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HARMONIC EXPLORER AND ITS CONVERGENCE TO SLE_4

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The harmonic explorer is a random grid path. Very roughly, at each step the harmonic explorer takes a turn to the right with probability equal to the discrete harmonic measure of the left-hand side of the path from a point near the end of the current path. We prove that the harmonic explorer converges to SLE_4 as the grid gets finer.

1. Introduction. Let D be a simply connected subset of the hexagonal faces in the planar honeycomb lattice. Two faces of D are considered adjacent if they share an edge. Suppose further that the boundary faces of D are partitioned into a "left boundary" component, colored black, and a "right boundary" component, colored white, in such a way that the set of interior faces remains simply connected. (See Figure 1.) Given any black–white coloring of the faces of D, there will be a unique interface γ separating the cluster of black hexagons containing the left boundary from the cluster of white hexagons containing the right boundary.

If the colors are chosen via independent Bernoulli percolation, we may view γ as being generated dynamically as follows: simply begin the path γ at an edge separating the left and right boundary components; when γ hits a black hexagon, it turns right, and when it hits a white hexagon, it turns left. Each time it hits a hexagon whose color has yet to be determined, we choose that hexagon's color with a coin toss.

The *harmonic explorer* (HE) is a random interface generated the same way, except that each time γ hits a hexagon f whose color has yet to be determined, we perform a simple random walk on the space of hexagons, beginning at f, and let f assume the color of the first black or white hexagon hit by that walk. (See Figure 1.) In other words, we color f black with probability equal to the value at f of the function which is equal to 1 on the black faces and 0 on the white faces, and is *discrete harmonic* at the undetermined faces (i.e., its value at each such face is the mean of the values on the six neighboring faces).

Denote by h_n the value of this function after n steps of the harmonic explorer process; that is, $h_n(f)$ is 1 if f is black, 0 if f is white, and discrete harmonic on the faces of undetermined color. Note that $h_n(f)$ is also the probability that a random walk on faces, started at f, hits a black face before hitting a white face. It is easy to see (and proved below) that for any fixed f, $h_n(f)$ is a martingale—and

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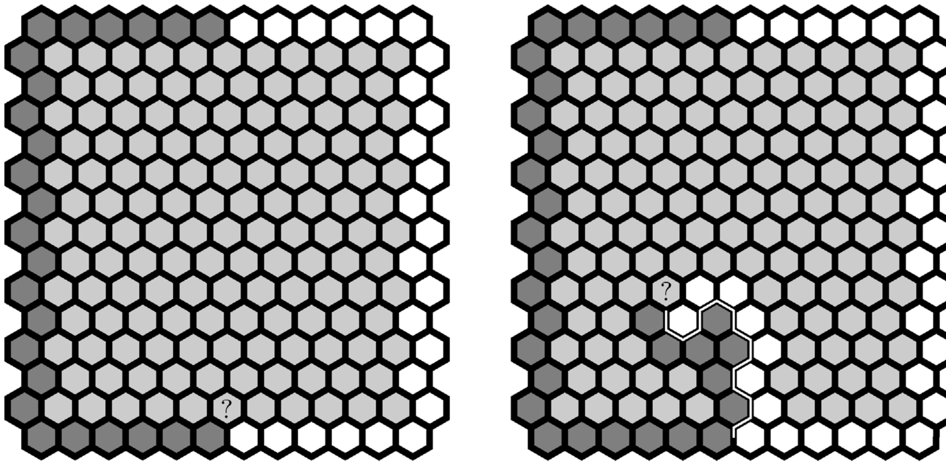

(a) (b)

FIG. 1. (a) *Initial left boundary faces* (*black*), *right boundary faces* (*white*) *and undetermined interior faces* (*gray*). (b) *A possible HE configuration several steps later.*

that the harmonic explorer is the only random path with this property. We will see later that SLE_4 is the only random path with a certain continuous analog of this property.

It was conjectured in [16] and proved in [18] that if the interior hexagons are each colored via critical Bernoulli percolation (i.e., $p = p_c = 1/2$), then, in a certain well-defined sense, the random paths γ tend to the stochastic Loewner evolution with parameter $\kappa = 6$ (SLE_6) as the hexagonal mesh gets finer. (See the survey [19] for background on SLE.) It has been further conjectured [13] that if colors are instead chosen from a critical FK cluster model (where one weights configurations according to the total number of clusters and the lengths of their interfaces), then γ will converge to some SLE_κ with $4 < \kappa < 8$, where κ depends on the weight parameters. We will prove that, as the mesh gets finer, the harmonic explorer converges to chordal SLE_4.

There are also natural variants of the harmonic explorer; for example, we might replace the honeycomb lattice with another three-regular lattice or replace the simple random walk on faces with a different periodic Markov chain. One may even use a non-three-regular lattice provided one fixes an appropriate ordering (say, left to right) for determining the color of multiple undetermined faces that are "hit" simultaneously by the HE path. Provided the simple random walk converges to Brownian motion as the mesh gets finer, we see no barrier to extending our results to all of these settings. Our proofs are more like the LERW proofs in [9] (which hold for general lattices) than the percolation proof in [18] (which uses the invariance of the lattice under $2\pi/3$ rotation in an essential way). However, for simplicity, we will focus only on the hexagonal lattice in this paper.

Although physicists and mathematicians have conjectured that many models for random self-avoiding lattice walks have conformally invariant scaling limits [e.g., the infinite self-avoiding walk, critical percolation cluster boundaries on two-dimensional lattices, critical Ising model interfaces, critical FK cluster boundaries and $O(n)$ model strands, etc.], rigorous proofs are available only in the following cases: percolation interface on the hexagonal lattice (which converges to chordal SLE_6), harmonic explorer (chordal SLE_4), loop erased random walk (LERW) on a periodic planar graph (radial SLE_2 [9]), the uniform spanning tree (UST) boundary (chordal SLE_8 [9]) and the boundaries of simple random walks (essentially $\mathrm{SLE}_{8/3}$ [8]—here conformal invariance follows easily from the conformal invariance of Brownian motion).

The harmonic explorer is similar in spirit to the loop erased random walk (LERW) and diffusion limited aggregation (DLA). All three models are processes based on simple random walks, and their transition probabilities may all be computed using discrete harmonic functions with appropriate boundary conditions. Since simple random walks on two-dimensional lattices have a conformally invariant scaling limit (Brownian motion), and since harmonicity (in the continuous limit) is a conformally invariant property, one might expect that all three models would have conformally invariant scaling limits. However, simulations suggest that DLA is not conformally invariant.

This paper follows the strategy of [9], and uses some of the techniques from that paper. We will freely quote results from [9], and therefore advise the reader to have a copy of [9] on hand while reading the present paper.

2. A martingale property of chordal SLE_4. The purpose of this section is to briefly review some background about Loewner's equation and SLE, and then present the basic strategy of the paper. For more details, the reader is encouraged to consult [19] or [7].

Let $T > 0$. Suppose that $\gamma : [0, T] \to \overline{\mathbb{H}}$ is a continuous simple path in the closed upper half plane $\overline{\mathbb{H}}$ which satisfies $\gamma[0, T] \cap \mathbb{R} = \{\gamma(0)\} = \{0\}$. For every $t \in [0, T]$, there is a unique conformal homeomorphism $g_t : \mathbb{H} \setminus \gamma[0, t]$ which satisfies the so-called *hydrodynamic* normalization at infinity

$$\lim_{z\to\infty} g_t(z) - z = 0.$$

The limit

$$\mathrm{cap}_\infty(\gamma[0, t]) := \lim_{z\to\infty} z\big(g_t(z) - z\big)/2$$

is real and monotone increasing in t. It is called the (half plane) *capacity* of $\gamma[0, t]$ from ∞, or just capacity, for short. Since $\mathrm{cap}_\infty(\gamma[0, t])$ is also continuous in t, it is natural to reparameterize γ so that $\mathrm{cap}_\infty(\gamma[0, t]) = t$. Loewner's theorem states that in this case the maps g_t satisfy his differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}, \qquad g_0(z) = z, \tag{2.1}$$

where $W(t) = g_t(\gamma(t))$. (Since $\gamma(t)$ is not in the domain of definition of g_t, the expression $g_t(\gamma(t))$ should be interpreted as a limit of $g_t(z)$ as $z \to \gamma(t)$ inside $\mathbb{H} \setminus \gamma[0,t]$. This limit does exist.) The function $W(t)$ is continuous in t, and is called the *driving parameter* for γ.

One may also try to reverse the above procedure. Consider the Loewner evolution defined by the ODE (2.1), where $W(t)$ is a continuous, real-valued function. The path of the evolution is defined as $\gamma(t) = \lim_{z\to W(t)} g_t^{-1}(z)$, where z tends to $W(t)$ from within the upper half plane $\mathbb{H}$, provided that the limit exists. The process (chordal) SLE_κ in the upper half plane, beginning at 0 and ending at ∞, is the path $\gamma(t)$ when $W(t)$ is $\sqrt{\kappa} B_t$, where $B_t = B(t)$ is a standard one-dimensional Brownian motion. ("Standard" means $B(0) = 0$ and $\mathbf{E}[B(t)^2] = t$, $t \geq 0$. Since $(\sqrt{\kappa} B_t : t \geq 0)$ has the same distribution as $(B_{\kappa t} : t \geq 0)$, taking $W(t) = B_{\kappa t}$ is equivalent.) In this case a.s. $\gamma(t)$ does exist and is a continuous path. See [13] ($\kappa \neq 8$) and [9] ($\kappa = 8$).

Fix $\kappa > 0$, and assume now that $W(t) = \sqrt{\kappa} B_t$ and γ is SLE_κ. Write $X = X(t,z) = g_t(z) - W(t)$. Then $\arg X(t,z)$ gives π times the probability that a two-dimensional Brownian motion starting at z first exits $\mathbb{H} \setminus \gamma[0,t]$ either in $(-\infty, 0)$ or on the left-hand side of $\gamma[0,t]$. (This follows from conformal invariance of a Brownian motion, run until its first exit point, and the fact that the probability that a Brownian motion started at $z \in \mathbb{H}$ first hits $\mathbb{R}$ at $(-\infty, 0]$ is $\arg(z)/\pi$. The latter fact may be seen by conformally mapping the half plane to a strip using the function $z \to \log(z)$.) In other words, for fixed t, $\arg X(t,z)$ is the harmonic function that is equal to π on one side of $\gamma[0,t]$ and 0 on the other. For short, we will sometimes refer to the quantity $\arg X(t,z)$ as simply the *angle* of z at time t.

Now, using Itô's formula, we compute the Itô derivatives of X and $\log X$:

$$
\begin{aligned}
dX &= \frac{2}{X}\,dt - \sqrt{\kappa}\,dB_t,\\
d\log X &= \frac{2}{X^2}\,dt - \frac{\sqrt{\kappa}}{X}\,dB_t - \frac{\kappa}{2X^2}\,dt = \frac{4-k}{2X^2}\,dt - \frac{\sqrt{\kappa}}{X}\,dB_t.
\end{aligned}
$$

When $\kappa = 4$, we have $d\log X = -2X^{-1}\,dB_t$, and hence $d\arg X = -\operatorname{Im}(2X^{-1})\,dB_t$. In particular, this implies that for any fixed value of z, $\arg X(t,z)/\pi$ is a martingale which is bounded in the interval $[0,1]$. The value of this martingale a.s. tends to either zero or 1 as t tends to infinity, depending on whether z is on the left or the right side of the path (see [15], Lemma 3). Hence, at a fixed time t, $\arg X(t,z)/\pi$ represents the *probability* that, conditioned on the SLE path up until time t, the point z will lie to the left of the path.

It is easy to see (and shown below) that a discrete version of this property holds for the harmonic explorer. The strategy of our SLE_4 proof will be, roughly speaking, to show that the fact that this property holds at two distinct values of z is enough to force the Loewner driving process for the path traced by the harmonic explorer to converge to Brownian motion. This is because the fact that $\arg X(t,z)$

is a martingale at z gives a linear constraint on the drift and diffusion terms at that point, and using two values of z gives two linear constraints, from which it is possible to calculate the drift and diffusion exactly. The arguments and error bounds needed to make this reasoning precise are essentially the same as those given in [9] (but the martingales considered there are different). The fact that the Loewner driving process converges to Brownian motion will enable us to conclude that HE converges to SLE_4 in the Hausdorff topology. We will then employ additional arguments to show that the convergence holds in a stronger topology.

We remark that we will reuse this strategy in [17] to prove that a certain zero level set of the discrete Gaussian free field (defined on the vertices of a triangular lattice, with boundary conditions equal to an appropriately chosen constant λ on the left boundary and $-\lambda$ on the right boundary) converges to chordal SLE_4. To keep notation consistent with [17] (which will cite the present paper), we will use the dual formulation (representing hexagons by vertices of the triangular lattice) in our precise statements and proofs below.

3. Statements of main results.

3.1. *Notation and basic properties of HE.* We now introduce the precise combinatorial notation for HE that we will use in our proofs. First, the triangular grid in the plane will be denoted by TG. Its vertices, denoted by $V(TG)$, are the sublattice of $\mathbb{C}$ spanned by 1 and $e^{2\pi i/6}$; two vertices are adjacent if their difference is a sixth root of unity. If $D \subset \mathbb{C}$, and $z \in \mathbb{C}$, let $\mathrm{rad}_z(D)$ denote the *inradius* of D about z; that is, $\mathrm{rad}_z(D) := \inf\{|w - z| : w \notin D\}$. Let $\mathfrak{D}$ denote the set of domains $D \subset \mathbb{C}$ whose boundary is a simple closed curve which is a union of edges from the lattice TG.

If V_0 is any set of vertices in $V(TG)$, and $h : V_0 \to \mathbb{R}$ is a bounded function, then there exists a unique bounded function $\bar{h} : V(TG) \to \mathbb{R}$ which agrees with h in V_0 and is harmonic at every vertex in $V(TG) \setminus V_0$. This function is called the *discrete harmonic extension* of h. [In fact, $\bar{h}(v)$ is the expected value of h at the point at which a simple random walk started at v hits V_0. Uniqueness is easily established using the maximum principle.]

Let $D \in \mathfrak{D}$. Let $V_0 := V(TG) \cap \partial D$ denote the set of vertices in ∂D. Let $\bar{v}_0$ and $\bar{v}_{\mathrm{end}}$ be the centers of two distinct edges of the grid TG on ∂D. (See Figure 2.) Let A_+ (resp. A_-) be the positively (resp. negatively) oriented arc of ∂D from $\bar{v}_0$ to $\bar{v}_{\mathrm{end}}$. Define $\mathfrak{h}_0 : V_0 \to \{0, 1\}$ to be 1 on $V_0 \cap A_+$, and 0 on $V_0 \cap A_-$. The HE [depending on the triple $(D, \bar{v}_0, \bar{v}_{\mathrm{end}})$] is a random simple path from $\bar{v}_0$ to $\bar{v}_{\mathrm{end}}$ in $\overline{D}$. Let $X_1, X_2, \dots$ be i.i.d. random variables, uniform in the interval $[0, 1]$. (These will be the "coin flips" needed to generate the HE.) Let $T_1 \subset D$ be the triangle of TG whose boundary contains $\bar{v}_0$ let v_1 be the vertex of T_1 that is not on the edge containing $\bar{v}_0$ and let $V_1 := V_0 \cup \{v_1\}$. Let v_1' be the middle of the edge of T_1 which is on the positively oriented arc from $\bar{v}_0$ to v_1, and let v_1'' be the middle of the edge of T_1 which is on the positively oriented arc from v_1 to $\bar{v}_0$. Let

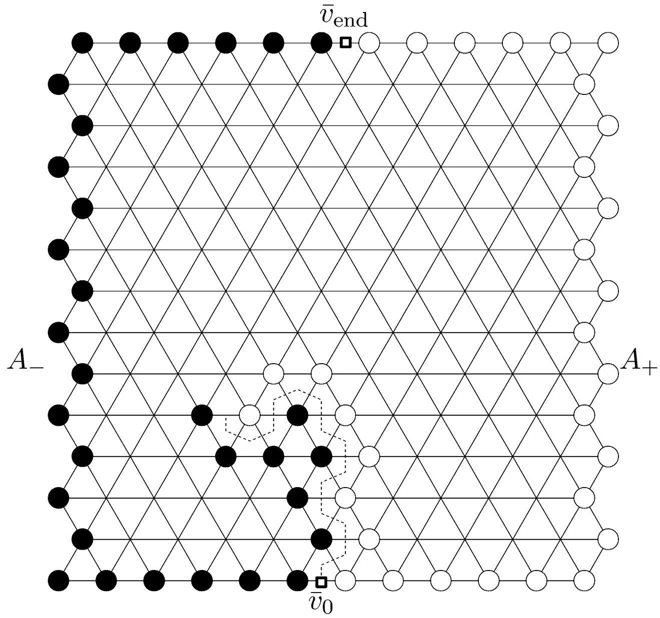

FIG. 2. *A dual perspective on Figure 1.*

p_1 be the value at v_1 of the discrete harmonic extension of $\mathfrak{h}_0$. If $X_1 \le p_1$, we let $\bar{v}_1 := v_1''$, and otherwise $\bar{v}_1 := v_1'$. The beginning of the HE path is chosen as the union of the two line segments from $\bar{v}_0$ to the center of the triangle T_1 and then to $\bar{v}_1$. Now define $\mathfrak{h}_1 : V_1 \to \{0, 1\}$ to equal $\mathfrak{h}_0$ on V_0 and set $\mathfrak{h}_1(v_1) := \mathbb{1}_{X_1 \le p_1}$ if $v_1 \notin V_0$. This defines the first step of the HE.

The process continues inductively. Assuming that $n \ge 1$ and $\bar{v}_n \notin \partial D$, let T_{n+1} be the triangle of *TG* containing $\bar{v}_n$ but not $\bar{v}_{n-1}$. Let v_{n+1} be the vertex of T_{n+1} which is not on the edge containing $\bar{v}_n$, and let $V_{n+1} := V_n \cup \{v_{n+1}\}$. Let p_{n+1} be the value at v_{n+1} of the discrete harmonic extension of $\mathfrak{h}_n$. Let v_{n+1}' and v_{n+1}'' be the two midpoints of edges of T_{n+1} that lie on the positively oriented arcs of ∂T_{n+1} from $\bar{v}_n$ to v_{n+1} and from v_{n+1} to $\bar{v}_n$, respectively. If $X_{n+1} \le p_{n+1}$ let $\bar{v}_{n+1} := v_{n+1}''$, and otherwise $\bar{v}_{n+1} := v_{n+1}'$. Let the next step of the HE consist of the segments from $\bar{v}_n$ to the center of T_{n+1} and from the center of T_{n+1} to $\bar{v}_{n+1}$. Also, let $\mathfrak{h}_{n+1}$ agree with $\mathfrak{h}_n$, where $\mathfrak{h}_n$ is defined and set $\mathfrak{h}_{n+1}(v_{n+1}) := \mathbb{1}_{X_{n+1} \le p_{n+1}}$ if $v_{n+1} \notin V_n$.

It is easy to verify that this procedure a.s. terminates when $\bar{v}_n = \bar{v}_{\text{end}}$, and that the HE so defined is a simple path from $\bar{v}_0$ to $\bar{v}_{\text{end}}$. Let N denote the termination time; that is, the n such that $\bar{v}_n = \bar{v}_{\text{end}}$.

LEMMA 3.1. *Let h_n denote the discrete harmonic extension of $\mathfrak{h}_n$, and let $v \in V(TG) \cap D$. Then $h_n(v)$ is a martingale and $h_N(v) \in \{0, 1\}$.*

PROOF. Given $X_1, \dots, X_n$, we have $h_{n+1}(v_{n+1}) = 1$ with probability $h_n(v_{n+1})$ and otherwise $h_{n+1}(v_{n+1}) = 0$. Consequently, $\mathbf{E}[h_{n+1}(v_{n+1})|X_1, \dots, X_n] = h_n(v_{n+1})$. Note that h_n is also the discrete harmonic extension of its restriction to V_{n+1}, and similarly for h_{n+1}. Since the harmonic extension is a linear operation and $\mathbf{E}[h_{n+1}(v)|X_1, \dots, X_n] = h_n(v)$ for $v \in V_{n+1}$, the same relation holds for every v. Thus $h_n(v)$ is a martingale. The claim that $h_N(v) \in \{0, 1\}$ is clear. $\square$

REMARK 3.2. The evolution of the HE path may be viewed as a Markov chain on the collection of appropriately marked domains. At the nth step, the chain is at $(D_n, \bar{v}_n, \bar{v}_{\mathrm{end}})$, where D_n is the connected component of $D \setminus \bigcup_{j=1}^n T_j$ that has $\bar{v}_{\mathrm{end}}$ on its boundary.

3.2. *Convergence of HE*: *statement.* Let $D \in \mathfrak{D}$. We assume the setup and notation of Section 3.1. Let $\gamma : [0, N] \to D \cup \{\bar{v}_0, \bar{v}_{\mathrm{end}}\}$ be the HE path with the parameterization proportional to arclength, where $\gamma(n) = \bar{v}_n$ for $n \in \{0, 1, \dots, N\}$. Let $\phi : D \to \mathbb{H}$ be a conformal map onto $\mathbb{H}$ that takes $\bar{v}_0$ to 0 and $\bar{v}_{\mathrm{end}}$ to ∞. Note that ϕ is unique up to positive scaling, $\phi(A_+) = (0, \infty)$ and $\phi(A_-) = (-\infty, 0)$. Let $p_0 := \phi^{-1}(i)$.

Instead of rescaling the grid, we consider larger and larger domains D. The quantity $\rho = \rho(D, \phi) := \mathrm{rad}_{p_0}(D)$ turns out to be the appropriate indicator of the size of D, from the perspective of the map ϕ. Indeed, if ρ is small, then the image under ϕ of the grid TG in D is not fine near i, and we cannot expect $\phi \circ \gamma$ to look like SLE_4. As we will see, $\phi \circ \gamma$ does approach SLE_4 when $\rho \to \infty$. Let γ^ϕ be the path $\phi \circ \gamma$, parameterized by capacity from ∞ in $\overline{\mathbb{H}}$, and let $\tilde{\gamma}$ be the SLE_4 path in $\overline{\mathbb{H}}$.

Let $d_*(\cdot, \cdot)$ be the metric on $\overline{\mathbb{H}} \cup \{\infty\}$ given by $d_*(z, w) = |\Psi(z) - \Psi(w)|$, where $\Psi(z) := (z - i)/(z + i)$ maps $\overline{\mathbb{H}} \cup \{\infty\}$ onto $\overline{\mathbb{U}}$. If $z \in \overline{\mathbb{H}}$, then $d_*(z_n, z) \to 0$ is equivalent to $|z_n - z| \to 0$, and $d_*(z_n, \infty) \to 0$ is equivalent to $|z_n| \to \infty$.

Note that although we started by mapping our domain D to the half plane (with boundary points 0, ∞ and inradius measured from the preimage of i), the above metric corresponds to a mapping to the unit disc (with boundary points -1, 1 and inradius measure from the preimage of 0). The half plane is the most convenient setting for describing Loewner evolution and chordal SLE_4, but the metric derived from the unit disc map is more convenient because it is compact.

THEOREM 3.3. *As $\rho \to \infty$, the law of γ^ϕ tends to the law of the* SLE_4 *path $\tilde{\gamma}$, with respect to uniform convergence in the metric d_*. In other words, for every $\varepsilon > 0$ there is some $R = R(\varepsilon)$ such that if $\rho > R$, then there is a coupling of γ^ϕ and $\tilde{\gamma}$ such that*

$$\mathbf{P}\bigl[\sup\{d_*(\tilde{\gamma}(t), \gamma^\phi(t)) : t \in (0, \infty)\} > \varepsilon\bigr] < \varepsilon.$$

4. The driving process converges to BM. Let $W = W(t)$ denote the Loewner driving process for γ^ϕ. Let $B:[0,\infty) \to \mathbb{R}$ be a standard one-dimensional Brownian motion. A slightly weaker form of Theorem 3.3 will follow as a consequence of the fact that for every $T > 0$ the restriction of W to $[0, T]$ converges in law to the restriction of $t \mapsto 2B(t)$ to $[0, T]$. This, in turn, will be a consequence of the following local statement.

PROPOSITION 4.1. *For $n \in [0, N]$ let $t_n := \operatorname{cap}_\infty(\psi \circ \gamma[0, n])$, $\tilde{D}_n := D \setminus \gamma[0, n]$, and let $\phi_n : \tilde{D}_n \to \mathbb{H}$ be the conformal map normalized by $\phi_n \circ \phi^{-1}(z) - z \to 0$ as $z \to \infty$ in $\mathbb{H}$. For every $\delta \in (0, 1)$ there is an $R = R(\delta) > 0$ such that the following holds. Fix any $n \in \mathbb{N}$. On the event $\mathcal{A}_1 = \mathcal{A}_1(n) := \{n < N\}$, let m be the least integer larger than n such that $\max\{t_m - t_n, (W(t_m) - W(t_n))^2\} \geq \delta^2$. (Note that $m \leq N$, since $t_N = \infty$.) Set $p_n := \phi_n^{-1}(i + W(t_n))$ and let $\mathcal{A}_2 = \mathcal{A}_2(n)$ be the event $\{\operatorname{rad}_{p_n}(D) \geq R\}$. Then*

$$\mathbf{E}\bigl[W(t_m)|\gamma[0, n]\bigr] = W(t_n) + O(\delta^3) \tag{4.1}$$

and

$$\mathbf{E}\bigl[\bigl(W(t_m) - W(t_n)\bigr)^2|\gamma[0, n]\bigr] = 4\mathbf{E}\bigl[t_m - t_n|\gamma[0, n]\bigr] + O(\delta^3) \tag{4.2}$$

both hold on the event $\mathcal{A}_1 \cap \mathcal{A}_2$.

Here, and below, $O(f)$ represents any quantity whose absolute value is bounded by cf, where c is any fixed constant.

The strategy for proving the proposition is as follows. We use Lemma 3.1 to conclude that $\mathbf{E}[h_m(v) - h_n(v)|\gamma[0, n]] = 0$. Since $h_j(v)$ is discrete-harmonic, it is approximately equal to the harmonic function on $\tilde{D}_j$ with the corresponding boundary values. The difference $h_m(v) - h_n(v)$ can then be approximated by a function of $t_m - t_n$ and $W(t_m) - W(t_n)$. Applying $\mathbf{E}[h_m(v) - h_n(v)|\gamma[0, n]] = 0$ for two distinct choices of v then gives the relations (4.1) and (4.2).

We start with a lemma describing the approximation of h_j by a (nondiscrete) harmonic function.

LEMMA 4.2. *Given any $\varepsilon > 0$ there is an $r = r(\varepsilon) > 0$ such that for every vertex $v \in V(TG)$ and every $j < N$, if $\operatorname{rad}_v(\tilde{D}_j) > r$, then*

$$\bigl|h_j(v) - \tilde{h}\bigl(\phi_j(v) - W(t_j)\bigr)\bigr| < \varepsilon, \tag{4.3}$$

where $\tilde{h}(z) = 1 - (1/\pi)\arg z$.

PROOF. Note that $\tilde{h} : \mathbb{H} \to (0, 1)$ is harmonic and has the boundary values 0 on $(-\infty, 0)$ and 1 on $(0, \infty)$. Since $W(t_j) = \phi_j(\gamma(j))$, $z \mapsto \tilde{h}(\phi_j(z) - W(t_j))$ is harmonic in $\tilde{D}_j$, and has boundary values 0 on A_- and on the "left side" of $\gamma[0, j]$

and 1 on A_+ and the "right side" of $\gamma[0, j]$. Since h_j is a discrete harmonic function with similar boundary conditions, the statement of the lemma can be obtained as a consequence of the convergence of random walk on TG to Brownian motion. We leave the details to the reader. (Also note that more delicate but similar estimates are given in [9], Section 5.) $\square$

PROOF OF PROPOSITION 4.1. Assume $\mathcal{A}_1$. We claim that there is an absolute constant $\delta_0 > 0$ such that $\mathrm{rad}_{p_n}(\tilde{D}_m) \geq \frac{1}{2}\mathrm{rad}_{p_n}(\tilde{D}_n) - 1$ if $\delta < \delta_0$. Let z be on the circle $|z| = \mathrm{rad}_{p_n}(\tilde{D}_n)/2$. Since $\mathrm{Im}\,\phi_n(p_n) = 1$, the Koebe distortion theorem implies a positive constant lower bound for $\mathrm{Im}\,\phi_n(z)$ (see, e.g., [12], Section 1.3). Let g_t be the Loewner chain driven by $W(t)$. Then $\phi_j = g_{t_j} \circ \phi$. By Loewner's equation (2.1), $\frac{d}{dt}\mathrm{Im}\,g_t(z) \geq -2/\mathrm{Im}\,g_t(z)$, which implies $\frac{d}{dt}(\mathrm{Im}\,g_t(z))^2 \geq -4$. Thus, $\tau(z) \geq t_n + (\mathrm{Im}\,\phi_n(z))^2/4$. Since $t_{m-1} - t_n \leq \delta^2$, it follows that $z \notin \gamma[0, m-1]$ if $\delta < \delta_0$, where δ_0 is the infimum of all possible values for $\mathrm{Im}\,\phi_n(z)/2$. In that case, $\mathrm{rad}_{p_n}(\tilde{D}_{m-1}) \geq \mathrm{rad}_{p_n}(\tilde{D}_n)/2$, which implies our claim $\mathrm{rad}_{p_n}(\tilde{D}_m) \geq \frac{1}{2}\mathrm{rad}_{p_n}(\tilde{D}_n) - 1$. We will henceforth assume, with no loss of generality, that $\delta < \delta_0$. Note that the above argument also gives a positive lower bound on $\mathrm{Im}\,\phi_{m-1}(p_n)$.

Now fix some vertex $w_0 \in \tilde{D}_n \cap V(TG)$ satisfying $|w_0 - p_n| < \mathrm{rad}_{p_n}(\tilde{D}_n)/6$. Let R be larger than $100\max\{1, r(\delta^3)\}$, in the notation of Lemma 4.2. Assume now that $\mathcal{A}_2$ holds. Then we may apply (4.3) with $j = n, m$, $v = w_0$ and $\varepsilon = \delta^3$. Since h_j is a martingale, it satisfies $\mathbf{E}[h_m(w_0)|\gamma[0, n]] = h_n(w_0)$, and so we get from (4.3)

$$\mathbf{E}\big[\tilde{h}\big(\phi_m(w_0) - W(t_m)\big)|\gamma[0, n]\big] = \tilde{h}\big(\phi_n(w_0) - W(t_n)\big) + O(\delta^3). \tag{4.4}$$

Below, we need the relations

$$\forall\, t \in [t_n, t_m] \qquad |W(t) - W(t_n)| = O(\delta), \qquad t_m - t_n = O(\delta^2). \tag{4.5}$$

By our choice of m, we have the first relation when $t = t_n, t_{n+1}, \dots, t_{m-1}$ and the second relation when t_{m-1} replaces t_m. The relations (4.5) will follow by assuming that R is large enough. Indeed, if $j \in \{n, \dots, m-1\}$ and R is large, then the harmonic measure from p_n of $\gamma[j, j+1]$ in $\tilde{D}_j$ is $O(\delta)$. (The Beurling projection theorem [1] tells us that $R > \delta^{-2}$ suffices.) By conformal invariance of harmonic measure, the harmonic measure from $\phi_j(p_n)$ of $\phi_j \circ \gamma[j, j+1]$ in $\mathbb{H}$ is $O(\delta)$. We want to use this to conclude that $\mathrm{diam}(\phi_j \circ \gamma[j, j+1]) = O(\delta)$. Note that $\phi_j(p_n) = g_{t_j} \circ \phi(p_n)$. Above, we have seen that there is a constant positive lower bound for $\mathrm{Im}\,\phi_{m-1}(p_n)$. By (2.1), $\mathrm{Im}\,g_t(z)$ is monotone decreasing in t. Hence, $\mathrm{Im}\,g_t \circ \phi(p_n)$ has a constant positive lower bound for $t \leq t_{m-1}$. By (2.1), we get $|\partial_t(g_t \circ \phi(p_n))| = O(1)$ for $t \leq t_{m-1}$. Integrating then gives $|\phi_j(p_n) - \phi_n(p_n)| \leq O(\delta^2)$ for $j = n, n+1, \dots, m-1$. As $W_j = \phi_j(\bar{v}_j) \in \phi_j \circ \gamma[j, j+1]$ the distance from $\phi_j(p_n)$ to $\phi_j \circ \gamma[j, j+1]$ is $O(1)$. Consequently, the harmonic measure estimate gives the bound $\mathrm{diam}\,\phi_j \circ \gamma[j, j+1] = O(\delta)$. The needed estimates (4.5) now follow from [9], Lemma 2.1, since $\phi_j(\gamma[j, j+1])$ is the set of points hitting the real line under Loewner's evolution (2.1) in the time interval $[t_j, t_{j+1}]$.

Let $z_t := g_t \circ \phi(w_0)$. Since we have $\phi_j(w_0) = z_{t_j}$, we may obtain $\phi_m(w_0) = z_{t_m}$ from $\phi_n(w_0) = z_{t_n}$ by flowing according to Loewner's equation (2.1) between the times t_n and t_m. As before, we get the bound $|z_t - z_{t_n}| = O(\delta^2)$ for $t \in [t_n, t_m]$. Since $|W(t) - W(t_n)| = O(\delta)$, we have

$$\frac{2}{z_t - W(t)} = \frac{2}{z_{t_n} - W(t_n)} + O(\delta), \qquad t \in [t_n, t_m].$$

By integrating this relation over $[t_n, t_m]$, (2.1) gives

$$z_{t_m} - z_{t_n} = \phi_m(w_0) - \phi_n(w_0) = \frac{2(t_m - t_n)}{\phi_n(w_0) - W(t_n)} + O(\delta^3). \tag{4.6}$$

Consider now $F(z, W) := \tilde{h}(z - W)$. We want an estimate for

$$F(z_{t_m}, W(t_m)) = \tilde{h}(\phi_m(w_0) - W(t_m))$$

up to $O(\delta^3)$ terms. For that purpose, we use a Taylor expansion of F about $(z_{t_n}, W(t_n))$. Since $z_{t_m} - z_t = O(\delta^2)$ and $W(t_m) - W(t_n) = O(\delta)$, it suffices to take the terms up to the first derivative of F with respect to z and the second derivative of F with respect to W, and no mixed terms. Hence,

$$\begin{aligned}
&\tilde{h}(\phi_m(w_0) - W(t_m)) - \tilde{h}(\phi_n(w_0) - W(t_n)) \\
&\quad = \partial_z F_{(z_{t_n}, W(t_n))}(z_{t_m} - z_{t_n}) + \partial_W F_{(z_{t_n}, W(t_n))}(W(t_m) - W(t_n)) \\
&\qquad + \tfrac{1}{2}\partial_W^2 F_{(z_{t_n}, W(t_n))}(W(t_m) - W(t_n))^2 + O(\delta^3).
\end{aligned}$$

(Since z is complex, $\partial_z F_{(z_{t_n}, W(t_n))}$ is actually a linear map from $\mathbb{C}$ to $\mathbb{R}$.) By (4.4), the conditional expectation of the left-hand side given $\gamma[0, n]$ is $O(\delta^3)$. After calculating the derivatives and applying (4.6), we get

$$\begin{aligned}
O(\delta^3) &= 2\,\mathrm{Im}\big((\phi_n(w_0) - W(t_n))^{-2}\big)\mathbf{E}[t_m - t_n | \gamma[0, n]] \\
&\quad - \mathrm{Im}\big((\phi_n(w_0) - W(t_n))^{-1}\big)\mathbf{E}[W(t_m) - W(t_n) | \gamma[0, n]] \\
&\quad - \tfrac{1}{2}\,\mathrm{Im}\big((\phi_n(w_0) - W(t_n))^{-2}\big)\mathbf{E}[(W(t_m) - W(t_n))^2 | \gamma[0, n]].
\end{aligned} \tag{4.7}$$

We now assume that $R > \delta^{-2}$. The Koebe distortion theorem (again, see [2], Section 1.3) then implies that a vertex $w_1 \in V(TG)$ closest to p_n satisfies $|\phi_n(w_1) - i - W(t_n)| = |\phi_n(w_1) - \phi_n(p_n)| = O(\delta^2)$. The Koebe distortion theorem also shows that a vertex $w_2 \in V(TG)$ closest to $\phi_n^{-1}(i + W(t_n) + 1/100)$ satisfies $|\phi_n(w_2) - i - W(t_n) - 1/100| = O(\delta^2)$ and $|w_2 - p_n| < \mathrm{rad}_{p_n}(D)/6$. Consequently, we may apply (4.7) with w_0 replaced by each of w_1, w_2. With $w_0 = w_1$, we get (4.1). Now eliminating the term $\mathbf{E}[W(t_m) - W(t_n) | \gamma[0, n]]$ from (4.7) [since it is $O(\delta^3)$] and applying (4.7) with $w_0 = w_1$ gives (4.2). $\square$

COROLLARY 4.3. *Fix $T \geq 1$. As $\rho \to \infty$, the restriction of $t \mapsto W(t/4)$ to $[0, T]$ converges in law to the corresponding restriction of standard Brownian motion.*

PROOF. Let $\varepsilon \in (0, 1)$, and let $\tilde{T} := \sup\{t \in [0, T] : |W(t/4)| \leq \varepsilon^{-1}\}$. Let $I := \{n \in \mathbb{N} : t_n \leq \tilde{T}\}$. In order to apply Proposition 4.1 at every $n \in I$, we need to verify $\mathcal{A}_1(n) \cap \mathcal{A}_2(n)$ for such n. From [9], Lemma 2.1, we get that $t_N = \infty$ or $\{W(t) : t \in [0, t_N]\}$ is unbounded, which implies $\mathcal{A}_1(n)$ for $n \in I$. Since $\phi_n(p_n) = i + W(t_n)$ and $g_{t_n} = \phi_n \circ \phi^{-1}$, we have $g_{t_n} \circ \phi(p_n) = i + W(t_n)$. We claim that there is a compact subset $K \subset \mathbb{H}$, which depends only on ε and T, such that $\phi(p_n) \in K$ holds for each $n \in I$. Indeed, $g_t \circ \phi(p_n)$ flows according to (2.1) starting from $\phi(p_n)$ at $t = 0$ to $i + W(t_n)$ at $t = t_n$. For every $t \in [0, t_n]$, we have $\operatorname{Im} g_t \circ \phi(p_n) \geq 1$, by the monotonicity of $\operatorname{Im} g_t$ with respect to t. By (2.1), this shows that $|\partial_t g_t \circ \phi(p_n)| = O(1)$. Hence, the bound $|\phi(p_n)| \leq 1 + |W(t_n)| + O(T) \leq 1 + \varepsilon^{-1} + O(T)$. We may therefore take $K = \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 1, |z| \leq O(T + \varepsilon^{-1})\}$. Since $\phi(p_n)$ lies in a compact subset of $\mathbb{H}$, the Koebe distortion theorem implies that $\rho \leq O(1) \operatorname{rad}_{p_n}(D)$. Thus we may assume that $\mathcal{A}_2(n)$ holds for every $n \in I$ provided we take $\rho \geq R'$ for some constant $R' = R'(\varepsilon, T, \delta)$.

Now the proof that the restriction of $t \mapsto W(t/4)$ to $[0, \tilde{T}]$ converges in law to the corresponding stopped Brownian motion follows from the proposition and the Skorokhod embedding theorem, as in [9], Section 3.3. Standard Brownian motion is unlikely to hit $\{-\varepsilon^{-1}, \varepsilon^{-1}\}$ before time T if ε is small. Thus, we obtain the corollary by taking a limit as $\varepsilon \searrow 0$. $\square$

5. Local Hausdorff convergence to SLE$_4$. Let $\mathrm{d}_H(\cdot, \cdot)$ denote the Hausdorff distance; that is, for two nonempty sets $A, B \subset \mathbb{C}$,

$$\mathrm{d}_H(A, B) := \max\left\{\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |b - a|\right\}.$$

LEMMA 5.1. *For every $T \geq 1$ and $\varepsilon > 0$ there is some $R = R(\varepsilon, T)$ so that if $\rho > R$, then there is a coupling of γ^ϕ and $\tilde{\gamma}$ so that*

$$\mathbf{P}\big[\sup\{\mathrm{d}_H(\gamma^\phi[0, t], \tilde{\gamma}[0, t]) : 0 \leq t \leq T\} > \varepsilon\big] < \varepsilon.$$

PROOF. We know that $\tilde{\gamma}$ is a simple path, from [13]. Let $\tilde{g}_t$ be the SLE$_4$ Loewner chain corresponding to $\tilde{\gamma}$, and let B be the Brownian motion so that the driving process for $\tilde{g}_t$ is $B(4t)$. Then $\tilde{g}_t$ is obtained by solving (2.1) with $W(t)$ replaced by $B(4t)$.

Let $S(T, \varepsilon)$ denote the set of points in $\overline{\mathbb{H}}$ whose distance from $\tilde{\gamma}[0, T]$ is ε. Let $s \in S(T, \varepsilon)$. By continuity of solutions of differential equations, there is some $\delta = \delta(s, B) > 0$ such that if $W : [0, T] \to \mathbb{R}$ is measurable and satisfies $\sup\{|W(t) - B(4t)| : t \in [0, T]\} \leq \delta$, then the Loewner chain corresponding to W

satisfies $\tau(s) > T$ (this is also easy to verify directly). Moreover, the same δ would apply to every $s' \in \overline{\mathbb{H}}$ sufficiently close to s. By compactness of $S(T, \varepsilon)$, there is some $\delta = \delta(B) > 0$, which would work for every $s \in S(T, \varepsilon)$. Now, this δ is random, as it depends on B, but it is a.s. positive. Therefore, there is a nonrandom $\delta_0 > 0$, depending only on T and ε, such that δ_0 would work for B with probability at least $1 - \varepsilon/2$.

By Corollary 4.3 (and the well-known relation between convergence in law and a.s. convergence [3], Theorem 11.7.2), when ρ is sufficiently large, we may couple γ with standard Brownian motion $B(t)$ so that

$$\mathbf{P}\bigl[\sup\{|W(t) - B(4t)| : t \in [0, T]\} \geq \delta_0\bigr] < \varepsilon/2,$$

where now $W(t)$ is the driving process for γ^{ϕ}. Consequently, when ρ is sufficiently large,

$$\mathbf{P}\bigl[\gamma^{\phi}[0, T] \cap S(T, \varepsilon) = \varnothing\bigr] \geq 1 - \varepsilon.$$

Since $\gamma^{\phi}[0, T]$ is connected and contains 0, when $\gamma^{\phi}[0, T]$ is disjoint from $S(T, \varepsilon)$ every point in $\gamma^{\phi}[0, T]$ is within distance ε from $\tilde{\gamma}[0, T]$.

Now consider a sequence of pairs (D, ϕ) such that $\rho \to \infty$. For each such pair we take a coupling of the corresponding $\gamma^{\phi}[0, T]$ and B such that $\sup\{|W(t) - B(4t)| : t \in [0, T]\} \to 0$ in probability. Fix some $t \in [0, T]$. Since the collection of probability measures on the (compact) Hausdorff space of closed nonempty subsets of $\overline{\mathbb{H}} \cup \{\infty\}$ is compact under convergence in law, by passing to a subsequence, if necessary, we get a coupling of B and a Hausdorff limit Γ_t of $\gamma^{\phi}[0, t]$. By the above, Γ_t is contained in $\tilde{\gamma}[0, T]$. Moreover, it is clearly connected. Note that the Carathéodory kernel theorem ([12], Theorem 1.8) implies that the maps $g_t : \mathbb{H} \setminus \gamma^{\phi}[0, t] \to \mathbb{H}$ converge to the normalized conformal map from $\mathbb{H} \setminus \Gamma_t$ to $\mathbb{H}$. Consequently, the capacity of Γ_t is t a.s. Thus, we conclude that $\Gamma_t = \tilde{\gamma}[0, t]$ a.s. Since the limit does not depend on the subsequence, it follows that $\tilde{\gamma}[0, t]$ is a.s. the Hausdorff limit Γ_t of $\gamma^{\phi}[0, t]$. As $t \in [0, T]$ is arbitrary, we conclude that a.s. $\Gamma_t = \tilde{\gamma}[0, t]$ for every rational t in $[0, T]$. The lemma now follows, since $\tilde{\gamma}$ is continuous and $\Gamma_t \supset \Gamma_{t'}$ when $t > t' \geq 0$. $\square$

6. Improving the topology. Lemma 5.1 gives some form of convergence of γ^{ϕ} to $\tilde{\gamma}$. Our goal now is to improve the quality of the convergence, in two ways. First, we want to show that the convergence is locally uniform (i.e., uniform on compact intervals $[0, T] \subset [0, \infty)$). Later, it will be shown that the convergence is uniform when we use the metric d_* on $\overline{\mathbb{H}} \cup \{\infty\}$.

To understand the issues here, we describe two examples where one form of convergence holds and another fails. We start with an example similar to one appearing in [9], Section 3.4. Let $\varepsilon > 0$, $a_j := i\varepsilon(1 - j^{-1})$ and $b_j := ij\varepsilon$. Let α_ε be the polygonal path determined by the points $a_1, b_1 + \varepsilon, a_2, b_2 - \varepsilon, a_3, b_3 + \varepsilon, a_4, b_4 - \varepsilon$, etc. Let α_0 be the path $t \mapsto it$, $t \geq 0$, reparameterized by capacity. Then the path α_ε reparameterized by capacity converges to α_0 in the sense of Lemma 5.1.

Moreover, the Loewner driving process for α_ε converges locally uniformly to the constant 0, which is the driving process for α_0. However, one cannot reparameterize α_ε so that $\alpha_\varepsilon \to \alpha_0$ locally uniformly.

To illustrate the second issue, consider the polygonal path β_ε determined by the points $0, i\varepsilon^{-1}, i+\varepsilon, \infty$, where the last segment can be chosen as any ray from $i+\varepsilon$ to ∞ in $\mathbb{H}$. Then β_ε, reparameterized by capacity, does converge locally uniformly to α_0. However, it does not converge uniformly with respect to the metric d_*.

6.1. *Discrete excursions.* The purpose of this subsection is to develop a tool which will be handy for proving some upper bounds on probabilities of rare events for the HE, the discrete excursion measure. It is a discrete analogue of the (two-dimensional) Brownian excursion as introduced in [10], Section 2.4. A slightly different variant of the continuous Brownian excursion was studied in [8].

Let D be a domain in the plane whose boundary is a subgraph of the triangular lattice TG. (We work here with the simple random walk on TG, but the results apply more generally to other walks on other lattices.) Let V_∂ denote the set of vertices in $V(TG) \cap \partial D$. A directed edge of TG is just an edge of TG with a particular choice of orientation (i.e., a choice of the initial vertex). If $e = [u, v]$ is a directed edge of TG, then $\operatorname{rev}(e) = [v, u]$ will denote the same edge with the reversed orientation. Let $\overrightarrow{E} = \overrightarrow{E}(D)$ denote the set of directed edges of TG whose interiors intersect D and whose initial vertex is in V_∂. Let $\overleftarrow{E} = \overleftarrow{E}(D)$ denote the set of directed edges of TG whose interiors intersect D and whose terminal vertex is in V_∂; that is, $\overleftarrow{E} = \operatorname{rev}(\overrightarrow{E})$. Let $E_1 \subset \overrightarrow{E}$ and $E_2 \subset \overleftarrow{E}$. For every $v \in V_\partial$, let X^v be a simple random walk on TG that starts at v and is stopped at the first time $t \geq 1$ such that $X^v(t) \notin D$. Let ν^v denote the restriction of the law of X^v to those walks that use an edge of E_1 as the first step and use an edge of E_2 as the last step. (This is zero if v is not adjacent to an edge in E_1, and generally it is not a probability measure.) Finally, let $\nu = \nu_{(D,E_1,E_2)} := \sum_{v \in V_\partial} \nu^v$. This is a measure on paths starting with an edge in E_1, ending with an edge in E_2 and staying in D in between. It will be called the discrete excursion measure from E_1 to E_2 in D. When $E_2 = \overleftarrow{E}$, we will often abbreviate $\nu_{(D,E_1)} = \nu_{(D,E_1,\overleftarrow{E})}$.

LEMMA 6.1. *Let D be as above, and let $E_1 \subset \overrightarrow{E}$. Fix $v \in V(TG) \cap D$, and for every path ω let $n_v(\omega)$ be the number of times ω visits v. Then*

$$\int n_v(\omega)\, d\nu_{(D,E_1)}(\omega) = H\bigl(v, \operatorname{rev}(E_1)\bigr), \tag{6.1}$$

where $H(v, E) = H_D(v, E)$ denotes the probability that a simple random walk started from v will first exit D through an edge in E. In particular, $\int n_v\, d\nu_{(D,\overrightarrow{E})} = 1$.

PROOF. Let (Ω, μ) denote the probability space of random walks starting at v and stopped when they first exit D. For a pair $(\omega_1, \omega_2) \in \Omega^2$, let $f(\omega_1, \omega_2)$ denote the reversal of ω_1 followed by ω_2. Then f is a map from Ω^2 to the support of $\nu = \nu_{(D,\overrightarrow{E})}$. Clearly, $\mu \times \mu(\{\omega_1, \omega_2\}) = \nu(\{f(\omega_1, \omega_2)\})$. If ω' is in Ω', the support of ν, then the cardinality of the preimage $f^{-1}(\omega') \subset \Omega^2$ is precisely $n_v(\omega')$. Consequently, we have

$$1 = \mu \times \mu(\Omega^2) = \sum_{\omega' \in \Omega'} |f^{-1}(\omega')| \nu(\{\omega'\}) = \int n_v \, d\nu.$$

This proves the claim in the case $E_1 = \overrightarrow{E}$. The general case is similarly established. $\square$

COROLLARY 6.2. *Let D and E_1 be as above, and let $v \in V(TG) \cap D$. Assume that ∂D is connected. Let B be the ball centered at v whose radius is $\frac{1}{2}\mathrm{rad}_v(D)$, and let Γ_B be the set of paths that visit B. Then*

$$c^{-1} H_D\bigl(v, \mathrm{rev}(E_1)\bigr) < \nu_{(D,E_1)}(\Gamma_B) < c H_D\bigl(v, \mathrm{rev}(E_1)\bigr)$$

for some absolute constant c.

PROOF. It is well known that there is an absolute constant c with $c^{-1} < G_D(w, v) < c$ for every $w \in B$ such that $|v - w|$ is at least half the radius of B, where G_D is Green's function. See, for example, [9], (3.5), where the radius of the ball B is different, but the same proof applies. Consequently, given that a random walk hits B, its expected number of visits to v before exiting D is between c^{-1} and c. Thus, the corollary follows from (6.1). $\square$

It is also important to note that $\|\nu_{(D,E_1,E_2)}\| = \|\nu_{(D,\mathrm{rev}(E_2),\mathrm{rev}(E_1))}\|$; that is, the total mass of $\nu_{(D,E_1,E_2)}$ is equal to that of $\nu_{(D,\mathrm{rev}(E_2),\mathrm{rev}(E_1))}$. This is proved by reversing the paths, which gives a measure-preserving bijection between the support of these two measures.

6.2. *Revisit probability estimate.* We now return to the setup and notation of Section 3. Fix some ball B that intersects $\gamma[0, n]$. The present goal is to get an upper bound for the conditional probability $\mathbf{P}[\gamma[n, N] \cap B \neq \varnothing | \gamma[0, n]]$, under some simple geometric assumptions. To simplify notation, instead of discussing conditional probabilities, we shall instead obtain a bound on $\mathbf{P}[\gamma[0, N] \cap B \neq \varnothing]$ for a ball B intersecting ∂D. The conditional probability estimate then readily follows, because of the Markovian property mentioned in Remark 3.2.

PROPOSITION 6.3. *Let $0 < r < R < \infty$. Let $B(z, r)$ be a ball of radius r intersecting ∂D, and let $B(z, R)$ be the concentric ball with radius R. Suppose that there is no component of $B(z, R) \cap D$ whose boundary intersects both A_- and A_+. Then*

$$\mathbf{P}\bigl[\gamma[0, N] \cap B(z, r) \neq \varnothing\bigr] \leq O(1)(r/R)^{\hat{c}}, \tag{6.2}$$

where $\hat{c} > 0$ is a universal constant.

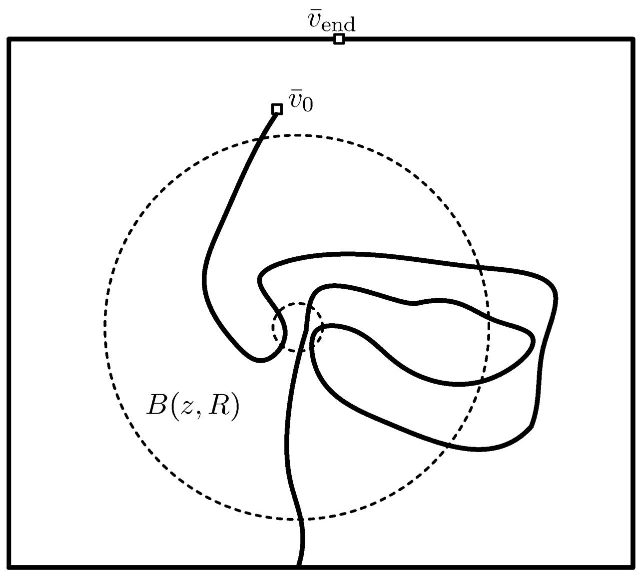

FIG. 3. *A possible boundary of the domain D.*

The reader should think about the case where ∂D is rather wild geometrically, as in Figure 3. In particular, we need to include the case where $B(z, r)$ intersects both A_- and A_+. The precise form of the right-hand side of (6.2) will not be important in the following. What is essential is only that it tends to zero with r/R. If necessary, one can probably show that $\hat{c} = 1/2$ works, by using the following proof and a discrete version of the Beurling projection theorem. (Some discrete version of the Beurling projection theorem is given in [6], Theorem 2.5.2, but it is not precisely the same statement that would apply here.)

PROOF OF PROPOSITION 6.3. We will assume that $r \geq 5$. This involves no loss of generality, because the distance from point on $\gamma[0, N]$ to ∂D is bounded from zero, except near $\bar{v}_0$ and $\bar{v}_{\text{end}}$, and the result clearly holds when R is bounded.

Let $\mathcal{Q}$ be the event that there is a j such that $\bar{v}_j \in B(z, r)$, and on $\mathcal{Q}$ let σ be the least such j. We assume, with no loss of generality, that $R > 9r$, say, since otherwise the statement of the proposition is trivial. Consider the ball $B(z, 3r)$. By our assumptions, each component of $B(z, 3r) \cap D$ has boundary entirely in A_- or entirely in A_+. On the event $\mathcal{Q}$, let S be the connected component of $B(z, 3r) \cap D$ intersecting $\gamma[\sigma - 1, \sigma]$. Let $\mathcal{Q}_- \subset \mathcal{Q}$ be the event that $\partial S \subset A_-$. By symmetry, it is enough to prove that $\mathbf{P}[\mathcal{Q}_-] \leq O(1)(r/R)^{\hat{c}}$.

Let E_- denote the set of directed edges in $\overrightarrow{E} = \overrightarrow{E}(D)$ whose initial vertex is in $B(z, 3r) \cap \mathfrak{h}_0^{-1}(0)$. Let $D_n := D \setminus V_n$, where V_n is as in Section 3.1 and let E_+^n denote the set of directed edges connecting vertices in D_n to vertices in $\partial D_n \cap \mathfrak{h}_n^{-1}(1)$. The reason that the measures ν are useful here is because the total mass $\|\nu_n\|$ of $\nu_n := \nu_{(D_n, E_-, E_+^n)}$ is a martingale. One easy way to deduce this is by considering walks that hit the vertex v_n before any other vertex in ∂D_n (except for the initial vertex of the walk). Given the part of such a walk up to its first visit to v_n, the probability that it first exits D_{n-1} using an edge from E_+^{n-1} is precisely $h_{n-1}(v_n)$, which is just the probability that $\mathfrak{h}_n(v_n) = 1$. Alternatively, Lemma 3.1 implies that $\|\nu_n\|$ is a martingale, because the total mass of $\|\nu_n\|$ is just a linear combination of the values of h_n on the terminal vertices of E_-.

Since $\|\nu_n\|$ is a nonnegative martingale, the optional stopping theorem implies that

$$\|\nu_0\| \geq \mathbf{E}[\mathbb{1}_{\mathcal{Q}_-} \|\nu_\sigma\|]. \tag{6.3}$$

The bound on $\mathbf{P}[\mathcal{Q}_-]$ will be proven by estimating $\|\nu_0\|$ and estimating $\|\nu_\sigma\|$ on the event $\mathcal{Q}_-$. By our assumption, every path starting from an edge in E_- and ending with an edge in E_+ which stays in D in between must exit $B(z, R)$. Consequently, $\|\nu_0\| \leq \|\nu_{(D^R, E_-, E_R)}\|$, where D^R is the intersection of D with $B(z, R)$ (adjusted to have as its boundary a subgraph of the grid TG), and E_R are the edges connecting vertices in D^R to vertices in $D \setminus D^R$. By Corollary 6.2, under the measure $\nu_{(B(z,R), \mathrm{rev}(E_R))}$, the expected number of paths that hit $B(z, R/2)$ is $O(1)$. For any $r' \in (r, R)$, a random walk starting near the boundary of $B(z, r')$ has probability bounded away from 0 to exit D before hitting $B(z, r'/2)$. Thus, the probability that a random walk started near the boundary of $B(z, R/2)$ will hit $B(z, 3r)$ before exiting D is $O(1)(r/R)^{\hat{c}}$, for some constant $\hat{c} > 0$. Hence,

$$\|\nu_0\| \leq \left\|\nu_{(D^R, E_-, E_R)}\right\| = \left\|\nu_{(D^R, \mathrm{rev}(E_R), \mathrm{rev}(E_-))}\right\| \leq O(1)(r/R)^{\hat{c}}. \tag{6.4}$$

Consider now the case where $\mathcal{Q}_-$ happens. As $\gamma[0, \sigma]$ crosses the annulus $B(z, 3r) \setminus B(z, r)$, there must be an arc among the connected components of $\partial B(z, 2r) \cap D \setminus \gamma[0, \sigma]$ that connects the right side of $\gamma[0, \sigma]$ (where $\mathfrak{h}_n$ takes the value 1) to the boundary of S (where $\mathfrak{h}_n$ takes the value 0). By considering vertices along this arc, we can find a vertex v close to $\partial B(z, 2r)$ from which the ratio between $H_{D_\sigma}(v, \mathrm{rev}(E_-))$ and $H_{D_\sigma}(v, E_+^\sigma)$ is bounded and bounded away from zero by universal constants (because these quantities do not vary by more than a constant factor when moving from a vertex to its neighbor), or else there is an edge in $E_- \cap E_+^\sigma$. In the latter case, clearly $\|\nu_{(D_\sigma, E_-, E_+^\sigma)}\|$ is bounded away from zero. Consider therefore the case where such a v exists. Since random walk starting from v has probability bounded away from zero to complete a loop going around the annulus $B(z, 3r) \setminus B(z, r)$ before exiting it, we have $H_{D_\sigma}(v, \mathrm{rev}(E_-)) + H_{D_\sigma}(v, E_+^\sigma)$ bounded away from zero. Consequently, each of these summands is bounded away from zero. Let B be the ball centered at v whose radius is half the distance from v to ∂D_σ. By Corollary 6.2, the measure under $\nu_{(D_\sigma, E_-)}$ of the set of paths hitting B is bounded away from zero. Since $H_{D_\sigma}(v, E_+^\sigma)$ is bounded away from zero, a random walk started at any vertex in B has probability bounded away from zero to exit D_σ in E_+^σ, by the Harnack principle (e.g., $k = 0$ in [9], Lemma 5.2). Consequently, we see that also in this case $\|\nu_{(D_\sigma, E_-, E_+^\sigma)}\|$ is bounded away from zero on the event $\mathcal{Q}_-$ by an absolute constant. Combining this with (6.3) and (6.4) establishes $\mathbf{P}[\mathcal{Q}_-] = O(r/R)^{\hat{c}}$. The proof for the event $\mathcal{Q} \setminus \mathcal{Q}_-$ is entirely symmetric. $\square$

6.3. *Local uniform convergence.*

PROPOSITION 6.4 (Local uniform convergence). *In the setting of Theorem 3.3, for every fixed $T > 0$, there is a coupling of γ^ϕ and $\tilde{\gamma}$ so that*

$$\sup\{|\gamma^\phi(t) - \tilde{\gamma}(t)| : 0 \le t \le T\} \to 0$$

in probability as $\rho \to \infty$.

PROOF. Consider a sequence of pairs (D, ϕ), with $\rho \to \infty$. Using Lemma 5.1, we couple $\tilde{\gamma}$ and the sequence γ^ϕ so that for each $t \ge 0$ the set $\tilde{\gamma}[0,t]$ is a.s. the Hausdorff limit of $\gamma^\phi[0,t]$.

Our strategy will be to prove that the curves $\gamma^\phi[t,t'']$ converge to $\tilde{\gamma}[t,t'']$ in the Hausdorff sense for all rational pairs $0 \le t, t'' \le T$ (for notational convenience, we may assume T is also rational). We will prove this, in turn, by showing that for all rationals $t_0 < t < t' < t'' < t_1 < T$, the Hausdorff limit of $\gamma^\phi[t,t'']$ is a.s. disjoint from both $\tilde{\gamma}[0,t_0]$ and $\tilde{\gamma}[t_1,T]$. We will begin by restricting our attention to a large compact set and using Proposition 6.3 to derive upper bounds for the probability that $\gamma^\phi[s(s(t)),\infty)$ comes close to $\gamma^\phi(s(t))$, where $t < s(t) < s(s(t))$ are chosen below so that $\gamma^\phi(s(t))$ is "well exposed" and so that the assumptions of Proposition 6.3 apply.

Let $\varepsilon > 0$, and let F be some compact subset of $\mathbb{H}$. For $t \in [0,\infty)$, let $s(t)$ be the first $s > t$ such that $\gamma^\phi(s)$ is in the unbounded connected component of $\{z \in \mathbb{H} : \operatorname{dist}(z, \gamma^\phi[0,t]) \ge \varepsilon\}$. Fix some $t, \delta > 0$, and let $\mathcal{A} = \mathcal{A}(t,\delta)$ be the event that $B(\gamma^\phi(s(t)), 2\varepsilon) \subset F$ and $\gamma^\phi[s(s(t)),\infty) \cap B(\gamma^\phi(s(t)),\delta) \neq \varnothing$. We claim that if ρ is sufficiently large, then

$$\mathbf{P}[\mathcal{A}] \le C_F(\delta/\varepsilon)^{\hat{c}}, \tag{6.5}$$

where $\hat{c}$ is the same constant as in Proposition 6.3, and C_F is a constant depending only on F.

By the Koebe distortion theorem, there is a constant $c = c(F) > 0$ such that for every $z, z' \in F$ with $z \neq z'$ we have

$$c^{-1}\rho \le \frac{|\phi^{-1}(z) - \phi^{-1}(z')|}{|z - z'|} \le c\rho. \tag{6.6}$$

[In other words, the metric in F is comparable to the metric in $\phi^{-1}(F) \subset D$ scaled by ρ^{-1}.] Let n be the least integer such that $\phi(\gamma[0,n]) \supset \gamma^\phi[0, s(s(t))]$. (Recall that the parameterization of γ is not by capacity, but is proportional to arclength.) Now condition on $\gamma[0,n]$, and assume that $B(\gamma^\phi(s(t)), 2\varepsilon) \subset F$. (If that has zero probability, then $\mathbf{P}[\mathcal{A}]=0$.) Let $z_0 := \phi^{-1}(\gamma^\phi(s(t)))$. Assume that ρ is sufficiently large so that $|\phi(\gamma(n)) - \gamma^\phi(s(s(t)))| < \varepsilon/3$; indeed, how large ρ is required to be can be determined from the constant c in (6.6). The metric comparison (6.6) implies that $\phi(B(z_0, \rho\varepsilon/(2c))) \subset B(\gamma^\phi(s(t)), \varepsilon/2)$. This and the definition of $s(s(t))$ imply that there is a path in $D \setminus (\gamma[0,n] \cup B(z_0, \rho\varepsilon/(2c)))$ connecting $\gamma(n)$ and $\bar{v}_{\mathrm{end}}$. We may then apply Proposition 6.3 with D replaced by the D_n of Remark 3.2 to conclude that conditioned on $\gamma[0,n]$ the probability that $\gamma[n,N]$ intersects $B(z_0, \delta'\rho)$ is $O(1)(2c\delta'/\varepsilon)^{\hat{c}}$. Now (6.5) follows by another application of (6.6).

Let $\Gamma_t^{t'}$ denote the Hausdorff limit of $\gamma^\phi[t,t']$, when it exists. Note that we may pass to a subsequence of pairs (D,ϕ) so that $\gamma^\phi[t,t']$ converges in law. Consequently, by using [3], Theorem 11.7.2 again, we may assume that almost surely the limits $\Gamma_t^{t'}$, $\overline{s}_t := \lim_{(D,\phi)} s(t)$, $\overline{ss}_t := \lim_{(D,\phi)} s(s(t))$ and $p_t := \lim_{(D,\phi)} \gamma^\phi(s(t))$ exist for every pair of rationals $0 < t < t' < \infty$.

Let $t > 0$ be rational, and let $S(t,b)$ be the set of $t' > t$ such that $\tilde{\gamma}(t')$ is in the unbounded component of $\{z \in \mathbb{H} : \mathrm{dist}(z, \tilde{\gamma}[0,t]) \geq b\}$. By construction, it is clear that $\mathrm{dist}(p_t, \tilde{\gamma}[0,t]) = \varepsilon$, $p_t \in \tilde{\gamma}[0, \overline{ss}_t]$, $\overline{s}_t < \inf S(t, 2\varepsilon)$ and $\overline{ss}_t < \inf S(t, 3\varepsilon)$. Let t' be a rational satisfying $\overline{ss}_t < t' < \inf S(t, 3\varepsilon)$.

Now, the following a.s. statements will hold on the event $\inf S(t,3\varepsilon) < T$ and $B(p_t, 3\varepsilon) \subset F$. First, by (6.5), we have a.s. $p_t \notin \Gamma_{t'}^T$. Note that $p_t \in \tilde{\gamma}[0,t'] \setminus \tilde{\gamma}[0,t]$. Since $\tilde{\gamma}$ is a simple path, $\tilde{\gamma}[0,T] \setminus p_t$ has two disjoint components, one containing $\tilde{\gamma}[0,t]$, and the other containing $\tilde{\gamma}(T)$. Since $\Gamma_{t'}^T$ is connected and contains $\tilde{\gamma}(T)$, we conclude that a.s. $\Gamma_{t'}^T \cap \tilde{\gamma}[0,t] = \varnothing$ [on the event $B(p_t, 3\varepsilon) \subset F$, $\inf S(t,3\varepsilon) < T$]. Now let t'' be any rational in (t,T). Since $\tilde{\gamma}$ is a simple path, there is some small $\varepsilon > 0$ such that $t'' \in S(t, 3\varepsilon)$.

Because F is an arbitrary compact subset of $\mathbb{H}$, $p_t \in \mathbb{H}$ a.s. and in the above discussion $\varepsilon > 0$ was arbitrary, it follows that $\Gamma_{t''}^T \cap \tilde{\gamma}[0,t] = \varnothing$ a.s. Since $\Gamma_t^{t''} \subset \tilde{\gamma}[0,T]$ and is a.s. disjoint from $\tilde{\gamma}[0,t_0] \cup \tilde{\gamma}[t_1,T]$ whenever $t_0 < t < t'' < t_1$ (by the above), it follows that $\Gamma_t^{t''} = \tilde{\gamma}[t,t'']$ a.s. for every pair of rationals $0 < t < t'' < T$. Consequently, $\Gamma_t^{t''} = \tilde{\gamma}[t,t'']$ a.s. for every pair $0 \leq t \leq t'' \leq T$ (the cases $t=0$ and $t''=T$ are similarly treated). Thus, $\lim_{(D,\phi)} \sup_{t \in [0,T]} |\gamma^\phi(t) - \tilde{\gamma}(t)| = 0$. Since every sequence of (D,ϕ) with $\rho \to \infty$ has a subsequence such that this holds, this also holds without passing to a subsequence. The proposition is thus established. $\square$

6.4. *Uniform transience and conclusion.* Since $\phi(\bar{v}_{\mathrm{end}}) = \infty$, γ^ϕ is transient; that is, $\lim_{t\to\infty} |\gamma^\phi(t)| = \infty$. The following is a uniform version of this statement.

PROPOSITION 6.5. *For every $\varepsilon > 0$ and $R > 0$ there is a $T = T(R,\varepsilon)$ such that*

$$\mathbf{P}\big[\gamma^\phi[T,\infty) \cap B(0,R) \neq \varnothing\big] < \varepsilon$$

if ρ is sufficiently large.

The reader may note the clear similarity with Proposition 6.3. The main difference is that there the path considered was in the domain D, whereas here the path γ^ϕ is in the image under the conformal map, that is, in the upper half plane $\mathbb{H}$. Indeed, the proof is quite similar.

The following lemma about the excursion measures will be needed.

LEMMA 6.6. *Let $R > 0$. Suppose that at each vertex $v \in V(TG) \cap \phi^{-1}(\overline{\mathbb{H} \cap B(0,R)})$ a simple random walk X^v is started, and the walk is stopped at the first time $t \geq 1$ such that $X^v(t)$ exits D or $|\phi(X^v(t))| \notin (R, 2R)$. Then the expected number of walks which stop when $|\phi(X^v(t))| \geq 2R$ is $O(1)$.*

PROOF. Let $V_D := V(TG) \cap \overline{D}$. For $v \in V_D$, let Y^v denote a simple random walk on V_D, where at each step the walk jumps with equal probability along each of the edges e with $e \subset \overline{D}$ and $Y^v(t) \in e$. Let $V_0 := \{v \in V_D : |\phi(v)| \leq R\}$ and $V_1 := \{v \in V_D : |\phi(v)| \geq 2R\}$. Let M be the expected number of walks Y^v with $v \in V_0$ such that $Y^v(\sigma) \in V_1$, where $\sigma := \inf\{t \geq 1 : Y^v(t) \in V_0 \cup V_1\}$. It clearly suffices to show that $M = O(1)$. (The difference from the X^v is that the Y^v are reflected off of ∂D, rather than killed there.)

For a function f on V_D and $v \in V_D$, let $\Delta f(v) := \sum_{[v,u]}(f(u) - f(v))$, where the sum extends over edges $e = [v,u]$ containing v such that $e \subset \overline{D}$. Now let $g : V_D \to [0,1]$ be the unique function such that $\Delta g(v) = 0$ for $v \in V_D \setminus (V_1 \cup V_2)$, $g = 0$ on V_0 and $g = 1$ on V_1; that is, g is harmonic in $V_D \setminus (V_0 \cup V_1)$ with the appropriate boundary values. It is immediate to verify that

$$M = \sum_{v \in V_0} \Delta g(v)/d_v \leq \sum_{v \in V_0} \Delta g(v),$$

where d_v is the number of edges containing v in $\overline{D}$, since $g(v)$ is the probability that Y^v hits V_1 before V_0. Because $\sum_{v \in V_D} \Delta g(v) = 0$, we have

$$M \leq -\sum_{v \in V_1} \Delta g(v).$$

Let $E(f) := \sum_{[v,u]}(f(v) - f(u))^2$, where the sum runs over all edges in $\overline{D}$. It is well known (and simple to show) that g minimizes E among functions mapping V_1 to 1 and V_0 to 0. For each edge $[v,u]$, we may distribute the quantity $(f(v) - f(u))^2$ by giving $f(v)(f(v) - f(u))$ to the vertex v and $f(u)(f(u) - f(v))$ to the vertex u. Consequently, by summing over the contributions to each vertex, we find

$$E(f) = -\sum_v f(v)\Delta f(v).$$

Hence, $E(g) = -\sum_{v \in V_1} \Delta g(v)$, which gives $M \leq E(g)$.

There are several different ways to estimate $E(g)$ and complete the proof. We opt for an easy and short argument, which unfortunately does require terminology and results from the literature. (Similar arguments appear, e.g., in [2, 5, 11].)

Let A be the annulus $A := B(0, 2R) \setminus B(0, R)$. Let $\tilde{g}(v) := (|\phi(v)| - R)/R$ when $v \in V_D \setminus (V_0 \cup V_1)$ and $\tilde{g}(v) := j$ when $v \in V_j$, $j = 0, 1$. Then $E(g) \le E(\tilde{g})$, by the characterization of g as a minimizer. Following [14], we say that a set $F \subset \mathbb{C}$ is s-fat if for every disk $B = B(z, r)$ with $z \in F$ and $F \not\subset B$, we have $\operatorname{area}(F \cap B) \ge s \operatorname{area}(B)$. Consider a triangle $\triangle$ of the grid *TG* with $\triangle \subset \overline{D}$. Then $\phi(\triangle)$ is a K-quasidisk, for some constant K, by [4]. [There, it is required that $\triangle \subset D$, but $\triangle \subset \overline{D}$ works too, by standard compactness properties. Besides, it suffices for the argument given below that $\phi(\triangle')$ is a K-quasidisk when $\triangle'$ is a slightly rescaled copy of $\triangle$ contained in the interior of $\triangle$, provided that K does not depend on the scaling factor.) Consequently, by [14], Corollary 2.3, $\phi(\triangle)$ is s-fat for some constant $s > 0$. Thus, $\operatorname{diam}(\phi(\triangle) \cap A)^2 \le O(1) \operatorname{area}(\phi(\triangle) \cap B(0, 3R))$. Hence, $\sum_\triangle \operatorname{diam}(\phi(\triangle) \cap A)^2 = O(R^2)$, where the sum extends over all the triangles of the grid *TG* that are contained in D. Now, if $[u, v]$ is an edge in $\overline{D}$, then there is a grid triangle $\triangle \subset \overline{D}$ with $[u, v] \subset \partial\triangle$. We have then $(\tilde{g}(v) - \tilde{g}(u))^2 \le \operatorname{diam}(\triangle \cap A)^2/R^2$. Since each triangle has three edges, we get

$$M \le E(g) \le E(\tilde{g}) \le 3 \sum_\triangle \operatorname{diam}(\triangle \cap A)^2/R^2 = O(1).$$

This completes the proof. $\square$

PROOF OF PROPOSITION 6.5. We choose $R_1 = R_1(\varepsilon, R)$ much larger than R. Let $t_1 := \inf\{t \ge 0 : |\gamma^\phi(t)| = R_1\}$, and let T be a constant such that

$$\mathbf{P}[T - 1 < \inf\{t \ge 0 : |\tilde{\gamma}(t)| = R_1\}] < \varepsilon/3.$$

Then, by Lemma 5.1, when ρ is sufficiently large, we have $\mathbf{P}[T < t_1] < \varepsilon/3$. Consequently, it suffices to show that

$$\mathbf{P}\big[\gamma^\phi[t_1, \infty) \cap B(0, R) \ne \varnothing\big] < \varepsilon/2. \tag{6.7}$$

Let m be the least integer such that $\phi \circ \gamma[0, m] \supset \gamma^\phi[0, t_1]$. The proof now proceeds as in Proposition 6.3, with only minor changes, which will be henceforth described. Let $\mathcal{Q}$ be the event that there is a $j > m$ such that $|\phi(\gamma(j))| < R$, and on $\mathcal{Q}$ let σ be the least such j. As in the proof of Proposition 6.3, the event $\mathcal{Q}_- \subset \mathcal{Q}$ is defined. For integer $n \in [m, N]$, we consider the excursion measure in $D \setminus V_n$ with excursions started at the vertices in $\alpha := \mathfrak{h}_m^{-1}(0) \cap \phi^{-1}(B(0, 3R))$ and terminating at vertices in $\beta_n := \mathfrak{h}_n^{-1}(1)$. The total mass of this measure is a martingale. It suffices to show that this is very small at $n = m$, but is bounded away from zero at $n = \sigma$ on the event $\mathcal{Q}_-$.

We first do the estimate for $n = m$. The expected number of excursions in D from α that hit $\phi^{-1}(\partial B(0, 6R))$ is the same as the number of excursions in the domain which consists of the grid triangles intersecting $\phi^{-1}(B(0, 6R))$ starting at vertices in $\phi^{-1}(\overline{\mathbb{H}} \setminus B(0, 6R))$ that hit α, by symmetry, and this quantity is bounded by the number of excursions in the domain which is essentially $\phi^{-1}(B(0, 6R) \setminus B(0, 3R))$ starting at vertices in $\phi^{-1}(\overline{\mathbb{H}} \setminus B(0, 6R))$ that hit $\phi^{-1}(B(0, 3R))$. By symmetry again, this is the same as the expected number of excursions in the reverse direction. This quantity is $O(1)$, by Lemma 6.6. Consequently, the expected number of excursions from α in D that cross $\phi^{-1}(\partial B(0, 6R))$ is $O(1)$. It is not hard to see that when ρ is sufficiently large, there will not be any grid edge crossing both $\phi^{-1}(\partial B(0, 6R))$ and $\phi^{-1}(\partial B(0, 7R))$, for example, by considering the harmonic measure from p_0 of such an edge. Now, [9], Lemma 5.4 tells us that a random walk started in $\phi^{-1}(B(0, 7R))$ has probability $o(1)$ to exit $\phi^{-1}(B(0, R_1))$ before exiting D, uniformly as $R_1/R \to \infty$. (That lemma refers to the square grid, but the proof applies here as well. Also, in that lemma the image conformal map is onto the unit disk $\mathbb{U}$, but this is simply handled by choosing an appropriate conformal homeomorphism from $\mathbb{U}$ to $\mathbb{H}$.)

It remains to prove a bound from below for measure of excursions in $D \setminus V_\sigma$ from α to β_σ, on the event $\mathcal{Q}_-$. As in the proof of Proposition 6.3, it suffices to find a vertex v such that the discrete harmonic measure in $D \setminus \gamma[0, \sigma]$ from v of each of the sets α and the left side of $\gamma[0, \sigma]$ is bounded from below. Consider any vertex w near $\phi^{-1}(\partial B(0, 3R))$. The continuous harmonic measure from $\phi(w)$ of $\mathbb{R}$ in the domain $\mathbb{H} \cap (B(0, 4R) \setminus B(0, 2R))$ is bounded from below. By the convergence of discrete harmonic measure to continuous harmonic measure, when ρ is large, a random walk started at w will have probability bounded from below to hit ∂D before exiting $\phi^{-1}(B(0, 4R) \setminus B(0, 2R))$. (Specifically, while not close to the boundary, the random walk behaves like Brownian motion, which is conformally invariant. Once it does get close to the boundary, we may apply [9], Lemma 5.4, say.) As in the proof of Proposition 6.3, on $\mathcal{Q}_-$ we can find a vertex v near $\phi^{-1}(\partial B(0, 3R))$ where the discrete harmonic measure of α is comparable to that of β_σ. Hence, both are bounded away from zero. This completes the proof. $\square$

PROOF OF THEOREM 3.3. The theorem follows immediately from Propositions 6.4 and 6.5. $\square$

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