The Annals of Probability
2010, Vol. 38, No. 2, 770–793
DOI: 10.1214/09-AOP495
© Institute of Mathematical Statistics, 2010

ERGODIC THEORY, ABELIAN GROUPS AND POINT PROCESSES INDUCED BY STABLE RANDOM FIELDS

BY PARTHANIL ROY[1]

Michigan State University

We consider a point process sequence induced by a stationary symmetric α-stable ($0 < \alpha < 2$) discrete parameter random field. It is easy to prove, following the arguments in the one-dimensional case in [*Stochastic Process. Appl.* **114** (2004) 191–210], that if the random field is generated by a dissipative group action then the point process sequence converges weakly to a cluster Poisson process. For the conservative case, no general result is known even in the one-dimensional case. We look at a specific class of stable random fields generated by conservative actions whose effective dimensions can be computed using the structure theorem of finitely generated Abelian groups. The corresponding point processes sequence is not tight, and hence needs to be properly normalized in order to ensure weak convergence. This weak limit is computed using extreme value theory and some counting techniques.

1. Introduction. Suppose that $\mathbf{X} := \{X_t\}_{t\in\mathbb{Z}^d}$ is a stationary symmetric α-stable ($S\alpha S$) discrete-parameter random field. In other words, every finite linear combination $\sum_{i=1}^{k} c_i X_{t_i+s}$ follows an $S\alpha S$ distribution which does not depend on $s \in \mathbb{Z}^d$. We consider the following sequence of point processes on $[-\infty,\infty] \setminus \{0\}$

$$N_n = \sum_{\|t\|_\infty \le n} \delta_{b_n^{-1} X_t}, \qquad n = 1, 2, 3, \ldots, \tag{1.1}$$

induced by the random field $\mathbf{X}$ with an aptly chosen sequence of scaling constants $b_n \uparrow \infty$. Here, δ_x denotes the point mass at x. We are interested in the weak convergence of this point process sequence in the space $\mathcal{M}$ of Radon measures on $[-\infty,\infty] \setminus \{0\}$ equipped with the vague topology. This is important in extreme value theory because a number of limit theorems for various functionals of $S\alpha S$ random fields can be obtained by continuous mapping arguments on the associated point process sequence. See, for example, Resnick (1987) and Balkema and Embrechts (2007) for a background on weak convergence of point processes and its applications to extreme value theory. See also Neveu (1977), Kallenberg (1983) and Resnick (2007).

Received December 2007; revised July 2009.

[1]Supported in part by NSF Grant DMS-03-03493 and NSF training grant "Graduate and Postdoctoral Training in Probability and Its Applications" at Cornell University and by the RiskLab of the Department of Mathematics, ETH Zurich.

AMS 2000 subject classifications. Primary 60G55; secondary 60G60, 60G70, 37A40.

Key words and phrases. Stable process, random field, point process, random measure, weak convergence, extreme value theory, ergodic theory, group action.

If $\{X_t\}_{t\in\mathbb{Z}^d}$ is an i.i.d. collection of random variables with tails decaying like those of a symmetric α stable distribution, then $\{b_n\}$ can be chosen as follows:

$$b_n = n^{d/\alpha}. \tag{1.2}$$

With the above choice, the sequence $\{N_n\}$ converges weakly in the space $\mathcal{M}$ to a Poisson random measure, whose intensity blows up near zero (this is the reason why we exclude zero from the state space) due to clustering of normalized observations. See, once again, Resnick (1987). Cluster Poisson processes are obtained as weak limits also for the point processes induced by a stationary stochastic process with the marginal distributions having balanced regularly varying tail probabilities provided the process is a moving average [see Davis and Resnick (1985)] or it satisfies some mild mixing conditions [see Davis and Hsing (1995)]. In these works, weak limits of various functionals of the process were computed from the point process convergence by clever use of the continuous mapping theorem. See also Mori (1977) for various possible weak limits of a two-dimensional point process induced by strong mixing sequences.

When the dependence structure is not necessarily local or mild, finding a suitable scaling sequence and computation of the weak limit both become challenging. As in the one-dimensional case in Resnick and Samorodnitsky (2004), we will observe that for point processes induced by stable random fields the choice of $\{b_n\}$ depends on the heaviness of the tails of the marginal distributions as well as on the length of memory. In the short memory case, the choice (1.2) of normalizing constants is appropriate whereas in the long memory case, it is not. Furthermore, the observations may cluster so much due to long memory that one may need to normalize the sequence $\{N_n\}$ itself to ensure weak convergence. This phenomenon was also observed in the one-dimensional case in Resnick and Samorodnitsky (2004).

This paper is organized as follows. We present some background materials on stationary symmetric α-stable random fields in Section 2. Section 3 deals with the point processes associated with dissipative actions, that is, point processes based on mixed moving averages. In Section 4, we state our main result on the weak convergence of the point process sequence induced by a class of random fields generated by conservative actions whose effective dimensions can be computed using group theory. This result is proved in Section 5 using extreme value theory and counting techniques. Finally, an example is discussed in Section 6. Throughout this paper, we use the notation $c_n \sim d_n$ to mean that c_n/d_n converges to a positive number as $n \to \infty$.

2. Preliminaries. It is well known that every $S\alpha S$ random field $\mathbf{X}$ has an integral representation of the form

$$X_t \stackrel{d}{=} \int_S f_t(s) M(ds), \qquad t \in \mathbb{Z}^d, \tag{2.1}$$

where M is an $S\alpha S$ random measure on some standard Borel space $(S, \mathcal{S})$ with σ-finite control measure μ and $f_t \in L^\alpha(S, \mu)$ for all $t \in \mathbb{Z}^d$. See, for example, Theorem 13.1.2 of Samorodnitsky and Taqqu (1994). The representation (2.1) is called an integral representation of $\{X_t\}$. Without loss of generality, we can also assume that the family $\{f_t\}$ satisfies the full support assumption

$$\text{Support}(f_t, t \in \mathbb{Z}^d) = S, \tag{2.2}$$

because we can always replace S by $S_0 = \text{Support}(f_t, t \in \mathbb{Z}^d)$ in (2.1).

For a stationary $\{X_t\}$, using the fact that the action of the group $\mathbb{Z}^d$ on $\{X_t\}_{t\in\mathbb{Z}^d}$ by translation of indices preserves the law together with certain rigidity properties of the spaces L^α, $\alpha < 2$, it has been shown in Rosiński (1995) (for $d = 1$) and Rosiński (2000) (for a general d) that there always exists an integral representation of the form

$$f_t(s) = c_t(s)\left(\frac{d\mu \circ \phi_t}{d\mu}(s)\right)^{1/\alpha} f \circ \phi_t(s), \qquad t \in \mathbb{Z}^d, \tag{2.3}$$

where $f \in L^\alpha(S, \mu)$, $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is a nonsingular $\mathbb{Z}^d$-action on (S, μ) (i.e., each $\phi_t : S \to S$ is measurable, ϕ_0 is the identity map on S, $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$ for all $t_1, t_2 \in \mathbb{Z}^d$ and each $\mu \circ \phi_t^{-1}$ is an equivalent measure of μ), and $\{c_t\}_{t\in\mathbb{Z}^d}$ is a measurable cocycle for $\{\phi_t\}$ taking values in $\{-1, +1\}$ [i.e., each c_t is a measurable map $c_t : S \to \{-1, +1\}$ such that for all $t_1, t_2 \in \mathbb{Z}^d$, $c_{t_1+t_2}(s) = c_{t_2}(s)c_{t_1}(\phi_{t_2}(s))$ for μ-a.a. $s \in S$].

Conversely, if $\{f_t\}$ is of the form (2.3) then $\{X_t\}$ defined by (2.1) is a stationary $S\alpha S$ random field. We will say that a stationary $S\alpha S$ random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is generated by a nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}$ on (S, μ) if it has an integral representation of the form (2.3) satisfying (2.2).

A measurable set $W \subseteq S$ is called a wandering set for the nonsingular $\mathbb{Z}^d$-action $\{\phi_t\}_{t\in\mathbb{Z}^d}$ (as defined above) if $\{\phi_t(W) : t \in \mathbb{Z}^d\}$ is a pairwise disjoint collection. Proposition 1.6.1 of Aaronson (1997) gives a decomposition of S into two disjoint and invariant parts as follows: $S = \mathcal{C} \cup \mathcal{D}$ where $\mathcal{D} = \bigcup_{t\in\mathbb{Z}^d} \phi_t(W)$ for some wandering set $W \subseteq S$, and $\mathcal{C}$ has no wandering subset of positive μ-measure. $\mathcal{D}$ is called the dissipative part, and $\mathcal{C}$ is called the conservative part of the action. The action $\{\phi_t\}$ is called conservative if $S = \mathcal{C}$ and dissipative if $S = \mathcal{D}$. The reader is suggested to read Aaronson (1997) and Krengel (1985) for various ergodic theoretical notions used in this paper. Following the notation of Rosiński (1995), Rosiński (2000) and Roy and Samorodnitsky (2008), we can obtain the following unique (in law) decomposition of the random field $\mathbf{X}$:

$$X_t \stackrel{d}{=} \int_{\mathcal{C}} f_t(s)M(ds) + \int_{\mathcal{D}} f_t(s)M(ds) =: X_t^{\mathcal{C}} + X_t^{\mathcal{D}}, \qquad t \in \mathbb{Z}^d, \tag{2.4}$$

into a sum of two independent random fields $\mathbf{X}^{\mathcal{C}}$ and $\mathbf{X}^{\mathcal{D}}$ generated by conservative and dissipative $\mathbb{Z}^d$-actions, respectively. This decomposition implies that it is enough to study stationary $S\alpha S$ random fields generated by conservative and dissipative actions.

Stationary stable random fields generated by conservative actions are expected to have longer memory than those generated by dissipative actions because a conservative action "does not wander too much," and so the same values of the random measure M in (2.3) contribute to observations X_t far separated in t. The length of memory of stable random fields determines, among other things, the rate of growth of the partial maxima sequence

$$M_n := \max_{\|t\|_\infty \le n} |X_t|, \qquad n = 0, 1, 2, \ldots. \tag{2.5}$$

If X_t is generated by a conservative action, the partial maxima sequence (2.5) grows at a slower rate because longer memory prevents erratic changes in X_t even when t becomes "large." More specifically,

$$n^{-d/\alpha} M_n \Rightarrow \begin{cases} c_{\mathbf{X}} Z_\alpha, & \text{if } \mathbf{X} \text{ is generated by a dissipative action,} \\ 0, & \text{if } \mathbf{X} \text{ is generated by a conservative action,} \end{cases} \tag{2.6}$$

weakly as $n \to \infty$. Here, Z_α is a standard Frechét type extreme value random variable with distribution function

$$P(Z_\alpha \le x) = e^{-x^{-\alpha}}, \qquad x > 0, \tag{2.7}$$

and $c_{\mathbf{X}}$ is a positive constant depending on the random field $\mathbf{X}$. The above dichotomy, which was established in the $d = 1$ case by Samorodnitsky (2004) and in the general case by Roy and Samorodnitsky (2008), implies that the choice of scaling sequence (1.2) is not appropriate in the conservative case since all the points in the sequence $\{N_n\}$ will be driven to zero by this normalization. On the other hand, we will see in the next section that (1.2) is indeed a good choice when the underlying action is dissipative.

3. The dissipative case.

Assume, in this section, that $\mathbf{X}$ is a stationary $S\alpha S$ discrete parameter random field generated by a dissipative $\mathbb{Z}^d$-action. In this case, $\mathbf{X}$ has the following mixed moving average representation:

$$\mathbf{X} \stackrel{d}{=} \left\{ \int_{W \times \mathbb{Z}^d} f(v, t+s) M(dv, ds) \right\}_{t \in \mathbb{Z}^d}, \tag{3.1}$$

where $f \in L^\alpha(W \times \mathbb{Z}^d, \nu \otimes \zeta)$, ζ is the counting measure on $\mathbb{Z}^d$, ν is a σ-finite measure on a standard Borel space $(W, \mathcal{W})$, and M is a $S\alpha S$ random measure on $W \times \mathbb{Z}^d$ with control measure $\nu \otimes \zeta$. Mixed moving averages were first introduced by Surgailis et al. (1993). The above representation was established in the $d = 1$ case by Rosiński (1995) and in the general case by Roy and Samorodnitsky (2008) based on a previous work by Rosiński (2000).

Suppose ν_α is the symmetric measure on $[-\infty, \infty] \setminus \{0\}$ given by

$$\nu_\alpha(x, \infty] = \nu_\alpha[-\infty, -x) = x^{-\alpha}, \qquad x > 0.$$

Without loss of generality, we assume that the original stable random field is of the form given in (3.1). Let

$$N = \sum_i \delta_{(j_i, v_i, u_i)} \sim \text{PRM}(\nu_\alpha \otimes \nu \otimes \zeta) \tag{3.2}$$

be a Poisson random measure on $([-\infty, \infty] \setminus \{0\}) \times W \times \mathbb{Z}^d$ with mean measure $\nu_\alpha \otimes \nu \otimes \zeta$. Then from the assumption above, it follows that $\mathbf{X}$ has the following series representation:

$$X_t = C_\alpha{}^{1/\alpha} \sum_i j_i f(v_i, u_i + t), \qquad t \in \mathbb{Z}^d, \tag{3.3}$$

where C_α is the stable tail constant given by

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx\right)^{-1} = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha \neq 1, \\ \dfrac{2}{\pi}, & \text{if } \alpha = 1. \end{cases} \tag{3.4}$$

See, for example, Samorodnitsky and Taqqu (1994).

It follows from (2.6) that the partial maxima sequence (2.5) grows exactly at the rate $n^{d/\alpha}$. As expected, $b_n \sim n^{d/\alpha}$ turns out to be the right normalization for the point process (1.1) in this case. The following theorem, which is an extension of Theorem 3.1 in Resnick and Samorodnitsky (2004) to the $d > 1$ case, states that with this choice of $\{b_n\}$ the limiting random measure is a cluster Poisson random measure even though the dependence structure is no longer weak or local. The proof is parallel to the one-dimensional case, and hence omitted.

THEOREM 3.1. *Let $\mathbf{X}$ be the mixed moving average (3.1), and define the point process $N_n = \sum_{\|t\|_\infty \le n} \delta_{(2n)^{-d/\alpha} X_t}$, $n = 1, 2, \ldots$. Then $N_n \Rightarrow N_*$ as $n \to \infty$, weakly in the space $\mathcal{M}$, where N_* is a cluster Poisson random measure with representation*

$$N_* = \sum_{i=1}^\infty \sum_{t \in \mathbb{Z}^d} \delta_{j_i f(v_i, t)}, \tag{3.5}$$

where j_i, v_i are as in (3.2). Furthermore, N_* is Radon on $[-\infty, \infty] \setminus \{0\}$ with Laplace functional ($g \ge 0$ continuous with compact support)

$$\begin{aligned} \psi_{N_*}(g) &= E\big(e^{-N_*(g)}\big) \\ &= \exp\left\{-\iint_{([-\infty,\infty]\setminus\{0\}) \times W} \big(1 - e^{-\sum_{t \in \mathbb{Z}^d} g(x f(v,t))}\big) \nu_\alpha(dx) \nu(dv)\right\}. \end{aligned} \tag{3.6}$$

REMARK 3.2. The above result is true as long as the underlying action is not conservative because Theorem 4.3 in Roy and Samorodnitsky (2008) ensures that the conservative part of the random field [see (2.4)] will be killed by the normalization (1.2) and hence the mixed moving average part will determine the convergence.

4. Point processes and group theory. This section deals with the longer memory case, that is, the random field $\mathbf{X}$ is now generated by a conservative action. In this case, we know from (2.6) that the partial maxima sequence (2.5) of the random field grows at a rate slower than $n^{d/\alpha}$. Hence, (1.2) is inappropriate in this case. In general, there may or may not exist a normalizing sequence $\{b_n\}$ that ensures weak convergence of $\{N_n\}$. See Resnick and Samorodnitsky (2004) for examples of both kinds in the $d = 1$ case.

We will work with a specific class of stable random fields generated by conservative actions for which the effective dimension $p \le d$ is known. For this class of random fields, the point process $\{N_n\}$ will not converge weakly to a nontrivial limit for any choice of the scaling sequence. Even for the most appropriate choice of $\{b_n\}$, the associated point process will not even be tight (see Remark 4.4 below) because of the clustering effect of extreme observations due to longer memory of the random field. Hence, in order to ensure weak convergence, we have to normalize the point process sequence $\{N_n\}$ in addition to using a normalizing sequence $\{b_n\}$ different from (1.2) for the points. This phenomenon was also observed in Example 4.2 in the one-dimensional case in Resnick and Samorodnitsky (2004).

Without loss of generality, we may assume that the original stable random field is of the form given in (2.1) and (2.3). Following the approach of Roy and Samorodnitsky (2008), we view the underlying action as a group of invertible nonsingular transformations on (S, μ) and use some basic counting arguments to analyze the point process $\{N_n\}$. We start with introducing the appropriate notation.

Consider $A := \{\phi_t : t \in \mathbb{Z}^d\}$ as a subgroup of the group of invertible nonsingular transformations on (S, μ) and define a group homomorphism

$$\Phi : \mathbb{Z}^d \to A$$

by $\Phi(t) = \phi_t$ for all $t \in \mathbb{Z}^d$. Let $K := \mathrm{Ker}(\Phi) = \{t \in \mathbb{Z}^d : \phi_t = 1_S\}$, where 1_S denote the identity map on S. Then K is a free Abelian group and by the first isomorphism theorem of groups [see, e.g., Lang (2002)] we have

$$A \simeq \mathbb{Z}^d / K.$$

Hence, by the structure theorem for finitely generated Abelian groups [see Theorem 8.5 in Chapter I of Lang (2002)], we get

$$A = \bar{F} \oplus \bar{N},$$

where $\bar{F}$ is a free Abelian group and $\bar{N}$ is a finite group. Assume $\mathrm{rank}(\bar{F}) = p \ge 1$ and $|\bar{N}| = l$. Since $\bar{F}$ is free, there exists an injective group homomorphism

$$\Psi : \bar{F} \to \mathbb{Z}^d$$

such that $\Phi \circ \Psi = 1_{\bar{F}}$. Let $F = \Psi(\bar{F})$. Then F is a free subgroup of $\mathbb{Z}^d$ of rank p. In particular, $p \le d$.

The rank p is the effective dimension of the random field, giving more precise information on the choice of normalizing sequence $\{b_n\}$ than the nominal dimension d. Theorem 5.4 in Roy and Samorodnitsky (2008) yields a better estimate on the rate of growth of the partial maxima (2.5) than (2.6), namely

$$n^{-p/\alpha} M_n \Rightarrow \begin{cases} c'_{\mathbf{X}} Z_\alpha, & \text{if } \{\phi_t\}_{t\in F} \text{ is a dissipative action,} \\ 0, & \text{if } \{\phi_t\}_{t\in F} \text{ is a conservative action.} \end{cases} \tag{4.1}$$

Here, $c'_{\mathbf{X}}$ is another positive constant depending on $\mathbf{X}$ and Z_α is as in (2.7). Hence, we can guess that $b_n \sim n^{p/\alpha}$ is a legitimate choice of the scaling sequence provided $\{\phi_t\}_{t\in F}$ is dissipative.

It is easy to check that the sum $F + K$ is direct and

$$\mathbb{Z}^d/G \simeq \bar{N}, \tag{4.2}$$

where $G = F \oplus K$. Let $x_1 + G, x_2 + G, \ldots, x_l + G$ be all the cosets of G in $\mathbb{Z}^d$. We give a group structure to

$$H := \bigcup_{k=1}^{l} (x_k + F) \tag{4.3}$$

as follows. For all $u_1, u_2 \in H$, there exists unique $u \in H$ such that $(u_1 + u_2) - u \in K$. We define this u to be $u_1 \oplus u_2$. Clearly, H becomes a countable Abelian group isomorphic to $\mathbb{Z}^d/K$ under the operation $\oplus$ ("addition modulo K").

Define a map $N : H \to \{0, 1, \ldots\}$ as,

$$N(u) := \min\{\|u + v\|_\infty : v \in K\}.$$

It is easy to check that $N(\cdot)$ satisfies "symmetry": for all $u \in H$,

$$N(u^{-1}) = N(u), \tag{4.4}$$

where u^{-1} is the inverse of u in $(H, \oplus)$, and the "triangle inequality": for all $u_1, u_2 \in H$,

$$N(u_1 \oplus u_2) \le N(u_1) + N(u_2). \tag{4.5}$$

Define

$$H_n = \{u \in H : N(u) \le n\}. \tag{4.6}$$

It has been shown in Roy and Samorodnitsky (2008) that the H_n's are finite and

$$|H_n| \sim n^p. \tag{4.7}$$

Also, clearly $H_n \uparrow H$.

If $\{\phi_t\}_{t\in F}$ is a dissipative group action, then we get a dissipative H-action $\{\psi_u\}_{u\in H}$ defined by

$$\psi_u = \phi_u \qquad \text{for all } u \in H. \tag{4.8}$$

See, once again, Roy and Samorodnitsky (2008). In this case, if we further assume that the cocycle in (2.3) satisfies

$$c_t \equiv 1 \qquad \text{for all } t \in K, \tag{4.9}$$

then it will follow that $\{c_u\}_{u\in H}$ is an H-cocycle for $\{\psi_u\}_{u\in H}$, i.e., for all u_1, $u_2 \in H$,

$$c_{u_1 \oplus u_2}(s) = c_{u_1}(s) c_{u_2}(\psi_{u_1}(s)) \qquad \text{for } \mu\text{-a.a. } s \in S.$$

Hence, the subfield $\{X_u\}_{u\in H}$ is H-stationary and is generated by the dissipative action $\{\psi_u\}_{u\in H}$. This implies, in particular, that there is a standard Borel space $(W, \mathcal{W})$ with a σ-finite measure ν on it such that

$$X_u \stackrel{d}{=} \int_{W\times H} h(w, u \oplus s) M'(dw, ds), \qquad u \in H, \tag{4.10}$$

for some $h \in L^\alpha(W \times H, \nu \otimes \tau)$, where τ is the counting measure on H, and M' is a $S\alpha S$ random measure on $W \times H$ with control measure $\nu \otimes \tau$ [see, for example, Remark 2.4.2 in (2008)].

Once again, we may assume, without loss of generality, that the original subfield $\{X_u\}_{u\in H}$ is given in the form (4.10). Let

$$N' = \sum_i \delta_{(j_i, v_i, u_i)} \sim \mathrm{PRM}(\nu_\alpha \otimes \nu \otimes \tau) \tag{4.11}$$

be a Poisson random measure on $([-\infty, \infty] \setminus \{0\}) \times W \times H$ with mean measure $\nu_\alpha \otimes \nu \otimes \tau$. The following series representation holds in parallel to (3.3):

$$X_u = C_\alpha^{1/\alpha} \sum_{i=1}^{\infty} j_i h(v_i, u_i \oplus u), \qquad u \in H, \tag{4.12}$$

where C_α is the stable tail constant (3.4).

Let $\operatorname{rank}(K) = q \geq 1$ (we can also allow $q = 0$ provided we follow the convention mentioned in Remark 4.2). Note that from (4.2) it follows that $q = d - p$. Choose a basis $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_p\}$ of F and a basis $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_q\}$ of K. Let U be the $d \times p$ matrix with $\bar{u}_i$ as its ith column and V be the $d \times q$ matrix with $\bar{v}_j$ as its jth column. Define

$$C = \{y \in \mathbb{R}^p\text{: there exists } \lambda \in \mathbb{R}^q \text{ such that } \|Uy + V\lambda\|_\infty \leq 1\}.$$

Let $|C|$ denote the p-dimensional volume of C, and for $y \in C$ denote by $\mathcal{V}(y)$ the q-dimensional volume of the polytope

$$P_y := \{\lambda \in \mathbb{R}^q : \|Uy + V\lambda\|_\infty \leq 1\}.$$

Define, for $t \in H$,

$$m(t, n) := |[-n\mathbf{1}, n\mathbf{1}] \cap (t + K)|. \tag{4.13}$$

Here, $|B|$ denotes the cardinality of the finite set B, $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{Z}^d$, and for $u = (u^{(1)}, u^{(2)}, \dots, u^{(d)})$ and $v = (v^{(1)}, v^{(2)}, \dots, v^{(d)})$,

$$[u, v] := \{(t^{(1)}, t^{(2)}, \dots, t^{(d)}) \in \mathbb{Z}^d : u^{(i)} \le t^{(i)} \le v^{(i)} \text{ for all } 1 \le i \le d\}.$$

The following result, which is an extension of Theorem 3.1 (see Remark 4.2 below), states that the weak limit of properly scaled $\{N_n\}$ is a random measure which is not a point process.

THEOREM 4.1. *Suppose $\{\phi_t\}_{t\in F}$ is a dissipative group action and (4.9) holds. Let $\tilde{N}_n = n^{-q} \sum_{\|t\|_\infty \le n} \delta_{(cn)^{-p/\alpha} X_t}$, $n = 1, 2, \dots$, where $c = (l|C|)^{1/p}$. Then $\tilde{N}_n \Rightarrow \tilde{N}_*$ weakly in $\mathcal{M}$, where $\tilde{N}_*$ is a random measure with the following representation*:

$$\tilde{N}_* = \sum_{i=1}^{\infty} \sum_{u \in H} \mathcal{V}(\xi_i) \delta_{j_i h(v_i, u)}, \tag{4.14}$$

where $\{j_i\}$ and $\{v_i\}$ are as in (4.11), $\{\xi_i\}$ is a sequence of i.i.d. p-dimensional random vectors uniformly distributed in C independent of $\{j_i\}$ and $\{v_i\}$, and $\mathcal{V}$ is the continuous function defined on C as above. Furthermore, $\tilde{N}_*$ is Radon on $[-\infty, \infty] \setminus \{0\}$ with Laplace functional ($g \ge 0$ *continuous with compact support*)

$$\begin{aligned}\psi_{\tilde{N}_*}(g) &= E(e^{-\tilde{N}_*(g)}) \\ &= \exp\biggl\{-\frac{1}{|C|} \int_C \int_{|x|>0} \int_W \bigl(1 - e^{-\mathcal{V}(y) \sum_{w \in H} g(x h(v, w))}\bigr) \\ &\qquad\qquad \nu(dv) \nu_\alpha(dx)\, dy\biggr\}. \end{aligned} \tag{4.15}$$

REMARK 4.2. In the above theorem, we can also allow q to be equal to 0 provided we follow the convention $\mathbb{R}^0 = \{0\}$, which is assumed to have 0-dimensional volume equal to 1. With these conventions, Theorem 4.1 reduces to Theorem 3.1 when $q = 0$. Also, by a reasoning similar to Remark 3.2 and using Theorem 5.4 in Roy and Samorodnitsky (2008), one can extend this result to the case when $\{\phi_t\}_{t\in F}$ is not conservative.

REMARK 4.3 (Due to Jan Rosiński). Suppose that $\{\phi_t\}_{t\in\mathbb{Z}^d}$ in (2.3) is measure-preserving. Define $\tilde{S} := \{-1, 1\} \times S$ and $\tilde{\mu} := \frac{\delta_{-1}+\delta_1}{2} \otimes \mu$. Then $\psi_t(\varepsilon, s) := (\varepsilon c_t(s), \phi_t(s))$, $t \in \mathbb{Z}^d$, is a measure-preserving action on $\tilde{S}$ and

$$X_t \stackrel{d}{=} \int_{\tilde{S}} \tilde{f}(\psi_t(\varepsilon, s)) \tilde{M}(d\varepsilon, ds), \qquad t \in \mathbb{Z}^d,$$

where $\tilde{f}(\varepsilon, s) := \varepsilon f(s) \in L^\alpha(\tilde{S}, \tilde{\mu})$ and $\tilde{M}$ is an $S\alpha S$ random measure on $\tilde{S}$ with control measure $\tilde{\mu}$. This means, in particular, that (4.9) holds. Since all the known stationary $S\alpha S$ random fields are generated by actions that preserve the underlying measure (or an equivalent measure), it follows that (4.9) is not at all a big restriction.

REMARK 4.4. Note that the above theorem together with Lemma 3.20 in Resnick (1987) implies that the sequence of point process (1.1) with the choice $b_n \sim n^{p/\alpha}$ is not tight and hence does not converge weakly in $\mathcal{M}$. Furthermore, $\{N_n\}$ will not converge weakly to a nontrivial limit for any other choice of normalizing sequence $\{b_n\}$. All the points of $\{N_n\}$ will be driven to zero if b_n grows faster than $n^{p/\alpha}$. This follows from (4.1), which also implies that if we select b_n to grow slower than $n^{p/\alpha}$ then we will see an accumulation of mass at infinity. Only $b_n \sim n^{p/\alpha}$ places the points at the right scale, but they repeat so much due to long memory, that the point process itself has to be normalized by n^q (the order of the cluster sizes) to ensure weak convergence.

5. Proof of Theorem 4.1. The major steps of the proof of Theorem 4.1 are similar to those of the proof of Theorem 3.1 in Resnick and Samorodnitsky (2004). However, Theorem 4.1 needs some counting which is taken care of mostly by the following lemma about C, $\mathcal{V}(y)$ and $m(t,n)$ defined in Section 4.

LEMMA 5.1. *With the notation introduced above, we have*:

(i) *C is compact and convex.*
(ii) *$\mathcal{V}(y)$ is a continuous function of y.*
(iii) *For all $1 \le k \le l$, the functions $m_{k,n} : C \to \mathbb{R}$ defined by*

$$m_{k,n}(y) := \frac{m(x_k + \sum_{i=1}^{p}[ny_i]u_i, n)}{n^q}, \qquad n = 1, 2, \ldots$$

$[y = (y_1, \ldots, y_p)]$ *are uniformly bounded on C and converge* (*as $n \to \infty$*) *to $\mathcal{V}(y)$ for all $y \in C$.*

(iv) *There is a constant $\kappa_0 > 0$ such that $m(t,n)/n^q \le \kappa_0$ for all $t \in H$ and for all $n \ge 1$. Also,*

$$\frac{1}{n^p} \sum_{u \in H_n} \frac{m(u,n)}{n^q} \to l \int_C \mathcal{V}(y)\, dy < \infty$$

as $n \to \infty$. Here, H_n is as in (4.6).

PROOF. (i) Let $W = [U : V]$ and $z = \begin{bmatrix} y \\ \lambda \end{bmatrix}$. Then C is a projection of the closed and convex set

$$P := \{z \in \mathbb{R}^{p+q} : \|Wz\|_\infty \le 1\}.$$

To complete the proof of part (i) it is enough to establish that P is bounded. To this end note that the columns of W are independent over $\mathbb{Z}$ and hence over $\mathbb{Q}$ which means that there is a $(p+q) \times d$ matrix Z over $\mathbb{Q}$ such that $ZW = I_{p+q}$, the identity matrix of order $p+q$. From the string of inequalities (for $z \in P$),

$$\|z\|_\infty = \|ZWz\|_\infty \le \|Z\|_\infty \|Wz\|_\infty \le \|Z\|_\infty$$

the boundedness of P follows.

(ii) Take $\{y^{(n)}\} \subseteq C$ such that $y^{(n)} \to y$. Fixing an integer $m \geq 1$ we get that for large enough n, $\|y^{(n)} - y\| \leq \frac{1}{m}$, and hence

$$\left\{\lambda \in \mathbb{R}^q : \|Uy + V\lambda\|_\infty \leq 1 - \frac{\|U\|_\infty}{m}\right\} \subseteq P_{y^{(n)}} \subseteq \left\{\lambda \in \mathbb{R}^q : \|Uy + V\lambda\|_\infty \leq 1 + \frac{\|U\|_\infty}{m}\right\}.$$

First, taking the lim sup (and lim inf) as $n \to \infty$ and then taking the limit as $m \to \infty$ we get that

$$\mathcal{V}(y) \leq \liminf_{n\to\infty} \mathcal{V}(y^{(n)}) \leq \limsup_{n\to\infty} \mathcal{V}(y^{(n)}) \leq \mathcal{V}(y),$$

which proves part (ii).

(iii) Fix $1 \leq k \leq l$. Let $L = \max_{1\leq k\leq l} \|x_k\|_\infty$. We start by showing that for all $y \in C$

$$m_{k,n}(y) \to \mathcal{V}(y) \tag{5.1}$$

as $n \to \infty$. Let

$$B_n := \left\{\nu \in \mathbb{Z}^q : \left\|x_k + \sum_{i=1}^{p} [ny_i]\bar{u}_i + V\nu\right\|_\infty \leq n\right\}, \qquad n \geq 1.$$

Since the columns of V are linearly independent over $\mathbb{Z}$, we have

$$|B_n| = \left|[-n\mathbf{1}, n\mathbf{1}] \cap \left(x_k + \sum_{i=1}^{p} [ny_i]\bar{u}_i + K\right)\right| = n^q m_{k,n}(y). \tag{5.2}$$

Define

$$C_m := \left\{\lambda \in \mathbb{R}^q : \|Uy + V\lambda\|_\infty \leq 1 - \frac{1}{m}\left(\sum_{i=1}^{p} \|\bar{u}_i\|_\infty + L\right)\right\}, \qquad m \geq 1.$$

We first fix $m \geq 1$ and claim that for all $n \geq m$

$$\mathbb{Z}^q \cap nC_m \subseteq \mathbb{Z}^q \cap nC_n \subseteq B_n. \tag{5.3}$$

The first inclusion is obvious. To prove the second one, take

$$\tilde{\nu} \in \mathbb{Z}^q \cap nC_n = \left\{\nu \in \mathbb{Z}^q : \left\|\sum_{i=1}^{p} ny_i\bar{u}_i + V\nu\right\|_\infty \leq n - \sum_{i=1}^{p} \|\bar{u}_i\|_\infty - L\right\}$$

and observe that

$$\left\|x_k + \sum_{i=1}^{p} [ny_i]\bar{u}_i + V\tilde{\nu}\right\|_\infty \leq \|x_k\|_\infty + \left\|\sum_{i=1}^{p} ny_i\bar{u}_i + V\tilde{\nu}\right\|_\infty + \sum_{i=1}^{p} \|\bar{u}_i\|_\infty \leq n.$$

It follows from (5.3) and (5.2) that

$$\frac{|\mathbb{Z}^q \cap nC_m|}{n^q} \le \frac{|B_n|}{n^q} = m_{k,n}(y) \tag{5.4}$$

for all $n \ge m$. Since C_m is a rational polytope (i.e., a polytope whose vertices have rational coordinates) the left-hand side of (5.4) converges to Volume(C_m), the q-dimensional volume of C_m by Theorem 1 of De Loera (2005). Hence, (5.4) yields

$$\text{Volume}(C_m) \le \liminf_{n\to\infty} m_{k,n}(y).$$

Now taking another limit as $m \to \infty$, we get

$$\mathcal{V}(y) \le \liminf_{n\to\infty} m_{k,n}(y) \tag{5.5}$$

since $C_m \uparrow P_y$. Defining another sequence of rational polytopes

$$C'_m := \left\{\lambda \in \mathbb{R}^q : \|Uy + V\lambda\|_\infty \le 1 + \frac{1}{m}\left(\sum_{i=1}^{p} \|\bar{u}_i\|_\infty + L\right)\right\}, \qquad m \ge 1,$$

and observing that $C'_m \downarrow P_y$ as $m \to \infty$ we can conclude using a similar argument that

$$\limsup_{n\to\infty} m_{k,n}(y) \le \mathcal{V}(y). \tag{5.6}$$

(5.1) follows from (5.5) and (5.6).

To establish the uniform boundedness let $R := \sup_{y\in C} \|y\|_\infty < \infty$ by part (i). Once again fixing $y \in C$ observe that for C'_1 defined above, we have

$$C'_1 \subseteq \left\{\lambda \in \mathbb{R}^q : \|V\lambda\|_\infty \le 1 + \sum_{i=1}^{p} \|\bar{u}_i\|_\infty + L + R\|U\|_\infty\right\} =: C',$$

which is another rational polytope. Hence,

$$m_{k,n}(y) \le \frac{|\mathbb{Z}^q \cap nC'_1|}{n^q} \le \frac{|\mathbb{Z}^q \cap nC'|}{n^q}$$

from which the uniform boundedness follows by another application of Theorem 1 of De Loera (2005).

(iv) To establish this part, we start by proving two set inclusions which will be useful once more later in this section. For $1 \le k \le l$ and $n \ge 1$, define

$$F_{k,n} = \{u \in x_k + F\text{: there exists } v \in K \text{ such that } u + v \in [-n\mathbf{1}, n\mathbf{1}]\}$$

and

$$Q_n^{(k)} = \{\alpha \in \mathbb{Z}^p : x_k + U\alpha \in F_{k,n}\}.$$

Clearly,

$$H_n = \bigcup_{k=1}^{l} F_{k,n}. \tag{5.7}$$

Let $L = \max_{1\le k\le l} \|x_k\|_\infty$ as before and $L' = L + \sum_{i=1}^{p} \|u_i\|_\infty + \sum_{j=1}^{q} \|v_j\|_\infty$. We claim that for all $n > L'$,

$$\begin{aligned} &\{([(n-L')y_1], \dots, [(n-L')y_p]) : y \in C\} \\ &\quad \subseteq Q_n^{(k)} \subseteq \{([(n+L)y_1], \dots, [(n+L)y_p]) : y \in C\}. \end{aligned} \tag{5.8}$$

To prove the first inclusion, let $y \in C$. Find $\lambda \in \mathbb{R}^q$ be such that

$$\|Uy + V\lambda\|_\infty \le 1.$$

Then we have

$$\begin{aligned} &\left\| x_k + \sum_{i=1}^{p} [(n-L')y_i]\bar{u}_i + \sum_{j=1}^{q} [(n-L')\lambda_j]\bar{v}_j \right\|_\infty \\ &\qquad \le L + (n-L') \left\| \sum_{i=1}^{p} y_i \bar{u}_i + \sum_{j=1}^{q} \lambda_j \bar{v}_j \right\|_\infty + \sum_{i=1}^{p} \|\bar{u}_i\|_\infty + \sum_{j=1}^{q} \|\bar{v}_j\|_\infty \le n \end{aligned}$$

proving $x_k + \sum_{i=1}^{p} [(n-L')y_i]\bar{u}_i \in F_{k,n}$ and hence the first inclusion in (5.8). The second one is easy. If $\alpha \in Q_n^{(k)}$, then for some $\beta \in \mathbb{Z}^q$

$$\|x_k + U\alpha + V\beta\|_\infty \le n,$$

and hence

$$\|U\alpha + V\beta\|_\infty \le n + L,$$

which yields $y = (1/(n+L))\alpha \in C$ and establishes the second set inclusion in (5.8).

To prove the uniform boundedness in part (iv), we use (5.8) as follows:

$$\begin{aligned} &\sup_{n\ge 1} \sup_{t\in H} \frac{m(t,n)}{n^q} \\ &\qquad = \sup_{n\ge 1} \max_{t\in H_n} \frac{m(t,n)}{n^q} \\ &\qquad \le \max_{1\le k\le l} \sup_{n\ge 1} \max_{\alpha\in Q_n^{(k)}} \frac{m(x_k + U\alpha, n+L)}{n^q} \\ &\qquad \le \max_{1\le k\le l} \sup_{n\ge 1} \sup_{y\in C} \left(1 + \frac{L}{n}\right)^q \frac{m(x_k + \sum_{i=1}^{p} [(n+L)y_i]\bar{u}_i, n+L)}{(n+L)^q}, \end{aligned}$$

and this is bounded above by

$$\kappa_0 = (1+L)^q \max_{1\le k\le l} \sup_{n\ge 1} \sup_{y\in C} m_{k,n}(y), \tag{5.9}$$

which is finite by part (iii).

Now we prove the convergence in part (iv). Because of (5.7), it is enough to show that for all $1 \le k \le l$

$$\frac{1}{n^p} \sum_{u\in F_{k,n}} \frac{m(u,n)}{n^q} \to \int_C \mathcal{V}(y)\,dy \qquad (n\to\infty). \tag{5.10}$$

To prove (5.10), we use (5.8) once again to get the following bound:

$$\begin{aligned}
&\frac{1}{n^p} \sum_{u\in F_{k,n}} \frac{m(u,n)}{n^q} \\
&\quad\le \left(\frac{n+L}{n}\right)^p \frac{1}{(n+L)^p} \sum_{\alpha\in Q_n^{(k)}} \frac{m(x_k+U\alpha, n+L)}{n^q} \\
&\quad\le \left(\frac{n+L}{n}\right)^{p+q} \int_C \frac{m(x_k+\sum_{i=1}^p [(n+L)y_i]\bar{u}_i, n+L)}{(n+L)^q}\,dy + o(1)
\end{aligned}$$

from which using part (iii) and the dominated convergence theorem we get

$$\limsup_{n\to\infty} \frac{1}{n^p} \sum_{u\in F_{k,n}} \frac{m(u,n)}{n^q} \le \int_C \mathcal{V}(y)\,dy.$$

Similarly, we can also prove

$$\liminf_{n\to\infty} \frac{1}{n^p} \sum_{u\in F_{k,n}} \frac{m(u,n)}{n^q} \ge \int_C \mathcal{V}(y)\,dy,$$

(5.10) follows from the above two inequalities. This completes the proof of Lemma 5.1. □

With the above lemma, we are now well prepared to prove Theorem 4.1. Following Resnick and Samorodnitsky (2004), we start with the Laplace functional of $\tilde{N}_*$,

$$\begin{aligned}
\psi_{\tilde{N}_*}(g) &= E\big(e^{-\tilde{N}_*(g)}\big) \\
&= E\exp\left\{-\sum_{i=1}^{\infty}\sum_{u\in H} \mathcal{V}(\xi_i) g(j_i h(v_i,u))\right\},
\end{aligned}$$

which can be shown to be equal to (4.15) using

$$\sum_i \delta_{(j_i,v_i,\xi_i)} \sim \mathrm{PRM}\left(\nu_\alpha \otimes \nu \otimes \frac{1}{|C|}\mathrm{Leb}|_C\right)$$

and by the argument used in the computation of the Laplace functional of the limiting point process in Theorem 3.1 of Resnick and Samorodnitsky (2004).

To prove that $\tilde{N}_*$ is Radon, we take $\eta(x) = I_{[-\infty,-\delta]\cup[\delta,\infty]}$, $\delta > 0$ and look at

$$E(\tilde{N}_*(\eta)) = E\sum_{i=1}^{\infty}\sum_{u\in H}\mathcal{V}(\xi_i)\eta(j_i h(v_i,u)) \\ \le \|\mathcal{V}\|_\infty E\sum_{i=1}^{\infty}\sum_{u\in H}\eta(j_i h(v_i,u)), \tag{5.11}$$

where $\|\mathcal{V}\|_\infty := \sup_{y\in C}\mathcal{V}(y) < \infty$ by Lemma 5.1. It is enough to show that $E(\tilde{N}_*(\eta)) < \infty$ which follows from (5.11) by the exact same argument used to establish that the limiting point process in Theorem 3.1 of Resnick and Samorodnitsky (2004) is Radon.

Observe that because of (4.9) and the assumption that the original stable random field is of the form given in (2.1) and (2.3) it follows that for all $u \in H$ and for all $v \in K$

$$X_{u+v} \overset{\text{a.s.}}{=} X_u.$$

As a consequence, $\tilde{N}_n$ can also be written as

$$\tilde{N}_n = \sum_{t\in H_n}\frac{m(t,n)}{n^q}\delta_{(cn)^{-p/\alpha}X_t},$$

where $m(t,n)$ is as in (4.13) and H_n is as in (4.6). The weak convergence of $\tilde{N}_n$ is established in two steps in parallel to the proof of Theorem 3.1 in Resnick and Samorodnitsky (2004) as follows: we first show that

$$\tilde{N}_n^{(2)} := \sum_{i=1}^{\infty}\sum_{t\in H_n}\frac{m(t,n)}{n^q}\delta_{(cn)^{-p/\alpha}j_i h(v_i,u_i\oplus t)}$$

converges to $\tilde{N}_*$ weakly in $\mathcal{M}$ and then show that $\tilde{N}_n$ must have the same weak limit as $\tilde{N}_n^{(2)}$.

We start by proving the weak convergence of $\tilde{N}_n^{(2)}$. The scaling property of ν_α yields the Laplace functional of $\tilde{N}_n^{(2)}$ ($g \ge 0$ continuous with compact support) as

$$E\big(e^{-\tilde{N}_n^{(2)}(g)}\big) \\ = \exp\bigg\{-\frac{1}{(cn)^p}\int_{|x|>0}\int_W\sum_{u\in H}\big(1-e^{-1/n^q\sum_{t\in H_n}m(t,n)g(xh(v,u\oplus t))}\big) \\ \nu(dv)\nu_\alpha(dx)\bigg\}, \tag{5.12}$$

which needs to be shown to converge to (4.15). As in Resnick and Samorodnitsky (2004), we first assume that h is compactly supported, that is, for some positive integer M

$$h(v,u)I_{W\times H_M^c}(v,u)\equiv 0. \tag{5.13}$$

Recall that each H_M is finite and $H_M \uparrow H$ as $M\to\infty$. Using properties (4.4), (4.5) and the compact support assumption (5.13), the integral in (5.12) becomes

$$\frac{1}{(cn)^p}\iint \sum_{u\in H_{n+M}}\biggl(1-\exp\biggl(-\sum_{t\in H_n}\frac{m(t,n)}{n^q}g\bigl(xh(v,u\oplus t)\bigr)\biggr)\biggr)\nu(dv)\nu_\alpha(dx),$$

which, by a change of variable, equals

$$\frac{1}{(cn)^p}\iint \sum_{u\in H_{n+M}}\biggl(1-\exp\biggl(-\sum_{w\in A_n'}\frac{m(w\ominus u,n)}{n^q}g(xh(v,w))\biggr)\biggr)$$
$$\nu(dv)\nu_\alpha(dx)=:\mathcal{I}_n.$$

Here, $w\ominus u := w\oplus u^{-1}$, u^{-1} is the inverse of u in $(H,\oplus)$, and $A_n' = H_M\cap \{w' : w'\ominus u\in H_n\}$.

We claim that for all $n>M$

$$m(u^{-1},n-M)\le m(w\ominus u,n)\le m(u^{-1},n+M). \tag{5.14}$$

The first inequality follows, for example, because

$$\tau\in[-(n-M)\mathbf{1},(n-M)\mathbf{1}]\cap(u^{-1}+K)$$

if and only if

$$\tau+w\in[-n\mathbf{1},n\mathbf{1}]\cap\bigl((w\ominus u)+K\bigr).$$

Similarly, we can prove the second inequality in (5.14).

We bound $\mathcal{I}_n$ using (5.14) by

$$\frac{1}{(cn)^p}\iint \sum_{u\in H_{n+M}}\biggl(1-\exp\biggl(-\sum_{w\in A_n'}\frac{m(u^{-1},n+M)}{n^q}g(xh(v,w))\biggr)\biggr)\nu(dv)\nu_\alpha(dx),$$

which we claim to be equal to

$$\begin{aligned}&=\frac{1}{(cn)^p}\iint \sum_{u\in H_{n+M}}\biggl(1-\exp\biggl(-\sum_{w\in A_n'}\frac{m(u^{-1},n)}{n^q}g(xh(v,w))\biggr)\biggr)\\ &\qquad \nu(dv)\nu_\alpha(dx)+o(1)\\ &=:\mathcal{I}_n'+o(1).\end{aligned} \tag{5.15}$$

To prove this claim, observe that using the inequality $|e^{-a}-e^{-b}|\le|a-b|$, $(a,b>0)$ the difference of the two integrals above can be bounded by

$$\frac{1}{(cn)^p}\sum_{u\in H_{n+M}}\left(\frac{m(u^{-1},n+M)-m(u^{-1},n)}{n^q}\right)\times\iint\sum_{w\in H_M}g(xh(v,w))\nu(dv)\nu_\alpha(dx),$$

which needs to be shown to converge to 0 as $n\to\infty$. This is easy because $g\le CI_{[-\infty,-\delta]\cup[\delta,\infty]}$ for some $C,\delta>0$ (since $g\ge 0$ has compact support on $[-\infty,\infty]\setminus\{0\}$) which implies

$$\begin{aligned}&\iint\sum_{w\in H_M}g(xh(v,w))\nu(dv)\nu_\alpha(dx)\\ &\quad\le C\iint\sum_{w\in H_M}I_{(|x|\ge\delta/|h(v,w)|)}\nu_\alpha(dx)\nu(dv)\\ &\quad=C\delta^{-\alpha}\int_W\sum_{w\in H_M}|h(v,w)|^\alpha\nu(dv)<\infty\end{aligned}\tag{5.16}$$

and Lemma 5.1 together with (4.7) implies

$$\begin{aligned}&\left|\frac{1}{(cn)^p}\sum_{u\in H_{n+M}}\left(\frac{m(u^{-1},n+M)-m(u^{-1},n)}{n^q}\right)\right|\\ &\quad=\frac{1}{(cn)^p}\sum_{u\in H_{n+M}}\left(\left(\frac{n+M}{n}\right)^q\frac{m(u,n+M)}{(n+M)^q}-\frac{m(u,n)}{n^q}\right)\\ &\quad=o(1)+\frac{1}{c^p}\left[\left(\frac{n+M}{n}\right)^{p+q}\frac{1}{(n+M)^p}\sum_{u\in H_{n+M}}\frac{m(u,n+M)}{(n+M)^q}\right.\\ &\qquad\left.-\frac{1}{n^p}\sum_{u\in H_n}\frac{m(u,n)}{n^q}\right]\to 0.\end{aligned}$$

This proves claim (5.15) which yields $\mathcal{I}_n\le\mathcal{I}_n'+o(1)$. Similarly, we can also get a lower bound of $\mathcal{I}_n$ and establish that $\mathcal{I}_n\ge\mathcal{I}_n'+o(1)$. Hence, in order to complete the proof of weak convergence of $\tilde{N}_n^{(2)}$ to $\tilde{N}_*$ under the compact support assumption (5.13), it is enough to show that

$$\mathcal{I}_n'=\frac{1}{(cn)^p}\iint\sum_{u\in H_{n+M}}\left(1-\exp\left(-\frac{m(u,n)}{n^q}\sum_{w\in A_n'}g(xh(v,w))\right)\right)\nu(dv)\nu_\alpha(dx)$$

converges to

$$l\frac{1}{c^p}\int_C\int_{|x|>0}\int_W\left(1-\exp\left(-\mathcal{V}(y)\sum_{w\in H_M}g(xh(v,w))\right)\right)\nu(dv)\nu_\alpha(dx)\,dy.\tag{5.17}$$

To this end, we decompose the integral $\mathcal{I}_n'$ into two parts as follows:

$$\begin{aligned}\mathcal{I}_n' &= \frac{1}{(cn)^p}\iint \sum_{u\in H_{n-M}}\biggl(1-\exp\biggl(-\frac{m(u,n)}{n^q}\sum_{w\in H_M} g(xh(v,w))\biggr)\biggr)\nu(dv)\nu_\alpha(dx)\\ &\quad+\frac{1}{(cn)^p}\iint \sum_{u\in B_n'}\biggl(1-\exp\biggl(-\frac{m(u,n)}{n^q}\sum_{w\in A_n'} g(xh(v,w))\biggr)\biggr)\nu(dv)\nu_\alpha(dx)\\ &=: J_n' + L_n'\end{aligned}$$

for all $n > M$. Here, $B_n' = H_{n+M}\cap H_{n-M}^c$. For $1\le k\le l$ let

$$J_{k,n}' = \frac{1}{(cn)^p}\iint \sum_{u\in F_{k,n-M}}\biggl(1-\exp\biggl(-\frac{m(u,n)}{n^q}\sum_{w\in H_M} g(xh(v,w))\biggr)\biggr)\\ \nu(dv)\nu_\alpha(dx).$$

Clearly, by (5.7), $J_n' = \sum_{k=1}^l J_{k,n}'$. We will show that each $J_{k,n}'$, $1\le k\le l$, converges to (5.17) except for the factor l.

Fix $k\in\{1,2,\dots,l\}$. Repeating the argument in the proof of (5.15), we obtain for all $n > M$,

$$\begin{aligned}J_{k,n}' &= o(1) + \iint \frac{1}{(cn)^p}\\ &\qquad\times\sum_{u\in F_{k,n-M}}\bigl(1-e^{-m(u,n-M+L)/n^q\sum_{w\in H_M} g(xh(v,w))}\bigr)\\ &\qquad\qquad \nu(dv)\nu_\alpha(dx)\\ &= o(1) + \biggl(\frac{n-M+L}{cn}\biggr)^p\\ &\qquad\times\iint\frac{1}{(n-M+L)^p}\\ &\qquad\qquad\times\sum_{\alpha\in Q_{n-M}^{(k)}}\bigl(1-e^{-m(x_k+U\alpha,n-M+L)/n^q\sum_{w\in H_M} g(xh(v,w))}\bigr)\\ &\qquad\qquad\qquad \nu(dv)\nu_\alpha(dx),\end{aligned}$$

which can be estimated using (5.8) as follows:

$$\begin{aligned}&\le o(1) + \biggl(\frac{n-M+L}{cn}\biggr)^p\\ &\quad\times\int_{|x|>0}\int_W\int_C\bigl(1-e^{-m(x_k+\sum_{i=1}^p[(n-M+L)y_i]\bar{u}_i,n-M+L)/n^q\sum_{w\in H_M} g(xh(v,w))}\bigr)\\ &\qquad\qquad \nu(dv)\nu_\alpha(dx)\,dy.\end{aligned}$$

By Lemma 5.1, there is a constant $\kappa > 0$ such that the above integrand sequence is dominated by

$$1 - \exp\left\{-\kappa \sum_{w \in H_M} g(xh(v, w))\right\},$$

which can be shown to be integrable using the inequality $1 - e^{-x} \le x$, $x > 0$, and the arguments given in (5.16). Hence, Lemma 5.1 together with the dominated convergence theorem yields

$$\int_{|x|>0} \int_W \int_C \left(1 - e^{-m(x_k + \sum_{i=1}^{p} [(n-M+L)y_i]\bar{u}_i, n-M+L)/n^q \sum_{w \in H_M} g(xh(v,w))}\right) \nu(dv)\nu_\alpha(dx)\, dy$$
$$\to \int_C \int_{|x|>0} \int_W \left(1 - \exp\left(-\mathcal{V}(y) \sum_{w \in H_M} g(xh(v, w))\right)\right) \nu(dv)\nu_\alpha(dx)\, dy.$$

This shows

$$\limsup_{n\to\infty} J'_{k,n} \le \frac{1}{c^p} \int_C \int_{|x|>0} \int_W \left(1 - \exp\left(-\mathcal{V}(y) \sum_{w \in H_M} g(xh(v, w))\right)\right) \nu(dv)\nu_\alpha(dx)\, dy.$$

Similarly, we can also prove that

$$\liminf_{n\to\infty} J'_{k,n} \ge \frac{1}{c^p} \int_C \int_{|x|>0} \int_W \left(1 - \exp\left(-\mathcal{V}(y) \sum_{w \in H_M} g(xh(v, w))\right)\right) \nu(dv)\nu_\alpha(dx)\, dy.$$

Hence, J'_n converges to (5.17) as $n \to \infty$. To establish the weak convergence of $\tilde{N}_n^{(2)}$ when h is compactly supported it remains to prove that $L'_n \to 0$ as $n \to \infty$. This is easy because

$$\begin{aligned} L'_n &\le \frac{1}{(cn)^p} \iint \sum_{u \in B'_n} \left(1 - \exp\left(-\frac{m(u, n)}{n^q} \sum_{w \in H_M} g(xh(v, w))\right)\right) \nu(dv)\nu_\alpha(dx) \\ &= \frac{1}{(cn)^p} \iint \sum_{u \in H_{n+M}} \left(1 - \exp\left(-\frac{m(u, n)}{n^q} \sum_{w \in H_M} g(xh(v, w))\right)\right) \nu(dv)\nu_\alpha(dx) \\ &\quad - \frac{1}{(cn)^p} \iint \sum_{u \in H_{n-M}} \left(1 - \exp\left(-\frac{m(u, n)}{n^q} \sum_{w \in H_M} g(xh(v, w))\right)\right) \nu(dv)\nu_\alpha(dx) \to 0 \end{aligned}$$

since the first term can also be shown to converge to the same limit as the second term by the exact same argument as above.

To remove the assumption of compact support on the function h, for a general $h \in L^\alpha(\nu \otimes \tau)$ define

$$h_M(v,u) = h(v,u) I_{H_M}(u), \qquad M \geq 1. \tag{5.18}$$

Notice that each h_M satisfies (5.13) and that $h_M \to h$ almost surely as well as in $L^\alpha(\nu \otimes \tau)$ as $M \to \infty$. Denote

$$\tilde{N}_n^{(2,M)} = \sum_{i=1}^{\infty} \sum_{t \in H_n} \frac{m(t,n)}{n^q} \delta_{(cn)^{-p/\alpha} j_i h_M(v_i, u_i \oplus t)} \tag{5.19}$$

for $M, n \geq 1$, and

$$\tilde{N}_*^{(M)} = \sum_{i=1}^{\infty} \sum_{u \in H} \mathcal{V}(\xi_i) \delta_{j_i h_M(v_i, u)}, \qquad M \geq 1, \tag{5.20}$$

with the notation as above. We already know that for every $M \geq 1$, $\tilde{N}_n^{(2,M)} \Rightarrow \tilde{N}_*^{(M)}$ weakly in the space $\mathcal{M}$ as $n \to \infty$. Therefore, to establish $\tilde{N}_n^{(2)} \Rightarrow \tilde{N}_*$, it is enough to show two things:

$$\tilde{N}_*^{(M)} \Rightarrow \tilde{N}_* \qquad \text{weakly as } M \to \infty, \tag{5.21}$$

and

$$\lim_{M \to \infty} \limsup_{n \to \infty} P\big(\big|\tilde{N}_n^{(2,M)}(g) - \tilde{N}_n^{(2)}(g)\big| > \epsilon\big) = 0 \tag{5.22}$$

for all $\epsilon > 0$ and for every nonnegative continuous function g with compact support on $[-\infty, \infty] \setminus \{0\}$.

Claim (5.21) is easy since the Laplace functional of $\tilde{N}_*^{(M)}$, which is obtained by replacing h in (4.15) by h_M, converges by the dominated convergence theorem to (4.15) for every nonnegative continuous function g with compact support on $[-\infty, \infty] \setminus \{0\}$. The proof of (5.22) is along the same lines as the proof of the corresponding limit [namely (3.13)] in Resnick and Samorodnitsky (2004). Using similar calculations, we have

$$\begin{aligned} E&\big|\tilde{N}_n^{(2,M)}(g) - \tilde{N}_n^{(2)}(g)\big| \\ &= \sum_{t \in H_n} \frac{m(t,n)}{n^q} E\left(\sum_{i=1}^{\infty} g\big((cn)^{-p/\alpha} j_i h(v_i, u_i \oplus t)\big) I\big(N(u_i \oplus t) > M\big) \right) \\ &= \left(\frac{1}{(cn)^p} \sum_{t \in H_n} \frac{m(t,n)}{n^q} \right) \int_W \int_{|x|>0} \sum_{u \in H_M^c} g(xh(v,u)) \nu_\alpha(dx) \nu(dv). \end{aligned}$$

Repeating the argument in (5.16), the integral

$$\int_W \int_{|x|>0} \sum_{u \in H_M^c} g(xh(v,u)) \nu_\alpha(dx) \nu(dv)$$

can be shown to be bounded by

$$C\delta^{-\alpha}\int_W \sum_{u\in H_M^c} |h(u,v)|^\alpha \nu(dv),$$

which converges to 0 as $M\to\infty$. Hence, by Lemma 5.1, (5.22) follows and so does $\tilde N_n^{(2)} \Rightarrow \tilde N_*$ without the assumption of compact support.

To complete the proof of the theorem, we need to prove (with ρ being the vague metric on $\mathcal{M}$) that for all $\epsilon > 0$

$$P\big[\rho(\tilde N_n, \tilde N_n^{(2)}) > \epsilon\big] \to 0 \qquad (n\to\infty)$$

and for this, it suffices to show that for every nonnegative continuous function g with compact support on $[-\infty,\infty]\setminus\{0\}$,

$$\begin{aligned} &P\big(\big|\tilde N_n(g) - \tilde N_n^{(2)}(g)\big| > \epsilon\big)\\ &\quad = P\left(\left|\sum_{t\in H_n}\frac{m(t,n)}{n^q}\left(g\left(\frac{X_t}{(cn)^{p/\alpha}}\right) - \sum_{i=1}^\infty g\left(\frac{j_i h(v_i, u_i\oplus t)}{(cn)^{p/\alpha}}\right)\right)\right| > \epsilon\right) \\ &\quad \to 0 \end{aligned} \tag{5.23}$$

as $n\to\infty$. By Lemma 5.1, (5.23) would follow from

$$P\left(\left|\sum_{t\in H_n}\left(g\left(\frac{X_t}{(cn)^{p/\alpha}}\right) - \sum_{i=1}^\infty g\left(\frac{j_i h(v_i, u_i\oplus t)}{(cn)^{p/\alpha}}\right)\right)\right| > \epsilon/\kappa_0\right) \to 0. \tag{5.24}$$

Here, κ_0 is as in (5.9). Once again, following verbatim the proof of (3.14) in Resnick and Samorodnitsky (2004), we can establish (5.24) and complete the proof of Theorem 4.1.

6. An example.

We end this paper by considering a simple example and computing the weak limit of the corresponding random measure (properly normalized $\{N_n\}$) using Theorem 4.1. This will help us understand the result as well as get used to the notation.

EXAMPLE 6.1. Suppose $d=2$, and define the $\mathbb{Z}^2$-action $\{\phi_{(t_1,t_2)}\}$ on $S=\mathbb{R}$ as

$$\phi_{(t_1,t_2)}(x) = x + t_1 - t_2.$$

Take any $f\in L^\alpha(S,\mu)$ where μ is the Lebesgue measure on $\mathbb{R}$ and define a stationary $S\alpha S$ random field $\{X_{(t_1,t_2)}\}$ as follows:

$$X_{(t_1,t_2)} = \int_{\mathbb{R}} f\big(\phi_{(t_1,t_2)}(x)\big)M(dx), \qquad t_1, t_2\in\mathbb{Z},$$

where M is an $S\alpha S$ random measure on $\mathbb{R}$ with control measure μ. Note that the above representation of $\{X_{(t_1,t_2)}\}$ is of the form (2.3) generated by a measure preserving conservative action with $c_{(t_1,t_2)} \equiv 1$.

In this case, using the notation as above, we have

$$K = \{(t_1, t_2) \in \mathbb{Z}^2 : t_1 = t_2\},$$

which implies $A \simeq \mathbb{Z}^2/K \simeq \mathbb{Z}$, and

$$F = \{(t_1, 0) : t_1 \in \mathbb{Z}\}.$$

In particular, we have $p = q = l = 1$, and

$$U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that

$$\begin{aligned} C &= \{y \in \mathbb{R}: \text{there exists } \lambda \in \mathbb{R} \text{ such that } \|Uy + V\lambda\|_\infty \le 1\} \\ &= \{y \in \mathbb{R} : |y + \lambda| \le 1 \text{ for some } \lambda \in [-1, 1]\} = [-2, 2]. \end{aligned}$$

For all $y \in C = [-2, 2]$, we have

$$P_y = \{\lambda \in [-1, 1] : |y + \lambda| \le 1\} = \begin{cases} [-(1 + y), 1], & y \in [-2, 0), \\ [-1, 1 - y], & y \in [0, 2], \end{cases}$$

which yields

$$\mathcal{V}(y) = 2 - |y|, \qquad y \in [-2, 2].$$

Clearly, $\{X_{(t_1,0)}\}_{t_1\in\mathbb{Z}}$ is a stationary $S\alpha S$ process generated by a dissipative $\mathbb{Z}$-action $\{\phi_{(t_1,0)}\}_{t_1\in\mathbb{Z}}$. Hence, by Theorem 4.4 in Rosiński (1995), there is a σ-finite standard measure space (W, ν) and a function $h \in L^\alpha(W \times \mathbb{Z}, \nu \otimes \zeta_{\mathbb{Z}})$ such that

$$X_{(t_1,0)} \stackrel{d}{=} \int_{W\times\mathbb{Z}} h(v, t_1 + s) M(dv, ds), \qquad t_1 \in \mathbb{Z}.$$

Here, $\zeta_{\mathbb{Z}}$ is the counting measure on $\mathbb{Z}$, and M is an $S\alpha S$ random measure on $W \times \mathbb{Z}$ with control measure $\nu \otimes \zeta_{\mathbb{Z}}$. Let

$$\sum_{i=1}^{\infty} \delta_{(j_i, v_i, \xi_i)} \sim \mathrm{PRM}\left(\nu_\alpha \otimes \nu \otimes \frac{1}{4} \mathrm{Leb}|_{[-2,2]}\right)$$

be a Poisson random measure on $([-\infty, \infty] \setminus \{0\}) \times W \times [-2, 2]$. In this example, $c = (l|C|)^{1/p} = 4$ and

$$\tilde{N}_n = n^{-1} \sum_{|t_1|,|t_2|\le n} \delta_{(4n)^{-1/\alpha} X_{(t_1,t_2)}}, \qquad n = 1, 2, \ldots.$$

Since $\{\phi_u\}_{u\in F}$ is a dissipative group action and (4.9) holds in this case, we can use Theorem 4.1 and conclude that

$$\tilde{N}_n \Rightarrow \sum_{i=1}^{\infty} \sum_{t_1\in\mathbb{Z}} (2 - |\xi_i|) \delta_{j_i h(v_i, t_1)}$$

weakly in the space $\mathcal{M}$.

REMARK 6.1. Note that $\tilde{N}_n$ can also be written as follows:

$$\tilde{N}_n = \sum_{k=-2n}^{2n} \left(2 - \frac{|k|}{n} + \frac{1}{n}\right) \delta_{(4n)^{-1/\alpha} Y_k},$$

where $Y_k = X_{(k,0)}$. Only a few (a Poisson number) of the Y_k's are not driven to zero by the normalization $b_n = (4n)^{-1/\alpha}$. By stationarity, each of these rare k's should be distributed uniformly in $\{-2n, -2n+1, \dots, 2n\}$ which along with Theorem 3.1 provides an intuitive justification of the above weak limit of $\tilde{N}_n$.

Acknowledgments. The author is immensely grateful to Gennady Samorodnitsky for the extremely useful discussions and valuable suggestions which contributed a lot to this work, to Paul Embrechts for the support and the suggestions during the author's stay at RiskLab, and to Jan Rosiński for Remark 4.3 which significantly improved this paper. He is also thankful to Johanna Neslehova, Augusto Teixeira and the anonymous referees for their comments.

REFERENCES

AARONSON, J. (1997). *An Introduction to Infinite Ergodic Theory*. *Mathematical Surveys and Monographs* **50**. Amer. Math. Soc., Providence, RI. MR1450400

BALKEMA, G. and EMBRECHTS, P. (2007). *High Risk Scenarios and Extremes*: *A Geometric Approach*. Eur. Math. Soc., Zürich. MR2372552

DAVIS, R. A. and HSING, T. (1995). Point process and partial sum convergence for weakly dependent random variables with infinite variance. *Ann. Probab.* **23** 879–917. MR1334176

DAVIS, R. and RESNICK, S. (1985). Limit theory for moving averages of random variables with regularly varying tail probabilities. *Ann. Probab.* **13** 179–195. MR770636

DE LOERA, J. A. (2005). The many aspects of counting lattice points in polytopes. *Math. Semesterber.* **52** 175–195. MR2159956

KALLENBERG, O. (1983). *Random Measures*, 3rd ed. Akademie-Verlag, Berlin. MR818219

KRENGEL, U. (1985). *Ergodic Theorems*. *De Gruyter Studies in Mathematics* **6**. De Gruyter, Berlin. MR797411

LANG, S. (2002). *Algebra*, 3rd ed. *Graduate Texts in Mathematics* **211**. Springer, New York. MR1878556

MORI, T. (1977). Limit distributions of two-dimensional point processes generated by strong-mixing sequences. *Yokohama Math. J.* **25** 155–168. MR0467887

NEVEU, J. (1977). Processus ponctuels. In *École D'Été de Probabilités de Saint-Flour, VI*—1976. *Lecture Notes in Math.* **598** 249–445. Springer, Berlin. MR0474493

RESNICK, S. I. (1987). *Extreme Values*, *Regular Variation*, *and Point Processes*. Springer, New York. MR900810

RESNICK, S. I. (2007). *Heavy-Tail Phenomena*: *Probabilistic and Statistical Modeling*. Springer, New York. MR2271424

RESNICK, S. and SAMORODNITSKY, G. (2004). Point processes associated with stationary stable processes. *Stochastic Process. Appl.* **114** 191–209. MR2101240

ROSIŃSKI, J. (1995). On the structure of stationary stable processes. *Ann. Probab.* **23** 1163–1187. MR1349166

ROSIŃSKI, J. (2000). Decomposition of stationary α-stable random fields. *Ann. Probab.* **28** 1797–1813. MR1813849

ROY, P. (2008). Stable random fields. Ph.D. dissertation, School of Operations Research and Industrial Engineering, Cornell Univ., Ithaca, NY.

ROY, P. and SAMORODNITSKY, G. (2008). Stationary symmetric α-stable discrete parameter random fields. *J. Theoret. Probab.* **21** 212–233. MR2384479

SAMORODNITSKY, G. (2004). Extreme value theory, ergodic theory and the boundary between short memory and long memory for stationary stable processes. *Ann. Probab.* **32** 1438–1468. MR2060304

SAMORODNITSKY, G. and TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models With Infinite Variance*. Chapman & Hall, New York. MR1280932

SURGAILIS, D., ROSIŃSKI, J., MANDREKAR, V. and CAMBANIS, S. (1993). Stable mixed moving averages. *Probab. Theory Related Fields* **97** 543–558. MR1246979

DEPARTMENT OF STATISTICS
AND PROBABILITY
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824-1027
USA
E-MAIL: roy@stt.msu.edu