CONFORMAL INVARIANCE OF PLANAR LOOP-ERASED RANDOM WALKS AND UNIFORM SPANNING TREES

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This paper proves that the scaling limit of a loop-erased random walk in a simply connected domain $D \subsetneqq \mathbb{C}$ is equal to the radial SLE_2 path. In particular, the limit exists and is conformally invariant. It follows that the scaling limit of the uniform spanning tree in a Jordan domain exists and is conformally invariant. Assuming that ∂D is a C^1-simple closed curve, the same method is applied to show that the scaling limit of the uniform spanning tree Peano curve, where the tree is wired along a proper arc $A \subset \partial D$, is the chordal SLE_8 path in $\overline{D}$ joining the endpoints of A. A by-product of this result is that SLE_8 is almost surely generated by a continuous path. The results and proofs are not restricted to a particular choice of lattice.

1. Introduction.

1.1. *Motivation from statistical physics.* One of the main goals of both probability theory and statistical physics is to understand the asymptotic behavior of random systems when the number of microscopic random inputs goes to ∞. These random inputs can be independent, such as a sequence of independent random variables, or dependent, as in the Ising model. Often, one wishes to understand these systems via some relevant "observables" that can be of a geometric or analytic nature. In order to understand this asymptotic behavior, one can attempt to prove convergence toward a suitable continuous model. The simplest and most important example of such random continuous models is Brownian motion, which is the scaling limit of random walks. In particular, a simple random walk on any lattice in $\mathbb{R}^d$ converges to (a linear image of) Brownian motion in the scaling limit.

Physicists and chemists have observed that critical systems (i.e., systems at their phase transition point) can exhibit macroscopic randomness. Hence, various quantities related to the corresponding lattice models should converge as the mesh is refined. In fact, one of the important starting points for theoretical physicists working on two-dimensional critical models is the assumption that the continuous limit is independent of the lattice and, furthermore, displays conformal invariance. This assumption has enabled them to develop and use techniques from conformal field theory to predict the exact values of certain critical exponents. Until very

recently, the existence of the limit, its conformal invariance and the derivation of the exponents assuming conformal invariance remained beyond mathematical justification for the basic lattice models in critical phenomena, such as percolation, the Ising model and random cluster measures. Although there are many interesting questions about higher-dimensional systems, we will limit our discussion to two dimensions where conformal invariance plays an essential role.

1.2. *Recent progress.* In [38], a one-parameter family of random growth processes (loosely speaking, random curves) in two dimensions was introduced. The growth process is based on Loewner's differential equation, where the driving term is time-scaled one-dimensional Brownian motion, and is therefore called stochastic Loewner evolution, or SLE_κ. The parameter $\kappa \geq 0$ of SLE is the time scaling constant for the driving Brownian motion. It was conjectured that the scaling limit of the loop-erased random walk (LERW) is SLE_2, and this conjecture was proved to be equivalent to the conformal invariance of the LERW scaling limit (see [38]). The argument given was quite general and shows that a conformally invariant random path satisfying a mild Markovian property, which will be described below, must be SLE. On this basis, it was also conjectured there that the scaling limits of the critical percolation interface and the uniform spanning tree Peano curve are the paths of SLE_6 and SLE_8, respectively, and it was claimed that conformal invariance is sufficient to establish these conjectures. (For additional conjectures regarding curves tending to SLE, including the interfaces in critical random cluster models—also called FK percolation models—for $q \in [0, 4]$, see [36].)

At some values of the parameter κ, SLE has some remarkable properties. For instance, SLE_6 has a locality property (see [27]) that makes it possible to relate its outer boundary to that of planar Brownian motion. This has led to the proof of conjectures concerning planar Brownian motion and simple random walks (see [27]–[29]).

In [40] and [41], Smirnov recently proved the existence and conformal invariance of the scaling limit of critical site percolation on the two-dimensional triangular lattice: he managed to prove Cardy's formula (see [5]) which is a formula for the limit of the probability of a percolation crossing between two arcs on the boundary of the domain. Combining this information with the independence properties of percolation, Smirnov then showed that the scaling limit of the percolation interface is SLE_6. This has led to the rigorous determination of critical exponents for this percolation model (see [30] and [42]).

1.3. *LERW and UST defined.* The uniform spanning tree (UST), which can be interpreted as the $q = 0$ critical random cluster model (see [15]), is a dependent model that has many remarkable features. In particular, it is very closely related to the loop-erased random walk, whose definition (see [24]) we now briefly recall.

Consider any finite or recurrent connected graph G, a vertex a and a set of vertices V. A *loop-erased random walk* (LERW) from a to V is a random simple curve joining a to V obtained by erasing the loops in chronological order from a simple random walk started at a and stopped upon hitting V. In other words, if $(\Gamma(n), 0 \leq n \leq T)$ is a simple random walk on G started from a and stopped at its first hitting time T of V, the loop erasure $\beta = (\beta_0, \ldots, \beta_\ell)$ is defined inductively as follows: $\beta_0 = a$; if $\beta_n \in V$, then $n = \ell$; and otherwise $\beta_{n+1} = \Gamma(k)$, where $k = 1 + \max\{m \leq T : \Gamma(m) = \beta_n\}$.

A spanning tree T of a connected graph G is a subgraph of G such that for every pair of vertices v, u in G there is a unique simple path (i.e., self-avoiding) in T with these vertices as endpoints. A *uniform spanning tree* (UST) in a finite, connected graph G is a sample from the uniform probability measure on spanning trees of G. It has been shown in [34] that the law of the self-avoiding path with endpoints a and b in the UST is the same as that of the LERW from a to $\{b\}$. See Figure 1.

Wilson [44] established an even stronger connection between LERW and UST by giving an algorithm to generate UST's using LERW. Wilson's algorithm runs as follows. Pick an arbitrary ordering $v_0, v_1, \ldots, v_m$ for the vertices in G. Let $T_0 = \{v_0\}$. Inductively, for $n = 1, 2, \ldots, m$, define T_n to be the union of T_{n-1} and a (conditionally independent) LERW path from v_n to T_{n-1}. (If $v_n \in T_{n-1}$, then $T_n = T_{n-1}$.) Then, regardless of the chosen order of the vertices, T_m is a UST on G.

Wilson's algorithm gives a natural extension of the definition of UST to infinite recurrent graphs. In fact, for transient graphs, there are two natural definitions which often coincide, but this interesting theory is somewhat removed from the topic of this paper. Many striking properties of UST and LERW have been
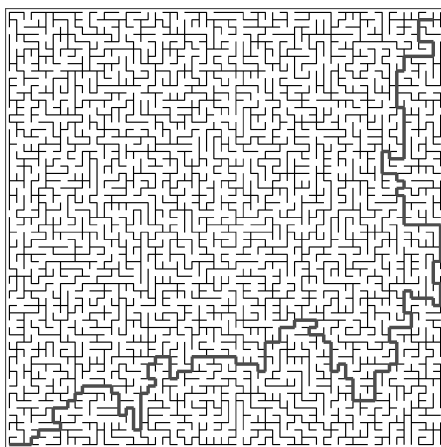

FIG. 1.   *The LERW in the UST.*

discovered. See [32] for a survey of UST's and [26] for a survey of properties of LERW in $\mathbb{Z}^d$, $d > 2$.

Exploiting a link with domino tilings and deriving discrete analogs of Cauchy–Riemann equations, Kenyon (see [20] and [21]) rigorously established the values of various critical exponents predicted for the LERW (see [11], [14] and [32]) in two dimensions. In particular, he showed that the expected number of steps of a LERW joining two corners of the $N \times N$ square in the square grid $\mathbb{Z}^2$ is of the order of magnitude of $N^{5/4}$. He also showed conformal invariance for the leading term in the asymptotics of the probability that the LERW contains a given edge. This was the first mathematical evidence for full conformal invariance of the LERW scaling limit.

In [3] and [1] subsequential scaling limits of the UST measures in $\mathbb{Z}^d$ were shown to exist, using a compactness argument. Moreover, these papers prove that all the paths in the scaling limit that intersect a fixed bounded region are uniformly Hölder continuous. In [38] the topology of subsequential scaling limits of the UST on $\mathbb{Z}^2$ was determined. In particular, it was shown that every subsequential scaling limit of the LERW is a simple path.

1.4. *A short description of SLE.* We now briefly describe SLE; precise definitions are deferred to Section 2.1. *Chordal* SLE is a random growing family of compact sets K_t, $t \in [0, \infty)$, in the closure $\overline{\mathbb{H}}$ of the upper half plane $\mathbb{H}$. The evolution of K_t is given by the Loewner differential equation with "driving function" Brownian motion. From [36], it is known that when $\kappa \neq 8$ the process is described by a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$, in the sense that, for every $t \geq 0$, $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. A corollary of our results is that this holds for $\kappa = 8$ as well. The curve γ satisfies $\gamma(0) = 0$ and $\lim_{t\to\infty} \gamma(t) = \infty$. If $\kappa \leq 4$, then γ is a simple curve and $K_t = \gamma[0, t]$.

There is another version of SLE called *radial* SLE. Radial SLE also satisfies the description above, except that the upper half plane $\mathbb{H}$ is replaced by the unit disk $\mathbb{U}$, $\gamma(0)$ is on the unit circle $\partial \mathbb{U}$ and $\lim_{t\to\infty} \gamma(t) = 0$.

Both radial and chordal versions of SLE may be defined in an arbitrary simply connected domain $D \subsetneq \mathbb{C}$ by mapping over to D using a fixed conformal map ϕ from $\mathbb{H}$ or $\mathbb{U}$ to D.

1.5. *The main results of the paper.* Let $D \subsetneq \mathbb{C}$ be a simply connected domain with $0 \in D$. For $\delta > 0$, let μ_δ be the law of the loop erasure of a simple random walk on the grid $\delta \mathbb{Z}^2$, started at 0 and stopped when it hits ∂D. See Figure 2. Let ν be the law of the image of the radial SLE_2 path under a conformal map from the unit disk $\mathbb{U}$ to D fixing 0. When the boundary of D is very rough, the conformal map from $\mathbb{U}$ to D might not extend continuously to the boundary, but the proof of the following theorem in fact shows that even in this case the image of the SLE_2 path has a unique endpoint on ∂D.
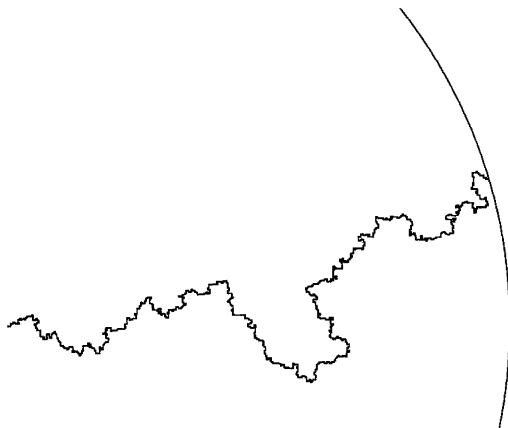
FIG. 2. *A sample of the loop-erased random walk; proved to converge to radial* SLE$_2$.

On the space of unparameterized paths in $\mathbb{C}$, consider the metric $\rho(\beta, \gamma) = \inf \sup_{t \in [0,1]} |\hat{\beta}(t) - \hat{\gamma}(t)|$, where the infimum is over all choices of parameterizations $\hat{\beta}$ and $\hat{\gamma}$ in $[0, 1]$ of β and γ.

THEOREM 1.1 (LERW scaling limit). *The measures μ_δ converge weakly to ν as $\delta \to 0$ with respect to the metric ρ on the space of curves.*

Since SLE is conformally invariant by definition, this theorem implies conformal invariance of the LERW. The theorem and proof apply also to some other walks on lattices in the plane where the scaling limit of the walk is isotropic Brownian motion. It even applies in the nonreversible setting. See Section 6 for further details.

There are two distinct definitions for the UST corresponding to a domain $D \subsetneq \mathbb{C}$, as follows. Let $G_F(D)$ denote the subgraph of $\mathbb{Z}^2$ consisting of all the edges and vertices which are contained in D. If $G_F(D)$ is connected, then we refer to the UST on $G_F(D)$ as the UST on D with *free* boundary conditions. Let $G_W(D)$ denote the graph obtained from $\mathbb{Z}^2$ by contracting all the vertices outside of D to a single vertex (and removing edges which become loops). Then the UST on $G_W(D)$ is the UST on D with *wired* boundary conditions.

Since the UST is built from the LERW via Wilson's algorithm, it is not surprising that conformal invariance of the UST scaling limit should follow from that of the LERW scaling limit. In fact, [38], Theorem 11.3, says just that.

COROLLARY 1.2 (UST scaling limit). *The wired and free UST scaling limits (as defined in* [38]) *in a simply connected domain $D \subset \mathbb{C}$ whose boundary is a C^1-smooth simple closed curve exist and are conformally invariant.*

One can easily show, using [38], Theorem 11.1(i), that the wired tree depends continuously on the domain, and hence for that case D may be an arbitrary simply connected domain. However, some regularity assumption is needed for the free UST scaling limit: conformal invariance fails for the domain whose boundary contains the topologist's sine curve (the closure of $\{x + i \sin(1/x) : x \in (0, 1]\}$).

The UST Peano curve is an entirely different curve derived from the UST in two dimensions. The curve is rather remarkable, as it is a natural random path visiting every vertex in an appropriate graph or lattice. We now roughly describe two natural definitions of this curve; further details appear in Section 4.

Let G be a finite planar graph, with a particular embedding in the plane, and let $G^\dagger$ denote its planar dual, again with a particular embedding. Then there is a bijection $e \leftrightarrow e^\dagger$ between the edges of G and those of $G^\dagger$ such that, for every edge e in G, $e \cap e^\dagger$ is a single point, and e does not intersect any other edge of $G^\dagger$. Given a spanning tree T of G, let $T^\dagger$ denote the graph whose vertices are the vertices of $G^\dagger$ and whose edges are those edges $e^\dagger$ such that $e \notin T$. It is then easy to verify that $T^\dagger$ is a spanning tree for $G^\dagger$. Therefore, if T is a UST on G, then $T^\dagger$ is a UST on $G^\dagger$.

The UST Peano curve is a curve that winds between T and $T^\dagger$ and separates them. More precisely, consider the graph $\hat{G}$ drawn in the plane by taking the union of G and $G^\dagger$, where each edge e or $e^\dagger$ is subdivided into two edges by introducing a vertex at $e \cap e^\dagger$. The subgraph of the planar dual $\hat{G}^\dagger$ of $\hat{G}$ containing all edges which do not intersect $T \cup T^\dagger$ is a simple closed path—the *UST Peano path*. See Figure 3.

Some properties of the UST Peano path on $\mathbb{Z}^2$ have been studied in the physics literature; see, for example, [10] and [18]. There, it has been called the Hamiltonian path on the Manhattan lattice. The reason for this name is as follows. On $\mathbb{Z}^2$, say, orient each horizontal edge whose y-coordinate is even to the right and each horizontal edge whose y-coordinate is odd to the left. Similarly, orient down each vertical edge whose x-coordinate is even and orient up each vertical edge whose
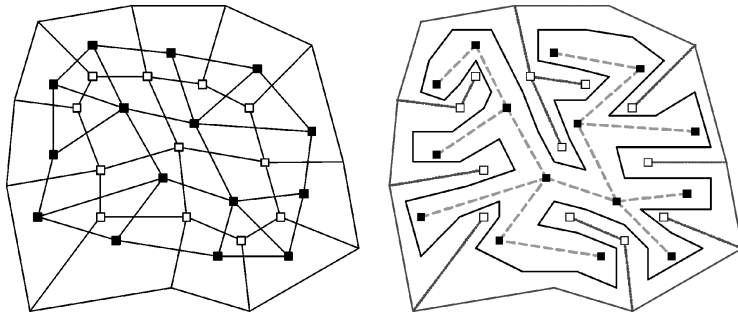

FIG. 3. *The graph, dual graph, tree, dual tree and Peano curve. The vertex of the dual graph corresponding to the unbounded face is drawn as a cycle.*

x-coordinate is odd. Now rescale the resulting oriented graph by $1/2$ and translate it by $(1/4, 1/4)$. It is easy to check that a Hamiltonian path (a path visiting every vertex exactly once) respecting the orientation on the resulting oriented graph is the same as the UST Peano path of $\mathbb{Z}^2$. It should be expected that the uniform measure on Hamiltonian paths in $\mathbb{Z}^2$ has the same scaling limit as that of the UST Peano path.

Given a domain D, one can consider the UST Peano curve for the wired or for the free UST (which is essentially the same as the wired, by duality). However, the conjecture from [38] regarding the convergence to chordal SLE pertains to the UST Peano curve associated with the tree with mixed wired and free conditions.

Let $D \subset \mathbb{C}$ be a domain whose boundary is a C^1-smooth simple closed curve and let $a, b \in \partial D$ be distinct boundary points. Let α and β denote the two complementary arcs of ∂D whose endpoints are a and b. For all $\delta > 0$, consider an approximation G_δ of the domain D in the grid $\delta\mathbb{Z}^2$. (A precise statement of what it means for G_δ to be an approximation of D will be given in Section 4.) Let γ_δ denote the Peano curve associated to the UST on G_δ with wired boundary near α and free boundary near β. Then γ_δ may be considered as a path in D from a point near a to a point near b.

THEOREM 1.3 (UST Peano path scaling limit). *The UST Peano curve scaling limit in D with wired boundary on α and free boundary on β exists and is equal to the image of the chordal* SLE_8 *path under any conformal map from $\mathbb{H}$ to D mapping 0 to a and ∞ to b.*

Again, the convergence is weak convergence of measures with respect to the metric ρ. Figure 4 shows a sample of the UST Peano path on a fine grid.

As explained above, it was proved in [36] that each SLE_κ is generated by a path, except for $\kappa = 8$. In Section 4.4, the remaining case $\kappa = 8$ is proved, using the convergence of the Peano curve.
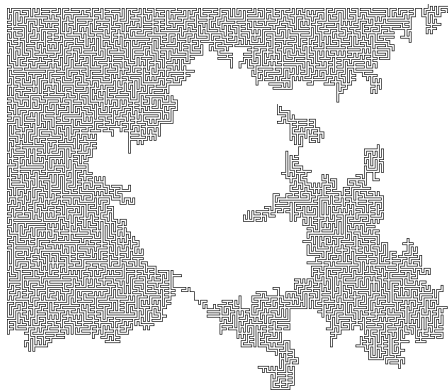

FIG. 4. *An arc from a sample of the UST Peano path*; *proved to converge to chordal* SLE_8.

Corollary 1.2 and Theorem 1.3 (and their proofs) apply to other reversible walks on planar lattices (the self-duality of $\mathbb{Z}^2$ does not play an important role); see Section 6.

To add perspective, we note that the convergence to SLE of the LERW and the UST Peano curve are two boundary cases of the conjectured convergence in [36] of the critical FK random cluster measures with parameter $q \in (0, 4)$. For these parameter values, the scaling limit of the interface of a critical cluster with mixed boundary values is conjectured to converge to chordal $\text{SLE}_\kappa(q)$, where $\kappa(q) = 4\pi/\cos^{-1}(-\sqrt{q}/2)$. The boundary case $\kappa(0) = 8$ corresponds to the convergence of the UST Peano path to SLE_8.

The outer boundary of the scaling limit of a macroscopic critical cluster is not the same as the scaling limit of a critical cluster outer boundary, because of "fjords" which are pinched off in the limit. The former is conjectured to "look like" $\text{SLE}_{16/\kappa(q)}$, but a precise form of this conjecture is not yet known. In the case $q = 0$, however, such a correspondence is easy to explain. In $\mathbb{Z}^2$, an arc of the Peano curve is surrounded on one side by a simple path in the tree, and on the other side by a simple path in the dual tree. Both these paths are LERWs. Similar correspondences exist for the UST in a subdomain of $\mathbb{R}^2$, but one has to set appropriate boundary conditions. Thus, the convergence of LERW to SLE_2 also corresponds to the case $q = 0$, as $16/\kappa(0) = 2$.

Suppose that $0 \in D$ and $\alpha, \beta \subset \partial D$, as before. Consider the simple random walk on $\delta\mathbb{Z}^2$ which is reflected off β and stopped when it hits α. Using an analogous method to that of the present paper, one could handle the scaling limit of the loop erasure of this walk. It is described by a variant of SLE_2 where the driving term is Brownian motion with time scaled by 2, but having an additional drift. The drift is not constant, but can be explicitly computed.

The identification of the scaling limit as one of the SLEs should facilitate the derivation of critical exponents and also the asymptotic probabilities of various events, including some results which have not been predicted by arguments from physics. This was the case for critical site percolation on the triangular grid; see [30], [39], [40] and [42].

1.6. *Some comments about the proof.* Since a loop-erased random walk is obtained in a deterministic way from a simple random walk (by erasing its loops) and since a simple random walk converges to Brownian motion in the scaling limit, it is natural to think that the scaling limit of the LERW should simply be the process obtained by erasing the loops from a planar Brownian motion. The problem with this approach is that planar Brownian motion has loops at every scale, so that there is no simple algorithm to erase loops. In particular, there is no "first" loop. Our proof does use the relation between the LERW and simple random walks, combined with the fact that quantities related to simple random walks, such as hitting probabilities, converge to their continuous conformally invariant counterparts.

The proof of each of our main theorems is naturally divided into two parts. The first part establishes the convergence to SLE with respect to a weaker topology than the topology induced by the metric ρ of paths; namely, we show that the Loewner driving process for the discrete random path converges to a Brownian motion. This part of the proof, which we consider to be the more important one, is essentially self-contained. The second part uses some regularity properties of the discrete processes from [38] to prove convergence with respect to the stronger topology.

The method for the first part can be considered as a rather general method for identifying the scaling limit of a dependent system that is conjectured to be conformally invariant. It requires having some "observable" quantity that can be estimated well and a mild Markovian property, which we now describe. Suppose that to every simply connected domain D containing 0 there is associated a random path γ from ∂D to 0 (e.g., the orientation reversal of the LERW). The required property is that if β is an arc with one endpoint in ∂D and we condition on $\beta \subset \gamma$ (assuming this has positive probability, say), then the conditioned distribution of $\gamma \setminus \beta$ is the same as the random path in the domain $D \setminus \beta$ conditioned to start at the other endpoint q of β. [Thus, $(D \setminus \beta, q)$ is the state of a Markov chain whose transitions correspond to adding edges from γ to β and modifying q appropriately.] Interestingly, among the discrete processes conjectured to converge to SLE, the LERW is the only one where the verification of this property is not completely trivial. [For the LERW it is not trivial, but not difficult; see part (iii) of Lemma 3.2.] The statement of this property for the UST Peano curve is given in Lemma 4.1. The fact that SLE satisfies this property follows from the Markovian property of its driving Brownian motion.

The particular choice of observable is not so important. What is essential is that one can conveniently calculate the asymptotics of the observable for appropriate large-scale configurations. The particular observable that we have chosen for the LERW convergence is the expected number of visits to a vertex v by the simple random walk generating the LERW. Conformal invariance is not assumed but comes out of the calculation—hitting probabilities for random walks are discrete harmonic functions, which converge to continuous harmonic functions. One technical issue is to establish this convergence without any boundary smoothness assumption. Once the observable has been approximated, the conditional expectation and variance of increments of the Loewner driving function for the discrete process can be estimated, and standard techniques (the Skorohod embedding) can be used to show that this random function approaches the appropriate Brownian motion.

Although Theorem 1.3 can probably be derived with some work from Corollary 1.2, instead, to illustrate our method, we prove it by applying again the same general strategy of the proof of Theorem 1.1, with the choice of a different observable.

Actually, it is easier to explain the main ideas behind the proof of Theorem 1.3. Fix some vertex v in D and a subarc $\alpha_1 \subset \alpha$. Let $\mathcal{A}$ be the event that the UST path (not the Peano path, but the path contained in the UST) from v to α hits α_1. By Wilson's algorithm the probability of $\mathcal{A}$ is the same as the probability that a simple random walk started at v reflected off β first hits α in α_1. The latter probability can be estimated directly. If $\gamma[0, n]$ denotes the restriction of the Peano path to its first n steps, then $\mathbf{P}[\mathcal{A}|\gamma[0, n]]$, the probability of $\mathcal{A}$ conditioned on $\gamma[0, n]$, is clearly a martingale with respect to n. But, by the Markovian property discussed above, the value of $\mathbf{P}[\mathcal{A}|\gamma[0, n]]$ may be estimated in precisely the same way that $\mathbf{P}[\mathcal{A}]$ is estimated. The estimate turns out to be a function of the conformal geometry of the configuration $(v, D \setminus \gamma[0, n], \gamma(n), \alpha_1, \beta)$. Knowing that this is a martingale for two appropriately chosen vertices v is sufficient to characterize the large-scale behavior of γ.

As mentioned above, in the case of the LERW, the observable we chose to look at is the expected number of visits to a fixed vertex v by the simple random walk Γ generating the LERW γ. The walk Γ can be considered as the union of γ with a sequence of loops Γ^j based at vertices of γ. We look at the conditioned expectation of the number of visits of Γ to v given an arc $\tilde{\gamma}$ of γ adjacent to the boundary of the domain. This is clearly a martingale with respect to the filtration obtained by taking larger and larger arcs $\tilde{\gamma} \subset \gamma$. This quantity falls into two parts: the visits to v in the loops Γ^j based at $\tilde{\gamma}$, and those that are not. Each of these two parts can be well estimated by random walk calculations. Translating the fact that this is a martingale to information about the Loewner driving process for γ inevitably leads to the identification of this driving process as appropriately scaled Brownian motion.

Actually, we first had a longer proof of convergence of the LERW to the SLE$_2$, based on the fact that it is possible to construct the hull of a Brownian motion by adding Brownian loops to SLE$_2$. This can be viewed as a particular case of the restriction properties of SLE$_\kappa$ with Brownian loops added, which we study in [31]. Let us also mention the following related open question. Consider a sequence of simple random walks $S^k(n)$ on a lattice with lattice spacing $\delta_k \to 0$, from $S^k(0) = 0$ to $\partial\mathbb{U}$, and let γ^k denote the corresponding loop-erased paths. Theorem 1.1 shows that one can find a subsequence such that the law of the pair (γ^k, S^k) converges to a coupling of SLE$_2$ with Brownian motion [i.e., a law for a pair (X, Y), where X has the same distribution as the SLE$_2$ path and Y has the same distribution as Brownian motion]. The question is whether, in this coupling, SLE$_2$ is a deterministic function of the Brownian motion. In other words, is it possible to show that this is not a deterministic procedure to erase loops from a Brownian motion?

2. Preliminaries. The reading of this paper requires some background knowledge in several different fields. Some background about Loewner's equation and SLE is reviewed in the next section. It is assumed that the reader is familiar

with some of the basic properties of Brownian motion (definition, strong Markov property, etc.). Some of the basic properties of conformal maps (Riemann's mapping theorem, compactness, Koebe distortion) are also needed for the proof. This material may be learned from the first two chapters of [35], for example. In terms of the theory of conformal mappings, this suffices for understanding the argument showing that the driving process of the LERW converges to Brownian motion. For improving the topology of convergence, some familiarity with the notion of extremal length (a.k.a. extremal distance) is also required. A possible source for that is [2]. The reader also needs to know some of the very basic properties of harmonic measure.

2.1. *Loewner's equation and SLE.* We now review some facts concerning Loewner's equations and stochastic Loewner evolutions. For more details, see, for example, [27], [28], [36] and [38].

Suppose that $D \subsetneq \mathbb{C}$ is a simply connected domain with $0 \in D$. Then there is a unique conformal homeomorphism $\psi = \psi_D : D \to \mathbb{U}$ which is onto the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ such that $\psi_D(0) = 0$ and $\psi'_D(0)$ is a positive real. If $D \subset \mathbb{U}$, then $\psi'_D(0) \geq 1$, and $\log \psi'_D(0)$ is called the *capacity* of $\overline{\mathbb{U}} \setminus D$ from 0.

Now suppose that $\eta : [0, \infty] \to \overline{\mathbb{U}}$ is a continuous simple curve in the unit disk with $\eta(0) \in \partial \mathbb{U}$, $\eta(\infty) = 0$ and $\eta(0, \infty] \subset \mathbb{U}$. For each $t \geq 0$, set $K_t := \eta[0, t]$, $U_t := \mathbb{U} \setminus K_t$ and $g_t := \psi_{U_t}$. Since $t \mapsto g'_t(0)$ is increasing (by the Schwarz lemma, say), one can reparameterize the path in such a way that $g'_t(0) = \exp(t)$. If that is the case, we say that η is parameterized by *capacity* from 0. By standard properties of conformal maps (see [35], Proposition 2.5), for each $t \in [0, \infty)$ the limit

$$W(t) := \lim_{z \to \eta(t)} g_t(z),$$

where z tends to $\eta(t)$ from within $\mathbb{U} \setminus \eta[0, t]$, exists. One can also verify that

$$W : [0, \infty) \to \partial \mathbb{U}$$

is continuous. Assuming the parameterization by capacity, Loewner's theorem states that g_t satisfies the differential equation

$$(2.1) \qquad \partial_t g_t(z) = -g_t(z) \frac{g_t(z) + W(t)}{g_t(z) - W(t)}.$$

It is also clear that

$$(2.2) \qquad \forall z \in \mathbb{U}, \qquad g_0(z) = z.$$

We call $(W(t), t \geq 0)$ the *driving function* of the curve η.

The driving function W is sufficient to recover the two-dimensional path η, because the procedure may be reversed, as follows. Suppose that $W : [0, \infty) \to \partial \mathbb{U}$ is continuous. Then for every $z \in \overline{\mathbb{U}}$ there is a solution $g_t(z)$ of the ODE (2.1) with initial value $g_0(z) = z$ up to some time $\tau(z) \in (0, \infty]$, beyond which the solution

does not exist. In fact, if $\tau(z) < \infty$ and $z \neq W(0)$, then we have $\lim_{t \uparrow \tau(z)} g_t(z) - W(t) = 0$, since this is the only possible reason the ODE cannot be solved beyond time $\tau(z)$. Then one defines $K_t := \{z \in \overline{\mathbb{U}} : \tau(z) \leq t\}$ and $D_t := \mathbb{U} \setminus K_t$ is the domain of definition of g_t. The set K_t is called the *hull* at time t. If W arises from a simple path η as described in the previous section, then we can recover η from W by using $\eta(t) = g_t^{-1}(W(t))$. However, if $W : [0, \infty) \to \mathbb{U}$ is an arbitrary continuous driving function, then, in general, K_t need not be a path, and even if it is a path, it does not have to be a simple path.

Radial SLE$_\kappa$ is the process $(K_t, t \geq 0)$, where the driving function $W(t)$ is set to be $W(t) := \exp(i B_{\kappa t})$, where $B : [0, \infty) \to \mathbb{R}$ is Brownian motion. Often, one takes the starting point B_0 to be random uniform in $[0, 2\pi]$. It has been shown in [36] that the hull K_t is a.s. a simple curve for every $t > 0$ if $\kappa \leq 4$ and that, a.s. for every $t > 0$, K_t is not a simple curve if $\kappa > 4$. For every $\kappa \geq 0$, there is a.s. some random continuous path $\eta : [0, \infty) \to \overline{\mathbb{U}}$ such that, for all $t > 0$, D_t is the component of $\mathbb{U} \setminus \eta[0, t]$ containing 0. When $\kappa \neq 8$, this was proved in [36], while for $\kappa = 8$ this will be proven in the current paper. This path is called the radial SLE path.

Suppose that D is a simply connected domain containing 0. If γ is a continuous simple curve joining ∂D to 0 with only an endpoint in ∂D, one can reparameterize the path $\eta := \psi \circ \gamma$ according to capacity and find its driving function W, as before. The conformal map

$$\hat{g}_t = \psi_{D \setminus \gamma[0,t]} : D \setminus \gamma[0, t] \to \mathbb{U}$$

still satisfies (2.1), but this time, $\hat{g}_0 = \psi_D$. (Here, the parameterization chosen for γ is according to the capacity of $\psi \circ \gamma[0, t]$.) Radial SLE in D is then simply the image under ψ_D^{-1} of radial SLE in the unit disk.

Similarly, one can encode continuous simple curves η from 0 to ∞ in the closed upper half plane $\overline{\mathbb{H}}$ via a variant of Loewner's equation. For each time $t \geq 0$, there is a unique conformal map g_t from $H_t := \mathbb{H} \setminus \eta[0, t]$ onto $\mathbb{H}$ satisfying the so-called *hydrodynamic normalization*

(2.3) $$\lim_{z \to \infty} g_t(z) - z = 0,$$

where $z \to \infty$ in $\mathbb{H}$. If we write $g_t(z) = z + a(t) z^{-1} + o(z^{-1})$ near ∞, it turns out that $a(t)$ is monotone. Consequently, one can reparameterize η in such a way that $a(t) = 2t$, that is, $g_t(z) = z + 2t z^{-1} + o(z^{-1})$ when $z \to \infty$. This parameterization of η is called the parameterization by *capacity* from ∞. (This notion of capacity is analogous to the notion of capacity in the radial setting; however, these are two distinct notions and should not be confused.) If $g : \mathbb{H} \setminus K \to \mathbb{H}$ is the conformal homeomorphism satisfying the hydrodynamic normalization, then $\lim_{z \to \infty}(g_t(z) - z) z/2$ is called the *capacity* of K from ∞. Assuming that η is parameterized by capacity, the following analog of Loewner's

equation holds:

$$(2.4) \qquad \forall t > 0, \ \forall z \in H_t, \qquad \partial_t g_t(z) = \frac{2}{g_t(z) - W(t)},$$

where the driving function W is again defined by $W(t) := g_t(\eta(t))$. As above, η is determined by W.

Conversely, suppose that W is a real-valued continuous function. For $z \in \overline{\mathbb{H}}$, one can solve the differential equation (2.4) starting with $g_0(z) = z$, up to the first time $\tau(z)$ where $g_t(z)$ and $W(z)$ collide [possibly, $\tau(z) = \infty$]. Let the hull be defined by $K_t := \{z \in \overline{\mathbb{H}} : \tau(z) \le t\}$. Then $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ is a conformal map onto $\mathbb{H}$, and $g_0(z) = z$. In general, K_t is not necessarily a simple curve. If $W(t) = B_{\kappa t}$, then $(K_t, t \ge 0)$ is called *chordal* SLE_κ.

It turns out (see [28], Section 4.1) that the local properties of chordal SLE_κ and of radial SLE_κ are essentially the same. [That is the reason the normalization $a(t) = 2t$ was chosen over the seemingly more natural $a(t) = t$.] In particular, for every κ, chordal SLE_κ is generated by a random continuous path, called the chordal SLE_κ path.

At some points in our proofs, we will need the following simple observation.

LEMMA 2.1 (Diameter bounds on K_t).   *There is a constant $C > 0$ such that the following always holds. Let $W : [0, \infty) \to \mathbb{R}$ be continuous and let $(K_t, t \ge 0)$ be the corresponding hull for Loewner's chordal equation* (2.4) *with driving function W. Set*

$$k(t) := \sqrt{t} + \max\{|W(s) - W(0)| : s \in [0, t]\}.$$

Then

$$\forall t \ge 0, \qquad C^{-1} k(t) \le \operatorname{diam} K_t \le C k(t).$$

Similarly, when $K_t \subset \overline{\mathbb{U}}$ is the radial hull for a continuous driving function $W : [0, \infty) \to \partial \mathbb{U}$, then

$$\forall t \ge 0, \qquad C^{-1} \min\{k(t), 1\} \le \operatorname{diam} K_t \le C k(t).$$

PROOF.   This lemma can be derived by various means. We will only give a detailed argument in the radial case. The chordal case is actually easier and can be derived using the same methods. It can also be seen as a consequence of the result in the radial setting (because chordal Loewner equations can be interpreted as scaling limits of radial Loewner equations).

We start by proving the upper bound on $\operatorname{diam} K_t$. Let $\delta \ge \max\{|W(s) - W(0)| : s \in [0, t]\}$. Then, as long as $|g_t(z) - W(0)| \ge 3\delta$, we have $|\partial_t g_t(z)| \le 1/\delta$. Hence, if $|z - W(0)| \ge 4\delta$, then, for all $t \le \delta^2$, $|g_t(z) - z| \le \delta$ and therefore $z \notin K_t$. Hence, $\operatorname{diam} K_t \le 8k(t)$.

In order to derive the lower bound, we will compare capacity with harmonic measure. It is sufficient to consider the case where diam $K_t < 1/10$. Let μ denote the harmonic measure on $K_t \cup \partial\mathbb{U}$ from 0. Because K_t is contained in the disk of radius diam K_t with center $W(0) \in K_t \cap \partial\mathbb{U}$, there is a universal constant c such that $\mu(K_t) \leq c \operatorname{diam} K_t$. Hence, it suffices to give a lower bound for $\mu(K_t)$.

Since $g_t(z)/z$ is analytic and nonzero in a neighborhood of 0, the function $h(z) = \log|g_t(z)| - \log|z|$ is harmonic in $U_t := \mathbb{U} \setminus K_t$. Note that $h(0) = t$. Because $|g_t(z)| \to 1$ as z tends to the boundary of U_t, the mean value property of $h \circ g_t^{-1}$ implies the following relation between harmonic measure and capacity: $t = h(0) = \int \log(1/|z|) \, d\mu(z)$. Since K_t contains points in $\partial\mathbb{U}$ and diam $K_t \leq 1/10$, we have $\log(1/|z|) \leq c' \operatorname{diam} K_t$ for all $z \in K_t$. Therefore, $t \leq c'\mu(K_t) \operatorname{diam} K_t \leq c''(\operatorname{diam} K_t)^2$.

It now remains to compare $\mu(K_t)$ and $|W(t) - W(0)|$. We still assume that diam $K_t < 1/10$. Let $A_t := \partial\mathbb{U} \setminus g_t(\overline{\mathbb{U}} \setminus K_t)$. If $z \in \partial\mathbb{U} \setminus K_s$ and $s \leq t$, then (2.1) shows that $\partial_u|g_{s+u}(z) - W(s)| \geq 0$ at $u = 0$. This implies that $(A_s, s \leq t)$ is nondecreasing. Hence, for all $s \leq t$, we have $W(0) \in A_0 \subset A_t$ and $W(s) \in A_s \subset A_t$ so that $|W(s) - W(0)|$ is bounded by the length of A_t, which is equal to $2\pi\mu(K_t)$. This completes the proof of the lemma. $\quad\square$

2.2. *A discrete harmonic measure estimate.*   In this section, we introduce some notation and state an estimate relating discrete harmonic measure and continuous harmonic measure in domains in the plane. In order to get more quickly to the core of our method in Section 3.2, we postpone the proof of the harmonic measure estimate to Section 5.

A *grid domain* D is a domain whose boundary consists of edges of the grid $\mathbb{Z}^2$. For an arbitrary domain $D \subset \mathbb{C}$, and $p \in D$ define the inner radius of D with respect to p,

$$\operatorname{rad}_p(D) := \inf\{|z - p| : z \notin D\}.$$

Let $\mathfrak{D}$ denote the set of all simply connected grid domains such that $0 < \operatorname{rad}_0(D) < \infty$ (i.e., $D \neq \mathbb{C}$ and $0 \in D$).

Points in $\mathbb{R}^2 = \mathbb{C}$ with integer coordinates will be called *vertices*, or *lattice points*. Let $V(D) := D \cap \mathbb{Z}^2$ denote the lattice points in D.

Let $D \in \mathfrak{D}$ and let v be a vertex in ∂D. If ∂D contains more than one edge incident with v, then it may happen that the intersection of D with a small disk centered at v will not be connected. Hence, as viewed from D, v appears as more than one vertex. In particular, $\psi = \psi_D$ does not extend continuously to v. This is a standard issue in conformal mapping theory, which is often resolved by introducing the notion of prime ends. But in the present case, there is a simpler solution which suffices for our purposes. Suppose $v \in \mathbb{Z}^2 \cap \partial D$ and e is an edge incident with v that intersects D. The set of such pairs $w = (v, e)$ will be denoted $V_\partial(D)$. If $\psi : D \to \mathbb{U}$ is conformal, then $\psi(w)$ will be shorthand for the

limit of $\psi(z)$ as $z \to v$ along e (which always exists, by [35], Proposition 2.14). Similarly, if a random walk first exits D at v, we say that it exited D at w if the edge e was used when first hitting v. A reader of this paper who chooses to be sloppy and not distinguish between v and w will not lose anything in the way of substance. We will not always be so careful to make this distinction.

If $a \in V(D)$ and $b \in V(D) \cup V_\partial(D)$, define $H(a, b) = H_D(a, b)$ as the probability that the simple random walk started from a and stopped at its first exit time of D visits b.

For any $w \in D$ and $u \in V_\partial(D)$, we define

$$(2.5) \qquad \lambda = \lambda(w, u; D) := \frac{1 - |\psi(w)|^2}{|\psi(w) - \psi(u)|^2} = \mathrm{Re}\left(\frac{\psi(u) + \psi(w)}{\psi(u) - \psi(w)}\right).$$

Note that λ is also equal to the imaginary part of the image of w by the conformal map from D onto the upper half plane that maps 0 onto i and u to ∞. It is also the limit when $\epsilon \to 0$ of the ratio between the harmonic measure in D of the ϵ neighborhood of u in ∂D, taken, respectively, at w and at 0 (i.e., it corresponds to the Poisson kernel). Therefore, λ can be viewed as the continuous analog of $H(w, u)/H(0, u)$. Note that the function $h(w) = H(w, u)/H(0, u)$ is *discrete harmonic* on $V(D)$, which means that $h(w)$ is equal to the average of h on the neighbors of w when $w \in V(D)$.

PROPOSITION 2.2 (Hitting probability). *For every $\epsilon > 0$ there is some $r_0 > 0$ such that the following holds. Let $D \in \mathfrak{D}$ satisfy $\mathrm{rad}_0(D) > r_0$ and let $u \in V_\partial(D)$ and $w \in V(D)$. Suppose $|\psi_D(w)| \le 1 - \epsilon$ and $H(0, u) \ne 0$. Then*

$$(2.6) \qquad \left| \frac{H(w, u)}{H(0, u)} - \lambda(w, u; D) \right| < \epsilon.$$

The proof is given in Section 5.

3. Conformal invariance of the LERW.

3.1. *Loop-erased random walk background.* We now recall some well-known facts concerning loop-erased random walks.

LEMMA 3.1 (LERW reversal). *Let $D \in \mathfrak{D}$ and let Γ be a simple random walk from 0 stopped when it hits ∂D. Let β be the loop erasure of Γ and let γ be the loop erasure of the time reversal of Γ. Then γ has the same distribution as the time reversal of β.*

See [25]. A simpler proof follows immediately from the symmetry of (12.2.3) in [26]. This result (and the proofs) also holds if we condition Γ to exit ∂D at a prescribed $u \in V_\partial(D)$, which corresponds to the event $\{\gamma \cap \partial D = \{u\}\} = \{\beta \cap \partial D = \{u\}\}$ (assuming this has positive probability).

Throughout our proof, we will use the simple random walk Γ and the loop erasure $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_\ell)$ of its time reversal (so that $\gamma_0 \in \partial D$ and $\gamma_\ell = 0$). We use D_j to denote the grid domains $D_j := D \setminus \bigcup_{i=0}^{j-1}[\gamma_i, \gamma_{i+1}]$. Define, for $j \in \{0, 1, \ldots, \ell\}$,

$$n_j := \min\{n \geq 0 : \Gamma(n) = \gamma_j\}$$

and note that $n_{j+1} < n_j$ for $j = 0, 1, \ldots, \ell - 1$ by the definition of γ. Also set

$$\Gamma^{j+1} := \Gamma[n_{j+1}, n_j].$$

More precisely, consider Γ^j as the grid path given by

$$\Gamma^j(m) := \Gamma(m + n_j), \qquad m = 0, 1, \ldots, n_{j-1} - n_j.$$

LEMMA 3.2 (Markovian property). *Let $j \in \mathbb{N}$ and let $u_0, \ldots, u_j \in \mathbb{Z}^2$. Suppose that the probability of the event $(\gamma_0, \ldots, \gamma_j) = (u_0, \ldots, u_j)$ is positive. Conditioned on this event, the following hold*:

(i) *The paths $\Gamma^1, \ldots, \Gamma^j$ and $\Gamma[0, n_j]$ are conditionally independent.*

(ii) *For $k \in \{1, \ldots, j\}$, the conditional law of Γ^k is that of a simple random walk in D_{k-1} started from u_k and conditioned to leave D_{k-1} through the edge $[u_k, u_{k-1}]$.*

(iii) *The conditional law of $\Gamma[0, n_j]$ is that of a simple random walk started from 0 conditioned to leave D_j at u_j, and $\gamma[j, \ell]$ is the loop erasure of the time reversal of $\Gamma[0, n_j]$.*

PROOF. Since γ is the loop erasure of the reversal of Γ, the event $(\gamma_0, \ldots, \gamma_j) = (u_0, \ldots, u_j)$ is equivalent to the statement that, for each $k = 0, 1, \ldots, j-1$, the first hit of Γ to $\{u_0, \ldots, u_k\} \cup \partial D$ is through the edge $[u_{k+1}, u_k]$. Let $\tau_k := \min\{n : \Gamma(n) \in \{u_0, \ldots, u_k\} \cup \partial D\}$, $k = 0, \ldots, j$. The strong Markov property of Γ with the stopping times τ_k now implies the lemma. $\square$

The following simple lemma will also be needed.

LEMMA 3.3 (Expected visits). *Suppose that $v \in V(D)$ and that u_0 and u_1 are two vertices satisfying $\mathbf{P}[\gamma_0 = u_0, \gamma_1 = u_1] > 0$. Conditioned on $\gamma_0 = u_0$ and $\gamma_1 = u_1$, the expected number of visits to v by Γ^1 is $G(u_1, v)H(v, u_1)$.*

Here $G(u, v)$ denotes the discrete Green's function, that is, the expected number of visits to v by a simple random walk started at u, which is stopped on exiting D.

PROOF OF LEMMA 3.3. Let X be a simple random walk from u_1 stopped on exiting D and let k be the last time such that $X(k) = u_1$. Then Γ^1 conditioned on $\gamma_0 = u_0$ and $\gamma_1 = u_1$ has the same distribution as X conditioned on $X(k + 1) = u_0$. But the path $j \mapsto X(k + j)$ is independent from $X[0, k]$. Consequently, the expected number of visits of X to v conditioned on $X(k + 1) = u_0$ is equal to the expected number of visits to v of $X[0, k]$. The lemma follows. $\square$

3.2. *The core argument.* We keep the previous notation and also use the conformal maps $\psi_j : D_j \to \mathbb{U}$ satisfying $\psi_j(0) = 0$ and $\psi_j'(0) > 0$. Set $U_j := \psi_j(\gamma_j)$ and $U := U_0$. Note that γ can also be viewed as a continuously growing simple curve from ∂D to 0 and therefore can be represented by Loewner's equation. Let $W : [0, \infty) \to \partial \mathbb{U}$ denote the (unique) continuous function such that solving the radial Loewner equation with driving function $W(t)$ gives the path γ. Note that $U_j = W(t_j)$, where t_j is the continuous capacity of $\gamma[0, j]$ from 0 in D (i.e., the capacity of $\psi(\gamma[0, j])$ from 0 in $\mathbb{U}$). We denote by $(\vartheta(t), t \geq 0)$ the continuous real-valued function with $\vartheta(0) = 0$ such that $W(t) = W(0) \exp(i\vartheta(t))$. We also define $\Delta_j = \vartheta(t_j)$, so that $U_j = U \exp(i\Delta_j)$.

PROPOSITION 3.4 (The key estimate). *There exists a positive constant C such that, for all small positive δ, there exists $r_0 = r_0(\delta)$ such that the following holds. Let $D \in \mathfrak{D}$ satisfy $\mathrm{rad}_0(D) > r_0$. For every $u_0 \in V_\partial(D)$ with $\mathbf{P}[\gamma_0 = u_0] > 0$, let γ denote the random path from u_0 to 0 obtained by loop erasure of the time reversal of a simple random walk from 0 to ∂D conditioned to hit ∂D in u_0. Let*

$$m := \min\{j \geq 1 : t_j \geq \delta^2 \ or \ |\Delta_j| \geq \delta\},$$

where Δ_j and t_j are as described above. Then

(3.1) $$|\mathbf{E}[\Delta_m]| \leq C\delta^3$$

and

(3.2) $$|\mathbf{E}[\Delta_m^2] - 2\mathbf{E}[t_m]| \leq C\delta^3.$$

Recall that Lemma 3.1 says that γ has the same distribution as the *chronological* loop erasure of a random walk from 0 to ∂D conditioned to hit ∂D at u_0.

Here is a rough sketch of the proof. Let $v \in V(D)$ satisfy

(3.3) $$\mathrm{rad}_0(D)/200 < |v| < \mathrm{rad}_0(D)/5.$$

Let h_0^+ denote the number of visits to v by Γ. (This is the quantity which we referred to in Section 1 as the "observable.") The proof is based on estimating the two sides of the equality

(3.4) $$\mathbf{E}[h_0^+] = \mathbf{E}[\mathbf{E}[h_0^+ | \gamma[0, m]]].$$

The estimate for the right-hand side will involve the distribution of t_m and Δ_m. We get the two relations (3.1) and (3.2) by considering two different choices for such a v.

The estimates for the two sides of (3.4) are rather straightforward. Basically, each side is translated into expressions involving the Green's functions G_j and the hitting probabilities H_j. These are then translated into analytic quantities using (2.6). Earlier versions of the proof required other estimates, somewhat more delicate, in addition to (2.6). Fortunately, it turned out that (2.6) is sufficient. Since

we came across several different variants for the proof, based on choosing different observables, it may be said that the proof is inevitable, rather than accidental (and this also applies to Theorem 1.3). Basically, the reason the proof works is that the expected number of visits to v in $\bigcup_{j=1}^{m} \Gamma^j$ given $\gamma[0, m]$ can be estimated rather well given the rough geometry of $\gamma[0, m]$ in a scale much coarser than the scale of the grid. Similarly, it is important that $\mathbf{E}[h_0^+]$ can be estimated given the rough geometry of D, but this fact is not surprising.

In the following, we abbreviate the Green's functions and hitting probabilities in D_j by $G_j := G_{D_j}$ and $H_j := H_{D_j}$. The following lemma will be needed.

LEMMA 3.5 (Green's function bounds). *There is a constant $C > 0$ such that, for every $D \in \mathfrak{D}$ and $v \in V(D)$ satisfying* (3.3),

$$(3.5) \qquad\qquad 1/C \leq G_D(0, v) \leq C$$

holds. Also, given $\delta > 0$, there is an $r = r(\delta)$ such that, if $\mathrm{rad}_0(D) > r$, then, with the notation of Proposition 3.4,

$$(3.6) \qquad\qquad G_0(0, v) - G_m(0, v) \leq C\delta^2.$$

PROOF OF PROPOSITION 3.4.   Since $t_{m-1} < \delta^2$, it follows from the Koebe $1/4$ theorem that $\mathrm{rad}_0(D_{m-1}) > \mathrm{rad}_0(D)/5 \geq r_0/5$ if δ is small. (Apply [35], Corollary 1.4, with $z = 0$ to ψ_0^{-1} and ψ_{m-1}^{-1}.) Moreover, the continuous harmonic measure in D at 0 of any edge e with a vertex on ∂D can be made arbitrarily small by requiring $\mathrm{rad}_0(D)$ to be large. [A Brownian motion started at 0 has probability going to 1 to surround the disk $\mathrm{rad}_0(D)\mathbb{U}$ before hitting e, as $\mathrm{rad}_0(D) \to \infty$.] By the conformal invariance of the harmonic measure, this implies that the diameter of $\psi(e)$ can be made arbitrarily small. Applying this to the domains D_j and using Lemma 2.1, we see that we may take r_0 large enough so that, for all $j < m$, for all $t \in [t_j, t_{j+1}]$, $|\vartheta(t) - \vartheta(t_j)| \leq \delta^3$ and $|t_{j+1} - t_j| \leq \delta^3$. In particular, $t_m \leq \delta^2 + \delta^3$ and $|\Delta_m| \leq \delta + \delta^3$. We also require $r_0/8$ to be larger than the $r(\delta)$ of Lemma 3.5.

Suppose $v \in V(D)$ satisfies (3.3). Set $Z_j := \psi_j(v)$ and $Z := Z_0$. For each $j \in \{1, \ldots, \ell\}$, let h_j denote the number of visits to v by Γ^j. Also let

$$h_j^+ := \sum_{k=j+1}^{\ell} h_k,$$

which is the number of visits of v by $\Gamma[0, n_j]$. Let $\lambda_j := \lambda(v, \gamma_j; D_j)$, where $\lambda(v, v'; D_j)$ is as in (2.5). Since, conditionally on $\gamma[0, j]$, $\Gamma[0, n_j]$ is a random walk in D_j conditioned to leave D_j at γ_j,

$$\mathbf{E}[h_j^+ | \gamma[0, j]] = \frac{G_j(0, v) H_j(v, \gamma_j)}{H_j(0, \gamma_j)}.$$

and Proposition 2.2 [together with (3.5)] implies that if $r_0(\delta)$ is sufficiently large, then, for every $j \in \{0, 1, \ldots, m\}$,

$$\mathbf{E}[h_j^+|\gamma[0, j]] = G_j(0, v)\lambda_j + O(\delta^3).$$

[This $O(\cdot)$ notation is shorthand for the statement that there is an absolute constant C such that $|\mathbf{E}[h_j^+|\gamma[0, j]] - G_j(0, v)\lambda_j| \leq C\delta^3$. We freely use this shorthand below.] In particular,

$$(3.7) \quad \mathbf{E}\left[\sum_{j=1}^m h_j\right] = \mathbf{E}[h_0^+ - h_m^+] = \mathbf{E}[G_0(0, v)\lambda_0 - G_m(0, v)\lambda_m] + O(\delta^3).$$

We will now get a different approximation for the left-hand side. Applying Lemma 3.3 to the domain D_{j-1} gives

$$\mathbf{E}[h_j|\gamma[0, j]] = G_{j-1}(\gamma_j, v)H_{j-1}(v, \gamma_j).$$

Proposition 2.2 implies that, for $r_0(\delta)$ large enough,

$$(3.8) \qquad \mathbf{E}[h_j|\gamma[0, j]] = (\lambda_{j-1} + O(\delta))G_{j-1}(\gamma_j, v)H_{j-1}(0, \gamma_j).$$

Considering the same simple random walk starting at 0 and stopped when it exits D_j or D_{j-1} shows that

$$(3.9) \qquad G_{j-1}(0, v) - G_j(0, v) = H_{j-1}(0, \gamma_j)G_{j-1}(\gamma_j, v).$$

We now derive an a priori bound on $\max\{|\lambda_j - \lambda_m| : j \leq m\}$. Recall that

$$(3.10) \qquad\qquad \lambda_j - \lambda_0 = \mathrm{Re}\left(\frac{U_j + Z_j}{U_j - Z_j} - \frac{U + Z}{U - Z}\right).$$

But $|U_j - U| \leq O(\delta)$ for $j \leq m$, and Loewner's equation shows that

$$\forall j \leq m, \qquad Z_j = Z + t_j Z \frac{U + Z}{U - Z} + t_j O(\delta)$$

$$(3.11)$$

$$= Z + t_j Z \frac{U + Z}{U - Z} + O(\delta^3),$$

and, in particular, $Z_j = Z + O(\delta^2)$. (The equation blows up when $|U - Z|$ is small, and such estimates would not be valid in such a situation. However, this is not a problem here. First, $\psi_0'(0) \leq 1/\mathrm{rad}_0(D)$ by the Schwarz lemma applied to the restriction of ψ_0 to $\mathrm{rad}_0(D)\mathbb{U}$. Now the Koebe 1/4 theorem (the case $z = 0$ in the left-hand inequality in [35], Corollary 1.4) gives $\psi_0^{-1}((4/5)\mathbb{U}) \supset (1/4)|\psi_0'(0)|^{-1} (4/5)\mathbb{U} \supset (\mathrm{rad}_0(D)/5)\mathbb{U}$. In particular, $|Z| = |\psi_0(v)| \leq 4/5$ by (3.3). Since $t_m = O(\delta^2)$, it is clear that if δ is small and one starts flowing from Z according to Loewner's equation, it is impossible for Z to get close to $\partial\mathbb{U}$ up to time t_m.) Thus, we get our bound,

$$\forall j \leq m, \qquad |\lambda_j - \lambda_m| \leq O(\delta).$$

Using (3.8), this implies

$$\mathbf{E}[h_j|\gamma[0, j]] = (\lambda_m + O(\delta))G_{j-1}(\gamma_j, v)H_{j-1}(0, \gamma_j).$$

Now applying (3.9) yields

$$\mathbf{E}\left[\sum_{j=1}^{m} h_j\right] = \mathbf{E}[(\lambda_m + O(\delta))(G_0(0, v) - G_m(0, v))],$$

and hence (3.6) implies

$$\mathbf{E}\left[\sum_{j=1}^{m} h_j\right] = \mathbf{E}[\lambda_m(G_0(0, v) - G_m(0, v))] + O(\delta^3).$$

Comparing with (3.7) gives $G_0(0, v)\mathbf{E}[\lambda_m - \lambda_0] = O(\delta^3)$, and hence (3.5) implies

$$(3.12) \qquad\qquad \mathbf{E}[\lambda_m - \lambda_0] = O(\delta^3).$$

[The reader may wonder about the apparent miracle happening here, namely, that λ_j turns out to be "almost" a martingale. In fact, this is not important for identifying the scaling limit. If the right-hand side in (3.12) turned out to be any other explicit quantity, up to δ^3 error terms, the proof would still work, but give a different limiting process. In Remark 3.6 below, we give a short proof of (3.12) and further comments.]

Recall that this equation is valid uniformly over all choices of v. We now Taylor-expand $\lambda_m - \lambda_0$ with respect to $U_m - U$ and $Z_m - Z$, up to $O(\delta^3)$ error terms. As we have seen, $U_m - U = O(\delta)$ and $Z_m - Z = O(\delta^2)$, and hence only the first-order derivative with respect to $Z_m - Z$ and the first two derivatives with respect to

$$U_m - U = (e^{i\Delta_m} - 1)U = iU\Delta_m - U\Delta_m^2/2 + O(\delta^3)$$

come into play (the mixed derivatives can be ignored). Using (3.10) and (3.11), we get

$$\lambda_m - \lambda_0 = \Delta_m \operatorname{Im}\left(\frac{2ZU}{(U - Z)^2}\right) + (2t_m - \Delta_m^2)\operatorname{Re}\left(\frac{ZU(U + Z)}{(U - Z)^3}\right) + O(\delta^3),$$

and therefore (3.12) gives

$$(3.13) \quad \operatorname{Im}\left(\frac{2ZU}{(U - Z)^2}\right)\mathbf{E}[\Delta_m] + \operatorname{Re}\left(\frac{ZU(U + Z)}{(U - Z)^3}\right)\mathbf{E}[2t_m - \Delta_m^2] = O(\delta^3).$$

We claim that when $r_0(\delta)$ is large enough we may find $v_1, v_2 \in V(D)$ in the range (3.3) satisfying $|\psi(v_1) - U/30| < \delta^3$ and $|\psi(v_2) - iU/30| < \delta^3$. Indeed, by Theorem 1.3 and Corollary 1.4 from [35], for every $R \in (0, 1)$, there is a $C = C(R) < \infty$ such that $|\psi'(z)| \le C/\operatorname{rad}_z(D)$ and $\operatorname{rad}_z(D) \ge C^{-1}\operatorname{rad}_0(D)$ hold for all $z \in \psi^{-1}(R\mathbb{U})$. Let v_1 be a vertex closest to $\psi^{-1}(U/30)$. By integrating

the above bound on ψ' along the line segment from $\psi^{-1}(U/30)$ to v_1 (whose length is less than 1), we get $|\psi(v_1) - U/30| < \delta^3$ if $\mathrm{rad}_0(D)$ is large enough. Another application of Theorem 1.3 and Corollary 1.4 from [35] now shows that v_1 satisfies (3.3). An entirely similar argument produces v_2.

Consequently, (3.13) holds with $Z \in \{U/30, i\,U/30\}$. Plugging in these two values for Z produces two linearly independent equations in the variables $\mathbf{E}[2t_m - \Delta_m^2]$ and $\mathbf{E}[\Delta_m]$ and thereby proves (3.1) and (3.2). $\quad\square$

REMARK 3.6. Here is another proof of (3.12). Given a vertex $v \in V(D)$, let $\beta = (\beta_0, \beta_1, \dots)$ denote the loop erasure of the reversal of the simple random walk Γ_v started from v and stopped on exiting D (i.e., the analog of γ, but starting from v instead of 0). Abbreviate $\gamma^n := (\gamma_0, \dots, \gamma_n)$ and similarly $\beta^n := (\beta_0, \dots, \beta_n)$. For a sequence of vertices $u = (u_0, u_1, \dots, u_n)$, let $a_n(u) := \mathbf{P}[\gamma^n = u]$ and $b_n(u) := \mathbf{P}[\beta^n = u]$. Set $M_n := b_n(\gamma^n)/a_n(\gamma^n)$. (In other words, M_n is the Radon–Nikodym derivative of the law of β^n with respect to the law of γ^n.) It is easy to verify that M_n is a martingale:

$$\mathbf{E}[M_{n+1}|\gamma^n] = \sum_w \frac{b_{n+1}(\gamma^n w)}{a_{n+1}(\gamma^n w)} \frac{a_{n+1}(\gamma^n w)}{a_n(\gamma^n)} = \frac{\sum_w b_{n+1}(\gamma^n w)}{a_n(\gamma^n)} = M_n.$$

Lemma 3.2 implies that $M_n = H_n(v, \gamma_n)/H_n(0, \gamma_n)$, since, on the event that Γ_v and Γ first hit $\{u_0, \dots, u_n\} \cup \partial D$ at u_n, we may couple them to agree after that first visit to u_n. Now (2.6) implies (3.12).

Although this proof is shorter than the first proof of (3.12), it is harder to motivate and less natural. For this reason, we chose to stress the first proof.

Let us finally note that (as opposed to the martingale that shows up in the analysis of the UST Peano curve), the quantity corresponding to this martingale in the scaling limit is unbounded and converges almost surely to 0 (it is not uniformly integrable), so that it cannot be interpreted as a conditional probability or a conditional expectation. Correspondingly, in the discrete setting, M_n is very large when the path hits v (if it does) and M_n is very small when the path hits 0.

3.3. *Recognizing the driving process.* The objective in this section is to show that W of the previous section is close to a time-scaled Brownian motion on the unit circle.

THEOREM 3.7 (Driving process convergence). *For every $T > 0$ and $\epsilon > 0$, there is an $r_1 = r_1(\epsilon, T) > 0$ such that, for all $D \in \mathfrak{D}$ with $\mathrm{rad}_0(D) > r_1$, there is a coupling of γ with Brownian motion $B(t)$ starting at a random uniform point in $[0, 2\pi]$ such that*

$$\mathbf{P}\big[\sup\{|\vartheta(t) - B(2t)| : t \in [0, T]\} > \epsilon\big] < \epsilon.$$

Recall that a coupling of two random variables (or random processes) A and B is a probability space with two random variables A' and B', where A' has the same distribution as A and B' has the same distribution as B. In the above statement (as is customary), we do not distinguish between A and A' and between B and B'.

In order to deduce this theorem from Proposition 3.4, we will use the Skorohod embedding theorem, which is one of the standard tools for proving convergence to Brownian motion (one could work out a more direct proof but the following proof seems cleaner).

LEMMA 3.8 (Skorohod embedding). *If $(M_n)_{n \leq N}$ is an $(\mathcal{F}_n)_{n \leq N}$ martingale, with $\|M_n - M_{n-1}\|_\infty \leq 2\delta$ and $M_0 = 0$ a.s., then there are stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_N$ for standard Brownian motion $(B_t, t \geq 0)$ such that $(M_0, M_1, \ldots, M_N)$ and $(B_{\tau_0}, B_{\tau_1}, \ldots, B_{\tau_N})$ have the same law. Moreover, one can impose, for $n = 0, 1, \ldots, N - 1$,*

$$(3.14) \qquad \mathbf{E}[\tau_{n+1} - \tau_n | B[0, \tau_n]] = \mathbf{E}[(B_{\tau_{n+1}} - B_{\tau_n})^2 | B[0, \tau_n]]$$

and

$$(3.15) \qquad \tau_{n+1} \leq \inf\{t \geq \tau_n : |B_t - B_{\tau_n}| \geq 2\delta\}.$$

The proof can be found in many probability textbooks, including [9] and [37]. Often, it is stated for just one random variable M_1; for a statement in terms of martingales, see, for instance, [8] and [43]. The relation (3.15) is not stated explicitly in these references (since the assumption that the increments of M_n are bounded is weakened), but is a consequence of the proof. It can also be derived a posteriori from $\mathbf{E}[\tau_{n+1} - \tau_n] = \mathbf{E}[(M_{n+1} - M_n)^2] < \infty$, since the expected time for Brownian motion started outside an interval to hit the interval is infinite.

PROOF OF THEOREM 3.7. Since the hitting measure of a simple random walk from 0 is close to the hitting measure for Brownian motion when $\mathrm{rad}_0(D)$ is large (see, e.g., Section 5), it is clear that $W(0)$ is nearly uniform in $\partial \mathbb{U}$. It is therefore enough to show that $\vartheta(t/2)$ is close to standard Brownian motion.

Assume, without loss of generality, that $T \geq 1$. Pick $\delta = \delta(\epsilon, T) > 0$ small. Let r_0 be as in Proposition 3.4 and take $r_1 := 8 \exp(20T) r_0$. Let γ^t denote the initial segment of γ such that $\psi_D(\gamma^t)$ has capacity t from 0. By the Schwarz lemma, $\psi'_D(0) \leq \mathrm{rad}_0(D)^{-1}$. Therefore, the Koebe $1/4$ theorem implies $\mathrm{rad}_0(D \setminus \gamma^t) \geq \exp(-t) \mathrm{rad}_0(D)/4$. Hence, if $\mathrm{rad}_0(D) \geq r_1$, Proposition 3.4 is valid not only for the initial domain D, but also for the domain D slitted by subarcs of γ, up to capacity $20T$.

As in Proposition 3.4, define m to be the first $j = 1, 2, \ldots$ such that $|\Delta_j| \geq \delta$ or $t_j \geq \delta^2$. Set $m_0 := 0$, $m_1 := m$ and inductively let m_{n+1} be the first $j \geq m_n + 1$ such that $|\Delta_j - \Delta_{m_n}| \geq \delta$ or $t_j - t_{m_n} \geq \delta^2$, whichever happens first. Let $\mathcal{F}_n$ denote the σ-field generated by $\gamma[0, m_n]$. Set

$$N := \lceil 10T\delta^{-2} \rceil.$$

Our choice of r_1 ensures that $t_{j+1} - t_j \leq 2\delta^2$ for all $j < N$ and that $t_N \leq 20T$. Hence, Proposition 3.4 holds for all domains D_{m_n} with $n < N$. Applying clause (iii) of Lemma 3.2 therefore gives

$$(3.16) \qquad \mathbf{E}[\Delta_{m_{n+1}} - \Delta_{m_n}|\mathcal{F}_n] = O(\delta^3)$$

and

$$(3.17) \qquad \mathbf{E}[(\Delta_{m_{n+1}} - \Delta_{m_n})^2|\mathcal{F}_n] = 2\mathbf{E}[t_{m_{n+1}} - t_{m_n}|\mathcal{F}_n] + O(\delta^3).$$

For $n \leq N$, set

$$(3.18) \qquad M_n := \sum_{j=0}^{n-1}(\Delta_{m_{j+1}} - \Delta_{m_j} - \mathbf{E}[\Delta_{m_{j+1}} - \Delta_{m_j}|\mathcal{F}_j]).$$

Clearly, $M_0, \ldots, M_N$ is a martingale for $\mathcal{F}_0, \ldots, \mathcal{F}_N$. The definition of m_n and the choice of r_1 imply that $\|M_{n+1} - M_n\|_\infty \leq 2\delta$.

By Lemma 3.8, we may couple $(M_0, \ldots, M_N)$ with a standard Brownian motion with stopping times $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_N$ such that $B_{\tau_n} = M_n$ and (3.14) hold. Extend the coupling to include γ (this clearly can be done).

Note that the definition of t_{m_n} and (3.15) ensure that, for all $n < N$,

$$(3.19) \qquad \begin{aligned} \sup\{|B_t - B_{\tau_n}| : t \in [\tau_n, \tau_{n+1}]\} &\leq 2\delta, \\ \sup\{|\vartheta(t) - \Delta_{t_{m_n}}| : t \in [t_{m_n}, t_{m_{n+1}}]\} &\leq 2\delta \end{aligned}$$

and (3.16) shows that

$$(3.20) \qquad \sup\{|\Delta_{t_{m_n}} - M_n| : n \leq N\} = O(\delta^3 N) = O(\delta T).$$

Hence, as $M_n = B_{\tau_n}$ and B_t is a.s. continuous, it remains to relate the capacities t_{m_n} with the stopping times τ_n and verify that $t_{m_N} > T$ with high probability. For this purpose, define

$$Y_n = \sum_{j=0}^{n-1}(M_{j+1} - M_j)^2.$$

We first show that Y_n is close to $2t_{m_n}$. Let $Z_n := Y_n - 2t_{m_n}$. By (3.18) and (3.16), we have, for $n < N$, $|M_{n+1} - M_n - \Delta_{t_{m_{n+1}}} + \Delta_{t_{m_n}}| = O(\delta^3)$. This implies $|M_{n+1} - M_n| = O(\delta)$ and hence also

$$Y_{n+1} - Y_n = (M_{n+1} - M_n)^2 = (\Delta_{t_{m_{n+1}}} - \Delta_{t_{m_n}})^2 + O(\delta^4).$$

Consequently, (3.17) gives

$$\mathbf{E}[Z_{n+1} - Z_n|\mathcal{F}_n] \leq O(\delta^3).$$

From the fact that the increments of t_{m_n} and those of Y_n are bounded by $O(\delta^2)$,

we also have $\mathbf{E}[(Z_{n+1} - Z_n)^2 | \mathcal{F}_n] \leq O(\delta^4)$. Set $Z'_n := Z_n - \sum_{j=1}^{n} \mathbf{E}[Z_j - Z_{j-1} | \mathcal{F}_{j-1}]$. Since this is an $\mathcal{F}_n$-martingale, we have $\mathbf{E}[Z'^2_N] = \sum_{j=1}^{N} \mathbf{E}[(Z'_j - Z'_{j-1})^2]$ and the above estimates give $\mathbf{E}[Z'^2_N] = O(N \delta^4)$. Assuming $N\delta^3 < \delta^{1/2}/2$, without loss of generality, and applying Doob's maximal inequality (see [37], II.1.7) for L^2-martingales to Z'_n, we get

$$(3.21) \qquad \mathbb{P}\left[\max_{n \leq N} |Y_n - 2t_{m_n}| > \delta^{1/2}\right] = O(N\delta^3) = O(T\delta).$$

By the definition of the t_{m_n}, we have $Y_{n+1} - Y_n + t_{m_{n+1}} - t_{m_n} \geq \delta^2$. Summing gives $Y_N + t_{m_N} \geq N\delta^2 \geq 10T$. Therefore, (3.21) implies

$$(3.22) \qquad \mathbf{P}[t_{m_N} < 2T] = O(T\delta).$$

We now show that with high probability τ_n is also close to Y_n for every $n \leq N$. By (3.15), it is clear that $\mathbf{E}[(\tau_{n+1} - \tau_n)^2 | B[0, \tau_n]] = O(\delta^4)$ and therefore

$$\mathbf{E}\big[\big((\tau_{n+1} - Y_{n+1}) - (\tau_n - Y_n)\big)^2 \big| B[0, \tau_n]\big] = O(\delta^4).$$

Also, (3.14) gives

$$\mathbf{E}\big[(\tau_{n+1} - Y_{n+1}) - (\tau_n - Y_n) | B[0, \tau_n]\big] = 0.$$

Doob's inequality therefore implies

$$\mathbf{P}\left[\max_{n \leq N} |\tau_n - Y_n| > \delta^{1/2}\right] = O(T\delta).$$

Combining this with (3.21) leads to

$$\mathbf{P}\left[\max_{n \leq N} |\tau_n - 2t_{m_n}| > \delta^{1/2}\right] = O(T\delta).$$

Since B_t is a.s. continuous, together with (3.22), (3.19) and (3.20), this completes the proof. $\square$

3.4. *Convergence with respect to a stronger topology.* Theorem 3.7 provides a kind of convergence of the loop-erased random walk to SLE$_2$. As we will see in this section, this kind of convergence suffices, for example, to show that the scaling limit with respect to the Hausdorff metric of the union of $\partial \mathbb{U}$ and the LERW in $\mathbb{U}$ is the union of $\partial \mathbb{U}$ and the SLE$_2$ path.

Let $\alpha : [0, 1] \to \mathbb{C}$ and $\beta : [0, 1] \to \mathbb{C}$ be two continuous paths. Define

$$\rho(\alpha, \beta) := \inf_{\phi \in \Phi} \sup_{t \in [0,1]} |\alpha(t) - \beta \circ \phi(t)|,$$

where Φ is the collection of all monotone nondecreasing continuous maps from $[0, 1]$ onto $[0, 1]$. It is an easy well-known fact that ρ is a metric on equivalence classes of paths, where two paths α and β are equivalent if $\alpha \circ \phi_1 = \beta \circ \phi_2$, where

$\phi_1, \phi_2 \in \Phi$. Since $\rho(\alpha, \beta)$ does not depend on the particular parameterization of α or β, the metric ρ is also defined for paths on intervals other than $[0, 1]$.

To explain our present goal, let us point out that there is a sequence of paths α_n from 1 to 0 in $\overline{\mathbb{U}}$ such that their Loewner driving functions $W_n(t)$ converge uniformly to the constant 1 but α_n does not converge to the path $\alpha(t) = 1 - t$, $t \in [0, 1]$, in the metric ρ, although the driving function for α (reparameterized by capacity) is the constant 1. For example, we may take α_n as the polygonal path through the points $a_1, b_1 + in^{-2}, a_2, b_2 - in^{-2}, a_3, b_3 + in^{-2}, \ldots, a_{\lfloor n/2 \rfloor}, 0$, where $a_j := 1 - n^{-1} + (jn)^{-1}$ and $b_j := 1 - j/n$.

THEOREM 3.9 (LERW image in $\overline{\mathbb{U}}$ converges). *For any sequence $D_n \in \mathfrak{D}$ with $\mathrm{rad}_0(D_n) \to \infty$, if μ_n denotes the law of $\tilde{\gamma}^n := \psi_{D_n} \circ \gamma^n$, where γ^n is the time reversal of the LERW from 0 to ∂D_n, then μ_n converges weakly (with respect to the metric ρ) to the law of the radial SLE_2 path started uniformly on the unit circle.*

The outline of the proof is as follows. We define a suitable family of compact subsets of the space of simple paths from $\partial \mathbb{U}$ to 0 in $\overline{\mathbb{U}}$, which we can use to show that the sequence μ_n is tight. (See, e.g., [9] for background on weak convergence and the notion of tightness.) This implies that a subsequence of μ_n converges weakly to some probability measure. Theorem 3.9 then shows that the law of SLE_2 is the unique possible subsequential limit.

In order to prove tightness, we will use properties of the loop-erased random walk proved in [38]. The actual details will require some background in the geometric theory of conformal maps. In particular, some properties of extremal distance (a.k.a. extremal length) will be used. See, for example, [2] for background. The basic ideas that are used in the proof are taken from [3] and [38].

For a simply connected $D \subsetneq \mathbb{C}$ containing 0, let $\mathcal{X}_0(D)$ denote the space of all simple paths $\gamma : [0, \infty] \to \overline{D}$ from ∂D to 0 in $\overline{D}$, which intersect ∂D only at the starting point. Given a monotone nondecreasing function $\Upsilon : (0, \infty) \to (0, 1]$, let $\mathcal{X}_\Upsilon(D) \subset \mathcal{X}_0(D)$ denote the space of all simple paths $\gamma \in \mathcal{X}_0(D)$ such that, for every $0 \leq s_1 < s_2$,

$$\mathrm{dist}(\gamma[0, s_1] \cup \partial D, \gamma[s_2, \infty]) / \mathrm{rad}_0(D) \geq \Upsilon\big(\mathrm{diam}(\gamma[s_1, s_2]) / \mathrm{rad}_0(D)\big).$$

Note that whether $\gamma \in \mathcal{X}_\Upsilon(D)$ does not depend on the parameterization of γ and is scaling invariant.

LEMMA 3.10 (Compactness). *Let $\Upsilon : (0, \infty) \to (0, 1]$ be monotone nondecreasing. Then $\mathcal{X}_\Upsilon(\mathbb{U})$ is compact in the topology of convergence with respect to ρ.*

PROOF. We use an idea from [3]. For all $n \in \mathbb{N}$, let Z_n be a finite collection of points such that the open balls $\mathcal{B}(z, 2^{-n})$, $z \in Z_n$, cover $\mathbb{U}$. Given a set $K \subset \overline{\mathbb{U}}$

and a point $z \in Z_n$, let $s(K, z, n)$ denote the diameter of $K \cap \mathcal{B}(z, 2^{1-n})$ in the metric obtained from the Euclidean metric on the disk $\overline{\mathcal{B}(z, 2^{1-n})}$ by collapsing the boundary $\partial \mathcal{B}(z, 2^{1-n})$ to a single point. [In other words, this metric $d(x, y)$ is defined as $d(x, y) = \min\{|x - y|, \operatorname{dist}(x, \partial \mathcal{B}) + \operatorname{dist}(y, \partial \mathcal{B})\}$, where $\mathcal{B} = \mathcal{B}(z, 2^{1-n})$.]

Fix $\gamma \in \mathcal{X}_\Upsilon(\mathbb{U})$. Given $t \geq 0$, let

$$s(t) = s_\gamma(t) := \sum_{n \in \mathbb{N}} \sum_{z \in Z_n} \frac{s(\gamma[0, t], z, n)}{|Z_n|}.$$

Clearly, $s_\gamma(t) \leq \sum_{n \geq 0} 2^{2-n} = 8$, and $s : [0, \infty] \to [0, \infty)$ is continuous and strictly monotone increasing. (To verify that s is strictly monotone increasing, note that, if $t_2 > t_1 \geq 0$, then there is some $n \in \mathbb{N}$ such that $\operatorname{dist}(\gamma(t_2), \gamma[0, t_1]) \geq 2^{2-n}$, and so $s(\gamma[0, t_2], z, n) \geq s(\gamma[0, t_1], z, n) + 2^{-n}$ if $z \in Z_n$ satisfies $\gamma(t_2) \in \mathcal{B}(z, 2^{-n})$.) Let $\hat{\gamma}(s)$ be γ parameterized by s; that is, $\hat{\gamma} = \gamma \circ s^{-1}$. Let $s_1 < s_2$ and set $\epsilon := \operatorname{diam} \hat{\gamma}[s_1, s_2] > 0$. Then $\operatorname{dist}(\hat{\gamma}(s_2), \hat{\gamma}[0, s_1] \cup \partial \mathbb{U}) \geq \Upsilon(\epsilon)$. By the argument for strict monotonicity given above, this shows that $s_2 - s_1 \geq 2^{-n}/|Z_n|$, where $n := \min\{k \in \mathbb{N} : 2^{2-k} \leq \Upsilon(\epsilon)\}$. Therefore, $\hat{\gamma}$ satisfies an equicontinuity estimate. By the Arzela–Ascoli theorem, it follows that the closure of $\mathcal{X}_\Upsilon(\mathbb{U})$ is compact in the ρ metric. It is also clear that $\mathcal{X}_\Upsilon(\mathbb{U})$ is closed. $\square$

Our next goal is to use these compact sets to prove tightness, and we start by observing that the diameter is tight.

LEMMA 3.11 (Diameter is tight). *There are constants $c, C > 0$ such that for every $D \in \mathfrak{D}$ and every $r \geq 1$ the simple random walk Γ starting from 0 and stopped on hitting ∂D satisfies*

$$\mathbf{P}[\operatorname{diam}(\Gamma) \geq r \operatorname{rad}_0(D)] \leq Cr^{-c}.$$

Consequently, the same estimate holds for the loop erasure γ.

The first statement is an easy well-known fact. Since the complement of D is connected and unbounded, if the random walk makes a loop separating the circle $\operatorname{rad}_0(D) \partial \mathbb{U}$ from the circle $(r/2) \operatorname{rad}_0(D) \partial \mathbb{U}$ before hitting the latter circle, then it must hit ∂D before $(r/2) \operatorname{rad}_0(D) \partial \mathbb{U}$. Thus, the lemma is easily proved directly and also follows from the convergence of a simple random walk to Brownian motion. A rather precise form of this estimate for the random walk, where $c = 1/2$, is known as the discrete Beurling theorem (see [25], Theorem 2.5.2).

LEMMA 3.12 (Tameness). *For every $\epsilon > 0$, there is some monotone nondecreasing $\Upsilon : (0, \infty) \to (0, 1]$ and some $r_0 > 0$ such that for every $D \in \mathfrak{D}$ with $\operatorname{rad}_0(D) \geq r_0$ its time-reversed loop-erased walk $\gamma = \gamma_D$ satisfies*

$$\mathbf{P}[\gamma \in \mathcal{X}_\Upsilon(D)] \geq 1 - \epsilon.$$

PROOF. The proof is essentially contained in the proof of [38], Theorem 1.1, where it is established that every subsequential scaling limit of LERW is a.s. a simple path. We will not repeat the complete proof from [38] here, but indicate how it may be adapted to yield the statement of the lemma.

Let $\epsilon > 0$. Clearly, $\gamma \in \mathcal{X}_0(D)$. If $\gamma \notin \mathcal{X}_\Upsilon(D)$, then there are $0 \leq s_1 < s_2 < \infty$ such that the distance between $\gamma[0, s_1] \cup \partial D$ and $\gamma[s_2, \infty]$ is smaller than $\mathrm{rad}_0(D)\Upsilon(\mathrm{diam}\,\gamma[s_1, s_2]/\mathrm{rad}_0(D))$. Let us first deal with the case where the distance between $\gamma[s_2, \infty]$ and ∂D is small. Let Γ be the walk generating the time reversal of γ, and let t_n be the first time t where the distance from $\Gamma(t)$ to ∂D is smaller than $2^{-n}\,\mathrm{rad}_0(D)$ and let $\tau = \inf\{t : \Gamma(t) \in \partial D\}$. By the Markov property of Γ at time t_n and Lemma 3.11,

$$\mathbf{P}\big[\mathrm{diam}\,\Gamma[t_n, \tau] > 2^{-n/2}\,\mathrm{rad}_0(D)\big] \leq C2^{-cn/2}.$$

Consequently, there is an $N = N(\epsilon)$ such that with probability $1 - \epsilon/2$ for every integer $n \geq N$ we have $\mathrm{diam}\,\Gamma[t_n, \tau] \leq 2^{-n/2}\,\mathrm{rad}_0(D)$. In this case, if $\mathrm{diam}\,\gamma[0, s_2] > 2^{-n/2}\,\mathrm{rad}_0(D)$, where $n > N$, then $\gamma[0, s_2]$ is not contained in $\Gamma[t_n, \tau]$, which implies that $\gamma[s_2, \infty] \subset \Gamma[0, t_n]$ and gives $\mathrm{dist}(\gamma[s_2, \infty], \partial D) \geq 2^{-n}\,\mathrm{rad}_0(D)$. In other words, if Υ satisfies

$$(3.23) \qquad \Upsilon(t) < \min\{t^2, 2^{-2N}\}/4,$$

then with probability at least $1 - \epsilon/2$, for every $s_1, s_2 \in [0, \infty]$,

$$(3.24) \qquad \mathrm{dist}(\partial D, \gamma[s_2, \infty]) \geq \mathrm{rad}_0(D)\Upsilon(\mathrm{diam}\,\gamma[s_1, s_2]/\mathrm{rad}_0(D)).$$

We now focus on the case where the distance between $\gamma[0, s_1]$ and $\gamma[s_2, \infty)$ is small. We shall say that γ has a (β, α)-quasi-loop if there are $0 < s_1 < s_2 < \infty$ such that $|\gamma(s_1) - \gamma(s_2)| \leq \alpha\,\mathrm{rad}_0(D)$ but $\mathrm{diam}\,\gamma[s_1, s_2] \geq \beta\,\mathrm{rad}_0(D)$. Note that if there are $0 < s_1 < s_2 < \infty$ such that $\mathrm{dist}(\gamma[0, s_1], \gamma[s_2, \infty)) < \alpha\,\mathrm{rad}_0(D)$ and $\mathrm{diam}\,\gamma[s_1, s_2] \geq \beta\,\mathrm{rad}_0(D)$, then γ has a (β, α)-quasi-loop. Let $\mathcal{A}(\beta, \alpha)$ denote the event that γ has a (β, α)-quasi-loop. Assume, for the moment, that, for all $n \geq 0$,

$$(3.25) \qquad \lim_{\alpha \searrow 0} \mathbf{P}[\mathcal{A}(2^{-n}, \alpha)] = 0,$$

uniformly in D. Then we may take a decreasing sequence $\alpha_n \searrow 0$ such that $\sum_{n=1}^{\infty} \mathbf{P}[\mathcal{A}(2^{-n}, \alpha_n)] < \epsilon/2$ holds for every $D \in \mathfrak{D}$. Then with probability at least $1 - \epsilon/2$, γ has no $(2^{-n}, \alpha_n)$-quasi-loop for any $n = 1, 2, \ldots$. Assuming that $\Upsilon(t) < \alpha_n$ holds whenever $t \leq 2^{1-n}$, $n \in \mathbb{N}$, and $\Upsilon(t) < \alpha_1$ for all t, on this event we also have

$$\mathrm{dist}(\gamma[0, s_1], \gamma[s_2, \infty]) \leq \mathrm{rad}_0(D)\Upsilon(\mathrm{diam}\,\gamma[s_1, s_2]/\mathrm{rad}_0(D))$$

for all $0 < s_1 < s_2 < \infty$. If we also assume (3.23), then together with (3.24) we get $\mathbf{P}[\gamma \in \mathcal{X}_\Upsilon(D)] \geq 1 - \epsilon$, completing the proof of the lemma. Thus, it remains to verify (3.25).

Let $\mathcal{A}(z_0, \beta, \alpha)$ denote the event that there are $0 < s_1 < s_2 < \infty$ such that $|\gamma(s_1) - \gamma(s_2)| \leq \alpha \operatorname{rad}_0(D)$, $\gamma(s_1), \gamma(s_2) \in \mathcal{B}(z_0, \beta \operatorname{rad}_0(D)/4)$ and $\operatorname{diam}(\gamma[s_1, s_2]) \geq \beta \operatorname{rad}_0(D)$. In particular, this implies that $\gamma[s_1, s_2]$ is not contained in the interior of $\mathcal{B}(z_0, \beta \operatorname{rad}_0(D)/2)$. Assume that $8\alpha < \beta$. By Lemma 3.11, there is an $R = R(\epsilon) > 0$ such that with probability at least $1 - \epsilon/2$ we have $\gamma[0, \infty] \subset \mathcal{B}(0, R \operatorname{rad}_0(D))$. There is a collection $\{z_1, z_2, \ldots, z_k\}$ of points such that every disk of radius $2\alpha \operatorname{rad}_0(D)$ with center in $\mathcal{B}(0, R \operatorname{rad}_0(D))$ is contained in one of the k balls $\mathcal{B}(z_j, \beta \operatorname{rad}_0(D)/2)$, $j = 1, 2, \ldots, k$, and we may take $k < c((R/\beta)^2 + 1)$, where c is an absolute constant. On the event $\gamma[0, \infty] \subset \mathcal{B}(0, R \operatorname{rad}_0(D))$, we have $\mathcal{A}(\beta, \alpha) \subset \bigcup_{j=1}^{k} \mathcal{A}(z_j, \beta, \alpha)$. Since $\epsilon > 0$ was arbitrary and $\mathbf{P}[\gamma[0, \infty] \subset \mathcal{B}(0, R \operatorname{rad}_0(D))] \geq 1 - \epsilon/2$, it is therefore sufficient to show that $\mathbf{P}[\mathcal{A}(z_j, \beta, \alpha)] \to 0$ as $\alpha \to 0$ uniformly in D. The proof of this statement is given (with minor changes in the setup) in [38], Theorem 1.1. □

Let $\mathcal{X}_\Upsilon^r(D)$ denote the set of paths $\gamma \in \mathcal{X}_\Upsilon(D)$ that are contained in the ball of radius $r \operatorname{rad}_0(D)$ about 0. Given $\gamma \in \mathcal{X}_0(D)$, let $\gamma^* : [0, \infty) \to \overline{\mathbb{U}}$ denote the path $\psi_D \circ \gamma$, parameterized by capacity.

LEMMA 3.13 (Tameness invariance). *For every monotone nondecreasing* $\Upsilon : (0, \infty) \to (0, 1]$ *and every* $r > 1$, *there is a monotone nondecreasing* $\Upsilon^* : (0, \infty) \to (0, 1]$ *such that, for all* $D \in \mathfrak{D}$ *and* $\gamma \in \mathcal{X}_\Upsilon^r(D)$, $\gamma^* \in \mathcal{X}_{\Upsilon^*}(\mathbb{U})$.

PROOF. Let $D \in \mathfrak{D}$, $\gamma \in \mathcal{X}_\Upsilon^r(D)$ and $0 \leq s_1' < s_2' \leq \infty$. Note that there exist s_1 and s_2 satisfying $s_1' \leq s_1 \leq s_2 \leq s_2'$ such that

$$\operatorname{diam}(\gamma^*[s_1, s_2]) \geq \operatorname{diam}(\gamma^*[s_1', s_2'])/4$$

and

$$(3.26) \qquad \operatorname{dist}(0, \gamma^*[s_1, s_2]) \geq \operatorname{diam}(\gamma^*[s_1, s_2]).$$

Since

$$(3.27) \quad \operatorname{dist}(\gamma^*[0, s_1'] \cup \partial \mathbb{U}, \gamma^*[s_2', \infty)) \geq \operatorname{dist}(\gamma^*[0, s_1] \cup \partial \mathbb{U}, \gamma^*[s_2, \infty)),$$

it is sufficient to give a lower bound of the right-hand side of (3.27) in terms of $\epsilon := \operatorname{diam}(\gamma^*[s_1, s_2])$.

The Schwarz lemma gives $\psi_D'(0) \leq 1/\operatorname{rad}_0(D)$. Therefore, by the Koebe 1/4 theorem (applied to the restriction of ψ_D^{-1} to $\epsilon \mathbb{U}$) and (3.26), $\operatorname{dist}(0, \gamma[s_1, s_2]) > c_1 \operatorname{rad}_0(D)$, where $c_1 = \epsilon/4$. On the other hand, the harmonic measure in $\mathbb{U}$ from 0 of $\gamma^*[s_1, s_2]$ is at least c_2, where $c_2 = c_2(\epsilon) > 0$, so that the harmonic measure in D from 0 of $\gamma[s_1, s_2]$ is at least c_2. Hence,

$$(3.28) \qquad \operatorname{diam} \gamma[s_1, s_2] \geq c_3 \operatorname{rad}_0(D),$$

where $c_3 = c_3(\epsilon)$.

Also set $\delta := \mathrm{dist}(\gamma^*[0, s_1] \cup \partial \mathbb{U}, \gamma^*[s_2, \infty])$. Since

$$\mathrm{diam}\, \gamma^*[s_2, \infty] \geq \mathrm{dist}(0, \gamma^*[s_1, s_2]) \geq \epsilon,$$

the extremal distance between $\gamma^*[0, s_1] \cup \partial \mathbb{U}$ and $\gamma^*[s_2, \infty]$ is at most $\phi_1(\delta, \epsilon) > 0$, where ϕ_1 is some function satisfying $\phi_1(\delta, \epsilon) \to 0$ as $\delta \downarrow 0$. By the conformal invariance of the extremal distance, this implies that the extremal distance between $\gamma[0, s_1] \cup \partial D$ and $\gamma[s_2, \infty]$ is at most $\phi_1(\delta, \epsilon)$. Because γ is contained in the disk of radius $r \,\mathrm{rad}_0(D)$ about 0, this implies that

$$\mathrm{dist}(\gamma[0, s_1] \cup \partial D, \gamma[s_2, \infty]) \leq \phi_2(\delta, \epsilon) r \,\mathrm{rad}_0(D),$$

where $\phi_2 \to 0$ as $\delta \downarrow 0$. Because $\gamma \in \mathcal{X}_\Upsilon(D)$, (3.28) and this together imply

$$\phi_2(\delta, \epsilon) r \geq \Upsilon(c_3(\epsilon)),$$

which gives a positive lower bound for $\delta = \mathrm{dist}(\gamma^*[0, s_1] \cup \partial \mathbb{U}, \gamma^*[s_2, \infty])$ in terms of Υ, r and $\epsilon = \mathrm{diam}(\gamma^*[s_1, s_2])$. This completes the proof. $\quad\square$

LEMMA 3.14 (Convergence relations). *Suppose W^n, W are continuous functions from $[0, \infty)$ to $\partial \mathbb{U}$ such that $W^n \to W$ locally uniformly. Let g_t^n, g_t be the corresponding solutions to Loewner's radial equation and set $f_t^n = (g_t^n)^{-1}$, $f_t = g_t^{-1}$. Then $f_t^n \to f_t$ locally uniformly on $[0, \infty) \times \mathbb{U}$. If there are continuous curves $\gamma^n : [0, \infty) \to \overline{\mathbb{U}}$ such that, for all $t \geq 0$, the image of f_t^n is the component of 0 in $\mathbb{U} \setminus \gamma^n[0, t]$ and there is a $\gamma : [0, \infty) \to \overline{\mathbb{U}}$ such that $\gamma^n \to \gamma$ locally uniformly on $[0, \infty)$, then for all $t \geq 0$ the image of f_t is the component of 0 in $\mathbb{U} \setminus \gamma[0, t]$.*

PROOF. Since g_t is obtained by flowing along a vector field depending on W, the inverse f_t is obtained by flowing along the opposite field, with the time reversed. Hence, the first statement is an immediate consequence of the principle that solutions of ODE depend continuously on the parameters of the ODE. The second statement is an immediate consequence of the Carathéodory kernel theorem (see [35], Theorem 1.8). $\quad\square$

PROOF OF THEOREM 3.9. Let W^n denote the Loewner parameter of $\tilde{\gamma}^n$ and let $\hat{\mu}_n$ denote the law of the pair $(\tilde{\gamma}^n, W^n)$. By Theorem 3.7, we know that the law of W^n tends weakly to the law of Brownian motion. Lemmas 3.10–3.13 show that the set of measures $\{\mu_n\}$ is tight with respect to the metric ρ. Consequently, the sequence $\hat{\mu}_n$ is also tight. Prokhorov's theorem (e.g., [9] and [37]) implies that there is a subsequence such that $\hat{\mu}_n$ converges weakly along the subsequence. Let $\hat{\mu}$ be any subsequential weak limit and let $(\tilde{\gamma}, W)$ be a sample from $\hat{\mu}$. The lemmas show that $\tilde{\gamma}$ is a.s. a simple path, and Theorem 3.7 shows that W is Brownian motion (with time scaled). By the properties of weak convergence, we may couple the subsequence of pairs $(\tilde{\gamma}^n, W^n)$ and $(\tilde{\gamma}, W)$ so that a.s. $\rho(\tilde{\gamma}^n, \tilde{\gamma}) \to 0$ and $W^n \to W$ locally uniformly.

Recall that the capacity is continuous with respect to the metric ρ; that is, if $\beta, \beta_n : [0, 1] \to \overline{\mathbb{U}} \setminus \{0\}$ and $\rho(\beta_n, \beta) \to 0$, then the capacity of $\beta_n[0, 1]$ tends to the capacity of $\beta[0, 1]$. (In fact, it is enough that $\beta_n[0, 1]$ tends to $\beta[0, 1]$ in the Hausdorff metric.) Indeed, this follows immediately from Carathéodory's kernel theorem (see [35], Theorem 1.8) and the fact that the local uniform convergence of conformal maps implies the convergence of the derivatives (by Cauchy's formula for the derivative).

Since $\tilde{\gamma}$ is almost surely a simple path, the capacity of $\tilde{\gamma}$ increases strictly, and one can parameterize the path continuously by its capacity. We also parameterize the paths $\tilde{\gamma}^n$ by capacity. The next goal is to show that $\tilde{\gamma}^n \to \tilde{\gamma}$ locally uniformly on $[0, \infty)$. Since $\rho(\tilde{\gamma}, \tilde{\gamma}^n) \to 0$, there are strictly monotone continuous onto maps $\varepsilon_n : [0, \infty) \to [0, \infty)$ so that $\tilde{\gamma}^n \circ \varepsilon_n \to \tilde{\gamma}$ locally uniformly. If $t_n \in [0, \infty)$ and $t_n \to t \in [0, \infty)$, then it follows from the continuity of capacity with respect to ρ that $\varepsilon_n(t_n) \to t$ [because if s is a subsequential limit of $\varepsilon_n(t_n)$, then the capacity of $\tilde{\gamma}(s)$ must be t; i.e., $s = t$]. This implies that ε_n converges to the identity map $t \mapsto t$, locally uniformly. By the continuity of $\tilde{\gamma}$, it follows that $\tilde{\gamma} \circ \varepsilon_n^{-1} \to \tilde{\gamma}$ locally uniformly. This gives $\tilde{\gamma}^n \to \tilde{\gamma}$ locally uniformly.

We can now finally apply Lemma 3.14 to show that $\tilde{\gamma}$ is the SLE$_2$ path. As the law of the limit $\tilde{\gamma}$ does not depend on the subsequence, the theorem follows. □

In the following proof of Theorem 1.1, the main technical point is that we do not make any smoothness assumptions on ∂D. If ∂D is a simple closed path, the theorem follows easily from Theorem 3.9, because the suitably normalized conformal maps from $\mathbb{U}$ to the discrete approximations of D converge uniformly to the conformal map onto D.

PROOF OF THEOREM 1.1. Let D_δ be the component of 0 in the complement of all the closed square faces of the grid $\delta \mathbb{Z}^2$ intersecting ∂D. Let γ_δ be the time reversal of the loop-erased random walk from 0 to ∂D_δ and let β be the radial SLE$_2$ path in $\overline{\mathbb{U}}$. Let $\phi_\delta : \mathbb{U} \to D_\delta$ be the conformal map satisfying $\phi_\delta(0) = 0$ and $\phi_\delta'(0) > 0$ and let $\phi : \mathbb{U} \to D$ be the conformal map satisfying $\phi(0) = 0, \phi'(0) > 0$. Theorem 3.9 tells us that we may couple β with each of the paths γ_δ such that $\rho(\phi_\delta^{-1} \circ \gamma_\delta, \beta) \to 0$ in probability as $\delta \downarrow 0$. Moreover, the proof shows that if we use the capacity parameterization for both, then, in probability,

$$\sup\{|\phi_\delta^{-1} \circ \gamma_\delta(t) - \beta(t)| : t \geq 0\} \to 0.$$

(There is no problem with convergence in a neighborhood of $t = \infty$, because we know that the weak limit of $\phi_\delta^{-1} \circ \gamma_\delta$ with respect to ρ is a simple path tending to 0 as $t \to \infty$.)

The Carathéodory kernel theorem (see [35], Theorem 1.8) implies that $\phi_\delta \to \phi$ uniformly on compact subsets of $\mathbb{U}$ as $\delta \searrow 0$. Consequently, the above gives

$$(3.29) \qquad \forall\, t_0 > 0, \qquad \sup\{|\gamma_\delta(t) - \phi \circ \beta(t)| : t \geq t_0\} \to 0,$$

in probability. Let $\epsilon > 0$ be small. Then, by Lemma 3.11, there is an $\epsilon' > 0$ such that for every $D' \in \mathfrak{D}$ the probability that a simple random walk from 0 gets to distance $\mathrm{rad}_0(D')/(2\epsilon')$ before hitting $\partial D'$ is less than $\epsilon/2$. Let A be the connected component of 0 in the set of points in $D \cap (\mathrm{rad}_0(D)/\epsilon')\mathbb{U}$ having distance at least $\epsilon\epsilon' \mathrm{rad}_0(D)$ from ∂D. By considering the first point where the random walk generating γ_δ exits A, it follows that, with probability at least $1 - \epsilon$, the diameter of $\gamma_\delta[0, \infty] \setminus A$ is at most $\epsilon \, \mathrm{rad}_0(D) + \delta$. Now note that there is a compact $A' \subset \mathbb{U}$ such that $\phi_\delta^{-1}(A) \subset A'$ for all sufficiently small δ, since $\phi_\delta \to \phi$ uniformly on compacts. Therefore, there is some $t_1 > 0$ such that $\gamma_\delta[0, t_1] \cap A = \varnothing$ a.s. for all sufficiently small $\delta > 0$. In particular,

$$(3.30) \qquad \mathbf{P}\big[\mathrm{diam}\, \gamma_\delta[0, t_1] > \epsilon\, \mathrm{rad}_0(D) + \delta\big] < \epsilon.$$

If we take $t_2 \in (0, t_1)$, then taking $\delta \searrow 0$ in (3.29) implies

$$\mathbf{P}\big[\mathrm{diam}\, \phi \circ \beta[t_2, t_1] > 2\epsilon\, \mathrm{rad}_0(D)\big] < \epsilon.$$

Since this holds for every t_2, it follows that

$$\mathbf{P}\big[\mathrm{diam}\, \phi \circ \beta(0, t_1] > 2\epsilon\, \mathrm{rad}_0(D)\big] < \epsilon.$$

Using this with (3.30) and choosing $t_0 = t_1$ in (3.29) gives

$$\mathbf{P}\big[\sup\{|\gamma_\delta(t) - \phi \circ \beta(t)| : t > 0\} < 3\epsilon\, \mathrm{rad}_0(D)\big] \to 1.$$

Since this holds for every $\epsilon > 0$, the theorem follows. $\quad\square$

4. The UST Peano curve.

4.1. *Setup.* The UST Peano curve is obtained as the interface between the UST and the dual UST. The setup which corresponds to chordal SLE_8 is where there is symmetry between the UST and the dual UST. Loosely speaking, the UST is the uniform spanning tree on the grid inside a domain D but with an entire arc $\alpha \subset \partial D$ on the boundary identified (wired) as a single vertex, and the dual UST also has an arc $\beta \subset \partial D$ on the boundary which is identified. The arcs α and β are essentially complementary arcs. See Figure 5, where D is approximately a rectangle. As mentioned in Section 1, it was conjectured in [36] that, for an analogous setup, the interface defined for the critical random cluster models with $q \in (0, 4]$ converges to SLE_κ, where $\kappa = \kappa(q) \in [4, 8)$.

A combinatorial framework is necessary in order to be more precise. There are several different possible setups that would work, and the following is somewhat arbitrary.

If a tree T lies in the grid $\mathbb{Z}^2$, then its dual tree $T^\dagger$ will lie in the dual grid $(\mathbb{Z} + 1/2)^2$, and the Peano path γ will lie in the graph G whose vertices are $(1/4 + \mathbb{Z}/2)^2$ and where v, u neighbor iff $|v - u| = 1/2$. We have three kinds of vertices: elements of $\mathbb{Z}^2$ are the *primal* vertices, elements of $(1/2 + \mathbb{Z})^2$ are the *dual*
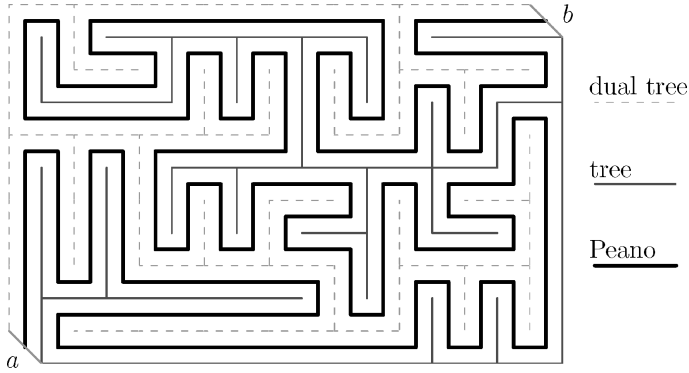
FIG. 5. *The tree, dual tree and Peano UST path γ.*

vertices and elements of $(1/4 + \mathbb{Z}/2)^2$ are the *Peano* vertices. If $w \neq v$ are vertices of any kind, not necessarily the same, we say that they are *adjacent* if the distance between them is as small as it can be for distinct vertices of these particular kinds. In other words, if they are of the same kind, this means that they are neighbors, if $v \in (1/4 + \mathbb{Z}/2)^2$ and $w \in \mathbb{Z}^2 \cup (1/2 + \mathbb{Z})^2$, this means $\|v - w\|_\infty = 1/4$, while if $v \in \mathbb{Z}^2$ and $w \in (1/2 + \mathbb{Z})^2$, this means $\|v - w\|_\infty = 1/2$.

Since there is no added complication, we consider a more general case where α and β are trees, rather than arcs. Let α be some finite tree in the primal grid $\mathbb{Z}^2$ and let β be a finite tree in the dual grid $(1/2 + \mathbb{Z})^2$. Suppose that no edge of α intersects an edge of β. Further suppose that there are two Peano vertices $a, b \in (1/4 + \mathbb{Z}/2)^2$ such that a is adjacent to both a primal vertex $\alpha_a \in \alpha$ and a dual vertex $\beta_a \in \beta$, and b is adjacent to both a primal vertex $\alpha_b \in \alpha$ and a dual vertex $\beta_b \in \beta$. See Figure 6. Note that the line segment $[\alpha_a, \beta_a]$ has a as its midpoint, and the line segment $[\alpha_b, \beta_b]$ has b as its midpoint. Let $D = D(\alpha, \beta, a, b)$ be the (unique) bounded
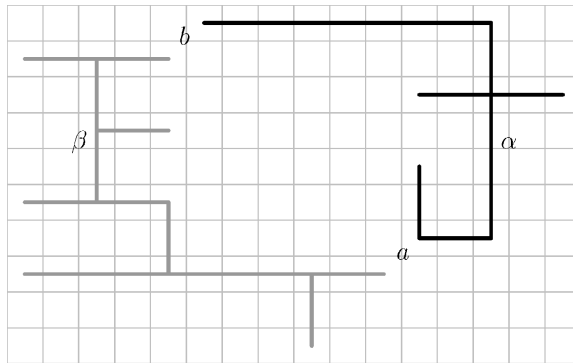

FIG. 6. *The boundary data and the Peano grid.*

connected component of $\mathbb{C} \setminus (\alpha \cup [\alpha_b, \beta_b] \cup \beta \cup [\beta_a, \alpha_a])$. Let $V_P = V_P(D)$ denote the collection of all Peano vertices in $\overline{D}$, and, as before, $V(D)$ denotes the collection of all primal vertices in D. Let $\ell = \ell(D)$ denote the cardinality of $V_P \setminus \{a, b\}$. By switching the role of a and b, if necessary, assume that D lies to the immediate right of the oriented segment $[\alpha_a, \beta_a]$. Let $\mathfrak{D}^*$ denote the collection of all domains obtained in this way.

Let $H = H(D)$ denote the subgraph of $\mathbb{Z}^2$ whose vertices are the vertices of α and $V(D)$ and whose edges are those edges on this set of vertices which do not intersect β. Since β is a tree, H is connected.

Since H is connected, there is at least one spanning tree T of H which contains α. If we replace α by T and apply the dual argument, it follows that there is also a tree $T^\dagger$ in the dual grid $(1/2 + \mathbb{Z})^2$, which is disjoint from T, contains β and whose vertices are the dual vertices in β and the dual vertices in D. In fact, $T^\dagger$ contains every dual edge lying in $\overline{D} \setminus T$.

We now need to give an orientation to the Peano grid G. Every edge in G is either on the boundary of a square face of G centered on a primal vertex or on the boundary of a square face of G centered on a dual vertex, and these two possibilities are exclusive. We orient the edges of G by specifying that the square faces of G containing a primal vertex are oriented clockwise, while those containing a dual vertex are oriented counterclockwise. When we want to emphasize the orientation of the edges, we write $G_\rightarrow$ instead of G. Note that the edges of G contained in a horizontal or vertical line all get the same direction in $G_\rightarrow$, and consecutive parallel lines get opposite orientations. For this reason, $G_\rightarrow$ is often called the *Manhattan lattice*.

Let $\gamma = \gamma(T)$ denote the set of all edges of $G_\rightarrow$ which do not intersect $T \cup T^\dagger$ and which have at least one endpoint in D. Let $v \in V_P \setminus \{a, b\}$ be some Peano vertex in D. Note that there are precisely two oriented edges of $G_\rightarrow$ with initial point v, say e_1 and e_2, where one of these, say e_1, intersects an edge f_1 of the primal grid $\mathbb{Z}^2$, and the other intersects an edge f_2 of the dual grid $(1/2 + \mathbb{Z})^2$. Note also that $f_1 \cap f_2 \neq \varnothing$. It therefore follows that exactly one of the edges f_1, f_2 is in $T \cup T^\dagger$. Consequently, exactly one of the edges e_1, e_2 is in γ. This shows that γ has out-degree 1 at every $v \in V_P \setminus \{a, b\}$. An entirely similar argument shows that γ has in-degree 1 at every such v. In particular, this shows that γ does not contain the entire boundary of a square face of G that does not contain a primal or dual vertex. If γ had a cycle, the cycle therefore would have to surround some primal or dual vertex. But as T and $T^\dagger$ are connected and disjoint from γ, this is impossible. It therefore follows that γ is an oriented simple path (i.e., self-avoiding path), and the endpoints of γ are a and b. Since we are assuming that D lies to the right of $[\alpha_a, \beta_a]$, the initial point of γ is a and the terminal point is b.

Conversely, suppose that $\gamma^* = (\gamma_0^*, \ldots, \gamma_{\ell+1}^*)$ is any oriented simple path in $G_\rightarrow$, respecting the orientation of $G_\rightarrow$, from a to b, whose vertices are V_P. For $n \in \{0, \ldots, \ell + 1\}$, let v_n be the (unique) primal vertex adjacent to γ_n^* and

let $v_n^\dagger$ be the dual vertex adjacent to γ_n^*. Note that v_n and v_{n+1} are either the same vertex or adjacent vertices when $n = \{0, \ldots, \ell\}$. Let $\alpha_n = \alpha_n(\gamma^*)$ denote the union of α with the collection of all edges $[v_k, v_{k+1}]$ for $k < n$ such that $v_k \neq v_{k+1}$ and, similarly, let $\beta = \beta_n(\gamma^*)$ denote the union of β with the collection of all dual edges $[v_k^\dagger, v_{k+1}^\dagger]$ for $k < n$ such that $v_k^\dagger \neq v_{k+1}^\dagger$. Then $T(\gamma^*) := \alpha_{\ell+1}(\gamma^*)$ and $T^\dagger(\gamma^*) := \beta_{\ell+1}(\gamma^*)$ are obviously connected, and there are no edges in $T(\gamma^*)$ intersecting edges in $T^\dagger(\gamma^*)$. Now, $T(\gamma^*)$ cannot contain a cycle, for such a cycle would have to separate $T^\dagger(\gamma^*)$. Hence, $T(\gamma^*)$ is a spanning tree of H containing α. It is also clear that $\gamma^* = \gamma(T(\gamma^*))$. That is, $T \mapsto \gamma(T)$ is a bijection between the set of spanning trees of H containing α and the set of oriented paths in $G_\rightarrow \cap \overline{D}$ from a to b containing V_P. Hence, when T is the UST on H conditioned to contain α, γ is uniformly distributed among such Peano paths; it is the *UST Peano path* associated with (α, β, a, b).

Let $(a = w_0, w_1, \ldots, w_{\ell+1} = b)$ be the order of the vertices in the UST Peano path γ. For $n \in \{0, 1, \ldots, \ell\}$, let $\gamma[0, n]$ denote the initial arc of γ from w_0 to w_n. Since γ is uniformly distributed among simple oriented paths in $G_\rightarrow$ from a to b which contain V_P, we immediately get the following Markov property.

LEMMA 4.1 (Markovian property).   *Fix any $n \in \{1, 2, \ldots, \ell\}$. Conditioned on $\gamma[0, n]$, the distribution of $(\gamma \setminus \gamma[0, n]) \cup \{w_n\}$ is the same as that of the UST Peano curve associated with $(\alpha_n(\gamma), \beta_n(\gamma), w_n, b)$.*

This lemma will play the same role in the proof as Lemma 3.2 in the case of the LERW. We will also use the convergence of certain discrete harmonic functions toward their continuous counterparts. To facilitate this, we have to set the combinatorial notation for the discrete Dirichlet–Neumann problem.

Let H be a finite nonempty connected subgraph of $\mathbb{Z}^2$ with vertices V_H and let E_∂ denote the set of oriented edges in $\mathbb{Z}^2$ whose initial endpoint is in V_H, but whose unoriented version is not in H. Suppose $E_\partial = E_0 \cup E_1 \cup E_2$ is a disjoint union, where $E_0 \cup E_1 \neq \varnothing$. Suppose also that $\hat{h} : V_H \rightarrow [0, 1]$ is some function. For $v \in V_H$, set $\Delta_{H,E_0,E_1,E_2}\hat{h}(v) := \sum d\hat{h}[v, u]$, where the sum is over all neighbors u of v in $\mathbb{Z}^2$, and $d\hat{h}[v, u] := \hat{h}(u) - \hat{h}(v)$ when $[v, u] \notin E_\partial$, $d\hat{h}[v, u] := 0 - \hat{h}(v)$ when $[v, u] \in E_0$, $d\hat{h}[v, u] := 1 - \hat{h}(v)$ when $[v, u] \in E_1$ and $d\hat{h}[v, u] := 0$ when $[v, u] \in E_2$. Note that there is a unique $\hat{h} : V_H \rightarrow [0, 1]$ such that $\Delta_{H,E_0,E_1,E_2}\hat{h}(v) = 0$ in $V_H : \hat{h}(v)$ is the probability that a simple random walk on $H \cup E_0 \cup E_1$ started from v will use an edge in E_1 before using an edge of E_0. This $\hat{h}$ will be called the Δ_{H,E_0,E_1,E_2}-harmonic function.

PROPOSITION 4.2 (Dirichlet–Neumann approximation).   *For every $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$ such that the following holds. Let $D \subset \mathbb{C}$ be a simply connected domain satisfying $\mathrm{rad}_0(D) \geq r_0$. Let $A_0, A_1 \subset \partial\mathbb{U}$ be two disjoint arcs, each of length at least ϵ, and set $A_2 := \partial\mathbb{U} \setminus (A_0 \cup A_1)$. Let $\eta \subset D$ be a simple*

closed path which surrounds 0, *such that each point of η is within distance* 5 *from ∂D. Suppose that $A'_0, A'_1 \subset \eta$ are two disjoint arcs, $A'_2 := \eta \setminus (A'_0 \cup A'_1)$ and the triple (A'_0, A'_1, A'_2) corresponds to (A_0, A_1, A_2) under ψ_D, in the sense that for each $j = 0, 1, 2$ and each $p \in A'_j$ there is a continuous path $\sigma : [0, 1) \to D$ satisfying* $\operatorname{diam} \sigma[0, 1) \le 5$, $\sigma(0) = p$, *and* $\lim_{s \uparrow 1} \psi_D \circ \sigma(s)$ *exists and is in A_j.*

Let H be the component of 0 *in the set of edges of $\mathbb{Z}^2$ that do not intersect η. For $j = 0, 1, 2$, let E_j denote the set of oriented edges $[v, u]$ intersecting η, where v is in H, and the first point of intersection from the direction of v is in A'_j. Let $\hat{h}$ denote the Δ_{H,E_0,E_1,E_2}-harmonic function. Let $h : \mathbb{U} \to [0, 1]$ be the continuous harmonic function which has boundary value* 0 *on A_0,* 1 *on A_1, and satisfies the Neumann boundary condition on A_2. Then $|\hat{h}(0) - h(0)| < \epsilon$.*

The proof will be given in Section 5.4.

4.2. *Driving process convergence.* Let α, β, a, b and $D = D(\alpha, \beta, a, b)$ be as above and suppose now that $0 \in D$. As before, let ℓ denote the number of Peano vertices in D and let $\gamma = (\gamma(0), \ldots, \gamma(\ell + 1))$ be the UST Peano path from a to b in $G_\to \cap \overline{D}$. For each $n \le \ell$, there are two domains that are naturally associated to $\gamma[0, n]$. The first one (as in Lemma 4.1) is $\tilde{D}_n := D(\alpha_n, \beta_n, \gamma(n), b)$ (see Figure 7). But $\tilde{D}_n$ is not so useful if we want to make estimates using Loewner's equation. We therefore also define $D_n := D \setminus \gamma[0, n]$. Let $\phi_0 = \phi : D \to \mathbb{H}$ be the conformal map which takes D to $\mathbb{H}$, takes b to ∞, takes a to 0 and satisfies $|\phi'(0)| = 1$. Let $\phi_n : D_n \to \mathbb{H}$ be the conformal maps satisfying $\phi_n(z) - \phi_0(z) \to 0$ as $z \to b$ within D_n. Define $W_n := \phi_n(\gamma(n)) \in \mathbb{R}$. Also let t_n denote the capacity from ∞ in $\mathbb{H}$ of $\phi_0 \circ \gamma[0, n]$, so that $\phi_n \circ \phi_0^{-1}(z) = z + 2t_n/z + o(1/z)$ when $z \to \infty$ in $\mathbb{H}$.
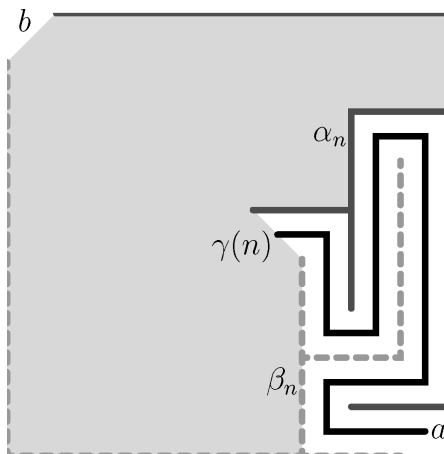

FIG. 7. *The domain $\tilde{D}_n$ is shaded.*

We now prove the analog of Proposition 3.4 for the UST Peano curve. Let $\mathfrak{H}_D(z, A)$ denote the continuous harmonic measure of A from z in the domain $D \setminus A$.

PROPOSITION 4.3 (The key estimate). *For every sufficiently small* $\delta, \epsilon > 0$, *there is some* $r_0 = r_0(\delta, \epsilon)$ *such that the following holds. Let* γ, D_n, ϕ_n, W_n *and* t_n *be as above, let* $k \in \mathbb{N}$ *and let* m *be the first* $n \geq k$ *such that* $|W_n - W_k| \geq \delta$ *or* $t_n - t_k \geq \delta^2$. *Then*

(4.1) $$\mathbf{E}[W_m - W_k | D_k] = O(\delta^3)$$

and

(4.2) $$\mathbf{E}[(W_m - W_k)^2 | D_k] = 8\mathbf{E}[t_m - t_k | D_k] + O(\delta^3),$$

provided that $\mathrm{rad}_0(D_k) \geq r_0$ *and* $\mathfrak{H}_{D_k}(0, \alpha_k(\gamma)) \in [\epsilon, 1 - \epsilon]$.

PROOF.   Assume first $k = 0$. Let $v_0 \in V(D)$ be some vertex such that $|\phi(v_0)| \leq 2$ and $\mathrm{Im}\,\phi(v_0) \geq 1/2$, say. [As we have seen in Section 3.2, there is such a v_0 when $\mathrm{rad}_0(D)$ is large.] If $Q = [q, q']$ is a line segment where $q \in D$ is a dual vertex and $q' \in \alpha$ is the midpoint of a dual edge containing q, then let $\phi^*(Q) \in \mathbb{R}_+$ denote the limit of $\phi(z)$ as z tends to ∂D along Q (which always exists by [35], Proposition 2.14). Fix such a Q_0 satisfying $U := \phi^*(Q_0) \in [1/2, 2]$; there clearly is such Q_0 when r_0 is large, because the harmonic measure from 0 of any square of the dual grid adjacent to the boundary of D is small. Let $\eta \subset D$ be the set of points within distance $1/10$ from ∂D. Then η is a simple closed path. Consider it as oriented counterclockwise around the bounded domain of $\mathbb{C} \setminus \eta$. Let p_0 be the point of η closest to a, p_1 the point in $\eta \cap Q_0$ and p_2 the point of η closest to b. Let A_0' be the positively oriented subarc of η from p_0 and p_1, A_1' the positively oriented arc from p_1 to p_2 and A_2' the positively oriented arc from p_2 to p_0.

Let $\mathcal{A}$ be the event that the path in the tree $T(\gamma) = \alpha_{\ell+1}(\gamma)$ from v_0 to α hits A_1'. We will now estimate both sides of the identity

(4.3) $$\mathbf{P}[\mathcal{A}] = \mathbf{E}\big[\mathbf{P}[\mathcal{A}|D_m]\big]$$

using Proposition 4.2. By Wilson's algorithm, $\mathbf{P}[\mathcal{A}]$ is the probability that a simple random walk on the graph $H(D)$ started at v_0 stopped on hitting α will cross A_1'. This is exactly $\hat{h}(v_0)$, where the function $\hat{h}$ is as defined in Proposition 4.2. Set

$$h(z) := \frac{1}{\pi} \cot^{-1}\left(\frac{1 - |z|}{2\,\mathrm{Im}\,\sqrt{z}}\right) = \frac{1}{\pi}\cot^{-1}\left(\frac{1 - r}{2\sqrt{r}\sin(\theta/2)}\right),$$

where $z = re^{i\theta}$ and we take the value of $\cot^{-1}$ between 0 and π. Note that h is harmonic in $\mathbb{H}$, is equal to 0 on $(0, 1)$, is equal to 1 on $(1, \infty)$, and $\partial_y h = 0$ on $(-\infty, 0)$. (Of course, we found the map h satisfying these boundary conditions by reflecting the domain along the negative real axis, mapping this larger domain to $\mathbb{H}$

with $z \mapsto \sqrt{z}$ and then using a conformal map from $\mathbb{H}$ to $\mathbb{U}$ to calculate the hitting probabilities.) Consequently, Proposition 4.2 shows that if r_0 is sufficiently large, then

$$(4.4) \qquad \mathbf{P}[\mathcal{A}] = h\big(\phi(v_0)/U\big) + O(\delta^3).$$

Set $V_j := \phi_j(v_0)$ and $U_j := \phi_j \circ \phi_0^{-1}(U)$. By the chordal version of Loewner's equation and the definition of m, we have

$$(4.5) \qquad V_m = V_0 + \frac{2t_m}{V_0} + O(\delta^3), \qquad U_m = U_0 + \frac{2\,t_m}{U_0} + O(\delta^3).$$

Note also that $\mathrm{rad}_0(D_m) > \mathrm{rad}_0(D)/2$, $v_0 \in D_m$ and $U_m \in [1/4, 4]$ provided that δ is small enough.

We now employ a similar argument to estimate $\mathbf{P}[\mathcal{A}|D_m]$. Recall that $\tilde{D}_n = D(\alpha_n, \beta_n, \gamma(n), b)$. Assume that Q_0 intersects $\tilde{D}_n$, which will be the case if $U_n > 1/4$, say. Let η_n be the set of points in $\tilde{D}_n$ at distance $1/10$ from $\partial \tilde{D}_n$. Again, η_n is a simple closed path, and we write $\eta_n = A_0'(n) \cup A_1'(n) \cup A_2'(n)$, where $A_0'(n)$ is the arc of η_n from the closest point to $\gamma(n)$ to the point of intersection of Q_0 with η_n, $A_1'(n)$ is the arc of η_n from the point in $Q_0 \cap \eta_n$ to the point of η_n closest to b and $A_2'(n)$ is the remaining part of η_n. By Lemma 4.1, $\mathbf{P}[\mathcal{A}|D_n]$ is the same as the quantity $\hat{h}_n(v_0)$, where $\hat{h}_n$ is the function $\hat{h}$ defined in Proposition 4.2, but with $A_0'(n)$, $A_1'(n)$, $A_2'(n)$ and η_n replacing A_0', A_1', A_2' and η and D_n replacing D. (It is D_n replacing D, not $\tilde{D}_n$. The conditions of Proposition 4.2 hold for either of these, but the conformal map we consider is defined on D_n.) Proposition 4.2 therefore gives

$$(4.6) \qquad \mathbf{P}[\mathcal{A}|D_m] = h\left(\frac{V_m - W_m}{U_m - W_m}\right) + O(\delta^3).$$

Write $f(U, V, W) := h((V - W)/(U - W))$. We Taylor-expand the right-hand side in (4.6) to second order in W_m and to first order in $V_m - V_0$ and $U_m - U_0$. Together with (4.3)–(4.5) this gives

$$0 = \mathbf{E}\big[\mathbf{P}[\mathcal{A}|D_m]\big] - \mathbf{P}[\mathcal{A}]$$

$$= \frac{1}{2}\partial_W^2 f \, \mathbf{E}[W_m^2] + \partial_W f \, \mathbf{E}[W_m] + \partial_V f \frac{2\mathbf{E}[t_m]}{V_0} + \partial_U f \frac{2\mathbf{E}[t_m]}{U_0} + O(\delta^3).$$

Here, the derivatives of f are evaluated at (V_0, W_0, U_0). (Note that V is complex valued, and we interpret $\partial_V f$ as an $\mathbb{R}$-linear map from $\mathbb{C}$ to $\mathbb{R}$.) If we plug in $V_0 = i + O(\delta^3)$ and $U_0 = 1 + O(\delta^3)$ [as we have seen in Section 3.2, one can certainly find v_0 and u_0 satisfying $\phi(v_0) = i + O(\delta^3)$ and $\phi(u_0) = 1 + O(\delta^3)$ if r_0 is large], then after some tedious but straightforward computations the above equality simplifies to

$$\mathbf{E}[W_m^2] + 2\mathbf{E}[W_m] - 8\mathbf{E}[t_m] = O(\delta^3),$$

while $V_0 = 2i + O(\delta^3)$ and $U_0 = 1 + O(\delta^3)$ give

$$3\mathbf{E}[W_m^2] + 8\mathbf{E}[W_m] - 24\mathbf{E}[t_m] = O\delta^3).$$

Combining these two relations implies (4.1) and (4.2) in the case $k = 0$. For $k > 0$, the proof is basically the same; the only essential difference is that one must use η_k in place of η.   □

THEOREM 4.4 (Driving process convergence).   *For every positive $\epsilon_1, \epsilon_2, \epsilon_3$ and $\bar{t}$, there is some positive $r_1 = r_1(\epsilon_1, \epsilon_2, \epsilon_3, \bar{t})$ such that the following holds. Let $D = D(\alpha, \beta, a, b) \in \mathfrak{D}^*$ satisfy $\mathrm{rad}_0(D) > r_1$ and $\mathfrak{H}_D(0, \alpha) \in [\epsilon_1, 1 - \epsilon_1]$. Let γ be the corresponding UST Peano path, let $\phi : D \to \mathbb{H}$ denote the conformal map which takes a to 0, b to ∞ and satisfies $|\phi(0)| = 1$, let $\hat{\gamma} := \phi \circ \gamma$, parameterized according to capacity from ∞, and let $W(t)$ denote the Loewner driving process for $\hat{\gamma}$. Then there is a coupling of standard Brownian motion $B : [0, \bar{t}] \to \mathbb{R}$ and W such that*

$$\mathbf{P}\big[\sup\{|W(t) - B(8t)| : t \in [0, \bar{t}]\} > \epsilon_2\big] < \epsilon_3.$$

PROOF.   The proof is almost identical to the proof of Theorem 3.7, where we used Skorohod's embedding, but one has to be a little careful because it may happen that 0 is "swallowed" before time $\bar{t}$.

Let us first assume that $\phi(0)$ is close to i, say $|\phi(0) - i| < 1/100$, and that $\bar{t}$ is small enough so that

(4.7) $\bar{t} \le 1/100$ and $\mathbf{P}\big[B[0, \bar{t}] \subset [-1/10, 1/10]\big] > 1 - \epsilon_3/3$,

where B is standard Brownian motion. Take $\delta = \delta(\epsilon_1, \epsilon_2, \epsilon_3) > 0$ small and $\epsilon = 1/10$. Define $r_0(\epsilon, \delta)$ as in Proposition 4.3. Let $k \in \mathbb{N}$ be the first integer where $\mathrm{rad}_0(D_k) \le r_0$ or $\mathfrak{H}_{D_k}(0, \alpha_k) \notin [\epsilon, 1 - \epsilon]$ and define $\bar{t}_0 := \min\{\bar{t}, t_k\}$, where t_n is as in the proposition. Exactly as in the proof of Theorem 3.7, Proposition 4.3 implies that we may couple W with a Brownian motion B in such a way that

$$\mathbf{P}\big[\sup\{|W(t) - B(8t)| : t \in [0, \bar{t}_0]\} > \epsilon_2/3\big] < \epsilon_3/3,$$

if $\mathrm{rad}_0(D) \ge r_1$ and r_1 is large enough. By our assumptions regarding $\bar{t}$, we have with high probability that, for all $t \in [0, \bar{t}_0]$, $W(t) \in [-1/5, 1/5]$. If we choose r_1 large enough, this guarantees that $\mathbf{P}[\bar{t}_0 \neq \bar{t}] < \epsilon_3/3$ and proves the theorem when (4.7) is satisfied and $\phi(0)$ is close to i.

Consider now a general $\bar{t} < \infty$. Let $\bar{t}_1 > 0$ be some constant satisfying (4.7) and let $z_0 := \phi^{-1}(2i\bar{t}^2/\bar{t}_1^2)$. From the Koebe distortion theorem, it follows that there is a constant $c = c(\bar{t}, \epsilon_1)$ such that $\mathrm{rad}_{z_0}(D) \ge c\, \mathrm{rad}_0(D)$. (See, e.g., Theorem 1.3 and Corollary 1.4 in [35].) Consequently, by choosing r_1 appropriately larger, we may invoke the above argument with the basepoint moved from 0 to a vertex near z_0 and with a smaller ϵ_2. Rescaling now completes the proof of the theorem.   □

4.3. *Uniform continuity.* In order to prove convergence with respect to a stronger topology, tightness will be needed, and we therefore derive in this section some regularity estimates for UST Peano curves with respect to the capacity parameterization. Some results from [38] will be used.

Let $D \subset \mathbb{C}$ be a simply connected domain containing 0, whose boundary is a C^1-simple closed path. Let a and b be two distinct points on ∂D. In this section, we consider for large R the UST Peano curve from a point near Ra to a point near Rb on a grid approximation of RD. One reason not to consider arbitrary domains is that we need to partially adapt to the framework of [38] in order to quote results from there. Also, it is natural (since the UST Peano curve is asymptotically space filling) to impose regularity conditions on ∂D in order to get uniform regularity estimates for the UST Peano curve.

Let α_D and β_D be, respectively, the clockwise and anticlockwise arcs of ∂D from b to a. Given R large, let $D^R = D(\alpha^R, \beta^R, a^R, b^R) \in \mathfrak{D}^*$ be an approximation of $(RD, R\alpha_D, R\beta_D)$ in the following sense. Fix some sufficiently large constant $C > 0$; for example, $C = 10$ would do. We require α^R to be a simple path in $\mathbb{Z}^2$ satisfying $\rho(\alpha^R, R\alpha_D) \leq C$ and require β^R to be a simple path in the dual grid $(1/2 + \mathbb{Z})^2$ satisfying $\rho(\beta^R, R\beta_D) \leq C$. We also require that $\beta^R \cap \alpha^R = \varnothing$, of course, and that each of a^R, b^R is a Peano vertex adjacent to an endpoint of α^R and an endpoint of β^R.

Let $\gamma = \gamma^R$ be the UST Peano path in D^R. Let $\phi : D \to \mathbb{H}$ be the conformal homeomorphism satisfying $\phi(a) = 0$, $\phi(b) = \infty$ and $|\phi(0)| = 1$. Let $\phi_R : D^R \to \mathbb{H}$ be the conformal homeomorphism satisfying $|\phi_R(0)| = 1$, taking a^R to 0 and b^R to ∞. Then $\lim_{R \to \infty} R^{-1} \phi_R^{-1}(z) = \phi^{-1}(z)$ uniformly in $\mathbb{H}$. (This follows, e.g., from Corollary 2.4 in [35].) Let $\hat{\gamma} := \phi_R \circ \gamma$, parameterized according to capacity from ∞. Let $g_t : \mathbb{H} \setminus \hat{\gamma}[0, t] \to \mathbb{H}$ be the conformal map with the usual normalization $g_t(z) - z \to 0$ when $|z| \to \infty$.

PROPOSITION 4.5 (Uniform continuity estimate). *For every $\epsilon > 0$ and $\bar{t} > 0$, there are some positive $R_0 = R_0(D, \bar{t}, \epsilon)$ and $\delta = \delta(D, \bar{t}, \epsilon)$ such that, for all $R > R_0$,*

$$\mathbf{P}[\sup\{|\hat{\gamma}(t_2) - \hat{\gamma}(t_1)| : t_1, t_2 \in [0, \bar{t}], |t_2 - t_1| \leq \delta\} > \epsilon] < \epsilon.$$

We first prove a slightly modified version of this proposition.

LEMMA 4.6. *For $0 < t_1 < t_2 < \infty$, let $Y(t_1, t_2) := \operatorname{diam}(g_{t_1} \circ \hat{\gamma}[t_1, t_2])$. For every $\epsilon > 0$, there is a $\delta = \delta(D, \epsilon) > 0$ and an $R_0 = R_0(D, \epsilon) > 0$ such that, for all $R \geq R_0$,*

(4.8) $\mathbf{P}[\sup\{|\hat{\gamma}(t_2) - \hat{\gamma}(t_1)| : 0 \leq t_1 \leq t_2 \leq \tau, Y(t_1, t_2) \leq \delta\} \geq \epsilon] < \epsilon,$

where $\tau := \inf\{t \geq 0 : |\hat{\gamma}(t)| = \epsilon^{-1}\}$.

The proof will use Theorems 10.7 and 11.1(ii) of [38]. As explained there, the proofs of these theorems are now easier, because we have established the conformal invariance of the UST; see Corollary 1.2.

PROOF OF LEMMA 4.6. Let δ_R be a positive function of R such that $\lim_{R \to \infty} \delta_R = 0$. It suffices to show that (4.8) holds for all sufficiently large R with δ_R in place of δ. Let Z denote the semicircle $2\epsilon^{-1} \partial \mathbb{U} \cap \mathbb{H}$, say. For R large, let t_1 and t_2 be such that $|\hat{\gamma}(t_2) - \hat{\gamma}(t_1)|$ is maximal subject to the constraints $0 \le t_1 \le t_2 \le \tau$ and $Y(t_1, t_2) \le \delta_R$. Note that $\min_{t \le \tau} \mathrm{dist}(g_t(Z), g_t \circ \hat{\gamma}[t, \tau])$ is bounded from below, as $\hat{\gamma}[0, \tau] \subset \epsilon^{-1} \overline{\mathbb{U}}$. [Considering the harmonic measure from a point near ∞, one deduces that the diameter of $g_t((1/2)Z)$ is bounded below. The extremal distance between $g_t(Z)$ and $g_t((1/2)Z)$ is the same as the extremal distance between Z and $(1/2)Z$. This shows that $\mathrm{dist}(g_t(Z), g_t \circ \hat{\gamma}[t, \tau]) \ge \mathrm{dist}(g_t(Z), g_t((1/2)Z))$ is bounded from below.] Since $Y(t_1, t_2) \le \delta_R \to 0$ as $R \to \infty$, the extremal length of the collection of simple arcs in $\mathbb{H} \setminus g_{t_1} \circ \hat{\gamma}[t_1, t_2]$ which separate $g_{t_1} \circ \hat{\gamma}[t_1, t_2]$ from $g_{t_1}(Z)$ goes to 0. By the conformal invariance of the extremal length, it follows that the extremal length of the collection of simple arcs in $\mathbb{H} \setminus \hat{\gamma}[t_1, t_2]$ which separate $\hat{\gamma}[t_1, t_2]$ from Z in $\mathbb{H} \setminus \hat{\gamma}[0, t_1]$ tends to 0 as well. In particular, the shortest such arc for the Euclidean metric, say η, satisfies $\lim_{R \to \infty} \mathrm{length}(\eta) = 0$.

We are going to study separately the three cases where η is close to the origin, close to the real line but not to the origin and not close to the real line. In each case, we will see that the existence of such an η is very unlikely. Let $\mathcal{A}$ be the event $|\hat{\gamma}(t_1) - \hat{\gamma}(t_2)| \ge \epsilon$. For $s > 0$, let $\mathcal{X}_0(s)$ be the event $\mathrm{dist}(0, \eta) < s$ and let $\mathcal{X}_1(s)$ be the event $\mathrm{dist}(\mathbb{R}, \eta) < s$. We will prove

(4.9) $\forall s_1 > 0, \ \exists R_0 > 0, \ \forall R > R_0, \qquad \mathbf{P}[\mathcal{A} \setminus \mathcal{X}_1(s_1)] < \epsilon,$

          $\forall s_0 > 0, \ \exists s_1 > 0, \ \exists R_0 > 0, \ \forall R > R_0,$

(4.10)

                    $\mathbf{P}[\mathcal{A} \cap \mathcal{X}_1(s_1) \setminus \mathcal{X}_0(s_0)] < \epsilon,$

(4.11) $\exists s_0 > 0, \ \exists R_0 > 0, \ \forall R > R_0, \qquad \mathbf{P}[\mathcal{A} \cap \mathcal{X}_0(s_0)] < \epsilon.$

Using these statements, the proof of the lemma is completed by choosing s_0 according to (4.11), then choosing s_1 according to (4.10) and, finally, choosing R_0 according to (4.9), (4.10) and (4.11).

We start with (4.9). Fix some $s_1 > 0$ and assume that $\mathcal{A} \setminus \mathcal{X}_1(s_1)$ holds. We also assume that $\epsilon < s_1$. There is no loss of generality in that assumption, since $\mathcal{A}$ is monotone decreasing in ϵ. Since $\lim_{R \to \infty} \mathrm{length}(\eta) = 0$, for large R the two endpoints of η must be in $\hat{\gamma}[0, t_1]$. Because $\hat{\gamma}$ tends to ∞ with t, it is clear that $\hat{\gamma}[t_2, \infty) \cap \eta \ne \varnothing$. In fact, the crossing number of $\hat{\gamma}[t_2, \infty)$ and η must be ± 1, since $\hat{\gamma}$ and η are simple curves. Consider the concentric annulus A whose inner circle is the smallest circle surrounding η and whose outer circle has radius $\epsilon/4$. Let $\mathcal{B}$ denote the open disk bounded by the outer circle of A and note that $\mathcal{B} \subset \mathbb{H}$, by our assumption $\epsilon < s_1$. On the event $\mathcal{A}$, there is a $t^* \in [t_1, t_2]$ such

that the distance from $\hat{\gamma}(t^*)$ to η is at least $\epsilon/2$. In particular, $\hat{\gamma}(t^*) \notin \mathcal{B}$. Now, η separates $\hat{\gamma}(t^*)$ from ∞ in $\mathbb{H} \setminus \hat{\gamma}[0, t_1]$. Therefore, if $\mathcal{A} \setminus X_1(s)$ holds, then $\hat{\gamma}[0, t_1] \cup \eta$ separates $\hat{\gamma}(t^*)$ from ∞. Since $\hat{\gamma}$ is a simple path, this implies that the arc of $\hat{\gamma}[0, t_1]$ between the two points $\overline{\eta} \cap \hat{\gamma}[0, t_1]$ does not stay in $\mathcal{B}$. Hence, $\hat{\gamma}[0, \infty) \cap \mathcal{B}$ has three distinct connected components, say $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3$, each of which intersects the inner circle of A such that $\hat{\gamma}_1, \hat{\gamma}_3 \subset \hat{\gamma}[0, t^*]$ and $\hat{\gamma}_2 \subset \hat{\gamma}[t^*, \infty)$ and $\hat{\gamma}_2$ separates $\hat{\gamma}_1$ from $\hat{\gamma}_3$ within $\mathcal{B}$. See Figure 8.

Note that adjacent to one side of $\phi_R^{-1}(\hat{\gamma}_2)$ lies T, the UST, and $T^\dagger$, the dual UST, is adjacent to the other side. Both are connected, and they do not intersect $\phi_R^{-1}(\hat{\gamma})$. It follows that there are paths $\chi_1 \subset T$ and $\chi_2 \subset T^\dagger$ with endpoints in $\phi_R^{-1}(\partial \mathcal{B})$, each of which intersects the inner boundary of $\phi_R^{-1}(A)$. But the diameter of the inner boundary of $R^{-1}\phi_R^{-1}(A)$ goes to 0 as $R \to \infty$, and the distance between the two boundary components of $R^{-1}\phi_R^{-1}(A)$ does not. Hence, by [38], Theorem 10.7, the probability that such a configuration appears somewhere goes to 0 with R. (Although the result from [38] refers to the UST in the whole plane, the proof is local, and since we are bounded away from the boundary, the result is applicable here.) This proves (4.9).

Now fix $s_0 > 0$ and let $s_1 > 0$ be much smaller. Assume that $\mathcal{A} \cap X_1(s_1) \setminus X_0(s_0)$ holds, $\epsilon < s_1$ and R is large. Also assume that η is closer to $[0, \infty)$ than to $(-\infty, 0]$. Note that η is then bounded away from $(-\infty, 0]$. Let A be defined as above and let $\mathcal{B}$ be the intersection of $\mathbb{H}$ with the disk bounded by the outer boundary component of A. We now need to consider two distinct possibilities. Either both endpoints of η are on $\hat{\gamma}[0, t_1]$, and then the configuration is topologically as in the argument for (4.9), or one endpoint of η is on $[0, \infty)$. But it is easy to see that in either case there is a simple path in $T^\dagger$ which intersects $\phi_R^{-1}(\eta)$ whose endpoints are in $\phi_R^{-1}(\partial \mathcal{B})$, by an argument very similar to the one given above. Now [38], Theorem 11.1(ii), shows that these events have small probabilities if s_1 is small. The case where η is closer to $(-\infty, 0]$ is treated similarly, with the roles of the tree and the dual tree switched. Thus, (4.10) is established.
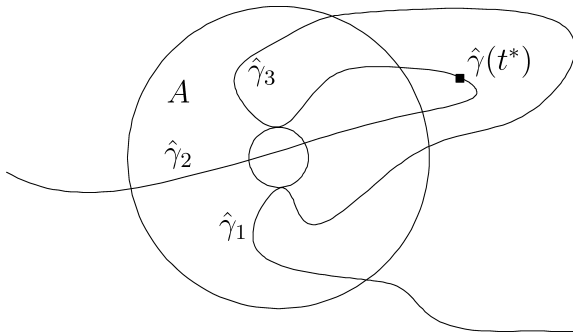

FIG. 8.   *The paths $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\gamma}_3$ and the annulus A.*

To prove (4.11), let $s > 0$ and let $v \in \mathbb{Z}^2$ be a vertex closest to $\phi_R^{-1}(is)$. Let $v^\dagger \in (\mathbb{Z} + 1/2)^2$ be a dual vertex adjacent to v. Let χ be the simple path from v to α^R in T and let $\chi^\dagger$ be the simple path from $v^\dagger$ to β^R in $T^\dagger$. We may sample χ by running a simple random walk from v on $H(D^R)$ stopped on hitting α^R and loop-erasing it. It therefore follows by Proposition 4.2 that if s is sufficiently small, then the diameter of $\phi_R(\chi)$ is smaller than $\epsilon/10$ with probability at least $1 - \epsilon/10$. Moreover, there is some $s_0 > 0$ such that for all sufficiently large R with probability at least $1 - \epsilon/10$ the distance from $\phi_R(\chi)$ to 0 is at least s_0, and the same two estimates will hold for $\chi^\dagger$. Let D' be the domain bounded by $[v, v^\dagger] \cup \chi \cup \chi^\dagger \cup \partial D^R$ which has the initial point a^R of γ on its boundary. Note that γ crosses the boundary of D' exactly once, through the segment $[v, v^\dagger]$. In particular, if $\operatorname{diam}(\phi_R(D')) < \epsilon$ and $\mathcal{A}$ holds, then η is not contained in $\phi_R(D')$. This proves (4.11) and completes the proof of the lemma. $\square$

PROOF OF PROPOSITION 4.5.   Theorem 4.4 implies that we may find some $r > 0$ such that $\mathbf{P}[\sup\{|W(t)| : t \in [0, \bar{t}]\} \geq r] < \epsilon/4$ for all sufficiently large R. Let $\epsilon' := \min\{\epsilon, r^{-1}\}$ and let δ' denote the δ obtained by using Lemma 4.6 with ϵ' in place of ϵ. Since Brownian motion is a.s. continuous, Theorem 4.4 implies that there is some $\delta > 0$ such that if R is large enough we have

$$\mathbf{P}\big[\sup\{|W(t_1) - W(t_2)| : t_1, t_2 \in [0, \bar{t}], |t_1 - t_2| < \delta\} \geq \delta'\big] < \epsilon/4.$$

Lemma 2.1 applied to the path $t \mapsto g_{t_1} \circ \gamma(t - t_1) - W(t_1)$ now implies

$$\mathbf{P}\big[\sup\{Y(t_1, t_2) : 0 \leq t_1 \leq t_2 \leq \bar{t}, |t_1 - t_2| < \delta\} \geq C(\delta^{1/2} + \delta')\big] < \epsilon/4.$$

Now the proof is completed by using Lemma 4.6. $\square$

4.4. *Consequences.*   In this section we gather some consequences, starting with the following two theorems.

THEOREM 4.7 (Chordal SLE₈ traces a path).   *Let $\tilde{g}_t$ denote the chordal* SLE₈ *process driven by $B(8t)$, where $B(t)$ is standard Brownian motion. Then, a.s. for every $t > 0$, the map $\tilde{g}_t^{-1}$ extends continuously to $\overline{\mathbb{H}}$ and $\tilde{\gamma}(t) := \tilde{g}_t^{-1}(B(8t))$ is a.s. continuous. Moreover, a.s. $\tilde{g}_t^{-1}(\mathbb{H})$ is the unbounded component of $\mathbb{H} \setminus \tilde{\gamma}[0, t]$ for every $t \geq 0$.*

THEOREM 4.8 (Peano path convergence).   *Let $D \subset \mathbb{C}$ be a domain containing 0 such that ∂D is a C^1-smooth simple closed path. Let $\partial D = \alpha_D \cup \beta_D$ be a partition of the boundary of D into two nontrivial complementary arcs. For $R > 0$, let (D^R, α^R, β^R) be an approximation of $(RD, R\alpha, R\beta)$, as described in Section 4.3. Let $\gamma = \gamma^R$ denote the UST Peano curve in D^R with the corresponding boundary conditions. Let $\phi_R : D^R \to \mathbb{H}$ denote the conformal map which takes the initial point of γ to 0, the terminal point to ∞ and satisfies $|\phi_R(0)| = 1$. Let $\hat{\gamma} := \phi_R \circ \gamma$, parameterized by capacity from ∞. Then the law of $\hat{\gamma}$ tends weakly to the law of $\tilde{\gamma}$ from Theorem 4.7.*

Here, we think of $\hat{\gamma}$ and $\tilde{\gamma}$ as elements of the space of continuous maps from $[0, \infty)$ to $\overline{\mathbb{H}}$, with the topology of locally uniform convergence.

A consequence of the theorem is that $R^{-1}\gamma$ is close to $R^{-1}\phi_R^{-1} \circ \tilde{\gamma}$. That is, we may approximate the UST Peano path γ by the image of chordal SLE$_8$ in D^R.

The analog of Theorem 4.7 was proven in [36] for all $\kappa \neq 8$, but the particular case $\kappa = 8$ could not be handled there. It is fortunate that the convergence of the UST Peano path to SLE$_8$ settles this problem. By Remark 7.5 in [36], it follows that with the notation of Theorem 4.7 for $\kappa \geq 8$ we have $\tilde{g}_t^{-1}(\mathbb{H}) = \mathbb{H} \setminus \tilde{\gamma}[0, t]$ for every $t \geq 0$ a.s.

PROOF OF THEOREMS 4.7 AND 4.8.   Let $W(t) = W_R(t)$ denote the chordal Loewner driving process for $\hat{\gamma}$. Fix a sequence $R_n \to \infty$. First, note that the family of laws of $\hat{\gamma}$ is tight, because of Proposition 4.5 and the Arzela–Ascoli theorem (see, for instance, [17], Theorem 2.4.10). Also, Theorem 4.4 implies that the law of W converges weakly to the law of $B(8t)$. Hence, there is a subsequence of R_n such that the law of the pair $(\hat{\gamma}, W)$ converges weakly to some probability measure μ. Let (γ^*, W^*) be random with law μ. Then we may identify $W^*(t)$ with $B(8t)$. By the chordal analog of Lemma 3.14, which is valid with the same proof, it follows that, for all $t > 0$, $\tilde{g}_t^{-1}(\mathbb{H})$ is the unbounded component of $\mathbb{H} \setminus \gamma^*[0, t]$. Since γ^* is continuous, elementary properties of conformal maps imply that $\tilde{g}_t^{-1}$ extends continuously to $\overline{\mathbb{H}}$ (e.g., Theorem 2.1 in [35]). It is easy to verify that, a.s. for every $t > 0$, $\tilde{g}_t^{-1}(B(8t)) = \gamma^*(t)$, using the fact that $\gamma^*[t, t']$ is contained in a small neighborhood of $\gamma^*(t)$ when $t' - t > 0$ is small. This proves Theorem 4.7. Because the law of the limit path γ^* does not depend on the subsequence, the original sequence converges, and so Theorem 4.8 is proved as well.   □

We now list some easy consequences of Theorems 4.7 and 4.8.

COROLLARY 4.9 (Radial SLE$_8$ traces a path).   *Let $\tilde{g}_t$ denote a radial SLE$_8$ process driven by $W(t) = \exp(iB(8t))$, where B is standard Brownian motion. Then, almost surely, for every $t > 0$, the map $\tilde{g}_t^{-1}$ extends continuously to $\overline{\mathbb{U}}$. Moreover, $\tilde{g}_t^{-1}(W(t))$ is almost surely continuous.*

PROOF.   This follows readily from Theorem 4.7 and the absolute continuity relation between radial and chordal SLE$_8$ derived in [28], Proposition 4.2.   □

PROOF OF THEOREM 1.3.   Define $G_\delta = \delta D^{1/\delta}$, where $D^{1/\delta}$ is defined as in Theorem 4.8. Consider the situation of Theorem 4.8. As previously remarked, it follows from [35], Corollary 2.4, that $\lim_{R\to\infty} R^{-1}\phi_R^{-1}(z) = \phi^{-1}(z)$ uniformly in $\mathbb{H}$. Consequently, Theorem 4.8 shows that, for all $\bar{t} > 0$, the UST Peano curve scaling limit up to capacity $\bar{t}$ from b is equal to $\phi^{-1} \circ \tilde{\gamma}$ up to time $\bar{t}$. It therefore suffices to prove that for all $\epsilon > 0$ there is an $\epsilon' > 0$ such that for all sufficiently

large R with probability at least $1 - \epsilon$ the part of γ after the first time it hits the ϵ'-neighborhood of b stays within the ϵ-neighborhood of b. This is easily proved by the same argument used to prove (4.11) applied to the reversal of the UST Peano path, which is also a UST Peano path. □

COROLLARY 4.10 (Path reversal). *The law of the chordal* SLE_8 *curve is invariant under simultaneously reversing time and inverting in the unit circle, up to a monotone increasing time change. More precisely, if $\tilde{\gamma}$ is the chordal SLE_8 curve from 0 to ∞ defined in Theorem 4.7, then a time change of $(-1/\tilde{\gamma}(1/t), t \geq 0)$ has the same law as $\tilde{\gamma}$.*

PROOF. This follows immediately from the fact that the reversed UST Peano curve is also a UST Peano curve. □

5. Random walk estimates. The goal of this section is to prove the remaining random walk estimates and thereby complete the proofs of the theorems. Basically, we show that, under certain boundary conditions, discrete harmonic functions converge to continuous harmonic functions satisfying corresponding boundary conditions, as the mesh of the grid goes to 0. This general principle is not new, of course (see, e.g., [6]), but it seems that the precise statements which are needed here do not appear in the literature. In particular, our results make no smoothness assumptions on the boundary. It should perhaps be noted that some of the following proofs (and most likely the results, too) are special to two dimensions.

5.1. *Preliminary lemmas.* We now state some lemmas on discrete harmonic functions, which will be helpful in the proofs of Proposition 2.2, Lemma 3.5 and Proposition 4.2.

For $\delta > 0$, define the discrete derivatives

$$\partial_x^\delta f(v) := \delta^{-1}\big(f(v + \delta) - f(v)\big),$$

$$\partial_y^\delta f(v) := \delta^{-1}\big(f(v + i\delta) - f(v)\big).$$

Let $\mathfrak{D}_\delta := \{\delta D : D \in \mathfrak{D}\}$, that is, domains adapted to the grid $\delta\mathbb{Z}^2$. Similarly, for $D_\delta = \delta D \in \mathfrak{D}_\delta$, define $V^\delta(D_\delta) := D_\delta \cap \delta\mathbb{Z}^2 = \delta V(D)$ and $V_\partial^\delta(D_\delta) := \delta V_\partial(D)$.

LEMMA 5.1 (Discrete derivative estimate). *There is a constant $C > 0$ such that, for every $D \in \mathfrak{D}$ and every bounded function $h : V(D) \cup V_\partial(D) \to \mathbb{R}$ that is harmonic in $V(D)$,*

$$(5.1) \qquad \partial_x^1 h(0) \leq C \operatorname{rad}_0(D)^{-1} \|h\|_\infty, \qquad \partial_y^1 h(0) \leq C \operatorname{rad}_0(D)^{-1} \|h\|_\infty.$$

This lemma is proved using the Green's functions in [25], Theorem 1.7.1; see also [16], Lemma 7.1, for a proof of the analogous statement in the triangular lattice using the maximum principle. In Section 6, we rewrite and adapt the proof from [25] to more general walks on planar lattices. One can also rather easily prove the lemma using coupling.

LEMMA 5.2. *For all $\epsilon > 0$ and $k \in \mathbb{N}$, there exists a $c = c_k(\epsilon) > 0$ such that the following always holds. Let $\delta \in (0, c^{-1})$ and let $D \in \mathfrak{D}_\delta$ satisfy $\text{rad}_0(D) \geq 1/2$. Let $\partial_{a_1}^\delta, \ldots, \partial_{a_k}^\delta \in \{\partial_x^\delta, \partial_y^\delta\}$. Let $h : V^\delta(D) \cup V_\partial^\delta(D) \to [0, \infty)$ be nonnegative and harmonic in $V^\delta(D)$. If $v \in V^\delta(D)$ satisfies $|\psi_D(v)| \leq 1 - \epsilon$, then*

$$(5.2) \qquad \left| \partial_{a_1}^\delta \partial_{a_2}^\delta \cdots \partial_{a_k}^\delta h(v) \right| \leq ch(0).$$

Note that the case $k = 0$, which is included, is a kind of Harnack inequality.

It is easy to give quantitative estimates for $c_k(\epsilon)$, but they will not be needed here. Only $k \leq 3$ will be used in the rest of the paper.

In the proof of the lemma, the following simple conformal geometry consequences of the Koebe distortion theorem (see [35], Theorem 1.3) will be needed. Let D, ϵ and v be as in the statement of the lemma. First, note that $1/4 \leq \text{rad}_0(D)\psi_D'(0) \leq 1$ follows from the Koebe $1/4$ theorem and the Schwarz lemma, respectively. Let $\ell = \ell(\epsilon)$ be large and set $z_j := j\,\psi_D(v)/\ell$ and $w_j := \psi_D^{-1}(z_j)$, $j = 0, 1, \ldots, \ell$. The Koebe distortion theorem gives upper and lower bounds for $\text{rad}_0(D)\,|\psi_D'|$ on the preimage of the line segment $[0, z_\ell]$. This implies that there is a constant $c_1 = c_1(\epsilon) > 0$ such that $\text{rad}_{w_j}(D) \geq c_1 \text{rad}_0(D)$ and that if $\ell = \ell(\epsilon)$ is large, then, $|w_j - w_{j-1}| \leq c_1 \text{rad}_0(D)/20$, $j = 1, \ldots, \ell$. In particular, if v_j is the vertex in $V^\delta(D)$ closest to w_j, then, provided that δ is sufficiently small, $|v_j - v_{j-1}| \leq \text{rad}_{v_{j-1}}(D)/10$.

PROOF OF LEMMA 5.2. We start with $k = 0$. Suppose first that $|v| \leq \text{rad}_0(D)/10$. Let $W \subset V^\delta(D)$ be the set of vertices w satisfying $h(w) \geq h(v)$. Then W contains a path from v to ∂D. But the probability p that the path traced by a simple random walk from 0 before exiting D separates v from ∂D is bounded away from 0. On that event, the simple random walk hits W before exiting D. Consequently, $h(0) \geq ph(v)$, as needed. For arbitrary $v \in V^\delta(D)$ satisfying $|\psi_D(v)| \leq 1 - \epsilon$, as we have noted, the Koebe distortion theorem implies that there is an $\ell = \ell(\epsilon)$ depending only on ϵ and a sequence $0 = v_0, v_1, \ldots, v_\ell = v$ in $V^\delta(D)$ with $\ell \leq \ell(\epsilon)$ such that $|v_j - v_{j-1}| \leq \text{rad}_{v_j}(D)/10$ for each $j = 1, \ldots, \ell$. Consequently, iterating the above result gives $h(0) \geq p^\ell h(v)$ and proves the case $k = 0$.

Using the above, we know that $h(w) \leq c'h(0)$ on the set of vertices $w \in V^\delta(D)$ such that $|w - v| \leq \text{rad}_v(D)/10$, where $c' = c'(\epsilon)$ is some constant depending only on ϵ. Consequently, the case $k = 1$ now follows from Lemma 5.1 applied with v translated to 0.

For $k > 1$, the proof is by induction. By the above, we may assume $v = 0$. Let M be the maximum of $|\partial_{a_k}^\delta h(w)|/h(0)$ on the set V of vertices $w \in V^\delta(D)$ satisfying $|w| \leq \text{rad}_0(D)/10$. The above shows that M is bounded by a universal constant. Since $\partial_{a_k}^\delta h$ is discrete harmonic on V, the proof is completed by applying the inductive hypotheses to the function $\partial_{a_k}^\delta h(w) + Mh(0)$. $\square$

LEMMA 5.3 (Continuous harmonic approximation). *For every $\epsilon > 0$, there is some $r_0 = r_0(\epsilon) > 0$ such that the following holds. If $D \in \mathfrak{D}$ satisfies $\mathrm{rad}_0(D) \geq r_0$ and $h : V(D) \cup V_\partial(D) \to [0, \infty)$ is discrete harmonic in $V(D)$, then there exists a harmonic function $h^* : D \to [0, \infty)$ such that*

$$(5.3) \qquad |h^*(v) - h(v)| \leq \epsilon h(0)$$

holds for every vertex $v \in V(D)$ satisfying $|\psi_D(v)| < 1 - \epsilon$.

PROOF. Suppose that the lemma is not true. Then there exists $\epsilon > 0$ and a sequence of pairs (D_n, h_n), where $D_n \in \mathfrak{D}$ satisfies $\mathrm{rad}_0(D_n) \geq n$ and $h_n > 0$ is discrete harmonic in $V(D_n)$, satisfies $h_n(0) = 1$, but (5.3) fails for every harmonic function $\hat{h}$.

Set $\delta = \delta_n := 1/\mathrm{rad}_0(D_n)$. Our objective is to apply compactness to show that the maps $h_n \circ \psi_D^{-1}$ converge locally uniformly in $\mathbb{U}$ as $n \to \infty$ along some subsequence to some harmonic $\hat{h}$, so that (5.3) does hold for some n. We put $h^n(v) := h_n(v/\delta_n)$.

First, standard compactness properties of conformal maps say that one can take a subsequence such that the maps $\delta_n \psi_{D_n}^{-1}$ converge locally uniformly in $\mathbb{U}$ to some conformal map, say ϕ. (This follows, e.g., from the Arzela–Ascoli theorem, together with [35], Corollary 1.4, with $z = 0$ and part two of [35], Theorem 1.3.) If $K \subset \mathbb{U}$ is compact, then Lemma 5.2 shows that there is a constant $C > 0$ such that, for all sufficiently large n in the subsequence, the discrete derivatives $|\partial_x^\delta h^n|$ and $|\partial_y^\delta h^n|$ are bounded by C in $\phi(K) \cap V(\delta D_n)$. By a variant of the Arzela–Ascoli theorem, it then follows that there is some continuous $h^* : \phi(\mathbb{U}) \to [0, \infty)$ and a further subsequence such that, for every compact $K \subset \phi(\mathbb{U})$,

$$\sup\{|h^n(v) - h^*(v)| : v \in K \cap \delta \mathbb{Z}^2\} \to 0$$

along the subsequence. The same argument may also be applied to prove the convergence of the discrete derivatives of h^n to arbitrary order, possibly in a further subsequence. Obviously, the discrete derivatives of h^n will converge to the corresponding continuous derivatives of h^*; that is,

$$(5.4) \qquad \sup\{|\partial_{a_1}^\delta \cdots \partial_{a_k}^\delta h^n(v) - \partial_{a_1} \cdots \partial_{a_k} h^*(v)| : v \in K \cap \delta \mathbb{Z}^2\} \to 0,$$

where $\partial_{a_j}^\delta \in \{\partial_x^\delta, \partial_y^\delta\}$ and $\partial_{a_j} \in \{\partial_x, \partial_y\}$ is the corresponding continuous derivative, $j = 1, 2, \ldots, k$. The fact that h^n is discrete harmonic translates to $(\partial_x^\delta)^2 h^n(v - \delta) + (\partial_y^\delta)^2 h^n(v - i\delta) = 0$. Therefore, (5.4) shows that h^* is harmonic. This completes the proof. $\square$

LEMMA 5.4 (Boundary hitting). *For every $\epsilon_1, \epsilon_2 > 0$, there is a $\delta = \delta(\epsilon_1, \epsilon_2) > 0$ such that if $D \in \mathfrak{D}$ and $w \in V(D)$ is a vertex satisfying $|\psi_D(w)| \geq 1 - \delta$, then the probability that the simple random walk started at w will hit*

$$\{v \in V(D) : |\psi_D(v) - \psi_D(w)| > \epsilon_1\}$$

before hitting ∂D is at most ϵ_2.

PROOF. We first prove the lemma in the case where ϵ_2 is very close to 1. Let $\delta > 0$ be much smaller than ϵ_1. Fix some vertex $w \in V(D)$ and suppose that $|\psi_D(w)| \geq 1 - \delta$. Let

$$\alpha := \{z \in D : |\psi_D(z) - \psi_D(w)| = \epsilon_1\}.$$

Let z_1 be a point in ∂D closest to w and set $r := \mathrm{dist}(w, \partial D) = |z_1 - w|$. Let A_1 be the line segment $[w, z_1]$. Let Q be the connected component of $C(z_1, r) \cap D$ which contains w, where $C(z, r)$ denotes the circle of radius r and center z. Then Q is an arc of a circle. Let A_2 and A_3 denote the two connected components of $Q \setminus \{w\}$. See Figure 9. For $j = 1, 2, 3$, let K_j be the connected component of $D \setminus (A_1 \cup A_2 \cup A_3)$ which does not have A_j as a subset of its boundary.

Because δ is small compared to ϵ_1, the Koebe distortion theorem (e.g., Corollaries 1.4 and 1.5 in [35]) shows that $\alpha \cap C(w, r/8) = \varnothing$. For $j = 1, 2, 3$, let $\mathcal{S}_j$ be the collection of all paths which stay in K_j from the first time they hit $C(w, r/8)$ until they first exit from D. Let $B(t)$ denote Brownian motion started from w. It is easy to see that there is a universal constant $c_1 > 0$ such that $\mathbf{P}[B \in \mathcal{S}_j] > c_1$ for $j = 1, 2, 3$. For example, to prove this for $j = 3$, observe that the collection of Brownian paths which first hit $C(w, r/8)$ in K_3 and later hit A_3 before $A_1 \cup A_2$ has probability bounded away from 0.

Suppose for the moment that α intersects A_1 and A_2. Consider a subarc $\alpha' \subset \alpha$ whose endpoints are in A_1 and A_2, which is minimal with respect to inclusion. Then $\alpha' \subset \overline{K_3}$ or $\alpha' \subset \overline{K_1 \cup K_2}$. If $\alpha' \subset \overline{K_3}$, then α' separates $C(w, r/8)$ from ∂D in K_3. Consequently, on the event $B \in \mathcal{S}_j$, B hits α before hitting ∂D. However, by choosing δ to be sufficiently small and invoking the conformal invariance of the harmonic measure, we may ensure that the latter event has probability smaller than c_1. An entirely similar argument rules out the possibility that $\alpha' \subset \overline{K_1 \cup K_2}$. Similarly, it is not possible that α intersects both A_1 and A_3 or that α intersects
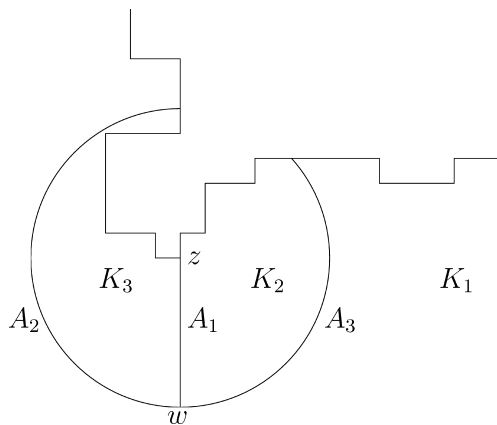

FIG. 9. *The arcs A_j and the components K_j.*

both A_2 and A_3. Hence, there is some $j \in \{1, 2, 3\}$ such that $\alpha \cap K_j = \varnothing$. Let j' be such a j.

By the convergence of a simple random walk to Brownian motion, it is clear that there is some universal constant $r_0 > 0$ such that if $r > r_0$, then the probability that the simple random walk started from w is in $\mathcal{S}_{j'}$ is at least $c_1/2$. This establishes the lemma in the case where $\epsilon_2 \in (1 - c_1/2, 1]$ and $r > r_0$. Suppose $r \leq r_0$. Then there are two grid paths of bounded length starting from w to ∂D that are disjoint except at w. If α intersects both these paths, then this gives a lower bound for the continuous harmonic measure of α from w. Consequently, by making δ small enough, we can make sure that this does not happen. Thus, again, with probability bounded away from 0, the random walk from w hits ∂D before α, since it may follow any one of these two paths. This proves the lemma in the case where $\epsilon_2 \in (c_2, 1]$, where c_2 is some universal constant.

The Koebe distortion theorem implies that there is a constant $c > 0$ such that if $v_1, v_2 \in V(D)$ are neighbors, then $1 - |\psi_D(v_2)| \leq c(1 - |\psi_D(v_1)|)$. (See, e.g., Corollaries 1.4 and 1.5 in [35].) Consequently, we may iterate the above restricted case of the lemma and use the Markov property, thereby proving the lemma for arbitrary $\epsilon_2 > 0$. $\square$

5.2. *The hitting probability estimate.*

PROOF OF PROPOSITION 2.2.   Let $\epsilon_1 > 0$ be much smaller than ϵ. We consider the discrete harmonic function $h(w) := H(w, u)/H(0, u)$. For $\delta > 0$, let

$$V(\delta, \epsilon_1) := \{z \in V(D) : |\psi_D(z)| \geq 1 - \delta, |\psi_D(z) - \psi_D(u)| > \epsilon_1\}.$$

Our first goal is to show that, for every $\epsilon_1 \in (0, 1/4)$, there is some $\delta = \delta(\epsilon_1) > 0$ and some $r_0 = r_0(\epsilon_1) > 0$ such that

$$(5.5) \qquad\qquad \max\{h(z) : z \in V(\delta, \epsilon_1)\} < \epsilon_1,$$

provided that $\mathrm{rad}_0(D) > r_0$. This will be achieved by first showing that h is not too large on the set

$$W := \{z \in V(D) : \epsilon_1/2 \geq |\psi_D(z) - \psi_D(u)| \geq \epsilon_1/3\}$$

and then letting δ go to 0 and appealing to Lemma 5.4.

Assume that $\mathrm{rad}_0(D)$ is sufficiently large so that any nearest neighbor path from 0 to u in D has a vertex in W. Let M denote the maximum of h on W. We claim that M is bounded by a constant $c = c(\epsilon_1)$ depending only on ϵ_1. Indeed, let K be the set of all $v \in V(D)$ satisfying $h(v) \geq M/2$ and let K' be the union of all edges where both endpoints are in $K \cup \{u\}$. Then the maximum principle shows that K' is connected and contains a simple nearest neighbor path J joining W to u whose vertices are in $\{z \in V(D) : |\psi_D(z) - \psi_D(u)| \leq \epsilon_1/2\} \cup \{u\}$. Note that, in particular, $\mathrm{diam}(\psi_D(J)) \geq \epsilon_1/3$. Consequently, the continuous harmonic measure from 0 of J in D is bounded from below by some constant $c_1(\epsilon_1) > 0$.

We claim that the discrete harmonic measure $H_D(0, J)$ of J at the origin is also bounded away from 0 if r_0 is large enough. Indeed, let $D' = D \setminus J$ and let A be the arc on $\partial \mathbb{U}$ corresponding to J under the map $\psi_{D'}$. The length of A is bounded from below, since it is equal to 2π times the harmonic measure of J. Let A' denote the middle subarc of A having half the length of A. By Lemma 5.4 applied to the domain D', it follows that there is a $c_2 = c_2(\epsilon) \in (0, 1/10)$, such that $H_{D'}(v, J) \geq 1/2$ on vertices v such that $\psi_{D'}(v)$ is within distance c_2 of A'. Using Lemma 5.3 with ϵ replaced by $c_2/4$, we find that if r_0 is large, there is a nonnegative harmonic function $h_J^* : D' \to [0, \infty)$ such that $|h_J^*(v) - H_{D'}(v, J)| \leq c_2/4$ for all $v \in V(D')$ satisfying $|\psi_{D'}(v)| < 1 - c_2/4$. Take $a \in A'$ and $z := (1 - c_2/2)a$. Then it follows from the Koebe distortion theorem (as in the argument toward the end of the proof of Proposition 3.4) that we may find a vertex $v \in V(D')$ such that $|\psi_{D'}(v) - z| < c_2/4$, assuming that r_0 is large enough. Thus, $h_J^*(v) > H_{D'}(v, J) - c_2/4 \geq 1/2 - c_2/4 > 1/4$. By the Harnack principle applied to h_J^*, there is a universal constant $c_3 > 0$ such that $h_J^*(z) \geq c_3$. Since this applies to every $z \in (1 - c_2/2)A'$, the mean value property for h_J^* gives $h_J^*(0) \geq c_3 \operatorname{length}(A')/(2\pi)$. Since $|h_J^*(0) - H_{D'}(0, J)| < c_2/4$, our claim that $H_D'(0, J)$ is bounded away from 0 is established. Since h is positive, harmonic $h(0) = 1$ and $h \geq M/2$ on J, this also gives the bound $M \leq c(\epsilon)$.

Since h is harmonic, Lemma 5.4 with $\tilde{\epsilon}_1 = \epsilon_1/2$ and $\epsilon_2 := \tilde{\epsilon}_1/c(\epsilon)$ instead of ϵ_1, ϵ_2 implies that if $\delta = \delta(\epsilon_1)$ is sufficiently small, and $\operatorname{rad}_0(D)$ is large enough to guarantee that W separates $V(\delta, \epsilon_1)$ from u (in the graph-connectivity sense), then $h(z) \leq M\epsilon_2 < \epsilon_1$ for all $z \in V(\delta, \epsilon_1)$: (5.5) holds.

Now apply Lemma 5.3 again to conclude that there is a harmonic function $\hat{h} : \mathbb{U} \to [0, \infty)$ such that

$$|\hat{h} \circ \psi_D(z) - h(z)| < \epsilon_1$$

for all $z \in V(D)$ such that $|\psi_D(z)| < 1 - \delta/4$. Set $\tilde{h}(z) := \hat{h}((1 - \delta/2)z)$. We know that $\tilde{h} \geq 0$ in $\partial \mathbb{U}$, $\tilde{h}(z') \leq 2\epsilon_1$ on the set $S := \{z' \in \partial \mathbb{U} : |z' - \psi_D(u)| \geq 2\epsilon_1\}$. Consequently, the Poisson representation of $\tilde{h}$ gives

$$\tilde{h}(z) = O(\epsilon_1) + \int_{\partial \mathbb{U} \setminus S} \tilde{h}(z') \frac{1 - |z|^2}{|z - z'|^2} |dz'|.$$

Since $\tilde{h}(0) = 1 + O(\epsilon_1)$ and ϵ_1 is arbitrary, the proposition follows. $\square$

5.3. *Some Green's function estimates.* As opposed to Proposition 2.2, Lemma 3.5 requires only crude bounds. It is actually possible to prove that $G_0(0, 0) - G_m(0, 0)$ is close to t_m, but we do not need this result here.

PROOF OF LEMMA 3.5. We start with (3.5). Let S be the set of vertices in $V(D)$ satisfying (3.3) and assume $S \neq \varnothing$. For a random walk starting from a vertex

in S, there is probability bounded away from 0 that within $\mathrm{rad}_0(D)^2$ steps it will exit D. This gives

$$(5.6) \qquad \sum_{w \in S} G_D(0, w) \leq O(1)\,\mathrm{rad}_0(D)^2.$$

On the other hand, with probability bounded away from 0, the number of steps into vertices in S for the random walk started at 0 that is stopped on exiting D is greater than $\mathrm{rad}_0(D)^2$. Therefore,

$$(5.7) \qquad O(1) \sum_{w \in S} G_D(0, w) \geq \mathrm{rad}_0(D)^2.$$

By reversing the walk, we know that $G_D(0, w) = G_D(w, 0)$. Since $G_D(w, 0)$ is harmonic on $V(D) \setminus \{0\}$, the Harnack principle [i.e., $k = 0$ in (5.2)] can be used to show that $G_D(w, 0)/G_D(w', 0) = O(1)$ when $w, w' \in S$. Combining this with $G_D(0, w) = G_D(w, 0)$ and the estimates (5.6), (5.7) gives (3.5).

By Lemma 2.1, we have

$$(5.8) \qquad \mathrm{diam}(\psi_D \circ \gamma[0, m]) = O(\delta).$$

In the following, we fix $\gamma[0, m]$ (i.e., it will be considered deterministic). Let z be the vertex where a simple random walk from 0 first exits D_m. By considering what happens to the random walk after first hitting z, we get the identity $G_0(0, v) - G_m(0, v) = \mathbf{E}[G_0(z, v)]$ [where $G_0(z, v) = 0$ for $z \notin V(D)$, by definition]. Consequently,

$$G_0(0, v) - G_m(0, v) \leq \mathbf{P}[z \in \gamma[0, m]] \max\{\mathbf{E}[G_0(\gamma_j, v)] : j = 1, \ldots, m\}.$$

By (5.8), the continuous harmonic measure from 0 of $\psi_D \circ \gamma[0, m]$ in $\mathbb{U}$ is $O(\delta)$. Therefore, the continuous harmonic measure from 0 of $\gamma[0, m]$ in D is also $O(\delta)$. As in the argument given in Section 5.2, this implies that if $\mathrm{rad}_0(D)$ is large enough, $\mathbf{P}[z \in \gamma[0, m]] = O(\delta)$.

Let K denote the disk $\{w \in D : |w - v| < \mathrm{rad}_0(D)/10\}$ and fix some $j \in \{1, 2, \ldots, m\}$. Since $\psi_D \circ \gamma[0, m]$ is contained in $\overline{\mathbb{U}} \setminus (1 - O(\delta))\mathbb{U}$. It follows that the continuous harmonic measure of K from γ_j in D is $O(\delta)$. If $\psi_D(\gamma_j)$ is sufficiently close to $\partial\mathbb{U}$ (how close may depend on δ), then we can make sure that the corresponding discrete harmonic measure $H_D(\gamma_j, K)$ is less than δ by Lemma 5.4. If $\psi_D(\gamma_j)$ is not close to $\partial\mathbb{U}$, then when $\mathrm{rad}_0(D)$ is large the bound $H_D(\gamma_j, K) \leq O(\delta)$ follows by the convergence of the discrete harmonic measure to the continuous harmonic measure, as we have seen before. If $w \in V(D) \cap K$ neighbors with a vertex outside of K, then $G_0(w, v) = O(1)$ follows from (5.7) by translating w to 0. Hence, $G_0(\gamma_j, v) = O(1) H_D(\gamma_j, K) = O(\delta)$. Putting these estimates together completes the proof. $\square$

5.4. *Mixed boundary conditions.* Recall that Proposition 4.2, which we will now prove, is not used in the proof of Theorem 1.1.

PROOF OF PROPOSITION 4.2. Suppose first that the distance between A_0 and A_1 is at least ϵ. Let A_0^* and A_1^* denote the two connected components of $\eta \setminus (A_0' \cup A_1')$ such that the sequence $(A_0', A_0^*, A_1', A_1^*)$ conforms to the counterclockwise order along η. This induces a corresponding partition $E_2 = E_0^* \cup E_1^*$ of E_2, according to whether or not the first point on the edge is in A_0^* or in A_1^*.

We need to use the discrete harmonic conjugate function $\hat{k}$ of $\hat{h}$. To be perfectly precise, it is necessary to set some combinatorial infrastructure: we first define a (multi-) graph $\hat{H}$, and $\hat{k}$ will be defined on the planar dual $\hat{H}^\dagger$ of $\hat{H}$. The vertices of $\hat{H}$ are $V_H \cup \{v_0, v_1\}$ (where v_0 and v_1 are new symbols not appearing in V_H). As edges of $\hat{H}$, we take all the edges of H, and, additionally, for every $j = 0, 1$ and every directed edge $[v, u]$ in E_j, there is a corresponding edge $[v, v_j]$ in $\hat{H}$. Finally, there is also the edge $[v_0, v_1]$ in $\hat{H}$. Consider a planar embedding of $\hat{H}$ which extends the planar embedding of H such that v_0 and v_1 are in the unbounded component of $\mathbb{C} \setminus H$. Let $\hat{H}^\dagger$ denote the planar dual of $\hat{H}$. Then there is a unique edge $[v_0^\dagger, v_1^\dagger]$ in $\hat{H}^\dagger$ which crosses $[v_0, v_1]$. We choose the labels so that $v_j^\dagger$ naturally corresponds to A_j^*, $j = 0, 1$. Set $\hat{h}(v_j) := j$, $j = 0, 1$. If we consider $\hat{h}$ as a function on $\hat{H}$, then it is discrete harmonic except at v_0 and v_1. This easily implies (see, e.g., [7] or, more explicitly, [4]) that there is a discrete harmonic conjugate $\hat{k}$ defined on the vertices of $\hat{H}^\dagger$; that is, for every directed edge $e = [u, v]$ in $\hat{H}$ if $\{u, v\} \neq \{v_0, v_1\}$, then the discrete Cauchy–Riemann equation $\hat{h}(v) - \hat{h}(u) = \hat{k}(v^\dagger) - \hat{k}(u^\dagger)$ holds, where $[u^\dagger, v^\dagger]$ is the edge of $\hat{H}^\dagger$ intersecting e from right to left. In fact, $\hat{k}$ is harmonic in $\hat{H}^\dagger$ except at $v_0^\dagger$ and $v_1^\dagger$. The function $\hat{k}$ is unique, up to an additive constant. We choose the additive constant so that $\hat{k}(v_0^\dagger) = 0$. Since $\hat{h} \geq 0$, by considering the neighbors of v_0 and the orientation, it follows that $\hat{k}(v_1^\dagger) \geq 0$.

Consider a sequence D_n of such domains satisfying $\mathrm{rad}_0(D_n) \geq n$, with arcs $\eta = \eta_n$ and such harmonic functions $\hat{h}_n, \hat{k}_n$. Let L_n denote the maximum value of $\hat{k}_n$, which is the value of $\hat{k}_n$ on $v_1^\dagger$.

Since $\epsilon > 0$ is fixed, we can consider a subsequence of $n \to \infty$ such that the arcs A_0 and A_1 converge to arcs $\tilde{A}_0$ and $\tilde{A}_1$ of length at least ϵ, and the distance between them is at least ϵ. Let $\tilde{A}_0^*$ and $\tilde{A}_1^*$ denote the two components of $\partial \mathbb{U} \setminus (\tilde{A}_0 \cup \tilde{A}_1)$, so that $\tilde{A}_0, \tilde{A}_0^*, \tilde{A}_1, \tilde{A}_1^*$ is the positive order along $\partial \mathbb{U}$ of these arcs.

We now separate the argument into two cases according to whether or not $L_n > 1$. Suppose that $L_n > 1$ for infinitely many n and take a further subsequence of n such that $L_n > 1$ along that subsequence. Then $\hat{k}_n/L_n$ and $\hat{h}_n/L_n$ are both bounded by 1. It follows from Lemma 5.3 that after taking a further subsequence, if necessary, there are harmonic functions h and k on $\mathbb{U}$

such that $L_n^{-1}\hat{h}_n \circ \psi_{D_n}^{-1} \to h$ and $L_n^{-1}\hat{k}_n \circ \psi_{D_n}^{-1} \to k$ uniformly on compact subsets of $\mathbb{U}$ (appropriately interpreted, since $\hat{h}_n$ and $\hat{k}_n$ are only defined on vertices and dual vertices, not on every point of D_n). Moreover, (5.4) shows that h and k are harmonic conjugates, because the discrete Cauchy–Riemann equations tend to the continuous Cauchy–Riemann equations.

By Lemma 5.4, it follows that k is, respectively, equal to 0 and 1 in the relative interior of $\tilde{A}_0^*, \tilde{A}_1^*$, and similarly h has boundary values 0 and $1/\tilde{L}$ in $\tilde{A}_0$ and $\tilde{A}_1$, where $\tilde{L} := \lim_{n\to\infty} L_n$ (where the limit is along the subsequence and must exist and be finite). By Schwarz reflection, say, this implies that h and k satisfy Neumann boundary conditions in $\tilde{A}_0^* \cup \tilde{A}_1^*$ and $\tilde{A}_0 \cup \tilde{A}_1$, respectively. It now easily follows (e.g., from the maximum principle) that $h + ik$ is the (unique) conformal map taking $\mathbb{U}$ to the rectangle $[0, 1/\tilde{L}] \times [0, 1]$ which takes the four arcs $\tilde{A}_0, \tilde{A}_0^*, \tilde{A}_1, \tilde{A}_1^*$ to the corresponding sides of the rectangle.

The argument in the case where $L_n \leq 1$ for infinitely many n proceeds in the same manner, except that one should not divide $\hat{h}_n \circ \psi_{D_n}^{-1}$ and $\hat{k}_n \circ \psi_{D_n}^{-1}$ by L_n.

It remains to remove the assumption that the distance between A_0 and A_1 is at least ϵ. Observe that the probabilistic description of $\hat{h}$ shows that it is monotone increasing in A_1' and monotone decreasing in A_0'. Take $\epsilon' > 0$ much smaller than ϵ. Then $\hat{h}(0)$ for the given configuration is bounded from above by the value of $\hat{h}(0)$ for the configuration where arcs of length ϵ' are removed at the two ends of A_1, and A_1' is adjusted accordingly. Similarly, $\hat{h}(0)$ is bounded from below by the value of $\hat{h}(0)$ for the configuration where such arcs are removed at the two ends of A_0. The difference between the value of h for the original versus any of the modified configurations goes to 0 as $\epsilon' \to 0$, since h depends continuously on (A_1, A_2), as long as the length of $A_1 \cup A_2$ is not 0. Consequently, we get the proposition by applying the restricted version proved above with ϵ' in place of ϵ and by "sandwiching." $\quad \square$

6. Other lattices.

For convenience and simplicity, the proofs up to now have been written for the loop-erased random walk and UST Peano curve on the square grid. The purpose of this section is to briefly indicate how to adapt the proofs to more general walks on more general grids. In order to keep this section short, we will not try to consider the most general cases.

Let L be a (strictly two-dimensional) lattice in $\mathbb{R}^2$; that is, L is a discrete additive subgroup of $\mathbb{R}^2$ that is not contained in a line. Discrete means that there is some neighborhood of 0 whose intersection with L is $\{0\}$. Suppose that G is a planar graph whose vertices are the elements of L, and G is invariant under translation by elements of L. That is, if $u, v \in L$ are neighbors in G and $\ell \in L$, then $\ell + u$ neighbors $\ell + v$. It is not hard to verify that there is a linear map taking L to the triangular lattice such that neighbors in G are mapped to vertices at distance 1. In particular, as a graph, G is isomorphic to the triangular grid or to the square grid.

Let N be the set of neighbors of 0 in G and let $N' := \{0\} \cup N$. Let X be an N'-valued random variable and let $X_1, X_2, \ldots$ be an i.i.d. sequence where each X_n has the same law as X. Consider the random walk

$$S_n := \sum_{j=1}^{n} X_j$$

on G. We are interested in the situation where the scaling limit of S_n is standard Brownian motion. For this purpose, we require that $\mathbf{E}[X] = 0$ and that the covariance matrix of X is the identity matrix. (Note that if the covariance matrix of X is nondegenerate but not equal to the identity, we can always apply a linear transformation to the system to convert to the above situation. Therefore, what we say below also applies in that case, provided that we appropriately modify the linear complex structure on $\mathbb{R}^2$.)

Note that, under these assumptions, the Markov chain corresponding to the walk S_n does not need to be reversible. An interesting particular example the reader may wish to keep in mind is where $\mathbf{P}[X = \exp(2\pi i j/3)] = 1/3$ for $j = 0, 1, 2$.

THEOREM 6.1. *Theorem* 1.1 *applies to the loop erasure of the random walk* S_n.

PROOF. An inspection of the proof of Theorem 1.1, including all the necessary lemmas, shows that only the generalization of the proof of Lemma 5.1 to the present framework requires special justification, which is given below. $\square$

LEMMA 6.2. *Let* τ_r *denote the first time* n *with* $|S_n| \geq r$. *There exists a constant* C, *depending on* X *but not on* r, *such that, for all* $r \geq C$, $w \in N$ *and* $y \in L$,

$$|\mathbf{P}^0[S_{\tau_r} = y] - \mathbf{P}^w[S_{\tau_r} = y]| \leq Cr^{-1}\mathbf{P}^0[S_{\tau_r} = y].$$

Here, $\mathbf{P}^w$ denotes the law of the Markov chain started from w; that is, the law of $(S_n + w : n \in \mathbb{N})$ under $\mathbf{P} = \mathbf{P}^0$. This lemma is clearly sufficient to provide the necessary analog of Lemma 5.1 for S_n.

PROOF OF LEMMA 6.2. There are various ways to prove the lemma (via coupling, for instance). We give here a proof based on the Green's functions, as in [25]. Without loss of generality, we assume that $\mathbf{P}[X = 0] > 0$ and that L is the minimal lattice containing $\{w \in N : p_w > 0\}$. Then the random walk is irreducible on L. The discrete Laplacian Δ_X associated with X is defined by

$$\Delta_X f(z) := \mathbf{E}[f(z + X)] - f(z).$$

Let a be the potential kernel for the random walk,

$$a(z) := \sum_{j=0}^{\infty} (\mathbf{P}^0[S_j = 0] - \mathbf{P}^0[S_j = -z]).$$

It is known that the series converges and, in fact,

$$a(z) = c_1 \log |z| + c_2 + O(|z|^{-1}) \tag{6.1}$$

as $|z| \to \infty$, $z \in L$ (where c_1, c_2 depend on the law of X). This is proved in [12] for the lattice $\mathbb{Z}^2$ with arbitrary nondegenerate covariance matrix (with an appropriate dependence on the matrix), so the above follows for other L by applying a linear transformation. Since $\mathbf{P}^0[S_j = -z] = \mathbf{P}^z[S_j = 0]$, it follows that

$$\Delta_X a(z) = \begin{cases} 1, & z = 0, \\ 0, & z \neq 0. \end{cases}$$

Let G_r denote the Green's function for the walk in $L \cap r\mathbb{U}$, that is, $G_r(z, z') := \sum_{j \in \mathbb{N}} \mathbf{P}^z[j < \tau_r, S_j = z']$. Note that, for all $z, w \in L \cap r\mathbb{U}$,

$$a(z - w) + G_r(z, w) = \mathbf{E}^z[a(S_{\tau_r} - w)], \tag{6.2}$$

since for fixed w both sides are Δ_X-harmonic for $z \in L \cap r\mathbb{U}$ and equality holds for $z \in L \setminus r\mathbb{U}$. Set $M := \max\{|w| : w \in N\}$ and $Z = \{z \in L : r/2 \leq |z| < M + r/2\}$. By (6.1) and (6.2), $G_r(z, w) = c_1 \log 2 + O(r^{-1})$ for $z \in Z$ and $w \in N'$. The same argument applied to the reverse walk $-S_j$, which has potential kernel $\bar{a}(z) = a(-z)$ and Green's function $\bar{G}_r(z, w) = G_r(w, z)$, gives

$$\forall w \in N', \ \forall z \in Z, \qquad G_r(w, z) = c_1 \log 2 + O(r^{-1}). \tag{6.3}$$

Assuming $r > 4M$, by considering the last vertex in Z visited by the walk before time τ_r, we obtain, for all $w \in N'$ and all $y \in L$,

$$\mathbf{P}^w[S_{\tau_r} = y] = \sum_{z \in Z} G_r(w, z)\mathbf{P}^z[S_{\tau_r} = y, \min\{j \geq 1 : S_j \in Z\} > \tau_r].$$

Together with (6.3), this completes the proof of the lemma. $\quad\square$

Observe that Theorem 1.1 also holds for the simple random walk on the honeycomb grid, because two steps on the honeycomb lattice are the same as a single step on a triangular grid containing every other vertex on the honeycomb grid, and so Lemma 6.2 may be applied.

We now turn our attention to spanning trees and the generalizations of Corollary 1.2 and Theorem 1.3. Suppose that X and $-X$ have the same distribution, so that the walk S is reversible. For an edge $e = [x, y]$, define $p_e = \mathbf{P}[X = y - x] = \mathbf{P}[X = x - y]$. In this case, it is easy to generalize the definition of UST to a measure on trees related to the law of X. This can be done either by using Wilson's algorithm or, equivalently, by giving to each tree T a probability that is proportional to the product of the transition probabilities along the edges of T. In other words, $\mathbf{P}[T] = Z^{-1} \prod_{e \in T} p_e$, where Z is a normalizing constant. (The equivalence is proved in [44]; see also [26].) We call this the UST corresponding to the walk S (even if this probability measure is not uniform). Note that Lemma 4.1 also holds in the present setting because the probability $\mathbf{P}[T]$ is given in terms of a product.

THEOREM 6.3. *Assuming that $-X$ has the same distribution as X (i.e., S_j is reversible), Corollary 1.2 and Theorem 1.3 hold for the UST corresponding to the walk S.*

PROOF. The proof of Corollary 1.2 holds in this generality. In the proof of Theorem 1.3, the only significant changes concern the discrete harmonic conjugate function, used in the proof of Proposition 4.2. Recall that there is an appropriate definition for the discrete harmonic conjugate for reversible walks on planar graphs, where the discrete Cauchy–Riemann equation is modified (see [7] or [19], Section 6.1). If G is graph isomorphic to the square grid, the same is true for the dual graph. If G is graph isomorphic to the triangular grid, then the dual is graph isomorphic to the honeycomb grid. As pointed out above, Lemma 6.2 may therefore be applied to the harmonic conjugate. The details are left to the reader. □

In the nonreversible setting, instead of a spanning tree, one should consider a spanning arborescence, which is an oriented tree with a root and the edges are oriented toward the root. Fix a finite Markov chain with state space V and a root $o \in V$. Consider the measure on spanning arborescences of V with root o, where the probability for T is proportional to the product of the transition probabilities along the directed edges of T. This is the analog of the UST in the nonreversible setting. Wilson's algorithm holds in this generality (see [44]); however, the choice of the root o clearly matters.

If we consider a finite piece of the lattice L, and we wire part or all of the boundary, it is natural to pick the wired vertex as the root. With this convention, Corollary 1.2 holds for the wired tree. It would be interesting to see if the free tree with root chosen at $0 \in D$ is invariant under conformal maps preserving 0, say (in the nonreversible setting). Of course, one needs to choose a grid approximation of D where there is an oriented path from each vertex to the root 0.

In the proof of Theorem 1.3, we have used reversibility in two places. The proofs of Theorems 10.7 and 11.1 of [38], which we quoted, currently require reversibility. However, these results were only used to improve the topology of convergence to SLE. More seriously, Section 5.4 uses the conjugate harmonic function, whose definition in the nonreversible setting is not clear. Notwithstanding the obstacles, it seems likely that these results can be proven in the nonreversible setting, too.

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