) = |s - \hat{s}| + |z - \hat{z}| + |\gamma - \hat{\gamma}| \tag{5.9}$$

denote the (Euclidean) distance between $(s, z, \gamma)$ and $(\hat{s}, \hat{z}, \hat{\gamma})$. Furthermore, based on $E$, we define, given $F_1, F_2 \subseteq [0, T] \times B_R(0) \times S_1(0)$ and $(s, z, \gamma) \in [0, T] \times B_R(0) \times S_1(0)$, the distances $E((s, z, \gamma), F_1)$, $E((s, z, \gamma), F_2)$ and $E(F_1, F_2)$ in the natural way. Furthermore, for $F_1$ and $F_2$ as above, we introduce a Hausdorff distance between $F_1$ and $F_2$ as

$$\begin{aligned} H(F_1, F_2) &= \max\{A, B\}, \\ A &= \sup\{E((s, z, \gamma), F_2) : (s, z, \gamma) \in F_1\}, \\ B &= \sup\{E((\hat{s}, \hat{z}, \hat{\gamma}), F_1) : (\hat{s}, \hat{z}, \hat{\gamma}) \in F_2\}. \end{aligned} \tag{5.10}$$

In the following we say that $G_T^n$ converges to $G_T$ if

$$H(G_T^n, G_T) \to 0 \qquad \text{as } n \to \infty. \tag{5.11}$$

Imposing the assumptions on $D$, $\Gamma$ stated above and assuming (5.11), we can, for example, ensure that if $\{(s_n, z_n)\}$ is a sequence of points in $\mathbb{R}^{d+1}$, $s_n \in [0, T]$, $z_n \in \partial D_{s_n}^n$, $\lim_{n\to\infty} s_n = s \in [0, T]$, $\lim_{n\to\infty} z_n = z \in \partial D_s$, then

$$\lim_{n\to\infty} h(\Gamma_{s_n}^n(z_n), \Gamma_s(z)) = 0. \tag{5.12}$$

To see this, we consider, for $\{(s_n, z_n)\}$ and $(s, z)$ given as above, $(s_n, z_n, \gamma_{s_n}^n) \in G_T^n$ and $(s, z, \gamma_s) \in G_T$. Given $(s_n, z_n, \gamma_{s_n}^n) \in G_T^n$, we let $(\hat{s}_n, \hat{z}_n, \hat{\gamma}_{\hat{s}_n}^n) \in G_T$ be a point on $G_T$ which minimizes the distance, as defined in (5.9), from $(s_n, z_n, \gamma^n_{s_n})$ to $G_T$. Then,

$$
\begin{aligned}
|\gamma^n_{s_n} - \gamma_s| &\le E((s_n, z_n, \gamma^n_{s_n}), (s, z, \gamma_s)) \\
&\le E((s_n, z_n, \gamma^n_{s_n}), (\hat{s}_n, \hat{z}_n, \hat{\gamma}^n_{\hat{s}_n})) + E((\hat{s}_n, \hat{z}_n, \hat{\gamma}^n_{\hat{s}_n}), (s, z, \gamma_s)) \\
&\le H(G_T, G^n_T) + E((\hat{s}_n, \hat{z}_n, \hat{\gamma}^n_{\hat{s}_n}), (s, z, \gamma_s)).
\end{aligned} \tag{5.13}
$$

Hence,

$$
h(\Gamma^n_{s_n}(z_n), \Gamma_s(z)) \le H(G_T, G^n_T) + R_n, \tag{5.14}
$$

where

$$
\begin{aligned}
R_n &= \max\{A_n, B_n\}, \\
A_n &= \sup\{E((\hat{s}_n, \hat{z}_n, \hat{\gamma}^n_{\hat{s}_n}), \{(s, z, \Gamma_s(z))\}) : \hat{\gamma}^n_{\hat{s}_n} \in \Gamma^n_{\hat{s}_n}(\hat{z}_n)\}, \\
B_n &= \sup\{E(\{(\hat{s}_n, \hat{z}_n, \Gamma^n_{\hat{s}_n}(\hat{z}_n))\}, (s, z, \gamma_s)) : \gamma_s \in \Gamma_s(z)\}.
\end{aligned} \tag{5.15}
$$

As, by assumption, $G_T$ is closed, we can now first conclude that $R_n \to 0$ as $n \to \infty$, and then we find, using (5.11), that $h(\Gamma^n_{s_n}(z_n), \Gamma_s(z)) \to 0$ as $n \to \infty$. This completes the proof of (5.12). We are now ready to formulate our convergence result.

THEOREM 5.1. *Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2). *Let $\Gamma = \Gamma_t(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_t$, $t \in [0, T]$, and assume that $\Gamma$ satisfies* (1.11) *and* (1.12). *Let $\{D^n\}_{n=1}^\infty$ be a sequence of time-dependent domains $D^n \subset \mathbb{R}^{d+1}$ satisfying* (1.2) *and a uniform exterior sphere condition in time with radius $r_0$ in the sense of* (1.8). *Let $\{\Gamma^n\}_{n=1}^\infty = \{\Gamma^n_t(z)\}_{n=1}^\infty$ be a sequence of closed convex cones $\Gamma^n = \Gamma^n_t(z)$ of vectors in $\mathbb{R}^d$ for every $z \in \partial D^n_t$, $t \in [0, T]$. For all $n \ge 1$, $D^n$ and $\Gamma^n$ satisfy* (5.1) *and* (5.2) *for some $0 < \rho_0 < r_0$, $\eta_0 > 0$, $a_n$, $e_n$ and, moreover, $((0, T) \times \mathbb{R}^d) \setminus \overline{D^n}$ has the $(\delta_0, h_0)$-property of good projections along $\Gamma^n$, for some $0 < \delta_0 < \rho_0$, $h_0 > 1$. Assume that $\inf_n\{a_n\} > 0$, $\sup_n\{e_n\} < 1$ and* (5.4) *hold. Regarding the convergence $D^n \to D$ and $\Gamma^n \to \Gamma$, assume that*

$$
\lim_{n\to\infty} \sup_{t\in[0,T]} h(D^n_t, D_t) = 0, \tag{5.16}
$$

$$
\lim_{n\to\infty} \sup_{t\in[0,T]} h(\partial D^n_t, \partial D_t) = 0, \tag{5.17}
$$

*and, with $G_T$ and $G^n_T$ defined as in* (5.8), *that*

$$
G^n_T \text{ converges to } G_T \text{ in the sense of (5.11)}. \tag{5.18}
$$

*Let $w^n \in \mathcal{D}([0, T], \mathbb{R}^d)$ with $w^n_0 \in \overline{D^n_0}$ and assume that there exists a solution $(x^n, \lambda^n)$ to the Skorohod problem for $(D^n, \Gamma^n, w^n)$ such that $x^n \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$*

*for all $n \geq 1$. Assume that $\{w^n\}$ is relatively compact in the Skorohod topology and that $\{w^n\}$ converges to $w \in \mathcal{D}([0,T],\mathbb{R}^d)$. Then $\{(x^n,\lambda^n)\}$ converges to $(x,\lambda) \in \mathcal{D}([0,T],\mathbb{R}^d) \times \mathcal{BV}([0,T],\mathbb{R}^d)$ and $(x,\lambda)$ is a solution to the Skorohod problem for $(D,\Gamma,w)$ with $x \in \mathcal{D}^{\rho_0}([0,T],\mathbb{R}^d)$.*

REMARK 5.2. We note that the formulation of Theorem 5.1 contains several subtle points. First, we do not have to assume that the elements in the sequence $\{\Gamma^n\}_{n=1}^{\infty}$ satisfy (1.11), (1.12) and (1.14). The reason for this (see Remark 4.4) is that Theorem 4.2 holds, with constants independent of $n$, for each element in the sequence $\{(w^n,x^n,\lambda^n)\}$ even without these assumptions. Second, we only have to impose very modest restrictions on $D$ but, as can be seen in the proof below, we have to assume that $\Gamma = \Gamma_t(z)$ satisfies (1.11) and (1.12).

PROOF OF THEOREM 5.1. As $\{w^n\}$ is relatively compact in the Skorohod topology, we first note that Theorem 4.2 can be used to conclude the existence of positive constants $L_1$, $L_2$, $L_3$ and $L_4$, independent of $n$, such that

$$\begin{aligned} \|x^n\|_{t_1,t_2} &\leq L_1\|w^n\|_{t_1,t_2} + L_2 l_n(t_2-t_1), \\ |\lambda^n|_{t_2} - |\lambda^n|_{t_1} &\leq L_3\|w^n\|_{t_1,t_2} + L_4 l_n(t_2-t_1), \end{aligned} \tag{5.19}$$

whenever $0 \leq t_1 \leq t_2 \leq T$. As $\{w^n\}$ converges to $w \in \mathcal{D}([0,T],\mathbb{R}^d)$, we also see, using (5.4) and (5.19), that $\{(w^n,x^n,\lambda^n,|\lambda^n|)\}$ is relatively compact in $\mathcal{D}([0,T],\mathbb{R}^d) \times \mathcal{D}([0,T],\mathbb{R}^d) \times \mathcal{D}([0,T],\mathbb{R}^d) \times \mathcal{D}([0,T],\mathbb{R}_+)$. Furthermore, we know that $x_t^n \in \overline{D^n}$ for all $t \in [0,T]$, $n \geq 1$. Hence, $\{(x^n,\lambda^n)\}$ converges to some $(x,\lambda) \in \mathcal{D}([0,T],\mathbb{R}^d) \times \mathcal{D}([0,T],\mathbb{R}^d)$. We intend to prove that $(x,\lambda) \in \mathcal{D}([0,T],\mathbb{R}^d) \times \mathcal{BV}([0,T],\mathbb{R}^d)$ solves the Skorohod problem for $(D,\Gamma,w)$ and to do this, we have to prove that

$$\lambda \in \mathcal{BV}([0,T],\mathbb{R}^d), \tag{5.20}$$

and we have to verify that

(5.21) $(D,\Gamma,w)$ and $(x,\lambda)$ satisfy properties (1.5)–(1.7) in Definition 1.1.

We begin by verifying (1.5) in Definition 1.1. To do this, we first note, using the convergence properties of the Skorohod topology and the fact that $(x^n,\lambda^n)$ solves the Skorohod problem for $(D^n,\Gamma^n,w^n)$, that

$$x_t = w_t + \lambda_t \tag{5.22}$$

for all points of continuity and, hence, since $w$, $x$ and $\lambda$ are càdlàg functions, that (5.22) holds for all $t \in [0,T]$. Hence, to verify (1.5) in Definition 1.1, it only remains to ensure that $x_t \in \overline{D_t}$ for all $t \in [0,T]$. To do this, we first note, using

Proposition 5.3 and Remark 5.4 in Chapter 3 of [30], that there exists a sequence $\{\tilde{t}_n\}$ such that

$$
\begin{aligned}
&\tilde{t}_n \to t, \qquad x^n_{\tilde{t}_n} \to x_t, \qquad x^n_{\tilde{t}_n^-} \to x_{t^-}, \\
&\lambda^n_{\tilde{t}_n} \to \lambda_t, \qquad \lambda^n_{\tilde{t}_n^-} \to \lambda_{t^-} \qquad \text{as } n \to \infty.
\end{aligned} \tag{5.23}
$$

Furthermore, using the triangle inequality, (1.10), (5.16) and (5.23), we obtain

$$
\begin{aligned}
d(x_t, \overline{D_t}) &\leq |x_t - x^n_{\tilde{t}_n}| + d(x^n_{\tilde{t}_n}, \overline{D^n_{\tilde{t}_n}}) + h(\overline{D^n_{\tilde{t}_n}}, \overline{D_{\tilde{t}_n}}) + h(\overline{D_{\tilde{t}_n}}, \overline{D_t}) \\
&\leq |x_t - x^n_{\tilde{t}_n}| + h(\overline{D^n_{\tilde{t}_n}}, \overline{D_{\tilde{t}_n}}) + l(|\tilde{t}_n - t|) \to 0, \qquad \text{as } n \to \infty.
\end{aligned} \tag{5.24}
$$

This proves that $x_t \in \overline{D_t}$ for all $t \in [0, T]$ and, hence, we have verified that $(w_t, x_t, \lambda_t)$ satisfies (1.5). We next prove (5.20), that is, that $\lambda \in \mathcal{BV}([0, T], \mathbb{R}^d)$. To do this, we use an argument similar to the proof of Theorem 3.1 in [15], but, as described below, our argument is more subtle due to the fact that we consider sequences $(D^n, \Gamma^n, w^n)$ where, in particular, $D^n$ is time-dependent. Recall that with $R$ as introduced in (5.6), we have

$$
\sup_n \sup_{t\in[0,T]} |x^n_t| < R, \qquad \sup_{t\in[0,T]} |x_t| < R. \tag{5.25}
$$

Let $G^{\Gamma^n}$, $G_t$ and $G^n_t$ be as in (5.7) and (5.8). By the prerequisites of Theorem 5.1 (see Remark 5.2), we have that $G^\Gamma$ is closed. We next define a positive measure $\mu^n$ on $[0, T] \times B_R(0) \times S_1(0)$ by setting, for every Borel set $A \subset [0, T] \times B_R(0) \times S_1(0)$,

$$
\mu^n(A) = \int_0^T \chi_{A\cap G^n_T}(s, x^n_s, \gamma^n_s)\, d|\lambda^n|_s, \tag{5.26}
$$

where $\gamma^n_s \in \Gamma^{n,1}_s(x^n_s)$ is as in (1.6)–(1.7) for the solution $(x^n, \lambda^n)$ to the Skorohod problem for $(D^n, \Gamma^n, w^n)$ and $\chi_{A\cap G^n_T}$ is the characteristic functions for the set $A \cap G^n_T$. We then first note that

$$
|\lambda^n|_t = \mu^n(G^n_t) \qquad \text{whenever } t \in [0, T]. \tag{5.27}
$$

We also note that the support of $\mu^n$ is contained in $G^n_T$ in the sense that $\mu^n(A) = 0$ whenever $A \subset [0, T] \times B_R(0) \times S_1(0)$ is such that $A \cap G^n_T = \varnothing$. Using this, and the fact that (1.6) holds for $\lambda^n$, we see that (5.27) implies that

$$
\lambda^n_t = \int_{[0,t]\times B_R(0)\times S_1(0)} \gamma\, d\mu^n(s, z, \gamma) \qquad \text{whenever } t \in [0, T]. \tag{5.28}
$$

Next, using (5.4), (5.19), (5.27) and the fact that $\{w^n\}$ converges to $w \in \mathcal{D}([0, T], \mathbb{R}^d)$, we conclude that

$$
\sup_n \mu^n(G^n_T) < \infty, \tag{5.29}
$$

which implies that $\{\mu^n\}$ is a compact set of measures, on $[0,T]\times B_R(0)\times S_1(0)$, in the sense of the weak$*$-topology. Therefore, by the Banach–Alaoglu theorem, we can conclude that $\{\mu^n\}$ converges in the weak$*$-topology to a measure $\mu$ such that

$$\mu\big([0,T]\times B_R(0)\times S_1(0)\big)<\infty. \tag{5.30}$$

Moreover, since $(x^n,\lambda^n)$ converges to $(x,\lambda)$ in the sense of the Skorohod topology, we obtain, using (5.28), that

$$\lambda_t=\int_{[0,t]\times B_R(0)\times S_1(0)}\gamma\,d\mu(s,z,\gamma) \tag{5.31}$$

for all $t\in[0,T]$ such that $\lambda_t=\lambda_{t-}$. However, as both sides of (5.31) are right continuous, (5.31) holds for all $t\in[0,T]$. Having proved (5.31), we see, also using (5.30), that $\lambda$ is of bounded variation and, hence, (5.20) is proved. We next claim that

$$\lambda_t=\int_{G_t}\gamma\,d\mu(s,z,\gamma), \tag{5.32}$$

that is, we claim that the support of the measure $\mu$ is the set $G_T$ in the sense that if $A\subset[0,T]\times B_R(0)\times S_1(0)$ is such that $A\cap G_T=\varnothing$, then $\mu(A)=0$. To see this, we let $(\hat s,\hat z,\hat\gamma)\in[0,T]\times B_R(0)\times S_1(0)\setminus G_T$ and we see, as $G_T$ is closed, that if we define, for $\eta>0$, $B((\hat s,\hat z,\hat\gamma),\eta):=\{(s,z,\gamma):E((s,z,\gamma),(\hat s,\hat z,\hat\gamma))<\eta\}\cap[0,T]\times B_R(0)\times S_1(0)$, then there exists $\eta_0>0$ such that $B((\hat s,\hat z,\hat\gamma),2\eta_0)\cap G_T=\varnothing$. Recall that $E$ is the distance function introduced in (5.9). Furthermore, the above setup implies that $E(B((\hat s,\hat z,\hat\gamma),\eta_0),G_T)>\eta_0$ and since

$$E(B((\hat s,\hat z,\hat\gamma),\eta_0),G_T^n)\ge E(B((\hat s,\hat z,\hat\gamma),\eta_0),G_T)-H(G_T,G_T^n), \tag{5.33}$$

we can use the assumption in (5.18) to conclude that there exists $n_0\in\mathbb{N}$ such that

$$E(B((\hat s,\hat z,\hat\gamma),\eta_0),G_T^n)\ge\eta_0/2 \tag{5.34}$$

for all $n\ge n_0$. In particular, $B((\hat s,\hat z,\hat\gamma),\eta_0)\cap G_T^n=\varnothing$ for all $n\ge n_0$. Hence, $\mu^n(B((\hat s,\hat z,\hat\gamma),\eta_0))=0$, for all $n\ge n_0$, and $\mu(B((\hat s,\hat z,\hat\gamma),\eta_0))=0$ by the weak$*$-convergence of $\mu^n$ to $\mu$. This completes the proof of (5.32). Having proved (5.20) and (5.32), we see that

$$\lambda_t=\int_0^t\gamma_s\,d|\lambda|_s\qquad\text{whenever } t\in[0,T] \tag{5.35}$$

for some $S_1(0)$-valued Borel measurable function $\gamma_s$ and, to prove (1.6), we have to prove that $\gamma_s\in\Gamma_s^1(x_s)$ for all $s\in[0,T]$. To prove this and to verify (1.7), we consider the following two cases:

$$\begin{aligned}&\text{Case 1.}\quad t\in[0,T]\text{ is such that }x_t-x_{t-}\neq0,\\&\text{Case 2.}\quad t\in[0,T]\text{ is such that }x_t-x_{t-}=0.\end{aligned} \tag{5.36}$$

*Case* 1. Note that Case 1 occurs for an at most countable set of jump times of $x$. Moreover, in Case 1 it is enough to prove that

$$\lambda_t - \lambda_{t-} \neq 0 \text{ implies that } x_t \in \partial D_t \text{ and that } \lambda_t - \lambda_{t-} \in \Gamma_t(x_t). \tag{5.37}$$

We first note, as we are assuming $\lambda_t - \lambda_{t-} \neq 0$, that $|\lambda^n_{\tilde{t}_n} - \lambda^n_{\tilde{t}_n-}| > 0$ for $n$ sufficiently large; see (5.23) . Furthermore, since $(x^n, \lambda^n)$ solves the Skorohod problem for $(D^n, \Gamma^n, w^n)$, we have that

$$x^n_{\tilde{t}_n} \in \partial D^n_{\tilde{t}_n}, \qquad \lambda^n_{\tilde{t}_n} - \lambda^n_{\tilde{t}_n-} \in \Gamma^n_{\tilde{t}_n}(x^n_{\tilde{t}_n}). \tag{5.38}$$

Combining (5.5), (5.17), (5.23) and (5.38), we obtain

$$\begin{aligned} d(x_t, \partial D_t) &\le |x_t - x^n_{\tilde{t}_n}| + d(x^n_{\tilde{t}_n}, \partial D^n_{\tilde{t}_n}) + h(\partial D^n_{\tilde{t}_n}, \partial D_{\tilde{t}_n}) + h(\partial D_{\tilde{t}_n}, \partial D_t) \\ &\le |x_t - x^n_{\tilde{t}_n}| + h(\partial D^n_{\tilde{t}_n}, \partial D_{\tilde{t}_n}) + \hat{l}(|\tilde{t}_n - t|) \to 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{5.39}$$

Hence, using (5.39), we can, since $\partial D_t$ is closed, conclude that

$$x_t \in \partial D_t. \tag{5.40}$$

We next recall that the set $G^\Gamma$, defined in (1.12), is, by assumption, closed. Furthermore, arguing as in (5.39), we first see that

$$\begin{aligned} d(\lambda_t - \lambda_{t-}, \Gamma_t(x_t)) &\le |(\lambda_t - \lambda_{t-}) - (\lambda^n_{\tilde{t}_n} - \lambda^n_{\tilde{t}_n-})| + d(\lambda^n_{\tilde{t}_n} - \lambda^n_{\tilde{t}_n-}, \Gamma^n_{\tilde{t}_n}(x^n_{\tilde{t}_n})) \\ &\quad + h(\Gamma^n_{\tilde{t}_n}(x^n_{\tilde{t}_n}), \Gamma_t(x_t)) \\ &\le |\lambda_t - \lambda^n_{\tilde{t}_n}| + |\lambda_{t-} - \lambda^n_{\tilde{t}_n-}| + h(\Gamma^n_{\tilde{t}_n}(x^n_{\tilde{t}_n}), \Gamma_t(x_t)), \end{aligned} \tag{5.41}$$

and then, letting $n \to \infty$, it follows, using (5.12), that $d(\lambda_t - \lambda_{t-}, \Gamma_t(x_t)) = 0$. Applying the fact that the set $G^\Gamma$ is closed, we can therefore conclude that

$$\lambda_t - \lambda_{t-} \in \Gamma_t(x_t). \tag{5.42}$$

This concludes the proof of (5.37) and, hence, we have verified (1.6)–(1.7) in Case 1.

*Case* 2. To verify (1.6)–(1.7) in Case 2, we first see, by combining (5.32) and (5.35), that

$$\int_0^t \gamma_s \, d|\lambda|_s = \int_{G_t} \gamma \, d\mu(s, z, \gamma) \qquad \text{whenever } t \in [0, T]. \tag{5.43}$$

We next introduce a measure $\nu$ on $[0, T]$ by setting

$$\nu([0, t]) = \mu(G_t) \qquad \text{whenever } t \in [0, T]. \tag{5.44}$$

Combining (5.43) and (5.44), it is clear that $\nu([0, t]) = 0$ implies $|\lambda|_t = 0$, showing that $|\lambda|$ is absolutely continuous with respect to $\nu$. To simplify the notation, in the following we let, for $k \in \mathbb{N}$,

$$\Omega_k = \left\{ (t, z, \gamma) \in G_T : \inf_{s \in [0,T]} \bigl(|t - s| + (|z - x_s| \wedge |z - x_{s-}|)\bigr) > \frac{1}{k} \right\}. \tag{5.45}$$

Then, using Theorem 1.2.1(iii) in [32], the fact that $\mu(U) \le \underline{\lim}_{n\to\infty}\mu^n(U)$ for all open sets $U$ and the fact that $x_t^n$ converges either to $x_t$ or $x_{t-}$, we can conclude that

$$\begin{aligned}
&\mu(\{(t,z,\gamma)\in G_T : z\neq x_t, z\neq x_{t-}\}) \\
&\quad = \lim_{k\to\infty}\mu(\Omega_k) \le \lim_{k\to\infty}\underline{\lim}_{n\to\infty}\mu^n(\Omega_k) \qquad (5.46)\\
&\quad \le \lim_{k\to\infty}\underline{\lim}_{n\to\infty}\mu^n\left(\left\{(t,z,\gamma)\in G_T : |z-x_t^n| > \frac{1}{2k}\right\}\right) = 0.
\end{aligned}$$

If $x_t = x_{t-}\in D_t$, then, since $z\in\partial D_t$ for all $(t,z,\gamma)\in G_T$, we deduce that $z\neq x_t$ and $z\neq x_{t-}$. Hence, using (5.44) and (5.46), we first see that

$$\nu(\{t\in[0,T] : x_t = x_{t-}, x_t\in D_t\}) = 0, \qquad (5.47)$$

and then, by the absolute continuity of $|\lambda|$ with respect to $\nu$, we can conclude that

$$|\lambda|(\{t\in[0,T] : x_t = x_{t-}, x_t\in D_t\}) = 0. \qquad (5.48)$$

In particular, (5.48) proves (1.7). Hence, it only remains to prove that $\gamma_s$, as defined in (5.35), satisfies $\gamma_s\in\Gamma_s^1(x_s)$ for all $s\in[0,t]$ and $t\in[0,T]$. From (5.43) and the fact that (5.46) implies that

$$\mu(\{(s,z,\gamma)\in G_t : x_s = x_{s-}\neq z\}) = 0, \qquad (5.49)$$

we deduce that

$$\begin{aligned}
\int_{\{s\in[0,t] : x_s = x_{s-}\}}\gamma_s\,d|\lambda|_s &= \int_{\{(s,z,\gamma)\in G_t : z = x_s = x_{s-}\}}\gamma\,d\mu(s,z,\gamma) \\
&= \int_{\{s\in[0,t] : x_s = x_{s-}\}}\int_{\Gamma_s^1(x_s)}\gamma p(s,x_s,d\gamma)\,d\nu, \qquad (5.50)
\end{aligned}$$

whenever $t\in[0,T]$. Note that the last equality in (5.50) follows from the definition of $\nu$ in (5.44). Here $p(s,x_s,\cdot)$ is a measure on the Borel $\sigma$-algebra of $S_1(0)$, concentrated on $\Gamma_s^1(x_s)$ for $d\nu$-almost all $s\in[0,T]$ such that $x_s = x_{s-}$, and $p(\cdot,\cdot,A)$ is a nonnegative Borel measurable function for every Borel set $A$. Then, since $|\lambda|$ is absolutely continuous with respect to $\nu$, the Radon–Nikodym theorem asserts the existence of a nonnegative Borel measurable function $f$ such that

$$\begin{aligned}
f(s)\gamma_s &= \int_{\Gamma_s^1(x_s)}\gamma p(s,x_s,d\gamma), \qquad (5.51)\\
&\qquad d\nu\text{-a.e. for all } s\in\{s\in[0,t] : x_s = x_{s-}\}.
\end{aligned}$$

From the assumption in (1.11) we deduce that $f$ is strictly positive. Thus, by the convexity of $\Gamma_s(x_s)$ and the absolute continuity of $|\lambda|$ with respect to $\nu$, we conclude that

$$\gamma_s\in\Gamma_s^1(x_s), \qquad d|\lambda|\text{-a.e. for all } s\in\{s\in[0,t] : x_s = x_{s-}\}. \qquad (5.52)$$

In particular, (5.52) verifies the second part in (1.6) and, hence, the proof in Case 2 is also complete.

Having completed the proof of Cases 1 and 2, we conclude that the proofs of (5.20) and (5.21) are complete. Hence, to complete the proof of Theorem 5.1, we only have to ensure that $x \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R})$. However, this follows from the assumption that $x^n \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R})$ for all $n \geq 1$ and from the fact that $x^n \to x$ in the Skorohod topology. $\square$

5.2. *Convergence of a sequence of solutions to approximating Skorohod problems*. Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2), (1.10) and a uniform exterior sphere condition in time with radium $r_0$ in the sense of (1.8). Let $\Gamma = \Gamma_t(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_t$, $t \in [0, T]$, and assume that $\Gamma$ satisfies (1.11), (1.12) and (1.14). Assume that (1.18) and (1.19) hold for some $0 < \rho_0 < r_0$, $\eta_0 > 0$, $a$ and $e$. Finally, assume that $([0, T] \times \mathbb{R}^d) \setminus \overline{D}$ has the $(\delta_0, h_0)$-property of good projections along $\Gamma$, for some $0 < \delta_0 < \rho_0$, $h_0 > 1$ and let $w \in \mathcal{D}^{(\delta_0/4 \wedge \rho_0/(4h_0))}([0, T], \mathbb{R}^d)$ with $w_0 \in \overline{D_0}$. The purpose of this subsection is to construct a sequence of solutions to Skorohod problems which approximate the Skorohod problem for $(D, \Gamma, w)$. Based on this sequence, in the next section we conclude the existence of a solution $(x, \lambda)$ to the Skorohod problem for $(D, \Gamma, w)$, in the sense of Definition 1.1, with $x \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$. This will then complete the proof of Theorem 1.2. To proceed, we let $n \in \mathbb{N}$, $n \gg 1$, and we let $\{\epsilon_n\}$ be a sequence of real numbers which tends to 0 as $n \to \infty$. Then, for each $n$, we can find a partition $\Delta_n = \{\tau_k^n\}_{k=0}^{N_n}$ of the interval $[0, T]$, that is, $0 = \tau_0^n < \tau_1^n < \cdots < \tau_{N_n-1}^n < \tau_{N_n}^n = T$, such that (5.53)–(5.56) stated below hold. In particular,

$$\lim_{n\to\infty} \Delta_n^* = 0 \qquad \text{where } \Delta_n^* := \max_{k\in\{1,\dots,N_n-1\}} \tau_{k+1}^n - \tau_k^n, \tag{5.53}$$

and, for some $n_0 \gg 1$,

$$\|w\|_{\tau_k^n, \tau_{k+1}^n} + l(\Delta_n^*) < \min\left\{\frac{\delta_0}{2}, \frac{\rho_0}{2h_0}\right\} \quad \text{whenever } n \geq n_0, k \in \{0, \dots, N_n - 1\}. \tag{5.54}$$

Furthermore, we define $w^{\Delta_n} = w_t^{\Delta_n}$, $t \in [0, T]$, as

$$w_t^{\Delta_n} = w_{\tau_k^{\Delta_n}} \qquad \text{whenever } t \in [\tau_k^n, \tau_{k+1}^n), k \in \{0, \dots, N_n - 1\}, \tag{5.55}$$

and $w_T^{\Delta_n} = w_T$, so that

$$\begin{aligned} &w^{\Delta_n} \in \mathcal{D}^{(\delta_0/4 \wedge \rho_0/(4h_0))}([0, T], \mathbb{R}^d), \\ &w_0^{\Delta_n} \in \overline{D_0} \quad \text{and} \quad d_{\mathcal{D}}([0, T], w^{\Delta_n}, w) \leq \epsilon_n. \end{aligned} \tag{5.56}$$

In particular, $w^{\Delta_n} \in \mathcal{D}([0, T], \mathbb{R}^d)$ is a step function which approximates $w$ in the Skorohod topology. Given $\Delta_n$ and $w^{\Delta_n}$, we define $D^{\Delta_n}$ and $\Gamma^{\Delta_n}$ as in (2.6).

Furthermore, to obtain a more simple notation, from now on we write $w^n$, $D^n$ and $\Gamma^n$ for $w^{\Delta_n}$, $D^{\Delta_n}$ and $\Gamma^{\Delta_n}$. Then, given $w^n$, $D^n$ and $\Gamma^n$ as above, we next define a pair of processes $(x^n, \lambda^n)$ as follows. Let

$$x_t^n = w_0, \qquad \lambda_t^n = 0 \qquad \text{for } t \in [0, \tau_1^n). \tag{5.57}$$

If $x^n_{\tau^n_{k-1}} \in \overline{D^n_{\tau^n_{k-1}}}$ for some $k \in \{1, \dots, N_n\}$, then, by the triangle inequality and (5.54),

$$d(x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}, \overline{D^n_{\tau^n_k}}) \le \|w^n\|_{\tau^n_{k-1}, \tau^n_k} + l(\Delta_n^*) < \delta_0. \tag{5.58}$$

Hence, by the $(\delta_0, h_0)$-property of good projections, it follows that if $x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}} \notin \overline{D^n_{\tau^n_k}}$, then there exists a point

$$\pi^{\Gamma^n_{\tau^n_k}}_{\partial D^n_{\tau^n_k}}(x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}) \in \partial D^n_{\tau^n_k}, \tag{5.59}$$

which is the projection of $x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}$ onto $\partial D^n_{\tau^n_k}$ along $\Gamma_{\tau^n_k}$. Furthermore, if $x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}} \in \overline{D^n_{\tau^n_k}}$, then we let

$$\pi^{\Gamma^n_{\tau^n_k}}_{\partial D^n_{\tau^n_k}}(x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}) = x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}. \tag{5.60}$$

Based on this argument, we define, whenever $t \in [\tau^n_k, \tau^n_{k+1})$, $k \in \{1, \dots, N_n - 1\}$,

$$\begin{aligned} x^n_t &= \pi^{\Gamma^n_{\tau^n_k}}_{\partial D^n_{\tau^n_k}}(x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}), \\ \lambda^n_t &= \lambda^n_{\tau^n_{k-1}} + \bigl(x^n_t - (x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}})\bigr), \end{aligned} \tag{5.61}$$

and, finally, we define $x^n_T$ and $\lambda^n_T$ using (5.61) by simply setting $k = N_n$ in (5.61). Note that in this way we have $x^n_{\tau^n_{k-1}} \in \overline{D^n_{\tau^n_{k-1}}}$ for all $k \in \{1, \dots, N_n\}$. Next, again using the $(\delta_0, h_0)$-property of good projections, we see that

$$\begin{aligned} |x^n_{\tau^n_k} - x^n_{\tau^n_{k-1}}| &\le |\pi^{\Gamma^n_{\tau^n_k}}_{\partial D^n_{\tau^n_k}}(x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}) - (x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}})| \\ &\quad + |w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}| \\ &\le h_0 d(x^n_{\tau^n_{k-1}} + w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}, \overline{D^n_{\tau^n_k}}) + |w^n_{\tau^n_k} - w^n_{\tau^n_{k-1}}| \\ &\le h_0\bigl(\|w^n\|_{\tau^n_{k-1}, \tau^n_k} + l(\Delta^*_n)\bigr) + \|w^n\|_{\tau^n_{k-1}, \tau^n_k} \\ &\le h_0\left(\frac{\rho_0}{2h_0}\right) + \frac{\delta_0}{4} < \rho_0. \end{aligned} \tag{5.62}$$

Hence, $x^n \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$. Using this notation, we next prove the following theorem.

THEOREM 5.3. *Let $T > 0$, $D \subset \mathbb{R}^{d+1}$, $r_0$, $\Gamma = \Gamma_t(z)$, $0 < \rho_0 < r_0$, $\eta_0 > 0$, $a$, $e$, $\delta_0$ and $h_0$ be as in the statement of Theorem 1.2. Let $w$ be as in the statement of Theorem 1.2 and let $w^n$, $D^n$, $\Gamma^n$, $x^n$ and $\lambda^n$ be defined as above for $n \geq 1$. Then $(x^n, \lambda^n)$ is a solution to the Skorohod problem for $(D^n, \Gamma^n, w^n)$ and $x^n \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Moreover, $\{(x^n, \lambda^n)\}$ converges to $(x, \lambda) \in \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{BV}([0, T], \mathbb{R}^d)$ and $(x, \lambda)$ is a solution to the Skorohod problem for $(D, \Gamma, w)$. Furthermore, $x \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$.*

REMARK 5.4. Note that for Theorem 5.3 we, in contrast to in Theorem 5.1, also need to assume (1.14) in order to be able complete the proof [see (5.69) below].

PROOF OF THEOREM 5.3. $(x^n, \lambda^n)$ is, by construction, a solution to the Skorohod problem for $(D^n, \Gamma^n, w^n)$ and the statement that $x^n \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$, is proved in (5.62). Next, using Theorem 4.6, we can conclude the existence of some positive constants $\hat{L}_1(w, T)$, $\hat{L}_2(w, T)$, $\hat{L}_3(w, T)$ and $\hat{L}_4(w, T)$ such that

$$
\begin{aligned}
\|x^n\|_{t_1,t_2} &\leq \hat{L}_1(w, T)\|w\|_{t_1,t_2} + \hat{L}_2(w, T)\big(l(t_2 - t_1) + l(\Delta_n^*)\big), \\
|\lambda^n|_{t_2} - |\lambda^n|_{t_1} &\leq \hat{L}_3(w, T)\|w\|_{t_1,t_2} + \hat{L}_4(w, T)\big(l(t_2 - t_1) + l(\Delta_n^*)\big),
\end{aligned}
\tag{5.63}
$$

whenever $0 \leq t_1 \leq t_2 \leq T$. In particular, note that by choosing $n_0$ sufficiently large we can ensure, using (5.53), that $l(\Delta_n^*) \leq \rho_0/(4(K_2(a, e) + 1))$ and that (4.68) holds for all $n \geq n_0$. Hence, Theorem 4.6 is applicable. Based on (5.63), we can now argue as in the proof of Theorem 5.1. In particular, as $\{w^n\}$ converges to $w \in \mathcal{D}([0, T], \mathbb{R}^d)$, we see, using (5.63), that $\{(w^n, x^n, \lambda^n, |\lambda|^n)\}$ is relatively compact in $\mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}_+)$. Furthermore, we know that $x_t^n \in \overline{D_n}$ for all $t \in [0, T]$, $n \geq 1$. Hence, $\{(x^n, \lambda^n)\}$ converges to some $(x, \lambda) \in \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}^d)$. We intend to prove that $(x, \lambda) \in \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{BV}([0, T], \mathbb{R}^d)$ solves the Skorohod problem for $(D, \Gamma, w)$ and, to do this, we have to prove that

$$
\lambda \in \mathcal{BV}([0, T], \mathbb{R}^d), \tag{5.64}
$$

and we have to verify

(5.65) properties (1.5)–(1.7) in Definition 1.1.

The proof of (5.64) and (5.65) follows along the lines of the proof of (5.20) and (5.21) in Theorem 5.1 and we shall only outline the main differences between the proofs. To start with, the statements in (5.22) and (5.23) remain true. However, the argument in (5.24) has to be changed. In particular, in this case we see, using (5.23) and (5.53), that

$$
\begin{aligned}
d(x_t, \overline{D_t}) &\leq |x_t - x_{\tilde{t}_n}^n| + h(\overline{D_{\tilde{t}_n}^n}, \overline{D_{\tilde{t}_n}}) \\
&\quad + l(|\tilde{t}_n - t|) + l(\Delta_n^*) \to 0 \qquad \text{as } n \to \infty,
\end{aligned}
\tag{5.66}
$$

which, since $\overline{D_t}$ is closed, proves that $x_t \in \overline{D_t}$, for all $t \in [0, T]$. Hence, we have verified that $(w_t, x_t, \lambda_t)$ satisfies (1.5). Next, arguing as in (5.25)–(5.35), using (5.63) to conclude (5.29), we can conclude that (5.64) holds and that

$$\lambda_t = \int_0^t \gamma_s \, d|\lambda|_s = \int_{G_t} \gamma \, d\mu(s, z, \gamma) \qquad \text{whenever } t \in [0, T] \tag{5.67}$$

for some $S_1(0)$-valued Borel measurable function $\gamma_s$. Hence, to prove (1.6), we again have to prove that $\gamma_s \in \Gamma_s^1(x_s)$ for all $t \in [0, T]$. As in the proof of Theorem 5.1, we consider Case 1 and Case 2. In fact, Case 2 can be handled exactly as in the proof of Theorem 5.1 and hence shall only discuss the proof of Case 1. To prove Case 1, we first see that the statements in (5.37) and (5.38) can be repeated and, arguing as in (5.66), we obtain

$$\begin{aligned} d(x_t, \partial D_t) \le |x_t - x_{\tilde{t}_n}^n| + h(\partial D_{\tilde{t}_n}^n, \partial D_{\tilde{t}_n}) \\ + \hat{l}(|\tilde{t}_n - t|) + \hat{l}(\Delta_n^*) \to 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{5.68}$$

Hence, since $\partial D_t$ is closed, we can conclude that $x_t \in \partial D_t$. To proceed, we deduce as in (5.41) that

$$d(\lambda_t - \lambda_{t-}, \Gamma_t(x_t)) \le |\lambda_t - \lambda_{\tilde{t}_n}^n| + |\lambda_{t-} - \lambda_{\tilde{t}_n^-}^n| + h(\Gamma_{\tilde{t}_n}^n(x_{\tilde{t}_n}^n), \Gamma_t(x_t)). \tag{5.69}$$

Obviously the first two terms on the right-hand side in (5.69) tend to zero as $n \to \infty$. Concerning the third term, we first note that there exists, for $n$ large enough, some integer $k(n)$ such that

$$\Gamma_{\tilde{t}_n}^n(x_{\tilde{t}_n}^n) = \Gamma_{\tau_{k(n)}^n}^n(x_{\tau_{k(n)}^n}^n). \tag{5.70}$$

Hence, as $|\tilde{t}_n - \tau_{k(n)}^n| \le l(\Delta_n^*)$, we can conclude, using (5.23), that $\tau_{k(n)}^n \to t$ as $n \to \infty$. Moreover, as $x_{\tau_{k(n)}^n}^n = x_{\tilde{t}_n}^n$, we can also use (5.23) to conclude that $x_{\tau_{k(n)}^n}^n \to x_t$ as $n \to \infty$. In particular, based on these conclusions, it follows from (1.14) that also the third term on the right-hand side in (5.69) tends to zero as $n \to \infty$. Hence, having proved that $d(\lambda_t - \lambda_{t-}, \Gamma_t(x_t)) = 0$, the proof of Case 1 can now be completed as in Theorem 5.1.

Having completed the proof of Cases 1 and 2, we can conclude that the proofs of (5.64) and (5.65) are complete. Hence, to complete the proof of Theorem 5.3, we only have to ensure that $x \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$. However, again this follows from the fact that $x^n \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$ for all $n \ge n_0$ and from the fact that $x^n \to x$ in the Skorohod topology. □

## 6. Proof of Theorems 1.2, 1.3 and 1.9.

PROOF OF THEOREM 1.2. Theorem 1.2 now follows immediately from Theorem 5.3. □

PROOF OF THEOREM 1.3. Using Theorem 4.2 and (1.10), we see that

$$
\begin{aligned}
\lim_{t_2\to t_1} |x_{t_2}-x_{t_1}| &\le \lim_{t_2\to t_1} \|x\|_{t_1,t_2} \\
&\le \lim_{t_2\to t_1} \bigl(L_1\|w\|_{t_1,t_2}+L_2 l(|t_2-t_1|)\bigr) \\
&\le 0+L_2\lim_{t_2\to t_1} l(|t_2-t_1|)=0.
\end{aligned}
\tag{6.1}
$$

This proves that $x$ is continuous. □

PROOF OF THEOREM 1.9. Let $W$ be a $m$-dimensional Wiener process on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P)$ and in the following let $W_t$, $t\in[0,T]$, be a continuous path of $W$. We define, for $n\in\mathbb{N}$, $n\gg 1$, $k\in\{0,1,\dots,2^n-1\}$,

$$
\begin{aligned}
D_t^n &= D_{kT/2^n}, \\
\Gamma_t^n &= \Gamma_{kT/2^n} \qquad \text{whenever } t\in[kT/2^n,(k+1)T/2^n),
\end{aligned}
\tag{6.2}
$$

and $D_T^n=D_T$, $\Gamma_T^n=\Gamma_T$. Furthermore, we recursively define three processes $X^n=X_t^n$, $Z^n=Z_t^n$ and $\Lambda^n=\Lambda_t^n$, for $t\in[0,T]$, in the following way. Let

$$
X_0^n=\hat{z}, \qquad Z_0^n=\hat{z}, \qquad \Lambda_0^n=0,
\tag{6.3}
$$

and let, for $k\in\{0,1,\dots,2^n-1\}$,

$$
\begin{aligned}
Z_{(k+1)T/2^n}^n &= Z_{kT/2^n}^n+\frac{T}{2^n}b(kT/2^n,X_{kT/2^n}^n) \\
&\quad +\sigma(kT/2^n,X_{kT/2^n}^n)\bigl(W_{(k+1)T/2^n}-W_{kT/2^n}\bigr), \\
X_{(k+1)T/2^n}^n &= \pi_{\partial D_{(k+1)T/2^n}}^{\Gamma_{(k+1)T/2^n}}\bigl(X_{kT/2^n}^n+Z_{(k+1)T/2^n}^n-Z_{kT/2^n}^n\bigr).
\end{aligned}
\tag{6.4}
$$

We here have to make sure that $X_{(k+1)T/2^n}^n$ is well defined. To do this, we note that, either $X_{kT/2^n}^n+Z_{(k+1)T/2^n}^n-Z_{kT/2^n}^n\in\overline{D_{(k+1)T/2^n}^n}$ or $X_{kT/2^n}^n+Z_{(k+1)T/2^n}^n-Z_{kT/2^n}^n\in\mathbb{R}^d\setminus\overline{D_{(k+1)T/2^n}^n}$. In the first case we identify the projection with the point itself, whereas, in the latter case, we have to assert the existence of appropriate projections onto $\partial D_{(k+1)T/2^n}^n$. However, assuming $X_{kT/2^n}^n\in D_{kT/2^n}^n$, we see that

$$
\begin{aligned}
& d\bigl(X_{kT/2^n}^n+Z_{(k+1)T/2^n}^n-Z_{kT/2^n}^n,D_{(k+1)T/2^n}^n\bigr) \\
&\quad\le l(T/2^n)+\bigl|Z_{(k+1)T/2^n}^n-Z_{kT/2^n}^n\bigr| \\
&\quad\le l(T/2^n)+(T/2^n)\Bigl(\sup_{\overline{D}}|b|\Bigr)+\Bigl(\sup_{\overline{D}}\|\sigma\|\Bigr)\bigl|W_{(k+1)T/2^n}-W_{kT/2^n}\bigr|.
\end{aligned}
\tag{6.5}
$$

Hence, since $W_t$ is a continuous path, there must exist some $n_0\in\mathbb{N}$ such that

$$
d\bigl(X_{kT/2^n}^n+Z_{(k+1)T/2^n}^n-Z_{kT/2^n}^n,D_{(k+1)T/2^n}^n\bigr)<\delta_0,
\tag{6.6}
$$

whenever $n \geq n_0$ and $k \in \{0, 1, \ldots, 2^n - 1\}$. By (6.6) and the $(\delta_0, h_0)$-property of good projections, it follows that the projection $\pi_{\partial D_{(k+1)T/2^n}}^{\Gamma_{(k+1)T/2^n}}(X^n_{kT/2^n} + Z^n_{(k+1)T/2^n} - Z^n_{kT/2^n})$ is well defined, for $n \geq n_0$ , whenever $X^n_{kT/2^n} + Z^n_{(k+1)T/2^n} - Z^n_{kT/2^n} \in \mathbb{R}^d \setminus \overline{D^n_{(k+1)T/2^n}}$. Furthermore, using the definition of $Z^n_{(k+1)T/2^n}$ in (6.4), we also see that

$$
\begin{aligned}
(6.7)\qquad |Z^n_{(k+1)T/2^n} - Z^n_{kT/2^n}| &\leq l(T/2^n) + (T/2^n)\Bigl(\sup_{\overline{D}} |b|\Bigr) \\
&\quad + \Bigl(\sup_{\overline{D}} \|\sigma\|\Bigr)|W_{(k+1)T/2^n} - W_{kT/2^n}|,
\end{aligned}
$$

and, hence, once more using that $W_t$ is a continuous path, we can ensure that

$$
(6.8)\qquad
\begin{aligned}
&\text{(i)}\quad Z^n \in \mathcal{D}^{(\delta_0/4\wedge\rho_0/(4h_0))}([0, T], \mathbb{R}^d), \\
&\text{(ii)}\quad h_0\bigl(\|Z^n\|_{kT/2^n,(k+1)T/2^n} + l(\Delta_n^*)\bigr) + \|Z^n\|_{kT/2^n,(k+1)T/2^n} \leq \rho_0,
\end{aligned}
$$

whenever $n \geq n_0$ and $k \in \{0, 1, \ldots, 2^n - 1\}$. We next let, for $k \in \{0, 1, \ldots, 2^n - 1\}$,

$$
(6.9)\qquad \Lambda^n_{(k+1)T/2^n} = \Lambda^n_{kT/2^n} + X^n_{(k+1)/2^n} - X^n_{kT/2^n} - Z^n_{(k+1)T/2^n} + Z^n_{kT/2^n}.
$$

Finally, we define, for $kT/2^n \leq t < (k + 1)T/2^n$, $k \in \{0, 1, \ldots, 2^n - 1\}$,

$$
(6.10)\qquad X^n_t = X^n_{kT/2^n}, \qquad Z^n_t = Z^n_{kT/2^n}, \qquad \Lambda^n_t = \Lambda^n_{kT/2^n}.
$$

Then, by arguing as in the proof of (5.62), using (i) and (ii) in (6.8), we can conclude that

$$
(6.11)\qquad X^n \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d) \qquad \text{whenever } n \geq n_0.
$$

Furthermore, using the definitions above, it is clear that

$$
(6.12)\qquad Z^n_t = \hat{z} + \int_0^t b(s, X^n_s)\,ds + \int_0^t \sigma(s, X^n_s)\,dW_s - \varepsilon^n(t),
$$

where

$$
(6.13)\qquad \sup_{t\in[0,T]} |\varepsilon^n(t)| \leq \Bigl(\sup_{\overline{D}} |b|\Bigr)\frac{T}{2^n} + \Bigl(\sup_{\overline{D}} \|\sigma\|\Bigr) \sup_{\substack{0\leq s\leq t\leq T,\\ |s-t|\leq T/2^n}} |W_t - W_s|.
$$

By construction, $(X^n, \Lambda^n)$ solves the Skorohod problem for $(D^n, \Gamma^n, Z^n)$ and using Theorem 4.6, we can conclude that there exist positive constants $\hat{L}_1(Z, T)$, $\hat{L}_2(Z, T)$, $\hat{L}_3(Z, T)$ and $\hat{L}_4(Z, T)$, independent of $n$, for $n \geq n_0$, such that

$$
(6.14)\qquad
\begin{aligned}
\|X^n\|_{t_1,t_2} &\leq \hat{L}_1(Z, T)\|Z\|_{t_1,t_2} + \hat{L}_2(Z, T)\bigl(l(t_2 - t_1) + l(T/2^n)\bigr), \\
|\Lambda^n|_{t_2} - |\Lambda^n|_{t_1} &\leq \hat{L}_3(Z, T)\|Z\|_{t_1,t_2} + \hat{L}_4(Z, T)\bigl(l(t_2 - t_1) + l(T/2^n)\bigr),
\end{aligned}
$$

whenever $0 \leq t_1 \leq t_2 \leq T$ and $n \geq n_0$. Hence, the sequence $\{(Z^n, X^n, \Lambda^n)\}$ is relatively compact in the Skorohod topology and we can conclude, by the construction of $Z^n$ and (6.12)–(6.13), that also the sequence $\{(W, Z^n, X^n, \Lambda^n, \varepsilon^n)\}$

is relatively compact in the Skorohod topology. In fact, as this argument can be repeated for each continuous path of $W_t$, it follows, by considering convergent subsequences if necessary, that the sequence of vector valued processes $\{(W, Z^n, X^n, \Lambda^n, \varepsilon^n)\}$ defined on $(\Omega, \mathcal{F}, P)$ converges in law to a stochastic process $(W, Z, X, \Lambda, 0)$. Furthermore, using the Skorohod representation theorem (see, e.g., [7] and [30]), there exists a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and versions $\{(\tilde{W}^n, \tilde{Z}^n, \tilde{X}^n, \tilde{\Lambda}^n, \tilde{\varepsilon}^n)\}$ and $(\tilde{W}, \tilde{Z}, \tilde{X}, \tilde{\Lambda}, 0)$ of $\{(W, Z^n, X^n, \Lambda^n, \varepsilon^n)\}$ and $(W, Z, X, \Lambda, 0)$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, such that $\{(\tilde{W}^n, \tilde{Z}^n, \tilde{X}^n, \tilde{\Lambda}^n, \tilde{\varepsilon}^n)\}$ converges to $(\tilde{W}, \tilde{Z}, \tilde{X}, \tilde{\Lambda}, 0)$ $\tilde{P}$-almost surely. Moreover, using Theorem 5.3 and the fact that $(\tilde{X}^n, \tilde{\Lambda}^n)$ solves, $\tilde{P}$-almost surely, the Skorohod problem for $(D^n, \Gamma^n, \tilde{Z}^n)$, it follows that $(\tilde{X}, \tilde{\Lambda})$ solves, $\tilde{P}$-almost surely, the Skorohod problem for $(D, \Gamma, \tilde{Z})$. In particular, $(\tilde{X}, \tilde{\Lambda}) \in \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{BV}([0, T], \mathbb{R}^d)$ and

$$\tilde{X}_t = \hat{z} + \tilde{Z}_t + \tilde{\Lambda}_t, \tag{6.15}$$

$$\tilde{\Lambda}_t = \int_0^t \gamma_s \, d|\tilde{\Lambda}|_s, \qquad \gamma_s \in \Gamma_s(\tilde{X}_s) \cap S_1(0), \ d|\tilde{\Lambda}|\text{-a.e.}, \tag{6.16}$$

$$\tilde{X}_t \in \overline{D_t}, \qquad d|\tilde{\Lambda}|(\{t \in [0, T] : \tilde{X}_t \in D_t\}) = 0 \tag{6.17}$$

holds $\tilde{P}$-almost surely whenever $t \in [0, T]$. We next want to verify that

$$\tilde{Z}_t = \int_0^t b(s, \tilde{X}_s) \, ds + \int_0^t \sigma(s, \tilde{X}_s) \, d\tilde{W}_s \tag{6.18}$$

holds $\tilde{P}$-almost surely whenever $t \in [0, T]$. Indeed, using (6.12) and (6.13), we can, following [15], simply quote Theorem 2.2 in [39], which in our case states that since $\{(\sigma(s, \tilde{X}_s^n), \tilde{W}_s^n)\}$ converges to $(\sigma(s, \tilde{X}_s), \tilde{W}_s)$ $\tilde{P}$-almost surely whenever $s \in [0, T]$,

$$\int_0^t \sigma(s, \tilde{X}_s^n) \, d\tilde{W}_s^n \quad \text{converges to} \quad \int_0^t \sigma(s, \tilde{X}_s) \, d\tilde{W}_s, \qquad \tilde{P}\text{-almost surely}, \tag{6.19}$$

whenever $t \in [0, T]$, as $n \to \infty$. This proves (6.18). Now let $\tilde{\mathcal{F}}_t$ and $\tilde{\mathcal{F}}_t^n$ be the $\sigma$-algebras generated by $\{\tilde{W}_s : s \leq t\}$ and $\{\tilde{W}_s^n : s \leq t\}$, respectively. We next prove that $\tilde{W}$ is a $m$-dimensional Wiener process on the filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})$. To obtain this, we first note that the $\sigma$-algebra generated by $\{\tilde{W}_s^n - \tilde{W}_t^n : s \geq t\}$, for $t \in [0, T]$, is independent of $\tilde{\mathcal{F}}_t^n$. Furthermore, since $\tilde{W}^n \to \tilde{W}$ $\tilde{P}$-almost surely, it follows that the $\sigma$-algebra generated by $\{\tilde{W}_s - \tilde{W}_t : s \geq t\}$, for $t \in [0, T]$, is independent of $\tilde{\mathcal{F}}_t$. In particular, $\{\tilde{W}_t : t \in [0, T]\}$ is a martingale with respect to $\{\tilde{\mathcal{F}}_t : t \in [0, T]\}$ and $\tilde{P}$. Now let $\tilde{W}_t^n = (\tilde{W}_t^{n,1}, \ldots, \tilde{W}_t^{n,m})$ and $\tilde{W}_t = (\tilde{W}_t^1, \ldots, \tilde{W}_t^m)$. Then, using essentially the same argument as in (6.19), we also see that $\tilde{W}^{n,i} \tilde{W}^{n,j} \to \tilde{W}^i \tilde{W}^j$, $\tilde{P}$-almost surely, for all $i, j \in \{1, \ldots, m\}$, and, hence, as above, we can conclude that $\{\tilde{W}_t^i \tilde{W}_t^j - \delta_{ij} t : t \in [0, T]\}$, with $\delta_{ij}$ being the Kronecker delta, is a martingale with respect to $\{\tilde{\mathcal{F}}_t : t \in [0, T]\}$ and $\tilde{P}$. By the Lévy characterization of Wiener processes (see, e.g., Theorem II.6.1 in [34]), we can thus conclude that $\tilde{W}$ is a $m$-dimensional Wiener process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})$. To finally conclude that $(\tilde{X}, \tilde{\Lambda})$ is a weak solution in the sense of Definition 1.8, and hence to complete the proof of Theorem 1.9, it only remains to prove the existence of a version of $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, denoted $\hat{X}$, such that $\hat{X} \in \mathcal{C}([0, T], \mathbb{R}^d)$, $\tilde{P}$-almost surely. However, using standard arguments, we first note that there exists a version of $\tilde{Z}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, denoted $\hat{Z}$, such that $\hat{Z} \in \mathcal{C}([0, T], \mathbb{R}^d)$, $\tilde{P}$-almost surely, and such that $(\tilde{X}, \tilde{\Lambda})$ solves, $\tilde{P}$-almost surely, the Skorohod problem for $(D, \Gamma, \hat{Z})$. Furthermore, by (6.11), it is clear that $\tilde{X} \in \mathcal{D}^{\rho_0}([0, T], \mathbb{R}^d)$. Hence, by Theorem 1.3, $\tilde{X}$ is continuous $\tilde{P}$-almost surely. This completes the proof of Theorem 1.9. $\square$

## APPENDIX: GEOMETRY OF TIME-DEPENDENT DOMAINS

Concerning the $(\delta_0, h_0)$-property of good projections along $\Gamma$, the following result follows immediately from Theorem 4.1 in [15].

LEMMA A.1. *Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2) *and a uniform exterior sphere condition in time with radius $r_0$ in the sense of* (1.8). *Let $\Gamma = \Gamma_t(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_t$, $t \in [0, T]$ and assume that $\Gamma$ satisfies* (1.11) *and* (1.12). *Assume that there exists a continuous map $Q : G^N \to \mathbb{R}^d$ such that*

$$\text{(A.1)}\qquad \begin{aligned} Q(t, z, N_t(z)) &= \Gamma_t(z) \qquad \textit{for all } z \in \partial D_t, t \in [0, T], \\ Q(t, z, \lambda v) &= \lambda Q(t, z, v) \\ &\qquad \textit{for all } \lambda \geq 0, v \in N_t(z), z \in \partial D_t, t \in [0, T]. \end{aligned}$$

*Moreover, assume that*

$$\text{(A.2)}\qquad \begin{aligned} \sup_{t\in[0,T], z\in\partial D_t} \max_{v\in N_t^1(z)} |Q(t, z, v)| &:= \|Q\| < \infty, \\ \inf_{t\in[0,T], z\in\partial D_t} \min_{v\in N_t^1(z)} v \cdot Q(t, z, v) &:= q > 0. \end{aligned}$$

*Let*

$$\text{(A.3)}\qquad \begin{aligned} \delta_0 &:= r_0\bigl(1 - \sqrt{1 - (q/\|Q\|)^2}\bigr), \\ h_0 &:= \frac{q/\|Q\|}{1 - \sqrt{1 - (q/\|Q\|)^2}}. \end{aligned}$$

*Then $([0, T] \times \mathbb{R}^d) \setminus \overline{D}$ has the $(\delta_0, h_0)$-property of good projections along $\Gamma$.*

Note that in Lemma A.1 we have that $q/\|Q\| < 1$, $0 < \delta_0 < r_0$ and $h_0 > 1$ by construction. To continue, given $T > 0$ and $D$ as above, we say that $D$ is a $\mathcal{H}_{1+\alpha}$-domain if we can find a $\rho > 0$ such that, for all $z_0 \in \partial D_{t_0}$, $t_0 \in [0, T]$, there exists a function $\psi(t, z)$, $\psi \in \mathcal{H}_{1+\alpha}(C_\rho(t_0, z_0))$, with the properties

$$\begin{aligned} D \cap C_\rho(t_0, z_0) &= \{\psi(t, z) > 0\} \cap C_\rho(t_0, z_0), \\ \partial D \cap C_\rho(t_0, z_0) &= \{\psi(t, z) = 0\} \cap C_\rho(t_0, z_0), \\ \inf_{(t,z)\in \partial D\cap C_\rho(t_0,z_0)} |\nabla_z \psi(t, z)| &> 0 \end{aligned} \tag{A.4}$$

for all $(t, z) \in (0, T) \times \mathbb{R}^d$.

LEMMA A.2. *Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2) and a uniform exterior sphere condition in time with radius $r_0$ in the sense of (1.8). Assume, in addition, that $D$ is a $\mathcal{H}_{1+\alpha}$-domain for some $\alpha \in (0, 1]$. Let $\Gamma = \Gamma_t(z)$ be, for every $z \in \partial D_t$, $t \in [0, T]$, a closed convex cone of vectors in $\mathbb{R}^d$ with the specific form $\{\lambda\gamma_t(z) : \lambda > 0\}$, for some $S_1(0)$-valued function $\gamma_t(z)$ which is uniformly continuous, in both space and time, and satisfies*

$$\beta = \inf_{t\in[0,T]} \inf_{z\in\partial D_t} \langle \gamma_t(z), n_t(z)\rangle > 0. \tag{A.5}$$

*Then $D$ satisfies (1.18) and (1.19) and*

$$l(r) \le L r^{\tilde{\alpha}} \qquad \textit{whenever } r \in [0, T] \tag{A.6}$$

*for some $0 < L < \infty$ and with $\tilde{\alpha} = (1 + \alpha)/2 \in (0, 1]$.*

PROOF. By the uniform continuity of $\gamma_t(z)$ in space and time, it is clear that the variation of $\gamma_t(z)$ can be made arbitrarily small on temporal neighborhoods. Following Proposition 2.5 in [15], it therefore immediately follows that criteria (1.18) and (1.19) are satisfied for some $0 < \rho_0 < r_0$ and $\eta_0 > 0$. Hence, it remains to prove (A.6) and we note that it suffices to prove (A.6) for small values of $r$. Let $z \in \mathbb{R}^d$ be arbitrary and let $z_s \in \partial D_s$ be such that $|z - z_s| = d(z, \partial D_s)$. We now claim that

$$z - z_s \parallel n_s(z_s) \tag{A.7}$$

or, in other words, that $z - z_s$ and $n_s(z_s)$ are parallel. To prove this claim, we can assume, without loss of generality, that $z_s = 0$. As $D$ is a time-dependent domain of class $\mathcal{H}_{1+\alpha}$, we can assume the existence of a function $\psi$, with property (A.4), such that $\{\psi(s, y) = 0 : y \in \overline{B_\rho(0)} \cap \partial D_s\}$, where $B_\rho(0)$ is a (spatial) neighborhood of the origin with the radius $\rho$ as given in the definition of $\mathcal{H}_{1+\alpha}$-domains. We consider the minimization problem

$$\min_{y\in \overline{B}_\rho(0)\cap\partial D_s} |z - y|^2. \tag{A.8}$$

Then, as the minimum in (A.8) is realized at the origin, we see that $z = \lambda\nabla\psi(s, 0)$ for some Lagrange multiplier $\lambda$. Obviously, this proves (A.7). Next, by Lemma 3.1, we see that, for $r < \rho \wedge r_0$,

$$l(r) = \sup_{\substack{s,t\in[0,T]\\|s-t|\le r}} \sup_{z\in\overline{D_s}} d(z, D_t) = \sup_{\substack{0\le s\le t\le T\\|s-t|\le r}} |z_t - z_s| \tag{A.9}$$

for some $z_s \in \partial D_s$, $z_t \in \partial D_t$ such that

$$z_t - z_s \parallel n_s(z_s) \tag{A.10}$$

and $|z_t - z_s| < \rho \wedge r_0$. Furthermore, employing once more the fact that $D$ is a time-dependent domain of class $\mathcal{H}_{1+\alpha}$, we conclude the existence of a function $\psi$, with the property $\psi(s, z_s) = \psi(t, z_t) = 0$, which is continuously differentiable in space. Taylor expanding $\psi$ up to the first order in spatial coordinates, we obtain, by the $\mathcal{H}_{1+\alpha}$-regularity of $\psi$,

$$\begin{aligned}\psi(t, z_t) - \psi(s, z_s) &= \psi(t, z_t) - \psi(s, z_t) + \psi(s, z_t) - \psi(s, z_s)\\ &= \psi(t, z_t) - \psi(s, z_t) + \langle z_t - z_s, \nabla_z\psi(s, z_s)\rangle\\ &\quad + \mathcal{O}(|z_t - z_s|^{1+\alpha}).\end{aligned} \tag{A.11}$$

As $n_s(z_s) = \frac{\nabla_z\psi(s,z_s)}{|\nabla_z\psi(s,z_s)|}$, it follows from (A.9)–(A.11) that

$$\begin{aligned} l(r) &= \sup_{\substack{0\le s\le t\le T\\|s-t|\le r}} |z_t - z_s| = \sup_{\substack{0\le s\le t\le T\\|s-t|\le r}} \frac{|\langle z_t - z_s, \nabla_z\psi(s, z_s)\rangle|}{|\nabla_z\psi(s, z_s)|}\\ &\le \sup_{\substack{0\le s\le t\le T\\|s-t|\le r}} \frac{|\psi(t, z_t) - \psi(s, z_t)|}{|\nabla_z\psi(s, z_s)|} + \sup_{\substack{0\le s\le t\le T\\|s-t|\le r}} \frac{\mathcal{O}(|z_t - z_s|^{1+\alpha})}{|\nabla_z\psi(s, z_s)|}.\end{aligned} \tag{A.12}$$

Furthermore, for small $r$, we can assume, without loss of generality, that for some $\delta > 0$ independent of $(s, z_s)$ we have $|\nabla_z\psi(s, z_s)| \ge \delta$. Hence, by the $\mathcal{H}_{1+\alpha}$-regularity of $\psi$ and the definition of $l$, we see that

$$l(r) \le c\delta^{-1}\bigl(r^{(\alpha+1)/2} + l(r)^{1+\alpha}\bigr), \tag{A.13}$$

provided $r < \rho \wedge r_0$. Finally, since $l(r) \to 0$ as $r \to 0$, there exists $\epsilon > 0$ such that if $r \le \epsilon$, then $c\delta^{-1}l(r)^\alpha \le 1/2$. Combining these facts, we conclude that

$$l(r) \le Lr^{(\alpha+1)/2} \tag{A.14}$$

for some constant $L$. Hence, (A.6) holds with $\tilde{\alpha} = (\alpha + 1)/2$. □

REMARK A.3. In the following let $\mathcal{C}_b^1$ and $\mathcal{C}_b^2$ be spaces containing all functions with bounded derivatives up to orders one and two, respectively. Consider a bounded spatial domain $\Omega \subset \mathbb{R}^d$ which is $\mathcal{C}_b^1$-smooth and satisfies a uniform exterior sphere condition. Moreover, assume that the cone of directions of reflection

$\Gamma(z)$ has the specific form $\{\lambda\gamma(z) : \lambda > 0\}$, for some $S_1(0)$-valued function $\gamma(z)$ which is continuous and satisfies

$$\beta := \inf_{z \in \partial\Omega} \langle \gamma(z), n(z) \rangle > 0. \tag{A.15}$$

Then Proposition 2.5 in [15] states that the time-independent counterparts of criteria (1.18) and (1.19) are satisfied. Furthermore, Theorem 4.5 in [15] states that the time-independent counterparts of (1.18) and (1.19) are also satisfied for piecewise $\mathcal{C}_b^1$-smooth domains $\Omega$ if the function $\gamma(z)$ is uniformly continuous on each face of $\partial\Omega$ and satisfies some nondegeneracy and consistency criteria. Finally, we also mention Theorem 4.6 in [15] which states that unique projections may be found if $\Omega$ is a piecewise $\mathcal{C}_b^2$-smooth domain and if $\gamma(z)$ is Lipschitz continuous on each face of $\partial\Omega$ and satisfies some other nondegeneracy and consistency criteria.

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