THICK POINTS OF THE GAUSSIAN FREE FIELD

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Let $U \subseteq \mathbf{C}$ be a bounded domain with smooth boundary and let F be an instance of the continuum Gaussian free field on U with respect to the Dirichlet inner product $\int_U \nabla f(x) \cdot \nabla g(x) \, dx$. The set $T(a; U)$ of a-thick points of F consists of those $z \in U$ such that the average of F on a disk of radius r centered at z has growth $\sqrt{a/\pi} \log \frac{1}{r}$ as $r \to 0$. We show that for each $0 \leq a \leq 2$ the Hausdorff dimension of $T(a; U)$ is almost surely $2 - a$, that $\nu_{2-a}(T(a; U)) = \infty$ when $0 < a \leq 2$ and $\nu_2(T(0; U)) = \nu_2(U)$ almost surely, where ν_α is the Hausdorff-α measure, and that $T(a; U)$ is almost surely empty when $a > 2$. Furthermore, we prove that $T(a; U)$ is invariant under conformal transformations in an appropriate sense. The notion of a thick point is connected to the Liouville quantum gravity measure with parameter γ given formally by $\Gamma(dz) = e^{\sqrt{2\pi} \gamma F(z)} \, dz$ considered by Duplantier and Sheffield.

1. Introduction. Let $U \subseteq \mathbf{C}$ be a bounded domain with smooth boundary and for $f, g \in C_0^\infty(U)$ let

$$(f, g)_\nabla = \int_U \nabla f(x) \cdot \nabla g(x) \, dx$$

denote the Dirichlet inner product of f and g where dx is the Lebesgue measure. Let (f_n) be an orthonormal basis of the Hilbert space closure $H_0^1(U)$ of $C_0^\infty(U)$ under $(\cdot, \cdot)_\nabla$. The continuum Gaussian free field (GFF) $F = F_U$ on U is given formally as a random linear combination

$$(1.1) \qquad F = \sum_n \alpha_n f_n,$$

where (α_n) is an i.i.d. Gaussian sequence.

The GFF is a 2-time dimensional analog of the Brownian motion. Just as the Brownian motion can be realized as the scaling limit of many random curve ensembles, the GFF arises as the scaling limit of a number of random surface ensembles [1, 14, 15, 19]. The purpose of this article is to study the fractal geometry and conformal invariance properties of its extremal points. It is not possible to make

sense of F as a function since the sum in (1.1) does not converge in a topology that would allow us to do so. However, it does converge almost surely in the space of distributions and is sufficiently regular that there is no difficulty in interpreting its integral with respect to Lebesgue measure over sufficiently nice Borel sets. This class includes, for example, disks, squares and the conformal images of such. Thus, to make the notion of an extremal point precise, we first regularize by averaging the field over disks of radius r and then study those points where the average is unusually large as $r \to 0$.

With this is in mind, we say that z is an *a-thick point* provided

$$(1.2) \qquad \lim_{r \to 0} \frac{\mu(D(z,r))}{\pi r^2 \log 1/r} = \sqrt{\frac{a}{\pi}},$$

where $D(z,r)$ denotes the disk of radius r centered at $z \in U$ and $\mu(A) = \int_A F(x)\,dx$. While integration against the GFF does not define a measure, we can still interpret the quantity $\mu(A)$ as measuring the signed mass the GFF associates with A, whenever it is defined. This motivates our usage of the term "thick point," which has become standard terminology in the literature which studies the extremes of the occupation measure of a stochastic process [3, 5–7].

Let $T(a; U)$ denote the set of a-thick points of F and let ν_α denote the Hausdorff-α measure.

THEOREM 1.1. *For any $0 \le a \le 2$, the Hausdorff dimension of $T(a; U)$ is almost surely $2 - a$. Moreover,*

$$\mathbf{P}[\nu_2(T(0; U)) = \nu_2(U)] = 1 \quad and \quad \mathbf{P}[\nu_{2-a}(T(a; U)) = \infty] = 1,$$

when $0 < a \le 2$. In particular, $\mathbf{P}[|T(2; U)| = \infty] = 1$. Finally, if $a > 2$ then $T(a; U)$ is almost surely empty.

The proof of this theorem easily extends to the five other cases where one replaces the limit in (1.2) with either lim inf or lim sup and the equality with "not less than." The particular choice of normalization in (1.2) is so that the dimension is a linear function of a.

It is possible to make sense of the *circle average process*

$$F(z, r) = \frac{1}{2\pi r} \int_{\partial D(z,r)} F(x)\sigma(dx),$$

$\sigma(dx)$ the length measure, in such a way that it is a continuous function in (z, r) [9, 20]. We will describe this construction in the next section and, furthermore, argue that almost surely

$$(1.3) \qquad \int_0^r 2\pi s\, F(z, s)\,ds = \int_{D(z,r)} F(x)\,dx \qquad \text{for all } (z, r).$$

This gives rise to another collection of thick points, namely the set $T^C(a; U)$ consisting of those $z \in U$ satisfying

$$\lim_{r \to 0} \frac{1}{\log 1/r} F(z, r) = \sqrt{\frac{a}{\pi}}.$$

Our proof implies that the Hausdorff dimension of $T^C(a; U)$ is $2 - a$ almost surely, and we include this result as a separate theorem.

THEOREM 1.2.   *For any $0 \le a \le 2$, the Hausdorff dimension of $T^C(a; U)$ is almost surely $2 - a$. Moreover,*

$$\mathbf{P}[v_2(T^C(0; U)) = v_2(U)] = 1 \quad and \quad \mathbf{P}[v_{2-a}(T^C(a; U)) = \infty] = 1,$$

when $0 < a \le 2$. In particular, $\mathbf{P}[|T^C(2; U)| = \infty] = 1$. Finally, if $a > 2$, then $T^C(a; U)$ is almost surely empty.

As before, our proof also extends to the cases where one replaces the limit with either a lim inf or lim sup and the equality with "not less than."

Suppose that V is another domain, $\varphi : U \to V$ is a conformal transformation, and for $A \subseteq V$, formally set

$$(1.4) \qquad\qquad F_V = F_U \circ \varphi^{-1} \quad and \quad \mu_V(A) = \int_A F_V(x) \, dx.$$

As the Dirichlet inner product is invariant under precomposition by conformal maps, it follows that F_V has the law of a GFF on V. Our next theorem is a uniform estimate on the difference between $\mu_U(D(\xi, r))$ and $\mu_V(D(\varphi(\xi), |\varphi'(\xi)|r)) \times |\varphi'(\xi)|^{-2}$.

THEOREM 1.3.   *If $K \subseteq U$ is compact, then almost surely*

$$\limsup_{r \to 0} \sup_{\xi \in K} \frac{1}{\pi r^2 \log 1/r} \big| \mu_U(D(\xi, r))$$

$$(1.5)$$

$$- \mu_V(D(\varphi(\xi), |\varphi'(\xi)|r)) |\varphi'(\xi)|^{-2} \big| = 0.$$

An immediate consequence of this is the conformal invariance of the thick points.

COROLLARY 1.4.   *The set of thick points is a conformal invariant. More precisely, if $T(a; V)$ denotes the a-thick points of μ_V as in (1.4), then*

$$\mathbf{P}\big(\varphi(T(a; U)) = T(a; V) \text{ for all } 0 \le a \le 2\big) = 1.$$

Let $G = (V, E)$ be a finite graph with distinguished subset $V_\partial \subseteq V$. The law of the discrete GFF (DGFF) is given by the Gibbs measure with Hamiltonian $H(f) = \frac{1}{2} \sum_{x \sim y} (f(x) - f(y))^2$ for $f|_{V_\partial} \equiv 0$. The Ginzburg–Landau (GL) $\nabla\phi$ interface model, also known as the anharmonic crystal, is a non-Gaussian analog of the DGFF and arises by replacing $|\cdot|^2$ in $H(f)$ with a symmetric, convex function with quadratic growth. The behavior of the extremal points of the DGFF and GL model are studied in [2, 4, 8] in the special case that V is a lattice approximation of a smooth subset in $\mathbf{C}$. In Theorem 1.3 of [4], Daviaud shows that if for each $\varepsilon > 0$ one lets F_ε have the law of the DGFF on the induced subgraph $U_\varepsilon = U \cap \varepsilon \mathbf{Z}^2$ then the cardinality of the set $\mathcal{H}_\varepsilon(a) = \{z \in U_\varepsilon : F_\varepsilon(x) \geq \sqrt{a/\pi} \log \frac{1}{\varepsilon}\}$ of "a-high points" has growth $\varepsilon^{-(2-a)}$ as $\varepsilon \to 0$. This growth exponent represents a sort of discrete Hausdorff dimension so that this is the natural discrete analog of Theorem 1.1. An interesting open question is to see if this result for the DGFF or analogous results for the models considered in [14, 15, 19] can be deduced from Theorem 1.1. A natural starting point to answering this question is the coupling of the DGFF and the GFF suggested in [20]. A proof of the reverse implication seems less hopeful since, intuitively, a point z is a-thick if and only if it is an a-high point for F_ε, all sufficiently small $\varepsilon > 0$ and the result of [4] only provides estimates of the number of a-high points for a fixed $\varepsilon > 0$.

Suppose that U is simply connected. If (S, g) is a Riemann surface homeomorphic to U, then the classical uniformization theorem implies that (S, g) is conformally equivalent either to U or $\mathbf{C}$. The former case is in turn equivalent to the existence of global coordinates with respect to which the metric g takes the form $e^{\lambda(z)} dz$ for some $\lambda : U \to \mathbf{R}$. One natural construction of a random surface is to select $\lambda : U \to \mathbf{R}$ randomly and then take the surface with metric $e^{\lambda(z)} dz$. Fix $0 \leq \gamma < 2$. Formally, the Liouville quantum gravity with parameter γ corresponds to the case $\lambda(z) = \sqrt{2\pi} \gamma F(z)$. This, however, does not make sense since F is not even pointwise defined. To make this rigorous, Duplantier and Sheffield in [9] consider the random surfaces with continuous metric $r^{\gamma^2/2} e^{\sqrt{2\pi} \gamma F(z, r)} dz$ and study their behavior as $r \to 0$. Although understanding the limiting object as a metric space is still out of reach, they show that the associated random area measures Γ_r have a weak limit Γ as $r \to 0$. For A Borel, the quantity $\Gamma(A)$ is referred to as the γ-quantum area of A. It is shown in Proposition 3.4 of [9] and the discussion thereafter that Γ is almost surely supported on

$$T_\geq^{C,i}(a; U) = \left\{ z \in U : \liminf_{r \to 0} \frac{1}{\log 1/r} F(z, r) \geq \sqrt{\frac{a}{\pi}} \right\},$$

where $a = \gamma^2/2$. Thus, developing an understanding of the geometry of the thick points leads to an understanding of the geometry of the support of Γ. Note that our definition is slightly different from that appearing in [9] because of a difference in a choice of normalization of the Dirichlet inner product. Denote by $\tilde{D}(z, r) = \sup\{s : \Gamma(D(z, s)) \leq r\}$ the quantum ball of radius r centered at z. Let $X \subseteq U$ be a

random Borel set independent of F and let $X^r = \bigcup_{z \in X} \tilde{D}(z, r)$ be the r-quantum neighborhood of X. Then X is said to have quantum scaling expectation exponent Δ provided

$$\lim_{r \to 0} \frac{\mathbf{E}\Gamma(X^r)}{\log r} = \Delta.$$

Duplantier and Sheffield speculate ([9], page 26) that if X has quantum expectation scaling exponent Δ, then its quantum support is concentrated on $T_{\geq}^{C,i}(\alpha; U)$ where $\alpha = (\gamma - \gamma \Delta)^2/2$.

The remainder of the paper is organized as follows. In Section 2, we will give a brief overview of the basic properties of the GFF; see [20] for a more thorough introduction and [11] for more on the closely related notion of a Gaussian Hilbert space. Next, in Section 3, we prove Theorems 1.1 and 1.2. The first step is to establish the identity (1.3) which is a consequence of the fact that the Lebesgue measure on a disk can be written as a limit of Riemann sums of the length measure where the convergence is an appropriate Sobolev space. This allows us to sandwich the sets considered in Theorems 1.1 and 1.2 between $T^C(a; U)$ and

$$T_{\geq}^{C,s}(a; U) = \left\{ \limsup_{r \to 0} \frac{1}{\log 1/r} F(z, r) \geq \sqrt{\frac{a}{\pi}} \right\}$$

so that we need only show $\dim_H T^C(a; U) \geq 2 - a$ and $\dim_H T_{\geq}^{C,s}(a; U) \leq 2 - a$. We prove the more difficult lower bound using a multi-scale refinement of the second moment method, similar to the techniques employed in [7]. Roughly speaking, the crucial estimate that one needs is a quantitative bound on the degree to which the events that two given points are a-thick are approximately independent. We address this by considering a special subset which we term "perfect thick points." These are defined in such a way so that the approximate independence is a consequence of the Markov property of the field. The upper bound follows from an estimate of the modulus of continuity of $F(z, r)$ and that for fixed z the processes $r \mapsto F(z, e^{-r})$ evolve as Brownian motions.

Finally, in Section 4, we prove Theorem 1.3. It is easy to predict that the conformal invariance result is true since the GFF itself is conformally invariant, thick points are defined in terms of averages over small disks, and conformal maps send disks to disks at infinitesimal scales. This intuition, however, is far from a proof since integration against the GFF does not define a measure, much less a measure that is absolutely continuous with respect to Lebesgue measure. In particular, the GFF can assign large mass to a set that is small in the Lebesgue sense precisely due to the presence of thick points. The basic idea of our proof is as follows. We use the Markov property of the field to reduce to the case that $V = [0, 1]^2$. This choice is particularly convenient because the $H_0^1([0, 1]^2)$ orthonormal basis (f_n) given by the eigenvectors of the Laplacian is given by products of sine functions; this makes many of our computations elementary and explicit. We then show that if A

is a small dyadic square centered at z or the image of such centered at $\xi = \varphi^{-1}(z)$ under φ then then $\mu_V(A)$ is sufficiently well approximated by $\sum_{n=1}^{N} \alpha_n f_n(z)|A|$. Using a covering argument, we then deduce that an analogous estimate also holds for disks $D(z, r)$ and conformal images of disks $\varphi(D(\xi, r))$ with small radii. This argument is sensitive to the geometry of a disk since we need that the number $N(t; D(z, r))$ of maximal dyadic squares of side length t in $D(z, r)$ does not grow too quickly as $t \to 0$. Theorem 1.3 then follows from a bound on the Lebesgue measure of the symmetric difference $\varphi(D(\xi, r)) \Delta D(\varphi(\xi), |\varphi'(\xi)|r)$ and some Gaussian estimates.

Throughout the paper, we will make use of the following notation. If f, g are two functions, then we write $f \sim g$ provided that $f(t)/g(t) \to 1$ as either $t \to \infty$ or $t \to 0$, the case being clear from the context. If f_α, g_α are one-parameter families of functions, then $f_\alpha \sim g_\alpha$ uniformly means that $f_\alpha(t)/g_\alpha(t) \to 1$ uniformly in α. We say that $f = O(g)$ if there exists a constant $C > 0$ such that $|f(t)| \le C|g(t)|$ for all t and that $f = o(g)$ provided that $|f(t)|/|g(t)| \to 0$ as either $t \to 0$ or $t \to \infty$, the case being clear from the context. Finally, we say that $f_\alpha = O(g_\alpha)$ and $f_\alpha = o(g_\alpha)$ uniformly in α if the constant and convergence are uniform in α, respectively.

2. The Gaussian free field.

The purpose of this section is to recall the basic properties of the GFF. Let U be a bounded domain in $\mathbf{C}$ with smooth boundary and let $C_0^\infty(U)$ denote the set of C^∞ functions compactly supported in U. We begin with a short discussion of Sobolev spaces; the reader is referred to Chapter 5 of [10] or Chapter 4 of [21] for a more thorough introduction. With $\mathbf{N}_0 = \{0, 1, \ldots\}$, the nonnegative integers when $f \in C_0^\infty(U)$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}_0^2$ we let $D^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} f$. For $k \in \mathbf{N}_0$, we define the $H^k(U)$-norm

$$(2.1) \qquad \|f\|_{H^k(U)}^2 = \sum_{|\alpha| \le k} \int_U |D^\alpha f(x)|^2 \, dx,$$

where $|\alpha| = \alpha_1 + \alpha_2$. The Sobolev space $H_0^k(U)$ is given by the Banach space closure of $C_0^\infty(U)$ under $\|\cdot\|_{H^k(U)}$. If $s \ge 0$ is not necessarily an integer, then $H_0^s(U)$ can be constructed via the complex interpolation of $H_0^0(U) = L^2(U)$ and $H_0^k(U)$ where $k \ge s$ is any positive integer (see Chapter 4 Section 2 of [21] for more on this construction and also Chapter 4 of [13] for more on interpolation). A consequence of this is that if $T : C_0^\infty(U) \to C_0^\infty(U)$ is a linear map continuous with respect to the $L^2(U)$ and $H^k(U)$ topologies, then it is also continuous with respect to $H^s(U)$ for all $0 \le s \le k$. For $s \ge 0$, we define $H^{-s}(U)$ to be the Banach space dual of $H_0^s(U)$ where the dual pairing of $f \in H^{-s}(U)$ and $g \in H_0^s(U)$ is given formally by the usual $L^2(U)$ inner product

$$(f, g) = (f, g)_{L^2(U)} = \int_U f(x)g(x) \, dx.$$

More generally, for any $s \in \mathbf{R}$ the $H^s(U)$-topology can be constructed explicitly via the norm

$$(2.2) \qquad \|f\|_s^2 = \int (1 + \xi_1^2 + \xi_2^2)^s (\widehat{f}(\xi))^2 \, d\xi,$$

where

$$\widehat{f}(\xi) = \int e^{-i\xi \cdot x} f(x) \, dx$$

is the Fourier transform of f. We will be most interested in the space $H_0^1(U)$. An application of the Poincare inequality (Chapter 4, Proposition 5.2) gives that the norm induced by the Dirichlet inner product

$$(f, g)_\nabla = \int_U \nabla f \cdot \nabla g \qquad \text{for } f, g \in C_0^\infty(U),$$

is equivalent to $\| \cdot \|_{H^1(U)}$. This choice of inner product is particularly convenient because it is invariant under precomposition by conformal transformations.

The GFF $F = F_U$ on U is given formally as a random linear combination of an orthonormal basis (f_n) of $H_0^1(U)$

$$F = \sum_n \alpha_n f_n,$$

where (α_n) is an i.i.d. sequence of standard Gaussian. Although the sum defining F does not converge in $H_0^1(U)$, for each $\varepsilon > 0$ it does converge almost surely in $H^{-\varepsilon}(U)$ ([20], Proposition 2.7 and the discussion thereafter) and, in particular, $H^{-1}(U)$. If $f, g \in C_0^\infty(U)$, then an integration by parts gives $(f, g)_\nabla = -(f, \Delta g)$. Using this, we define

$$(F, f)_\nabla = -(F, \Delta f) \qquad \text{for } f \in C_0^\infty(U).$$

Observe that $(F, f)_\nabla$ is a Gaussian random variable with mean zero and variance $(f, f)_\nabla$. Hence, by polarization, F induces a map $C_0^\infty(U) \to \mathcal{G}$, $\mathcal{G}$ a Gaussian Hilbert space, that preserves the Dirichlet inner product. This map extends uniquely to $H_0^1(U)$ which allows us to make sense of $(F, f)_\nabla$ for all $f \in H_0^1(U)$. We are careful to point out, however, that while $(F, \cdot)_\nabla$ is well defined off of a set of measure zero as a linear functional on $C_0^\infty(U)$ this is not the case for general $f \in H_0^1(U)$. This is a technical point that we will touch on a bit later. It is not hard to see that the law of F is independent of the choice of (f_n); the eigenvectors of the Laplacian serve as a convenient choice since they are also orthogonal in $L^2(U)$. In particular, when $U = [0, 1]^2$ then $F_{[0,1]^2}$ admits the explicit representation

$$(2.3) \quad F_{[0,1]^2}(x, y) = \sum_{i,j \geq 1} \frac{2\alpha_{ij}}{\pi \sqrt{i^2 + j^2}} \sin(\pi i x) \sin(\pi j y) \qquad \text{for } (x, y) \in [0, 1]^2.$$

If $V \subseteq \mathbf{C}$ is another domain, $\varphi \colon U \to V$ is a conformal transformation and $f, g \in C_0^\infty(U)$, then a change of variables shows that the Dirichlet inner product is invariant under precomposition by φ^{-1}:

$$\int_V \nabla(f \circ \varphi^{-1}) \cdot \nabla(g \circ \varphi^{-1}) = (f, g)_\nabla.$$

Thus if (f_n) is an orthonormal basis of $H_0^1(U)$, then $(f_n \circ \varphi^{-1})$ is an orthonormal basis of $H_0^1(V)$, so that if F is a GFF on U, then $F_V = F \circ \varphi^{-1}$ has the law of a GFF on V.

If $\eta \in H^{-1}(U)$ so that $-\Delta^{-1}\eta \in H_0^1(U)$, then $(F, -\Delta^{-1}\eta)_\nabla = (F, \eta)$. The particular case that will be of interest to us is when $\eta(z, r)$ is the uniform measure on the circle $\partial D(z, r)$ where we think of $F(z, r) = (F, \eta(z, r))$ as the mean value of F on $\partial D(z, r)$. Letting

$$G(x, y) = -\frac{1}{2\pi}\big(\log|x - y| - \phi^y(x)\big),$$

where for each fixed $y \in U$ we denote by $x \mapsto \phi^y(x)$ the harmonic extension of $\log|x - y|$ from ∂U to U, be the Green's function for the Dirichlet problem of the Laplacian on U with zero boundary conditions, observe

$$\mathbf{E}(F, -\Delta^{-1}f)_\nabla (F, -\Delta^{-1}g)_\nabla = -(f, \Delta^{-1}g)$$

$$= \int_U \int_U f(x)g(y)G(x, y)\, dx\, dy.$$

When $D(z, e^{-t_1}) \subseteq U$ and $s, t > t_1$ we have

$$\mathbf{E}F(z, e^{-s})F(z, e^{-t}) = \mathbf{E}(F, -\Delta^{-1}\eta(z, e^{-s}))_\nabla (F, -\Delta^{-1}\eta(z, e^{-t}))_\nabla$$

$$= \frac{s}{2\pi} + C(z),$$

where $C(z)$ is a constant depending only on z and not s, t. Hence, with $z \in U$ fixed and letting $B(z, t) = \sqrt{2\pi}\, F(z, e^{-t})$ the process $t \mapsto B(z, t) - B(z, t_1)$ is Gaussian with the mean and autocovariance of a standard Brownian motion.

Using the Kolmogorov–Centsov theorem one can show ([9], Proposition 3.1) that $(z, r) \mapsto F(z, r)$ has a locally γ-Hölder continuous modification whenever $\gamma < 1/2$ is fixed. We will need some control of the Hölder norm of $F(z, r)$ on compact intervals as $r \to 0$; we are able to do this using Lemma C.1, a refinement of the Kolmogorov–Centsov theorem.

PROPOSITION 2.1. *The circle average process $F(z, r)$ possesses a modification $\tilde{F}(z, r)$ such that for every $0 < \gamma < 1/2$ and $\varepsilon, \zeta > 0$ there exists $M = M(\gamma, \varepsilon, \zeta)$ such that*

$$(2.4) \qquad |\tilde{F}(z, r) - \tilde{F}(w, s)| \le M\left(\log\frac{1}{r}\right)^\zeta \frac{|(z, r) - (w, s)|^\gamma}{r^{\gamma+\varepsilon}}$$

for all $z, w \in U$ and $r, s \in (0, 1]$ with $1/2 \le r/s \le 2$.

PROOF.   Note that if $\tilde{F}$ and $\tilde{F}'$ are two different modifications satisfying (2.4), then they are almost surely equal by continuity. Thus, it suffices to show that F satisfies the hypotheses of Lemma C.1 for α, β arbitrarily large with β/α arbitrarily close to $1/2$. With $a = (z, w, r, s) \in \mathcal{U} = U^2 \times [0, \infty)^2$, we know that

$$\Upsilon(a) = \mathbf{E}F(z, r)F(w, s) = (\eta(z, r), -\Delta^{-1}\eta(w, s))$$
$$= (-\Delta^{-1}\eta(z, r), \eta(w, s)).$$

One can check directly (see the discussion after Proposition 3.1 of [9]) that $\xi_r^z = -\Delta^{-1}\eta(z, r)$ is given by

$$\xi_r^z(y) = \tau_r^z(y) - \psi_r^z(y),$$

where $\tau_r^z(y) = -\log\max(r, |z - y|)$ and ψ_r^z is the harmonic extension τ_r^z from ∂U to U. As $|\log\frac{x}{y}| \leq \frac{|x-y|}{x \wedge y}$ for $x, y > 0$, we have

$$(2.5) \qquad |\tau_r^z(y) - \tau_{r'}^{z'}(y')| \leq C\frac{|r - r'| + |z - z'| + |y - y'|}{r \wedge r'}.$$

In particular, this holds when $y_0 \in \partial U$. This implies that the partial derivatives $\partial_y, \partial_z, \partial_r$ of $\psi_r^z(y_0)$ are all $O(1/r)$ uniformly $z \in U$ and $y_0 \in \partial U$ when $r \in (0, 1]$. Since these partials are harmonic from the maximum principle we conclude that (2.5) holds with ψ_r^z in place of τ_r^z. This gives

$$|\xi_s^w(z + rx) - \xi_{s'}^{w'}(z' + r'x)| \leq \frac{C|a - a'|}{s \wedge s'}$$

for all $a, a' \in \mathcal{U}$ and $x \in \mathbf{S}^1$ so that

$$|\Upsilon(a) - \Upsilon(a')| \leq \int_{\mathbf{S}^1} |\xi_s^w(z + rx) - \xi_{s'}^{w'}(z' + t'x)|\sigma(dx) \leq \frac{C|a - a'|}{s \wedge s'}.$$

As everything is symmetric,

$$|\Upsilon(a) - \Upsilon(a')| \leq \frac{C|a - a'|}{(r \wedge r') \vee (s \wedge s')}.$$

Hence,

$$\mathbf{E}\big(F(z, r) - F(w, s)\big)^2$$
$$\leq |\mathbf{E}(F(z, r))^2 - \mathbf{E}F(z, r)F(w, s)| + |\mathbf{E}(F(w, s))^2 - \mathbf{E}F(z, r)F(w, s)|$$
$$= |\Upsilon(z, z, r, r) - \Upsilon(z, w, r, s)| + |\Upsilon(w, w, s, s) - \Upsilon(z, w, r, s)|$$
$$\leq \frac{C|(z, r) - (w, s)|}{r \wedge s}.$$

This implies that for any $\alpha > 1$, $z, w \in U$, and $r, s \in (0, 1]$ we have

$$\mathbf{E}|F(z, r) - F(w, s)|^\alpha \leq C\left(\frac{|(z, r) - (w, s)|}{r \wedge s}\right)^{\alpha/2},$$

which puts us exactly in the setting of Lemma C.1.   □

From now on, we assume that $F(z, r)$ is a modification as in Proposition 2.1.

The most natural way to make sense of $\int_A F(x)\, dx$ is to show that $1_A \in H_0^\varepsilon(U)$ for some $\varepsilon > 0$ and then to interpret the integral as the dual pairing of F and $f = 1_A$. To show $\|1_A\|_{H^\varepsilon(U)} < \infty$ it suffices to show that the asymptotics of the Fourier transform $\widehat{f}$ are sufficiently well-behaved so that the $\| \cdot \|_\varepsilon$ norm of (2.2) is finite. When A is a disk or square then the Fourier transform $\widehat{f}(r, \theta)$ of 1_A in polar coordinates satisfies

$$\sup_{r \geq 0} r^{3/2 - \varepsilon} |\widehat{f}(r, \theta)| \in L^2(\mathbf{S}^1)$$

for every $\varepsilon > 0$ (Theorems 1 and 2 of [17]). This implies $1_A \in H_0^\varepsilon(U)$ whenever $\varepsilon < 1/2$. If $\varphi : U \to V$ is a conformal transformation and $W \subseteq U$ is an open set such that $\overline{W} \subseteq U$ then $c \leq |\varphi'(z)| \leq C$ for all $z \in W$ and $0 < c \leq C < \infty$. It thus follows that precomposition by φ^{-1} induces a continuous linear map $L^2(W) \to L^2(\varphi(W))$ and $H_0^1(W) \to H_0^1(\varphi(W))$. Therefore by interpolation $1_{\varphi(A)} = 1_A \circ \varphi^{-1} \in H_0^\varepsilon(\varphi(W)) \subseteq H_0^\varepsilon(V)$ for all $\varepsilon < 1/2$ when $A \subseteq W$ is a square or disk.

The Lebesgue measure $\rho = \rho(z, r)$ on $D(z, r)$ can be expressed as the integral $\rho = \int_0^r 2\pi s \eta(z, s)\, ds$. This gives rise to two different interpretations of $\int_{D(z,r)} F(x)\, dx$ both of which will be important for us. The first is as the dual pairing we have already mentioned and the second is

$$(2.6) \qquad \int_0^r 2\pi s F(z, s)\, ds = \sqrt{2\pi} \int_{-\log r}^\infty B(z, t) e^{-2t}\, dt.$$

Thus, we must be careful to ensure that they agree in an appropriate sense. This does not represent a serious difficulty, however, since it is easy to see that the Riemann sums corresponding to $\int_0^r 2\pi s \eta(z, s)\, ds$ converge to $\rho(z, r)$ in $H^{-1}(U)$. If Π is any partition of $[0, r]$ then as random variables in $\mathcal{G}$

$$\sqrt{2\pi}\left(F, \sum_\Pi t_k \eta(z, t_k)(t_{k+1} - t_k) \right) \overset{\text{a.s.}}{=} \sqrt{2\pi} \sum_\Pi (F, t_k \eta(z, t_k))(t_{k+1} - t_k)$$

$$\overset{\text{a.s.}}{=} \sum_\Pi B(z, -\log t_k) t_k (t_{k+1} - t_k).$$

Therefore,

$$(F, \rho(z, r)) \overset{\text{a.s.}}{=} \int_{-\log r}^\infty B(z, t) e^{-2t}\, dt$$

as random variables in $\mathcal{G}$. As both sides of the equation are continuous in (z, r), we obtain the following proposition.

PROPOSITION 2.2. *Almost surely,*

$$(F, \rho(z, r)) = \int_{-\log r}^{\infty} B(z, t) e^{-2t} \, dt \qquad \textit{for all } (z, r).$$

In particular, z is an a-thick point if and only if

$$(2.7) \quad \lim_{r \to 0} \frac{\sqrt{2\pi} \int_{-\log r}^{\infty} B(z, t) e^{-2t} \, dt}{\sqrt{\pi} r^2 \log 1/r} = \lim_{r \to 0} \frac{\sqrt{2} \int_{-\log r}^{\infty} B(z, t) e^{-2t} \, dt}{r^2 \log 1/r} = \sqrt{a}.$$

Suppose that $W \subseteq U$ is an open set. Then there is a natural inclusion of $H_0^1(W)$ into $H_0^1(U)$ given by the extension by value zero. If $f \in C_0^\infty(W)$ and $g \in C_0^\infty(U)$, then as $(f, g)_\nabla = -(f, \Delta g)$ it is easy to see that $H_0^1(U)$ admits the orthogonal decomposition $\mathcal{M} \oplus \mathcal{N}$ where $\mathcal{M} = H_0^1(W)$ and $\mathcal{N}$ is the set of functions in $H_0^1(U)$ that are harmonic on W. Thus, we can write

$$F = F_W + H_W = \sum_n \alpha_n f_n + \sum_n \beta_n g_n,$$

where (α_n), (β_n) are independent i.i.d. sequences of standard Gaussians and (f_n), (g_n) are orthonormal bases of $\mathcal{M}$ and $\mathcal{N}$, respectively. Observe that F_W has the law of the GFF on W, H_W the harmonic extension of $F|\partial W$ to W, and F_W and H_W are independent. We arrive at the following proposition.

PROPOSITION 2.3 (Markov property). *The conditional law of $F|W$ given $F|U \setminus W$ is that of the GFF on W plus the harmonic extension of the restriction of F on ∂W to W. In particular, if $D(z, e^{-t_1}) \setminus D(z, e^{-t_2})$ and $D(w, e^{-s_1}) \setminus D(w, e^{-s_2})$ are disjoint annuli contained in U then the Brownian motions $B(z, t) - B(z, t_1)$ for $t_1 \le t \le t_2$ and $B(w, s) - B(w, s_1)$ for $s_1 \le s \le s_2$ are independent.*

3. The Hausdorff dimension. Let U be a bounded domain with smooth boundary. It follows from the discussion in the previous section that we can express $T^C(a; U)$ and $T_{\ge}^{C,s}(a; U)$ as

$$T^C(a; U) = \left\{ z \in U : \lim_{t \to \infty} \frac{B(z, t)}{\sqrt{2t}} = \sqrt{a} \right\},$$

$$T_{\ge}^{C,s}(a; U) = \left\{ z \in U : \limsup_{t \to \infty} \frac{B(z, t)}{\sqrt{2t}} \ge \sqrt{a} \right\}.$$

3.1. *The upper bound.*

LEMMA 3.1. *If $0 \le a \le 2$, then almost surely $\dim_H(T_{\ge}^{C,s}(a; U)) \le 2 - a$. If $a > 2$, then $T_{\ge}^{C,s}(a; U)$ is empty.*

PROOF. First, we suppose that $0 \le a \le 2$. Let $\varepsilon > 0$ be arbitrary and take $K = \varepsilon^{-1}$. For each n, let $r_n = n^{-K}$. With $\zeta \in (0, 1)$, $\gamma \in (0, 1/2)$ and $\tilde{\gamma} = (1+\varepsilon)\gamma$ fixed and $M = M(\gamma, \varepsilon, \zeta)$ as in (2.4), we have

$$\left| B(z, t) - B\left(z, \log \frac{1}{r_n}\right) \right| = \sqrt{2\pi} |F(z, e^{-t}) - F(z, r_n)|$$

$$\le MK^{\zeta}(\log n)^{\zeta} \frac{(r_{n+1} - r_n)^{\gamma}}{r_{n+1}^{\tilde{\gamma}}}$$

$$= O\left((\log n)^{\zeta} n^{K\tilde{\gamma} - (K+1)\gamma}\right)$$

$$= O((\log n)^{\zeta})$$

uniformly in $n \in \mathbf{N}$, $z \in U$, and $\log \frac{1}{r_n} < t \le \log \frac{1}{r_{n+1}}$. Therefore, $z \in T_{\ge}^{C,s}(a; U)$ if and only if

$$\limsup_{n \to \infty} \frac{B(z, \log 1/r_n)}{\sqrt{2} \log 1/r_n} \ge \sqrt{a}.$$

For each $n \in \mathbf{N}$, let (z_{nj}) be a maximal $r_n^{1+\varepsilon}$ net of U. If $z \in D(z_{nj}, r_n)$, then

$$\left| B\left(z, \log \frac{1}{r_n}\right) - B\left(z_{nj}, \log \frac{1}{r_n}\right) \right| \le O((\log n)^{\zeta}).$$

Let

$$\delta(n) = C(\log n)^{\zeta - 1} \quad \text{and} \quad \mathcal{I}_n = \left\{ j : \left| B\left(z_{nj}, \log \frac{1}{r_n}\right) \right| \ge \sqrt{2}(\sqrt{a} - \delta(n)) \log \frac{1}{r_n} \right\}.$$

Then we see that for each $N \ge 1$

$$I(a, N) = \bigcup_{n \ge N} \{D(z_{nj}, r_n) : j \in \mathcal{I}_n\}$$

is such that $z \in T_{\ge}^{C,s}(a; U)$ implies that there exists arbitrarily small balls in $I(a, N)$ containing z provided C is large enough.

Since $B(z, t)$ evolves as a Brownian motion Lemma A.4 implies

$$\mathbf{P}(j \in \mathcal{I}_n) = \mathbf{P}\left(\frac{|B(z_{nj}, \log 1/r_n)|}{\sqrt{\log 1/r_n}} \ge (\sqrt{a} - \delta(n))\sqrt{2 \log \frac{1}{r_n}} \right) = O(r_n^{a - o(1)}).$$

Hence,

$$(3.1) \qquad \mathbf{E}|\mathcal{I}_n| \le O\left(\frac{r_n^{a - o(1)}}{r_n^{2(1+\varepsilon)}} \right) = O(r_n^{a - o(1) - 2(1+\varepsilon)}).$$

Letting $\alpha = 2 - a + \frac{2+a}{1+\varepsilon}\varepsilon$, we thus have

$$\mathbf{E}\left[\sum_{n\geq N}\sum_{j\in\mathcal{I}_n}(\mathrm{diam}(D(z_{nj}, r_n^{1+\varepsilon})))^\alpha\right] = O\left(\sum_{n\geq N} r_n^{2\varepsilon - o(1)}\right)$$

$$= O\left(\sum_{n\geq N} n^{-2+o(1)}\right).$$

This proves that the Hausdorff-$[2 - a + \frac{2+a}{1+\varepsilon}\varepsilon]$ measure of $T_\geq^{C,s}(a; U)$ is 0.

If $a > 2$, then all of our analysis still applies. In particular, for $\varepsilon > 0$ such that $a > 2(1 + \varepsilon)$ (3.1) gives that $\mathbf{E}|\mathcal{I}_n| \to 0$ as $n \to \infty$. $\quad\square$

3.2. *The lower bound.* Let $s_n = \frac{1}{n!}$ and $t_n = -\log s_n$. Let $H \subseteq U$ be a fixed compact square. By rescaling, we may assume without loss of generality that if $z \in H$, then $D(z, s_n) \subseteq D(z, s_1) \subseteq U$ and that H has side length 1. We further assume $H = [0, 1]^2$ by translation. For $m \in \mathbf{N}$, let

$$E_m(z) = \left\{\sup_{t_m < t \leq t_{m+1}} \left|B(z, t) - B(z, t_m) - \sqrt{2a}(t - t_m)\right| \leq \sqrt{t_{m+1} - t_m}\right\},$$

$$F_m(z) = \left\{\sup_{t \geq t_m}|B(z, t) - B(z, t_m)| \leq (t - t_m) + 1\right\}.$$

We say that $z \in H$ is an n-perfect a-thick point provided that the event $E^n(z) = \bigcap_{m \leq n} E_m(z) \cap F_{n+1}(z)$ occurs. Note that on $E^n(z)$ for $t_m < t \leq t_{m+1}$ and $m \leq n$ we have

$$\left|B(z, t) - B(z, t_1) - \sqrt{2a}(t - t_1)\right|$$

$$\leq \sum_{k=1}^{m-1}\left|B(z, t_{k+1}) - B(z, t_k) - \sqrt{2a}(t_{k+1} - t_k)\right|$$

$$+ \left|B(z, t) - B(z, t_m) - \sqrt{2a}(t - t_m)\right|$$

$$\leq \sum_{k=1}^{m}\sqrt{\log(k + 1)} = o(m\log m) = o(t) \qquad \text{as } m \to \infty,$$

where we used $t_{n+1} - t_n = \log(n+1)$ in the last inequality. Furthermore, if $t \geq t_{n+1}$ then

$$|B(z, t) - B(z, t_1)| \leq \sum_{k=1}^{n}\sqrt{\log(k + 1)} + O(t) = O(t).$$

Divide H into s_n^{-2} squares of side length s_n. Let C_n denote the set of centers of these squares and $C_n(a)$ the set of centers in H that are n-perfect. Finally, we let

$$P(a) = \bigcap_{k\geq 1}\overline{\bigcup_{n\geq k}\bigcup_{z\in C_n(a)} S(z, s_n)}$$

be the set of "perfect a-thick points," with $S(z, r)$ denoting the square centered at z of side length r. We obtain the following lemma as an immediate consequence of the continuity of $B(z, r)$.

LEMMA 3.2. *Almost surely* $P(a) \subseteq T^C(a; U)$.

PROOF. Fix $z \in P(a)$. Then there exists a sequence $(z_{n_k})_{k=1}^\infty$ so that $z_{n_k} \in C_{n_k}(a)$ for every k and $|z_{n_k} - z| \le s_{n_k}$. Fix $t > 0$ and let m be such that $t_m < t \le t_{m+1}$. Uniformly in k such that $n_k > m + 1$ we know that

$$\big|B(z_{n_k}, t) - B(z_{n_k}, t_1) - \sqrt{2a}(t - t_1)\big| = o(t).$$

Thus taking a limit as $k \to \infty$ and using the spatial continuity of B, we have

$$\big|B(z, t) - B(z, t_1) - \sqrt{2a}(t - t_1)\big| = o(t).$$

Since $t > 0$ was arbitrary, dividing both sides by t we arrive at

$$\frac{|B(z, t) - \sqrt{2a}t|}{t} = o(1) \qquad \text{as } t \to \infty.$$

Therefore, $z \in T^C(a; U)$. □

From now on, we let $\gamma_n = \prod_{k=1}^n \exp(\frac{1}{2}\sqrt{\log k})$.

LEMMA 3.3. *Suppose* $z, w \in H$. *Let* $l \in \mathbf{N}$ *be such that* $w \in S(z, s_l) \setminus S(z, s_{l+1})$. *There exists* $C > 0$ *such that for every* $n \ge l$ *we have*

$$\mathbf{P}\big(E^n(z) \cap E^n(w)\big) \le C^l \gamma_l^{-a} s_l^{-a} \mathbf{P}(E^n(z))\mathbf{P}(E^n(w)).$$

PROOF. By making the constant sufficiently large the inequality holds uniformly when $l \le 2$, hence we assume that $l > 2$. Observe that the events $E_i(z), E_j(w)$ for $l + 1 < i \le n$ and $1 \le j \le n, j \ne l - 1, l, l + 1$ are independent. By adjusting $C > 0$ if necessary, Lemma A.3 gives us the bound

$$\mathbf{P}(E_m(z)), \qquad \mathbf{P}(E_m(w)) \ge C \frac{\exp(a/2\sqrt{\log m})}{m^a} \qquad \text{for all } 1 \le m \le n.$$

By further adjusting $C > 0$, it follows from Proposition 2.3 that

$$\mathbf{P}\bigg(\bigcap_{1 \le i \le l+1} E_i(z)\bigg)\mathbf{P}\bigg(\bigcap_{l-1 \le j \le l+1} E_j(w)\bigg) \ge C^l \gamma_l^a s_l^a.$$

Applying Proposition 2.3 again, we therefore have the inequality

$$\mathbf{P}\big(E^n(z) \cap E^n(w)\big)$$

$$\le \mathbf{P}\bigg(\bigcap_{l+2 \le i \le n} E_i(z) \cap \bigcap_{\substack{1 \le j \le n \\ j \ne l-1, l, l+1}} E_j(w)\bigg)$$

$$= \mathbf{P}\Big(\bigcap_{l+2 \leq i \leq n} E_i(z) \Big) \mathbf{P}\Big(\bigcap_{\substack{1 \leq j \leq n \\ j \neq l-1, l, l+1}} E_j(w) \Big)$$

$$\leq \frac{1}{C^l \gamma_l^a s_l^a} \mathbf{P}\Big(\bigcap_{1 \leq i \leq n} E_i(z) \Big) \mathbf{P}\Big(\bigcap_{1 \leq j \leq n} E_j(w) \Big).$$

The last step comes by multiplying the second to last expression by

$$\frac{1}{C^l \gamma_l^a s_l^a} \mathbf{P}\Big(\bigcap_{1 \leq i \leq l+1} E_i(z) \Big) \mathbf{P}\Big(\bigcap_{l-1 \leq j \leq l+1} E_j(w) \Big) \geq 1$$

and then using that each of the collections of events $E_i(z)$, $1 \leq i \leq n$ and $E_j(w)$, $1 \leq j \leq n$ are independent (though, of course, not from each other). The lemma now follows as, by Proposition 2.3, $F_{n+1}(z)$ is independent of $E_m(z)$ for $1 \leq m \leq n$ and $\mathbf{P}(F_{n+1}(z)) \geq c > 0$ and the same is also true for w with c uniform in n, z, w.
□

For $\alpha \geq 0$, let ν_α denote the Hausdorff-α measure.

LEMMA 3.4.   *We have* $\mathbf{P}[\nu_{2-a}(T^C(a; U)) = \infty] = 1$ *for all* $0 < a \leq 2$ *and* $\mathbf{P}[\nu_2(T^C(0; U)) = \nu_2(U)] = 1$. *In particular,* $\mathbf{P}[\dim_H(T^C(a; U)) \geq 2 - a] = 1$ *and* $\mathbf{P}[|T^C(2; U)| = \infty] = 1$.

PROOF.   Assume $0 < a \leq 2$. Let $M_n = |H \cap C_n|$ and, for $z_{nj} \in H \cap C_n$, let $p_{nj} = \mathbf{P}(z_{nj} \in C_n(a))$. For each $n \in \mathbf{N}$, define a random measure τ_n on H by

$$\tau_n(A) = \int_A \sum_{i=1}^{M_n} p_{ni}^{-1} 1_{C_n(a)}(z_{ni}) 1_{S(z_{ni}, s_n)}(z) \, dz \qquad \text{for } A \subseteq H.$$

Observe $\mathbf{E}\tau_n(H) = 1$ and

$$\mathbf{E}(\tau_n(H))^2 = s_n^4 \sum_{i,j=1}^{M_n} p_{ni}^{-1} p_{nj}^{-1} \mathbf{P}(z_{ni}, z_{nj} \in C_n(a))$$

$$\leq s_n^4 |M_n| \sum_{l \geq 1} \Big(\frac{s_l^2}{s_n^2} \Big) O(C^l \gamma_l^{-a} s_l^{-a})$$

$$= \sum_{l \geq 1} O(C^l \gamma_l^{-a} s_l^{2-a}) < \infty.$$

Let

$$I_\alpha(\tau_n) = \int_{[0,1]^2} \int_{[0,1]^2} \frac{d\tau_n(z_1) \, d\tau_n(z_2)}{|z_1 - z_2|^\alpha}$$

be the α-energy measure of τ_n. By a similar computation, we have

$$\mathbf{E}I_\alpha(\tau_n) = \sum_{i,j=1}^{M_n} p_{ni}^{-1} p_{nj}^{-1} \mathbf{P}(z_{ni}, z_{nj} \in C_n(a)) \int_{S(z_{ni}, s_n)} \int_{S(z_{nj}, s_n)} \frac{dz_1 \, dz_2}{|z_1 - z_2|^\alpha}$$

$$\leq \sum_{l \geq 1} O(C^l \gamma_l^{-a} s_l^{2-a} s_{l+1}^{-\alpha}),$$

hence $\mathbf{E}I_{2-a}(\tau_n) < \infty$ uniformly in n. This implies that there exists $d, b > 0$ such that with

$$G_n = \{b \leq \tau_n(H) \leq b^{-1}, I_{2-a}(\tau_n) \leq d\} \quad \text{and} \quad G = \limsup_n G_n$$

we have

$$\mathbf{P}(G) > 0.$$

As I_{2-a} is lower semi-continuous the set $\mathcal{M}_{2-a}(b, d)$ of measures τ on H such that $b \leq \tau(H) \leq b^{-1}$ and $I_{2-a}(\tau) \leq d$ is compact with respect to weak convergence. For each $\omega \in G$ there exists a sequence (n_k) such that $\tau_{n_k, \omega} \in \mathcal{M}_{2-a}(b, d)$ and hence has a weak limit $\tau \in \mathcal{M}_{2-a}(b, d)$ which is a finite measure supported on $P(a)(\omega)$ with positive mass and finite $(2 - a)$-energy. Therefore,

$$\mathbf{P}(\nu_{2-a}(P(a)) > 0) > 0.$$

A simple application of the Hewitt–Savage zero-one law implies that

$$\mathbf{P}[\nu_{2-a}(T^C(a; U)) > 0] = 1,$$

and, in particular, $\mathbf{P}[\dim_H(T^C(a; U)) \geq 2 - a] = 1$ (see [7], Lemma 3.2 for a similar argument).

We will now show that in fact $\mathbf{P}[\nu_{2-a}(T^C(a; U)) = \infty] = 1$. Consider the covering $S(z_{n_i}, s_n)$ of H by $(n!)^2$ disjoint squares. The Markov property implies that we can write $F|S(z_{n_i}, s_n) = F_{n_i} + H_{n_i}$, where the F_{n_i} are independent zero-boundary GFFs and H_{n_i} is the harmonic extension of $F|\partial S(z_{n_i}, s_n)$ to $S(z_{n_i}, s_n)$. It is not hard to see that H_{n_i} is negligible in the definition of a thick point, hence the set $T_{n_i}(a)$ of a-thick points of F_{n_i} in $S(z_{n_i}, s_n)$ is the same as $T(a; U) \cap S(z_{n_i}, s_n)$. Therefore, the random variables $\nu_{2-a}(T_{n_i}(a))$ are i.i.d. and $\nu_{2-a}(H) = \sum_i \nu_{2-a}(T_{n_i}(a))$. By the basic scaling properties of ν_{2-a}, we have that

$$\nu_{2-a}(T_{n_i}(a)) \overset{d}{=} (s_n^{2-a})\nu_{2-a}(H).$$

The statement of the lemma in the case that $a > 0$ is now immediate.

It is left to consider the case that $a = 0$. It is immediate that $\mathbf{P}[z \in T^C(0; U)] = 1$ for any nonrandom $z \in U$. Hence, by Fubini's theorem, we have that

$$\mathbf{E}\nu_2(T^C(0; U)) = \mathbf{E}\int_U 1_{T^C(0;U)}(z) \, d\nu_2(z)$$

$$= \int_U \mathbf{P}[z \in T^C(0; U)] \, d\nu_2(z) = \nu_2(U).$$

Combining this with the trivial bound $0 \le \nu_2(T^C(0; U)) \le \nu_2(U)$ implies $\nu_2(T^C(0; U)) = \nu_2(U)$ almost surely. $\quad\square$

4. Conformal invariance. The purpose of this section is to establish Theorem 1.3 and Corollary 1.4. The idea of the proof is to show that $\mu(A)$ is sufficiently well approximated by $\sum_{n=1}^{N} \alpha_n f_n(z)|A|$ where (f_n) is an ONB of $H_0^1(U)$ and A is a disk, square, or the conformal image of such. The proof is divided into two subsections. In the first subsection, we will compute the asymptotic variance of the GFF $F = F_{[0,1]^2}$ on $[0, 1]^2$ integrated over small disks, squares and the conformal images of small disks and squares. We will then combine these estimates with a covering argument and the Borel–Cantelli lemma to bound $\mu(A) - \sum_{n=1}^{N} \alpha_n f_n(z)|A|$. The reason that we restrict our attention to this case is that the $H_0^1([0, 1]^2)$ orthonormal basis given by the eigenvectors of the Laplacian is particularly convenient with which to work. In the second subsection, we will combine these with some Gaussian estimates to prove the theorem.

4.1. *Preliminary estimates.* Let $F = F_{[0,1]^2}$ and let $\mu = \mu_{[0,1]^2}$ be given by $\mu(A) = \int_A F(x)\,dx$. Throughout this section, we consider a fixed simply connected domain U and let $\varphi : U \to [0, 1]^2$ be a conformal transformation with inverse $\psi : [0, 1]^2 \to U$. Fix compact sets $K \subseteq U$ and $L \subseteq (0, 1)^2$. Let

$$G_{ij}(z, r) = \int_{D(z,r)} \sin(\pi i u) \sin(\pi j v)\,du\,dv \qquad \text{for } z \in [0, 1]^2,$$

and denote by $S(z, r)$ the square in $[0, 1]^2$ centered at z with side length r.

LEMMA 4.1. *Uniformly in $z \in L$ and as $r \to 0$,*

$$(4.1) \qquad\qquad \mathbf{E}(\mu(D(z, r)))^2 \sim \frac{\pi}{2} r^4 \log \frac{1}{r},$$

$$(4.2) \qquad\qquad \mathbf{E}(\mu(S(z, r)))^2 \sim \frac{1}{2\pi} r^4 \log \frac{1}{r}.$$

We remark that it is possible to give a short proof of (4.1) using (2.6) and a little bit of stochastic calculus. We give the following proof, however, because it easily generalizes to the case of (4.2) and the intermediate estimates will be important for us later on.

PROOF OF LEMMA 4.1. We will only prove (4.1) as (4.2) follows from the same argument. Using the representation (2.3), observe

$$\mathbf{E}(\mu(D(z, r)))^2 = \frac{4}{\pi^2} \sum_{i, j \ge 1} \frac{1}{i^2 + j^2} G_{ij}^2(z, r).$$

Let $g(r) = (r \log \log \frac{1}{r})^{-1}$,

$$\Sigma_1 = \sum_{i,j \leq g(r)} \frac{1}{i^2 + j^2} G_{ij}^2(z, r) \quad \text{and} \quad \Sigma_2 = \sum_{i \vee j > g(r)} \frac{1}{i^2 + j^2} G_{ij}^2(z, r).$$

With $z = (x, y)$ the symmetry of $D(0, r)$ implies

$$\begin{aligned} G_{ij}(z, r) &= \int_{D(0,r)} \sin(\pi i (u + x)) \sin(\pi j (v + y)) \, du \, dv \\ &= \int_{D(0,r)} \sin(\pi i x) \sin(\pi j y) \cos(\pi i u) \cos(\pi j v) \, du \, dv \\ &= \sin(\pi i x) \sin(\pi j y) \Big(\pi r^2 + \int_{D(0,r)} [\cos(\pi i u) \cos(\pi j v) - 1] \, du \, dv \Big) \\ &= \sin(\pi i x) \sin(\pi j y) (\pi r^2 + O(r^4 (i \vee j)^2)), \end{aligned}$$

(4.3)

so that for $i, j \leq g(r)$,

$$G_{ij}^2(z, r) = \sin^2(\pi i x) \sin^2(\pi j y) \Big(\pi^2 r^4 + O\Big(r^4 \Big(\log \log \frac{1}{r} \Big)^{-2} \Big) \Big).$$

Thus, by Lemma A.1,

$$\Sigma_1 \sim \frac{\pi^3}{8} r^4 \log \frac{1}{r}.$$

We have

$$|G_{ij}(z, r)| = \left| \int_{x-r}^{x+r} \int_{y - \sqrt{r^2 - x^2}}^{y + \sqrt{r^2 - x^2}} \sin(\pi i u) \sin(\pi j v) \, du \, dv \right| \leq \int_{x-r}^{x+r} \frac{2}{\pi i} \, dv = \frac{4r}{\pi i}.$$

Similarly, $|G_{ij}(z, r)| \leq \frac{4r}{\pi j}$ so that $G_{ij}(z, r) = O(\frac{r}{i \vee j})$. As

$$\sum_{i \geq 1} \sum_{j \geq g(r)} \frac{r^2}{(i^2 + j^2)(i \vee j)^2} = O\Big(\int_1^{\infty} \int_{g(r)}^{\infty} \frac{r^2}{(u^2 + v^2) u^2} \, du \, dv \Big)$$

and

$$\begin{aligned} \int_1^{\infty} \int_{g(r)}^{\infty} \frac{r^2}{(u^2 + v^2) u^2} \, du \, dv &= r^2 \int_{g(r)}^{\infty} \frac{1}{u^4} \int_1^{\infty} \frac{1}{1 + v^2/u^2} \, dv \, du \\ &= O\Big(r^4 \Big(\log \log \frac{1}{r} \Big)^2 \Big) \end{aligned}$$

(4.4)

it follows that Σ_2 is negligible compared to Σ_1 as $r \to 0$. Therefore,

$$\mathbf{E}(\mu(D(z, r)))^2 \sim \frac{4}{\pi^2} \Sigma_1 \sim \frac{\pi}{2} r^4 \log \frac{1}{r}. \qquad \square$$

The purpose of the next lemma is to show that the same estimates hold for *conformal images* of disks and squares, the proof by simple Fourier analysis. We will need to introduce some more notation. For $\xi \in U$, let $\rho(r) = \rho(\xi, r) = |\varphi'(\xi)|r$, $E(\xi, r) = \varphi(D(\xi, r))$, $T(\xi, r) = \varphi(S(\xi, r))$, and

$$H_{ij}(\xi, r) = \int_{E(\xi, r)} \sin(\pi i u) \sin(\pi j v) \, du \, dv.$$

In the case of the former, we will always write $\rho(r)$ since ξ will be clear from the context. Obviously, the collection of functions

$$(x, y) \mapsto 2 \sin(\pi i x) \sin(\pi j y)$$

is orthonormal in $L^2([0, 1]^2)$ so that with $z = \varphi(\xi)$ Lemma B.1 gives the bound

$$(4.5) \quad \sum_{i,j \geq 1} \left(G_{ij}(z, \rho(r)) - H_{ij}(\xi, r) \right)^2 \leq C \int_{[0,1]^2} \left| 1_{D(z, \rho(r))} - 1_{E(\xi, r)} \right|^2 = O(r^3),$$

which holds uniformly in $\xi \in K$. As another consequence of Lemma B.1, we have

$$(4.6) \qquad |G_{ij}(z, \rho(r)) - H_{ij}(\xi, r)| = O(r^3) \qquad \text{as } r \to 0,$$

uniformly in i, j and $\xi \in K$.

LEMMA 4.2.   *Uniformly in $\xi \in K$ we have*

$$(4.7) \qquad\qquad \mathbf{E}(\mu(E(\xi, r)))^2 \sim \frac{\pi}{2} \rho^4(r) \log \frac{1}{\rho(r)},$$

$$(4.8) \qquad\qquad \mathbf{E}(\mu(T(\xi, r)))^2 \sim \frac{1}{2\pi} \rho^4(r) \log \frac{1}{\rho(r)}.$$

PROOF.   As in the proof of Lemma 4.1, we will only show (4.7) since the justification of (4.8) is exactly the same. Fix $\xi \in K$. For $z = \varphi(\xi)$, let

$$\Gamma_1 = \sum_{i,j \leq g(r)} \frac{G_{ij}^2(z, \rho(r)) - H_{ij}^2(\xi, r)}{i^2 + j^2}$$

and

$$\Gamma_2 = \sum_{i \vee j > g(r)} \frac{G_{ij}^2(z, \rho(r)) - H_{ij}^2(\xi, r)}{i^2 + j^2}.$$

Using

$$\sum_{i,j \leq n} \frac{1}{i^2 + j^2} = O(\log n),$$

we see that (4.6) implies

$$\Gamma_1 = \sum_{i,j \le g(r)} \frac{1}{i^2 + j^2} (G_{ij}(z, \rho(r)) + H_{ij}(\xi, r))(G_{ij}(z, \rho(r)) - H_{ij}(\xi, r))$$

(4.9)

$$= O(\log g(r)) O(r^2) O(r^3) = O\left(r^5 \log \frac{1}{r}\right).$$

An application of (4.5) and the Cauchy–Schwarz inequality gives

$$\left| \sum_{i \vee j > g(r)} (G_{ij}^2(z, \rho(r)) - H_{ij}^2(\xi, r)) \right|$$

$$\le \left(\sum_{i,j \ge 1} (G_{ij}(z, \rho(r)) + H_{ij}(\xi, r))^2 \right)^{1/2}$$

$$\times \left(\sum_{i,j \ge 1} (G_{ij}(z, \rho(r)) - H_{ij}(\xi, r))^2 \right)^{1/2}$$

$$= [O(r^2) O(r^3)]^{1/2} = O(r^{5/2}).$$

Hence,

$$|\Gamma_2| \le \left(\sup_{i \vee j > g(r)} \frac{1}{i^2 + j^2} \right) \left| \sum_{i,j > g(r)} (G_{ij}^2(z, \rho(r)) - H_{ij}^2(\xi, r)) \right|$$

(4.10)

$$= O\left(r^4 \left(\log \log \frac{1}{r} \right)^2 \right).$$

Therefore, uniformly in $\xi \in K$ and with $z = \varphi(\xi)$,

$$\frac{1}{\rho^4(r) \log 1/\rho(r)} |\mathbf{E}(\mu(D(z, \rho(r))))^2 - \mathbf{E}(\mu(E(\xi, r)))^2| \to 0 \qquad \text{as } r \to 0. \qquad \square$$

Let (α_{ij}) be the coefficients of F as in (2.3) expressed in terms of the $H_0^1([0,1]^2)$ eigenbasis of Δ. Let $r_n = e^{-n}$. For $r_{n+1} < r \le r_n$, set $\zeta(r) = r_n$, $\widehat{g}(r) = g(\zeta(r))$, and define

$$\nu(A) = \mu(A) - \sum_{i,j \le \widehat{g}(r)} \frac{2\alpha_{ij}}{\pi \sqrt{i^2 + j^2}} |A| \sin(\pi i x) \sin(\pi j y),$$

where A is either a disk or a square centered at $z = (x, y) \in L$ of radius r. If A is the image of a disk or square centered at $\xi = \psi(z) \in K$ of radius r under φ, then we set

$$\nu(A) = \mu(A) - \sum_{i,j \le \widehat{g}(\rho(r))} \frac{2\alpha_{ij}}{\pi \sqrt{i^2 + j^2}} |A| \sin(\pi i x) \sin(\pi j y).$$

The estimates (4.3), (4.4) and (4.9), (4.10) imply

$$\mathbf{E}(\nu(A))^2 = O\left(r^4 \left(\log\log\frac{1}{r}\right)^2\right).$$

LEMMA 4.3. *Let A be either a disk or square in $[0, 1]^2$ centered in L or the image of such in U under φ centered in K. Then there exists $\alpha = \alpha(\omega) > 0$ such that almost surely $\mathrm{diam}(A) \leq \alpha$ implies uniformly*

$$(4.11) \qquad |\mu(A)| = O\left(|A|\log\frac{1}{|A|}\right),$$

$$(4.12) \qquad |\nu(A)| = o\left(|A|\log\frac{1}{|A|}\right).$$

We will not make use of (4.11) but record the result anyway because its proof is the same as that of (4.12).

PROOF OF LEMMA 4.3. We are going to give the complete proof in the case that A is a disk or square in $[0, 1]^2$ centered in L and then indicate the necessary modifications to show that the result also holds for conformal images. Lemma 4.1 implies

$$\mathbf{E}(\mu(S(z, 2^{-n})))^2 \sim \frac{1}{2\pi}(2^{-n})^4(\log 2^n)$$

so that for some $c_1 > 0$ and n large enough

$$\frac{(2^{-n})^2(\log 2^n)}{\sqrt{\mathbf{E}(\mu(S(z, 2^{-n})))^2}} \geq c_1\sqrt{n}.$$

Therefore, by Lemma A.4 with $c_2 = \sqrt{6}c_1^{-1}$, we have

$$\mathbf{P}\big(|\mu(S(z, 2^{-n}))| > c_2(2^{-n})^2(\log 2^n)\big) = O(2^{-3n}).$$

Fix $\varepsilon > 0$ so that L^ε, the ε-neighborhood of L, satisfies $\overline{L^\varepsilon} \subseteq (0, 1)^2$. Letting $\mathcal{S}_n$ be the set of dyadic squares in $[0, 1]^2$ contained in L^ε of side length 2^{-n} we see

$$\sum_{n \geq 1} \sum_{S \in \mathcal{S}_n} \mathbf{P}\big(|\mu(S)| > c_2(2^{-n})^2(\log 2^n)\big) < \infty.$$

By the Borel–Cantelli lemma, there exists $n_0 = n_0(\omega)$ such that for $n \geq n_0$ almost surely

$$(4.13) \qquad |\mu(S)| \leq c_2(2^{-n})^2(\log 2^n) \qquad \text{for all } S \in \mathcal{S}_n.$$

Suppose $R = [a, b] \times [c, d] \subseteq L^\varepsilon$ is a rectangle with length $l = d - c$ and width $w = b - a = 2^{-n}$ with $n > n_0$ and $a = i/2^n, b = (i + 1)/2^n$ dyadic rationals.

Assume further that $l \geq w$. Fit as many dyadic squares of side length 2^{-n} into R as possible. Visibly, the number of such squares is bounded by $l/2^{-n}$. The set that arises by removing these squares from R consists of two ends each of which contains at most

$$\frac{2^{-n}}{2^{-n-1}} \cdot \frac{2^{-n}}{2^{-n-1}} = 2^2$$

dyadic squares of side length 2^{-n-1}. After removing these, each end now contains at most

$$\frac{2^{-n}}{2^{-n-2}} \cdot \frac{2^{-n-1}}{2^{-n-2}} = 2^3$$

dyadic squares of side length 2^{-n-2}. Iterating this procedure, each end contains at most

$$\frac{2^{-n}}{2^{-n-k}} \cdot \frac{2^{-n-(k-1)}}{2^{-n-k}} = 2^{k+1}$$

squares of side length 2^{-n-k} at the kth step. Thus,

$$|\mu(R)| \leq c_2 \left(\frac{l}{2^{-n}} (2^{-n})^2 (\log 2^n) + 2 \sum_{k \geq 1} 2^{k+1} (2^{-n-k})^2 (\log 2^{n+k}) \right)$$

$$\leq c_3 l w \log \frac{1}{w}.$$

Now suppose that $2^{-n-1} < w \leq 2^{-n}$ is not necessarily dyadic and $l \geq 2^{-n-1}$. Then a maximal decomposition of R into rectangles of length l and with left- and right-hand sides located at rationals of the form $i/2^{n+k}, (i+1)/2^{n+k}$, always taking the largest possible such rectangle, contains at most two of width 2^{-n-k} for each $k \in \mathbf{N}$ so that

$$|\mu(R)| \leq c_4 l \left(\sum_{m \geq n+1} 2^{-m} \log 2^m \right) \leq c_5 l 2^{-n} n \leq c_6 l w \log \frac{1}{lw}$$

(4.14)
$$= c_6 |R| \log \frac{1}{|R|}.$$

Note that this argument also works with the roles of l and w reversed.

Let A be a disk contained in L^ε with radius $r < 2^{-n_0-1}$. Slice A vertically starting from the center to the right and left into equal pieces of width r^2 and then slice it once horizontally through the center. Let A_1 be the set consisting of the union of the largest rectangles that fit into each slice. Then, since there are at most $4r/r^2 = 4/r$ rectangles, each of area at most r^3, (4.14) gives us

$$|\mu(A_1)| \leq 12 c_6 r^2 \log \frac{1}{r}.$$

Slice the regions above and below each of the rectangles in A_1, including the degenerate rectangles on the left- and right-hand sides, into equal pieces of width r^3. Denote the union of all the largest rectangles contained in these slices by A_2 and note that the length of each rectangle is at most $\sqrt{2}r^3$. The reason for this is that the maximal length of such a rectangle with horizontal coordinates contained in the interval $[a, b]$, say with $a \geq 0$, is given by $f(a) - f(b)$ where $f(x) = \sqrt{r^2 - x^2}$. Obviously,

$$f(a) - f(b) \leq f(r - (b - a)) - f(r) = f(r - (b - a)).$$

In our case $b - a = r^2$ so that we have the bound $f(r - r^2) \leq \sqrt{2}r^3$. If we iterate this procedure so that at the nth step we slice out rectangles of width r^{n+1} then at most $4 \cdot \frac{r}{r^{n+1}} = 4r^{-n}$ rectangles each with length at most $f(r - r^n) \leq \sqrt{2}r^{(n+1)/2}$ and hence with area at most $\sqrt{2}r^{(n+1)/2} \cdot r^{n+1} = \sqrt{2}r^{3(n+1)/2}$. If A_n denotes the region from the nth step for $n \geq 2$, then

$$|\mu(A_n)| \leq 4\sqrt{2}c_6 r^{-n} \cdot r^{3(n+1)/2} \log \frac{1}{\sqrt{2}r^{3(n+1)/2}}$$

$$\leq 12(n+1)c_6 r^{(n+3)/2} \log \frac{1}{r}.$$

Therefore,

$$|\mu(A)| = \left| \mu\left(\bigcup_n A_n \right) \right| \leq c_7 r^{3/2} \log \frac{1}{r} \sum_{n \geq 1} n r^{n/2} \leq c_8 r^2 \log \frac{1}{r}.$$

This completes the proof of (4.11) when A is either a disk or square centered in L.

If A is a square centered in L of radius r, we know that

$$\mathbf{E}(\nu(A))^2 = O\left(r^4 \left(\log \log \frac{1}{r} \right)^2 \right) \qquad \text{as } r \to 0,$$

so that for some $d_1 > 0$ and n large enough,

$$\frac{(2^{-n})^2 (\log 2^n)(\log \log 2^n)^{-1}}{\sqrt{\mathbf{E}(\nu(S(z, 2^{-n})))^2}} \geq d_1 \frac{n}{(\log n)^2} \geq d_1 \sqrt{n}.$$

Hence for $d_2 > 0$ appropriately chosen and n large enough,

$$\mathbf{P}(|\nu(S(z, 2^{-n}))| \geq d_2(2^{-n})^2 (\log 2^n)(\log \log 2^n)^{-1}) = O(2^{-3n}).$$

With $a(r) = (\log \log \frac{1}{r})^{-1}$ it follows from the Borel–Cantelli lemma that on dyadic squares small enough and contained in L^ε we have

$$|\nu(S)| \leq d_2 |S| \left(\log \frac{1}{|S|} \right) a(|S|).$$

If we do the covering argument as before, we can bound from above the ν-mass of the intermediate dyadic squares S in our cover of A by

$$|\nu(S)| \leq d_2 |S| \left(\log \frac{1}{|S|} \right) a(|S|) \leq d_2 |S| \left(\log \frac{1}{|S|} \right) a(|A|).$$

Thus, (4.12) is now obvious.

To deduce the case when A is a conformal image one runs the same argument except instead of building coverings by dyadic squares in $[0,1]^2$ one works with coverings by *conformal images* of dyadic squares in U. Indeed, we know by Lemma 4.2 that the images of squares satisfy the same asymptotic variance bounds as those in L^ε up to a factor of $|\varphi'(\xi)|^2$. Hence, one only needs to keep uniform control on $|\varphi'(\xi)|$ which is easily accomplished by restricting to dyadic squares contained in a neighborhood K^δ of K such that $\overline{K^\delta} \subseteq U$. $\square$

4.2. *Proof of conformal invariance.*   Let $U, V \subseteq \mathbf{C}$ be bounded domains with smooth boundary and $\varphi : U \to V$ a conformal transformation with inverse ψ.

PROOF OF THEOREM 1.3.   Let (S_n), $S_n = S(z_n, r_n)$, be a covering of V by closed squares such that $S(z_n, 2r_n) \subseteq V$. Fix $K \subseteq U$ compact. With $R_n = \psi(S_n)$, we can find indices $i_1, \ldots, i_k$ such that $K \subseteq \bigcup_{1 \leq j \leq k} R_{i_j}$. Therefore, it suffices to show

$$(4.15) \quad \lim_{r \to 0} \sup_{\xi \in K \cap R_{i_j}} \frac{1}{h(r)} \left| \mu_U(D(\xi, r)) - \mu_V(D(\varphi(\xi), |\varphi'(\xi)|r)) |\psi'(z)|^2 \right| = 0$$

for each j where $h(r) = \pi r^2 \log \frac{1}{r}$. If we write $F_U|R_{i_j} = F_{i_j} + H_{i_j}$ with F_{i_j} a zero-boundary GFF and H_{i_j} harmonic on R_{i_j} then the term arising from H_{i_j} in (4.15) is negligible. As the same is also true for $F_V|S_{i_j}$, we therefore may assume without loss of generality that $U = \psi(S(z_{i_j}, 2r_{i_j}))$, which contains $R_{i_j} \cap K$, and $V = S(z_{i_j}, 2r_{i_j})$. By a translation and rescaling, we may further assume $V = [0,1]^2$.

For $\xi \in U$, let $E(\xi, r) = \varphi(D(\xi, r))$ be the image of the disk $D(\xi, r) \subseteq U$ under φ. With $\rho(r) = |\varphi'(\xi)|r$, Lemma B.1 implies $|E(\xi, r) \triangle D(\varphi(\xi), \rho(r))| = O(r^3)$ so that by Lemma A.2 we have

$$\mathbf{P}\left(\sup_{\xi \in K} \frac{1}{h(\rho(r))} \sum_{1 \leq i, j \leq g(r)} \frac{|\alpha_{ij}| |E(\xi, r) \triangle D(\varphi(\xi), \rho(r))|}{\sqrt{i^2 + j^2}} \geq t \right)$$

$$= \mathbf{P}\left(\sum_{1 \leq i, j \leq g(r)} \frac{|\alpha_{ij}| O(r)}{\sqrt{i^2 + j^2}} \geq t \log \frac{1}{r} \right)$$

$$\leq r^t e^{O(r^2 \log g(r))} \prod_{1 \leq i, j \leq g(r)} \left(1 + \frac{O(r)}{\sqrt{i^2 + j^2}} \right).$$

The inequality $\log(1 + x) \leq x$ yields

$$\log \prod_{1 \leq i, j \leq g(r)} \left(1 + \frac{O(r)}{\sqrt{i^2 + j^2}}\right) \leq O(r) \sum_{1 \leq i, j \leq g(r)} \frac{1}{\sqrt{i^2 + j^2}}$$

$$= O\left(\left(\log \log \frac{1}{r}\right)^{-1}\right).$$

Taking $r_n = e^{-n}$ and $t_n = n^{-1/2}$ it thus follows from the Borel–Cantelli lemma that there exists $n_0 = n_0(\omega)$ such that almost surely $n \geq n_0$ implies

$$\sup_{\xi \in K} \frac{1}{h(\rho(r_n))} \sum_{1 \leq i, j \leq g(r_n)} \frac{|\alpha_{ij}||E(\xi, r_n) \Delta D(\varphi(\xi), \rho(r_n))|}{\sqrt{i^2 + j^2}} \leq \frac{1}{\sqrt{n}}.$$

Combining this with Lemma 4.3 yields for all n sufficiently large and $r_{n+1} < \rho(r) \leq r_n$ that

$$|\mu(E(\xi, r)) - \mu(D(z, \rho(r)))|$$

$$\leq \sum_{1 \leq i, j \leq g(r_n)} \frac{2|\alpha_{ij}|||E(\xi, r)| - |D(z, \rho(r))||}{\pi \sqrt{i^2 + j^2}}$$

$$+ |\nu(E(\xi, r))| + |\nu(D(z, \rho(r)))|$$

$$\leq C \sum_{1 \leq i, j \leq g(r_n)} \frac{|\alpha_{ij}||E(\xi, r_n) \Delta D(z, \rho(r_n))|}{\sqrt{i^2 + j^2}} + o(h(r))$$

$$= o(h(r))$$

uniformly in $\xi \in K$. Therefore,

$$(4.16) \qquad \limsup_{r \to 0} \sup_{\xi \in K} \frac{1}{h(\rho(r))} |\mu(E(\xi, r)) - \mu(D(\varphi(\xi), \rho(r)))| = 0.$$

With $F_U = F \circ \varphi^{-1}$ the GFF on U and $\mu_U(A) = \int_A F_U(x)\,dx$, a change of variables gives

$$\mu_U(D(\xi, r)) = \int_{[0,1]^2} F 1_{E(\xi, r)} |\psi'|^2 = [|\psi'(z)|^2 + O(r)]\mu(E(\xi, r))$$

uniformly in $\xi \in K$. With $z = \varphi(\xi)$ we have

$$\left|\mu_U(D(\xi, r)) - \mu(D(z, \rho(r)))|\varphi'(\xi)|^{-2}\right|$$

$$= \left|\mu(E(\xi, r))[|\psi'(z)|^2 + O(r)] - \mu(D(z, \rho(r)))|\varphi'(\xi)|^{-2}\right|$$

$$\leq \left|\mu(E(\xi, r))|\psi'(z)|^2 - \mu(D(z, \rho(r)))|\varphi'(\xi)|^{-2}\right| + O(r)|\mu(E(\xi, r))|.$$

The theorem now follows as by (4.16), we have

$$\mu(E(\xi, r))|\psi'(z)|^2 = \mu(D(z, \rho(r)))|\psi'(z)|^2 + o(h(r))$$

$$= \mu(D(z, |\varphi'(\xi)|r))|\varphi'(\xi)|^{-2} + o(h(r)). \qquad \square$$

APPENDIX A: GAUSSIAN ESTIMATES

LEMMA A.1. *If $L \subseteq (0, 1)^2$ is compact, then*

$$\sum_{1 \le i, j \le n} \frac{\sin^2(\pi i x) \sin^2(\pi j y)}{i^2 + j^2} \sim \frac{\pi}{8} \log n \qquad as\ n \to \infty,$$

uniformly in $(x, y) \in L$.

PROOF. Observe there exists $c > 0$ such that

$$\sum_{1 \le i, j \le n} \frac{\sin^2(\pi i x) \sin^2(\pi j y)}{i^2 + j^2} \ge c \log n.$$

Hence, as far as the asymptotics of the summation are concerned, we may ignore terms that are $o(\log n)$. Thus as $n \to \infty$, we have

$$\sum_{1 \le i, j \le n} \frac{\sin^2(\pi i x) \sin^2(\pi j y)}{i^2 + j^2}$$

$$\sim \int_1^n \int_1^n \frac{\sin^2(\pi u x) \sin^2(\pi v y)}{u^2 + v^2}\, du\, dv$$

$$\sim \sum_{1 \le i, j \le n} \int_{i/x}^{(i+1)/x} \int_{j/y}^{(j+1)/y} \frac{\sin^2(\pi u x) \sin^2(\pi v y)}{u^2 + v^2}\, du\, dv$$

$$\sim \sum_{1 \le i, j \le n} \frac{1}{(i/x)^2 + (j/y)^2} \int_{i/x}^{(i+1)/x} \int_{j/y}^{(j+1)/y} \sin^2(\pi u x) \sin^2(\pi v y)\, du\, dv$$

$$= \frac{1}{4} \sum_{1 \le i, j \le n} \frac{1}{(i/x)^2 + (j/y)^2} \frac{1}{x} \cdot \frac{1}{y}$$

$$\sim \frac{1}{4} \sum_{1 \le i, j \le n} \int_{i/x}^{(i+1)/x} \int_{j/y}^{(j+1)/y} \frac{1}{u^2 + v^2}\, du\, dv$$

$$\sim \frac{1}{4} \int_1^n \int_1^n \frac{1}{u^2 + v^2}\, du\, dv$$

$$\sim \frac{\pi}{8} \log n. \qquad \square$$

LEMMA A.2. *If (X_n) is an i.i.d. sequence of standard normals and (β_n) a sequence of positive constants, then we have*

$$\mathbf{P}\left(\sum_n \beta_n |X_n| \ge t\right) \le e^{-t} \prod_n (1 + \beta_n) e^{\beta_n^2/2}.$$

PROOF.   Markov's inequality gives

$$\mathbf{P}\left(\sum_n \beta_n |X_n| \geq t\right) \leq e^{-t} \prod_n \mathbf{E}\exp(\beta_n |X_n|).$$

If $X \sim N(0, 1)$ and $\beta > 0$, we have

$$\mathbf{E}e^{\beta X} 1_{\{X \geq 0\}} = \frac{e^{\beta^2/2}}{\sqrt{2\pi}} \int_{-\beta}^{\infty} e^{-x^2/2}\,dx \leq \left(\frac{1}{2} + \frac{\beta}{\sqrt{2\pi}}\right)e^{\beta^2/2}.$$

Thus, $\mathbf{E}e^{\beta|X|} \leq (1 + \beta)e^{\beta^2/2}$. Combining everything gives the result.   □

LEMMA A.3.   *Let $B(t)$ be a standard Brownian motion, $\mu > 0$, and $T \geq 1$ fixed. Then*

$$\mathbf{P}\left(\max_{0 \leq t \leq T} |B(t) - \mu t| \leq \sqrt{T}\right) \geq C \exp\left(\frac{1}{2}(\mu\sqrt{T} - \mu^2 T)\right),$$

where $C > 0$ is a constant independent of T.

PROOF.   Let $E_T^{\mu} = \{\max_{0 \leq t \leq T} |B(t) - \mu t| \leq \sqrt{T}\}$. By the Girsanov theorem,

$$\mathbf{P}(E_T^{\mu}) = \mathbf{E}[e^{\mu B(T) - \mu^2 T/2} 1_{E_T^0}] \geq \exp(\tfrac{1}{2}(\mu\sqrt{T} - \mu^2 T))\mathbf{P}[E_T^0, B(T) \geq \sqrt{T}/2].$$

Taking

$$C = \mathbf{P}[E_T^0, B(T) \geq \sqrt{T}/2] = \mathbf{P}\left(\max_{0 \leq t \leq 1}|B(t)| \leq 1, B(1) \geq 1/2\right) > 0$$

proves the lemma.   □

LEMMA A.4.   *If $Z \sim N(0, 1)$, then*

$$\mathbf{P}(|Z| > \lambda) \sim \sqrt{\frac{2}{\pi}}\lambda^{-1}e^{-\lambda^2/2} \qquad as\ \lambda \to \infty.$$

PROOF.   See Lemma 1.1 of [16].   □

APPENDIX B: AREA DISTORTION UNDER CONFORMAL MAPS

LEMMA B.1.   *Suppose that $U, V \subseteq \mathbf{C}$ are domains with $K \subseteq U$ compact. If $\varphi : U \to V$ is a conformal transformation, then*

$$|E(\xi, r) \Delta D(\varphi(\xi), \rho(r))| = O(r^3)$$

uniformly in $\xi \in K$ where $E(\xi, r) = \varphi(D(\xi, r))$ and $\rho(r) = \rho(\xi, r) = |\varphi'(\xi)|r$.

PROOF. For $|\xi - \eta| \leq r$, we have

$$|\varphi(\xi) - \varphi(\eta)| = |\varphi'(\xi)(\xi - \eta) + O(r)(\xi - \eta)|$$

so that

$$(|\varphi'(\xi)| - O(r))|\xi - \eta| \leq |\varphi(\xi) - \varphi(\eta)| \leq (|\varphi'(\xi)| + O(r))|\xi - \eta|.$$

This implies

$$D(\varphi(\xi), \rho(r) - O(r^2)) \subseteq E(\xi, r) \subseteq D(\varphi(\xi), \rho(r) + O(r^2)),$$

which gives

$$|E(\xi, r) \triangle D(\varphi(\xi), \rho(r))| \leq |D(\xi, \rho(r) + O(r^2))| - |D(\xi, \rho(r) - O(r^2))|$$
$$= O(r^3) \qquad \text{as } r \to 0$$

uniformly in $z \in K$. $\square$

APPENDIX C: MODIFIED KOLMOGOROV–CENTSOV

LEMMA C.1. *Suppose that $U \subseteq \mathbf{R}^d$ is a bounded open set and that $X : U \times (0, 1] \to \mathbf{R}$ is a time-varying random field satisfying*

$$\mathbf{E}|X(z, r) - X(w, s)|^\alpha \leq C \left(\frac{|(z, r) - (w, s)|}{r \wedge s} \right)^{d+1+\beta}$$

for some $\alpha, \beta > 0$. For each $\zeta > \alpha^{-1}$ and $\gamma \in (0, \beta/\alpha)$, X has a modification $\tilde{X}$ satisfying

$$|\tilde{X}(z, r) - \tilde{X}(w, s)| \leq M \left(\log \frac{1}{r} \right)^\zeta \frac{|(z, r) - (w, s)|^\gamma}{r^{\tilde{\gamma}}},$$

where

$$\tilde{\gamma} = \frac{d + \beta}{\alpha},$$

$z, w \in U$ and $r, s \in (0, 1]$ with $1/2 \leq r/s \leq 2$.

The proof is almost exactly the same as the usual proof of the Kolmogorov–Centsov theorem [12, 18]. We will include a proof for the convenience of the reader which will follow very closely that given in [18].

PROOF OF LEMMA C.1. We may assume without loss of generality that $U \subseteq [0, 1]^d$ by rescaling. For each $n, T \in \mathbf{N}$, let

$$R_n^T = \{(\underline{i}, j)/2^n \in U \times (2^{-T}, 2^{1-T}] : \underline{i} \in \mathbf{Z}^d, j \in \mathbf{Z}\} \quad \text{and} \quad R^T = \bigcup_n R_n^T.$$

Let Δ_n^T be the set of pairs $a, b \in R_n^T$ such that $|a - b| = 2^{-n}$. Trivially, $|\Delta_n^T| = O(2^{(n+1)(d+1)-T})$. Let

$$K_i = \sup_{T \geq 1} \left(\frac{2^{-\tilde{\gamma} T}}{T^{\zeta}} \sup_{(a,b) \in \Delta_i^T} |X(a) - X(b)| \right).$$

We have

$$\mathbf{E} K_i^{\alpha} \leq \sum_{T \geq 1} \frac{2^{-\alpha \tilde{\gamma} T}}{T^{\alpha \zeta}} \sum_{a,b \in \Delta_i^T} \mathbf{E} |X(a) - X(b)|^{\alpha}$$

$$\leq \sum_{T \geq 1} \frac{C 2^{-(d+\beta) T}}{T^{\alpha \zeta}} 2^{(i+1)(d+1)-T} \cdot 2^{(T-i)(d+1+\beta)} = O(2^{-i\beta}).$$

For $a, b \in U \times (0, 1]$, we say that $a \leq b$ if the corresponding component-wise inequalities hold. If $a \in R^T$, then there exists an increasing sequence (a_n) in $U \times (2^{-T}, 2^{1-T}]$ such that $a_n \in R_n^T$ for every n, $a_n \leq a$, and $a_n = a$ for all n large enough. Let $b \in R^T$ and (b_n) be such a sequence for b. Assume $|a - b| \leq 2^{-m}$. Then

$$X(a) - X(b) = \sum_{i=m}^{\infty} \left(X(a_{i+1}) - X(a_i) \right) + X(a_m) - X(b_m)$$

$$+ \sum_{i=m}^{\infty} \left(X(b_i) - X(b_{i+1}) \right),$$

which implies

$$\frac{2^{-\tilde{\gamma} T}}{T^{\zeta}} |X(a) - X(b)| \leq K_m + 2 \sum_{i=m+1}^{\infty} K_i \leq 2 \sum_{i=m}^{\infty} K_i.$$

We have

$$A \equiv \sup_{T,m} \left(\sup \left\{ \frac{2^{-\tilde{\gamma} T} |X(a) - X(b)|}{T^{\zeta} |a - b|^{\gamma}} : a, b \in R^T, 2^{-(m+1)} \leq |a - b| \leq 2^{-m} \right\} \right)$$

$$\leq \sup_{m \in \mathbf{N}} \left(2^{\gamma(m+1)+1} \sum_{i=m}^{\infty} K_i \right) \leq 2 \sum_{i=0}^{\infty} 2^{\gamma i} K_i.$$

This implies $\mathbf{E} A^{\alpha} < \infty$ so that for some $M > 0$ when $(z, r), (w, s) \in R^T$ we have

$$|X(z, r) - X(w, s)| \leq M T^{\zeta} \frac{|(z, r) - (w, s)|^{\gamma}}{2^{-\tilde{\gamma} T}} \leq M \left(\log \frac{1}{r} \right)^{\zeta} \frac{|(z, r) - (w, s)|^{\gamma}}{r^{\tilde{\gamma}}}.$$

With $R = \bigcup_T R^T$,

$$\tilde{X}(a) = \lim_{\substack{b \to a \\ b \in R}} X(b)$$

is clearly is the desired modification. $\quad \square$

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