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GAUSSIAN MULTIPLICATIVE CHAOS REVISITED

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In this article, we extend the theory of multiplicative chaos for positive definite functions in $\mathbb{R}^d$ of the form $f(x) = \lambda^2 \ln^+ \frac{R}{|x|} + g(x)$, where g is a continuous and bounded function. The construction is simpler and more general than the one defined by Kahane in [*Ann. Sci. Math. Québec* **9** (1985) 105–150]. As a main application, we provide a rigorous mathematical meaning to the Kolmogorov–Obukhov model of energy dissipation in a turbulent flow.

1. Introduction. The theory of multiplicative chaos was first defined rigorously by Kahane in 1985 in the article [13]. More specifically, Kahane constructed a theory relying on the notion of a σ-positive-type kernel: a generalized function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ \cup \{\infty\}$ is of σ-positive type if there exists a sequence $K_k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ of continuous positive and positive definite kernels such that

$$K(x, y) = \sum_{k \geq 1} K_k(x, y). \tag{1.1}$$

If K is a σ-positive-type kernel with decomposition (1.1), one can consider a sequence of Gaussian processes $(X_n)_{n\geq 1}$ of covariance given by $\sum_{k=1}^n K_k$. It is proved in [13] that the sequence of random measures m_n given by

$$m_n(A) = \int_A e^{X_n(x) - (1/2)E[X_n(x)^2]}\, dx, \qquad A \in \mathcal{B}(\mathbb{R}^d), \tag{1.2}$$

converges almost surely in the space of Radon measures (equipped with the topology of weak convergence) to a random measure m and that the limit measure m obtained does not depend on the sequence $(K_k)_{k\geq 1}$ used in the decomposition (1.1) of K. Thus, the theory enables one to give a unique and mathematically rigorous definition to a random measure m in $\mathbb{R}^d$ defined formally by

$$m(A) = \int_A e^{X(x) - (1/2)E[X(x)^2]}\, dx, \qquad A \in \mathcal{B}(\mathbb{R}^d), \tag{1.3}$$

where $(X(x))_{x\in\mathbb{R}^d}$ is a "Gaussian field" whose covariance K is a σ-positive-type kernel. As it will appear later, the σ-positive-type condition is not easy to check in practice. Therefore it is convenient to avoid of this hypothesis.

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The main application of the theory is to give a meaning to the "limit-lognormal" model introduced by Mandelbrot in [17]. In the sequel, we define $\ln^+ x$ for $x > 0$ by means of the following formula:

$$\ln^+ x = \max(\ln(x), 0).$$

The "limit-lognormal" model corresponds to the choice of a homogeneous K given by

$$K(x, y) = \lambda^2 \ln^+(R/|x - y|) + O(1), \tag{1.4}$$

where λ^2, R are positive parameters and $O(1)$ is a bounded quantity as $|x - y| \to 0$. This model has many applications which we will review in the following subsections.

1.1. *Multplicative chaos in dimension* 1: *A model for the volatility of a financial asset.* If $(X(t))_{t\geq 0}$ is the logarithm of the price of a financial asset, the volatility m of the asset on the interval $[0, t]$ is, by definition, equal to the quadratic variation of X:

$$m[0, t] = \lim_{n\to\infty} \sum_{k=1}^{n} \bigl(X(tk/n) - X\bigl(t(k - 1)/n\bigr)\bigr)^2.$$

The volatility m can be viewed as a random measure on $\mathbb{R}$. The choice of m for multiplicative chaos associated with the kernel $K(s, t) = \lambda^2 \ln^+ \frac{T}{|t-s|}$ satisfies many empirical properties measured on financial markets, for example, lognormality of the volatility and long range correlations (see [6] for a study of the SP500 index and components, and [7] for a general review). Note that K is indeed of σ-positive type (see Example 2.3), so m is well defined. In the context of finance, λ^2 is called the *intermittency parameter*, in analogy with turbulence, and T is the correlation length. Volatility modeling and forecasting is an important area of financial mathematics since it is related to option pricing and risk forecasting; we refer to [9] for the problem of forecasting volatility with this choice of m.

Given the volatility m, the most natural way to construct a model for the (log) price X is to set

$$X(t) = B_{m[0,t]}, \tag{1.5}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion independent of m. Formula (1.5) defines the multifractal random walk (MRW) first introduced in [1] (see [2] for a recent review of the financial applications of the MRW model).

1.2. *Multiplicative chaos in dimension* 3: *A model for the energy dissipation in a turbulent fluid.* We refer to [10] for an introduction to the statistical theory of three-dimensional turbulence. Consider a stationary flow with high Reynolds number. It is believed that at small scales, the velocity field of the flow is homogeneous and isotropic in space. By "small scales," we mean scales much smaller than the integral scale R characteristic of the time stationary force driving the flow. In the work [15] and [19], Kolmogorov and Obukhov propose to model the mean energy dissipation per unit mass in a ball $B(x,l)$ of center x and radius $l \ll R$ by a random variable ε_l such that $\ln(\varepsilon_l)$ is normal with variance σ_l^2 given by

$$\sigma_l^2 = \lambda^2 \ln\left(\frac{R}{l}\right) + A,$$

where A is a constant and λ^2 is the intermittency parameter. As noted by Mandelbrot [17], the only way to define such a model is to construct a random measure ε by a limit procedure. Then, one can define ε_l by the formula

$$\varepsilon_l = \frac{3\langle\varepsilon\rangle}{4\pi l^3}\varepsilon(B(x,l)),$$

where $\langle\varepsilon\rangle$ is the average mean energy dissipation per unit mass. Formally, one is looking for a random measure ε such that

$$\forall A \in \mathcal{B}(\mathbb{R}^d) \qquad \varepsilon(A) = \int_A e^{X(x) - (1/2)E[X(x)^2]}\,dx, \tag{1.6}$$

where $(X(x))_{x\in\mathbb{R}^d}$ is a "Gaussian field" whose covariance K is given by $K(x,y) = \lambda^2 \ln^+ \frac{R}{|x-y|}$. The kernel $\lambda^2 \ln^+ \frac{R}{|x-y|}$ is positive definite when considered as a tempered distribution [see (2.1) below for a definition of positive definite distributions and Lemma 3.2 for a proof of this assertion]. Therefore, one can give a rigorous meaning to (1.6) by using Theorem 2.1 below.

However, it is not clear whether $\lambda^2 \ln^+ \frac{R}{|x-y|}$ is of σ-positive type in $\mathbb{R}^3$ and, therefore, in [13], Kahane considers the σ-positive-type kernel $K(x,y) = \int_{1/R}^{\infty} \frac{e^{-u|x-y|}}{u}\,du$ as an approximation of $\lambda^2 \ln^+ \frac{R}{|x-y|}$. Indeed, one can show that $\int_{1/R}^{\infty} \frac{e^{-u|x-y|}}{u}\,du = \ln^+ \frac{R}{|x-y|} + g(|x-y|)$, where g is a bounded continuous function. Nevertheless, it is important to work with $\lambda^2 \ln^+ \frac{R}{|x-y|}$ since this choice leads to measures which exhibit generalized scale invariance properties; see Proposition 3.3.

1.3. *Organization of the paper.* In Section 2, we recall the definition of positive definite tempered distributions and we state Theorem 2.1, wherein we define multiplicative chaos m associated with kernels of the type $\ln^+ \frac{R}{|x|} + O(1)$. In Section 3, we review the main properties of the measure m: existence of moments and density with respect to Lebesgue measure, multifractality and generalized scale invariance. In Sections 4 and 5, we supply the proofs for Sections 2 and 3, respectively.

2. Definition of multiplicative chaos.

2.1. *Positive definite tempered distributions.* Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of smooth, rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions (see [21]). A distribution f in $\mathcal{S}'(\mathbb{R}^d)$ is positive definite if

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d) \qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi(x)\overline{\varphi(y)}\,dx\,dy \geq 0. \tag{2.1}$$

On $\mathcal{S}'(\mathbb{R}^d)$, one can define the Fourier transform $\hat{f}$ of a tempered distribution via the formula

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d) \qquad \int_{\mathbb{R}^d} \hat{f}(\xi)\varphi(\xi)\,d\xi = \int_{\mathbb{R}^d} f(x)\hat{\varphi}(x)\,dx, \tag{2.2}$$

where $\hat{\varphi}(x) = \int_{\mathbb{R}^d} e^{-2i\pi x.\xi}\varphi(\xi)\,d\xi$ is the Fourier transform of φ. An extension of Bochner's theorem (Schwartz [21]) states that a tempered distribution f is positive definite if and only if its Fourier transform is a tempered positive measure.

By definition, a function f in $\mathcal{S}'(\mathbb{R}^d)$ is of σ-positive type if the associated kernel $K(x,y) = f(x-y)$ is of σ-positive type. As mentioned in the Introduction, Kahane's theory of multiplicative chaos is defined for σ-positive-type functions f. The main problem stems from the fact that definition (1.1) is not practical. A key question is whether there exists a simple characterization (like the computation of a Fourier transform) of functions whose associated kernel can be decomposed in the form (1.1). If such a characterization exists, there is the further question of how one finds the kernels K_n explicitly.

Finally, we recall the following simple implication: if f belongs to $\mathcal{S}'(\mathbb{R}^d)$ and is of σ-positive type, then f is positive and positive definite. However, the converse statement is not clear.

2.2. *A generalized theory of multiplicative chaos.* In this subsection, we construct a theory of multiplicative chaos for positive definite functions of type $\lambda^2 \ln^+ \frac{R}{|x|} + O(1)$, without the assumption of σ-positivity for the underlying function. The theory is therefore much easier to use.

We consider, in $\mathbb{R}^d$, a positive definite function f such that

$$f(x) = \lambda^2 \ln^+ \frac{R}{|x|} + g(x), \tag{2.3}$$

where $\lambda^2 \neq 2d$ and $g(x)$ is a bounded continuous function. Let $\theta : \mathbb{R}^d \to \mathbb{R}$ be some continuous function with the following properties:

(1) θ is positive definite;
(2) $|\theta(x)| \leq \frac{1}{1+|x|^{d+\gamma}}$ for some $\gamma > 0$;
(3) $\int_{\mathbb{R}^d} \theta(x)\,dx = 1$.

The following is the main theorem of the article.

THEOREM 2.1 (Definition of multiplicative chaos). *For all* $\varepsilon > 0$, *we consider the centered Gaussian field* $(X_\varepsilon(x))_{x\in\mathbb{R}^d}$ *defined by the convolution*

$$E[X_\varepsilon(x)X_\varepsilon(y)] = (\theta^\varepsilon * f)(y-x),$$

where $\theta^\varepsilon = \frac{1}{\varepsilon^d}\theta(\frac{\cdot}{\varepsilon})$. *The associated random measure* $m_\varepsilon(dx) = e^{X_\varepsilon(x)-(1/2)E[X_\varepsilon(x)^2]}\,dx$ *then converges in law in the space of Radon measures* (*equipped with the topology of weak convergence*), *as* ε *goes to* 0, *to a random measure* m, *independent of the choice of the regularizing function* θ *with properties* (1)–(3). *We call the measure* m *the* multiplicative chaos associated with the function f.

Below, we review two possible choices of the underlying function f. The first example is a d-dimensional generalization of the cone construction considered in [3]. The second example is $\lambda^2 \ln^+ \frac{R}{|x|}$ for $d = 1, 2, 3$ (the case $d = 2, 3$ seems never to have been considered in the literature). Both examples are, in fact, of σ-positive type (except perhaps the crucial example of $\lambda^2 \ln^+ \frac{R}{|x|}$ in dimension $d = 3$) and it is easy to show that in these cases, Theorem 2.1 and Kahane's theory lead to the same limit measure m.

EXAMPLE 2.2. One can construct a positive definite function f with decomposition (2.3) by generalizing the cone construction of [3] to dimension d. This was performed in [5]. For all x in $\mathbb{R}^d$, we define the cone $C(x)$ in $\mathbb{R}^d \times \mathbb{R}_+$:

$$C(x) = \left\{(y,t) \in \mathbb{R}^d \times \mathbb{R}_+; |y-x| \le \frac{t\wedge R}{2}\right\}.$$

The function f is given by

$$f(x) = \lambda^2 \int_{C(0)\cap C(x)} \frac{dy\,dt}{t^{d+1}}. \tag{2.4}$$

One can show that f has decomposition (2.3) (see [5]). The function f is of σ-positive type, in the sense of Kahane, since one can write $f = \sum_{n\ge 1} f_n$ with f_n given by

$$f_n(x) = \lambda^2 \int_{C(0)\cap C(x); 1/n \le t < 1/(n-1)} \frac{dy\,dt}{t^{d+1}}.$$

In dimension $d = 1$, we get the simple formula $f(x) = \lambda^2 \ln^+ \frac{R}{|x|}$.

EXAMPLE 2.3. In dimension $d = 1, 2$, the function $f(x) = \ln^+ \frac{R}{|x|}$ is of σ-positive type, in the sense of Kahane, and, in particular, positive definite. Indeed, one has, by straightforward calculations,

$$\ln^+ \frac{R}{|x|} = \int_0^\infty (t-|x|)_+ \nu_R(dt),$$

where $\nu_R(dt) = 1_{[0,R[}(t)\frac{dt}{t^2} + \frac{\delta_R}{R}$. For all $\mu > 0$, we have

$$\ln^+ \frac{R}{|x|} = \frac{1}{\mu} \ln^+ \frac{R^\mu}{|x|^\mu} = \frac{1}{\mu} \int_0^\infty (t - |x|^\mu)_+ \nu_{R^\mu}(dt).$$

We are therefore led to consider the $\mu > 0$ such that $(1 - |x|^\mu)_+$ is positive definite (the so-called Kuttner–Golubov problem; see [11] for an introduction).

For $d = 1$, it is straightforward to show that $(1 - |x|)_+$ is of σ-positive type. One can thus write $f = \sum_{n \geq 1} f_n$ with f_n given by

$$f_n(x) = \int_{R/n}^{R/(n-1)} (t - |x|)_+ \nu_R(dt).$$

For $d = 2$, the function $(1 - |x|^{1/2})$ is positive definite (Pasenchenko [20]). One can thus write $f = \sum_{n \geq 1} f_n$, with f_n given by

$$f_n(x) = \int_{R^{1/2} n}^{R^{1/2}/(n-1)} (t - |x|^{1/2})_+ \nu_{R^{1/2}}(dt).$$

In dimension $d = 3$, the function $\ln^+ \frac{R}{|x|}$ is positive definite (see Lemma 3.2), but it is an open question whether it is of σ-positive type.

3. Main properties of multiplicative chaos. In the sequel, we will consider the structure functions ζ_p defined for all p in $\mathbb{R}$ by

$$\zeta_p = \left(d + \frac{\lambda^2}{2}\right)p - \frac{\lambda^2 p^2}{2}. \tag{3.1}$$

3.1. *Multiplicative chaos is equal to* 0 *for* $\lambda^2 > 2d$. The following proposition shows that multiplicative chaos is nontrivial only for sufficiently small values of λ^2.

PROPOSITION 3.1. *If* $\lambda^2 > 2d$, *then the limit measure is equal to* 0.

3.2. *Generalized scale invariance.* In this subsection and the following, in view of Proposition 3.1, we will suppose that $\lambda^2 < 2d$.

Let m be a homogeneous random measure on $\mathbb{R}^d$; we recall that this means that for all x, the measures m and $m(x + \cdot)$ are equal in law. We denote by $B(0, R)$ the ball of center 0 and radius R in $\mathbb{R}^d$. We say that m has the *generalized scale invariance property with integral scale* $R > 0$ if, for all c in $]0, 1]$, the following equality in law holds:

$$(m(cA))_{A \subset B(0,R)} \overset{\text{(Law)}}{=} e^{\Omega_c} (m(A))_{A \subset B(0,R)}, \tag{3.2}$$

where Ω_c is a random variable independent of m. Let ν_t denote the law of $\Omega_{e^{-t}}$. If m is different from 0, then it is straightforward to prove that the laws $(\nu_t)_{t \geq 0}$

satisfy the convolution property $\nu_{t+t'} = \nu_t * \nu_{t'}$. Therefore, one can find a Lévy process $(L_t)_{t\geq 0}$ such that, for each t, ν_t is the law of L_t. In the context of Gaussian multiplicative chaos, the process $(L_t)_{t\geq 0}$ will be Brownian motion with drift.

In order to get scale invariance with integral scale R, one can choose $f = \ln^+ \frac{R}{|\cdot|}$. This is possible if and only if $\ln^+ \frac{R}{|\cdot|}$ is positive definite. This motivates the following lemma.

LEMMA 3.2. *Let $d \geq 1$ be the dimension of the space and $R > 0$ the integral scale. We consider the function $f : \mathbb{R}^d \to \mathbb{R}_+$ defined by*

$$f(x) = \ln^+ \frac{R}{|x|}.$$

The function f is positive definite if and only if $d \leq 3$.

The above choice of f leads to measures that have the generalized scale invariance property.

PROPOSITION 3.3. *Let d be less than or equal to 3 and m the Gaussian multiplicative chaos with kernel $\lambda^2 \ln^+ \frac{R}{|x|}$. Then m is scale invariant: for all c in $]0, 1]$, we have*

$$(m(cA))_{A\subset B(0,R)} \overset{\text{(Law)}}{=} e^{\Omega_c}(m(A))_{A\subset B(0,R)}, \tag{3.3}$$

where Ω_c is a Gaussian random variable independent of m with mean $-(d + \frac{\lambda^2}{2})\ln(1/c)$ and variance $\lambda^2 \ln(1/c)$.

The proof of the proposition is straightforward.

REMARK 3.4. *It remains an open problem to construct isotropic and homogeneous measures in dimension greater or equal to 4 which are scale invariant.*

3.3. *Existence of moments and multifractality.* We recall that we have supposed that $\lambda^2 < 2d$: this ensures the existence of $\varepsilon > 0$ such that $\zeta_{1+\varepsilon} > d$. Therefore, there exists a unique $p_* > 1$ such that $\zeta_{p_*} = d$. The following two propositions establish the existence of positive and negative moments for the limit measure.

PROPOSITION 3.5 (Positive moments). *Let p belong to $]0, p_*[$ and m be the Gaussian multiplicative chaos associated with the function f given by (2.3). For all bounded A in $\mathcal{B}(\mathbb{R}^d)$,*

$$E[m(A)^p] < \infty. \tag{3.4}$$

Let θ be some function satisfying the conditions (1)–(3) *of Section* 2.2. *With the notation of Theorem* 2.1, *we consider the random measure m_ε associated with θ. We have the following convergence for all bounded A in $\mathcal{B}(\mathbb{R}^d)$*:

$$E[m_\varepsilon(A)^p] \underset{\varepsilon\to 0}{\longrightarrow} E[m(A)^p]. \tag{3.5}$$

PROPOSITION 3.6 (Negative moments). *Let p belong to* $]-\infty, 0]$ *and m be the Gaussian multiplicative chaos associated with the function f given by* (2.3). *For all $c > 0$*,

$$E[m(B(0,c))^p] < \infty. \tag{3.6}$$

Let θ be some function satisfying the conditions (1)–(3) *of Section* 2.2. *With the notation of Theorem* 2.1, *we consider the random measure m_ε associated with θ. We have the following convergence for all $c > 0$*:

$$E[m_\varepsilon(B(0,c))^p] \underset{\varepsilon\to 0}{\longrightarrow} E[m(B(0,c))^p]. \tag{3.7}$$

The following proposition states the existence of the structure functions.

PROPOSITION 3.7. *Let p belong to* $]-\infty, p_*[$. *Let m be the Gaussian multiplicative chaos associated with the function f given by* (2.3). *There exists some $C_p > 0$* [*independent of g and R in decomposition* (2.3): $C_p = C_p(\lambda^2)$] *such that we have the following multifractal behavior*:

$$E[m([0,c]^d)^p] \underset{c\to 0}{\sim} e^{p(p-1)g(0)/2} C_p \left(\frac{c}{R}\right)^{\zeta_p}. \tag{3.8}$$

In the next proposition, we will suppose that $d \leq 3$ and that $f(x) = \lambda^2 \ln^+ \frac{R}{|x|}$. In this case, we can prove the existence of a C^∞ density.

PROPOSITION 3.8. *Let d be less than or equal to* 3 *and m the Gaussian multiplicative chaos with kernel $\lambda^2 \ln^+ \frac{R}{|x|}$. For all $c < R$, the variable $m(B(0,c))$ has a C^∞ density with respect to the Lebesgue measure.*

4. Proof of Theorem 2.1.

4.1. *A few intermediate lemmas.* In order to prove the theorem, we start by giving some lemmas we will need in the proof.

LEMMA 4.1. *Let θ be some function on $\mathbb{R}^d$ such that there exist $\gamma, C > 0$ with $|\theta(x)| \leq \frac{C}{1+|x|^{d+\gamma}}$. We then have the following convergence*:

$$\sup_{|z|>A} \left| \int_{\mathbb{R}^d} |\theta(v)| \ln\left|\frac{z}{z-v}\right| dv \right| \underset{A\to\infty}{\longrightarrow} 0. \tag{4.1}$$

PROOF. We have

$$\begin{aligned}\int_{\mathbb{R}^d} |\theta(v)| \ln\left|\frac{z}{z-v}\right| dv \\ = \int_{|v|\leq\sqrt{|z|}} |\theta(v)| \ln\left|\frac{z}{z-v}\right| dv + \int_{|v|>\sqrt{|z|}} |\theta(v)| \ln\left|\frac{z}{z-v}\right| dv.\end{aligned}$$

In the remainder of the proof, we will suppose that $|z| > 1$.

Considering the first term. We have $1 - \frac{|v|}{|z|} \leq \frac{|z-v|}{|z|} \leq 1 + \frac{|v|}{|z|}$ so that for $|v| \leq \sqrt{|z|}$,

$$1 - \frac{1}{\sqrt{|z|}} \leq \frac{|z-v|}{|z|} \leq 1 + \frac{1}{\sqrt{|z|}}.$$

Thus, we get $|\ln\frac{|z-v|}{|z|}| \leq \ln(\frac{1}{1-1/\sqrt{|z|}}) \leq \frac{1}{\sqrt{|z|}-1}$. We conclude that

$$\int_{|v|\leq\sqrt{|z|}} |\theta(v)| \ln\left|\frac{z}{z-v}\right| dv \leq \frac{1}{\sqrt{|z|}-1}\int_{\mathbb{R}^d} |\theta(v)|\, dv.$$

Considering the second term. We have

$$\begin{aligned}\int_{|v|>\sqrt{|z|}} |\theta(v)| \ln\left|\frac{z}{z-v}\right| dv \\ \leq \ln|z| \int_{|v|>\sqrt{|z|}} |\theta(v)|\, dv + \int_{|v|>\sqrt{|z|}} |\theta(v)| \big|\ln|z-v|\big|\, dv.\end{aligned}$$

The first term above is obvious. We decompose the second as follows:

$$\begin{aligned}\int_{|v|>\sqrt{|z|}} |\theta(v)| \big|\ln|z-v|\big|\, dv \\ = \int_{\sqrt{|z|}<|v|<|z|+1} |\theta(v)| \big|\ln|z-v|\big|\, dv + \int_{|v|\geq|z|+1} |\theta(v)| \big|\ln|z-v|\big|\, dv.\end{aligned}$$

For $|v| \geq |z| + 1$, we have $1 \leq |z-v| \leq |z||v|$ and thus

$$0 \leq \ln|z-v| \leq \ln|z| + \ln|v|,$$

which enables us to handle the corresponding integral. Let us now estimate the remaining term $I = \int_{\sqrt{|z|}<|v|<|z|+1} |\theta(v)||\ln|z-v||\, dv$. Applying Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ gives

$$I \leq \left(\int_{\sqrt{|z|}<|v|<|z|+1} |\theta(v)|^p\, dv\right)^{1/p} \left(\int_{\sqrt{|z|}<|v|<|z|+1} \big|\ln|z-v|\big|^q\, dv\right)^{1/q},$$

from which we straightforwardly get, if p is close to 1,

$$I \leq \frac{C\ln|z|}{|z|^{d/2+\gamma/2-d/2p-d/q}} \underset{|z|\to\infty}{\longrightarrow} 0. \qquad \square$$

We will also use the following lemma.

LEMMA 4.2. *Let λ be a positive number such that $\lambda^2 \neq 2$ and $(X_i)_{1\le i\le n}$ an i.i.d. sequence of centered Gaussian variables with variance $\lambda^2\ln(n)$. For all positive p such that $p < \max(\frac{2}{\lambda^2}, 1)$, there exists $0 < x < 1$ such that*

$$E\Big[\sup_{1\le i\le n} e^{pX_i - p(\lambda^2/2)\ln(n)}\Big] = O(n^{xp}). \tag{4.2}$$

PROOF. By Fubini, we get

$$\begin{aligned}
&E\Big[\sup_{1\le i\le n} e^{pX_i - p(\lambda^2/2)\ln(n)}\Big]\\
&\quad = \int_0^\infty P\Big(\sup_{1\le i\le n} e^{pX_i - p(\lambda^2/2)\ln(n)} > v\Big)\,dv\\
&\quad = \int_0^\infty P\Big(\sup_{1\le i\le n} X_i > \frac{\ln(v)}{p} + \frac{\lambda^2}{2}\ln(n)\Big)\,dv \qquad (4.3)\\
&\quad = \int_{-\infty}^\infty pe^{pu} P\Big(\sup_{1\le i\le n} X_i > u + \frac{\lambda^2}{2}\ln(n)\Big)\,du\\
&\quad \le 1 + \int_0^\infty pe^{pu} P\Big(\sup_{1\le i\le n} X_i > u + \frac{\lambda^2}{2}\ln(n)\Big)\,du,
\end{aligned}$$

where we have performed the change of variable $u = \frac{\ln(v)}{p}$ in the above identities. If we define $\bar{F}(u) = P(X_1 > u)$, then we have

$$P\Big(\sup_{1\le i\le n} X_i > u + \frac{\lambda^2}{2}\ln(n)\Big) = 1 - e^{n\ln(1-\bar{F}(u+(\lambda^2/2)\ln(n)))}.$$

Let x be some positive number such that $0 < x < 1$. Using (4.3), we get

$$\begin{aligned}
&E\Big[\sup_{1\le i\le n} e^{pX_i - p(\lambda^2/2)\ln(n)}\Big]\\
&\quad \le n^{xp} + p\int_{x\ln(n)}^\infty e^{pu}\big(1 - e^{n\ln(1-\bar{F}(u+(\lambda^2/2)\ln(n)))}\big)\,du \qquad (4.4)\\
&\quad \le n^{xp} + pn^{xp}\int_0^\infty e^{p\widetilde{u}}\big(1 - e^{n\ln(1-\bar{F}(\widetilde{u}+((\lambda^2/2)+x)\ln(n)))}\big)\,d\widetilde{u}.
\end{aligned}$$

We have

$$\begin{aligned}
\bar{F}\Big(\widetilde{u} + \Big(\frac{\lambda^2}{2} + x\Big)\ln(n)\Big) &= \frac{1}{\sqrt{2\pi}\lambda\sqrt{\ln(n)}}\int_{\widetilde{u}+(\lambda^2/2+x)\ln(n)}^\infty e^{-v^2/(2\lambda^2\ln(n))}\,dv\\
&= \frac{n^{-(\lambda^2/2+x)^2/(2\lambda^2)}}{\sqrt{2\pi}\lambda\sqrt{\ln(n)}}\int_{\widetilde{u}}^\infty e^{-(1/2+x/\lambda^2)\widetilde{v}-\widetilde{v}^2/(2\lambda^2\ln(n))}\,d\widetilde{v},
\end{aligned}$$

where we have performed the change of variable $\widetilde{v} = v - (\frac{\lambda^2}{2} + x)\ln(n)$. Thus, we get

$$
\begin{aligned}
& n^{xp} \int_0^{\infty} e^{p\widetilde{u}} \bigl(1 - e^{n \ln(1-\bar{F}(\widetilde{u}+((\lambda^2/2)+x)\ln(n)))}\bigr)\, d\widetilde{u} \\
& \quad \leq n^{xp+1} \int_0^{\infty} e^{p\widetilde{u}} \bar{F}\biggl(\widetilde{u} + \biggl(\frac{\lambda^2}{2} + x\biggr)\ln(n)\biggr)\, d\widetilde{u} \\
& \quad \leq \frac{n^{xp+1-(\lambda^2/2+x)^2/(2\lambda^2)}}{\sqrt{2\pi}\lambda\sqrt{\ln(n)}} \int_0^{\infty} e^{p\widetilde{u}} \biggl(\int_{\widetilde{u}}^{\infty} e^{-(1/2+x/\lambda^2)\widetilde{v}-\widetilde{v}^2/(2\lambda^2\ln(n))}\, d\widetilde{v}\biggr)\, d\widetilde{u} \\
& \quad \leq \frac{n^{xp+1-(\lambda^2/2+x)^2/(2\lambda^2)}}{p\sqrt{2\pi}\lambda\sqrt{\ln(n)}} \int_0^{\infty} e^{p\widetilde{v}-(1/2+x/\lambda^2)\widetilde{v}-\widetilde{v}^2/(2\lambda^2\ln(n))}\, d\widetilde{v} \\
& \quad \leq \frac{n^{xp+1-(\lambda^2/2+x)^2/(2\lambda^2)}}{p\sqrt{2\pi}\lambda\sqrt{\ln(n)}} \int_{-\infty}^{\infty} e^{p\widetilde{v}-(1/2+x/\lambda^2)\widetilde{v}-\widetilde{v}^2/(2\lambda^2\ln(n))}\, d\widetilde{v} \\
& \quad = \frac{n^{xp+\alpha(x,\lambda^2,p)}}{p},
\end{aligned}
\tag{4.5}
$$

with $\alpha(x, \lambda^2, p) = 1 - \frac{(\lambda^2/2+x)^2}{2\lambda^2} + (p - \frac{1}{2} - \frac{x}{\lambda^2})^2 \frac{\lambda^2}{2}$. We have, by combining (4.4) and (4.5),

$$
E\Bigl[\sup_{1 \leq i \leq n} e^{pX_i - p(\lambda^2/2)\ln(n)}\Bigr] \leq n^{xp} + n^{xp+\alpha(x,\lambda^2,p)}.
$$

We focus on the case $p \in]\frac{1}{2} + \frac{1}{\lambda^2}, \max(\frac{2}{\lambda^2}, 1)[$. This implies inequality (4.2) for $p \leq \frac{1}{2} + \frac{1}{\lambda^2}$; indeed, if inequality (4.2) holds for some p, then it holds for all $p' < p$ by applying Jensen's inequality to the concave function $u \to u^{p'/p}$.

First case: $\lambda^2 < 2$. Note that $\alpha(1, \lambda^2, \frac{2}{\lambda^2}) = 0$, so if $p < \frac{2}{\lambda^2}$, then there exists $0 < x < 1$ such that $\alpha(x, \lambda^2, p) < 0$.

Second case: $\lambda^2 > 2$. Note that $\alpha(1, \lambda^2, 1) = 0$, so if $p < 1$, then there exists $0 < x < 1$ such that $\alpha(x, \lambda^2, p) < 0$. □

4.2. *Proof of Theorem* 2.1. For the sake of simplicity, we give the proof in the case where $d = 1$, $R = 1$ and the function $f(x) = \lambda^2 \ln^+ \frac{1}{|x|}$. This is no restriction; indeed, the proof in the general case is an immediate adaptation of the following proof.

4.2.1. *Uniqueness*. Let $\alpha \in]0, 1/2[$. We consider θ and $\widetilde{\theta}$, two continuous functions satisfying properties (1)–(3). We note that

$$
m(dt) = e^{X(t)-(1/2)E[X(t)^2]}\, dt = \lim_{\varepsilon \to 0} e^{X_\varepsilon(t)-(1/2)E[X_\varepsilon(t)^2]}\, dt,
$$

where $(X_\varepsilon(t))_{t\in\mathbb{R}}$ is a Gaussian process of covariance $q_\varepsilon(|t-s|)$ with

$$q_\varepsilon(x) = (\theta^\varepsilon * f)(x) = \lambda^2 \int_{\mathbb{R}} \theta(v) \ln^+\left(\frac{1}{|x-\varepsilon v|}\right) dv.$$

We similarly define the measure $\widetilde{m}$, $\widetilde{X}_\varepsilon$ and $\widetilde{q}_\varepsilon$ associated with the function $\widetilde{\theta}$. Note that we suppose that the random measures $m_\varepsilon(dt) = e^{X_\varepsilon(t)-(1/2)E[X_\varepsilon(t)^2]}\,dt$ and $\widetilde{m}_\varepsilon(dt) = e^{\widetilde{X}_\varepsilon(t)-(1/2)E[X_\varepsilon(t)^2]}\,dt$ converge in law in the space of Radon measures. This is no restriction since, using Fubini and $E[e^{X_\varepsilon(t)-(1/2)E[X_\varepsilon(t)^2]}] = 1$, we get the equality $E[m_\varepsilon(A)] = E[\widetilde{m}_\varepsilon(A)] = |A|$ for all bounded A in $\mathcal{B}(\mathbb{R})$ which implies that the measures are tight (see Lemma 4.5 in [14]).

We will show that

$$E[m[0,1]^\alpha] = E[\widetilde{m}[0,1]^\alpha]$$

for α in the interval $]0,1/2[$. If we define $Z_\varepsilon(t)(u) = \sqrt{t}\widetilde{X}_\varepsilon(u) + \sqrt{1-t}X_\varepsilon(u)$ with $X_\varepsilon(u)$ and $\widetilde{X}_\varepsilon(u)$ independent, then we get, by using the continuous version of Lemma A.1,

$$E[\widetilde{m}_\varepsilon[0,1]^\alpha] - E[m_\varepsilon[0,1]^\alpha] = \frac{\alpha(\alpha-1)}{2}\int_0^1 \varphi_\varepsilon(t)\,dt, \tag{4.6}$$

with $\varphi_\varepsilon(t)$ defined by

$$\varphi_\varepsilon(t) = \int_{[0,1]^2} \big(\widetilde{q}_\varepsilon(|t_2-t_1|) - q_\varepsilon(|t_2-t_1|) E[\mathcal{X}_\varepsilon(t,t_1,t_2)]\big)\,dt_1\,dt_2,$$

where $\mathcal{X}_\varepsilon(t,t_1,t_2)$ is given by

$$\mathcal{X}_\varepsilon(t,t_1,t_2) = \frac{e^{Z_\varepsilon(t)(t_1)+Z_\varepsilon(t)(t_2)-(1/2)E[Z_\varepsilon(t)(t_1)^2]-(1/2)E[Z_\varepsilon(t)(t_2)^2]}}{(\int_0^1 e^{Z_\varepsilon(t)(u)-(1/2)E[Z_\varepsilon(t)(u)^2]}\,du)^{2-\alpha}}.$$

We now state and prove the following short lemma which we will need in the sequel.

LEMMA 4.3. *For $A>0$, we let $C_A^\varepsilon = \sup_{|x|\geq A\varepsilon}|q_\varepsilon(x) - \widetilde{q}_\varepsilon(x)|$. We have*

$$\lim_{A\to\infty}\Big(\overline{\lim_{\varepsilon\to 0}}\, C_A^\varepsilon\Big) = 0.$$

PROOF. Let $|x|\geq A\varepsilon$. If $|x|\geq 1/2$, then $q_\varepsilon(x)$ and $\widetilde{q}_\varepsilon(x)$ converge uniformly to $\lambda^2\ln^+\frac{1}{|x|}$, thus $q_\varepsilon(x) - \widetilde{q}_\varepsilon(x)$ converges uniformly to 0 (this a consequence of the fact that $\lambda^2\ln^+\frac{1}{|x|}$ is continuous and of compact support for $|x|\geq 1/2$). If $|x|<1/2$, then we write

$$q_\varepsilon(x) = \lambda^2\left(\ln\frac{1}{\varepsilon} + Q(x/\varepsilon) + R_\varepsilon(x)\right),$$

where $Q(x) = \int_{\mathbb{R}} \ln \frac{1}{|x-z|}\theta(z)\,dz$ and $R_\varepsilon(x)$ converges uniformly to 0 (for $|x| < 1/2$) as $\varepsilon \to 0$ [similarly, we can write $\widetilde{q}_\varepsilon(x) = \lambda^2(\ln \frac{1}{\varepsilon} + \widetilde{Q}(x/\varepsilon) + \widetilde{R}_\varepsilon(x))$]. This follows from straightforward calculations. Applying Lemma 4.1, we get that $Q(x) = \ln \frac{1}{|x|} + \Sigma(x)$ with $\Sigma(x) \to 0$ for $|x| \to \infty$. Thus, $Q(x) - \widetilde{Q}(x)$ is a continuous function such that, for $|x| \geq A\varepsilon$ and $|x| \leq 1/2$, we have

$$|q_\varepsilon(x) - \widetilde{q}_\varepsilon(x)| \leq \lambda^2 \sup_{|y|\geq A} |Q(y) - \widetilde{Q}(y)| + \lambda^2 \sup_{|x|\leq 1/2} |R_\varepsilon(x) - \widetilde{R}_\varepsilon(x)|.$$

The result follows. □

One can decompose expression (4.6) in the following way:

$$\begin{aligned} &E[\widetilde{m}_\varepsilon[0,1]^\alpha] - E[m_\varepsilon[0,1]^\alpha] \\ &\quad = \frac{\alpha(\alpha-1)}{2}\int_0^1 \varphi_\varepsilon^A(t)\,dt + \frac{\alpha(\alpha-1)}{2}\int_0^1 \bar{\varphi}_\varepsilon^A(t)\,dt, \end{aligned} \tag{4.7}$$

where

$$\varphi_\varepsilon^A(t) = \int_{[0,1]^2, |t_2-t_1|\leq A\varepsilon} \big(\widetilde{q}_\varepsilon(|t_2-t_1|) - q_\varepsilon(|t_2-t_1|)E[\mathcal{X}_\varepsilon(t,t_1,t_2)]\big)\,dt_1\,dt_2$$

and

$$\bar{\varphi}_\varepsilon^A(t) = \int_{[0,1]^2, |t_2-t_1|> A\varepsilon} \big(\widetilde{q}_\varepsilon(|t_2-t_1|) - q_\varepsilon(|t_2-t_1|)E[\mathcal{X}_\varepsilon(t,t_1,t_2)]\big)\,dt_1\,dt_2.$$

With the notation of Lemma 4.3, we have

$$\begin{aligned} |\bar{\varphi}_\varepsilon^A(t)| &\leq C_A^\varepsilon \int_{[0,1]^2, |t_2-t_1|>A\varepsilon} E[\mathcal{X}_\varepsilon(t,t_1,t_2)]\,dt_1\,dt_2 \\ &\leq C_A^\varepsilon \int_{[0,1]^2} E[\mathcal{X}_\varepsilon(t,t_1,t_2)]\,dt_1\,dt_2 \\ &= C_A^\varepsilon E\bigg[\bigg(\int_0^1 e^{Z_\varepsilon(t)(u) - (1/2)E[Z_\varepsilon(t)(u)^2]}\,du\bigg)^\alpha\bigg] \\ &\leq C_A^\varepsilon. \end{aligned}$$

Thus, taking the limit as ε goes to 0 in (4.7) gives

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}} |E[\widetilde{m}_\varepsilon[0,1]^\alpha] - E[m_\varepsilon[0,1]^\alpha]| \\ &\quad \leq \frac{\alpha(1-\alpha)}{2} \overline{\lim_{\varepsilon\to 0}} C_A^\varepsilon + \frac{\alpha(1-\alpha)}{2} \overline{\lim_{\varepsilon\to 0}} \int_0^1 |\varphi_\varepsilon^A(t)|\,dt. \end{aligned}$$

We will show that $\lim_{\varepsilon\to 0} \varphi_\varepsilon^A(0) = 0$ [the general case $\varphi_\varepsilon^A(t)$ is similar]. There exists a constant $\widetilde{C}_A > 0$, independent of ε, such that

$$\sup_{|x|\leq A\varepsilon} |\widetilde{q}_\varepsilon(x) - q_\varepsilon(x)| \leq \widetilde{C}_A.$$

Therefore, we have

$$
\begin{aligned}
|\varphi_\varepsilon^A(0)| &\le \widetilde{C}_A \int_0^1 \int_{t_1-A\varepsilon}^{t_1+A\varepsilon} E[\mathcal{X}_\varepsilon(0,t_1,t_2)]\,dt_2\,dt_1 \\
&= \widetilde{C}_A E\biggl[\frac{\int_0^1\int_{t_1-A\varepsilon}^{t_1+A\varepsilon} e^{X_\varepsilon(t_1)+X_\varepsilon(t_2)-(1/2)E[X_\varepsilon(t_1)^2]-(1/2)E[X_\varepsilon(t_2)^2]}\,dt_1\,dt_2}{(\int_0^1 e^{X_\varepsilon(u)-(1/2)E[X_\varepsilon(u)^2]}\,du)^{2-\alpha}}\biggr].
\end{aligned}
\tag{4.8}
$$

We now have

$$
\begin{aligned}
&\int_0^1 \int_{t_1-A\varepsilon}^{t_1+A\varepsilon} e^{X_\varepsilon(t_1)+X_\varepsilon(t_2)-(1/2)E[X_\varepsilon(t_1)^2]-(1/2)E[X_\varepsilon(t_2)^2]}\,dt_2\,dt_1 \\
&\quad\le \biggl(\sup_{t_1}\int_{t_1-A\varepsilon}^{t_1+A\varepsilon} e^{X_\varepsilon(t_2)-(1/2)E[X_\varepsilon(t_2)^2]}\,dt_2\biggr)\int_0^1 e^{X_\varepsilon(t_1)-(1/2)E[X_\varepsilon(t_1)^2]}\,dt_1 \\
&\quad\le 2\biggl(\sup_{0\le i<1/(2A\varepsilon)}\int_{2iA\varepsilon}^{2(i+1)A\varepsilon} e^{X_\varepsilon(t_2)-(1/2)E[X_\varepsilon(t_2)^2]}\,dt_2\biggr) \\
&\qquad\times\int_0^1 e^{X_\varepsilon(t_1)-(1/2)E[X_\varepsilon(t_1)^2]}\,dt_1.
\end{aligned}
$$

In view of (4.8), this implies that

$$
\begin{aligned}
|\varphi_\varepsilon^A(0)| &\le 2\widetilde{C}_A E\biggl[\biggl(\sup_{0\le i<1/(2A\varepsilon)}\int_{2iA\varepsilon}^{2(i+1)A\varepsilon} e^{X_\varepsilon(t_2)-(1/2)E[X_\varepsilon(t_2)^2]}\,dt_2\biggr) \\
&\qquad\qquad\times\biggl(\int_0^1 e^{X_\varepsilon(t_1)-(1/2)E[X_\varepsilon(t_1)^2]}\,dt_1\biggr)^{\alpha-1}\biggr] \\
&\le 2\widetilde{C}_A E\biggl[\biggl(\sup_{0\le i<1/(2A\varepsilon)}\int_{2iA\varepsilon}^{2(i+1)A\varepsilon} e^{X_\varepsilon(t_2)-(1/2)E[X_\varepsilon(t_2)^2]}\,dt_2\biggr)^{\alpha}\biggr],
\end{aligned}
$$

where we have used the inequality $\frac{\sup_i a_i}{(\sum_i a_i)^{1-\alpha}} \le (\sup_i a_i)^\alpha$. For the sake of simplicity, we now replace $2A$ by A.

To study the above supremum, the idea is to use the approximation $X_\varepsilon(t) \approx X_\varepsilon(Ai\varepsilon)$ for t in $[Ai\varepsilon, A(i+1)\varepsilon]$. We define $\mathcal{C}_\varepsilon$ by

$$
\mathcal{C}_\varepsilon = \sup_{\substack{0\le i<1/(A\varepsilon)\\ Ai\varepsilon\le u\le A(i+1)\varepsilon}} \bigl(X_\varepsilon(u)-X_\varepsilon(Ai\varepsilon)\bigr).
\tag{4.9}
$$

By the definition of $\mathcal{C}_\varepsilon$, we have $X_\varepsilon(t) \le X_\varepsilon(Ai\varepsilon)+\mathcal{C}_\varepsilon$ for all $i < \frac{1}{A\varepsilon}$ and all t in $[Ai\varepsilon, A(i+1)\varepsilon]$. We then get

$$
\begin{aligned}
&E\biggl[\biggl(\sup_{0\le i<1/(A\varepsilon)}\int_{Ai\varepsilon}^{A(i+1)\varepsilon} e^{X_\varepsilon(t)-(1/2)E[X_\varepsilon(t)^2]}\,dt\biggr)^{\alpha}\biggr] \\
&\quad\le E\biggl[\biggl(\sup_{0\le i<1/(A\varepsilon)}\int_{Ai\varepsilon}^{A(i+1)\varepsilon} e^{X_\varepsilon(Ai\varepsilon)-(1/2)E[X_\varepsilon(Ai\varepsilon)^2]}\,dt\biggr)^{\alpha} e^{\alpha\mathcal{C}_\varepsilon}\biggr]
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&= E\Big[\Big(\varepsilon A \sup_{0\le i<1/(A\varepsilon)} e^{X_\varepsilon(Ai\varepsilon)-(1/2)E[X_\varepsilon(Ai\varepsilon)^2]}\Big)^{\alpha} e^{\alpha \mathcal{C}_\varepsilon}\Big] \\
&\le (\varepsilon A)^{\alpha} E\Big[\Big(\sup_{0\le i<1/(A\varepsilon)} e^{X_\varepsilon(Ai\varepsilon)-(1/2)E[X_\varepsilon(Ai\varepsilon)^2]}\Big)^{2\alpha}\Big]^{1/2} E[e^{2\alpha \mathcal{C}_\varepsilon}]^{1/2}.
\end{aligned}
$$

There exists some $c \ge 0$ (independent of ε) such that for all s, t in $[0, 1]$,

$$
E[X_\varepsilon(s)X_\varepsilon(t)] = q_\varepsilon(|t-s|) \ge -c.
$$

Indeed, for simplicity, let us suppose that θ has compact support in $[-K, K]$ with $K > 0$. The function $q_\varepsilon(x)$ converges uniformly to $\lambda^2 \ln^+ \frac{1}{|x|}$ on $|x| \ge \frac{1}{2}$, so we can restrict to the case $|x| \le \frac{1}{2}$. If $x = \varepsilon\widetilde{x}$, then $|\widetilde{x}| \le \frac{1}{2\varepsilon}$ and we have

$$
\begin{aligned}
q_\varepsilon(x) &= \lambda^2 \int_{-K}^{K} \theta(v) \ln\Big(\frac{1}{|x-\varepsilon v|}\Big)\, dv \\
&= \lambda^2 \ln\Big(\frac{1}{\varepsilon}\Big) - \lambda^2 \int_{-K}^{K} \theta(v) \ln(|\widetilde{x}-v|)\, dv.
\end{aligned}
$$

The quantity $\lambda^2 \int_{-K}^{K} \theta(v)\ln(|\widetilde{x}-v|)\, dv$ is bounded for $|\widetilde{x}| \le K+1$ and for $|\widetilde{x}| > K+1$, it can be written

$$
\begin{aligned}
\lambda^2 \int_{-K}^{K} \theta(v)\ln(|\widetilde{x}-v|)\, dv &= \lambda^2 \ln|\widetilde{x}| + \lambda^2 \int_{-K}^{K} \theta(v) \ln\Big(\frac{|\widetilde{x}-v|}{|\widetilde{x}|}\Big)\, dv \\
&\le \lambda^2 \ln\Big(\frac{1}{2\varepsilon}\Big) + \lambda^2 \int_{-K}^{K} \theta(v) \ln\Big(\frac{|\widetilde{x}-v|}{|\widetilde{x}|}\Big)\, dv.
\end{aligned}
$$

The conclusion follows from the fact that the second term in the right-hand side above is bounded independently of ε.

We introduce a centered Gaussian random variable Z independent of X_ε and such that $E[Z^2] = c$. Let $(R_i^\varepsilon)_{1\le i<1/(A\varepsilon)}$ be a sequence of i.i.d. Gaussian random variables such that $E[(R_i^\varepsilon)^2] = E[X_\varepsilon(Ai\varepsilon)^2] + c$. By applying Corollary A.3, we get

$$
\begin{aligned}
&E\Big[\Big(\sup_{0\le i<1/(A\varepsilon)} e^{X_\varepsilon(Ai\varepsilon)-(1/2)E[X_\varepsilon(Ai\varepsilon)^2]}\Big)^{2\alpha}\Big] \\
&\quad = \frac{1}{e^{2\alpha^2 c-\alpha c}} E\Big[\Big(\sup_{0\le i<1/(A\varepsilon)} e^{X_\varepsilon(Ai\varepsilon)+Z-(1/2)E[X_\varepsilon(Ai\varepsilon)^2]-(c/2)}\Big)^{2\alpha}\Big] \\
&\quad \le \frac{1}{e^{2\alpha^2 c-\alpha c}} E\Big[\Big(\sup_{0\le i<1/(A\varepsilon)} e^{R_i^\varepsilon-(1/2)E[(R_i^\varepsilon)^2]}\Big)^{2\alpha}\Big].
\end{aligned}
$$

We have $E[(R_i^\varepsilon)^2] = \lambda^2 \ln\frac{1}{\varepsilon} + C(\varepsilon)$, with $C(\varepsilon)$ converging to some constant as ε goes to 0. Since $2\alpha < 1$, by applying Lemma 4.2, there exists some $0 < x < 1$ such that

$$E\Big[\Big(\sup_{0\le i<1/(A\varepsilon)} e^{R_i^\varepsilon-(1/2)E[(R_i^\varepsilon)^2]}\Big)^{2\alpha}\Big]\le C\Big(\frac{1}{\varepsilon}\Big)^{2\alpha x}$$

and we therefore have

$$|\varphi_\varepsilon^A(0)|\le C\varepsilon^\gamma E[e^{2\alpha\mathcal{C}_\varepsilon}]^{1/2}$$

with $\gamma=\alpha(1-x)>0$.

One can write $\mathcal{C}_\varepsilon=\sup_{0\le i<1/(A\varepsilon),0\le v\le 1} W_\varepsilon^i(v)$, where $W_\varepsilon^i(v)=X_\varepsilon(Ai\varepsilon+A\varepsilon v)-X_\varepsilon(Ai\varepsilon)$. We have

$$E[W_\varepsilon^i(v)W_\varepsilon^i(v')]=g_\varepsilon(v-v'),$$

where g_ε is a continuous function bounded by some constant M independent of ε. Let Y be a centered Gaussian random variable independent of W_ε^i such that $E[Y^2]=M$. Thus, we can write

$$E[e^{2\alpha\mathcal{C}_\varepsilon}]=\frac{E[e^{2\alpha\sup_{i,v}(W_\varepsilon^i(v)+Y)}]}{e^{2\alpha^2M}}.$$

Let us now consider a family $(\overline{W}_\varepsilon^i)_{1\le i<1/(A\varepsilon)}$ of centered i.i.d. Gaussian processes of law $(W_\varepsilon^0(v)+Y)_{0\le v\le 1}$. Applying Corollary A.3 from the Appendix, we get

$$E[e^{2\alpha\mathcal{C}_\varepsilon}]\le\frac{E[e^{2\alpha\sup_{i,v}\overline{W}_\varepsilon^i(v)}]}{e^{2\alpha^2M}}.$$

We now estimate $E[e^{2\alpha\sup_{i,v}\overline{W}_\varepsilon^i(v)}]$. Let us write $\mathcal{X}_i=\sup_{0\le v\le 1}\overline{W}_\varepsilon^i(v)$. Applying Corollary 3.2 of [16] to the continuous Gaussian process $(W_\varepsilon^0(v)+Y)_{0\le v\le 1}$, we get that the random variable has a Gaussian tail:

$$P(\mathcal{X}_i>z)\le Ce^{-z^2/(2\sigma^2)}\qquad\forall z>0$$

for some C and σ. Using computations similar to the ones used in the proof of Lemma 4.2, the above tail inequality gives the existence of some constant $C>0$ such that

$$E\big[e^{2\alpha\sup_{0\le i<1/(A\varepsilon)}\mathcal{X}_i}\big]\le Ce^{C\sqrt{\ln(1/\varepsilon)}}.$$

Therefore, we have $E[e^{2\alpha\mathcal{C}_\varepsilon}]\le Ce^{C\sqrt{\ln(1/\varepsilon)}}$ and then

$$|\varphi_\varepsilon^A(0)|\le C\varepsilon^\gamma e^{C\sqrt{\ln(1/\varepsilon)}}.$$

It follows that $\overline{\lim}_{\varepsilon\to0}|\varphi_\varepsilon^A(0)|=0$ so that for $\alpha<1/2$,

$$\overline{\lim_{\varepsilon\to0}}\,|E[\widetilde{m}_\varepsilon[0,1]^\alpha]-E[m_\varepsilon[0,1]^\alpha]|\le\frac{\alpha(1-\alpha)}{2}\,\overline{\lim_{\varepsilon\to0}}\,C_A^\varepsilon.$$

Since $\overline{\lim}_{\varepsilon\to 0} C_A^\varepsilon \to 0$ as A goes to infinity (Lemma 4.3), we conclude that

$$\overline{\lim_{\varepsilon\to 0}} |E[\widetilde{m}_\varepsilon[0,1]^\alpha] - E[m_\varepsilon[0,1]^\alpha]| = 0.$$

It is straightforward to check that the above proof can be generalized to show that for all positive $\lambda_1,\dots,\lambda_n$ and intervals $I_1,\dots,I_n$, we have

$$E\left[\left(\sum_{k=1}^n \lambda_k m(I_k)\right)^\alpha\right] = E\left[\left(\sum_{k=1}^n \lambda_k \widetilde{m}(I_k)\right)^\alpha\right].$$

This implies that the random measures m and $\widetilde{m}$ are equal (see [8]).

Existence. Let $f(x)$ be a real positive definite function on $\mathbb{R}^d$ (note that this implies that f is symmetric). Let us recall that a centered Gaussian field of correlation $f(x-y)$ can be constructed by means of the following formula:

$$X(x) = \int_{\mathbb{R}^d} \zeta(x,\xi)\sqrt{\hat{f}(\xi)}\,W(d\xi),$$

where $\zeta(x,\xi) = \cos(2\pi x.\xi) - \sin(2\pi x.\xi)$ and $W(d\xi)$ is the standard white noise on $\mathbb{R}^d$ (to see this, one can check, using the inverse Fourier formula, that the above X has the desired correlations). This can also be written as

$$X(x) = \int_{]0,\infty[\times\mathbb{R}^d} \zeta(x,\xi)\sqrt{\hat{f}(\xi)}\,g(t,\xi)\,W(dt,d\xi), \tag{4.11}$$

where $W(dt,d\xi)$ is the white noise on $]0,\infty[\times\mathbb{R}^d$ and $\int_0^\infty g(t,\xi)^2\,dt = 1$ for all ξ. The significance of the expression (4.11) should be evident in what follows. Let the function θ be radially symmetric and let $\hat{\theta}$ be a decreasing function of $|\xi|$ [e.g., take $\theta(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{d/2}}$]. Let us consider $g(t,\xi) = \sqrt{-\hat{\theta}'(t|\xi|)|\xi|}$ so that $\int_\varepsilon^\infty g(t,\xi)^2\,dt = \hat{\theta}(\varepsilon|\xi|)$ for $|\xi| \neq 0$. If we then consider the fields X_ε defined by

$$X_\varepsilon(x) = \int_{]\varepsilon,\infty[\times\mathbb{R}^d} \zeta(x,\xi)\sqrt{\hat{f}(\xi)}\,g(t,\xi)\,W(dt,d\xi), \tag{4.12}$$

then we will find

$$\begin{aligned} E[X_\varepsilon(x)X_\varepsilon(y)] &= \int_{\mathbb{R}^d} \cos\bigl(2\pi(x-y).\xi\bigr)\hat{f}(\xi)\hat{\theta}(\varepsilon|\xi|)\,d\xi \\ &= (f * \theta^\varepsilon)(x-y). \end{aligned}$$

The significance of (4.12) is to make the approximation process appear as a martingale. Indeed, if we define the filtration $\mathcal{F}_\varepsilon = \sigma\{W(A,B), A\subset]\varepsilon,\infty[, B\in\mathcal{B}(\mathbb{R}^d)$ and bounded$\}$, we have that for all $A\in\mathcal{B}(\mathbb{R}^d)$, $(m_\varepsilon(A))_{\varepsilon>0}$ is a positive $\mathcal{F}_\varepsilon$-martingale of expectation $|A|$, so it converges almost surely to a random variable $m(A)$ such that

$$E[m(A)] \leq |A|. \tag{4.13}$$

This defines a collection $(m(A))_{A\in\mathcal{B}(\mathbb{R}^d)}$ of random variables such that:

(1) for all disjoint and bounded sets A_1, A_2 in $\mathcal{B}(\mathbb{R}^d)$,

$$m(A_1 \cup A_2) = m(A_1) + m(A_2) \qquad \text{a.s.;}$$

(2) for any bounded sequence $(A_n)_{n\geq 1}$ decreasing to $\varnothing$,

$$m(A_n) \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{a.s.}$$

By Theorem 6.1.VI. in [8], one can consider a version of the collection $(m(A))_{A\in\mathcal{B}(\mathbb{R}^d)}$ such that m is a random measure. It is straightforward that m_ε converges almost surely to m in the space of Radon measures (equipped with the weak topology).

5. Proofs for Section 3.

5.1. *Proof of Proposition* 3.1. Since $\zeta_1 = d$, we note that $\lambda^2 > 2d$ is equivalent to the existence of $\alpha < 1$ such that $\zeta_\alpha > d$. Let α be fixed and such that $\zeta_\alpha > d$. We will show that $m[[0,1]^d] = 0$. We partition the cube $[0,1]^d$ into $\frac{1}{\varepsilon^d}$ subcubes $(I_j)_{1\leq j\leq 1/\varepsilon^d}$ of size ε. One has, by subadditivity and homogeneity,

$$\begin{aligned}
&E\biggl[\biggl(\int_{[0,1]^d} e^{X_\varepsilon(x)-(1/2)E[X_\varepsilon(x)^2]}\,dx\biggr)^\alpha\biggr]\\
&\quad= E\biggl[\biggl(\sum_{1\leq j\leq 1/\varepsilon^d}\int_{I_j} e^{X_\varepsilon(x)-(1/2)E[X_\varepsilon(x)^2]}\,dx\biggr)^\alpha\biggr]\\
&\quad\leq E\biggl[\sum_{1\leq j\leq 1/\varepsilon^d}\biggl(\int_{I_j} e^{X_\varepsilon(x)-(1/2)E[X_\varepsilon(x)^2]}\,dx\biggr)^\alpha\biggr]\\
&\quad= \frac{1}{\varepsilon^d}E\biggl[\biggl(\int_{[0,\varepsilon]^d} e^{X_\varepsilon(x)-(1/2)E[X_\varepsilon(x)^2]}\,dx\biggr)^\alpha\biggr].
\end{aligned}$$

Let Y_ε be a centered Gaussian random variable of variance $\lambda^2\ln(\frac{1}{\varepsilon}) + \lambda^2 c$, where c is such that

$$\theta^\varepsilon * \ln^+\frac{1}{|x|} \geq \ln\frac{1}{\varepsilon} + c$$

for $|x| \leq \varepsilon$ and ε small enough. By the definition of c, we have

$$\forall x, x' \in [0,\varepsilon]^d \qquad E[X_\varepsilon(x)X_\varepsilon(x')] \geq E[Y_\varepsilon^2].$$

Using Corollary (A.2) in the continuous version, this implies that

$$\begin{aligned}
&E\biggl[\biggl(\int_{[0,1]^d} e^{X_\varepsilon(x)-(1/2)E[X_\varepsilon(x)^2]}\,dx\biggr)^\alpha\biggr]\\
&\qquad\leq \frac{1}{\varepsilon^d}E\biggl[\biggl(\int_{[0,\varepsilon]^d} e^{Y_\varepsilon-(1/2)E[Y_\varepsilon^2]}\,dx\biggr)^\alpha\biggr]
\end{aligned}$$

$$= \frac{\varepsilon^{d\alpha}}{\varepsilon^d} E\big[\big(e^{Y_\varepsilon - (1/2)E[Y_\varepsilon^2]}\big)^\alpha\big]$$

$$= \frac{\varepsilon^{d\alpha}}{\varepsilon^d} e^{\alpha^2 E[Y_\varepsilon^2]/2 - \alpha E[Y_\varepsilon^2]/2}$$

$$= e^{((\alpha^2-\alpha)/2)c} \varepsilon^{\zeta_\alpha - d}.$$

Taking the limit as ε goes to 0 gives $m[[0,1]^d] = 0$.

5.2. *Proof of Lemma* 3.2. One has the following general formula for the Fourier transform of radial functions:

$$\hat{f}(\xi) = \frac{2\pi}{|\xi|^{(d-2)/2}} \int_0^\infty \rho^{d/2} J_{(d-2)/2}(2\pi|\xi|\rho) f(\rho)\, d\rho, \tag{5.1}$$

where J_ν is the Bessel function of order ν (see, e.g., [21]).

First case: $d \le 3$. It suffices to consider the case $d = 3$. Indeed, consider some function φ in $\mathcal{S}(\mathbb{R}^2)$. We introduce the family of functions $\psi_\varepsilon(x_1, x_2, x_3) = \varphi(x_1, x_2)\theta_\varepsilon(x_3)$, where θ_ε is a smooth function that converges to the Dirac mass δ_0 as ε goes to 0. If we take the limit as ε goes to 0 in inequality (2.1) applied to ψ_ε, then we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x-y,0)\varphi(x)\overline{\varphi(y)}\, dx\, dy \ge 0.$$

This shows that $(x_1, x_2) \to f(x_1, x_2, 0)$ is positive definite. Similarly, one can show that $x \to f(x, 0, 0)$ is positive definite.

Using the explicit formula $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$, we conclude, by integrating by parts, that

$$\begin{aligned}
\hat{f}(\xi) &= \frac{2}{|\xi|} \int_0^T \rho \sin(2\pi|\xi|\rho) \ln\left(\frac{T}{\rho}\right) d\rho \\
&= \frac{1}{\pi|\xi|^2} \int_0^T \cos(2\pi|\xi|\rho)\left(\ln\left(\frac{T}{\rho}\right) - 1\right) d\rho \\
&= \frac{1}{2\pi^2|\xi|^3}\left(\int_0^T \frac{\sin(2\pi|\xi|\rho)}{\rho}\, d\rho - \sin(2\pi|\xi|T)\right) \\
&= \frac{1}{2\pi^2|\xi|^3}\big(\operatorname{sinc}(2\pi|\xi|T) - \sin(2\pi|\xi|T)\big),
\end{aligned}$$

where "sinc" denotes the sinus cardinal function:

$$\operatorname{sinc}(x) = \int_0^x \frac{\sin(\rho)}{\rho}\, d\rho.$$

For $x \ge 0$, we introduce the function $l(x) = \operatorname{sinc}(x) - \sin(x)$. Since $\hat{f}(\xi) = \frac{l(2\pi|\xi|T)}{2\pi^2|\xi|^3}$, the nonnegativity of $\hat{f}$ is equivalent to the nonnegativity of l. We have

$l'(x) = \frac{\sin(x) - x\cos(x)}{x}$. Thus, there exists some α in $]\pi, 2\pi[$ such that l is increasing on $]0, \alpha[$ and decreasing on $]\alpha, 2\pi[$. Since $l(0) = 0$ and $l(2\pi) = \int_0^{2\pi} \frac{\sin(\rho)}{\rho}\, d\rho \geq 0$, we conclude that for all x in $[0, 2\pi]$, $l(x) \geq 0$. A classical computation (Dirichlet integral) gives $\int_0^\infty \frac{\sin(\rho)}{\rho}\, d\rho = \frac{\pi}{2}$. Thus, we have, by an integration by parts,

$$\begin{aligned}\int_0^{2\pi} \frac{\sin(\rho)}{\rho}\, d\rho &= \frac{\pi}{2} - \int_{2\pi}^{\infty} \frac{\sin(\rho)}{\rho}\, d\rho \\ &= \frac{\pi}{2} - \int_{2\pi}^{\infty} \frac{1-\cos(\rho)}{\rho^2}\, d\rho \\ &\geq \frac{\pi}{2} - \frac{1}{2\pi} \\ &\geq 1.\end{aligned}$$

Therefore, if $x \geq 2\pi$, then we have

$$\begin{aligned} l(x) &= \int_0^x \frac{\sin(\rho)}{\rho}\, d\rho - \sin(x) \\ &\geq \int_0^{2\pi} \frac{\sin(\rho)}{\rho}\, d\rho - \sin(x) \\ &\geq 0. \end{aligned}$$

Second case: $d \geq 4$. Combining (5.1) with the identity $\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu \times J_{\nu-1}(x)$, we get

$$\begin{aligned}\hat{f}(\xi) &= \frac{2\pi}{|\xi|^{(d-2)/2}} \int_0^T \rho^{d/2} J_{(d-2)/2}(2\pi|\xi|\rho) \ln\left(\frac{T}{\rho}\right) d\rho \\ &= \frac{1}{(2\pi)^{d/2}|\xi|^d} \int_0^{2\pi|\xi|T} x^{d/2} J_{(d-2)/2}(x) \ln\left(\frac{2\pi|\xi|T}{x}\right) dx \\ &= \frac{1}{(2\pi)^{d/2}|\xi|^d} \int_0^{2\pi|\xi|T} x^{d/2-1} J_{d/2}(x)\, dx.\end{aligned} \tag{5.2}$$

One has the following asymptotic expansion as x goes to ∞ [12]:

$$\begin{aligned} J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{(1+2\nu)\pi}{4}\right) \\ - \frac{(4\nu^2-1)\sqrt{2}}{8\sqrt{\pi} x^{3/2}} \sin\left(x - \frac{(1+2\nu)\pi}{4}\right) + O\left(\frac{1}{x^{5/2}}\right).\end{aligned} \tag{5.3}$$

Combining (5.2) with (5.3), we therefore get the following expansion as $|\xi|$ goes to infinity:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}|\xi|^d} \times \left(\sqrt{\frac{2}{\pi}}(2\pi|\xi|T)^{(d-3)/2}\sin\left(2\pi|\xi|T - \frac{(1+2\nu)\pi}{4}\right) + o\left(|\xi|^{(d-3)/2}\right)\right).$$

Thus, $\overline{\lim}_{|\xi|\to\infty}|\xi|^d\hat{f}(\xi) = -\underline{\lim}_{|\xi|\to\infty}|\xi|^d\hat{f}(\xi) = +\infty$. In particular, $\hat{f}(\xi)$ takes negative values for some ξ.

5.3. *Proofs for Section* 3.3.

PROOF OF PROPOSITIONS 3.5 AND 3.6. We suppose that p belongs to $]1, p_*[$ or $]-\infty, 0[$. Let θ be some function satisfying the conditions (1)–(3) of Section 2.2 and m_ε be the random measure associated with $\theta^\varepsilon * f$. Following the notation of Example 2.2 for $C(x)$, we consider $\widetilde{m}_\varepsilon$, the random measure associated with $\widetilde{f}_\varepsilon$, where $\widetilde{f}_\varepsilon$ is the function

$$\widetilde{f}_\varepsilon(x) = \lambda^2 \int_{C(0)\cap C(x);\varepsilon<t<\infty} \frac{dy\,dt}{t^{d+1}}.$$

One can show that there exists $c, C > 0$ such that for all x, we have (see Appendix B in [5])

$$\widetilde{f}_\varepsilon(x) - c \le (\theta^\varepsilon * f)(x) \le \widetilde{f}_\varepsilon(x) + C.$$

By using Corollary A.2 from the Appendix in the continuous version [with $F(x) = x^p$], we conclude that there exist $c, C > 0$ such that for all ε and all bounded A in $\mathcal{B}(\mathbb{R}^d)$,

$$cE[\widetilde{m}_\varepsilon(A)^p] \le E[m_\varepsilon(A)^p] \le CE[\widetilde{m}_\varepsilon(A)^p].$$

First case: p belongs to $]1, p_*[$. Proposition 3.5 is therefore established if we can show that

$$\sup_{\varepsilon>0} E[\widetilde{m}_\varepsilon(A)^p] < \infty.$$

To prove the above inequality for all bounded A, it is enough to suppose that $A = [0,1]^d$. This is proved in dimension 1 in [3], Theorem 3. One can adapt the dyadic decomposition performed in the proof of Theorem 3 in [3] to handle the d-dimensional case.

Second case: p belongs to $]-\infty, 0[$. Proposition 3.5 is therefore established if we can show that for all $c > 0$,

$$\sup_{\varepsilon>0} E[\widetilde{m}_\varepsilon(B(0,c))^p] < \infty.$$

The above bound can be proven by adapting the proof of Proposition 4 in [18] (this is done to prove Theorem 3 in [4], where a log-Poisson model is considered). □

PROOF OF PROPOSITION 3.7. For the sake of simplicity, we consider the case $R = 1$ and will consider the case $p \in [1, p_*[$. We consider θ, a continuous and positive function with compact support $B(0, A)$ satisfying properties (1)–(3) of Section 2.2. We note that

$$m_\varepsilon(dx) = e^{X_\varepsilon(x)-(1/2)E[X_\varepsilon(x)^2]}\,dx,$$

where $(X_\varepsilon(x))_{x\in\mathbb{R}^d}$ is a Gaussian field of covariance $q_\varepsilon(x-y)$ with

$$q_\varepsilon(x) = (\theta^\varepsilon * f)(x) = \int_{\mathbb{R}^d} \theta(z)\left(\lambda^2 \ln^+ \frac{1}{|x-\varepsilon z|} + g(x-\varepsilon z)\right) dz.$$

Let c, c' be two positive numbers in $]0, \frac{1}{2}[$ such that $c < c'$. If ε is sufficiently small and u, v belong to $[0,1]^d$, then we get

$$\begin{aligned} q_{c\varepsilon}\big(c(v-u)\big) &= \int_{\mathbb{R}^d} \theta(z)\left(\lambda^2 \ln \frac{1}{|c(v-u)-c\varepsilon z|} + g\big(c(v-u)-c\varepsilon z\big)\right) dz \\ &= \lambda^2 \ln\left(\frac{c'}{c}\right) + \int_{\mathbb{R}^d} \theta(z)\left(\lambda^2 \ln \frac{1}{|c'(v-u)-c'\varepsilon z|}\right. \\ &\qquad\qquad \left. + g\big(c(v-u)-c\varepsilon z\big)\right) dz \\ &\le \lambda^2 \ln\left(\frac{c'}{c}\right) + q_{c'\varepsilon}\big(c'(v-u)\big) + C_{c,c',\varepsilon}, \end{aligned}$$

where

$$C_{c,c',\varepsilon} = \sup_{\substack{|z|\le A \\ |v-u|\le 1}} \big|g\big(c(v-u)-c\varepsilon z\big) - g\big(c'(v-u)-c'\varepsilon z\big)\big|.$$

Let $Y_{c,c',\varepsilon}$ be some centered Gaussian variable with variance $C_{c,c',\varepsilon} + \lambda^2 \ln(\frac{c'}{c})$. By using Corollary A.2 from the Appendix in the continuous version, we conclude that

$$\begin{aligned} &E[m_{c\varepsilon}([0,c]^d)^p] \\ &\quad = E\left[\left(\int_{[0,c]^d} e^{X_{c\varepsilon}(x)-(1/2)E[X_{c\varepsilon}(x)^2]}\,dx\right)^p\right] \\ &\quad = c^{dp} E\left[\left(\int_{[0,1]^d} e^{X_{c\varepsilon}(cu)-(1/2)E[X_{c\varepsilon}(cu)^2]}\,du\right)^p\right] \\ &\quad \le c^{dp} E\left[\left(\int_{[0,1]^d} e^{X_{c'\varepsilon}(c'u)+Y_{c,c',\varepsilon}-(1/2)E[(X_{c'\varepsilon}(c'u)+Y_{c,c',\varepsilon})^2]}\,du\right)^p\right] \end{aligned}$$

$$= c^{dp}\left(\frac{c'}{c}\right)^{p(p-1)\lambda^2/2} e^{p(p-1)C_{c,c',\varepsilon}/2}$$

$$\times E\left[\left(\int_{[0,1]^d} e^{X_{c'\varepsilon}(c'u)-(1/2)E[X_{c'\varepsilon}(c'u)^2]}\,du\right)^p\right]$$

$$= \left(\frac{c}{c'}\right)^{dp-p(p-1)\lambda^2/2} e^{p(p-1)C_{c,c',\varepsilon}/2} E\left[\left(\int_{[0,c']^d} e^{X_{c'\varepsilon}(x)-(1/2)E[X_{c'\varepsilon}(x)^2]}\,dx\right)\right]$$

$$= \left(\frac{c}{c'}\right)^{\zeta_p} e^{p(p-1)C_{c,c',\varepsilon}/2} E[m_{c'\varepsilon}([0,c']^d)^p].$$

Taking the limit $\varepsilon \to 0$ in the above inequality leads to

$$\frac{E[m([0,c]^d)^p]}{c^{\zeta_p}} \leq e^{p(p-1)C_{c,c'}/2} \frac{E[m([0,c']^d)^p]}{c'^{\zeta_p}}, \tag{5.4}$$

where $C_{c,c'} = \sup_{|v-u|\leq 1} |g(c(v-u)) - g(c'(v-u))|$. Similarly, we have,

$$\frac{E[m([0,c']^d)^p]}{c'^{\zeta_p}} \leq e^{p(p-1)C_{c,c'}/2} \frac{E[m([0,c]^d)^p]}{c^{\zeta_p}}. \tag{5.5}$$

Since $C_{c,c'}$ goes to 0 as $c, c' \to 0$,we conclude by inequality (5.4) and (5.5) that $(\frac{E[m([0,c]^d)^p]}{c^{\zeta_p}})_{c>0}$ is a Cauchy sequence as $c \to 0$, bounded from below and above by positive constants. Therefore, there exists some $c_p > 0$ such that

$$E[m([0,c]^d)^p] \underset{c\to 0}{\sim} c_p c^{\zeta_p}.$$

The same method can be applied to show that $\frac{c_p}{e^{p(p-1)g(0)/2}}$ is independent of g. The proof is then concluded by setting $C_p = \frac{c_p}{e^{p(p-1)g(0)/2}}$. □

PROOF OF PROPOSITION 3.8. We use the scaling relation (3.3) to compute the characteristic function of $m(B(0,c))$ for all ξ in $\mathbb{R}$:

$$E[e^{i\xi m(B(0,c))}] = E[e^{i\xi e^{\Omega_c} m(B(0,R))}]$$
$$= E[\mathcal{F}(\xi m(B(0,R)))],$$

where $\mathcal{F}$ is the characteristic function of e^{Ω_c}. It is easy to show that for all $n \in \mathbb{N}$, there exists $C > 0$ such that

$$|\mathcal{F}(\xi)| \leq \frac{C}{|\xi|^n}.$$

From this, we conclude, by Proposition 3.6, that

$$E[e^{i\xi m(B(0,c))}] \leq \frac{C}{|\xi|^n} E\left[\frac{1}{m(B(0,R))^n}\right] \leq \frac{C'}{|\xi|^n}.$$

This implies the existence of a C^∞ density. □

APPENDIX

We give the following classical lemma, which was first derived in [13].

LEMMA A.1. *Let* $(X_i)_{1\le i\le n}$ *and* $(Y_i)_{1\le i\le n}$ *be two independent centered Gaussian vectors and* $(p_i)_{1\le i\le n}$ *a sequence of positive numbers. If* $\phi:\mathbb{R}_+\to\mathbb{R}$ *is some smooth function with polynomial growth at infinity, then we define*

$$\varphi(t)=E\left[\phi\left(\sum_{i=1}^n p_i e^{Z_i(t)-(1/2)E[Z_i(t)^2]}\right)\right],$$

with $Z_i(t)=\sqrt{t}X_i+\sqrt{1-t}Y_i$. *We then have the following formula for the derivative*:

$$\varphi'(t)=\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n p_i p_j (E[X_iX_j]-E[Y_iY_j]) \\ \times E\big[e^{Z_i(t)+Z_j(t)-(1/2)E[Z_i(t)^2]-(1/2)E[Z_j(t)^2]}\phi''(W_{n,t})\big], \tag{A.1}$$

where

$$W_{n,t}=\sum_{k=1}^n p_k e^{Z_k(t)-(1/2)E[Z_k(t)^2]}.$$

As a consequence of the above formula, we can derive a similar formula in the continuous case. Let I *be a bounded subinterval of* $\mathbb{R}^d$ *and let* $(X(u))_{u\in I}$, $(Y(u))_{u\in I}$ *be two independent centered continuous Gaussian processes. If we define*

$$\varphi(t)=E\left[\phi\left(\int_I e^{Z(t)(u)-(1/2)E[Z(t)(u)^2]}\,du\right)\right]$$

with $Z(t)(u)=\sqrt{t}X(u)+\sqrt{1-t}Y(u)$, *then we have the following formula for the derivative*:

$$\varphi'(t)=\frac{1}{2}\int_I\int_I\big(E[X(t_1)X(t_2)]-E[Y(t_1)Y(t_2)]\big) \\ \times E\big[e^{Z(t)(t_1)+Z(t)(t_2)-(1/2)E[Z(t)(t_1)^2]-(1/2)E[Z(t)(t_2)^2]} \\ \times \phi''(W_t)\big]\,dt_1\,dt_2,$$

where

$$W_t=\int_I e^{Z(t)(u)-(1/2)E[Z(t)(u)^2]}\,du.$$

As a consequence of the above lemma, one can derive the following classical comparison principle.

COROLLARY A.2. *Let $(p_i)_{1\leq i\leq n}$ be a sequence of positive numbers. Consider $(X_i)_{1\leq i\leq n}$ and $(Y_i)_{1\leq i\leq n}$, two centered Gaussian vectors such that*

$$\forall i,j \qquad E[X_iX_j]\leq E[Y_iY_j].$$

Then, for all convex function $F:\mathbb{R}\to\mathbb{R}$, we have

$$E\left[F\left(\sum_{i=1}^n p_i e^{X_i-(1/2)E[X_i^2]}\right)\right]\leq E\left[F\left(\sum_{i=1}^n p_i e^{Y_i-(1/2)E[Y_i^2]}\right)\right]. \tag{A.2}$$

Similarly, we get a comparison in the continuous case. Let I be a bounded subinterval of $\mathbb{R}^d$ and $(X(u))_{u\in I}$, $(Y(u))_{u\in I}$ be two independent centered continuous Gaussian processes such that

$$\forall u,u' \qquad E[X(u)X(u')]\leq E[Y(u)Y(u')].$$

Then, for all convex functions $F:\mathbb{R}\to\mathbb{R}$, we have

$$E\left[F\left(\int_I e^{X(u)-(1/2)E[X(u)^2]}\,du\right)\right]\leq E\left[F\left(\int_I e^{Y(u)-(1/2)E[Y(u)^2]}\,du\right)\right].$$

We will also use the following corollary.

COROLLARY A.3. *Let $(X_i)_{1\leq i\leq n}$ and $(Y_i)_{1\leq i\leq n}$ be two centered Gaussian vectors such that*:

- $\forall i$, $E[X_i^2]=E[Y_i^2]$;
- $\forall i\neq j$, $E[X_iX_j]\leq E[Y_iY_j]$.

Then, for all increasing functions $F:\mathbb{R}\to\mathbb{R}_+$, we have

$$E\Big[F\Big(\sup_{1\leq i\leq n} Y_i\Big)\Big]\leq E\Big[F\Big(\sup_{1\leq i\leq n} X_i\Big)\Big]. \tag{A.3}$$

PROOF. It is enough to show inequality (A.3) for $F=1_{]x,+\infty[}$, for some $x\in\mathbb{R}$. Let β be some positive parameter. Integrating equality (A.1) applied to the convex function $\phi:u\to e^{-e^{-\beta x}u}$ and the sequences (βX_i), (βY_i), $p_i=e^{(\beta^2/2)E[X_i^2]}$, we get

$$E\big[e^{-\sum_{i=1}^n e^{\beta(X_i-x)}}\big]\leq E\big[e^{-\sum_{i=1}^n e^{\beta(Y_i-x)}}\big].$$

By letting $\beta\to\infty$, we conclude that

$$P\Big(\sup_{1\leq i\leq n} X_i<x\Big)\leq P\Big(\sup_{1\leq i\leq n} Y_i<x\Big). \qquad \square$$

REFERENCES

[1] BACRY, E., DELOUR, J. and MUZY, J. F. (2001). Multifractal random walks. *Phys. Rev. E* **64** 026103–026106.

[2] BACRY, E., KOZHEMYAK, A. and MUZY, J.-F. (2008). Continuous cascade models for asset returns. *J. Econom. Dynam. Control* **32** 156–199. MR2381693

[3] BACRY, E. and MUZY, J. F. (2003). Log-infinitely divisible multifractal processes. *Comm. Math. Phys.* **236** 449–475. MR2021198

[4] BARRAL, J. and MANDELBROT, B. B. (2002). Multifractal products of cylindrical pulses. *Probab. Theory Related Fields* **124** 409–430. MR1939653

[5] CHAINAIS, P. (2006). Multidimensional infinitely divisible cascades. Application to the modelling of intermittency in turbulence. *Eur. Phys. J. B* **51** 229–243.

[6] CIZEAU, P., GOPIKRISHNAN, P., LIU, Y., MEYER, M., PENG, C. K. and STANLEY, E. (1999). Statistical properties of the volatility of price fluctuations. *Phys. Rev. E.* **60** 1390–1400.

[7] CONT, R. (2001). Empirical properties of asset returns: Stylized facts and statistical issues. *Quant. Finance* **1** 223–236.

[8] DALEY, D. J. and VERE-JONES, D. (1988). *An Introduction to the Theory of Point Processes*. Springer, New York. MR950166

[9] DUCHON, J., ROBERT, R. and VARGAS, V. (2008). Forecasting volatility with the multifractal random walk model. Available at http://arxiv.org/abs/0801.4220.

[10] FRISCH, U. (1995). *Turbulence*. Cambridge Univ. Press, Cambridge. MR1428905

[11] GNEITING, T. (2001). Criteria of Pólya type for radial positive definite functions. *Proc. Amer. Math. Soc.* **129** 2309–2318 (electronic). MR1823914

[12] GRAY, A., MATHEWS, G. B. and MACROBERT, T. (1922). *A Treatise on Bessel Functions*, 2nd ed. Macmillan, London.

[13] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR829798

[14] KALLENBERG, O. (1986). *Random Measures*, 4th ed. Akademie-Verlag, Berlin. MR854102

[15] KOLMOGOROV, A. N. (1962). A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **13** 82–85. MR0139329

[16] LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete* (3) [*Results in Mathematics and Related Areas* (3)] **23**. Springer, Berlin. MR1102015

[17] MANDELBROT, B. B. (1972). A possible refinement of the lognormal hypothesis concerning the distribution of energy in intermittent turbulence. In *Statistical Models and Turbulence. Lecture Notes in Phys.* **12** 333–351. Springer, Berlin. MR0438885

[18] MOLCHAN, G. M. (1996). Scaling exponents and multifractal dimensions for independent random cascades. *Comm. Math. Phys.* **179** 681–702. MR1400758

[19] OBOUKHOV, A. M. (1962). Some specific features of atmospheric turbulence. *J. Fluid Mech.* **13** 77–81. MR0139328

[20] PASENCHENKO, O. Y. (1996). Sufficient conditions for the characteristic function of a two-dimensional isotropic distribution. *Theory Probab. Math. Statist.* **53** 149–152. MR1449115

[21] SCHWARTZ, L. (1951). *Théorie des Distributions*. Hermann, Paris. MR041345

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