The Annals of Probability
2005, Vol. 33, No. 5, 1760–1781
DOI 10.1214/009117905000000279

DIFFUSION IN RANDOM ENVIRONMENT AND THE RENEWAL THEOREM[1]

BY DIMITRIOS CHELIOTIS

University of Toronto

According to a theorem of Schumacher and Brox, for a diffusion X in a Brownian environment, it holds that $(X_t - b_{\log t})/\log^2 t \to 0$ in probability, as $t \to \infty$, where b is a stochastic process having an explicit description and depending only on the environment. We compute the distribution of the number of sign changes for b on an interval $[1, x]$ and study some of the consequences of the computation; in particular, we get the probability of b keeping the same sign on that interval. These results have been announced in 1999 in a nonrigorous paper by Le Doussal, Monthus and Fisher [*Phys. Rev. E* **59** (1999) 4795–4840] and were treated with a Renormalization Group analysis. We prove that this analysis can be made rigorous using a path decomposition for the Brownian environment and renewal theory. Finally, we comment on the information these results give about the behavior of the diffusion.

1. Introduction. On the space $\mathcal{W} := C(\mathbb{R})$ consider the topology of uniform convergence on compact sets, the corresponding σ-field of the Borel sets and $\mathbb{P}$ the measure on $\mathcal{W}$ under which the coordinate of the processes $\{w(t) : t \geq 0\}$, $\{w(-t) : t \geq 0\}$ are independent standard Brownian motions.

Also let $\Omega := C([0, +\infty))$, and equip it with the σ-field of Borel sets derived from the topology of uniform convergence on compact sets. For $w \in \mathcal{W}$, we denote by $\mathbf{P}_w$ the probability measure on Ω such that $\{\omega(t) : t \geq 0\}$ is a diffusion with $\omega(0) = 0$ and generator

$$\frac{1}{2} e^{w(x)} \frac{d}{dx}\left(e^{-w(x)} \frac{d}{dx}\right).$$

The construction of such a diffusion is done with scale and time transformation from a one-dimensional Brownian motion (see, e.g., [14, 16]). Using this construction, it is easy to see that, for $\mathbb{P}$-almost all $w \in \mathcal{W}$, the diffusion does not explode in finite time; and on the same set of w's, it satisfies the formal SDE

$$\begin{aligned} d\omega(t) &= d\beta(t) - \tfrac{1}{2} w'(\omega(t))\, dt, \\ \omega(0) &= 0, \end{aligned} \tag{1}$$

Received August 2003; revised August 2004.
[1]Supported in part by an anonymous Stanford Graduate Fellowship, A. Onassis Scholarship and by NSF Grants DMS-00-72331 and DMS-FRG-02-44323.
AMS 2000 subject classifications. Primary 60K37; secondary 60G17, 60J65.
Key words and phrases. Diffusion, random environment, renewal theorem, Brownian motion, Sinai's walk, favorite point.

where β is a one-dimensional standard Brownian motion.

Then consider the space $\mathcal{W} \times \Omega$, equip it with the product σ-field, and take the probability measure defined by

$$d\mathcal{P}(w, \omega) = d\mathbf{P}_w(\omega)\, d\mathbb{P}(w).$$

The marginal of $\mathcal{P}$ in Ω gives a process that is known as diffusion in a random environment; the environment being the function w.

Schumacher [13, 14] proved the following result.

FACT 1. There is a process $b : [0, \infty) \times \mathcal{W} \to \mathbb{R}$ such that, for the formal solution ω of (1), it holds

$$\frac{\omega_t}{(\log t)^2} - b_1\big(w^{(\log t)}\big) \to 0 \qquad \text{in } \mathcal{P} \text{ as } t \to +\infty, \tag{2}$$

where, for $r > 0$, we let $w^{(r)}(s) = r^{-1} w(sr^2)$ for all $s \in \mathbb{R}$.

We will define the process b soon. This result shows the dominant effect of the environment, through the process b, on the asymptotic behavior of the diffusion. The results we prove in this paper concern the process b. In Section 1.2 we commend on their implications for the behavior of the diffusion itself.

Besides this diffusion model, there is a discrete time and space analog, known as Sinai's walk, which was studied first. Sinai's pioneering paper [17] identified the role of the process b in the analogous to (2) limit theorem for the walk. Then Schumacher proved in [13] (see also [14] for the results without the proofs) a more general statement than the above proposition, where the environment w was not necessarily a two-sided Brownian motion, while Brox [1] gave a different proof in the Brownian case. Kesten [8] computed the density of b_1 in the case we consider, and Tanaka [19] generalized the computation to the case that w is a two-sided symmetric stable process. Localization results have been given for the Sinai walk by Golosov ([6], actually, for the reflected walk) and for the diffusion model by Tanaka [21]. Also Tanaka [18, 19] studied the cases where the environment is nonpositive reflecting Brownian motion, nonnegative reflecting Brownian motion, or Brownian motion with drift. Finer results on the asymptotics of Sinai's have been obtained by Shi [16] and Hu [7]. A survey of some of them, as well as a connection between Sinai's walk and diffusion in random environment, is given in [16]. Another connection is established in [15].

In [10], Le Dousal, Monthus and Fisher proposed a new method for tackling questions related to asymptotic properties of Sinai's walk and, using it, they gave a host of results. The method is a Renormalization Group analysis and it has consequences agreeing with rigorously proved results (e.g., [3, 8]). This is the starting point of the present paper. In the context of diffusion in random environment, we show how one can justify the Renormalization Group method using two tools. The first is a path decomposition for a two-sided standard

Brownian motion; the second is the renewal theorem. Our main results illustrate the use of the method and the way we justify it.

The structure of the paper is as follows. In the remaining of the Introduction we state our results. In Section 2 we provide all the necessary machinery for the proofs, which are given in Section 3. Some technical lemmata that we use are proved in Section 4.

We begin by defining the process b.

For a function $w:\mathbb{R}\to\mathbb{R}, x>0$ and $y_0\in\mathbb{R}$, we say that w *admits an x-minimum at* y_0 if there are $\alpha,\beta\in\mathbb{R}$ with $\alpha<y_0<\beta$, $w(y_0)=\inf\{w(y): y\in[\alpha,\beta]\}$ and $w(\alpha)\geq w(y_0)+x$, $w(\beta)\geq w(y_0)+x$. We say that w *admits an x-maximum at* y_0 if $-w$ admits an x-minimum at y_0 (see Figure 1).

For convenience, we will call a point where w admits an x-maximum or x-minimum an x-maximum or an x-minimum respectively.

We denote by $R_x(w)$ the set of x-extrema of w and define

$$\mathcal{W}_1:=\left\{w\in\mathcal{W}:\begin{array}{c}\text{For every } x>0,\text{ the set } R_x(w)\text{ has no accumulation point}\\ \text{in }\mathbb{R},\text{ it is unbounded above and below,}\\ \text{and the points of } x\text{-maxima and } x\text{-minima alternate}\end{array}\right\}.$$

Thus, for $w\in\mathcal{W}_1$ and $x>0$, we can write $R_x(w)=\{x_k(w,x):k\in\mathbb{Z}\}$ with $(x_k(w,x))_{k\in\mathbb{Z}}$ strictly increasing, $x_0(w,x)\leq 0<x_1(w,x)$, $\lim_{k\to-\infty}x_k(w,x)=-\infty$, $\lim_{k\to\infty}x_k(w,x)=\infty$. It holds that $\mathbb{P}(\mathcal{W}_1)=1$, and the easy proof of this fact is given in Lemma 8.

DEFINITION 1. The process $b:[0,+\infty)\times\mathcal{W}\to\mathbb{R}$ is defined for $x>0$ and $w\in\mathcal{W}_1$ as

$$b_x(w):=\begin{cases}x_0(w,x), & \text{if } x_0(w,x)\text{ is an } x\text{-minimum},\\ x_1(w,x), & \text{otherwise},\end{cases}$$

and $b_x(w)=0$ if $x=0$ or $w\in\mathcal{W}\setminus\mathcal{W}_1$.

REMARK 1. In the definition of $b_x(w)$ we do not make use of the entire sequence of x-extrema. The reason we introduce this sequence is that we plan

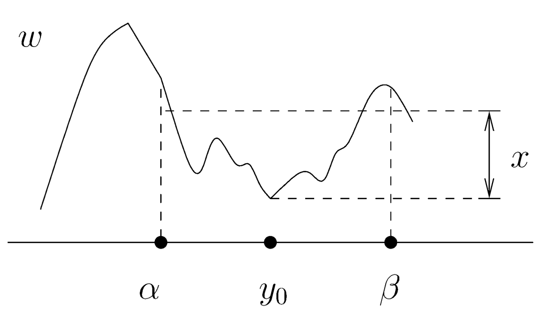

FIG. 1. *w admits an x-minimum at y_0.*

to study the evolution of the process b as x increases. Since $R_{\tilde{x}}(w) \subset R_x(w)$ for $x < \tilde{x}$, the later values of $b.(w)$ are elements of $R_x(w)$. For $\tilde{x}$ large enough, the points $x_0(w,x)$, $x_1(w,x)$ will not be $\tilde{x}$-extrema.

REMARK 2. We will decompose the process w at the endpoints of the intervals $\{[x_k(w,x), x_{k+1}(w,x)] : k \in \mathbb{Z}\}$ and study its restriction to each of them. Of course, $[x_0(w,x), x_1(w,x)]$ has a particular importance for the process b; and it is in the study of $w|[x_0(w,x), x_1(w,x)]$ that the renewal theorem enters (see Lemma 1).

REMARK 3. It is clear that b satisfies $b_x(\frac{1}{a}w(c\cdot)) = \frac{1}{c}b_{ax}(w)$ for all $a, c, x > 0$, and $w \in \mathcal{W}$. So that the quantity $b_1(w^{(\log t)})$ appearing in (2) equals also $b_{\log t}(w)/(\log t)^2$.

1.1. *Sign changes of b.* For $x \geq 1$, define on $\mathcal{W}_1$ the random variable

$$k(x) = \#\text{ times } b.(w) \text{ has changed sign in } [1, x].$$

The main result of the paper is the computation of the generating function of $k(x)$.

THEOREM. *For $x \geq 1$ and $z \in \mathbb{C}$ with $|z| < 1$, it holds*

$$\mathbb{E}\big(z^{k(x)}\big) = c_1(z)x^{\lambda_1(z)} + c_2(z)x^{\lambda_2(z)}, \tag{3}$$

where

$$\lambda_1(z) = \frac{-3 + \sqrt{5 + 4z}}{2}, \qquad \lambda_2(z) = \frac{-3 - \sqrt{5 + 4z}}{2}$$

and

$$c_1(z) = \big((z-1)/3 - \lambda_2(z)\big)/\big(\lambda_1(z) - \lambda_2(z)\big),$$
$$c_2(z) = \big(-(z-1)/3 + \lambda_1(z)\big)/\big(\lambda_1(z) - \lambda_2(z)\big).$$

From this we extract several corollaries. Corollary 1 and Corollary 3 are immediate, while the rest require some work and are proved in Section 3.

Corollary 1 follows by taking $z \to 0$ in (3).

COROLLARY 1.

$$\mathbb{P}\big(b.(w) \textit{ keeps the same sign in } [1, x]\big)/x^{(-3+\sqrt{5})/2} \to 1/2 + 7\sqrt{5}/30 \tag{4}$$

as $x \to +\infty$.

COROLLARY 2. *The increasing process of points* $(X_k)_{k\geq 1}$, *where* b *changes sign in* $[1,+\infty)$, *has the form* $X_k = X_1 r_1, \dots, r_{k-1}$, $k \geq 2$, *where* X_1 *is the smallest such point and the* r_i*'s are i.i.d. with density*

$$f(r) = \frac{1}{\lambda_1 - \lambda_2}(r^{\lambda_1 - 1} - r^{\lambda_2 - 1}), \qquad r \geq 1, \tag{5}$$

where $\lambda_1 = \lambda_1(0), \lambda_2 = \lambda_2(0)$.

Now observe that

$$\begin{aligned} k(t) &= \sup\{n \in \mathbb{N} : X_n \leq t\} \\ &= \sup\{n \in \mathbb{N} : \log X_1 + \log r_1 + \cdots + \log r_{n-1} \leq \log t\}. \end{aligned}$$

Since $\mathbb{E}(\log r_1) = 3$ and $\log X_1$ is finite a.s. [e.g., by Corollary 1, $\mathbb{E}(\log X_1) < +\infty$], the next statement follows from renewal theory.

COROLLARY 3. $k(t)/\log t \to 1/3$ *as* $t \to +\infty$ $\mathbb{P}$-*a.s.*

Corollary 2 allows the following strengthening of the above theorem.

COROLLARY 4. *Relation* (3) *holds for all* $z \in \mathbb{C} \setminus (-\infty, -5/4]$.

For $t \in (0, \infty)$, consider the random variable $k(e^t)/t$ and let μ_t be its distribution measure. Then the following holds.

COROLLARY 5. *The family of measures* $(\mu_t)_{t>0}$ *satisfies a large deviation principle with speed* t *and good rate function*

$$I(x) = \begin{cases} x\log\bigl(2x\bigl(x+\sqrt{x^2+5/4}\bigr)\bigr) + \frac{3}{2} - \bigl(x+\sqrt{x^2+5/4}\bigr), & \textit{if } x \in [0,+\infty), \\ +\infty, & \textit{if } x \in (-\infty, 0). \end{cases}$$

In [10], Corollary 2 appears in Paragraph IV.B with a different justification. We state it here because we need it for the proof of Corollary 5. The large deviation result of Corollary 5 is the precise mathematical interpretation of the discussion in Paragraph IV.A of the same paper.

In Corollary 1, the condition $k(x) = 0$ means that the process b tends to keep the diffusion away form zero in the time interval $[e, e^x]$ (since the diffusion localizes around b, and b keeps sign on $[1, x]$). For the event that the diffusion *hits* zero, there are two interesting relevant papers. The first one is by Hu [7] who treats the annealed asymptotics of the first time of hitting zero after time t as $t \to +\infty$. The second is by Comets and Popov [2] and refers to a related model. That is, it considers a process $(X_t)_{t\geq 0}$ on $\mathbb{Z}$ that runs in continuous time in an environment ω satisfying the conditions of the Sinai model and studies, among other things, the asymptotics of the quenched probability $\mathbf{P}_w(X_t = 0 | X_0 = 0)$ as $t \to +\infty$.

1.2. *The process b and the diffusion.* The results of the previous subsection concern the process b, which is a functional of the environment. And our motivation for studying b was the localization results involving this process (the simplest being Fact 1, and keep in mind Remark 3). An obvious question is what we can infer about the behavior of the diffusion from our results.

Using the representation of the diffusion as a time and scale change of Brownian motion, one can show easily that the diffusion is recurrent and 0 is a regular point for $(0, +\infty)$ and $(-\infty, 0)$. Consequently, for all $c > 0$, the diffusion visits 0 in the time interval $[c, +\infty)$ and in its first visit there it scores an infinite number of sign changes. So there can be no direct connection with the corresponding number for the process b. One can consider, say, the number of sign changes of the diffusion between times where the diffusion achieves a positive or negative record value. This number is finite on compact intervals of $(0, +\infty)$ but still not related with the sign changes of b (and it is not hard to see this).

When $b_{\log \cdot}$ jumps to a new value, what happens is not that just the diffusion goes through that value shortly before or after that. As it is known (see, e.g., [1, 20]), the impact of the jump is that the diffusion goes to that value and it is trapped in its neighborhood for a large amount of time. The way to detect the approximate location of such values when we observe the diffusion in real time (i.e., at time t we know $\omega|[0, t]$) is to find the site the diffusion has spent the most time thus far.

To make the last statement precise, we use the local time process $\{L_\omega(t, x) : t \geq 0, x \in \mathbb{R}\}$ that corresponds to the diffusion ω. This process is jointly continuous, and with probability one satisfies

$$\int_0^t f(\omega(s))\, ds = \int_{\mathbb{R}} f(x) L_\omega(t, x)\, dx$$

for all $t \geq 0$ and any bounded Borel function $f \in \mathbb{R}^{\mathbb{R}}$.

For a fixed $t > 0$, the set $\mathfrak{F}(t) := \{x \in \mathbb{R} : L_\omega(t, x) = \sup_{y \in \mathbb{R}} L_\omega(t, y)\}$ of the points with the most local time at time t is nonempty and compact. Any point there is called a *favorite point* of the diffusion at time t. Define $F : (0, +\infty) \to \mathbb{R}$ with $F(t) = \inf \mathfrak{F}(t)$, the smallest favorite point at time t. Pick any $c > 6$, and for any x with $|x| > 1$ define the interval $I(x) := (x - (\log |x|)^c, x + (\log |x|)^c)$.

In a work in progress we expect to prove the following:

CLAIM. With $\mathcal{P}$ probability one, there is a strictly increasing sequence $(t_n(\omega, w))_{n \geq 1}$ converging to infinity so that if we denote by $(x_n(w, t_1))_{n \geq 1}$ the sequence of consecutive values that $b_{\log \cdot}$ takes on the interval $(t_1, +\infty)$ (remember that b is a step function in any interval $[x, +\infty)$ with $x > 0$), then

$$F\big((t_n, t_{n+1})\big) \subset I(x_n) \qquad \text{for all } n \geq 1,$$

and $x_n = b_{\log t}$ for some $t \in (t_n, t_{n+1})$. We abbreviated $t_n(\omega, w), x_n(w, t_1)$ to t_n, x_n.

Observe that, for big x, the interval $I(x)$ is a relatively small neighborhood of x. Thus, the claim says that, after some point, the function F "almost tracks" the values of the process $b_{\log}$. with the same order and at about the same time. Consequently, the number of sign changes of $b_{\log}$. on an interval (s,t) (with s,t large) would correspond to the number of sign changes of F on approximately the same interval. Or, more precisely, the number of sign changes of F on (s,t) and the corresponding number for $b_{\log}$. differ by at most two. It is easy to see that the last statement follows from the Claim above.

2. Preliminaries. As a first step toward the study of the process b, we look at the law of the Brownian path between two consecutive x-extrema, as well as the way these pieces are put together to constitute the entire path.

The first piece of information is provided by the Proposition of Section 1 in [12].

FACT 2. For every $x > 0$, the times of x-extrema of a Brownian motion $(w_t : t \in \mathbb{R})$, $w_0 = 0$ build a stationary renewal process $R_x(w) = \{x_k : k \in \mathbb{Z}\}$ with $(x_k)_{k\in\mathbb{Z}}$ strictly increasing and $x_0 \le 0 < x_1$. The trajectories between consecutive x-extrema $(w_{x_k+t} - w_{x_k} : t \in [0, x_{k+1} - x_k])$, $k \in \mathbb{Z}$ are independent and for $k \neq 0$, identically distributed (up to changes of sign).

In the Lemma of Section 1 of [12] a description of each such trajectory is given which we quote (see Figure 2).

For $x, t \ge 0$, let

$$\begin{aligned} M_t &:= \sup\{w_s : s \in [0,t]\}, \\ \tau &:= \min\{t : M_t = w_t + x\}, \\ \beta &:= M_\tau, \\ \sigma &:= \max\{s \in [0,\tau] : w_s = \beta\}. \end{aligned}$$

FACT 3. The following hold:

(i) The two trajectories $(w_t : t \in [0,\sigma])$ and $(\beta - w_{\sigma+t} : t \in [0, \tau - \sigma])$ are independent.

(ii) β has exponential distribution with mean x.

(iii) The Laplace transform of the law of σ given β is

$$\mathbb{E}[e^{-s\sigma} | \beta = y] = \exp\{-(y/x)\phi(sx^2)\}, \qquad s > 0.$$

(iv) The Laplace transform of the law of $\tau - \sigma$ is

$$\mathbb{E}\big[e^{-s(\tau-\sigma)}\big] = \psi(sx^2), \qquad s > 0,$$

where $\phi(s) = \sqrt{2s}\coth\sqrt{2s} - 1$ and $\psi(s) = \sqrt{2s}/\sinh\sqrt{2s}$.

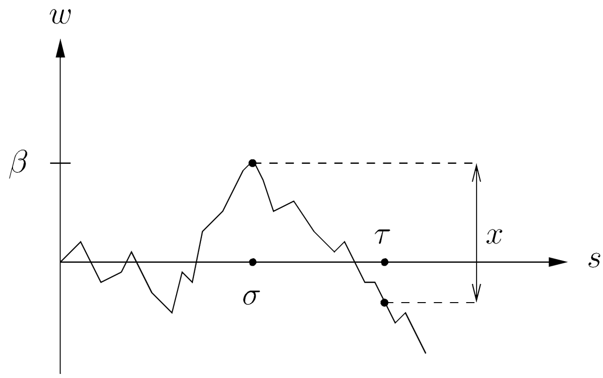

FIG. 2. *The graph of w until $M - w$ hits x.*

We call the translation $(w - w(x_k))|[x_k, x_{k+1}]$ of the trajectory of w between two consecutive x-extrema an *x-slope* (or a slope, when the value of x is clear or irrelevant), a slope that takes only nonnegative values an *upward slope*, and a slope taking only nonpositive values a *downward slope*. We call $(w - w(x_0))|[x_0, x_1]$ the *central x-slope*.

In the following we will use the operation of "gluing together" functions defined on compact intervals. For two functions $f : [\alpha, \beta] \to \mathbb{R}$, $g : [\gamma, \delta] \to \mathbb{R}$, by gluing g to the right of f, we mean that we define a new function $j : [\alpha, \beta + \delta - \gamma] \to \mathbb{R}$ with

$$j(x) = \begin{cases} f(x), & \text{for } x \in [\alpha, \beta], \\ f(\beta) + g(x - \beta + \gamma) - g(\gamma), & \text{for } x \in [\beta, \beta + \delta - \gamma]. \end{cases}$$

It is clear that, for all $k \neq 0$, if $(w - w(x_k))|[x_k, x_{k+1}]$ is an upward x-slope, then it is obtained by gluing together a trajectory of type $(w_t : t \in [0, \sigma])$ to the right of a trajectory of type $(\beta - w_{\sigma+t} : t \in [0, \tau - \sigma])$ and then translating the resulting path so that it starts at $(x_k, 0)$. Similarly for a downward slope.

For any x-slope $T : [\alpha, \beta] \to \mathbb{R}$, we call $l(T) := \beta - \alpha$, $h(T) := |T(\beta) - T(\alpha)|$ the *length* and the *height* of the slope, respectively, and we denote by $\eta(T) := h(T) - x$ the "*excess height*" of T. Also, we denote by $|T|$, $\theta(T)$, the slopes with domains $[\alpha, \beta]$, $[0, \beta - \alpha]$ and values $|T|(\cdot) = |T(\cdot)|$, $\theta(T)(\cdot) = T(\alpha + \cdot)$, respectively.

For any slope T, the slope $|\theta(T)|$ is in the set $\mathcal{S}$ defined by

$$\mathcal{S} := \left\{ f \subset [0, +\infty)^2 : \begin{array}{c} f \text{ is a function such that there is a } l(f) \geq 0 \text{ with} \\ \text{Domain}(f) = [0, l(f)],\ f \text{ continuous,} \\ \text{and } 0 = f(0) \leq f(x) \leq f(l(f))\, \forall x \in [0, l(f)] \end{array} \right\}.$$

On $\mathcal{S}$ we define a topology for which the base of neighborhoods of an element $f \in \mathcal{S}$ is the collection of all sets of the form

$$\{g \in \mathcal{S} : |l(g) - l(f)| < \varepsilon \text{ and } |f(tl(f)) - g(tl(g))| < \varepsilon \text{ for all } t \in [0, 1]\}.$$

With this topology, $\mathcal{S}$ is a Polish space. Equip $\mathcal{S}$ with the Borel σ-algebra, and define the measures m_x^r, m_x^c the first to be the distribution of $\theta(|(w - w(x_1))||[x_1, x_2])$ and the second to be the distribution of $\theta(|(w - w(x_0))||[x_0, x_1])$ (the superscripts r and c standing for renewal and central).

In the remaining part of this section we compute the distribution of the length and excess height of a slope picked from m_x^r or m_x^c. We assume $x = 1$ since the scaling property of Brownian motion gives the corresponding results for the case $x \neq 1$.

Let T be a slope picked from m_1^r. Earlier we described the way an upward slope is formed. Its length is the sum of two independent independent random variables Z_1, Z_2 with $Z_1 \overset{\text{law}}{=} \sigma, Z_2 \overset{\text{law}}{=} \tau - \sigma$ (where we take $x = 1$ in their definitions just before Fact 3). But from (i) of Fact 3, σ and $\tau - \sigma$ are independent. Thus, $l(T) \overset{\text{law}}{=} \tau$. By definition, τ is the time where the reflected process $M - w$ hits one. This reflected process has the same law as $|w|$, and the Laplace transform of the time it first hits one is known as

$$\mathbb{E}\big[e^{-\lambda l(T)}\big] = \big(\cosh\sqrt{2\lambda}\,\big)^{-1} \qquad \text{for } \lambda > 0. \tag{6}$$

Also, $\mathbb{E}(l(T)) = \mathbb{E}(\tau) = 1$. Using the Laplace inversion formula (see [11], page 531), we find that the density of $l = l(T)$ is

$$f_l(x) = \frac{\pi}{2}\sum_{k\in\mathbb{Z}}(-1)^k\left(k + \frac{1}{2}\right)\exp\left[-\frac{\pi^2}{2}\left(k + \frac{1}{2}\right)^2 x\right] \qquad x > 0. \tag{7}$$

We note for future reference that, by (ii) of Fact 3, for any $a > 0$, the excess height of a slope picked from m_a^r is exponential with mean a; that is, it has density

$$p_a(x) = a^{-1}e^{-x/a}, \qquad x > 0. \tag{8}$$

Now let T_0 be a slope picked from m_1^c. More specifically, take T_0 to be the central 1-slope. Observe that, by virtue of Fact 2, we can "start a renewal process at $-\infty$" with i.i.d. alternating upward and downward slopes, and ask what the characteristics are of the slope covering zero. The renewal theorem says that the length of the slope covering zero is picked from the distribution of l given in (7) with size-biased sampling. Once the length, say z, is picked, we expect that the remaining characteristics of the slope, ignoring direction (upward or downward), are determined by the law of $T|l(T) = z$ under m_1^r. We give a formal proof of this. Notice that a regular conditional distribution for the random variable T given the σ-field $\sigma(l(T))$ exists because the space $\mathcal{S}$ is Polish.

LEMMA 1. *For any measurable subset A of $\mathcal{S}$, it holds*

$$\mathbb{P}\big(|\theta(T_0)| \in A\big) = \int_0^\infty \mathbb{P}\big(T \in A|l(T) = z\big)zf_l(z)\,dz, \tag{9}$$

where T has under $\mathbb{P}$ distribution m_1^r.

PROOF. Let F_l be the distribution function of $l = l(T)$, and for $t \in \mathbb{R}$, let $T(t)$ be the 1-slope around t. That is, the slope whose domain contains t. Then $\mathbb{P}(|\theta(T_0)| \in A) = \mathbb{P}(|\theta(T(t))| \in A)$ for all $t > 0$ because $\theta(T_0)$ is the same as the image under θ of the slope around t for $(w_{s-t} - w_{-t} : s \in \mathbb{R})$, and the latter process is again a standard two-sided Brownian motion. Now let $(Y_n)_{n\geq 0}$ be an independent sequence of slopes with $Y_n \stackrel{\text{law}}{=} (-1)^{n+1}T$. Glue them sequentially to get a function f in $C([0,+\infty))$ with $f(0) = 0$, and denote by $\tilde{T}(t)$ the slope around t, for $t > 0$.

If we take σ as defined just before Fact 3 with $x = 1$ in the definition of τ there, then it holds

$$\begin{aligned}\mathbb{P}\big(|\theta(T_0)| \in A\big) &= \mathbb{P}\big(|\theta(T(t))| \in A\big)\\ &= \mathbb{P}\big(|\theta(T(t))| \in A, \sigma < t\big) + \mathbb{P}\big(|\theta(T(t))| \in A, \sigma > t\big)\end{aligned}$$

and

$$\mathbb{P}\big(|\theta(T(t))| \in A, \sigma < t\big) = \int_0^t \mathbb{P}\big(\big|\theta\big(\tilde{T}(t-y)\big)\big| \in A\big) f_\sigma(y)\,dy,$$

where f_σ is the density of σ. We will take $t \to +\infty$ and finish with the proof by showing that the limit $\lim_{t\to+\infty} P(|\theta(\tilde{T}(t))| \in A)$ exists and

$$\lim_{t\to+\infty} \mathbb{P}\big(|\theta(\tilde{T}(t))| \in A\big) = \int_0^\infty \mathbb{P}\big(T \in A | l(T) = z\big) z f_l(z)\,dz.$$

To see this, define $g(t) := \mathbb{P}(|\theta(\tilde{T}(t))| \in A)$ for $t \geq 0$. Then

$$\begin{aligned}g(t) &= \mathbb{P}\big(T \in A, l(T) > t\big) + \int_0^t \mathbb{P}\big(\big|\theta\big(\tilde{T}(t-s)\big)\big| \in A\big)\,dF_l(s)\\ &= \mathbb{P}\big(T \in A, l(T) > t\big) + \int_0^t g(t-s)\,dF_l(s).\end{aligned}$$

The distribution of l is nonarithmetic with mean value 1. By the renewal theorem ([5], Chapter 3, statement (4.9)), it follows that the $\lim_{t\to+\infty} g(t)$ exists and

$$\begin{aligned}\lim_{t\to+\infty} g(t) &= \int_0^\infty \mathbb{P}\big(T \in A, l(T) > s\big)\,ds\\ &= \int_0^\infty \int_s^\infty \mathbb{P}\big(T \in A | l(T) = z\big) f_l(z)\,dz\,ds\\ &= \int_0^\infty \int_0^z \mathbb{P}\big(T \in A | l(T) = z\big) f_l(z)\,ds\,dz\\ &= \int_0^\infty \mathbb{P}\big(T \in A | l(T) = z\big) z f_l(z)\,dz.\end{aligned}$$

□

Now we apply Lemma 1 to obtain the distribution of the length and height of the central 1-slope.

• Regarding the length of T_0, observe that, for $x > 0$, the set $A := \{T \in \mathcal{S} : l(T) < x\}$ is open and

$$\mathbb{P}\big(l(T_0) < x\big) = \int_0^\infty \mathbb{P}\big(l(T) < x | l(T) = z\big) z f_l(z)\, dz = \int_0^x z f_l(z)\, dz.$$

So that $l(T_0)$ has the density

$$f_{l(T_0)}(x) = x f_l(x) \qquad \text{for } x \geq 0, \tag{10}$$

which is the size-biased sampling formula from renewal theory.

• Regarding the excess height of T_0, observe that, for $x > 0$, the set $A := \{T \in \mathcal{S} : \eta(T) < x\}$ is open and

$$\begin{aligned} \mathbb{P}\big(\eta(T_0) < x\big) &= \int_0^\infty \mathbb{P}\big(\eta(T) < x | l(T) = z\big) z f_l(z)\, dz \\ &= \int_0^\infty \mathbb{P}\big(\eta(T) < x, l(T) = z\big) z\, dz. \end{aligned}$$

Differentiating with respect to x, we get the density of $\eta(T_0)$ as

$$\int_0^\infty \mathbb{P}\big(\eta(T) = x, l(T) = z\big) z\, dz = \mathbb{E}\big(l(T) | \eta(T) = x\big) e^{-x}.$$

From (iii) and (iv) of Fact 3,

$$\mathbb{E}\big(e^{-tl(T)} | \eta(T) = x\big) = \psi(t) e^{-x\phi(t)}.$$

After some calculations, $\frac{\partial}{\partial t}\psi(t)e^{-x\phi(t)}|_{t=0} = -(2x+1)/3$. The derivative can move inside the expectation on the left-hand side of the above relation, giving $-\mathbb{E}(l(T)|\eta(T) = x)$, due to the monotone convergence theorem; which applies because $l(T) > 0$ and the function $(x \mapsto (1 - e^{-ax})/x)$ is nonnegative and decreasing in $\mathbb{R}$ for any $a > 0$. Therefore, the density of $\eta(T_0)$ is

$$f_{\eta(T_0)}(x) = \frac{(2x+1)e^{-x}}{3} \qquad \text{for } x > 0. \tag{11}$$

The last bit of information needed to achieve our goal, stated at the first sentence of this section, is the direction of the central 1-slope (i.e., upward or downward) and its location with respect to zero. By symmetry, T_0 is an upward slope with probability 1/2, and from Exercise 3.4.7 of [5], it follows easily that given the length l of the slope $(w - w(x_0))|[x_0, x_1]$ around zero, the distance of zero from x_0 is uniformly distributed in $[0, l]$.

3. Proofs of the Theorem and Corollaries 2, 4 and 5.

For $x > 0$ and $w \in \mathcal{W}_1$ with set of x-extrema $R_x(w) = \{x_k : k \in \mathbb{Z}\}$, define

$$A_x(w) := \big\{\big(w - w(x_k)\big)|[x_k, x_{k+1}] : k \in \mathbb{Z}\big\}.$$

We refer to the parameter x as time since we are going to study the evolution of $A_x(w)$ as x increases. $A_x(w)$ is the set of slopes at time x. Roughly, as x increases, the slopes that have height smaller than x are absorbed into greater ones.

For any Lebesgue measurable set $S \subset [0, +\infty)$, $x \geq 1$, and $k \in \mathbb{N}$, we define

$$\mathcal{U}(x, S, k) = \mathbb{P}\begin{pmatrix} \text{In } A_x(w) \text{ the central slope has excess height } y \in S \\ \text{and } b_\cdot(w) \text{ has changed sign } k \text{ times in } [1, x] \end{pmatrix}.$$

It is important to note that the values of b up to time x are "encoded" in the central slope of $A_x(w)$. So the number of sign changes of $b_\cdot(w)$ in $[1, x]$ can be inferred by that slope.

For x, k fixed, $\mathcal{U}$ is a measure that satisfies

$$\mathcal{U}(x, S, k) \leq \mathbb{P}\big(\text{In } A_x(w) \text{ the central slope has excess height } y \in S\big),$$

and the right-hand side, considered as a function of S, is a measure absolutely continuous with respect to the Lebesgue measure with density $(1/x) f_{\eta(T_0)}(\cdot/x)$, where $f_{\eta(T_0)}$ is given in (11). Therefore, the measure on the left-hand side has a density also [an element of $L^1([0, +\infty))$], call it $u(x, y, k)$, and for Lebesgue almost all y, it holds

$$\sum_{k=0}^{+\infty} u(x, y, k) = \frac{1}{x} f_{\eta(T_0)}\left(\frac{y}{x}\right). \tag{12}$$

Define $U : [1, +\infty) \times [0, +\infty) \times \mathbb{N} \to [0, +\infty)$ with $U(x, y, k) := \mathcal{U}(x, [y, +\infty), k)$. U is continuous, as is proved in Lemma 7. We plan to establish a PDE for U. To do this, we look at $A_x(w)$ and try to predict how $A_{x+\varepsilon}(w)$ should look like.

In the transition from $A_x(w)$ to $A_{x+\varepsilon}(w)$, a slope around a point remains the same if the excess height of this and the two neighboring slopes are greater than ε. In case one slope has excess height in $[0, \varepsilon)$, it does not appear in $A_{x+\varepsilon}(w)$. For example, in Figure 3 the slope T_0 has a height, say, $x + h_0$ with $h_0 \in [0, \varepsilon)$, and the two neighboring slopes T_{-1}, T_1 have height greater than $x + \varepsilon$. Assume that T_{-1}, T_1 have heights $x + v_1$, $x + v_2$, respectively. In $A_{x+\varepsilon}(w)$ we know that the slopes T_{-1}, T_0, T_1 will merge to constitute a new slope with

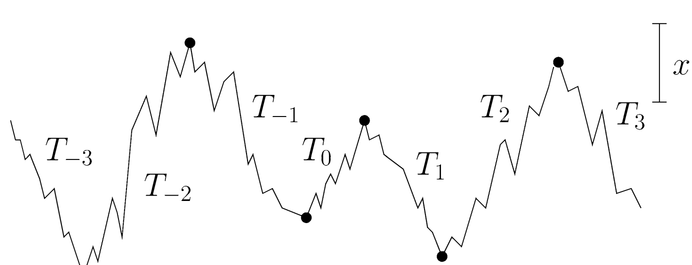

FIG. 3. *The decomposition of a piece of the Brownian path in x-slopes. The dots mark the points of x-extrema. The length x is shown on the side.*

height $x + v_1 + x + v_2 - x - h_0 = x + v_1 + v_2 - h_0$; that is, with excess height $v_1 + v_2 - h_0 - \varepsilon$. The slopes T_{-2}, T_2 can stay as they are in the transition from $A_x(w)$ to $A_{x+\varepsilon}$ or they can be extended if some of T_{-3}, T_3 has excess height in $[0, \varepsilon)$. In any case, they do not interfere with T_{-1}, T_0, T_1. This simple observation, combined with the renewal structure of the sets $A_x(w)$, is the basis for the next lemma, which is the first step towards establishing a PDE that U solves. We denote by $\partial_y U$ the y derivative of $U(x, y, k)$, and recall that $p_x(v)$ was defined in (8) as the density of an exponential with mean x.

LEMMA 2. *For $x \geq 1, y \geq 0, \varepsilon > 0, k \in \mathbb{N}$, we have*

$$
\begin{aligned}
&U(x+\varepsilon, y, k) \\
&\quad = U(x, y+\varepsilon, k)e^{-2\varepsilon/x} + \frac{2\varepsilon}{x}\int_0^{+\infty} p_x(v)U\bigl(x, (y-v)^+, k\bigr)\,dv \\
&\qquad - \mathbb{1}_{\{k\geq 1\}}\varepsilon\,\partial_y U(x, 0, k-1)(y/x+1)e^{-y/x} + o(\varepsilon),
\end{aligned}
\tag{13}
$$

where, for $k \geq 1$, we assume that $U(x, y, k-1)$ is differentiable in y with continuous derivative. The term $o(\varepsilon)$ depends on x, y, k.

PROOF. The left-hand side of the equation is the probability of an event referring to $A_{x+\varepsilon}(w)$, and we express it in terms of probabilities referring to $A_x(w)$. In $A_x(w)$ we focus our attention on the seven x-slopes closest to zero. Denote them by $T_i, i = -3, \dots, 3$, in the order they appear in the path of w from left to right, with T_0 being the central one. The slopes $|\theta(T_i)|, i = -3, \dots, 3$, are independent having for $i \neq 0$ law m_x^r, and $|\theta(T_0)|$ having law m_x^c. The probability of the event that at least two of them have excess height in $[0, \varepsilon)$ is bounded by $21\varepsilon^2/x^2$, and this is accounted for in the $o(\varepsilon)$ term in (13). In the complement of this event, the event whose probability appears in the left-hand side of (13) happens if and only if in $A_x(w)$ one of the following three holds (considering what slope, if any, among the seven has excess height in $[0, \varepsilon)$):

- At most, one of T_{-3}, T_{-2}, T_2, T_3 has excess height in $[0, \varepsilon)$, the excess height of each of T_{-1}, T_0, T_1 is at least ε, and b has changed sign k times in $[1, x]$. In this case the slope around zero is the same for both $A_x(w), A_{x+\varepsilon}(w)$.
- Exactly one of T_{-1}, T_1 has excess height in $[0, \varepsilon)$, T_0 has excess height at least ε, and b has changed sign k times in $[1, x]$. In this case $b_{x+\varepsilon}$ has the same sign as b_x. For example, assume that T_1 has excess height in $[0, \varepsilon)$. If $b_x > 0$, then $b_{x+\varepsilon} > b_x$, while if $b_x < 0$, then $b_{x+\varepsilon} = b_x$ and, simply, the central slope in $A_{x+\varepsilon}(w)$ results from merging T_0, T_1, T_2.
- $k \geq 1$, T_0 has excess height in $[0, \varepsilon)$, and b has changed sign $k-1$ times in $[1, x]$. In this case $b_{x+\varepsilon}b_x < 0$ because in $A_{x+\varepsilon}(w)$ the central slope results from merging T_{-1}, T_0, T_1 and it has different direction than T_0 [i.e., if, e.g., T_0 is an upward slope, then the central slope in $A_{x+\varepsilon}(w)$ is a downward slope].

The first case has probability

$$U(x, y+\varepsilon, k)\mathbb{P}\big(\eta_{-1} \in [\varepsilon, +\infty)\big)\mathbb{P}\big(\eta_1 \in [\varepsilon, +\infty)\big) = U(x, y+\varepsilon, k)\exp(-2\varepsilon/x),$$

with η_{-1} (resp. η_1) denoting the excess height of T_{-1} (resp. T_1). This follows by the independence mentioned above and by the fact that η_{-1} and η_1 have exponential distribution with mean x, and it expresses the demand that both of the x-slopes neighboring the central x-slope have excess height greater than ε. In this case, the excess heights of T_{-3}, T_{-2}, T_2, T_3 do not matter. They cannot influence the central slope in $A_{x+\varepsilon}(w)$.

The second case has probability

$$2\int_0^{\varepsilon}\int_{\varepsilon}^{+\infty} p_x(v_1)p_x(v_2)U\big(x, \varepsilon \vee (y+\varepsilon - v_2 + v_1), k\big)\,dv_2\,dv_1$$

because, say, in the event that T_1 has excess height in $[0, \varepsilon)$, the central slope in $A_{x+\varepsilon}(w)$ comes from merging T_0, T_1, T_2 of $A_x(w)$. Assume that they have excess heights u, v_1, v_2, respectively. Then the central slope in $A_{x+\varepsilon}(w)$ will have excess height $u - v_1 + v_2 - \varepsilon$, and the requirement that this is greater than y translates to u being greater than $(y + v_1 - v_2 + \varepsilon)^+$. And, of course, by assumption, T_0 and T_2 have excess height greater than ε.

U is continuous as it is proved in Lemma 7. Thus, dividing with ε the previous integral and taking $\varepsilon \to 0$, we get as limit

$$\frac{2}{x}\int_0^{+\infty} p_x(v_2)U\big(x, (y-v_2)^+, k\big)\,dv_2.$$

From this procedure we pick up another $o(\varepsilon)$ term.

The last case has probability

$$-\int_0^{\varepsilon}\int_{\varepsilon}^{+\infty}\int_{\varepsilon}^{+\infty} \partial_y U(x, z, k-1)p_x(v_1)p_x(v_2)\mathbb{1}_{\{v_1+v_2 \ge y+z+\varepsilon\}}\,dv_1\,dv_2\,dz.$$

By assumption, $-\partial_y U(x, y, k-1)$ exists and it is the density of the measure $\mathcal{U}(x, \cdot, k)$. The dummy variables v_1, v_2, z stand for the excess heights of T_{-1}, T_1, T_0, respectively, and in this case the central slope in $A_{x+\varepsilon}(w)$ has excess height $v_1 + v_2 - z - \varepsilon$, giving the restriction $v_1 + v_2 - z - \varepsilon \ge y$. Since $\partial_y U(x, \cdot, k-1)$ is continuous, dividing by ε and taking $\varepsilon \to 0$, we get

$$-\partial_y U(x, 0, k-1)\int_0^{+\infty}\int_0^{+\infty} \mathbb{1}_{\{v_1+v_2\ge y\}}p_x(v_1)p_x(v_2)\,dv_1\,dv_2.$$

And again we pick up an $o(\varepsilon)$ term. The double integral equals $(y/x + 1)\exp(-y/x)$. $\square$

Before getting to the actual proof of our theorem, we give a nonrigorous short derivation to illustrate its main steps. The main problem is that we do not know if U is differentiable in the x, y variables for every $k \in \mathbb{N}$. Assume for the moment that it is. For $f, g \in L^1([0,+\infty))$, as usual, we define $f * g \in C([0,+\infty))$ by $(f*g)(x) := \int_0^x f(x-y)g(y)\,dy$ for all $x \in [0,+\infty)$.

The above lemma would give for U the PDE

$$
\begin{aligned}
&(\partial_x - \partial_y)U(x,y,k) \\
&\qquad = -\frac{2}{x}U(x,y,k) + \frac{2}{x^2}\bigl(U(x,\cdot,k) * e^{-\cdot/x}\bigr)(y) \\
&\qquad\quad + \frac{2}{x}e^{-y/x}U(x,0,k) - \mathbb{1}_{\{k\geq 1\}}\,\partial_y U(x,0,k-1)(y/x+1)\exp(-y/x).
\end{aligned}
\tag{14}
$$

Let $f(x,y,k) = U(x,yx,k)$; that is, $U(x,y,k) = f(x,y/x,k)$. Then f should satisfy

$$
\begin{aligned}
&\bigl(x\partial_x - (1+y)\partial_y + 2\bigr)f(x,y,k) \\
&\qquad = 2\bigl(f(x,\cdot,k) * e^{-\cdot}\bigr)(y) + 2e^{-y}f(x,0,k) \\
&\qquad\quad - \mathbb{1}_{\{k\geq 1\}}(y+1)e^{-y}\partial_y f(x,0,k-1),
\end{aligned}
\tag{15}
$$

while the conditions at $x = 1$ are

$$
f(1,y,0) = (2y/3+1)e^{-y} \qquad \text{for } y \geq 0, \tag{16}
$$

$$
f(1,y,k) = 0 \qquad \text{for } y \geq 0, k \geq 1. \tag{17}
$$

The first equation comes from (11), the second is clear.

For $z \in D := \{z \in \mathbb{C} : |z| < 1\}$, the generating function

$$
M(x,y,z) := \sum_{k=0}^{\infty} f(x,y,k)z^k
$$

is well defined. Assuming that M is differentiable with respect to x, y and its x, y derivatives are obtained with term by term differentiation, we see that M satisfies the PDE problem

$$
\begin{aligned}
&\bigl(x\partial_x - (1+y)\partial_y + 2\bigr)M(x,y,z) \\
&\qquad = 2\bigl(M(x,\cdot,z) * e^{-\cdot}\bigr)(y) + 2e^{-y}M(x,0,z) \\
&\qquad\quad - (y+1)e^{-y}z\,\partial_y M(x,0,z) \qquad \text{in } (1,\infty)\times(0,\infty),
\end{aligned}
\tag{18}
$$

$$
M(1,y,z) = (2y/3+1)e^{-y} \qquad \text{for } y \geq 0. \tag{19}
$$

We try for a solution of the form

$$
M(x,y,z) = [a(x,z) + b(x,z)y]e^{-y}.
$$

Substituting this into (18), we see that e^{-y} factors out in both sides, and after cancellation, we arrive in an equality of two first degree polynomials in y with coefficients depending on x, z. Equating the coefficients in equal powers of y in the two sides of the equation, we arrive at the following system of ODEs for a, b:

$$x\,\partial_x a(x,z) + (z-1)\big(b(x,z) - a(x,z)\big) = 0,$$
$$x\,\partial_x b(x,z) + (2+z)b(x,z) - (z+1)a(x,z) = 0.$$

The initial condition (19) for M, expressed in terms of a, b, becomes $a(1,z) = 1$, $b(1,z) = 2/3$ for all $z \in D$. We easily see that the only solution of the system satisfying these conditions is

$$a(x,z) = c_1(z)x^{\lambda_1(z)} + c_2(z)x^{\lambda_2(z)},$$
$$b(x,z) = c_1(z)\left(1 + \frac{\lambda_1(z)}{1-z}\right)x^{\lambda_1(z)} + c_2(z)\left(1 + \frac{\lambda_2(z)}{1-z}\right)x^{\lambda_2(z)},$$

where

$$\lambda_1(z) = \frac{-3+\sqrt{5+4z}}{2}, \qquad \lambda_2(z) = \frac{-3-\sqrt{5+4z}}{2}$$

and

$$c_1(z) = \big((z-1)/3 - \lambda_2(z)\big)/\big(\lambda_1(z) - \lambda_2(z)\big),$$
$$c_2(z) = \big(-(z-1)/3 + \lambda_1(z)\big)/\big(\lambda_1(z) - \lambda_2(z)\big).$$

Now $E(z^{k(x)}) = M(x, 0, z) = a(x, z)$, which is what we want.

PROOF OF THE THEOREM. The proof is done by taking the steps of the above "proof" in reverse order. This time all the steps can be justified. We will need three lemmata. Two of them are nontrivial, and their proof is given in Section 4.

LEMMA 3. *The solution M of* (18) *and* (19) *obtained above is analytic as a function of z in D. The coefficients of its development as a power series around zero are differentiable with respect to x and y in* $(1, +\infty) \times (0, +\infty)$, *and its x, y-derivatives are continuous and can be found with term by term differentiation.*

Using this lemma, we write M as

$$M(x,y,z) := \sum_{k=0}^{\infty} g(x,y,k)z^k.$$

Differentiating M term by term and equating the coefficients of equal powers of z in the two sides of (18), we see that the sequence of functions $(g(\cdot,\cdot,k))_{k\geq 0}$ satisfies the PDEs (15) with conditions at $x = 1$ given by (16) and (17).

For $k \in \mathbb{N}$, define $\tilde{g}(\cdot,\cdot,k) : [1,+\infty) \times [0,+\infty) \to \mathbb{R}$ by $\tilde{g}(x,y,k) = g(x, y/x, k)$ for $(x,y,k) \in [1,+\infty) \times [0,+\infty)$. The sequence of functions $(\tilde{g}(\cdot,\cdot,k))_{k\geq 0}$ satisfies the PDEs (14) with conditions at $x = 1$ given by (16) and (17).

The proof is finished by showing that the sequence $(U(\cdot,\cdot,k))_{k\in\mathbb{N}}$ satisfies a weak form of these PDEs, and then a uniqueness result will identify U as $\tilde{g}$.

For fixed $c>1$, define $g_{c,k}(x)=U(x,c-x,k)$ for $x\in[1,c]$. We state as a lemma an equation that $g_{c,k}$ satisfies. The proof is straightforward from Lemma 2.

LEMMA 4. *The function $g_{c,k}$ is differentiable in $(1,c)$ and satisfies*

$$\begin{aligned} g'_{c,k}(x) &= -\frac{2}{x}g_{c,k}(x)+\frac{2}{x^2}\big(U(x,\cdot,k)*e^{-\cdot/x}\big)(c-x) \\ &\quad +\frac{2}{x}e^{-(c-x)/x}U(x,0,k)-\mathbb{1}_{\{k\geq 1\}}\partial_y U(x,0,k-1)(c/x)e^{-(c-x)/x}, \end{aligned} \tag{20}$$

where, for $k\geq 1$, we assume that $U(x,y,k-1)$ is differentiable in y with continuous derivative.

And the promised uniqueness result is the following.

LEMMA 5 (Uniqueness). *Let $f:[1,+\infty)\times[0,+\infty)\to[0,+\infty)$ be a continuous function such that, for every $c>1$, the function $g_c:[1,c]\to\mathbb{R}$ with $g_c(x):=f(x,c-x)$ is continuous on $[1,c]$ and differentiable on $(1,c)$, with*

$$g'_c(x)=-\frac{2}{x}g_c(x)+\frac{2}{x^2}\big(f(x,\cdot)*e^{-\cdot/x}\big)(c-x)+\frac{2}{x}e^{-(c-x)/x}f(x,0) \tag{21}$$

and $g_c(1)=0$. Then $f\equiv 0$.

Now using induction, we show that

$$U(\cdot,\cdot,k)=\tilde{g}(\cdot,\cdot,k)\qquad \forall k\in\mathbb{N}.$$

The function $U(\cdot,\cdot,0)-\tilde{g}(\cdot,\cdot,0)$ satisfies the assumptions of Lemma 5 because of the PDE problem that $\tilde{g}(\cdot,\cdot,0)$ solves and Lemma 4. For $k\geq 1$, assuming the statement true for $k-1$, the same argument works for $U(\cdot,\cdot,k)-\tilde{g}(\cdot,\cdot,k)$, where now the assumption on $U(\cdot,\cdot,k-1)$ required by Lemma 4 is provided by the inductive hypothesis.

Therefore, $\sum_{k=0}^{+\infty}U(x,xy,k)z^k=M(x,y,z)$ for $(x,y,z)\in[1,+\infty)\times[0,+\infty)\times D$ and $\mathbb{E}(z^{k(x)})=M(x,0,z)=a(x,z)$, proving (3) for $z\in D$. $\square$

PROOF OF COROLLARY 2. Our theorem gives that $\mathbb{P}(b$ does not change sign in $[1,x])=a(x,0)$, where the function a is defined on page 1775, and by scaling

$$\mathbb{P}(b \text{ does not change sign in } [x,y])=a(y/x,0)\qquad \text{for } 0<x<y.$$

Consequently, the density of the last point before y that we have sign change is $-yx^{-2}\,\partial_x a(y/x,0)$, where we use ∂_x here and below to denote derivative with respect to the first argument. Differentiating with respect to y, we get the density of the event that x, y are consecutive times of sign change as

$$x^{-2}\,\partial_x a(y/x,0)+yx^{-3}\,\partial_{xx}a(y/x,0),$$

which, after using the expression for $a(x,0)$, becomes

$$\frac{1}{3(\lambda_1-\lambda_2)xy}\big((y/x)^{\lambda_1}-(y/x)^{\lambda_2}\big).$$

The event that b changes sign at x translates to the central x-slope having excess height 0. This has density $1/(3x)$ due to the scaling property of the x-slopes and relation (11), which refers to the central 1-slope. Thus, the density of the time Y where the next sign change after x happens, given that there was a sign change at x, is

$$h(y)=\begin{cases}\dfrac{1}{(\lambda_1-\lambda_2)y}\big((y/x)^{\lambda_1}-(y/x)^{\lambda_2}\big), & \text{if } y\ge x,\\ 0, & \text{otherwise.}\end{cases}\tag{22}$$

The quotient Y/x, given that there was a sign change at x, has the density of the r_i's given in (5). $\square$

PROOF OF COROLLARY 4. Observe that the right-hand side of (3) is a function analytic in $\mathbb{C}\setminus(-\infty,-5/4]$. As for the left-hand side, we have the following lemma.

LEMMA 6. *For any $x>1$, the power series $\sum_{n=0}^{+\infty}\mathbb{P}(k(x)=n)z^n$ defines an entire function.*

And from a basic property of analytic functions, it follows that the quantities $M(x,0,z)$, $a(x,z)$ agree for all $z\in\mathbb{C}\setminus(-\infty,-5/4]$. $\square$

REMARK 4. Of course, Lemma 6 implies that the right-hand side of (3) can be extended to an entire function. However, the way (3) is written does not allow as to claim that it holds for all $z\in\mathbb{C}$ because the function $(z\to\sqrt{4+4z})$ does not have an entire extension.

PROOF OF COROLLARY 5. We apply the Gartner–Ellis theorem (Theorem 2.3.6 in [4]). The moment generating function of $k(e^t)/t$ is given for any $\lambda\in\mathbb{R}$ by $\Lambda_t(\lambda):=\log\mathbb{E}(\exp\{\lambda k(e^t)/t\})$. So $t^{-1}\Lambda_t(t\lambda)=t^{-1}\log\mathbb{E}(e^{\lambda k(e^t)})=t^{-1}\log M(e^t,0,e^\lambda)$ and using Corollary 4, we see that

$$\Lambda(\lambda):=\lim_{t\to\infty}\frac{1}{t}\Lambda_t(t\lambda)=\lambda_1(e^\lambda)=\frac{-3+\sqrt{5+4e^\lambda}}{2}.$$

The Fenchel–Legendre transform Λ^* of Λ, defined by $\Lambda^*(x)=\sup_{\lambda\in\mathbb{R}}\{\lambda x-\Lambda(\lambda)\}$ for all $x\in\mathbb{R}$, is found to be the function I defined in the statement of the proposition. Also, $D_\Lambda:=\{\lambda\in\mathbb{R}:\Lambda(\lambda)<\infty\}=\mathbb{R}$, and Λ is strictly convex and differentiable in D_Λ. The result follows from the Gartner–Ellis theorem. $\square$

4. Proofs of the lemmata.

LEMMA 7. *For any $x \geq 1$, $y \geq 0$, $k \in \mathbb{Z}$ and $\varepsilon > 0$, we have*

$$|U(x+\varepsilon, y, k) - U(x, y+\varepsilon, k)| \leq 3\varepsilon/x,$$
$$|U(x, y, k) - U(x, y+\varepsilon, k)| \leq \varepsilon/x.$$

In particular, U is continuous.

PROOF. Call A and B the two events whose probabilities are $U(x+\varepsilon, y, k)$ and $U(x, y+\varepsilon, k)$, respectively. Then $A \triangle B \subset$ [In $A_x(w)$ at least one of the three slopes neighboring zero has excess height $< \varepsilon$]. Denote by T_0, T_1 the central slope and the slope to the right to it in $A_x(w)$. Then

$$\mathbb{P}(A \triangle B) \leq 2\mathbb{P}(T_1 \text{ has excess height } < \varepsilon) + \mathbb{P}(T_0 \text{ has excess height } < \varepsilon)$$
$$= 2\int_0^{\varepsilon/x} e^{-z}\,dz + \int_0^{\varepsilon/x} (2z/3 + 1/3)e^{-z}\,dz < 3\varepsilon/x.$$

The other inequality follows because the density of the measure $\mathcal{U}(x, S, k)$ is bounded by $1/x$ [see (12)]. $\square$

PROOF OF LEMMA 3. Define $K : \mathbb{C} \times \mathbb{C} \times D \to \mathbb{C}$ with $K(z_1, z_2, z) := M(e^{z_1}, z_2, z)$ for $(z_1, z_2, z) \in \mathbb{C} \times \mathbb{C} \times D$. Clearly, K is a holomorphic function [choose an analytic branch of the square root function defined on $\mathbb{C} \setminus (-\infty, 0)$; the number $5 + 4z$ is there for $z \in D$] and has a power series development centered at zero that converges in $\mathbb{C} \times \mathbb{C} \times D$ (see, e.g., [9], Proposition 2.3.16). The claims of the lemma follow by the relation $M(x, y, z) = K(\log x, y, z)$ and standard properties of power series. $\square$

PROOF OF LEMMA 5. For $c > 1$ and $x \in [1, c]$, define $N(c, x) := \sup\{|f(z, c-z)| : z \in [1, x]\}$. N is well defined because f is bounded on compact sets.

From (21), for $x \in (1, c)$, one has

$$|g_c'(x)| \leq 2N(c, x) + 4 \sup_{1 \leq d \leq c} N(d, x),$$

and since $g_c(1) = 0$, we get after integrating

$$|f(x, c-x)| \leq 6 \int_1^x \sup_{1 \leq d \leq c} N(d, t)\,dt,$$

which implies

$$N(c, x) \leq 6 \int_1^x \sup_{1 \leq d \leq c} N(d, t)\,dt$$

and

$$\sup_{1\le d\le c} N(d,x) \le 6\int_1^x \sup_{1\le d\le c} N(d,t)\,dt.$$

The function $A(x)=\sup_{1\le d\le c} N(d,x)$ ($x\in[1,c]$) is continuous (because f is) and has $A(1)=0$. An application of Gronwall's lemma to A gives $N(d,x)=0$ for $x\in[1,c], d\in[1,c]$; that is, $f(x,y)=0$ for all $(x,y)\in[1,+\infty)\times[0,+\infty)$. $\square$

PROOF OF LEMMA 6. Let X_k be as in Corollary 2. Then

$$\begin{aligned}\mathbb{P}\bigl(k(x)=n+1\bigr) &\le \mathbb{P}\bigl(k(x)\ge n+1\bigr)=\mathbb{P}(X_{n+1}\le x)\\ &\le \mathbb{P}\bigl(r_1r_2\dots r_n\le x\bigr)=\mathbb{P}(\log r_1+\log r_2+\cdots+\log r_n\le \log x).\end{aligned}$$

For $i\ge 1$, set $Y_i=\log r_i$, $S_i=Y_1+Y_2+\cdots+Y_i$, and let m_i be the distribution measure of S_i/i. By Cramér's theorem ([4], Theorem 2.2.3), the sequence $(m_i)_{i\ge1}$ satisfies a large deviation principle with rate function $I(x)=\sup_{\lambda\in\mathbb{R}}\{\lambda x-\log\mathbb{E}(e^{\lambda\log r_1})\}$ for $x\in\mathbb{R}$.

Clearly, for any $\varepsilon>0$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\biggl(\frac{S_n}{n}<\frac{\log x}{n}\biggr)<-I(\varepsilon).$$

And $\lim_{\varepsilon\to0} I(\varepsilon)=+\infty$ because I is lower semicontinuous and $I(0)=+\infty$. To see the last point, observe that

$$I(0)=\sup_{\lambda\in\mathbb{R}}\{-\log\mathbb{E}(e^{\lambda\log r_1})\}\ge\limsup_{\lambda\to-\infty}\{-\log\mathbb{E}(e^{\lambda\log r_1})\}=+\infty$$

because $\lim_{\lambda\to-\infty}e^{\lambda\log r_1}=0$ with probability one and the bounded convergence theorem applies. Thus, $\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(k(x)=n+1)=-\infty$, proving that the radius of convergence for the power series is infinite. $\square$

LEMMA 8. *For $\mathbb{P}$ and $\mathcal{W}_1$ as defined in the Introduction, it holds that* $\mathbb{P}(\mathcal{W}_1)=1$.

PROOF. First we prove that, for fixed $x>0$, the set

$$C_x:=\{w\in C(\mathbb{R}):R_x(w)\text{ has the properties appearing in the definition of }\mathcal{W}_1\}$$

has $\mathbb{P}(C_x)=1$. Observe that, for z a point of x-minimum and $\alpha_z:=\sup\{\alpha<z:w(\alpha)\ge w(z)+x\}$, $\beta_z:=\inf\{\beta>z:w(\beta)\ge w(z)+x\}$, it holds that $\alpha_z<z<\beta_z$ because w is continuous at z. And there is no other point of x-minimum in (α_z,β_z) since if $\tilde z$ is such a point, say, in (α_z,z), then, assuming $\beta_{\tilde z}<z$, we get a contradiction with the definition of a_z, while, assuming $\beta_{\tilde z}>z$, we get that w takes the same value in two local minima, which has probability zero. Now assume that there is a strictly monotone, say, increasing, sequence $(z_n)_{n\ge1}$ of x-minima converging to $z_\infty\in\mathbb{R}$. By the above observations, we get $\limsup_{y,\tilde{y}\nearrow z_\infty}(w(y) - w(\tilde{y})) \geq x$, contradicting the continuity of w at z_∞. Similarly, if $(z_n)_{n\geq 1}$ is decreasing. So in a set of w's in $C(\mathbb{R})$ with probability 1, it holds that the set of x-minima of w has no accumulation point. The same holds for the set of x-maxima and as a result, also for $R_x(w)$. Since $\liminf_{t\to-\infty} w_t = \liminf_{t\to+\infty} w_t = -\infty$, $\limsup_{t\to-\infty} w_t = \limsup_{t\to+\infty} w_t = +\infty$, it follows that $\mathbb{P}(R_x(w)$ is unbounded above and below$) = 1$. It is clear that, between two consecutive x-maxima (resp. minima), there is exactly one x-minimum (resp. x-maximum). Consequently, $\mathbb{P}(C_x) = 1$.

Finally, note that, for all $n \in \mathbb{N} \setminus \{0\}$, we have $R_n(w) \subset R_x(w) \subset R_{1/n}(w)$ for $x \in [1/n, n]$, from which it follows that $\mathcal{W}_1 = \bigcap_{x\in(0,+\infty)} C_x = \bigcap_{n\in\mathbb{N}\setminus\{0\}}(C_n \cap C_{1/n})$. Thus, $\mathbb{P}(\mathcal{W}_1) = 1$. $\square$

Acknowledgments. I thank my advisor Amir Dembo for his help throughout the time this research was conducted and for his comments on previous drafts of the paper. I also thank Ofer Zeitouni for his comments on an earlier draft, Daniel Fisher for a useful discussion we had during a visit of his to Stanford and Francis Comets for providing me with a preprint of [2].

REFERENCES

[1] BROX, TH. (1986). A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.* **14** 1206–1218.

[2] COMETS, F. and POPOV, S. (2003). Limit law for transition probabilities and moderate deviations for Sinai's random walk in random environment. *Probab. Theory Related Fields* **126** 571–609.

[3] DEMBO, A., GUIONNET, A. and ZEITOUNI, O. (2001). Aging properties of Sinai's model of random walk in random environment. In *Saint Flour Summer School 2001 Lecture Notes* (O. Zeitouni, ed.). Available at www.math.umn.edu/ zeitouni/technion/.

[4] DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. Springer, New York.

[5] DURRETT, R. (1996). *Probability*: *Theory and Examples*, 2nd ed. Wadsworth, Pacific Grove, CA.

[6] GOLOSOV, A. O. (1984). Localization of random walks in one-dimensional random environments. *Comm. Math. Phys.* **92** 491–506.

[7] HU, Y. (2000). Tightness of localization and return time in random environment. *Stochastic Process. Appl.* **86** 81–101.

[8] KESTEN, H. (1986). The limit distribution of Sinai's random walk in random environment. *Phys. A* **138** 299–309.

[9] KRANTZ, S. (1992). *Function Theory of Several Complex Variables*. Wadsworth, Pacific Grove, CA.

[10] LE DOUSSAL, P., MONTHUS, C. and FISHER, D. (1999). Random walkers in one-dimensional random environments: Exact renormalization group analysis. *Phys. Rev. E* **59** 4795–4840.

[11] MARSDEN, J. and HOFFMAN, M. (1987). *Basic Complex Analysis*, 2nd ed. W. H. Freeman and Company, New York.

[12] NEVEU, J. and PITMAN, J. (1989). Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion. *Séminaire de Probabilités XXIII. Lecture Notes in Math.* **1372** 239–247. Springer, Berlin.

[13] SCHUMACHER, S. (1984). Diffusions with random coefficients. Ph.D. dissertation, UCLA.

[14] SCHUMACHER, S. (1985). Diffusions with random coefficients. In *Particle Systems*, *Random Media and Large Deviations* (R. Durett, ed.) 351–356. Amer. Math. Soc., Providence, RI.

[15] SEIGNOUREL, P. (2000). Discrete schemes for processes in random media. *Probab. Theory Related Fields* **118** 293–322.

[16] SHI, Z. (2001). Sinai's walk via stochastic calculus. In *Milieux Aléatoires*, *Panoramas et Synthèses* (F. Comets and E. Pardoux, eds.) **12** 53–74. Société Mathématique de France.

[17] SINAI, Y. G. (1982). The limiting behavior of a one-dimensional random walk in a random medium. *Theory Probab. Appl.* **27** 256–268.

[18] TANAKA, H. (1987). Limit distribution for 1-dimensional diffusion in a reflected Brownian medium. *Séminaire de Probabilités XXI. Lecture Notes in Math.* **1247** 246–261. Springer, New York.

[19] TANAKA, H. (1987). Limit distributions for one-dimensional diffusion processes in self-similar random environments. In *Hydrodynamic Behavior and Interacting Particle Systems* (G. Papanicolaou, ed.) 189–210. Springer, New York.

[20] TANAKA, H. (1988). Limit theorem for one-dimensional diffusion process in Brownian environment. *Stochastic Analysis*. *Lecture Notes in Math.* **1322** 157–172. Springer, New York.

[21] TANAKA, H. (1994). Localization of a diffusion process in a one-dimensional Brownian environment. *Comm. Pure Appl. Math.* **47** 755–766.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
100 ST. GEORGE STREET
TORONTO, ONTARIO
CANADA M5S 3G3
E-MAIL: dimitris@math.toronto.edu
URL: www.math.toronto.edu/dimitris/