

BEST CONSTANTS IN ROSENTHAL-TYPE INEQUALITIES AND THE KRUGLOV OPERATOR¹

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Let X be a symmetric Banach function space on $[0, 1]$ with the Kruglov property, and let $\mathbf{f} = \{f_k\}_{k=1}^n$, $n \geq 1$ be an arbitrary sequence of independent random variables in X . This paper presents sharp estimates in the deterministic characterization of the quantities

$$\left\| \sum_{k=1}^n f_k \right\|_X, \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X, \quad 1 \leq p < \infty,$$

in terms of the sum of disjoint copies of individual terms of \mathbf{f} . Our method is novel and based on the important recent advances in the study of the Kruglov property through an operator approach made earlier by the authors. In particular, we discover that the sharp constants in the characterization above are equivalent to the norm of the Kruglov operator in X .

1. Introduction. For an arbitrary sequence $\mathbf{f} := \{f_k\}_{k=1}^n \subset L_1[0, 1]$ consider its disjointification, that is, the function

$$(1) \quad F(u) := \sum_{i=1}^n \bar{f}_i(u) \quad (u > 0),$$

where the sequence $\{\bar{f}_k\}_{k=1}^n$ is a disjointly supported sequence of equimeasurable copies of the individual elements from the sequence \mathbf{f} (we always assume here that $[0, 1]$ is equipped with the Lebesgue measure λ). Denote by F^* the decreasing rearrangement of $|F|$ (see relevant definitions in Section 2). Let X be a symmetric Banach function space on $[0, 1]$ for which there exists a universal constant $C_X > 0$ such that the inequality

$$(2) \quad \left\| \sum_{k=1}^n f_k \right\|_X \leq C_X (\|F^* \chi_{[0,1]}\|_X + \|F^* \chi_{[1,\infty]}\|_{L_1})$$

(resp.,

$$(3) \quad \left\| \sum_{k=1}^n f_k \right\|_X \leq C_X (\|F^* \chi_{[0,1]}\|_X + \|F^* \chi_{[1,\infty]}\|_{L_2}))$$

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holds for every sequence $\mathbf{f} \subset X$ of independent random variables (i.r.v.'s) (resp., of mean zero i.r.v.'s) and for every $n \in \mathbb{N}$. The inequalities above can be viewed as a fundamental generalization of the famous Khintchine inequality and, in the case when $X = L_p$ (resp., $X \supset L_p$), $1 \leq p < \infty$, they may be found in [23] (resp., [12]). The original proof of Rosenthal in [23], as well as a subsequent proof of a more general result by Burkholder in [7] yielded only constants C_{L_p} in (2) and (3) which grow exponentially in p , as $p \rightarrow \infty$. The sharp result that $C_{L_p} \asymp \frac{p}{\ln(p+1)}$, that is, there are universal constants $0 < \alpha < \beta < \infty$ such that the ratio between C_{L_p} and $\frac{p}{\ln(p+1)}$ lies in the interval $[\alpha, \beta]$ for all $p \in [1, \infty)$ was obtained in [13] (see also subsequent alternative proofs in [14, 18]).

The main purpose of this paper is to provide a sharp estimate on the constant C_X in (2) in the more general setting of symmetric spaces X . At the same time, we also believe that our methods shed additional light on the well-studied case $X = L_p$. Indeed, the methods exploited in [13, 14, 18] have a distinct L_p -flavor and do not appear to extend to other symmetric function spaces for which (2) and (3) hold. Our approach here is linked with the so-called Kruglov property (see the definition in Section 2.2). Consider the special case of Rosenthal's inequalities (2) and (3) when i.r.v.'s $f_k, k = 1, 2, \dots, n$, satisfy the additional assumption that

$$(4) \quad \sum_{k=1}^n \lambda(\{f_k \neq 0\}) \leq 1, \quad n \in \mathbb{N}$$

[in this case, the right-hand sides of (2) and (3) become equal]. In this special case, it was first established by Braverman [5] that if X is a symmetric space with the Fatou property (see Section 2.1 below), then X has the Kruglov property if and only if (2) holds. Recently, in [2–4] we have developed a novel approach to the study of spaces with the Kruglov property that involves defining a positive linear operator $K : L_1[0, 1] \rightarrow L_1[0, 1]$ (see details in Section 2.3) which is bounded in a symmetric function space X with the Fatou property if and only if X has the Kruglov property. Furthermore, we have shown in those papers that in this case the (Kruglov) operator K is bounded in X if and only if (2) and (3) hold in full generality. The following key fact is an immediate consequence of Prohorov's familiar inequality [22] (see also the proof of Theorem 3.5 from [2]): if X is a symmetric space, then for every sequence $\{f_k\}_{k=1}^\infty \subset L_1[0, 1]$ of symmetrically distributed i.r.v.'s satisfying assumption (4) we have

$$\left(\sum_{k=1}^n f_k \right)^* \leq 16K(F)^* \quad (n \in \mathbb{N}),$$

where the function F is defined by (1) (the definition of the decreasing rearrangement f^* of a measurable function f is given in the next section). This observation has naturally led us to the conjecture that the best constant C_X in (2) and (3) should be equivalent to the norm of the operator K in X . We prove this conjecture in Section 3 and present computations of the norm $\|K\|$ in various classes of symmetric

spaces in Section 4. In the case of L_p -spaces, $1 \leq p < \infty$, our results, of course, yield the same estimates as in [13, 14, 18]. In the case of symmetric Lorentz and Marcinkiewicz spaces (and other classes of symmetric spaces in which we are able to compute the norm of the operator K) our results are new and appear to be unattainable by methods used in [13, 14, 18]. In the final section of this paper we provide two complements to Rosenthal's inequality (2).

2. Preliminaries.

2.1. *Symmetric function spaces and interpolation of operators.* In this subsection we present some definitions from the theory of symmetric spaces and interpolation of operators. For more details on the latter theory we refer to [16, 20].

We will denote by $S(\Omega, \mathcal{P})$ the linear space of all measurable finite a.e. functions on a given measure space (Ω, \mathcal{P}) equipped with the topology of convergence locally in measure.

Let I denote either $[0, 1]$ or $(0, \infty)$ with Lebesgue measure λ . If $f \in S(I, \lambda)$ we denote by f^* the decreasing rearrangement of f , that is,

$$f^*(t) = \inf_{\lambda(A)=t} \sup_{s \in I \setminus A} |f(s)|.$$

A Banach function space X on I is said to be *symmetric* if the conditions $f \in X$ and $g^* \leq f^*$ imply that $g \in X$ and $\|g\|_X \leq \|f\|_X$. We will assume always the normalization that $\|\chi_{(0,1)}\|_X = 1$, where χ_A is the characteristic function of the set $A \subset I$. Let $\varphi_X(t) = \|\chi_{(0,t)}\|_X$ be the *fundamental function* of X . A symmetric space X is said to have *the Fatou property* if for every sequence $(f_n)_{n=1}^\infty \subset X$ of nonnegative functions such that $f_n \uparrow f$ a.e. and $\lim_{n \rightarrow \infty} \|f_n\|_X < \infty$ we have $f \in X$ and $\|f\|_X = \lim_{n \rightarrow \infty} \|f_n\|_X$.

Let us recall some classical examples of symmetric spaces on $[0, 1]$.

Let $M(t)$ be an increasing convex function on $[0, \infty)$ such that $M(0) = 0$. By L_M we denote *the Orlicz space* on $[0, 1]$ (see, e.g., [16, 20]) endowed with the norm

$$\|x\|_{L_M} = \inf \left\{ \lambda > 0 : \int_0^1 M(|x(t)|/\lambda) dt \leq 1 \right\}.$$

We suppose that ψ is a positive concave function on $[0, 1]$ with $\psi(0+) = 0$. *The Lorentz space* $\Lambda(\psi)$ is the space of all measurable functions f on the interval $[0, 1]$ such that

$$\|f\|_{\Lambda(\psi)} = \int_0^1 f^*(s) d\psi(s) < \infty.$$

The Marcinkiewicz space $M(\psi)$ is the space of all measurable functions f on the interval $[0, 1]$ such that

$$\|f\|_{M(\psi)} = \sup_{0 < t \leq 1} \frac{\psi(t)}{t} \int_0^t f^*(s) ds < \infty.$$

It is easy to check that $\varphi_{\Lambda(\psi)}(t) = \varphi_{M(\psi)}(t) = \psi(t)$. In this paper, we mainly work with the case when $\psi(t) = t^{1/p}$, $1 \leq p < \infty$.

Let $\vec{X} = (X_0, X_1)$ be a Banach couple and X be a Banach space such that $X_0 \cap X_1 \subseteq X \subseteq X_0 + X_1$. We say that X is an interpolation space between X_0 and X_1 if any bounded linear operator $A : X_0 + X_1 \rightarrow X_0 + X_1$ which maps X_i boundedly into X_i ($i = 0, 1$) also maps X boundedly into X . Then $\|A\|_{X \rightarrow X} \leq C(\|A\|_{X_0 \rightarrow X_0}, \|A\|_{X_1 \rightarrow X_1})$ for some $C \geq 1$. If the last inequality holds with $C = 1$ we will refer X to an 1 -interpolation space between X_0 and X_1 .

In what follows $\text{supp } f$ is the support of a function f defined on Ω , that is, $\text{supp } f := \{\omega \in \Omega : f(\omega) \neq 0\}$, \mathcal{F}_ξ is the distribution function of a random variable ξ , and $[z]$ is the integral part of a real number z .

2.2. The Kruglov property of symmetric function spaces. Let f be a measurable function (a random variable) on $[0, 1]$. By $\pi(f)$ we denote the random variable $\sum_{i=1}^N f_i$, where f_i 's are independent copies of f , and N is a Poisson random variable with parameter 1 independent of the sequence $\{f_i\}$.

DEFINITION. A symmetric function space X is said to have the Kruglov property if and only if $f \in X \iff \pi(f) \in X$.

This property has been studied by Braverman [5], using some probabilistic constructions of Kruglov [15] and by the authors in [2–4] via an operator approach. We refer to the latter papers for various equivalent characterizations of the Kruglov property. Note that only the implication $f \in X \implies \pi(f) \in X$ is nontrivial, since the implication $\pi(f) \in X \implies f \in X$ is always satisfied [5], page 11. Moreover, a symmetric space X has the Kruglov property if $X \supseteq L_p$ for some $p < \infty$ [5], Theorem 1.2, and [2], Corollaries 5.4, 5.6. At the same time, some exponential Orlicz spaces which do not contain L_q for any $q < \infty$ also possess this property (see [2, 5]).

2.3. The Kruglov operator in symmetric function spaces. Let $\{B_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint measurable subsets of $[0, 1]$ and let $\lambda(B_n) = \frac{1}{en!}$. If $f \in L_1[0, 1]$, then we set

$$(5) \quad Kf(\omega_0, \omega_1, \dots) = \sum_{n=1}^\infty \sum_{k=1}^n f(\omega_k) \chi_{B_n}(\omega_0).$$

Then $K : L_1[0, 1] \rightarrow L_1(\Omega, \mathcal{P})$ is a positive linear operator. Here $(\Omega, \mathcal{P}) = \prod_{n=0}^\infty ([0, 1], \lambda_n)$, where λ_n is the Lebesgue measure on $[0, 1]$.

For convenience, by Kf we also denote another random variable defined on $[0, 1]$ and having the same distribution as the variable introduced in (5). If $f \in L_1[0, 1]$, $\{B_n\}$ is the same sequence of subsets of $[0, 1]$ as above, and, for each $n \in \mathbb{N}$ $f_{n,1}, f_{n,2}, \dots, f_{n,n}$ and χ_{B_n} form a set of independent functions such that $f_{n,k}^* =$

f^* for every $k = 1, \dots, n$, then $Kf(t)$ is defined as the decreasing rearrangement of the function

$$(6) \quad \sum_{n=1}^{\infty} \sum_{k=1}^n f_{n,k}(t) \chi_{B_n}(t) \quad (0 \leq t \leq 1).$$

As we have pointed out above K is a linear operator from $L_1[0, 1]$ to $L_1(\Omega, \mathcal{P})$. By saying that K maps boundedly a symmetric space X on $[0, 1]$ into symmetric space Y , we mean that K is bounded as a linear mapping from $X[0, 1]$ into $Y(\Omega, \mathcal{P})$. The representation of Kf given by (6) allows us, without any ambiguity, also speak about K as a bounded map from $X[0, 1]$ into $Y[0, 1]$. A direct computation (see, e.g., [2]) yields the following equality for the characteristic function φ_{Kf} of Kf :

$$(7) \quad \begin{aligned} \varphi_{Kf}(t) &= \exp\left(\int_{-\infty}^{\infty} (e^{itx} - 1) d\mathcal{F}_f(x)\right) \\ &= \exp(\varphi_f(t) - 1) = \varphi_{\pi(f)}(t), \quad t \in \mathbb{R}. \end{aligned}$$

Therefore, $\mathcal{F}_{Kf} = \mathcal{F}_{\pi(f)}$, and we can treat Kf as an explicit representation of $\pi(f)$. In particular, a symmetric space X has the Kruglov property if and only if K is bounded in X .

It follows from the definition of the operator K that for any symmetric spaces X and Y $\|K\|_{X \rightarrow Y} \geq 1/e$ provided that $\|\chi_{[0,1]}\|_X = \|\chi_{[0,1]}\|_Y = 1$ (see also [5], page 11). It is shown in [2–4] that the operator K plays an important role in estimating the norm of sums of i.r.v.’s through the norm of sums of their disjoint copies. In particular, in [2] the well-known results of Johnson and Schechtman from [12] have been strengthened. In the next section, we shall improve the main results of [2] and explain the role of this improvement in obtaining sharp constants in Rosenthal-type inequalities studied earlier in some special cases [12, 13, 18]. Subsequent sections contain explicit computation of the norm of the operator K in various classes of symmetric spaces X and further modifications of Rosenthal’s inequality (2).

3. Kruglov operator and Rosenthal’s inequalities. The main objective of the present section is the strengthening of [2], Theorem 3.5. For an arbitrary symmetric space X on $[0, 1]$ and an arbitrary $p \in [1, \infty]$, we defined in [2] a function space Z_X^p on $[0, \infty)$ by

$$Z_X^p := \{f \in L_1[0, \infty) + L_\infty[0, \infty) : \|f\|'_{Z_X^p} < \infty\},$$

where

$$\|f\|'_{Z_X^p} := \|f^* \chi_{[0,1]}\|_X + \|f^* \chi_{[1,\infty)}\|_p \asymp \|f^* \chi_{[0,1]}\|_X + \left(\sum_{k=1}^{\infty} f^*(k)^p\right)^{1/p}.$$

Clearly, $\|\cdot\|'_{Z^p_X}$ is a quasi-norm. It is easy to see that Z^p_X equipped with the equivalent norm

$$\|f\|_{Z^p_X} := \|f^* \chi_{[0,1]}\|_X + \|f\|_{(L_1+L_p)(0,\infty)}, \quad f \in Z^p_X,$$

is a symmetric space on $[0, \infty)$. The spaces Z^2_X were introduced in [11], and the spaces Z^1_X and Z^2_X were used in [12] in the study of Rosenthal-type inequalities. Following [20], page 46, we define the space $\widetilde{X}(l_p)$ as the set of all sequences $\mathbf{f} = \{f_k(\cdot)\}_{k=1}^\infty$, $f_k \in X$ ($k \geq 1$) such that

$$\|\mathbf{f}\|_{\widetilde{X}(l_p)} := \sup_{n=1,2,\dots} \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X < \infty$$

(with an obvious modification for $p = \infty$). The closed subspace of $\widetilde{X}(l_p)$ generated by all eventually vanishing sequences $\mathbf{f} \in \widetilde{X}(l_p)$ is denoted by $X(l_p)$.

Let X and Y be symmetric spaces on $[0, 1]$ such that $X \subseteq Y$. The main focus of [2–4] is on inequalities of the type

$$(8) \quad \left\| \sum_{i=1}^n f_i \right\|_Y \leq C \left\| \sum_{i=1}^n \tilde{f}_i \right\|_{Z_X}$$

and

$$(9) \quad \|\mathbf{f}\|_{Y(l_p)} \leq C \left\| \sum_{i=1}^n \tilde{f}_i \right\|_{Z^p_X},$$

where the sequence $\mathbf{f} := \{f_k\}_{k=1}^\infty \subset X$ consists of i.r.v.'s, and the sequence $\{\tilde{f}_k\}_{k=1}^\infty$ is a disjointly supported sequence of equimeasurable copies of the elements from the sequence \mathbf{f} .

Our first result in this section strengthens [2], Theorems 3.5 and 6.1, by establishing sharp estimates on the constant C in (8). Let F be the disjointification function related to the sequence $\mathbf{f} := \{f_k\}_{k=1}^\infty$ [see (1)].

THEOREM 1. *Let X and Y be symmetric spaces on $[0, 1]$ such that $X \subseteq Y$ and Y has the Fatou property.*

(i) *If there exists a constant C such that the estimate*

$$(10) \quad \left\| \sum_{i=1}^n f_i \right\|_Y \leq C \|F\|_{Z_X} \quad \left(= C \left\| \sum_{i=1}^n \tilde{f}_i \right\|_X \right),$$

holds for every sequence $\{f_k\}_{k=1}^\infty \subset X$ of i.r.v.'s satisfying the assumption (4) for all $n \in \mathbb{N}$, then the operator K acts boundedly from X into Y and $\|K\|_{X \rightarrow Y} \leq C$.

(ii) If the operator K acts boundedly from X into Y , then for every sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent random variables, we have

$$(11) \quad \left\| \sum_{i=1}^n f_i \right\|_Y \leq \alpha \|K\|_{X \rightarrow Y} \|F\|_{Z_X},$$

where $\alpha > 0$ is a universal constant which does not depend on X and Y .

PROOF. (i) The claim follows from the inspection of the first part of the proof of [2], Theorem 3.5. For the convenience of the reader, we include details of the argument. Fix $f \in X$ and $n \in \mathbb{N}$ and choose $h \in X$ such that $\mathcal{F}_h = \mathcal{F}_f$ and such that h and $\chi_{[0,1/n]}$ are independent. Set $h_n := h\chi_{[0,1/n]}$, and let $\{\chi_{[0,1/n]}, h_{n,k}\}_{k=1}^n$ be a set of $(n + 1)$ independent random variables such that $\mathcal{F}_{h_{n,k}} = \mathcal{F}_{h_n}$ for all $1 \leq k \leq n$. Since the functions $|\sum_{k=1}^n \bar{h}_{n,k}|$ and $|h|$ have the same distribution function, we conclude that the functions $|\sum_{k=1}^n \bar{h}_{n,k}|$ and $|f|$ are equidistributed. Observing now that the sequence $\{h_{n,k}\}_{k=1}^n$ satisfies (4), we obtain

$$(12) \quad \left\| \sum_{k=1}^n h_{n,k} \right\|_Y \leq C \left\| \sum_{k=1}^n \bar{h}_{n,k} \right\|_X = C \|f\|_X.$$

A direct computation shows that $\varphi_{h_n}(t) = n^{-1}\varphi_f(t) + (1 - n^{-1})$ for all $t \in \mathbb{R}$. Hence, the characteristic function of the sum $H_n := \sum_{k=1}^n h_{n,k}$ is given by

$$\varphi_{H_n}(t) = (n^{-1}(\varphi_f(t) - 1) + 1)^n \quad \forall t \in \mathbb{R}.$$

Since $\lim_{n \rightarrow \infty} \varphi_{H_n}(t) = \exp(\varphi_f(t) - 1) = \varphi_{\pi(f)}(t)$, for all $t \in \mathbb{R}$, we see that H_n converges weakly to Kf . Combining this with (12), (7), [5], Proposition 1.5, and with the fact that Y has the Fatou property, we conclude that

$$\|Kf\|_Y \leq C \|f\|_X.$$

This completes the proof of the first assertion.

(ii) Firstly, let us assume that a sequence $\{f_k\}_{k=1}^\infty \subset X$ consists of independent random variables satisfying assumption (4). Denote by $\{h_k\}_{k=1}^n$ a sequence of independent random variables such that $\mathcal{F}_{h_k} = \mathcal{F}_{\pi(f_k)}$ for all $k = 1, 2, \dots$. Consider the following two cases.

(a) f_k 's are symmetrically distributed r.v.'s. In [22], Prokhorov proved that in this case we have

$$\lambda \left\{ \left| \sum_{k=1}^n f_k \right| \geq x \right\} \leq 8\lambda \left\{ \left| \sum_{k=1}^n h_k \right| \geq \frac{x}{2} \right\} \quad (x > 0).$$

From this inequality (see, e.g., [16], Corollary II.4.2) it follows that

$$(13) \quad \left\| \sum_{k=1}^n f_k \right\|_Y \leq 16 \left\| \sum_{k=1}^n h_k \right\|_Y.$$

(b) $f_k = a_k \chi_{A_k}$, where $a_k \geq 0$ and A_k are arbitrary independent subsets of $[0, 1]$ ($k = 1, 2, \dots$). Without loss of generality, we may assume that f_k 's are defined on the measure space $\prod_{n=0}^{\infty}([0, 1], \lambda_n)$ by the formula $f_k(t) = a_k \chi_{[0, p_k]}(t_k)$, where $p_k = \lambda(A_k)$, $k \geq 1$. From the definition of the Kruglov operator [see (6)] it follows that $\lambda(\{Kf_k = a_k\}) = p_k/e \geq p_k/3$. Hence, by (7), we may assume that $h_k \geq a_k \chi_{[0, p_k/3]}$, and so

$$(14) \quad \left\| \sum_{k=1}^n f_k \right\|_Y \leq 3 \left\| \sum_{k=1}^n a_k \chi_{[0, p_k/3]}(t_k) \right\|_Y \leq 3 \left\| \sum_{k=1}^n h_k \right\|_Y.$$

Next, from $\bar{f}_k \bar{f}_m = 0$ ($k \neq m$) it follows that

$$e^{itF} - 1 = \sum_{k=1}^n (e^{it\bar{f}_k} - 1).$$

Therefore,

$$\varphi_F(t) - 1 = \int (e^{itF} - 1) = \sum_{k=1}^n \int (e^{itf_k} - 1) = \sum_{k=1}^n (\varphi_{f_k}(t) - 1)$$

and

$$\varphi_{K(F)} = \exp(\varphi_F - 1) = \prod_{k=1}^n \exp(\varphi_{f_k} - 1) = \prod_{k=1}^n \varphi_{h_k} = \varphi_{\sum_{k=1}^n h_k}.$$

Thus, the sum $\sum_{k=1}^n h_k$ is equidistributed with $K(F)$. Therefore, by inequalities (13) and (14), we have

$$(15) \quad \left\| \sum_{k=1}^n f_k \right\|_Y \leq 16 \left\| \sum_{k=1}^n h_k \right\|_Y = 16 \|K(F)\|_Y \leq 16 \|K\|_{X \rightarrow Y} \|F\|_X$$

in the case (a) and analogously

$$(16) \quad \left\| \sum_{k=1}^n f_k \right\|_Y \leq 3 \|K\|_{X \rightarrow Y} \|F\|_X$$

in the case (b).

Now, we consider the case when a sequence $\{f_k\}_{k=1}^{\infty} \subset X$ consists of mean zero i.r.v.'s satisfying assumption (4) with $1/2$ instead of 1 . We shall use the standard "symmetrization trick." Let $\{f'_k\}_{k=1}^n$ be a sequence of i.r.v.'s such that the sequence $\{f_k, f'_k\}_{k=1}^n$ consists of independent random variables and $\mathcal{F}_{f_k} = \mathcal{F}_{f'_k}$ for all $k \geq 1$.

Setting $g_k := f_k - f'_k$, we obtain a sequence $\{g_k\}_{k=1}^n$ of symmetrically distributed independent random variables satisfying assumption (4), and, by (15), we have

$$(17) \quad \left\| \sum_{k=1}^n g_k \right\|_Y \leq 16 \|K\|_{X \rightarrow Y} \|G\|_X,$$

where $G := \sum_{k=1}^n \bar{g}_k$. Let \mathcal{B} be the σ -subalgebra generated by the sequence $\{f_k\}_{k=1}^n$ and let $E_{\mathcal{B}}$ be the corresponding conditional expectation operator. Thanks to our assumption Y is an 1-interpolation space for the couple (L_1, L_∞) . Hence, since $E_{\mathcal{B}}$ is bounded in L_1 and L_∞ (with constant 1) [20], Theorem 2.a.4, we have $\|E_{\mathcal{B}}\|_{Y \rightarrow Y} = 1$. Therefore, due to the independence of f_k 's and f_k' 's, we have

$$(18) \quad \left\| \sum_{k=1}^n f_k \right\|_Y = \left\| E_{\mathcal{B}} \left(\sum_{k=1}^n g_k \right) \right\|_Y \leq \left\| \sum_{k=1}^n g_k \right\|_Y.$$

On the other hand, it is obvious that $\|G\|_X \leq 2\|F\|_X$. Combining (18) and (17), we see that (11) holds with $\alpha = 32$.

Next, let us consider an arbitrary sequence $\{f_k\}_{k=1}^\infty \subset X$ of i.r.v.'s satisfying assumption (4) with 1/2 instead of 1. In this case, set $u_k = f_k - v_k$, where

$$v_k := \frac{1}{\lambda(\text{supp } f_k)} \int_0^1 f_k(t) dt \cdot \chi_{\text{supp } f_k} \quad (k \geq 1).$$

Clearly, $\{u_k\}_{k \geq 1}$ is a mean zero sequence of i.r.v.'s satisfying assumption (4) with 1/2 instead of 1. Thus, the preceding argument yields

$$(19) \quad \left\| \sum_{k=1}^n u_k \right\|_Y \leq 32 \|K\|_{X \rightarrow Y} \|U\|_X,$$

where $U := \sum_{k=1}^n \bar{u}_k$. Moreover, by (16), we have

$$\left\| \sum_{k=1}^n f_k \right\|_Y \leq \left\| \sum_{k=1}^n u_k \right\|_Y + \left\| \sum_{k=1}^n v_k \right\|_Y \leq 32 \|K\|_{X \rightarrow Y} \|U\|_X + 3 \|K\|_{X \rightarrow Y} \|V\|_X.$$

Let \mathcal{C} be the σ -algebra generated by the supports of \bar{f}_k 's. It is clear that $V = E_{\mathcal{C}}(F)$, where $V := \sum_{k=1}^n \bar{v}_k$. Therefore, as above, we have $\|V\|_X \leq \|F\|_X$. Moreover, since $U = F - V$, we also have $\|U\|_X \leq \|F\|_X + \|V\|_X \leq 2\|F\|_X$. Thus,

$$\left\| \sum_{k=1}^n f_k \right\|_Y \leq 67 \|K\|_{X \rightarrow Y} \|F\|_X.$$

If $\{f_k\}_{k=1}^\infty \subset X$ is an arbitrary sequence of i.r.v.'s satisfying assumption (4), then we may represent $f_k = f'_k + f''_k$, where each of the sequences $\{f'_k\}_{k=1}^\infty$ and $\{f''_k\}_{k=1}^\infty$ consists of i.r.v.'s satisfying assumption (4) with 1/2 instead of 1 and moreover $|F'| \leq |F|$ and $|F''| \leq |F|$. In this case, using the preceding formulas, we obtain that

$$\left\| \sum_{k=1}^n f_k \right\|_Y \leq \left\| \sum_{k=1}^n f'_k \right\|_Y + \left\| \sum_{k=1}^n f''_k \right\|_Y \leq 134 \|K\|_{X \rightarrow Y} \|F\|_X.$$

Finally, repeating verbatim the proof of [2], Theorem 6.1, we obtain (11) for arbitrary sequences of i.r.v.'s [which do not necessarily satisfy assumption (4)] as a corollary of already considered special case when (4) holds. \square

Our next result strengthens [2], Theorem 6.7, by establishing sharp estimates on the constant C in (9). Before proceeding, we recall the following construction due to Calderon [8]. Let X_0 and X_1 be two Banach lattices of measurable functions on the same measure space (\mathcal{M}, m) and let $\theta \in (0, 1)$. The space $X_0^{1-\theta} X_1^\theta$ consists of all measurable functions f on (\mathcal{M}, m) such that for some $\lambda > 0$ and $f_i \in X_i$ with $\|f_i\|_{X_i} \leq 1, i = 0, 1$, we have

$$|f(x)| \leq \lambda |f_0(x)|^{1-\theta} |f_1(x)|^\theta, \quad x \in \mathcal{M}.$$

This space is equipped with the norm given by the greatest lower bound of all numbers λ taken over all possible such representations. Even though this construction is not an interpolation functor on general couples of Banach lattices (see [19]), it is still a convenient tool of interpolation theory. Indeed, if (X_0, X_1) is a Banach couple and if (Y_0, Y_1) is another Banach couple of lattices of measurable functions on some measure space (\mathcal{M}', m') , then any positive operator A from $S(\mathcal{M}, m)$ into $S(\mathcal{M}', m')$, which acts boundedly from the couple (X_0, X_1) into the couple (Y_0, Y_1) also maps boundedly $X_0^{1-\theta} X_1^\theta$ into $Y_0^{1-\theta} Y_1^\theta$ and, in addition, $\|A\|_{X_0^{1-\theta} X_1^\theta \rightarrow Y_0^{1-\theta} Y_1^\theta} \leq \|A\|_{X_0 \rightarrow Y_0}^{1-\theta} \|A\|_{X_1 \rightarrow Y_1}^\theta$ for all $\theta \in (0, 1)$. The proof of the latter claim follows by inspection of the standard arguments from [20], Proposition 1.d.2(i), page 43.

THEOREM 2. *Let X and Y be symmetric spaces on $[0, 1]$ such that $X \subseteq Y$. If K acts boundedly from X into Y , then there exists a universal constant $\beta > 0$ such that for every sequence $\{f_k\}_{k=1}^\infty \subset X$ of i.r.v.'s and for every $q \in [1, \infty]$, we have*

$$\|\mathbf{f}\|_{Y(l_q)} \leq \beta \|K\|_{X \rightarrow Y}^{1/q} \|F\|_{Z_X^q}.$$

PROOF. The case $q = 1$ has been treated in Theorem 1. If $q = \infty$, then it is sufficient to observe that

$$\|F^*\|_{Z_X^\infty} \asymp \|F^* \chi_{[0,1]}\|_X,$$

and that (see, e.g., [10], Proposition 2.1)

$$(20) \quad \frac{1}{2} \lambda \{F^* \chi_{[0,1]} > \tau\} \leq \lambda \left\{ \sup_{k=1,2,\dots} |f_k| > \tau \right\} \leq \lambda \{F^* \chi_{[0,1]} > \tau\} \quad (\tau > 0).$$

Therefore, for some constant γ (which does not depend on X and Y) we have

$$\|\mathbf{f}\|_{Y(l_\infty)} \leq \|\mathbf{f}\|_{X(l_\infty)} \leq \gamma \|F\|_{Z_X^\infty}.$$

The rest of the proof is based on methods from interpolation theory and is very similar to the arguments in [2], Theorem 6.7. Let $\delta: (\Omega, \mathcal{P}) \rightarrow ([0, 1], \lambda)$ be a measure preserving isomorphism, where $(\Omega, \mathcal{P}) := \prod_{k=0}^\infty ([0, 1], \lambda_k)$ (here, λ_k is the Lebesgue measure on $[0, 1]$ for every $k \geq 0$). For every $g \in S(\Omega, \mathcal{P})$, we set

$T(g)(x) := g(\delta^{-1}x)$, $x \in [0, 1]$. Note that T is a rearrangement-preserving mapping between $S(\Omega, \mathcal{P})$ and $S([0, 1], \lambda)$. We define the positive linear mapping Q from $S(0, \infty)$ into $S(\Omega, \mathcal{P})^{\mathbb{N} \cup \{0\}}$ by setting

$$Qf(\omega_0, \omega_1, \dots) := \{(Qf)_k\}_{k=0}^\infty, \quad f \in S(0, \infty),$$

where $(Qf)_k(\omega_0, \omega_1, \dots) := f(\omega_k + k)$ ($\omega_k \in [0, 1]$), $k \geq 0$. The arguments above show that the positive operator $Q'f := \{T(Qf)_k\}_{k=0}^\infty$ is bounded from Z_X^1 into $Y(l_1)$ with the norm not exceeding $\alpha \|K\|_{X \rightarrow Y}$ (where α is a universal constant) and also from Z_X^∞ into $Y(l_\infty)$ with the norm not exceeding γ . It follows that

$$Q' : (Z_X^1)^{1-\theta} (Z_X^\infty)^\theta \rightarrow (Y(l_1))^{1-\theta} (Y(l_\infty))^\theta$$

and

$$\|Q'\| \leq \|Q'\|_{Z_X^1 \rightarrow Y(l_1)}^{1-\theta} \cdot \|Q'\|_{Z_X^\infty \rightarrow Y(l_\infty)}^\theta, \quad \theta \in (0, 1).$$

Now, we shall use the following facts: for every $\theta \in (0, 1)$ we have

$$Z_X^q \subseteq (Z_X^1)^{1-\theta} (Z_X^\infty)^\theta, \quad (Y(l_1))^{1-\theta} (Y(l_\infty))^\theta \subseteq Y(l_q), \quad q = \frac{1}{1-\theta}.$$

To see the first embedding above, fix $g = g^* \in Z_X^q$, $\|g\|_{Z_X^q} = 1$, and set

$$g_1 := g\chi_{[0,1]} + g^q\chi_{[1,\infty)}, \quad g_\infty := g\chi_{[0,1]} + \chi_{[1,\infty)}.$$

Clearly, $g = (g_1)^{1-\theta} (g_\infty)^\theta$. Moreover, since $g(1) \leq 1$, then g_1 decreases, which implies that $g_i \in Z_X^i$ and $\|g_i\|_{Z_X^i} \leq 3$ ($i = 1, \infty$). The second embedding above (in fact, equality) is established in [6], Theorem 3. Now, we are in a position to conclude that

$$\|Q'\|_{Z_X^q \rightarrow Y(l_q)} \leq \alpha^{1/q} \|K\|_{X \rightarrow Y}^{1/q} \cdot \gamma^{1-1/q} \leq \max(1, \alpha, \gamma) \|K\|_{X \rightarrow Y}^{1/q}.$$

The proof is completed by noting that every sequence $\{f_k\}_{k=1}^\infty \subset X$ of i.r.v.'s may be represented in the form $\{f_k\} = Q'(\tilde{F})$, with some function \tilde{F} which is equidistributed with F . \square

The results presented in this section show that the sharp estimates in the deterministic estimates of expressions

$$\left\| \sum_{k=1}^n f_k \right\|_X, \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_X, \quad 1 \leq p < \infty,$$

in terms of the sum of disjoint copies of individual terms of \mathbf{f} are fully determined by the norm of the Kruglov operator K in X . In the next section, we shall present sharp estimates of this norm in a number of important cases, including the case $X = L_p$ ($1 \leq p < \infty$) studied earlier in [13, 18] and [14] by completely different methods. It does not seem that the methods used in those papers can be extended outside of the L_p -scale.

4. Norm of the Kruglov operator and sharp constants in Rosenthal's inequalities. Recall that a Banach lattice X is said to satisfy an *upper p -estimate*, if there exists a constant $C > 0$ such that for every finite sequence $(x_j)_{j=1}^n \subseteq X$ of pairwise disjoint elements,

$$\left\| \sum_{j=1}^n x_j \right\|_X \leq C \left(\sum_{j=1}^n \|x_j\|_X^p \right)^{1/p}.$$

Recall also that if $\tau > 0$, the *dilation operator* σ_τ is defined by setting

$$(\sigma_\tau x)(s) = \begin{cases} x(s/\tau), & s \leq \min\{1, \tau\}, \\ 0, & \tau < s \leq 1. \end{cases}$$

The operator σ_τ , $\tau > 0$ acts boundedly in every symmetric function space X [16], Theorem II.4.4.

First, we suppose that $1 \leq p < \infty$ and that X is a symmetric space satisfying the following two conditions:

- (i) X satisfies an upper p -estimate;
- (ii) $\|\sigma_t\|_{X \rightarrow X} \leq Ct^{1/p}$, $0 < t < 1$.

Conditions (i) and (ii) imply, in particular, that both Boyd indices of X (see, e.g., [20]) are equal to p .

The value of the constant C varies from line to line in this section.

PROPOSITION 3. *If a symmetric space X satisfies the above assumptions, then there exists a universal constant $\alpha > 0$ whose value depends only on the constants in (i) and (ii) above such that*

$$(21) \quad \|K\|_{X \rightarrow X} \leq \alpha \frac{p}{\ln(p+1)}, \quad p \geq 1.$$

PROOF. Let $0 \leq f \in X$, $p \geq 1$, and $n \in \mathbb{N}$. Let $f_{n,1}, f_{n,2}, \dots, f_{n,n}$ and χ_{B_n} have the same meaning as in Section 2.3. Then, by (6), the random variable Kf is equimeasurable with the random variable

$$\sum_{n=1}^{\infty} g_n \chi_{B_n} \quad \text{where } g_n = \sum_{k=1}^n f_{n,k} \quad (n = 1, 2, \dots).$$

Since g_n and χ_{B_n} are independent, then the assumptions on X imply

$$\begin{aligned} \|Kf\|_X &\leq C \left(\sum_{n=1}^{\infty} \|g_n \chi_{B_n}\|_X^p \right)^{1/p} = C \left(\sum_{n=1}^{\infty} \|\sigma_{\lambda(B_n)} g_n\|_X^p \right)^{1/p} \\ &\leq C \left(\sum_{n=1}^{\infty} \frac{1}{e \cdot n!} \|g_n\|_X^p \right)^{1/p} \leq C \left(\sum_{n=1}^{\infty} \frac{n^p}{n!} \right)^{1/p} \|f\|_X. \end{aligned}$$

It is clear that

$$\sum_{n=1}^{\infty} \frac{n^p}{n!} \leq \sup_n \frac{n^p}{p^n} \sum_{n=1}^{\infty} \frac{p^n}{n!} = e^p \sup_n \frac{n^p}{p^n}.$$

The function x^p/p^x takes its maximal value at $x = p/\log(p)$ and this maximum does not exceed $(p/\log(p))^p$. Therefore,

$$\left(\sum_{n=1}^{\infty} \frac{n^p}{n!}\right)^{1/p} \leq e \cdot \frac{p}{\log(p)}.$$

On the other side, if $1 \leq p < 2$, then

$$\left(\sum_{n=1}^{\infty} \frac{n^p}{n!}\right)^{1/p} \leq \sum_{n=1}^{\infty} \frac{n^2}{n!} = 2e. \quad \square$$

It is well known that an L_p -space, $1 \leq p < \infty$, satisfies an upper p -estimate and that $\|\sigma_t\|_{L_p \rightarrow L_p} = t^{1/p}$, $0 < t \leq 1$. The facts that similarly $M(t^{1/p})$, $1 < p < \infty$, satisfies an upper p -estimate and $\|\sigma_t\|_{M(t^{1/p}) \rightarrow M(t^{1/p})} \leq t^{1/p}$, $0 < t < 1$, follow from a combination of [9], Theorem 3.4(a)(i), and [20], Proposition 1.f.5, and from [16], Chapter II, Theorem 4.4. Combining these facts with Proposition 3, we obtain the following corollary.

COROLLARY 4. *There exists a constant $C > 0$ such that for all $p \geq 1$*

$$(22) \quad \|K\|_{L_p \rightarrow L_p} \leq \frac{Cp}{\ln(p+1)} \quad \text{and} \quad \|K\|_{M(t^{1/p}) \rightarrow M(t^{1/p})} \leq \frac{Cp}{\ln(p+1)}.$$

Although the Lorentz space $\Lambda(t^{1/p})$, $1 < p < \infty$, does not satisfy the assumptions of Proposition 3, nevertheless estimates similar to (22) also hold for the norm of the operator $K : \Lambda(t^{1/p}) \rightarrow \Lambda(t^{1/p})$, $1 \leq p < \infty$. The proof below is based on the properties of the Kruglov operator K in Lorentz spaces exposed in [2], Section 5.

PROPOSITION 5. *There exists a constant $C > 0$ such that for all $p \geq 1$*

$$(23) \quad \|K\|_{\Lambda(t^{1/p}) \rightarrow \Lambda(t^{1/p})} \leq \frac{Cp}{\ln(p+1)}, \quad p \geq 1.$$

PROOF. By [2], Theorem 5.1, we have

$$\|K\|_{\Lambda(\psi) \rightarrow \Lambda(\psi)} \leq 2 \sup_{u \in (0,1)} \frac{1}{\psi(u)} \sum_{k=1}^{\infty} \psi\left(\frac{u^k}{k!}\right).$$

If $\psi(t) = t^{1/p}$, then the latter supremum is equal to

$$\sum_{n=1}^{\infty} \left(\frac{1}{n!}\right)^{1/p} \leq 2 + \sum_{n=3}^{\infty} \left(\frac{1}{[n/3]}\right)^{3[n/3]/p} = 2 + 3 \sum_{k=1}^{\infty} e^{-3k \log(k)/p}.$$

Since the latter sequence decreases, we can replace sum with an integral and obtain

$$\|K\|_{\Lambda_{t^{1/p}} \rightarrow \Lambda_{t^{1/p}}} \leq 22 + 6 \int_e^{\infty} e^{-3s \log(s)/p} ds.$$

Substitute $t = s \log(s)$. It follows that $\frac{dt}{ds} = 1 + \log(s) \geq \frac{1}{2} \log(t)$. Therefore,

$$\|K\|_{\Lambda_{t^{1/p}} \rightarrow \Lambda_{t^{1/p}}} \leq 22 + 12 \int_e^{\infty} \frac{1}{\log(t)} e^{-3t/p} dt.$$

It is clear that

$$\int_e^p \frac{1}{\log(t)} e^{-3t/p} dt \leq \int_e^p \frac{1}{\log(t)} dt \leq \text{const} \cdot \frac{p}{\log(p)}.$$

On the other side,

$$\int_p^{\infty} \frac{1}{\log(t)} e^{-3t/p} dt \leq \frac{1}{\log(p)} \int_p^{\infty} e^{-3t/p} dt \leq \frac{p}{3 \log(p)}.$$

It follows that

$$\|K\|_{\Lambda_{t^{1/p}} \rightarrow \Lambda_{t^{1/p}}} \leq \text{const} \cdot \frac{p}{\log(p)}. \quad \square$$

We shall now estimate the norm of the operator $K : \Lambda(t^{1/p}) \rightarrow M(t^{1/p})$, $1 < p < \infty$.

LEMMA 6.

$$(24) \quad \|K\|_{\Lambda(t^{1/p}) \rightarrow M(t^{1/p})} \asymp \frac{p}{\ln(p+1)}, \quad 1 < p < \infty.$$

PROOF. Since $\Lambda(t^{1/p}) \subset L_p \subset M(t^{1/p})$ for all $1 < p < \infty$, the estimate from the above in (24) follows immediately from Corollary 4. Let us now concentrate on the converse inequality. Since $\|K1\|_{M(t^{1/p})} \geq 1 - 1/e$ for all $p \geq 1$, then it suffices to establish the estimate from below for sufficiently large p 's.

By [2], Remark 5.2, we have

$$(25) \quad \begin{aligned} \|K\|_{\Lambda(t^{1/p}) \rightarrow M(t^{1/p})} &\geq \frac{1}{e} \sup_{u \in (0,1], k \in \mathbb{N}} \frac{k(u^k/k!)^{1/p}}{u^{1/p}} = \frac{1}{e} \sup_{k \in \mathbb{N}} \frac{k}{(k!)^{1/p}} \\ &\geq \frac{1}{e} \sup_{x \in \mathbb{R}} \frac{x}{x^{x/p}}. \end{aligned}$$

Substitute $x = p/\log(p)$. We obtain

$$\|K\|_{\Lambda(t^{1/p}) \rightarrow M(t^{1/p})} \geq \frac{1}{e} (p/\log(p))^{1-1/\log(p)} \geq \frac{1}{e} \frac{p}{\log(p)}. \quad \square$$

The first main result of the present section is the following theorem and its corollary.

THEOREM 7. *We have*

$$(26) \quad \begin{aligned} \|K\|_{L_p \rightarrow L_p} &\asymp \|K\|_{\Lambda(t^{1/p}) \rightarrow \Lambda(t^{1/p})} \asymp \|K\|_{M(t^{1/p}) \rightarrow M(t^{1/p})} \\ &\asymp \frac{p}{\ln(p+1)}, \quad p > 1, \end{aligned}$$

with universal constants.

PROOF. The proof follows from combining Proposition 5 and Corollary 4 with Lemma 6. \square

COROLLARY 8. *The order of the constant $\frac{\alpha p}{\ln(p+1)}$ in Rosenthal’s inequality (2) with $X = L_p$ is optimal when $p \rightarrow \infty$.*

PROOF. Apply Theorem 1 to the case $X = Y = L_p$ and then apply Theorem 7. \square

REMARK 9. The same argument as above also shows that the order of the constant $\frac{\alpha p}{\ln(p+1)}$ is optimal in variants of Rosenthal’s inequality (2) for scales of Lorentz spaces $\Lambda(t^{1/p})$ ($1 < p < \infty$) and Marcinkiewicz spaces $M(t^{1/p})$ ($1 < p < \infty$).

REMARK 10. Earlier the result presented in Corollary 8 was established in [13, 18] and [14] by completely different methods. Our approach here shows that the order of the constant whether in the special L_p -case studied in the papers just cited, or in a more general case of various scales of symmetric spaces (as indicated in the preceding remark) is fully determined by the norm of the Kruglov operator.

REMARK 11. It is shown in [13], Theorem 4.1 and Proposition 4.3, that the constant $\frac{\alpha p}{1+\ln(p)}$, where α is an absolute constant, is also sharp in order when $p \rightarrow \infty$ in Rosenthal’s inequality (3) for mean zero i.r.v.’s. More precisely, using our notation, it is proved there that for any such sequence $\{f_k\}_{k=1}^n \subset L_p$ we have

$$(27) \quad \left\| \sum_{k=1}^n f_k \right\|_p \leq \frac{\alpha p}{1 + \ln(p)} \left(\|F^* \chi_{[0,1]}\|_p + \left(\sum_{k=1}^n F^*(k)^2 \right)^{1/2} \right).$$

Furthermore, if c_p is the least constant in similar inequality which would hold for any sequence $\{f_k\}_{k=1}^n \subset L_p$ of symmetrically and identically distributed i.r.v.'s, then $c_p \geq \frac{p}{\sqrt{2e(1+\ln(p))}}$. Without going into precise details, we observe that a careful inspection of the proof of [4], Theorem 3.1, shows that a similar result holds also in the case of an arbitrary symmetric space X with Kruglov's property (with an obvious replacement of the constant $\frac{\alpha p}{1+\ln(p)}$ with the constant $\|K\|_{X \rightarrow X}$). We state this result in full.

PROPOSITION 12. *There exists an absolute constant α such that if the Kruglov operator K acts boundedly in a symmetric space X , then for any $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of mean zero i.r.v.'s we have the following sharp estimate:*

$$(28) \quad \left\| \sum_{k=1}^n f_k \right\|_X \leq \alpha \|K\|_{X \rightarrow X} \left(\|F^* \chi_{[0,1]}\|_X + \left(\sum_{k=1}^n F^*(k)^2 \right)^{1/2} \right).$$

The second main result of this section yields optimal (in order) constant in a somewhat more general setting recently studied in [14, 21].

THEOREM 13. *If $p, q \in [1, \infty)$, then for an arbitrary sequence $\{f_k\}_{k=1}^\infty \subset L_p$ of i.r.v.'s we have*

$$(29) \quad \|\mathbf{f}\|_{l_q} \|L_p\| \leq \alpha \left(\frac{p}{\ln(p+1)} \right)^{1/q} \left(\|F^* \chi_{[0,1]}\|_{L_p} + \left(\sum_{k=1}^\infty F^*(k)^q \right)^{1/q} \right),$$

where α is a universal constant. Furthermore, the order $(\frac{p}{\ln(p+1)})^{1/q}$ is optimal when $p \rightarrow \infty$.

PROOF. The fact that (29) holds follows from a combination of Theorem 2 and Corollary 4 above. It remains to show that the order $(\frac{p}{\ln(p+1)})^{1/q}$ is optimal. Let $f := \chi_{[0,u]}$ ($0 < u \leq 1$), $n \in \mathbb{N}$ and let $\{f_{n,k}\}_{k=1}^n$ be a sequence of i.r.v.'s equidistributed with the function $\sigma_{1/n} f = \chi_{[0,u/n]}$. Setting $f_n := \sum_{k=1}^n f_{n,k}$, we see that

$$\lambda\{t \in [0, 1] : f_n(t) = k\} = \frac{n!}{k!(n-k)!} \cdot \left(\frac{u}{n}\right)^k \cdot \left(1 - \frac{u}{n}\right)^{n-k}, \quad k = 1, 2, \dots, n.$$

Hence,

$$f_n^*(t) = \sum_{k=1}^n \chi_{[0, \tau_k^n]} \quad \text{where } \tau_k^n := \sum_{i=k}^n \frac{n!}{i!(n-i)!} \cdot \left(\frac{u}{n}\right)^i \cdot \left(1 - \frac{u}{n}\right)^{n-i}.$$

Note that for every $1 \leq q < \infty$, we have

$$\left(\sum_{k=1}^n f_{n,k}^q \right)^{1/q} = \left(\sum_{k=1}^n f_{n,k} \right)^{1/q} = f_n^{1/q},$$

and so the function $(\sum_{k=1}^n f_{n,k}^q)^{1/q}$ is equidistributed with the function

$$(f_n^*)^{1/q} = \left(\sum_{k=1}^n \chi_{[0, \tau_k^n]} \right)^{1/q} = \sum_{k=1}^n (k^{1/q} - (k-1)^{1/q}) \chi_{[0, \tau_k^n]}.$$

By the definition of the norm in Marcinkiewicz spaces, we have

$$\begin{aligned} \|(f_{n,k})\|_{l_q} \|L_p\| &\geq \| (f_n^*)^{1/q} \|_{M(t^{1/p})} \geq \sup_{1 \leq k \leq n} \sum_{i=1}^k (i^{1/q} - (i-1)^{1/q}) (\tau_k^n)^{1/p} \\ (30) \qquad \qquad \qquad &= \sup_{1 \leq k \leq n} k^{1/q} (\tau_k^n)^{1/p}. \end{aligned}$$

Estimating τ_k^n via Stirling’s formula, we have

$$\begin{aligned} \tau_k^n &\geq \frac{n!}{k!(n-k)!} \cdot \left(\frac{u}{n}\right)^k \cdot \left(1 - \frac{u}{n}\right)^{n-k} \\ &\geq \frac{\sqrt{\pi n} e^{-n}}{k!(n-k)^{n-k} e^{-n+k} \sqrt{n-k}} \cdot \left(\frac{u}{n}\right)^k \cdot \left(1 - \frac{u}{n}\right)^{n-k} \\ &\geq \frac{n^{n-k} u^k e^{-1}}{k!(n-k)^{n-k} e^k} \geq \frac{1}{\sqrt{2\pi e}} \cdot \frac{u^k}{k^k \sqrt{k}} \geq \frac{1}{\sqrt{2\pi e}} \cdot \frac{u^k}{k^{2k}}. \end{aligned}$$

Using the latter estimate in (30), we obtain

$$(31) \quad \|(f_{n,k})\|_{l_q} \|L_p\| \geq \| (f_n^*)^{1/q} \|_{M(t^{1/p})} \geq \frac{1}{\sqrt{2\pi e}} \sup_{1 \leq k \leq n} k^{1/q} \left(\frac{u^k}{k^{2k}}\right)^{1/p}.$$

Observe further that for the sequence $\{f_{n,k}\}_{k=1}^n$ we have $(\sum_{k=1}^n \bar{f}_{n,k})^* = \chi_{[0,u]}$ and therefore the term $(\|F^* \chi_{[0,1]}\|_{L_p} + (\sum_{k=1}^\infty F^*(k)^q)^{1/q})$ in the right-hand side of (29) is equal in this case to $\|\chi_{[0,u]}\|_{L_p}$. Thus, by (31), we now estimate the constant in (29) from below as follows:

$$\begin{aligned} \sup_{0 < u \leq 1} \sup_{n \geq 1} \frac{\|(f_{n,k})\|_{l_q} \|L_p\|}{\|\chi_{[0,u]}\|_{L_p}} &\geq \sup_{0 < u \leq 1} \sup_{n \geq 1} \| (f_n^*)^{1/q} \|_{M(t^{1/p})} u^{-1/p} \\ &\geq \frac{1}{\sqrt{2\pi e}} \sup_{0 < u \leq 1} \sup_{n \geq 1} n^{1/q} \left(\frac{u^n}{n^{2n}}\right)^{1/p} u^{-1/p} \\ &= \frac{1}{\sqrt{2\pi e}} \sup_{n \geq 1} \frac{n^{1/q}}{(n^{2n})^{1/p}}. \end{aligned}$$

Choosing $n = \lfloor \frac{p}{\ln p} \rfloor$, we obtain for all sufficiently large p ’s

$$\sup_{n \geq 1} \frac{n^{1/q}}{(n^{2n})^{1/p}} \geq \left(\frac{p}{\ln p} - 1\right)^{1/q} \cdot \left(\frac{\ln p}{p}\right)^{2/\ln(p)} \geq \frac{1}{2e^2} \left(\frac{p}{\ln(p+1)}\right)^{1/q}.$$

The foregoing estimates show that the order $(\frac{p}{\ln(p+1)})^{1/q}$ in (29) is the best possible. \square

REMARK 14. Since $\Lambda(t^{1/p}) \subset L_p \subset M(t^{1/p})$ ($1 < p < \infty$), it follows from (29) that for an arbitrary sequence $\{f_k\}_{k=1}^\infty \subset \Lambda(t^{1/p})$ of i.r.v.'s

$$(32) \quad \begin{aligned} \|\mathbf{f}\|_{l_q} \|_{M(t^{1/p})} &\leq \alpha \left(\frac{p}{\ln(p+1)} \right)^{1/q} \\ &\times \left(\|F^* \chi_{[0,1]}\|_{\Lambda(t^{1/p})} + \left(\sum_{k=1}^\infty F^*(k)^q \right)^{1/q} \right), \end{aligned}$$

where α is a universal constant. The argument used in the proof of the preceding theorem shows that the order $(\frac{p}{\ln(p+1)})^{1/q}$ remains optimal when $p \rightarrow \infty$.

The following corollary from Theorem 13 strengthens [2], Theorem 4.4.

COROLLARY 15. *Let $1 \leq q < \infty$. If a sequence $\{f_k\}_{k \geq 1}$ of uniformly bounded i.r.v.'s satisfies the assumptions*

$$\sup_{k \geq 1} \sup_{t \in [0,1]} |f_k(t)| < \infty \quad \text{and} \quad \sum_{k=1}^\infty F^*(k)^q < \infty,$$

then the function $(\sum_{k=1}^\infty |f_k|^q)^{1/q} \in L_{N_q}$, where L_{N_q} is the Orlicz space on $[0, 1]$ generated by the function $N_q(t) := t^{t^q} - 1$, $N_q(0) = 0$.

PROOF. Due to (29), we have

$$\sup_{p \geq 1} \left(\frac{\ln(p+1)}{p} \right)^{1/q} \|\mathbf{f}\|_{l_q} \|_{L_p} \leq \alpha \left(\|F^* \chi_{[0,1]}\|_\infty + \left(\sum_{k=1}^\infty F^*(k)^q \right)^{1/q} \right),$$

and, by the assumptions, the right-hand side of this inequality is finite. Since the left-hand side coincides (up to equivalence) with the norm of the function $\|\mathbf{f}\|_{l_q} = (\sum_{k=1}^\infty |f_k|^q)^{1/q}$ in the Orlicz space L_{N_q} (see, e.g., [1], Corollary 1, or [13], Proposition 3.6), and we are done. \square

5. Two complements to Rosenthal's inequality. The main results of this section are Proposition 16 and Theorem 18 which complement Theorem 1.

Let $\{f_k\}_{k=1}^n$ be a sequence of i.r.v.'s on $[0, 1]$, $S_n := \sum_{k=1}^n f_k$ and, as before, F be the disjointification function related to the sequence $\{f_k\}_{k=1}^n$ [see (1)], which may be written in the form

$$F(t) := \sum_{k=1}^n f_k(t - k + 1) \chi_{[k-1, k]}(t), \quad t > 0.$$

PROPOSITION 16. *Let X be a symmetric space on $[0, 1]$ such that the Kruglov operator K acts boundedly on X . Then there exists a universal constant $\beta > 0$ such that for every $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s the following inequality holds:*

$$(33) \quad \|S_n\|_X \leq \beta \|K\|_{X \rightarrow X} (\|F^* \chi_{[0,1]}\|_X + \|S_n\|_{L_1}).$$

PROOF. First, we assume that

$$(34) \quad \int_0^1 f_k(s) ds = 0 \quad (k = 1, 2, \dots, n).$$

In this case, from Proposition 12 (see also [4], Theorem 3.1) we infer that

$$\|S_n\|_X \leq \gamma \|K\|_{X \rightarrow X} (\|F^* \chi_{[0,1]}\|_X + \|F^* \chi_{[1,\infty)}\|_{L_2}),$$

where γ is a universal constant. In addition, for the space L_1 , we have by [12], Theorem 1,

$$\|S_n\|_{L_1} \geq c (\|F^* \chi_{[0,1]}\|_{L_1} + \|F^* \chi_{[1,\infty)}\|_{L_2}).$$

Combining these two estimates we obtain (33) under assumption (34).

Suppose now that the sequence $\{f_k\}_{k=1}^n \subset X$ is an arbitrary sequence of i.r.v.'s. Setting

$$(35) \quad g_k := f_k - \int_0^1 f_k(s) ds \quad (k = 1, 2, \dots, n)$$

we obtain a sequence $\{g_k\}_{k=1}^n \subset X$ satisfying (34), and, therefore, by the above

$$(36) \quad \|\sigma_n\|_X \leq \gamma \|K\|_{X \rightarrow X} (\|G^* \chi_{[0,1]}\|_X + \|\sigma_n\|_{L_1}),$$

where

$$(37) \quad \sigma_n = \sum_{k=1}^n g_k, \quad G(t) := \sum_{k=1}^n g_k(t - k + 1) \chi_{[k-1,k]}(t), \quad t > 0.$$

On one hand,

$$\begin{aligned} G(t) &= \sum_{k=1}^n \left(f_k(t - k + 1) - \int_0^1 f_k(s) ds \right) \chi_{[k-1,k]}(t) \\ &= F(t) - \sum_{k=1}^n \int_0^1 f_k(s) ds \chi_{[k-1,k]}(t), \end{aligned}$$

which implies

$$G^* \chi_{[0,1]}(t) \leq F^* \chi_{[0,1]}(t) + \max_{k=1,2,\dots,n} \|f_k\|_{L_1},$$

and, therefore, in view of the embedding $X \subset L_1$ with the constant 1,

$$(38) \quad \|G^* \chi_{[0,1]}\|_X \leq \|F^* \chi_{[0,1]}\|_X + \max_{k=1,2,\dots,n} \|f_k\|_X \leq 2\|F^* \chi_{[0,1]}\|_X.$$

On the other hand,

$$\|\sigma_n\|_X \geq \|S_n\|_X - \left| \int_0^1 S_n(u) du \right| \geq \|S_n\|_X - \|S_n\|_{L_1}$$

and

$$\|\sigma_n\|_{L_1} \leq \|S_n\|_{L_1} + \left| \int_0^1 S_n(u) du \right| \leq 2\|S_n\|_{L_1}.$$

Combining these estimates with (36), we obtain

$$\|S_n\|_X \leq (2\gamma \|K\|_{X \rightarrow X} + 1)(\|F^* \chi_{[0,1]}\|_X + \|S_n\|_{L_1}),$$

and the assertion is established in view of the fact $\|K\|_{X \rightarrow X} \geq 1/e$. \square

REMARK 17. The converse inequality to (33) fails in general. However, if in addition $f_k \geq 0$, $k = 1, 2, \dots, n$, then for any symmetric space X

$$c(\|F^* \chi_{[0,1]}\|_X + \|S_n\|_{L_1}) \leq \|S_n\|_X$$

for some universal constant $c > 0$.

Denote

$$U_n(t) := \max_{k=1,2,\dots,n} |S_k(t)|, \quad t \in [0, 1].$$

THEOREM 18. *Let us assume that X is an interpolation space for the couple (L_1, L_∞) and that the Kruglov operator K acts boundedly on X . Then there exists a universal constant $\alpha > 0$ such that for all $n \in \mathbb{N}$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s the following inequality holds*

$$(39) \quad \begin{aligned} & \frac{1}{5}(\|F^* \chi_{[0,1]}\|_X + \|U_n\|_{L_1}) \\ & \leq \|U_n\|_X \leq \alpha \|K\|_{X \rightarrow X} (\|F^* \chi_{[0,1]}\|_X + \|U_n\|_{L_1}). \end{aligned}$$

PROOF. First, it is obvious that

$$2U_n(t) \geq M_n(t) := \max_{k=1,2,\dots,n} |f_k(t)|.$$

Appealing to (20), we see that

$$M_n^*(t/2) \geq F^*(t) \quad (0 < t \leq 1),$$

and so

$$\|U_n\|_X \geq \frac{1}{4} \|F^* \chi_{[0,1]}\|_X,$$

whence

$$\|U_n\|_X \geq \frac{1}{5}(\|F^* \chi_{[0,1]}\|_X + \|U_n\|_{L_1}).$$

Let us prove the right-hand side inequality in (39). It holds if f_k 's are symmetrically distributed. Indeed, by the well-known Levy theorem (see, e.g., [17], Proposition 1.1.1), we have

$$\lambda\{t \in [0, 1]: U_n(t) > \tau\} \leq 2\lambda\{t \in [0, 1]: |S_n(t)| > \tau\} \quad (\tau > 0),$$

and so the result follows from (33) (with the universal constant β) and the assumption that X is a symmetric space. Moreover, by the standard ‘‘symmetrization trick’’ and using the assumption that X is an interpolation space for the couple (L_1, L_∞) , it is not hard to extend this result to all sequences $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s satisfying condition (34).

Finally, let us consider the case of an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of i.r.v.'s. Suppose that the sequence $\{g_k\}_{k=1}^n$ is defined by formula (35). Applying the (already established) assertion to this sequence, we obtain

$$(40) \quad \|W_n\|_X \leq 2\beta \|K\|_{X \rightarrow X} (\|G^* \chi_{[0,1]}\|_X + \|W_n\|_{L_1}),$$

where G and σ_k are defined as in (37), and $W_n := \max_{k=1,2,\dots,n} |\sigma_k|$. Noting that

$$W_n(t) \leq U_n(t) + \max_{k=1,2,\dots,n} \int_0^1 |S_k(u)| du \leq U_n(t) + \|U_n\|_{L_1},$$

we infer $\|W_n\|_{L_1} \leq 2\|U_n\|_{L_1}$. Since

$$\begin{aligned} W_n(t) &= \max_{k=1,2,\dots,n} \left| S_k(t) - \int_0^1 S_k(u) du \right| \\ &\geq U_n(t) - \int_0^1 \max_{k=1,2,\dots,n} |S_k(u)| du = U_n(t) - \|U_n\|_{L_1}, \end{aligned}$$

we have

$$\|W_n\|_X \geq \|U_n\|_X - \|U_n\|_{L_1}.$$

By (40) and (38), this guarantees

$$\|U_n\|_X \leq (4\beta \|K\|_{X \rightarrow X} + 1)(\|F^* \chi_{[0,1]}\|_X + \|U_n\|_{L_1}),$$

and the assertion follows in view of the fact $\|K\|_{X \rightarrow X} \geq 1/e$. \square

REMARK 19. In the case when X is a symmetric space containing L_p for some finite p (this condition is more restrictive than the boundedness of the Kruglov operator in X , see [2]), the last result was also obtained in [10], Theorem 5. At the same time, it is established in [10] even for quasi-normed spaces.

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