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LIMIT THEOREMS IN FREE PROBABILITY THEORY. I[1]

BY G. P. CHISTYAKOV AND F. GÖTZE

Institute For Low Temperature Physics and Engineering and University of Bielefeld

Based on an analytical approach to the definition of additive free convolution on probability measures on the real line, we prove free analogues of limit theorems for sums for nonidentically distributed random variables in classical probability theory.

1. Introduction. In recent years a number of papers have investigated limit theorems for the free convolution of probability measures (p-measures) defined by D. Voiculescu.

The key concept of this definition is the notion of freeness, which can be interpreted as a kind of independence for noncommutative random variables. As in the classical probability where the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the p-measures on the real line, the free convolution. Many classical results in the theory of addition of independent random variables have their counterpart in this new theory, such as the law of large numbers, the central limit theorem, the Lévy–Khintchine formula and others. We refer to Voiculescu, Dykema and Nica [33] for an introduction to these topics. Bercovici and Pata [15] established the distributional behavior of sums of free identically distributed random variables and described explicitly the correspondence between limits laws for free and classical additive convolution. In this paper, using an analytical approach to the definition of the additive free convolution (see [20]), we generalize the results of Bercovici and Pata to the case of free nonidentically distributed random variables. We show that the parallelism between limits law for additive free and classical convolution found by Bercovici and Pata holds in the general case of free nonidentically distributed random variables. Our analytical approach to the definition of the additive free convolution allows us to obtain estimates of the rate of convergence of distribution functions of free sums. We prove the semicircle approximation theorem (an analogue of the Berry–Esseen inequality), the law of large numbers with estimates of the rate of convergence. We describe Lévy's class $\mathcal{L}_{\boxplus}$ of limiting distributions of normed sums of free random variables obeying infinitesimal conditions.

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As in the classical case we derive a canonical representation of the measures in the class $\mathcal{L}_{\boxplus}$. Furthermore, we shall give a characterization of the class $\mathcal{L}_{\boxplus}$ by means of the property of self-decomposability, extending results by Barndorff-Nielsen and Thorbjørsen [3].

The paper is organized as follows. In Section 2 we formulate and discuss the main results of the paper. In Section 3 we formulate auxiliary results. In Section 4 we prove the extended additive free central limit theorem for the general case of free *nonidentically* distributed random variables. This extends the Bercovici and Pata parallelism between additive free and classical additive infinite divisibility and limits laws for additive free and classical convolution to the general case. In Section 5, using results of Section 4, we describe an analogue of the Lévy class $\mathcal{L}_{\boxplus}$ for additive free convolution. We establish the Bercovici and Pata parallelism between the classical Lévy class $\mathcal{L}_*$ and the class $\mathcal{L}_{\boxplus}$. In Section 6, using our approach to the definition of the additive free convolution, we derive the semicircle approximation theorem (an analogue of the Berry–Esseen inequality) as well as a law of large numbers with estimates of convergence.

2. Results. Denote by $\mathcal{M}$ the family of all Borel p-measures defined on the real line $\mathbb{R}$. On $\mathcal{M}$ define the associative composition laws denoted $*$ and $\boxplus$ as follows. For $\mu_1, \mu_2 \in \mathcal{M}$ let the p-measure $\mu_1 * \mu_2$ denote the classical convolution of μ_1 and μ_2. In probabilistic terms, $\mu_1 * \mu_2$ is the probability distribution of $X + Y$, where X and Y are (commuting) independent random variables with probability distributions μ_1 and μ_2, respectively. The p-measure $\mu_1 \boxplus \mu_2$, on the other hand, denotes the free (additive) convolution of μ_1 and μ_2 introduced by Voiculescu [31] for compactly supported p-measures. Free convolution was extended by Maassen [26] to p-measures with finite variance and by Bercovici and Voiculescu [11] to the class $\mathcal{M}$. Thus, $\mu_1 \boxplus \mu_2$ is the distribution of $X + Y$, where X and Y are free random variables with the distributions μ_1 and μ_2, respectively. There are free analogues of multiplicative convolutions as well; these were first studied in Voiculescu [32].

Let $\mathbb{C}^+(\mathbb{C}^-)$ denote the open upper (lower) half of the complex plane. For $\mu \in \mathcal{M}$, define its Cauchy transform by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{\mu(dt)}{z - t}, \qquad z \in \mathbb{C}^+. \tag{2.1}$$

Following Maassen [26] and Bercovici and Voiculescu [11], we shall consider in the following the *reciprocal Cauchy transform*

$$F_\mu(z) = \frac{1}{G_\mu(z)}. \tag{2.2}$$

The corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$ will be denoted by $\mathcal{F}$. This class admits a simple description. Recall that the Nevanlinna class $\mathcal{N}$ is the class of analytic functions $F:\mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}$. The class $\mathcal{F}$ is the subclass of Nevanlinna functions F_μ for which $F_\mu(z)/z \to 1$ as $z \to \infty$ nontangentially to $\mathbb{R}$ (i.e., such that $\Re z/\Im z$ stays bounded), and this implies that F_μ has certain invertability properties. (See [1, 2, 17].) More precisely, for two numbers $\alpha > 0, \beta > 0$ define

$$\Gamma_\alpha = \{z = x + iy \in \mathbb{C}^+ : |x| < \alpha y\} \quad \text{and} \quad \Gamma_{\alpha,\beta} = \{z = x + iy \in \Gamma_\alpha : y > \beta\}.$$

Then for every $\alpha > 0$ there exists $\beta = \beta(\mu, \alpha)$ such that F_μ has a right inverse $F_\mu^{(-1)}$ defined on $\Gamma_{\alpha,\beta}$. The function $\phi_\mu(z) = F_\mu^{(-1)}(z) - z$ is called the Voiculescu transform of μ. It is not hard to show that $\phi_\mu(z)$ is an analytic function on $\Gamma_{\alpha,\beta}$ and $\Im\phi_\mu(z) \le 0$ for $z \in \Gamma_{\alpha,\beta}$, where ϕ_μ is defined. Furthermore, note that $\phi_\mu(z) = o(z)$ as $|z| \to \infty$, $z \in \Gamma_\alpha$.

Based on an alternative definition of free convolution developed in Chistyakov and Götze [20], we define the free convolution $\mu_1 \boxplus \mu_2$ of p-measures μ_1 and μ_2 as follows. Let $F_{\mu_1}(z)$ and $F_{\mu_2}(z)$ denote their reciprocal Cauchy transforms, respectively. We shall define the free convolution $\mu_1 \boxplus \mu_2$, using $F_{\mu_1}(z)$ and $F_{\mu_2}(z)$ only. It was proved in Chistyakov and Götze [20] that there exist unique functions $Z_1(z)$ and $Z_2(z)$ in the class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)). \tag{2.3}$$

The function $F_{\mu_1}(Z_1(z))$ belongs again to the class $\mathcal{F}$ and hence by Remark 3.1 (see Section 3) there exists a p-measure μ such that $F_{\mu_1}(Z_1(z)) = F_\mu(z)$, where $F_\mu(z) = 1/G_\mu(z)$ and $G_\mu(z)$ is the Cauchy transform as in (2.1). We define $\mu_1 \boxplus \mu_2 := \mu$. The measure μ depends on μ_1 and μ_2 only.

Thus, we define the additive free convolution by purely complex analytic methods.

The existence and uniqueness of the subordination functions $Z_j(z)$ in (2.3) have been studied earlier by other methods in Voiculescu [34–36], Biane [19], Maassen [26], Pastur and Vasilchuk [27], Vasilchuk [29].

On the domain $\Gamma_{\alpha,\beta}$, where the functions $\phi_{\mu_1\boxplus\mu_2}(z), \phi_{\mu_1}(z)$ and $\phi_{\mu_2}(z)$ are defined, we have

$$\phi_{\mu_1\boxplus\mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z). \tag{2.4}$$

This relation for the distribution $\mu_1 \boxplus \mu_2$ of $X + Y$, where X and Y are free random variables, is due to Voiculescu [31] for the case of compactly supported p-measures. The result was extended by Maassen [26] to p-measures with finite variance; the general case was proved by Bercovici and Voiculescu [11]. Note that Voiculescu and Bercovici's definition uses the operator context for the definition of $\mu_1 \boxplus \mu_2$, whereas Maassen's approach is closest to our analytical definition for the additive free convolution of arbitrary p-measures. Note that this approach extends as well to the case of multiplicative free convolutions (see [20]). By (2.4) it follows that our definition of $\mu_1 \boxplus \mu_2$ coincides with that of Voiculescu and Bercovici as well as Maassen's definition.

In their seminal paper [15] Bercovici and Pata proved free analogues of limit theorems for *identically* distributed random variables based on the relation (2.4). In this paper we use the relation (2.3) to obtain free analogues of limit theorems in the general case of *nonidentically* distributed random variables. The functions $F_{\mu_j}(z)$, $Z_j(z)$ from (2.3) belong to the class $\mathcal{F}$. This means that they are defined on the whole half-plane $\mathbb{C}^+$ and admit a special integral representation which allows us to study the limit behavior of the corresponding measures. It seems that free convolutions of nonidentical measures are easier to handle by these characterizing functions than the Voiculescu transforms $\phi_{\mu_j}(z)$.

The relation (2.3) has been used successfully in the papers [6–9, 13, 18] as well.

There is a notion of infinitely divisible p-measures for additive free convolution. As in the classical case, a p-measure μ is $\boxplus$-infinitely divisible if, for every natural number n, μ can be written as $\mu = \nu_n \boxplus \nu_n \boxplus \cdots \boxplus \nu_n$ (n times) with $\nu_n \in \mathcal{M}$. Such $\boxplus$-infinitely divisible p-measures were characterized by Voiculescu [31] for compactly supported measures. The $\boxplus$-infinitely divisible p-measures with finite variance were studied in Maassen [26] and Bercovichi and Voiculescu [11] extended these results to the general case. There is an analogue of the Lévy–Khintchine formula (see [10, 11, 33]) which states that a p-measure μ, on $\mathbb{R}$, is infinitely divisible if and only if the function $\phi_\mu(z)$ has an analytic continuation to $\mathbb{C}^+$, with values in $\mathbb{C}^- \cup \mathbb{R}$, such that

$$\lim_{y\to+\infty} \frac{\phi_\mu(iy)}{y} = 0. \tag{2.5}$$

By the Nevanlinna representation for such function (see Section 3), we know that there exist a real number α, and a finite nonnegative measure ν, on $\mathbb{R}$, such that

$$\phi_\mu(z) = \alpha + \int_{\mathbb{R}} \frac{1+uz}{z-u}\nu(du), \qquad z \in \mathbb{C}^+. \tag{2.6}$$

Since there is a one-to-one correspondence between functions $\phi_\mu(z)$ and pairs (α, ν), we shall write $\phi_\mu = (\alpha, \nu)$.

Formula (2.6) is an analogue of the well-known Lévy–Khintchine formula for characteristic functions $\varphi(t;\mu) := \int_{\mathbb{R}} e^{itu}\mu(du)$, $t \in \mathbb{R}$, of $*$-infinitely divisible measures $\mu \in \mathcal{M}$. A measure $\mu \in \mathcal{M}$ is $*$-infinitely divisible if and only if there exist a finite nonnegative Borel measure ν on $\mathbb{R}$, and a real number α such that

$$\begin{aligned}\varphi(t;\mu) &= \exp\{f_\mu(t)\} \\ &:= \exp\left\{i\alpha t + \int_{\mathbb{R}} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\nu(du)\right\}, \qquad t \in \mathbb{R},\end{aligned} \tag{2.7}$$

where $(e^{itu} - 1 - itu/(1+u^2))(1+u^2)/u^2$ is defined as $-t^2/2$ when $u = 0$. Since there is again a one-to-one correspondence between functions $f_\mu(t)$ and pairs (α, ν), we shall write $f_\mu = \{\alpha, \nu\}$.

Bercovici and Pata [15] determined the distributional behavior of sums of free *identically* distributed infinitesimal random variables. More precisely, they showed that, given a sequence μ_n of p-measures, and an increasing sequence k_n of positive integers, the free convolution product of k_n measures identical to μ_n converges weakly to a free infinitely divisible distribution if and only if the corresponding classical convolution product converges weakly to a classical infinitely divisible distribution. Moreover, the correspondence between the classical and free limits can be described explicitly.

In the classical case the precise formulation of the limit problem is as follows: Let $\{\mu_{nk} : n \geq 1, 1 \leq k \leq k_n\}$ be a triangular scheme of measures in $\mathcal{M}$ such that

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} \mu_{nk}(\{u : |u| > \varepsilon\}) = 0 \tag{2.8}$$

for every $\varepsilon > 0$, and let $\{a_n : n \geq 1\}$ be a sequence of real numbers. Such triangular schemes of measures μ_{nk} are called infinitesimal. Denote by δ_a a p-measure such that $\delta_a(\{a\}) = 1$. The basic limit problem arising in this context is:

(a) Find all $\mu \in \mathcal{M}$ such that $\mu^{(n)} = \delta_{-a_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$ converges to μ in the weak topology.
(b) Find conditions such that $\mu^{(n)}$ converges weakly to a given μ.

The complete solution of this problem has been obtained by the efforts of Kolmogorov, Lévy, Feller, de Finetti, Bawly, Khintchine, Marcinkewicz, Gnedenko and Doblin.

The limit problem in free probability theory has the same form for the p-measures $\mu^{(n)} = \delta_{-a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$. In the sequel we denote by $\widehat{\mu}_{nk}$ p-measures such that $\widehat{\mu}_{nk}((-\infty, u)) := \mu_{nk}((-\infty, u + a_{nk}))$, where $a_{nk} := \int_{(-\tau,\tau)} u\mu_{nk}(du)$ with finite $\tau > 0$ which is arbitrary, but fixed.

We provide a complete solution of this limit problem for free random variables. For the classical case see [22], Chapter 4 and [25], Section 22.

THEOREM 2.1. *Let μ_{nk} be a triangular scheme of infinitesimal probability measures. Then we have*:

(a) *The family of limit measures of sequences $\mu^{(n)} = \delta_{-a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ coincides with the family of $\boxplus$-infinitely divisible measures.*
(b) *There exist constants a_n such that the sequence $\mu^{(n)} = \delta_{-a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ converges weakly if, and only if, ν_n converges weakly to some finite nonnegative measure ν, where, for any Borel set S,*

$$\nu_n(S) := \sum_{k=1}^{k_n} \int_S \frac{u^2}{1+u^2} \widehat{\mu}_{nk}(du).$$

Then all admissible a_n are of the form $a_n = \alpha_n - \alpha + o(1)$, where α is an arbitrary finite number and

$$\alpha_n = \sum_{k=1}^{k_n}\left(a_{nk} + \int_{\mathbb{R}} \frac{u}{1+u^2}\widehat{\mu}_{nk}(du)\right).$$

Furthermore, all possible limit measures $\mu \in \mathcal{M}$ have a Voiculescu transform of type $\phi_\mu = (\alpha, \nu)$.

Note that the first statement of the theorem is due to Bercovici and Pata [16]. Another proof of this statement, based on the theory of Delphic semigroups, has been given by Chistyakov and Götze [20]. We see that this result is an obvious consequence of the second statement of the theorem.

Comparing the formulations of the second statement of Theorem 2.1 and of the second statement of the classical limit theorem (see [25], page 310), we see that these formulations coincide for $(\mathcal{M}, \boxplus)$ and $(\mathcal{M}, *)$. Therefore the following result holds, which for the case of identical measures μ_{nj}, $j = 1, \dots, k_n$, is known as Bercovici–Pata bijection [15].

THEOREM 2.2. *Let μ_{nk} be a triangular scheme of infinitesimal probability measures. There exist constants a_n such that the sequence $\delta_{a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ converges weakly to $\mu^{\boxplus} \in \mathcal{M}$ such that $\phi_{\mu^{\boxplus}} = (\alpha, \nu)$ if and only if the sequence $\delta_{a_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$ converges weakly to $\mu^* \in \mathcal{M}$ such that $f_{\mu^*} = \{\alpha, \nu\}$.*

Let $\mu \in \mathcal{M}$. Denote $\mu^{k*} := \mu * \cdots * \mu$ (k times) and $\mu^{k\boxplus} := \mu \boxplus \cdots \boxplus \mu$ (k times). Theorem 2.2 in the *identical* case $\mu_{n1} = \cdots = \mu_{nk_n}$ has the following form.

COROLLARY 2.3. *Let μ_n be a sequence of probability measures. The sequence $\mu_n^{k_n\boxplus}$ converges weakly to $\mu^{\boxplus} \in \mathcal{M}$ such that $\phi_{\mu^{\boxplus}} = (\alpha, \nu)$ if and only if the sequence $\mu_n^{k_n*}$ converges weakly to $\mu^* \in \mathcal{M}$ such that $f_{\mu^*} = \{\alpha, \nu\}$.*

Bercovici and Pata [15] characterized stable laws and domains of attraction in free probability theory for the case of identical p-measures μ_{nj} and established the so-called Bercovici–Pata bijection between infinitely divisible limits in $(\mathcal{M}, *)$ and $(\mathcal{M}, \boxplus)$. In particular they proved Corollary 2.3. Our approach allows us to study the case of *nonidentical* p-measures μ_{nj} as well and to obtain the results about limiting stable laws.

By Theorem 2.2, all results concerning the convergence of distribution functions of free sums can be reduced to the corresponding classical results. In particular, one obtains a criterion for the semicircle convergence [the case when $\phi_{\mu^{\boxplus}} = (\alpha, \lambda\delta_0)$, $\lambda > 0$], a criterion for the Marchenko–Pastur convergence [$\phi_{\mu^{\boxplus}} =$

$(\alpha, \lambda\delta_b), \lambda > 0, b \neq 0]$, as well as the degenerate convergence criterion $[\phi_{\mu^{\boxplus}} = (\alpha, \nu = 0)]$ for additive free convolution. These results generalize the corresponding results of Voiculescu [30], Bercovici and Voiculescu [12], Maassen [26], Pata [28], Bercovici and Pata [14] and of Lindsay and Pata [24] to the nonidentically distributed case.

Our analytical approach to the definition of the additive free convolution allows us to give explicit estimates for the rate of convergence of distribution functions of free sums. We shall demonstrate this by proving a semicircle approximation theorem (an analogue of the Berry–Esseen inequality; see [25], page 288), and a quantitative version of the law of large numbers, that is, including estimates of convergence.

To formulate the corresponding results we need the following notation. Let μ be a p-measure. Define $m_k(\mu) := \int_{\mathbb{R}} u^k \mu(du)$ and $\beta_k(\mu) := \int_{\mathbb{R}} |u|^k \mu(du)$, where $k = 0, 1, \dots$. We denote by μ_w the semicircle p-measure, that is, the measure with the density $\frac{1}{2\pi}\sqrt{(4-x^2)_+}$, $x \in \mathbb{R}$, where $a_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$.

Denote by $\Delta(\mu, \nu)$ the Kolmogorov distance between the p-measures μ and ν, that is,

$$\Delta(\mu, \nu) := \sup_{x \in \mathbb{R}} |\mu((-\infty, x)) - \nu((-\infty, x))|,$$

and by $L(\mu, \nu)$ the Lévy distance between these measures, that is,

$$L(\mu, \nu) := \inf\{h : \mu((-\infty, x-h)) - h \le \nu((-\infty, x)) \le \mu((-\infty, x+h)) + h, x \in \mathbb{R}\}.$$

As it is easy to see, $L(\mu, \nu) \le \Delta(\mu, \nu)$.

Let μ be a p-measure such that $m_1(\mu) = 0$ and $m_2(\mu) < \infty$. Denote $\mu_n((-\infty, x)) := \mu((-\infty, x\sqrt{m_2(\mu)n}))$, $x \in \mathbb{R}$.

The following theorem is an analogue of the well-known Berry–Esseen inequality (see [25], page 288) for the case of identically distributed free random variables assuming that the moment condition $m_4(\mu) < \infty$ holds.

THEOREM 2.4. *Let μ be a p-measure such that $m_1(\mu) = 0$ and $m_2(\mu) = 1$. If $m_4(\mu) < \infty$, there exists an absolute constant $c > 0$ such that*

$$\Delta(\mu_n^{n\boxplus}, \mu_w) \le c \frac{|m_3(\mu)| + (m_4(\mu))^{1/2}}{\sqrt{n}}. \tag{2.9}$$

The following proposition shows that estimate (2.9) is sharp.

PROPOSITION 2.5. *Let μ be a p-measure such that $\mu(\{-\sqrt{p/q}\}) = q$ and $\mu(\{\sqrt{q/p}\}) = p$, where $0 < p < 1$, $q = 1 - p$ and $p - q \neq 0$. Then*

$$\Delta(\mu_n^{n\boxplus}, \mu_w) \ge L(\mu_n^{n\boxplus}, \mu_w) \ge \frac{c(p)}{\sqrt{n}},$$

where $c(p)$ is a positive constant, depending on p only.

Now we shall consider the case of nonidentically distributed free random variables. Let $\{\mu_j\}_{j=1}^{\infty}$ be a sequence of measures in $\mathcal{M}$ such that $m_1(\mu_j)=0$ and $\beta_3(\mu_j)<\infty$ for all $j=1,\dots$. Denote

$$B_n^2=\sum_{k=1}^n m_2(\mu_k),\qquad A_n:=\sum_{k=1}^n \beta_3(\mu_k),\qquad L_n:=\frac{A_n}{B_n^3}.$$

Write $\mu_{nk}((-\infty,x)):=\mu_k((-\infty,B_nx)$, $x\in\mathbb{R}$, $k=1,\dots,n$, and $\mu^{(n)}:=\mu_{n1}\boxplus\cdots\boxplus\mu_{nn}$ as well.

THEOREM 2.6. *There exists an absolute constant $c>0$ such that*

$$\Delta(\mu^{(n)},\mu_w)\le cL_n^{1/2},\qquad n=1,\dots. \tag{2.10}$$

Finally we shall formulate the classical degenerate convergence criterion for additive free convolution with an estimate of the convergence.

Let $\{\mu_j\}_{j=1}^{\infty}$ be a sequence of measures in $\mathcal{M}$ and let $\mu_{nk}((-\infty,x)):=\mu_k((-\infty,nx))$, $x\in\mathbb{R}$, for $k=1,\dots,n$. Denote $\mu^{(n)}:=\mu_{n1}\boxplus\dots\boxplus\mu_{nn}$.

THEOREM 2.7. *In order that*

$$L(\mu^{(n)},\delta_0)\to 0 \tag{2.11}$$

as $n\to\infty$ it is necessary and sufficient that, for $n\to\infty$,

$$\eta_{n1}:=\sum_{k=1}^n\int_{\{|x|\ge n\}}\mu_k\to 0, \tag{2.12}$$

$$\eta_{n2}:=\frac{1}{n}\sum_{k=1}^n\int_{(-n,n)}x\,\mu_k(dx)\to 0, \tag{2.13}$$

$$\eta_{n3}:=\frac{1}{n^2}\sum_{k=1}^n\left\{\int_{(-n,n)}x^2\mu_k(dx)-\left(\int_{(-n,n)}x\,\mu_k(dx)\right)^2\right\}\to 0. \tag{2.14}$$

In addition, for some absolute positive constant c,

$$L(\mu^{(n)},\delta_0)\le c\big((\eta_{n1}+\eta_{n3})^{1/6}+\eta_{n2}\big),\qquad n=1,\dots. \tag{2.15}$$

Note that the statement of this theorem without the quantitative bound (2.15) is a simple consequence of Theorem 2.2 and the classical degenerate criterion (see [25], page 318). Therefore we need to prove (2.15) only.

Finally we shall describe Lévy's class $\mathcal{L}_{\boxplus}$ of limit laws of normed sums obeying the infinitesimal condition for the case of free summands. Let $\mu_1,\mu_2,\dots$ be a sequence of measures in $\mathcal{M}$ and sequences of real numbers $\{a_n\}$ and $\{b_n>0\}$. Denote by $\mu_{nk}: n\ge 1, 1\le k\le n$, the measures such that $\mu_{nk}(S):=\mu_k(b_nS)$ for every Borel set S.

Consider again the sequence of measures $\{\mu^{(n)} := \delta_{-a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nn}\}$.

As in the classical case the following problems arise:

(a) Given a sequence $\{\mu^{(n)}\}$ of measures in $\mathcal{M}$, find whether there exist sequences $\{a_n\}$ and $\{b_n > 0\}$ such that the $\mu_{nk}, n \ge 1, k = 1, \dots, n$, are infinitesimal and $\mu^{(n)} \to \mu$ weakly as $n \to \infty$, where μ is an infinitely divisible probability distribution such that $\phi_\mu = (\alpha, \nu)$. If such sequences exist, then characterize them.

(b) Characterize the family $\mathcal{L}_\boxplus$; in other words, characterize those functions $\phi_\mu(z)$ and the corresponding measures ν which represent limit measures of $\mu^{(n)}$.

It is convenient to exclude degenerate limit distributions from our consideration.

The solution of the problem (a) follows from the next result.

THEOREM 2.8. *There exist constants a_n and $b_n > 0$ such that μ_{nk}, $k = 1, \dots, n$, are infinitesimal and the sequence $\delta_{a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nn}$ converges weakly to $\mu^\boxplus \in \mathcal{M}$ such that $\phi_{\mu^\boxplus} = (\alpha, \nu)$ if and only if the sequence $\delta_{a_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nn}$ converges weakly to $\mu^* \in \mathcal{M}$ such that $f_{\mu^*} = \{\alpha, \nu\}$.*

This theorem establishes the Bercovici–Pata bijection for the case of infinitesimal measures μ_{nj}, which are rescaled versions of the measures μ_j. The proof follows immediately from Theorem 2.1 and the classical results for cumulative sums (see [22], Section 31 and [25], Section 23). The characterization of the sequences $\{a_n\}$ and $\{b_n\}$ follows immediately from the classical norming theorem (see [25], Section 23, pages 320–322).

The next result allows us to solve the problem (b).

Using the classical results about the class $\mathcal{L}_*$ (see [22], Section 30 and [25], Section 23), we obtain from Theorem 2.8 the canonical representation of the measures of the class $\mathcal{L}_\boxplus$.

THEOREM 2.9. *In order that $\mu \in \mathcal{M}$ belong to the class $\mathcal{L}_\boxplus$, it is necessary and sufficient that Voiculescu's transform of the measure μ has the form $\phi_\mu = (\alpha, \nu)$, where on $(-\infty, 0)$ and $(0, \infty)$ the left and right derivatives of the function $\nu(u) := \nu((-\infty, u))$, $u \in \mathbb{R}$, denoted indifferently by $\nu'(u)$, exist and $\frac{1+u^2}{u}\nu'(u)$ do not increase.*

The class $\mathcal{L}_\boxplus$ admits another description which does not follow in a straightforward way from the classical results. Let $\mu \in \mathcal{M}$. For any real constant $\gamma \neq 0$, we denote by $D_\gamma \mu$ the measure on $\mathbb{R}$ given by $D_\gamma \mu(S) = \mu(\gamma^{-1} S)$ for any Borel set S.

THEOREM 2.10. *In order that $\mu \in \mathcal{M}$ belong to the class $\mathcal{L}_\boxplus$, it is necessary and sufficient that for every γ, $0 < \gamma < 1$, $\mu = D_\gamma \mu \boxplus \mu_\gamma$, where $\mu_\gamma \in \mathcal{M}$.*

REMARK 2.11. The measure μ and, for any $\gamma \in (0,1)$, the measure μ_γ are $\boxplus$-infinitely divisible and $\phi_\mu = (\alpha, \nu)$ and $\phi_{\mu_\gamma} = (\alpha_\gamma, \nu_\gamma)$. Moreover, a measure μ^* such that $f_{\mu^*} = \{\alpha, \nu\}$ admits the representation $\mu^* = D_\gamma \mu^* * \mu_\gamma^*$ for any $\gamma \in (0,1)$, where μ_γ^* is $*$-infinitely divisible and $f_{\mu_\gamma^*} = \{\alpha_\gamma, \nu_\gamma\}$.

Barndorff-Nielsen and Thorbjørnsen in [3] and [5] studied the connection between the classes of infinitely divisible p-measures in classical and free probability. In [4] they studied the property of self-decomposability in free probability and, proving that such laws are infinitely divisible, studied Lévy processes in free probability and constructed stochastic integrals with respect to such processes. Our results allow us to extend the results of Biane [19] and of Barndorff-Nielsen and Thorbjørnsen [3]–[5].

3. Auxiliary results. We need results about some classes of analytic functions (see [1], Section 3, and [2], Section 6, §59).

The class $\mathcal{N}$ (R. Nevanlinna) is the class of analytic functions $f(z): \mathbb{C}^+ \to \{z : \Im z \geq 0\}$. For such functions there is an integral representation

$$(3.1)\qquad \begin{aligned} f(z) &= a + bz + \int_{\mathbb{R}} \frac{1+uz}{u-z} \tau(du) \\ &= a + bz + \int_{\mathbb{R}} \left(\frac{1}{u-z} - \frac{u}{1+u^2} \right)(1+u^2)\tau(du), \qquad z \in \mathbb{C}^+, \end{aligned}$$

where $b \geq 0$, $a \in \mathbb{R}$ and τ is a nonnegative finite measure. Moreover, $a = \Re f(i)$ and $\tau(\mathbb{R}) = \Im f(i) - b$. The measure τ and the parameter b are defined by the function $f(z)$ uniquely. From formula (3.1) it follows that

$$(3.2)\qquad f(z) = \big(b + o(1)\big) z$$

for $z \in \mathbb{C}^+$ such that $z \to \infty$ nontangentially.

A function $f \in \mathcal{N}$ admits the representation

$$(3.3)\qquad f(z) = \int_{\mathbb{R}} \frac{\sigma(du)}{u-z}, \qquad z \in \mathbb{C}^+,$$

where σ is a finite nonnegative measure, if and only if $\sup_{y \geq 1} |y f(iy)| < \infty$.

REMARK 3.1. Since the class $\mathcal{F}$ is the subclass of Nevanlinna functions $f(z)$ for which $f(z)/z \to 1$ as $z \to \infty$ nontangentially, we note that every $f \in \mathcal{F}$ admits representation (3.1), where $b = 1$. Moreover $-1/f(z)$ admits representation (3.3), where $\sigma \in \mathcal{M}$. Note as well that a function $f \in \mathcal{F}$ satisfies the obvious inequality

$$(3.4)\qquad \Im f(z) \geq \Im z, \qquad z \in \mathbb{C}^+.$$

The Stieltjes–Perron inversion formula for the functions f of class $\mathcal{N}$ has the following form. Let $\psi(u) := \int_0^u (1+t^2)\tau(dt)$. Then

$$\psi(u_2) - \psi(u_1) = \lim_{\eta\to 0} \frac{1}{\pi} \int_{u_1}^{u_2} \Im f(\xi + i\eta)\, d\xi, \tag{3.5}$$

where $u_1 < u_2$ denote two continuity points of the function $\psi(u)$.

Furthermore, we shall need the following inequality for the distance between distributions in terms of their Stieltjes transform.

LEMMA 3.2. *Let μ_w be the semicircle measure and let μ be a p-measure such that*

$$\int_{\mathbb{R}} |\mu_w((-\infty, x)) - \mu((-\infty, x))|\, dx < \infty. \tag{3.6}$$

Then there exists an absolute constant c such that, for any $0 < v < 1$,

$$\Delta(\mu_w, \mu) \le c \int_{\mathbb{R}} |G_{\mu_w}(u+i) - G_\mu(u+i)|\, du + cv$$
$$+ c \sup_{x\in[-2,2]} \left| \int_v^1 \big(G_{\mu_w}(x+iu) - G_\mu(x+iu)\big)\, du \right|,$$

where G_{μ_w} and G_μ are defined in (2.1).

This lemma is a simple consequence of Corollary 2.3 in Götze and Tikhomirov [23].

Let $\mu_j \in \mathcal{M}$, $j = 1, 2$. In Section 2 we defined the additive free convolution $\mu_1 \boxplus \mu_2$ by purely complex analytic methods.

As shown in [20], (2.3) admits the following consequences.

PROPOSITION 3.3. *Let $\mu_1, \dots, \mu_n \in \mathcal{M}$. There exist unique functions $Z_1(z), \dots, Z_n(z)$ of class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,*

$$z = Z_1(z) + \cdots + Z_n(z) - (n-1) F_{\mu_1}(Z_1(z))$$
$$\text{and} \tag{3.7}$$
$$F_{\mu_1}(Z_1(z)) = \cdots = F_{\mu_n}(Z_n(z)).$$

Moreover, $F_{\mu_1 \boxplus \cdots \boxplus \mu_n}(z) = F_{\mu_1}(Z_1(z))$ for all $z \in \mathbb{C}^+$.

Let $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$ and write $\mu_1 \boxplus \cdots \boxplus \mu_n = \mu^{n\boxplus}$.

PROPOSITION 3.4. *Let $\mu \in \mathcal{M}$. There exists a unique function $Z \in \mathcal{F}$ such that*

$$z = nZ(z) - (n-1) F_\mu(Z(z)), \qquad z \in \mathbb{C}^+, \tag{3.8}$$

and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z(z))$, $z \in \mathbb{C}^+$.

We need the following auxiliary results of Bercovici and Voiculescu [11].

PROPOSITION 3.5. *Let* $\{\mu_n\}_{n=1}^{\infty}$ *be a sequence of p-measures on* $\mathbb{R}$. *The following assertions are equivalent*:

(a) *The sequence* $\{\mu_n\}_{n=1}^{\infty}$ *converges weakly to a p-measure* μ.
(b) *There exist* $\alpha > 0$, $\beta > 0$, *and a function* ϕ, *such that all the functions* ϕ, ϕ_n *are defined on* $\Gamma_{\alpha,\beta}$, *and such that the sequence* $\{\phi_{\mu_n}\}_{n=1}^{\infty}$ *converges uniformly on compact subsets of* $\Gamma_{\alpha,\beta}$ *to a function* ϕ, *and* $\phi_{\mu_n}(iy) = o(y)$ *uniformly in* n *as* $y \to +\infty$.

Moreover, if (a) *and* (b) *are satisfied, we have* $\phi = \phi_\mu$ *in* $\Gamma_{\alpha,\beta}$.

PROPOSITION 3.6. *Let* $\{\mu_n\}_{n=1}^{\infty}$ *and* $\{\nu_n\}_{n=1}^{\infty}$ *be sequences of p-measures on* $\mathbb{R}$ *which converge weakly to p-measures* μ *and* ν, *respectively. Then* $\{\mu_n \boxplus \nu_n\}_{n=1}^{\infty}$ *converges weakly to the p-measure* $\mu \boxplus \nu$.

We also need the following two results which are due to Bercovici and Pata [16, 15].

PROPOSITION 3.7. *Let* $\alpha, \beta, \varepsilon$ *be positive numbers*, *and let* $\phi : \Gamma_{\alpha,\beta} \to \mathbb{C}$ *be an analytic function such that*

$$|\phi(z)| \le \varepsilon |z|, \qquad z \in \Gamma_{\alpha,\beta}.$$

For every $\alpha' < \alpha$ *and* $\beta' > \beta$ *there exists* $k > 0$ *such that*

$$|\phi'(z)| \le k\varepsilon, \qquad z \in \Gamma_{\alpha',\beta'}.$$

PROPOSITION 3.8. *For every* $\alpha, \beta > 0$ *there exists* $\varepsilon > 0$ *with the following property. If* $\mu \in \mathcal{M}$ *such that* $\int_{\mathbb{R}} u^2/(1+u^2)\mu(du) < \varepsilon$, *then* ϕ_μ *is defined on the region* $\Gamma_{\alpha,\beta}$ *and* $\phi_\mu(\Gamma_{\alpha,\beta}) \subset \mathbb{C}^- \cup \mathbb{R}$.

Let μ be a p-measure. Denote by $\bar{\mu}$ the measure defined by $\bar{\mu}(B) = \mu(-B)$ for any Borel set B. Write $\mu^s := \mu \boxplus \bar{\mu}$.

PROPOSITION 3.9. *A p-measure* μ *is symmetric if and only if the functions* $G_\mu(iy)$ *and* $F_\mu(iy)$ *take imaginary values for* $y > 0$ *and the function* $\phi_\mu(iy)$ *takes imaginary values on the set* $y \ge y_0 > 0$, *where it is defined.*

We omit the proof of this simple proposition.

We obtain, as an obvious consequence of Proposition 3.9, that μ^s is a symmetric p-measure. In addition, if μ_1 and μ_2 are symmetric p-measures, then $\mu_1 \boxplus \mu_2$ is a symmetric p-measure as well.

4. Additive free limit theorem. In this section we shall prove Theorem 2.1. In the sequel we denote by c positive absolute constants. For some measure ν and for some parameter τ we denote by $c(\nu)$, $c(\tau)$ and $c(\nu,\tau)$ positive constants which only depend on the measure ν, on the parameter τ, and on ν and τ, respectively. Before proving Theorem 2.1 we establish some properties of the measures $\{\mu_{nk}:n\ge 1, 1\le k\le k_n\}$, satisfying condition (2.8), and the corresponding reciprocal Cauchy transforms $\{F_{\mu_{nk}}(z):n\ge 1, 1\le k\le k_n\}$.

It is well known that condition (2.8) is equivalent to the following relation (see [25], page 302):

$$\max_{k=1,\dots,k_n}\int_{\mathbb{R}}\frac{u^2}{1+u^2}\mu_{nk}(du)\to 0, \qquad n\to\infty.$$

Recall that $\widehat{\mu}_{nk}((-\infty,u)):=\mu_{nk}((-\infty,u+a_{nk}))$, where $a_{nk}:=\int_{(-\tau,\tau)}x\,\mu_{nk}(dx)$, $k=1,\dots,k_n$, with arbitrary $\tau>0$ which is finite and fixed. Since obviously $\max_{k=1,\dots,k_n}|a_{nk}|\to 0$ as $n\to\infty$, we obtain

$$\varepsilon_n:=\max_{k=1,\dots,k_n}\varepsilon_{nk}\to 0, \qquad n\to\infty, \text{ where } \varepsilon_{nk}:=\int_{\mathbb{R}}\frac{u^2}{1+u^2}\widehat{\mu}_{nk}(du). \tag{4.1}$$

By Remark 3.1, for every $k=1,\dots,k_n$ the reciprocal of the Cauchy transform $G_{\widehat{\mu}_{nk}}(z)$ [see (2.1)] has the form

$$F_{\widehat{\mu}_{nk}}(z)=b_{nk}+z+\int_{\mathbb{R}}\frac{1+uz}{u-z}\sigma_{nk}(du), \tag{4.2}$$

where $b_{nk}:=\Re(G_{\widehat{\mu}_{nk}}(i))^{-1}$ and σ_{nk} is a nonnegative finite measure such that $\sigma_{nk}(\mathbb{R})=\Im(G_{\widehat{\mu}_{nk}}(i))^{-1}-1$. From (4.2) we deduce the following relation:

$$-\frac{\Im G_{\widehat{\mu}_{nk}}(iy)}{|G_{\widehat{\mu}_{nk}}(iy)|^2}=y\Big(1+\int_{\mathbb{R}}\frac{1+u^2}{u^2+y^2}\sigma_{nk}(du)\Big), \qquad y>0, k=1,\dots,k_n, \tag{4.3}$$

which yields

$$1+\int_{\mathbb{R}}\frac{1+u^2}{u^2+y^2}\sigma_{nk}(du)\le-\frac{1}{y\Im G_{\widehat{\mu}_{nk}}(iy)}, \qquad y>0, k=1,\dots,k_n. \tag{4.4}$$

On the other hand we see that, for $y>0$,

$$-y\Im G_{\widehat{\mu}_{nk}}(iy)=\int_{\mathbb{R}}\frac{y^2}{u^2+y^2}\widehat{\mu}_{nk}(du)=1-\int_{\mathbb{R}}\frac{u^2}{u^2+y^2}\widehat{\mu}_{nk}(du). \tag{4.5}$$

Hence, for sufficiently large $n\ge n_0$ and $k=1,\dots,k_n$, we obtain, by (4.4) and (4.5), the upper bound

$$\int_{\mathbb{R}}\frac{1+u^2}{u^2+y^2}\sigma_{nk}(du)\le 2\int_{\mathbb{R}}\frac{u^2}{u^2+y^2}\widehat{\mu}_{nk}(du)\le 32\varepsilon_{nk}, \qquad y\ge 1/4. \tag{4.6}$$

It follows from (4.6) that, for $n \ge n_0$,

$$\sigma_{nk}(\mathbb{R}) \le 2\varepsilon_{nk}, \qquad k = 1, \dots, k_n, \tag{4.7}$$

and $\max_{k=1,\dots,k_n} \sigma_{nk}(\mathbb{R}) \to 0$ as $n \to \infty$.

Now we deduce the relation

$$\begin{aligned}
\Re G_{\widehat{\mu}_{nk}}(i) &= \int_{\mathbb{R}} \frac{u - a_{nk}}{(u - a_{nk})^2 + 1} \mu_{nk}(du) \\
&= \int_{(-\tau,\tau)} \frac{u - a_{nk}}{(u - a_{nk})^2 + 1} \mu_{nk}(du) + \int_{|u| \ge \tau} \frac{u - a_{nk}}{(u - a_{nk})^2 + 1} \mu_{nk}(du) \\
&= \int_{(-\tau,\tau)} (u - a_{nk}) \mu_{nk}(du) - \int_{(-\tau,\tau)} \frac{(u - a_{nk})^3}{(u - a_{nk})^2 + 1} \mu_{nk}(du) \\
&\quad + \int_{|u| \ge \tau} \frac{u - a_{nk}}{(u - a_{nk})^2 + 1} \mu_{nk}(du) \\
&= - \int_{(-\tau,\tau)} \frac{(u - a_{nk})^3}{(u - a_{nk})^2 + 1} \mu_{nk}(du) \\
&\quad + \int_{|u| \ge \tau} \frac{u + a_{nk}(u - a_{nk})^2}{(u - a_{nk})^2 + 1} \mu_{nk}(du).
\end{aligned}$$

Using straightforward estimates we easily have, for sufficiently large $n \ge n_0$,

$$|\Re G_{\widehat{\mu}_{nk}}(i)| \le c(\tau)\varepsilon_{nk}, \qquad k = 1, \dots, k_n. \tag{4.8}$$

In view of (4.1), (4.5) and (4.8), we get, for $n \ge n_0$ and $k = 1, \dots, k_n$,

$$|b_{nk}| \le |\Re G_{\widehat{\mu}_{nk}}(i)| / (\Im G_{\widehat{\mu}_{nk}}(i))^2 \le c(\tau)\varepsilon_{nk}. \tag{4.9}$$

From (4.2), (4.7) and (4.9) we obtain, for $z \in \mathbb{C}^+$ and $n \ge n_0$, $k = 1, \dots, k_n$,

$$\begin{aligned}
|F_{\widehat{\mu}_{nk}}(z) - z| &\le |b_{nk}| + \int_{\mathbb{R}} \frac{\sigma_{nk}(du)}{|u - z|} + \int_{\mathbb{R}} \frac{|z||u|}{|u - z|} \sigma_{nk}(du) \\
&\le c(\tau)\varepsilon_{nk} \left(1 + \frac{1 + |z|^2}{\Im z} \right) \le c(\tau)\varepsilon_{nk} Q(z),
\end{aligned} \tag{4.10}$$

where $Q(z) := \frac{1+|z|^2}{\Im z}$. Using (4.6) we deduce the estimate, for $k = 1, \dots, k_n$ and $\Im z \ge 1/4$,

$$\begin{aligned}
\Im\big(F_{\widehat{\mu}_{nk}}(z) - z\big) &= \Im z \int_{\mathbb{R}} \frac{1 + u^2}{(u - \Re z)^2 + (\Im z)^2} \sigma_{nk}(du) \\
&\le 2 \left(\frac{|z|}{\Im z} \right)^2 \Im z \int_{\mathbb{R}} \frac{1 + u^2}{(\Im z)^2 + u^2} \sigma_{nk}(du) \\
&\le 4 \left(\frac{|z|}{\Im z} \right)^2 \Im z \int_{\mathbb{R}} \frac{1 + u^2}{(\Im z)^2 + u^2} \frac{u^2}{1 + u^2} \widehat{\mu}_{nk}(du) \\
&\le 64 \left(\frac{|z|}{\Im z} \right)^2 \eta_{nk}(\Im z) \Im z,
\end{aligned} \tag{4.11}$$

where

$$\eta_{nk}(\Im z) := \frac{1}{\Im z}\int_{[-\sqrt{\Im z},\sqrt{\Im z}]} \frac{u^2}{1+u^2}\widehat{\mu}_{nk}(du) + \int_{|u|>\sqrt{\Im z}} \frac{u^2}{1+u^2}\widehat{\mu}_{nk}(du).$$

Note that, for such $k = 1, \dots, k_n$ and $\Im z \geq 1/4$, $\eta_{nk}(\Im z) \leq 4\varepsilon_{nk}$.

We conclude from (4.10), (4.11) and Rouché's theorem that for every $y \geq 1$ there exists a neighborhood $|z - iy| \leq y/2$ such that the inverse function $F^{(-1)}_{\widehat{\mu}_{nk}}(z)$ with $n \geq n_0$ exists and is analytic in this domain. In addition, the following inequalities hold:

$$\begin{aligned} |\Re\phi_{\widehat{\mu}_{nk}}(z)| &= \big|\Re\big(F^{(-1)}_{\widehat{\mu}_{nk}}(z) - z\big)\big| \leq c(\tau)\varepsilon_{nk} y, \\ |\Im\phi_{\widehat{\mu}_{nk}}(z)| &= \big|\Im\big(F^{(-1)}_{\widehat{\mu}_{nk}}(z) - z\big)\big| \leq c\tilde{\eta}_{nk}(y) y, \end{aligned} \tag{4.12}$$

for $|z - iy| \leq y/2, n \geq n_0, k = 1, \dots, k_n$, where $\tilde{\eta}_{nk}(y) := \max_{t\in[y/4,2y]} \eta_{nk}(t)$.

PROOF OF THEOREM 2.1. *Sufficiency*. Consider the measure $\widehat{\mu}_n := \widehat{\mu}_{n1} \boxplus \cdots \boxplus \widehat{\mu}_{nk_n}$. It follows from Proposition 3.3 that there exist unique functions $Z_{n1}, \dots, Z_{nk_n}$ of class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,

$$\begin{aligned} F_{\widehat{\mu}_{n1}}(Z_{n1}(z)) - z &= F_{\widehat{\mu}_{n1}}(Z_{n1}(z)) - Z_{n1}(z) + \cdots \\ &\quad + F_{\widehat{\mu}_{nk_n}}(Z_{nk_n}(z)) - Z_{nk_n}(z) \end{aligned} \tag{4.13}$$

and

$$F_{\widehat{\mu}_{n1}}(Z_{n1}(z)) = F_{\widehat{\mu}_{n2}}(Z_{n2}(z)) = \cdots = F_{\widehat{\mu}_{nk_n}}(Z_{nk_n}(z)) = F_{\widehat{\mu}_n}(z). \tag{4.14}$$

Then, by (4.12)–(4.14), for $|z - iy| \leq y/2$, $y \geq c(\nu) \geq 1$, it follows that

$$\begin{aligned} |\Re\phi_{\widehat{\mu}_{n1}\boxplus\cdots\boxplus\widehat{\mu}_{nk_n}}(z)| &\leq \big|\Re\phi_{\widehat{\mu}_{n1}}(z)\big| + \cdots + |\Re\phi_{\widehat{\mu}_{nk_n}}(z)| \\ &\leq c(\tau)\eta_n y := c(\tau)\Big(\sum_{k=1}^{k_n} \varepsilon_{nk}\Big) y \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} |\Im\phi_{\widehat{\mu}_{n1}\boxplus\cdots\boxplus\widehat{\mu}_{nk_n}}(z)| &\leq \big|\Im\phi_{\widehat{\mu}_{n1}}(z)\big| + \cdots + |\Im\phi_{\widehat{\mu}_{nk_n}}(z)| \\ &\leq c\eta_n(y) y := c\Big(\sum_{k=1}^{k_n} \tilde{\eta}_{nk}(y)\Big) y. \end{aligned} \tag{4.16}$$

By the assumptions of the theorem, we have $\eta_n \leq \nu(\mathbb{R}) + 1$ for sufficiently large $n \geq n_0$. In addition, by (4.11) and the assumptions of the theorem, we see that

$$\eta_n(y) \leq \frac{16\nu(\mathbb{R}) + 1}{y} + 4\nu\big(\mathbb{R} \setminus [-\sqrt{y}/2, \sqrt{y}/2]\big) \tag{4.17}$$

for sufficiently large $n \ge n_1(y)$, where $-\sqrt{y}/2$ and $\sqrt{y}/2$ are continuity points of the function $\nu((-\infty, x))$, $x \in \mathbb{R}$. In the sequel we choose y so that $-\sqrt{y}/2$ and $\sqrt{y}/2$ are continuity points of $\nu((-\infty, x))$. Since

$$\begin{aligned}\phi_{\widehat{\mu}_{n1}\boxplus\cdots\boxplus\widehat{\mu}_{nk_n}}(z) &= (F_{\widehat{\mu}_{nk}}(Z_{nk}))^{(-1)}(z) - z\\ &= Z_{nk}^{(-1)}\big(F_{\widehat{\mu}_{nk}}^{(-1)}(z)\big) - z, \qquad k = 1, \dots, k_n,\end{aligned}$$

for $|z - iy| \le y/2$, we have, by (4.10), the relation

$$\phi_{\widehat{\mu}_{n1}\boxplus\cdots\boxplus\widehat{\mu}_{nk_n}}(F_{\widehat{\mu}_{nk}}(z)) = Z_{nk}^{(-1)}(z) - F_{\widehat{\mu}_{nk}}(z), \qquad k = 1, \dots, k_n,$$

for $|z - iy| \le y/4$. Therefore we conclude by (4.10)–(4.12) and (4.15)–(4.17) that the functions $Z_{nk}^{(-1)}(z)$ are analytic in the disk $|z - iy| < y/4$ and

$$\begin{aligned}&\big|\Re\big(Z_{nk}^{(-1)}(z) - z\big)\big| \le c(\tau)\big(\nu(\mathbb{R}) + 1\big)Q(y),\\ &\big|\Im\big(Z_{nk}^{(-1)}(z) - z\big)\big| \le c\big(1 + \nu(\mathbb{R}) + y\nu\big(\mathbb{R} \setminus [-\sqrt{y}/2, \sqrt{y}/2]\big)\big),\end{aligned} \tag{4.18}$$

for $|z - iy| \le y/4, n \ge n_1(y), k = 1, \dots, k_n$. We conclude from (4.18) that there exists $y_0 = y_0(\nu) \ge 4$ such that $Z_{nk}^{(-1)}(z) \in R_{y_0} := \{z : |\Re z| \le c(\tau)(\nu(\mathbb{R}) + 1)y_0, y_0/2 \le \Im z \le 3y_0/2\}$ for $|z - iy_0| \le y_0/4, n \ge n_1(y_0), k = 1, \dots, k_n$. Hence there exist points $z_{nk} \in R_{y_0}$ such that $|Z_{nk}(z_{nk}) - iy_0| \le y_0/4$ for $n \ge n_1(y_0), k = 1, \dots, k_n$.

The functions Z_{nk} are of class $\mathcal{F}$. Therefore

$$\begin{aligned}Z_{nk}(z) &= d_{nk} + z + \int_{\mathbb{R}} \frac{1 + uz}{u - z}\nu_{nk}(du)\\ &= d_{nk} + z + \int_{\mathbb{R}} \left(\frac{1}{u - z} - \frac{u}{1 + u^2}\right)(1 + u^2)\nu_{nk}(du)\end{aligned} \tag{4.19}$$

for $z \in \mathbb{C}^+$, where $d_{nk} \in \mathbb{R}$ and ν_{nk} are finite nonnegative measures. Since $\Im Z_{nk}(z_{nk}) - y_0 \le y_0/2$, we have

$$\begin{aligned}c(\nu, \tau)\nu_{nk}(\mathbb{R}) &\le \Im z_{nk} \int_{\mathbb{R}} \frac{1 + u^2}{(u - \Re z_{nk})^2 + (\Im z_{nk})^2}\nu_{nk}(du)\\ &\le \Im Z_{nk}(z_{nk}) \le \frac{3y_0}{2}.\end{aligned} \tag{4.20}$$

It is easy to see from (4.19) and (4.20) that $|Z_{nk}(z_{nk}) - d_{nk}| \le c(\nu, \tau)$. Hence, using the bound $|Z_{nk}(z_{nk})| \le 3y_0/2$, we conclude that $|d_{nk}| \le c(\nu, \tau) + 3y_0/2$. Hence we have

$$|d_{nk}| \le c(\nu, \tau) \quad \text{and} \quad \nu_{nk}(\mathbb{R}) \le c(\nu, \tau), \qquad n \ge n_1(y_0), k = 1, \dots, k_n. \tag{4.21}$$

In the sequel we assume that $n \ge n_1(y_0) + n_0$. As in (4.10) we obtain, for $z \in \mathbb{C}^+$ and $k = 1, \dots, k_n$,

$$|Z_{nk}(z) - z| \le c(\nu, \tau)Q(z). \tag{4.22}$$

Using (4.22) and the inequality $\Im Z_{nk}(z) \ge \Im z$, $z \in \mathbb{C}^+$ [see (3.4)], we deduce

$$Q(Z_{nk}(z)) = \frac{1 + |Z_{nk}(z)|^2}{\Im Z_{nk}(z)} \le c(\nu, \tau) \frac{1}{\Im z} Q^2(z) \tag{4.23}$$

for $z \in \mathbb{C}^+$ and $k = 1, \dots, k_n$. Therefore we obtain from (4.10)

$$|F_{\mu_{nk}}(Z_{nk}(z)) - Z_{nk}(z)| \le c(\tau)\varepsilon_{nk} Q(Z_{nk}(z)) \le c(\nu, \tau)\varepsilon_{nk} \frac{1}{\Im z} Q^2(z) \tag{4.24}$$

for $z \in \mathbb{C}^+$ and $k = 1, \dots, k_n$. Let us return to relation (4.14). In view of (4.24), we have, for $z \in \mathbb{C}^+$ and $k = 1, \dots, k_n$,

$$\begin{aligned} |Z_{n1}(z) - Z_{nk}(z)| &\le |F_{\widehat{\mu}_{n1}}(Z_{n1}(z)) - Z_{n1}(z)| + |F_{\widehat{\mu}_{nk}}(Z_{nk}(z)) - Z_{nk}(z)| \\ &\le c(\nu, \tau)\varepsilon_n \frac{1}{\Im z} Q^2(z). \end{aligned} \tag{4.25}$$

On the other hand, in view of (4.2) and (4.7), we conclude

$$\begin{aligned} &\big|\big(F_{\widehat{\mu}_{nk}}(Z_{nk}(z)) - Z_{nk}(z)\big) - \big(F_{\widehat{\mu}_{nk}}(Z_{n1}(z)) - Z_{n1}(z)\big)\big| \\ &\quad \le \int_{\mathbb{R}} \frac{|Z_{nk}(z) - Z_{n1}(z)|(1+u^2)\sigma_{nk}(du)}{\sqrt{(u - \Re Z_{nk}(z))^2 + (\Im Z_{nk}(z))^2}\sqrt{(u - \Re Z_{n1}(z))^2 + (\Im Z_{n1}(z))^2}} \\ &\quad \le c\varepsilon_{nk}|Z_{nk}(z) - Z_{n1}(z)| \frac{(1 + |Z_{n1}(z)|)(1 + |Z_{nk}(z)|)}{\Im Z_{n1}(z)\Im Z_{nk}(z)} \end{aligned}$$

for $z \in \mathbb{C}^+$ and $k = 1, \dots, k_n$. Thus, taking into account (4.22), (4.25) and the inequality $\Im Z_{nk}(z) \ge \Im z$, $z \in \mathbb{C}^+$, we have, for the same z and k as above,

$$\begin{aligned} &\big|\big(F_{\widehat{\mu}_{nk}}(Z_{nk}(z)) - Z_{nk}(z)\big) - \big(F_{\widehat{\mu}_{nk}}(Z_{n1}(z)) - Z_{n1}(z)\big)\big| \\ &\quad \le c(\nu, \tau)\varepsilon_{nk}\varepsilon_n \frac{1}{(\Im z)^3} Q^4(z). \end{aligned} \tag{4.26}$$

Consider the functions

$$\begin{aligned} f_{nk}(z) &:= z^2\left(G_{\widehat{\mu}_{nk}}(z) - \frac{1}{z}\right) \\ &= \gamma_{nk} + \int_{\mathbb{R}} \frac{1 + uz}{z - u} \rho_{nk}(du), \qquad z \in \mathbb{C}^+, k = 1, \dots, k_n, \end{aligned}$$

where

$$\gamma_{nk} := \int_{\mathbb{R}} \frac{u}{1+u^2} \widehat{\mu}_{nk}(du) \qquad \text{and} \qquad \rho_{nk}(du) := \frac{u^2}{1+u^2} \widehat{\mu}_{nk}(du).$$

By (4.8), the constants γ_{nk} admit the estimates $|\gamma_{nk}| \le c(\tau)\varepsilon_{nk}$ for $k = 1, \dots, k_n$. Hence, $\gamma_n := \sum_{k=1}^{k_n} \gamma_{nk}$ satisfies the inequality

$$|\gamma_n| \le c(\nu, \tau), \qquad n \ge n_0. \tag{4.27}$$

As in (4.10), we conclude that

$$|f_{nk}(z)| \le c(\tau)\varepsilon_{nk} Q(z), \qquad z \in \mathbb{C}^+, k = 1, \dots, k_n. \tag{4.28}$$

We have, for $z \in \mathbb{C}^+$ and $k = 1, \dots, k_n$,

$$F_{\widehat{\mu}_{nk}}(z) = \frac{z^2}{z + f_{nk}(z)} = z - f_{nk}(z) + \theta_{nk}(z), \tag{4.29}$$

where

$$\theta_{nk}(z) = \frac{f_{nk}^2(z)}{z + f_{nk}(z)} = f_{nk}^2(z) F_{\widehat{\mu}_{nk}}(z) z^{-2}.$$

Hence, by (4.10) and (4.28), we conclude, for those z, k,

$$|\theta_{nk}(z)| \le c(\tau)\varepsilon_{nk}^2 Q^2(z)\big(|z| + \varepsilon_{nk} Q(z)\big)\frac{1}{|z|^2}. \tag{4.30}$$

We see from (4.22), (4.23), (4.30) and from the inequality $\Im Z_{nk}(z) \ge \Im z$, $z \in \mathbb{C}^+$, that, for z, k as above,

$$\begin{aligned} |\theta_{nk}(Z_{n1}(z))| &\le c(\tau)\varepsilon_{n1}^2 Q^2(Z_{n1}(z))\big(|Z_{n1}(z)| + \varepsilon_{nk} Q(Z_{n1}(z))\big)\frac{1}{|Z_{nk}(z)|^2} \\ &\le c(\nu,\tau)\varepsilon_{nk}^2 \frac{1}{(\Im z)^4} Q^5(z)\bigg(1 + \varepsilon_{nk}\frac{1}{\Im z} Q(z)\bigg). \end{aligned} \tag{4.31}$$

Therefore (4.13), (4.26), (4.29) and (4.31) together yield the relation

$$\begin{aligned} &F_{\widehat{\mu}_{n1}}(Z_{n1}(z)) - z \\ &\quad = -f_{n1}(Z_{n1}(z)) + \cdots - f_{nk_n}(Z_{n1}(z)) + r_n(z), \qquad z \in \mathbb{C}^+, \end{aligned} \tag{4.32}$$

where the function

$$\begin{aligned} r_n(z) := &\sum_{k=1}^{k_n}\Big(\big(F_{\widehat{\mu}_{nk}}(Z_{nk}(z)) - Z_{nk}(z)\big) - \big(F_{\widehat{\mu}_{nk}}(Z_{n1}(z)) - Z_{n1}(z)\big)\Big) \\ &+ \sum_{k=1}^{k_n}\theta_{nk}(Z_{n1}(z)) \end{aligned}$$

is analytic in $\mathbb{C}^+$ and admits the estimate

$$|r_n(z)| \le c(\nu,\tau)\varepsilon_n \frac{1}{(\Im z)^4} Q^5(z)\bigg(1 + \varepsilon_n \frac{1}{\Im z} Q(z)\bigg). \tag{4.33}$$

From (4.33) it is easy to see that

$$|r_n(z)| \le c(\nu,\tau)\varepsilon_n^{1/20} \tag{4.34}$$

in the closed domain $D_n := \{z \in \mathbb{C}^+ : \varepsilon_n^{1/20} \le \Im z \le \varepsilon_n^{-1/20}, |\Re z| \le \varepsilon_n^{-1/20}\}$.

We return to the representation (4.19) for the functions $Z_{n1}(z)$.

By (4.21), (4.27) and the vague compactness theorem (see [25], page 179), we conclude that there exists a subsequence $\{n'\}$ such that

$$d_{n'1} \to d, \qquad \nu_{n'1}(\mathbb{R}) \to b, \qquad \gamma_{n'} \to \gamma, \qquad n' \to \infty,$$

where $d, b \geq 0, \gamma$ are real numbers, and $\{\nu_{n'1}\}$ converges in the vague topology to some nonnegative measure ν_1 such that $\nu_1(\mathbb{R}) \leq b$. Now we rewrite the formula (4.19) with $n = n'$ and $k = 1$ in the form

$$Z_{n'1}(z) = d_{n'1} + z + \int_{\mathbb{R}} \left(\frac{1+uz}{u-z} - z \right) \nu_{n'1}(du) + \nu_{n'1}(\mathbb{R}) z.$$

Since the kernel under the integral sign tends to 0 as $u \to \pm\infty$ uniformly in z from every compact set in $\mathbb{C}^+$, we obtain, by the Helly–Bray lemma (see [25], page 181),

$$\begin{aligned} Z_{n'1}(z) &\to d + z + \int_{\mathbb{R}} \left(\frac{1+uz}{u-z} - z \right) \nu_1(du) + bz \\ &= d + z + \int_{\mathbb{R}} \frac{1+uz}{u-z} \nu_1(du) + \big(b - \nu_1(\mathbb{R})\big) z, \qquad n' \to \infty, \end{aligned}$$

uniformly on every compact set in $\mathbb{C}^+$.

Finally we obtain from this relation that

$$Z_{n'1}(z + \gamma_{n'1}) \to Z(z) + az \tag{4.35}$$

as $n' \to \infty$, uniformly on every compact set in $\mathbb{C}^+$, where $Z(z) \in \mathcal{F}$ and $a \geq 0$.

Rewrite the relation (4.32) in the form

$$\begin{aligned} &F_{\widehat{\mu}_{n1}}\big(Z_{n1}(z + \gamma_n)\big) - z \\ &\qquad = \int_{\mathbb{R}} \frac{(Z_{n1}(z+\gamma_n) - Z(z) - az)(1+u^2)}{(u - Z_{n1}(z+\gamma_n))(u - Z(z) - az)} \nu_n(du) \\ &\qquad\quad + \int_{\mathbb{R}} \frac{1 + u(Z(z) + az)}{u - Z(z) - az} \nu_n(du) + r_n(z + \gamma_n), \qquad z \in \mathbb{C}^+. \end{aligned} \tag{4.36}$$

Note that, for every compact set S in $\mathbb{C}^+$,

$$\limsup_{n' \to \infty} \sup_{z \in S} \left| \int_{\mathbb{R}} \frac{1+u^2}{(u - Z_{n'1}(z + \gamma_{n'}))(u - Z(z) - az)} \nu_{n'}(du) \right| < \infty. \tag{4.37}$$

In view of the assumption that $\nu_n \to \nu$ weakly, we have, by the Helly–Bray theorem (see [25], page 182),

$$\lim_{n' \to \infty} \int_{\mathbb{R}} \frac{1 + u(Z(z) + az)}{u - Z(z) - az} (\nu_{n'} - \nu)(du) = 0, \qquad z \in \mathbb{C}^+. \tag{4.38}$$

Since $F_{\widehat{\mu}_{n1}}(z) \to z$ uniformly on every compact set in $\mathbb{C}^+$ and (4.34), (4.35), (4.37) and (4.38) hold, we easily deduce from (4.36) in the limit $n' \to \infty$ that

$$Z(z) + az - z = \int_{\mathbb{R}} \frac{1 + u(Z(z) + az)}{u - (Z(z) + az)} \nu(du), \qquad z \in \mathbb{C}^+. \tag{4.39}$$

It is easy to see that $Z(iy) - iy = o(y)$ and the integral on the right-hand side of (4.39) is a function which is $o(y)$ as $y \to \infty$ for $z = iy$. Therefore we conclude that $a = 0$. Thus the relation (4.39) holds with $a = 0$.

Since $Z \in \mathcal{F}$ has an inverse $Z^{(-1)}$ defined on $\Gamma_{\alpha,\beta}$ with some positive α and β, it is easy to see that (4.39) has a unique solution in the set $\mathcal{F}$. Now suppose that $\{Z_{n1}(z+\gamma_n)\}_{n=1}^{\infty}$ does not converge to $Z(z)$ on some compact set in $\mathbb{C}^+$. Then, as above there exists a subsequence $\{n''\}$ such that $Z_{n''1}(z+\gamma_{n''}) \to Z^*(z)$ as $n'' \to \infty$ on every compact set in $\mathbb{C}^+$, and $Z^*(z) \in \mathcal{F}$, $Z^*(z) \not\equiv Z(z), z \in \mathbb{C}^+$. But $Z^*(z)$ is a solution of (4.39). We arrive at a contradiction. Hence $\{Z_{n1}(z+\gamma_n)\}_{n=1}^{\infty}$ converges to $Z(z)$ uniformly on every compact set in $\mathbb{C}^+$. The relation (4.39) implies that $Z(z)$ is infinitely divisible with parameters $(0,\nu)$, since we may rewrite (4.39) via $z = Z^{(-1)}(w)$ for $w \in \Gamma_{\alpha,\beta}$ with some $\alpha, \beta > 0$. Since $F_{\widehat{\mu}_{n1}}(Z_{n1}(z+\gamma_n)) \to Z(z)$ uniformly on every compact set in $\mathbb{C}^+$, we see that $\widehat{\mu}_n \boxplus \delta_{-\gamma_n}$ converges weakly to a p-measure $\widehat{\mu}$ such that $\phi_{\widehat{\mu}} = (0,\nu)$. Recalling the definition of a_n, we finally conclude that $\mu^{(n)}$ converges weakly to p-measure μ such that $\phi_\mu = (\alpha,\nu)$.

Hence the sufficiency of the assumptions of Theorem 2.1(b) is proved.

Necessity. Denote $\mu_{nk}^s := \mu_{nk} \boxplus \bar{\mu}_{nk} = \widehat{\mu}_{nk} \boxplus \overline{\widehat{\mu}}_{nk}, n \ge 1, k = 1, \dots, k_n$. By Proposition 3.6, we obtain the convergence

$$\mu^{(n,s)} := \mu_{n1}^s \boxplus \mu_{n2}^s \boxplus \cdots \boxplus \mu_{nk_n}^s \to \mu^s \qquad \text{weakly as } n \to \infty. \tag{4.40}$$

For the measures $\mu_{k,n}^s, n \ge 1, k = 1, \dots, k_n$, relations (4.13) and (4.14) hold with the functions $F_{\mu_{nk}^s}(z), n \ge 1, k = 1, \dots, k_n$, replacing $F_{\mu_{nk}}(z), n \ge 1, k = 1, \dots, k_n$, and with some functions $Z_{nk,s}(z) \in \mathcal{F}, n \ge 1, k = 1, \dots, k_n$, replacing $Z_{nk}(z), n \ge 1, k = 1, \dots, k_n$. Rewrite (4.13) in the form

$$\begin{aligned} &F_{\mu_{n1}^s}(Z_{n1,s}(z)) - z \\ &\quad = F_{\mu_{n1}^s}(Z_{n1,s}(z)) - Z_{n1,s}(z) + \cdots \\ &\qquad + F_{\mu_{nk_n}^s}(Z_{nk_n,s}(z)) - Z_{nk_n,s}(z), \qquad z \in \mathbb{C}^+. \end{aligned} \tag{4.41}$$

By Proposition 3.9, the measures $\mu_{nk}^s, k = 1, \dots, k_n$, are symmetric and $\mu^{(n,s)} := \mu_{n1}^s \boxplus \mu_{n2}^s \boxplus \cdots \boxplus \mu_{nk_n}^s$ is symmetric as well. Since $F_{\mu_{nk}^s}(Z_{nk,s}(z)) = F_{\mu^{(n,s)}}(z), z \in \mathbb{C}^+$, and by Proposition 3.9, $F_{\mu^{(n,s)}}(iy), F_{\mu_{nk}^s}(iy), y > 0$, assume imaginary values, we conclude that $Z_{nk,s}(iy), y > 0, k = 1, \dots, k_n$, assume imaginary values as well. Hence, it is easy to see that $Z_{nk,s}(z), k = 1, \dots, k_n$, admit representation (4.19) with $d_{nk} = 0, k = 1, \dots, k_n$, and with finite nonnegative symmetric measures $\nu_{nk}^s, k = 1, \dots, k_n$, respectively.

Since

$$F_{\mu_{nk}^s}(z) = z + \int_{\mathbb{R}} \frac{1+uz}{u-z} \sigma_{nk}^s(du), \qquad z \in \mathbb{C}^+,\ k = 1, \dots, k_n,$$

where σ_{nk}^s is a finite nonnegative measure, we deduce from (4.41) that

$$\Im\big(F_{\mu_{n1}^s}(Z_{n1,s}(i))-i\big)=\sum_{k=1}^n \Im\big(F_{\mu_{nk}^s}(Z_{nk,s}(i))-Z_{nk,s}(i)\big)$$

(4.42)
$$=\sum_{k=1}^n \Im Z_{nk,s}(i)\int_{\mathbb{R}}\frac{(1+u^2)\sigma_{nk}^s(du)}{u^2+(\Im Z_{nk,s}(i))^2}.$$

Let us show that the measures $\mu_{n1}^s,\dots,\mu_{nk_n}^s$ are infinitesimal. Indeed, we deduce from (4.12) the estimate

$$-\Im\phi_{\mu_{nk}^s}(z)=-\Im\phi_{\widehat{\mu}_{nk}}(z)-\Im\phi_{\overline{\widehat{\mu}}_{nk}}(z)\le c(\varepsilon_{nk}+\overline{\varepsilon}_{nk})\le c\varepsilon_{nk},\qquad k=1,\dots,k_n,$$

for $|z-i|\le 1/4$, where $\overline{\varepsilon}_{nk}:=\int_{\mathbb{R}}u^2/(1+u^2)\overline{\widehat{\mu}}_{nk}(du)=\varepsilon_{nk}$. This implies

$$\int_{\mathbb{R}}\frac{u^2}{1+u^2}\mu_{nk}^s(du)\Big/\int_{\mathbb{R}}\frac{1}{1+u^2}\mu_{nk}^s(du)$$

(4.43)
$$=\Im\big(F_{\mu_{nk}^s}(i)-i\big)\le c\varepsilon_{nk},\qquad k=1,\dots,k_n,$$

as claimed.

The bounds (4.43) and (4.10) for the functions $F_{\mu_{nk}^s}(z), n\ge n_0, k=1,\dots,k_n$, imply the inequality

$$|Z_{nk,s}(i)|\le |Z_{nk,s}(i)-F_{\mu_{nk}^s}(Z_{nk,s}(i))|+\big|F_{\mu^{(n,s)}}(i)\big|$$

(4.44)
$$\le c\varepsilon_{nk}Q(Z_{nk,s}(i))+\big|F_{\mu^{(n,s)}}(i)\big|$$

for $n\ge n_0, k=1,\dots,k_n$. Since $Z_{nk,s}\in\mathcal{F}$ and takes imaginary values for $z=iy, y>0$, we see that $|Z_{nk,s}(i)|=\Im Z_{nk,s}(i)\ge 1$. We note from this that $Q(Z_{nk,s}(i))\le 2|Z_{nk,s}(i)|$ and, by (4.40), we easily conclude from (4.44) that

$$|Z_{nk,s}(i)|\le c(\mu^s),\qquad n\ge n_0,\ k=1,\dots,k_n.$$

Moreover,

$$\Im\big(F_{\mu_{n1}^s}(Z_{n1,s}(i))-i\big)=\Im\big(F_{\mu^{(n,s)}}(i)-i\big)\to\Im\big(F_{\mu^{(s)}}(i)-i\big),\qquad n\to\infty.$$

Therefore we obtain from (4.42) the relation

(4.45)
$$\sigma_{n1}^s(\mathbb{R})+\dots+\sigma_{nk_n}^s(\mathbb{R})\le c(\mu^s),\qquad n\to\infty.$$

Since $\mu_{nk}^s=\widehat{\mu}_{nk}\boxplus\overline{\widehat{\mu}}_{nk}$, we note, by definition of the free $\boxplus$-convolution (see Section 2), that there exist functions $W_{nk}(z)\in\mathcal{F}$ such that $F_{\mu_{nk}^s}(z)=F_{\widehat{\mu}_{nk}}(W_{nk}(z)), z\in\mathbb{C}^+$. Therefore we have $\Im F_{\mu_{nk}^s}(i)-1=\Im F_{\widehat{\mu}_{nk}}(W_{nk}(i))-1$. Rewrite this relation in the form

$$\sigma_{nk}^s(\mathbb{R})=\Im W_{nk}(i)-1$$

(4.46)
$$+\Im W_{nk}(i)\int_{\mathbb{R}}\frac{1+u^2}{(u-\Re W_{nk}(i))^2+(\Im W_{nk}(i))^2}\sigma_{nk}(du)$$

$$\ge \Im W_{nk}(i)\int_{\mathbb{R}}\frac{1+u^2}{(u-\Re W_{nk}(i))^2+(\Im W_{nk}(i))^2}\sigma_{nk}(du).$$

As in the proof of (4.10), we see that $F_{\mu^s_{nk}}(z)$ and $F_{\widehat{\mu}_{nk}}(z)$ tend to z as $n \to \infty$ uniformly in $k = 1, \dots, k_n$ and $|z - i| < 1/2$. Hence $W_{nk}(i) \to i$ as $n \to \infty$ uniformly in $k = 1, \dots, k_n$. Thus we obtain from (4.46) that, for sufficiently large $n \geq n_0$,

$$\sigma_{nk}(\mathbb{R}) \leq 2\sigma^s_{nk}(\mathbb{R}), \qquad k = 1, \dots, k_n. \tag{4.47}$$

Thus (4.45) and (4.47) imply the inequality

$$\sigma_{n1}(\mathbb{R}) + \dots + \sigma_{nk_n}(\mathbb{R}) \leq c(\mu^s), \qquad n \to \infty. \tag{4.48}$$

By (4.3), (4.5) with $y = 1$, and (4.8), we note that $\sigma_{nk}(\mathbb{R}) \geq \varepsilon_{nk}/2, k = 1, \dots, k_n$, for sufficiently large $n \geq n_0$ and we deduce from (4.48) the upper bound

$$\varepsilon_{n1} + \dots + \varepsilon_{nk_n} \leq c(\mu^s), \qquad n \to \infty. \tag{4.49}$$

Let us return to (4.13) and (4.14). Since $F_{\mu^{(n)}}(z) = F_{\widehat{\mu}_n}(z + a_n - b_n)$, where $b_n = \sum_{k=1}^{k_n} a_{nk}$, we see that $F_{\mu^{(n)}}(z) = F_{\widehat{\mu}_{nk}}(Z_{nk}(z + a_n - b_n)), z \in \mathbb{C}^+, k = 1, \dots, k_n$.

We shall show that $\{Z_{nk}(z + a_n - b_n)\}_{n=1}^{\infty}$ converges uniformly in $k = 1, \dots, k_n$ and on every compact set in $\mathbb{C}^+$ to the function $Z(z) := F_\mu(z) \in \mathcal{F}$.

Let S be a compact set in $\mathbb{C}^+$. Then there exist $\alpha_0 > 0$ and $\beta_0 > 0$ such that $S \subset \Gamma_{\alpha_0,\beta_0}$. It is clear that, for $n \geq n_1(\alpha_0, \beta_0)$, $F_{\mu^{(n)}}(\Gamma_{\alpha_0,\beta_0}) \subset \Gamma_{\alpha_1,\beta_1}$ with some $\alpha_1 > 0$ and $\beta_1 > 0$. It is well known (see Proposition 3.8 as well) that for every $\alpha > 0$, $\beta > 0$ there exists sufficiently large $n_1(\alpha, \beta)$ such that $F^{(-1)}_{\widehat{\mu}_{nk}}(z)$, $k = 1, \dots, k_n$, are defined on $\Gamma_{\alpha,\beta}$ for $n \geq n_1(\alpha, \beta)$. Therefore we can choose α, β such that $\Gamma_{\alpha_1,\beta_1} \subset \Gamma_{\alpha,\beta}$ and we will consider the functions $F^{(-1)}_{\widehat{\mu}_{nk}}(z), k = 1, \dots, k_n$, for $n \geq n_1(\alpha, \beta)$. Since, for sufficiently large $z \in \Gamma_{\alpha,\beta}$, the functions $F_{\widehat{\mu}_{nk}}(z)$ and $Z_{nk}(z + a_n - b_n)$ satisfy (3.2) with $b = 1$ and are univalent, and the functions $F^{(-1)}_{\widehat{\mu}_{nk}}(F_{\mu^{(n)}}(z)), k = 1, \dots, k_n$, are analytic in the domain $\Gamma_{\alpha_0,\beta_0}$ for $n \geq n_1(\alpha, \beta)$, the relation

$$Z_{nk}(z + a_n - b_n) = F^{(-1)}_{\widehat{\mu}_{nk}}\big(F_{\mu^{(n)}}(z)\big), \quad k = 1, \dots, k_n,\ z \in \Gamma_{\alpha_0,\beta_0},$$

holds for these n. Taking into account that $F^{(-1)}_{\widehat{\mu}_{nk}}(z)$ tend to z and $F_{\mu^{(n)}}(z)$ tend to $F_\mu(z)$ as $n \to \infty$ uniformly in $k = 1, \dots, k_n$ and uniformly on every compact set in $\mathbb{C}^+$, we obtain that $\{Z_{nk}(z + a_n - b_n)\}_{n=1}^{\infty}$ converges uniformly in $k = 1, \dots, k_n$ and on the compact set S to the function $Z(z) = F_\mu(z) \in \mathcal{F}$. Thus, the required assertion is proved.

Using relations (4.13) and (4.14) with $z + a_n - b_n$ instead of z and taking into account that the measures $\mu_{n1}, \dots, \mu_{nk_n}$ are infinitesimal and the upper bound (4.49) holds, we can repeat the arguments which we used for the proof of (4.32). We arrive at the following relation, for $z \in \mathbb{C}^+$:

$$\begin{aligned} &Z_{n1}(z + a_n - b_n) - (z + a_n - b_n) \\ &\quad = -f_{n2}\big(Z_{n1}(z + a_n - b_n)\big) - \dots - f_{nk_n}\big(Z_{n1}(z + a_n - b_n)\big) + r_n(z), \end{aligned} \tag{4.50}$$

where $r_n(z)$ is analytic in $\mathbb{C}^+$ and $r_n(z) \to 0$ on every compact set in $\mathbb{C}^+$. As above, $\{Z_{nk}(z + a_n - b_n)\}_{n=1}^{\infty}$ converges uniformly in $k = 1, \dots, k_n$ and on every compact set in $\mathbb{C}^+$ to $Z(z) \in \mathcal{F}$. Since, by (4.49), the sequence $\{\nu_n\}_{n=1}^{\infty}$ is tight in the vague topology, there exists a subsequence $\{n'\}$ such that there exists $\lim_{n' \to \infty} \nu_{n'}(\mathbb{R}) < \infty$ and $\{\nu_{n'}\}$ converges to some finite nonnegative measure ν in the vague topology. Now we conclude from (4.50) that $(a_{n'} - b_{n'} - \gamma_{n'}) \to a'$ as $n' \to \infty$, where $a' \in \mathbb{R}$, and the following relation holds:

$$Z(z) = z + a' + b'Z(z) + \int_{\mathbb{R}} \frac{1 + uZ(z)}{u - Z(z)} \nu(du), \qquad z \in \mathbb{C}^+, \tag{4.51}$$

with $b' = \lim_{n' \to \infty} \nu_{n'}(\mathbb{R}) - \nu(\mathbb{R}) \geq 0$. Recalling that $Z(z) \in \mathcal{F}$, we easily conclude from this relation that $b' = 0$. Indeed, it is not difficult to see that $Z(iy) - iy = o(y)$ and the integral in (4.51) for $z = iy$ is $o(y)$ as $y \to +\infty$. Comparing a behavior of all members in (4.51), we obtain the desired result.

We shall show that $\{\nu_n\}$ converges to the measure ν in the vague topology. Assume to the contrary that there exists a subsequence $\{n''\}$ such that there exists $\lim_{n'' \to \infty} \nu_{n''}(\mathbb{R}) < \infty$ and $\{\nu_{n''}\}$ converges in the vague topology to some finite measure $\nu'' \not\equiv \nu$. Then $(a_{n''} - b_{n''} - \gamma_{n''}) \to a''$ as $n'' \to \infty$, and (4.51) holds with a'' replacing a' and ν'' replacing ν. Comparing relations (4.51), we deduce the relation

$$a' + \int_{\mathbb{R}} \frac{1 + uz}{u - z} \nu(du) = a'' + \int_{\mathbb{R}} \frac{1 + uz}{u - z} \nu''(du), \qquad z \in \mathbb{C}^+.$$

Applying the Stieltjes–Perron inversion formula (3.5) (see Section 3), we get that $\nu = \nu''$ and then $a' = a''$, a contradiction. Since, as above, $\lim_{n \to \infty} \nu_n(\mathbb{R}) = \nu(\mathbb{R})$, we finally conclude that $\{\nu_n\}$ converges to the measure ν weakly. In addition, $a_n - b_n - \gamma_n$ tends to some real constant as $n \to \infty$. It remains to note that, by the relation $Z(z) = F_\mu(z)$, $z \in \mathbb{C}^+$, we see from (4.51) that the limit measure μ is infinitely divisible with parameters (a', ν).

This proves the necessity of the assumptions of Theorem 2.1(b) and thus Theorem 2.1. □

PROOF OF COROLLARY 2.3. In order to prove Corollary 2.3 using Theorem 2.2 we need to show that if $\mu_n^{k_n *}$ converges weakly to some $\mu^* \in \mathcal{M}$ or $\mu_n^{k_n \boxplus}$ converges to some $\mu^{\boxplus} \in \mathcal{M}$, then μ_n are infinitesimal. The first assertion is a well-known fact (see [25]). It remains to prove the second assertion only. By Proposition 3.5 and Lemma 2.2 from [15], $k_n \phi_{\mu_n}(z)$ converges uniformly on compact subsets of $\Gamma_{\alpha,\beta}$, with some $\alpha > 0$, $\beta > 0$, to the function $\phi_{\mu^{\boxplus}}(z)$ and $k_n \phi_{\mu_n}(iy) = o(y)$ uniformly in n as $y \to +\infty$. Hence $\phi_{\mu_n}(z) \to 0$ uniformly on compact subsets of $\Gamma_{\alpha,\beta}$, as $n \to \infty$, and $\phi_{\mu_n}(iy) = o(y)$ uniformly in n as $y \to +\infty$. By Proposition 3.5, μ_n converges weakly to δ_0 as $n \to \infty$. Therefore the p-measures μ_n are infinitesimal. □

5. The class $\mathcal{L}_{\boxplus}$ of infinitely divisible limits for free sums. In this section we shall prove Theorem 2.10.

Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures in $\mathcal{M}$ and let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, $b_n > 0$, be sequences of real numbers. In Section 5 we denote by $\mu_{nk} : n \geq 1, 1 \leq k \leq n$, the p-measures such that $\mu_{nk}(S) := \mu_k(b_n S)$ for every Borel set $S \in \mathbb{R}$. Consider the sequence of p-measures $\{\mu^{(n)} := \delta_{-a_n} \boxplus \mu_{n1} \boxplus \cdots \boxplus \mu_{nn}\}_{n=1}^{\infty}$.

PROOF OF THEOREM 2.10. *Sufficiency*. First we show that $\phi_\mu(z)$ admits an analytic continuation on $\mathbb{C}^+$. Indeed, the function $\phi_\mu(z)$ is regular on some domain $\Gamma_{\alpha,\beta}$ (see Section 2). Let us assume that there are singular points of $\phi_\mu(z)$ on the boundary of this domain. Let z_0 be one of such points with the largest modulus. By the definition of $\phi_\mu(z)$, it is easy to see that $|z_0| < \infty$. By the assumption and by Voiculescu's relation (2.4), we have

$$\phi_\mu(z) = \gamma\phi_\mu(z/\gamma) + \phi_\gamma(z), \qquad z \in \Gamma_{\alpha,\beta}, \tag{5.1}$$

for $0 < \gamma < 1$. Here $\phi_\gamma(z) := \phi_{\mu_\gamma}(z)$ and is defined on $\Gamma_{\alpha,\beta}$ as well (see Lemma 4.4 in [4]). Hence

$$\begin{aligned}\phi_\gamma(z) &= (1-\gamma)\phi_\mu(z) + \gamma\big(\phi_\mu(z) - \phi_\mu(z/\gamma)\big)\\ &= (1-\gamma)\phi_\mu(z) + \gamma\int_{z/\gamma}^{z}\phi'_\mu(\zeta)\,d\zeta, \qquad z \in \Gamma_{\alpha,\beta}.\end{aligned}$$

By Proposition 3.7 and the property $|\phi_\mu(z)| = o(|z|)$ as $z \to \infty$, $z \in \Gamma_{\alpha,\beta}$, we have the relation $\phi_\gamma(z) \to 0$ for $z \in \Gamma_{\alpha,\beta}$ and $\phi_\gamma(iy) = o(y)$, $y \to \infty$, uniformly in $\gamma \to 1$. If $\gamma \to 1$, then by Proposition 3.5, $\mu_\gamma \to \delta_0$ weakly and, by Proposition 3.8, $\phi_{\mu_\gamma}(z)$ is regular in the domain $\Gamma_{2\alpha,\beta/2}$ for γ close to 1. The functions $\phi_\mu(z/\gamma)$ and $\phi_\gamma(z)$ are regular on $\overline{\Gamma}_{\alpha,\Im z_0}$, therefore $\phi_\mu(z)$ is regular at the point z_0, a contradiction. Hence our assertion holds. Note that the function $\phi_\gamma(z)$ admits an analytic continuation on $\mathbb{C}^+$ for every $\gamma \in (0,1)$ as well and the relation (5.1) holds for all $\gamma \in (0,1)$ and $z \in \mathbb{C}^+$ for such functions. We again denote these functions, defined on $\mathbb{C}^+$, by $\phi_\mu(z)$ and $\phi_\gamma(z)$.

Consider the p-measures μ_k, $k = 1, \dots,$ determined via

$$\phi_{\mu_k}(z) := \frac{1}{\pi_k}\phi_{\gamma_k}(\pi_k z) = \frac{1}{\pi_k}\phi_\mu(\pi_k z) - \frac{1}{\pi_{k-1}}\phi_\mu(\pi_{k-1}z), \qquad z \in \mathbb{C}^+,$$

where $\pi_k = \prod_{l=1}^{k}\gamma_l$ and $\gamma_l = 1 - 1/(l+1)$, $\pi_0 := 1$.

Voiculescu's transform of the p-measure $\mu^{(n)} := \mu_{n1} \boxplus \ldots \boxplus \mu_{nn}$ with μ_{nk}, $k = 1, \dots, n$, defined at the beginning of this section using $b_n := 1/\pi_n$, has the form

$$\begin{aligned}\phi_{\mu^{(n)}}(z) &= \sum_{k=1}^{n}\left(\frac{\pi_n}{\pi_k}\phi_\mu\left(\frac{\pi_k}{\pi_n}z\right) - \frac{\pi_n}{\pi_{k-1}}\phi_\mu\left(\frac{\pi_{k-1}}{\pi_n}z\right)\right)\\ &= \phi_\mu(z) - \pi_n\phi_\mu\left(\frac{z}{\pi_n}\right), \qquad z \in \mathbb{C}^+.\end{aligned} \tag{5.2}$$

From (5.2) and Proposition 3.5 it follows that $\mu^{(n)} \to \mu$ weakly as $n \to \infty$. It is not difficult to verify that the functions $\phi_{\mu_{nk}}(z)$ converge to zero on every compact set of $\mathbb{C}^+$ uniformly in k, $1 \le k \le n$. This implies that $|F_{\mu_{nk}}(i) - i| \to 0$ as $n \to \infty$ uniformly in $k = 1, \dots, n$, and hence $|G_{\mu_{nk}}(i) + i| \to 0$ as $n \to \infty$ uniformly in $k = 1, \dots, n$. Finally note that the p-measures $\mu_{n1}, \dots, \mu_{nn}$ are infinitesimal. This follows directly from formula (4.5) for the measures μ_{nk}, $k = 1, \dots, n$.

Necessity. Let $\mu \in \mathcal{L}_{\boxplus}$. This means that there exists a sequence of p-measures $\{\mu_n\}$ such that for some suitably chosen sequences of constants $\{a_n\}$ and $\{b_n\}$, $b_n > 0$, the sequence of the measures $\{\mu^{(n)} = \delta_{-a_n} \boxplus \mu_{n1} \boxplus \cdots \boxplus \mu_{nn}\}$ converges weakly to a limit measure μ and that the measures $\mu_{nk}, n \ge 1, k = 1, \dots, n$, are infinitesimal. By Theorem 2.1, the measure μ is $\boxplus$-infinitely divisible and $\phi_\mu = (\alpha, \nu)$.

Consider the sequence of p-measures $\{\delta_{-a_n} * \mu_{n1} * \cdots * \mu_{nn}\}$. By Theorem 2.2, this sequence converges weakly to the p-measure $\mu^* \in \mathcal{L}_*$ such that $f_{\mu^*} = \{\alpha, \nu\}$. Using the classical arguments on pages 323–324 in [25], Theorem 2.2 and Proposition 3.6, we easily conclude that, for every $\gamma \in (0,1)$, $\mu = D_\gamma \mu \boxplus \mu_\gamma$, where $\mu_\gamma \in \mathcal{M}$ and is $\boxplus$-infinitely divisible. Hence $\phi_{\mu_\gamma} = (\alpha_\gamma, \nu_\gamma)$ with some $\alpha_\gamma \in \mathbb{R}$ and a finite nonnegative measure ν_γ.

Hence, by these arguments, we have $\mu^* = D_\gamma \mu^* * \mu_\gamma^*$ for every $\gamma \in (0,1)$, where $\mu_\gamma^* \in \mathcal{M}$ and μ_γ^* is $*$-infinitely divisible. Moreover $f_{\mu_\gamma^*} = \{\alpha_\gamma, \nu_\gamma\}$.

Hence, the theorem is proved. □

The assertion of Remark 2.11 follows from the arguments above.

6. Estimates of convergence in the free central limit theorem. In this section we prove Theorem 2.4, Proposition 2.5, Theorem 2.6 and Theorem 2.7. In the sequel we denote $c_1, c_2, \dots$ explicit positive absolute constants.

PROOF OF THEOREM 2.4. Denote $\mu^{(n)} := \mu_n^{n\boxplus}$. By Proposition 3.4, $G_{\mu^{(n)}}(z) = 1/F_{\mu^{(n)}}(z)$, $z \in \mathbb{C}^+$, where $F_{\mu^{(n)}}(z) := F_\mu(Z(\sqrt{n}z))/\sqrt{n}$. In this formula $Z(z) \in \mathcal{F}$ is the solution of (3.8). Consider the functions $S(z) := \frac{1}{2}(z + \sqrt{z^2 - 4})$ and $S_n(z) := Z(\sqrt{n}z)/\sqrt{n}$ for $z \in \mathbb{C}^+$. Note that $1/S(z) = G_{\mu_w}(z)$, where w denotes Wigner semicircle measure. Since $S_n \in \mathcal{F}$, we see by Remark 3.1 that there exists a p-measure $\nu^{(n)}$ such that $1/S_n(z) = G_{\nu^{(n)}}(z)$.

We obtain the estimate (2.9) for $n \ge n_2$, where $n_2 := [c_1(|m_3(\mu)|^2 + m_4(\mu))]$ with a sufficiently large positive absolute constant c_1. For $n \le n_2$, (2.9) holds obviously. Using (2.1), we may write

$$Z(z)G_\mu(Z(z)) = 1 + \frac{1}{Z^2(z)} + \frac{m_3(\mu)}{Z^3(z)} + \frac{1}{Z^3(z)} \int_{\mathbb{R}} \frac{u^4 \mu(du)}{Z(z) - u}, \qquad z \in \mathbb{C}^+. \tag{6.1}$$

Equation (3.8) may be rewritten as

$$G_\mu(Z(z))\big(Z(z) - z\big) = (n-1)\big(1 - Z(z)G_\mu(Z(z))\big), \qquad z \in \mathbb{C}^+. \tag{6.2}$$

By (6.1) and the definition of $S_n(z)$, (6.2) may be reformulated as

$$\begin{aligned}
&\Big(1+\frac{1}{Z^2(\sqrt{n}z)}+\frac{m_3(\mu)}{Z^3(\sqrt{n}z)}+\frac{1}{Z^3(\sqrt{n}z)}\int_{\mathbb{R}}\frac{u^4\mu(du)}{Z(\sqrt{n}z)-u}\Big)(S_n(z)-z)\\
&\quad=-\frac{n-1}{n}\Big(\frac{1}{S_n(z)}+\frac{m_3(\mu)}{S_n(z)Z(\sqrt{n}z)}\\
&\qquad+\frac{1}{S_n(z)Z(\sqrt{n}z)}\int_{\mathbb{R}}\frac{u^4\mu(du)}{Z(\sqrt{n}z)-u}\Big)
\end{aligned}\tag{6.3}$$

for $z\in\mathbb{C}^+$. Rewrite (6.3) in the form

$$(1+r_{n1}(z))(S_n(z)-z)=-\Big(1-\frac{1}{n}\Big)\frac{1}{S_n(z)}(1+r_{n2}(z)),\tag{6.4}$$

where $r_{n1}(z)$ and $r_{n2}(z)$ are analytic functions on $\mathbb{C}^+$ which, by the inequality $\Im Z(\sqrt{n}z)\ge\sqrt{n}\Im z$, $z\in\mathbb{C}^+$ [compare with (3.4)], admit the estimates

$$\begin{aligned}
|r_{n1}(z)|&\le\frac{1}{(\Im z\sqrt{n})^2}+\frac{|m_3(\mu)|}{(\Im z\sqrt{n})^3}+\frac{m_4(\mu)}{(\Im z\sqrt{n})^4},\\
|r_{n2}(z)|&\le\frac{|m_3(\mu)|}{\Im z\sqrt{n}}+\frac{m_4(\mu)}{(\Im z\sqrt{n})^2},\qquad z\in\mathbb{C}^+.
\end{aligned}\tag{6.5}$$

Introduce for every $\alpha>0$, $\mathbb{C}^+_\alpha:=\{z\in\mathbb{C}:\Im z>\alpha\}$ and $D_\alpha:=\{z\in\mathbb{C}:\alpha\le\Im z\le 1, |\Re z|\le 4\}$.

By (6.5), $|r_{n1}(z)|+|r_{n2}(z)|\le 1/10$ for $z\in\mathbb{C}^+_{a/2}$, where $a=:c_2(|m_3(\mu)|+m_4^{1/2}(\mu))/\sqrt{n}$ and $c_2>0$ is a sufficiently large absolute constant. Therefore we conclude from (6.4) that

$$10^{-1}\le|S_n(z)|\le 10,\qquad z\in D_a.\tag{6.6}$$

From (6.4) we see that the function $S_n(z)$ satisfies the approximate functional equation

$$S_n(z)-z=-\frac{1}{S_n(z)}+\frac{r_{n3}(z)}{S_n(z)},\tag{6.7}$$

for $z\in\mathbb{C}^+_{a/2}$, where

$$r_{n3}(z):=1-\Big(1-\frac{1}{n}\Big)\frac{1+r_{n2}(z)}{1+r_{n1}(z)}.$$

Here $r_{n3}(z)$ is an analytic function on $z\in\mathbb{C}^+_{a/2}$ which is bounded as follows:

$$|r_{n3}(z)|\le 2\Big(\frac{1}{n}+|r_{n1}(z)|+|r_{n2}(z)|\Big),\qquad z\in\mathbb{C}^+_{a/2}.\tag{6.8}$$

Recalling the definition of the functions $r_{n1}(z)$ and $r_{n2}(z)$, we obtain with the help of (6.6) and the inequality $|S_n(z)| \ge 1$, $z \in \overline{\mathbb{C}}_1^+$,

$$
\begin{aligned}
|r_{n1}(z)| &\le \frac{1}{(\sqrt{n}|S_n(z)|)^2} + \frac{|m_3(\mu)|}{(\sqrt{n}|S_n(z)|)^3} + \frac{m_4(\mu)}{(\sqrt{n}|S_n(z)|)^3\sqrt{n}\Im z} \\
&\le 10^3\Big(\frac{1}{n} + \frac{|m_3(\mu)|}{n^{3/2}} + \frac{m_4(\mu)}{n^2\Im z}\Big)
\end{aligned} \tag{6.9}
$$

and

$$
|r_{n2}(z)| \le \frac{|m_3(\mu)|}{\sqrt{n}|S_n(z)|} + \frac{m_4(\mu)}{n|S_n(z)|\Im z} \le 10\Big(\frac{|m_3(\mu)|}{\sqrt{n}} + \frac{m_4(\mu)}{n\Im z}\Big)
$$

for $z \in D_a \cup \overline{\mathbb{C}}_1^+$. Applying these estimates to (6.8), we finally have

$$
|r_{n3}(z)| \le 3 \cdot 10^3\Big(\frac{|m_3(\mu)|}{\sqrt{n}} + \frac{m_4(\mu)}{n\Im z} + \frac{1}{n}\Big), \qquad z \in D_a \cup \overline{\mathbb{C}}_1^+. \tag{6.10}
$$

Solving (6.7), we see that

$$
S_n(z) = \tfrac{1}{2}\big(z \pm \sqrt{\rho_n(z)}\,\big), \qquad z \in \mathbb{C}_{a/2}^+,
$$

where $\rho_n(z) := z^2 - 4 + 4r_{n3}(z)$. Note that the function $\rho_n(z)$ is nonzero on the half-plane $\mathbb{C}_{a/2}^+$. Indeed, let $\rho_n(w) = 0$ for some $w \in \mathbb{C}_{a/2}^+$. Then, by (6.7), $S_n^2(w) - wS_n(w) = -w^2/4$ and we have $S_n(w) = w/2$. But the function $S_n(z)$ satisfies the inequality $\Im S_n(z) \ge \Im z$, $z \in \mathbb{C}^+$, a contradiction. We define the function $\sqrt{\rho_n(z)}$ on $\mathbb{C}_{a/2}^+$, taking the branch of $\sqrt{\rho_n(z)}$ such that $\sqrt{\rho_n(i)} \in \mathbb{C}^+$. Since $S_n(z) \in \mathcal{N}$, we see that $S_n(z) = \frac{1}{2}(z + \sqrt{\rho_n(z)})$ for $z \in \mathbb{C}_{a/2}^+$.

For $z \in \mathbb{C}_{a/2}^+$, using the previous formula for $S_n(z)$ and $S(z) = \frac{1}{2}(z + \sqrt{z^2-4})$, we write

$$
\begin{aligned}
\frac{1}{S_n(z)} - \frac{1}{S(z)} &= \frac{S(z) - S_n(z)}{S(z)S_n(z)} \\
&= \frac{1}{S(z)S_n(z)} \cdot \frac{2r_{n3}(z)}{\sqrt{z^2-4} + \sqrt{z^2-4+4r_{n3}(z)}}.
\end{aligned} \tag{6.11}
$$

Since, for $z \in \mathbb{C}$, $0 < \Im z \le 1$, $|z^2-4| \ge m(z) := \max\{\Im z, ((\Re z)^2 - 5)_+\}$, where for $x \in \mathbb{R}$, $(x)_+ := \max\{0, x\}$, we obtain from (6.10) the following inequality:

$$
\Big|\frac{r_{n3}(z)}{z^2-4}\Big| \le \frac{3\cdot 10^3}{m(z)}\Big(\frac{|m_3(\mu)|}{\sqrt{n}} + \frac{2m_4(\mu)}{n\Im z}\Big) \le \frac{1}{10}, \qquad z \in D_a \cup \{z \in \mathbb{C} : \Im z = 1\}. \tag{6.12}
$$

Hence we get, for $z \in D_a$ or for $\Im z = 1$,

$$
\begin{aligned}
&\big|\sqrt{z^2-4} + \sqrt{z^2-4+4r_{n3}(z)}\big| \\
&\quad = \sqrt{|z^2-4|}\big|1 + \sqrt{1 + 4r_{n3}(z)/(z^2-4)}\big| \ge \sqrt{|z^2-4|}.
\end{aligned}
$$

Using this estimate we deduce from (6.10) and (6.11), for $z \in (D_a \cup \{z \in \mathbb{C} : \Im z = 1\})$,

$$\begin{aligned}\left|\frac{1}{S_n(z)} - \frac{1}{S(z)}\right| &\le 2\frac{|r_{n3}(z)|}{|\sqrt{z^2-4}|}\frac{1}{|S(z)||S_n(z)|} \\ &\le \frac{6\cdot 10^3}{\sqrt{m(z)}}\left(\frac{|m_3(\mu)|}{\sqrt{n}} + \frac{2m_4(\mu)}{n\Im z}\right)\frac{1}{|S(z)||S_n(z)|}.\end{aligned} \tag{6.13}$$

Recall that $1/S(z) = G_{\mu_w}(z)$ and $1/S_n(z) = G_{\nu^{(n)}}(z)$, where $\nu^{(n)}$ is a p-measure.

Since, for $u \in \mathbb{R}$, $m(u+i) = \max\{1, (u^2-5)_+\}$, $|S_n(u+i)| \ge 1$ and $|S(u+i)| \ge \frac{1}{2}\sqrt{1+((u-4)_+)^2}$, we conclude, using (6.13),

$$\begin{aligned}&\int_{\mathbb{R}} \left|G_{\mu_w}(u+i) - G_{\nu^{(n)}}(u+i)\right| du \\ &\qquad \le c\left(\frac{|m_3(\mu)|}{\sqrt{n}} + \frac{m_4(\mu)}{n}\right)\int_{\mathbb{R}} \frac{du}{1+u^2} \\ &\qquad \le \frac{c}{\sqrt{n}}\left(|m_3(\mu)| + \frac{m_4(\mu)}{\sqrt{n}}\right) \\ &\qquad \le \frac{c}{\sqrt{n}}\left(|m_3(\mu)| + (m_4(\mu))^{1/2}\right), \qquad n \ge n_2.\end{aligned} \tag{6.14}$$

Since, for $z \in D_a$, $\sqrt{m(z)} \ge \sqrt{\Im z}$, $|S_n(z)| \ge 1/10$ and $|S(z)| \ge 1/10$, we obtain from (6.13), for $x \in [-2, 2]$,

$$\begin{aligned}&\int_a^1 \left|G_{\mu_w}(x+iu) - G_{\nu^{(n)}}(x+iu)\right| du \\ &\qquad \le c\int_a^1 \left(\frac{|m_3(\mu)|}{\sqrt{nu}} + \frac{m_4(\mu)}{nu^{3/2}}\right) du \\ &\qquad \le \frac{c}{\sqrt{n}}\left(|m_3(\mu)| + \frac{m_4(\mu)}{\sqrt{na}}\right) \\ &\qquad \le \frac{c}{\sqrt{n}}\left(|m_3(\mu)| + (m_4(\mu))^{1/2}\right), \qquad n \ge n_2.\end{aligned} \tag{6.15}$$

Now we consider the representation

$$G_{\mu^{(n)}}(z) - G_{\nu^{(n)}}(z) = \frac{r_{n1}(z)}{S_n(z)}, \qquad z \in \mathbb{C}^+. \tag{6.16}$$

Relation (6.13) leads to the following estimate, for $z \in \mathbb{C}$ with $\Im z = 1$:

$$\frac{1}{2}\left(1 + (|\Re z| - 4)_+\right) \le |S(z)| - \frac{6\cdot 10^3}{\sqrt{nm(z)}}\left(|m_3(\mu)| + \frac{2m_4(\mu)}{\sqrt{n}}\right) \le |S_n(z)| \tag{6.17}$$

$$\le |S(z)| + \frac{6\cdot 10^3}{\sqrt{nm(z)}}\Big(|m_3(\mu)| + \frac{2m_4(\mu)}{\sqrt{n}}\Big)$$
$$\le 2\big(1+(|\Re z|-4)_+\big).$$

Using (6.9), (6.16) and (6.17), we easily obtain the following inequality:

$$\int_{\mathbb{R}} \big|G_{\mu^{(n)}}(u+i) - G_{\nu^{(n)}}(u+i)\big|\,du$$
$$\le \frac{c}{n}\Big(1+\frac{|m_3(\mu)|}{\sqrt{n}}+\frac{m_4(\mu)}{n}\Big)\int_{\mathbb{R}}\frac{du}{1+u^2} \tag{6.18}$$
$$\le \frac{c}{n}\Big(\frac{|m_3(\mu)|}{\sqrt{n}}+\frac{m_4(\mu)}{n}\Big) \le \frac{c}{n}, \qquad n\ge n_2,$$

and for $x\in[-2,2]$, using (6.9) and the estimate $|S_n(z)|\ge 1/10$, $z\in D_a$, we deduce

$$\int_a^1 \big|G_{\mu^{(n)}}(x+iu) - G_{\nu^{(n)}}(x+iu)\big|\,du$$
$$\le \frac{c}{n}\Big(1+\frac{|m_3(\mu)|}{\sqrt{n}}+\frac{m_4(\mu)|\log a|}{n}\Big) \tag{6.19}$$
$$\le \frac{c}{n}\big(|m_3(\mu)| + (m_4(\mu))^{1/2}\big), \qquad n\ge n_2.$$

In order to prove the upper bound of $\Delta(\mu^{(n)},\mu_w)$ for $n\ge n_2$, we apply Lemma 3.2 with $v=a$. Since $m_4(\mu)<\infty$, it is well known that $m_2(\mu^{(n)})<\infty$ and the assumption (3.6) obviously holds. Therefore Lemma 3.2, (6.14), (6.15), (6.18) and (6.19) together imply the estimate (2.9).

Hence, Theorem 2.4 is proved. □

PROOF OF PROPOSITION 2.5. Let μ be a measure satisfying the assumptions of Proposition 2.5. The corresponding transforms are given by

$$G_\mu(z) = \frac{q}{z+\sqrt{p/q}} + \frac{p}{z-\sqrt{q/p}} \quad\text{and}\quad F_{\mu_n}(z) = z - \frac{1}{n}\cdot\frac{1}{z+\tilde{c}/\sqrt{n}},$$

where $\tilde{c} := (p-q)/\sqrt{pq}$. With the help of simple calculations we find the explicit form of the functions $\phi_{\mu_n}(z)$ and thus of $\phi_{\mu_n^{n\boxplus}}(z) = n\phi_{\mu_n}(z)$. From this relation we obtain the explicit form of $G_{\mu_n^{n\boxplus}}(z)$ and, using the Stieltjes–Perron inversion formula (3.5), we have

$$\mu_n^{n\boxplus}((-\infty,u)) = \frac{1}{2\pi}\int_{x_1}^u p(x)\,dx, \qquad x_1<u<x_2 \tag{6.20}$$

where $p(x) := \dfrac{\sqrt{(x-x_1)(x_2-x)}}{1-x(x/n+\tilde{c}/\sqrt{n})}$.

In this formula,

$$x_1 := -\frac{\tilde{c}}{\sqrt{n}} - 2\sqrt{1-\frac{1}{n}}, \qquad x_2 := -\frac{\tilde{c}}{\sqrt{n}} + 2\sqrt{1-\frac{1}{n}}.$$

It is not difficult to verify that the assertion of the proposition follows from (6.20).
For details of these calculations, see [21]. □

PROOF OF THEOREM 2.6. By Proposition 3.3, $G_{\mu^{(n)}}(z) = 1/F_{\mu^{(n)}}(z), z \in \mathbb{C}^+$, where $F_{\mu^{(n)}}(z) := F_{\mu_1}(Z_1(B_n z))/B_n = \cdots = F_{\mu_n}(Z_n(B_n z))/B_n$. In this formula $Z_j(z)$, $j = 1, \dots, n$, are in the class $\mathcal{F}$ and are the solutions of the functional equations (3.7). Without loss of generality, we assume that $\min_{j=1,\dots,n} m_2(\mu_j) \geq 1$ and $\min_{j=1,\dots,n} m_2(\mu_j) = m_2(\mu_1)$. Denote $S_n(z) := Z_1(B_n z)/B_n$ and let, as in the proof Theorem 2.4, $S(z) := \frac{1}{2}(z + \sqrt{z^2-4})$. Note that $1/S_n(z) = G_{\nu^{(n)}}$ for some p-measure $\nu^{(n)}$.

We prove inequality (2.10) for $L_n \leq c$ with a sufficiently small positive absolute constant c. For $L_n \geq c$ (2.10) holds obviously. From (3.7) we have the relation

$$\begin{aligned} Z_1(z) - z = F_{\mu_2}(Z_2(z)) - Z_2(z) + F_{\mu_3}(Z_3(z)) - Z_3(z) + \cdots \\ + F_{\mu_n}(Z_n(z)) - Z_n(z) \end{aligned} \tag{6.21}$$

and

$$F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)) = \cdots = F_{\mu_n}(Z_n(z)), \qquad z \in \mathbb{C}^+. \tag{6.22}$$

By (6.1), we note that

$$\begin{aligned} F_{\mu_j}(Z_j(z)) - Z_j(z) &= \frac{1 - Z_j(z)G_{\mu_j}(Z_j(z))}{Z_j(z)G_{\mu_j}(Z_j(z))} Z_j(z) \\ &= -\frac{r_{n,j}(z)}{1 + r_{n,j}(z)} Z_j(z), \qquad z \in \mathbb{C}^+, \end{aligned} \tag{6.23}$$

where

$$r_{n,j}(z) := \frac{1}{Z_j(z)} \int_{\mathbb{R}} \frac{u^2 \mu_j(du)}{Z_j(z) - u} = \frac{m_2(\mu_j)}{Z_j^2(z)} + \frac{1}{Z_j^2(z)} \int_{\mathbb{R}} \frac{u^3 \mu_j(du)}{Z_j(z) - u}. \tag{6.24}$$

In addition, by (6.22), we have

$$\frac{Z_1(z)}{Z_j(z)} = \frac{Z_1(z)G_{\mu_1}(Z_1(z))}{Z_j(z)G_{\mu_j}(Z_j(z))} = \frac{1 + r_{n,1}(z)}{1 + r_{n,j}(z)}, \qquad z \in \mathbb{C}^+. \tag{6.25}$$

Since $\Im Z_j(B_n z) \geq B_n \Im z$, we obtain from (6.24) that $|r_{n,j}(B_n z)| \leq 1/10$, $j = 1, \dots, n$, for $\Im z \geq c_3 M_n$, where $M_n := (\max_{j=1,\dots,n} m_2(\mu_j))^{1/2}/B_n$ and c_3 is a sufficiently large absolute constant. Moreover, we deduce from (6.23) and (6.24) the following estimates:

$$\begin{aligned}\Big|F_{\mu_j}(Z_j(B_nz))-Z_j(B_nz)+\frac{m_2(\mu_j)}{Z_j(B_nz)}\Big|\\ \le \frac{\beta_3(\mu_j)}{|Z_j(B_nz)|B_n\Im z}\\ +\frac{2m_2(\mu_j)}{|Z_j(B_nz)|^2B_n\Im z}\Big(m_2(\mu_j)+\frac{\beta_3(\mu_j)}{B_n\Im z}\Big)\end{aligned}\tag{6.26}$$

and

$$\begin{aligned}&\Big|F_{\mu_j}(Z_j(B_nz))-Z_j(B_nz)+\frac{m_2(\mu_j)}{Z_j(B_nz)}\Big|\\ &\le \frac{\beta_3(\mu_j)}{|Z_j(B_nz)|B_n\Im z}+\frac{2}{|Z_j(B_nz)|^3}\Big(m_2(\mu_j)+\frac{\beta_3(\mu_j)}{B_n\Im z}\Big)^2\end{aligned}\tag{6.27}$$

for $\Im z\ge a_1:=c_3M_n$. In the same way we obtain from (6.25) the following inequalities:

$$\Big|\frac{Z_1(B_nz)}{Z_j(B_nz)}-1\Big|\le\frac{2}{B_n\Im z}\Big(\frac{m_2(\mu_1)}{|Z_1(B_nz)|}+\frac{m_2(\mu_j)}{|Z_j(B_nz)|}\Big)\le\frac{1}{10}\tag{6.28}$$

for $\Im z\ge a_1$ and $j=2,\dots,n$. Using (6.28) we conclude that, for $\Im z\ge a_1$,

$$\begin{aligned}&\Big|\frac{m_2(\mu_2)}{Z_2(B_nz)}+\dots+\frac{m_2(\mu_n)}{Z_n(B_nz)}-\frac{B_n^2-m_2(\mu_1)}{Z_1(B_nz)}\Big|\\ &\le\sum_{j=2}^n\frac{2m_2(\mu_j)}{|Z_1(B_nz)|B_n\Im z}\Big(\frac{m_2(\mu_1)}{|Z_1(B_nz)|}+\frac{m_2(\mu_j)}{|Z_j(B_nz)|}\Big)\\ &\le\frac{8}{|Z_1(B_nz)|^2B_n\Im z}\sum_{j=2}^n m_2^2(\mu_j).\end{aligned}\tag{6.29}$$

In view of (6.26), (6.28) and (6.29), (6.21) yields for $\Im z\ge a_1$ the functional equation

$$S_n(z)-z=-\frac{1-\widehat{r}_n(z)}{S_n(z)},\tag{6.30}$$

where $\widehat{r}_n(z)$ is an analytic function on $\mathbb{C}^+_{a_1}$ which admits the upper bound

$$\begin{aligned}|\widehat{r}_n(z)|\le\frac{2}{B_n^3\Im z}\sum_{j=1}^n\beta_3(\mu_j)+\frac{12}{(B_n^2\Im z)^2}\sum_{j=1}^n m_2^2(\mu_j)\\ +\frac{4}{B_n^5(\Im z)^3}\sum_{j=1}^n m_2(\mu_j)\beta_3(\mu_j)+\frac{m_2(\mu_1)}{B_n^2}\end{aligned}$$

for $\Im z \ge a_1$. Using the well-known inequalities

$$\frac{1}{B_n^4}\sum_{j=1}^n m_2^2(\mu_j) \le \min\{M_n^2, L_n^{4/3}\},$$

(6.31)

$$\frac{1}{B_n^5}\sum_{j=1}^n m_2(\mu_j)\beta_3(\mu_j) \le M_n^2 L_n, \qquad L_n \ge \frac{1}{\sqrt{n}},$$

we finally arrive at

$$|\widehat{r}_n(z)| \le \frac{2L_n}{\Im z} + \frac{12\min\{M_n^2, L_n^{4/3}\}}{(\Im z)^2} + \frac{4M_n^2 L_n}{(\Im z)^3} + L_n^2 \le \frac{20}{c_4} < \frac{1}{10} \tag{6.32}$$

for $\Im z \ge a_2 := c_4(L_n + \min\{M_n, L_n^{2/3}\} + M_n^{2/3}L_n^{1/3})$, where $c_4 > c_3$ is a sufficiently large absolute constant. It follows from (6.30) and (6.32) that

$$10^{-1} \le |S_n(z)| \le 10, \qquad z \in D_{a_2}, \tag{6.33}$$

where the closed domain D_{a_2} is defined in the proof of Theorem 2.4. Using this inequality and (6.27)–(6.29), we may improve the estimate (6.32) for $z \in D_{a_2}$. Using as well (6.31) and the well-known estimate

$$\frac{1}{B_n^6}\sum_{j=1}^n \beta_3^2(\mu_j) \le L_n^2,$$

we obtain the following bound:

$$\begin{aligned} |\widehat{r}_n(z)| &\le \frac{2}{B_n^3 \Im z}\sum_{j=1}^n \beta_3(\mu_j) \\ &\quad + \frac{10^4}{B_n^4 \Im z}\sum_{j=1}^n m_2^2(\mu_j) + \frac{10^4}{B_n^6(\Im z)^2}\sum_{j=1}^n \beta_3^2(\mu_j) + \frac{m_2(\mu_1)}{B_n^2} \\ &\le 5\frac{L_n}{\Im z}, \qquad z \in D_{a_2}. \end{aligned} \tag{6.34}$$

By (6.32), this estimate holds for $z \in \mathbb{C}^+$ such that $\Im z = 1$.

Now we repeat the arguments of the proof of Theorem 2.4. Solving (6.30) we see that

$$S_n(z) = \tfrac{1}{2}\big(z + \sqrt{\widehat{\rho}_n(z)}\big), \qquad \Im z \ge a_2, \tag{6.35}$$

where $\widehat{\rho}_n(z) := z^2 - 4 + 4\widehat{r}_n(z)$.

Write the formula, for $z \in \mathbb{C}^+_{a_2}$,

$$\begin{aligned} \frac{1}{S_n(z)} - \frac{1}{S(z)} &= \frac{S(z) - S_n(z)}{S(z)S_n(z)} \\ &= \frac{1}{S(z)S_n(z)} \cdot \frac{\widehat{r}_n(z)}{\sqrt{z^2-4} + \sqrt{z^2 - 4 + 4\widehat{r}_n(z)}}. \end{aligned} \tag{6.36}$$

Let $a_3 := 3c_4 L_n^{1/2}$. Note that the well-known inequality $M_n \le L_n^{1/3}$ implies $a_2 < a_3$. Recalling that $|z^2-4| \ge m(z) := \max\{\Im z, ((\Re z)^2-5)_+\}, 0 < \Im z \le 1$, we deduce from (6.34) that

$$\left|\frac{\widehat{r}_n(z)}{z^2-4}\right| \le \frac{5L_n}{m(z)\Im z} \le \frac{1}{10}, \qquad z \in D_{a_3}.$$

Therefore we easily get, for $z \in D_{a_3} \cup \{z \in \mathbb{C} : \Im z = 1\}$,

$$|\sqrt{z^2-4} + \sqrt{z^2-4+4\widehat{r}_n(z)}| \ge \sqrt{|z^2-4|} \ge \sqrt{m(z)}.$$

Applying this estimate together with (6.34) to (6.36), we conclude that, for $z \in D_{a_3} \cup \{z \in \mathbb{C} : \Im z = 1\}$,

$$\begin{aligned}\left|\frac{1}{S_n(z)} - \frac{1}{S(z)}\right| &\le \frac{|\widehat{r}_n(z)|}{|\sqrt{z^2-4}|}\frac{1}{|S(z)||S_n(z)|} \\ &\le 5\frac{L_n}{\sqrt{m(z)}\Im z |S(z)||S_n(z)|}.\end{aligned} \tag{6.37}$$

We conclude in the same way as in (6.14) and (6.15), using (6.37), that is,

$$\int_{\mathbb{R}} \big|G_{\mu_w}(u+i) - G_{\nu^{(n)}}(u+i)\big|\,du \le cL_n \int_{\mathbb{R}} \frac{du}{1+u^2} \le cL_n \tag{6.38}$$

and, for $x \in [-2,2]$,

$$\int_{a_3}^1 \big|G_{\mu_w}(x+iu) - G_{\nu^{(n)}}(x+iu)\big|\,du \le c\int_{a_3}^1 \frac{L_n}{u^{3/2}}\,du \le c\frac{L_n}{a_3^{1/2}} \le cL_n^{3/4}. \tag{6.39}$$

Now we write

$$G_{\mu^{(n)}}(z) - G_{\nu^{(n)}}(z) = \frac{r_{n,1}(B_n z)}{S_n(z)}, \qquad z \in \mathbb{C}^+. \tag{6.40}$$

We deduce from (6.37) the following estimate, for $z \in D_{a_3} \cup \{z \in \mathbb{C} : \Im z = 1\}$:

$$\begin{aligned}\frac{1}{2}\big(1 + (|\Re z| - 4)_+\big) &\le |S(z)| - \frac{5L_n}{\sqrt{m(z)}\Im z} \le |S_n(z)| \\ &\le |S(z)| + \frac{5L_n}{\sqrt{m(z)}\Im z} \le 2\big(1 + (|\Re z - 4)_+\big).\end{aligned} \tag{6.41}$$

In addition we have, by (6.24),

$$|r_{n,1}(B_n z)| \le \frac{1}{(B_n|S_n(z)|)^2}\left(m_2(\mu_1) + \frac{\beta_3(\mu_1)}{B_n \Im z}\right), \qquad z \in D_{a_3} \cup \{z \in \mathbb{C} : \Im z = 1\}. \tag{6.42}$$

Using (6.40)–(6.42), we easily obtain the following inequalities:

$$\int_{\mathbb{R}} \big|G_{\mu^{(n)}}(u+i) - G_{\nu^{(n)}}(u+i)\big|\,du \le c\left(\frac{1}{n} + L_n\right)\int_{\mathbb{R}} \frac{du}{1+u^2} \le cL_n \tag{6.43}$$

and, for $x \in [-2, 2]$,

$$\int_{a_3}^{1} \big|G_{\mu^{(n)}}(x+iu) - G_{\nu^{(n)}}(x+iu)\big|\, du \le c\Big(\frac{1}{n} + L_n|\log a_3|\Big) \le cL_n|\log L_n|. \tag{6.44}$$

In order to prove the upper estimate of $\Delta(\mu^{(n)}, \mu_w)$ we apply again Lemma 3.2 with $v = a_3$. Since $\beta_3(\mu_j) < \infty$, $j = 1, \dots,$ it is well known that $m_2(\mu^{(n)}) < \infty$ and the assumption (3.6) holds. Lemma 3.2, (6.38), (6.39), (6.43) and (6.44) together imply the estimate (2.10) and the theorem is proved. $\square$

PROOF OF THEOREM 2.7. For $k = 1, \dots, n$ denote $\widehat{\mu}_{nk}\times\ ((-\infty, x)) := \mu_k((-\infty, nx + a_{nk}))$, $x \in \mathbb{R}$, where $a_{nk} := \int_{(-n,n)} u\mu_k(du)$. We shall now verify the condition (4.1) with $k_n = n$ for the measures $\widehat{\mu}_{nk}$. We obtain

$$\begin{aligned}\varepsilon_{nk} &= \int_{\mathbb{R}} \frac{u^2}{1+u^2}\widehat{\mu}_{nk}(du) = \int_{\mathbb{R}} \frac{(u-a_{nk})^2}{n^2 + (u-a_{nk})^2}\mu_k(du) \\ &\le \frac{1}{n^2}\int_{(-n,n)} (u-a_{nk})^2\mu_k(du) + \int_{\{|u|\ge n\}} \mu_k(du).\end{aligned}$$

Therefore (4.1) follows from (2.12) and (2.14). Moreover, it follows from (2.12) and (2.14) that

$$\sum_{k=1}^{n} \varepsilon_{nk} \le \eta_n \qquad \text{where} \quad \eta_n := \eta_{n1} + \eta_{n3}. \tag{6.45}$$

In the proof of this theorem we use the notation of Section 4 with $k_n = n$ and $\tau = 1$.

From Proposition 3.3 we deduce the relations (4.13) and (4.14) with $k_n = n$. In addition $F_{\widehat{\mu}_n}(z) = F_{\widehat{\mu}_{n1}}(Z_{n1}(z))$, $z \in \mathbb{C}^+$, where $\widehat{\mu}_n := \widehat{\mu}_{n1} \boxplus \cdots \boxplus \widehat{\mu}_{nn}$. By (4.12) and (6.45), we get

$$\begin{aligned}|\phi_{\widehat{\mu}_{n1}\boxplus\cdots\boxplus\widehat{\mu}_{nn}}(z)| &\le |\phi_{\widehat{\mu}_{n1}}(z)| + \dots + |\phi_{\widehat{\mu}_{nn}}(z)| \\ &\le c\sum_{k=1}^{n} \varepsilon_{nk} \le c\eta_n, \qquad |z-i| \le 1/2.\end{aligned} \tag{6.46}$$

Since

$$\phi_{\widehat{\mu}_{n1}\boxplus\cdots\boxplus\widehat{\mu}_{nn}}(z) = (F_{\widehat{\mu}_{n1}}(Z_{n1}))^{(-1)}(z) - z = Z_{n1}^{(-1)}\big(F_{\widehat{\mu}_{n1}}^{(-1)}(z)\big) - z$$

for $|z - i| \le 1/2$, we have, by (4.10), the relation

$$\phi_{\widehat{\mu}_{n1}\boxplus\cdots\boxplus\widehat{\mu}_{nn}}(F_{\widehat{\mu}_{n1}}(z)) = Z_{n1}^{(-1)}(z) - F_{\widehat{\mu}_{n1}}(z)$$

for $|z-i|\le 1/4$. Therefore we conclude by (4.10) and (6.46) that the function $Z_{n1}^{(-1)}(z)$ is analytic in the disk $|z-i|<1/4$ and $|Z_{nk}^{(-1)}(z)-z)|\le c\eta_n$ for $|z-i|<1/4$. From this relation we see that

$$|Z_{n1}(z)-z|\le c\eta_n, \qquad |z-i|\le 1/8. \tag{6.47}$$

The function $Z_{n1}(z)$ admits the representation (4.19). By (6.47), $|d_{n1}|\le c\eta_n$ and $\nu_{n1}(\mathbb{R})\le c\eta_n$. Similarly to (4.10) we obtain

$$|Z_{n1}(z)-z|\le c\eta_n\Big(1+\frac{1+|z|^2}{\Im z}\Big), \qquad z\in\mathbb{C}^+. \tag{6.48}$$

Then we have, using (4.10) and (6.48),

$$|F_{\widehat{\mu}_{n1}}(Z_{n1}(z))-Z_{n1}(z)|\le c\eta_n\Big(1+\frac{1+|Z_{n1}(z)|^2}{\Im Z_{n1}(z)}\Big)\le c\eta_n^{2/3} \tag{6.49}$$

for $z=x+i\eta_n^{1/3}$, $\eta_n^{1/6}\le x\le \eta_n^{1/6}$. For such z we finally get

$$|F_{\widehat{\mu}_n}(z)-z|\le c\eta_n^{2/3}. \tag{6.50}$$

Since $F_{\widehat{\mu}_n}(z)\in\mathcal{F}$ and therefore $|F_{\widehat{\mu}_n}(z)|\ge\Im z$, $z\in\mathbb{C}^+$, we conclude from (6.50) that, for $z=x+i\eta_n^{1/3}$, $\eta_n^{1/6}\le x\le\eta_n^{1/6}$,

$$\Big|G_{\widehat{\mu}_n}(z)-\frac{1}{z}\Big|=\frac{|F_{\widehat{\mu}_n}(z)-z|}{|F_{\widehat{\mu}_n}(z)||z|}\le c. \tag{6.51}$$

From (6.51) we get, for sufficiently large $n\ge n_3\ge c$,

$$\begin{aligned}&-\frac{1}{\pi}\int_{\{|x|\le\eta_n^{1/6}\}}\Im G_{\widehat{\mu}_n}(x+i\eta_n^{1/3})\,dx\\ &\qquad\ge\frac{1}{\pi}\int_{\{|x|\le\eta_n^{1/6}\}}\frac{\eta_n^{1/3}}{x^2+\eta_n^{2/3}}\,dx-c\eta_n^{1/6}\ge 1-c\eta_n^{1/6}.\end{aligned} \tag{6.52}$$

On the other hand, we obtain

$$\begin{aligned}&-\frac{1}{\pi}\int_{|x|\le\eta_n^{1/6}}\Im G_{\widehat{\mu}_n}(x+i\eta_n^{1/3})\,dx\\ &\qquad=\frac{1}{\pi}\int_{\mathbb{R}}\Big(\arctan\frac{\eta_n^{1/6}-u}{\eta_n^{1/3}}+\arctan\frac{\eta_n^{1/6}+u}{\eta_n^{1/3}}\Big)\widehat{\mu}_n(du)\\ &\qquad\le\widehat{\mu}_n(\{|u|\le 2\eta_n^{1/6}\})+1-\frac{2}{\pi}\arctan\frac{1}{\eta_n^{1/6}}\\ &\qquad\le\widehat{\mu}_n(\{|u|\le 2\eta_n^{1/6}\})+c\eta_n^{1/6}.\end{aligned} \tag{6.53}$$

From (6.52) and (6.53), for sufficiently large $n\ge n_4\ge c$, we have

$$\widehat{\mu}_n(\{|u|\le 2\eta_n^{1/6}\})\ge 1-c\eta_n^{1/6}$$

which immediately implies $L(\widehat{\mu}_n, \delta_0) \leq c\eta_n^{1/6}$. By the definition of $\mu^{(n)}$ and $\widehat{\mu}_n$, we see that $L(\mu^{(n)}, \widehat{\mu}_n) \leq \eta_{n2}$. The estimate (2.15) is now an obvious consequence of the last two estimates.

Thus, Theorem 2.7 is proved. $\square$

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INSTITUTE FOR LOW TEMPERATURE PHYSICS
AND ENGINEERING
NATIONAL ACADEMY OF SCIENCES OF UKRAINE
47 LENIN AVE.
61103 KHARKOV
UKRAINE
E-MAIL: chistyakov@ilt.kharkov.ua
chistyak@math.uni-bielefeld.de

FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT BIELEFELD
POSTFACH 100131
33501 BIELEFELD 1
GERMANY
E-MAIL: goetze@math.uni-bielefeld.de